UNBIASEDNESS OF HOMOGENEITY TEST OF NORMAL MEAN VECTORS UNDER MULTIVARIATE ORDER RESTRICTIONS

A Dissertation by

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The following faculty members have examined the final copy of this dissertation for form and content, and recommend that it be accepted in partial fulfillment of the requirement for the degree of Doctor of Philosophy with a major in Applied Mathematics.

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DEDICATION

To my parents, siblings, nieces and nephews, my future wife and childern, my friends, teachers and all human beings ...

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ABSTRACT

This dissertation considers homogeneity test for comparing multivariate normal populations with generalized order restricted alternative hypothesis. The framework in this dissertation presents a generalized multivariate order for the mean vectors. This order is induced from a closed convex cone in Hilbert space without any specifics on particular structures of the cones. Such cones are used to express the generalized order restricted alternative hypothesis. This dissertation derives the restricted maximum likelihood estimators (RMLEs) for the mean vectors under multivariate order restrictions, develops the likelihood ratio tests (LRTs) for the hypotheses about the restricted mean vectors. The statistical procedures are described through the projections onto closed convex cones. These closed convex cones are used to describe both the null and alternative hypotheses. The main result of this work is establishing the unbiasedness of the homogeneity test.

TABLE OF CONTENTS

CHAPTER 1
INTRODUCTION

In statistical inference, the problem of comparing several normal populations is classical and of considerable interest. It is often desirable to test the null hypothesis that the means are all equal. Such hypothesis is refered to as the homogeneity hypothesis in the field of statistical inference. The classical tests have been developed for the extreme situation in which the alternative hypothesis is unrestricted i.e. at least two of the means of these populations are not equal. Using the likelihood ratio procedure, the classical χ^2-tests have been developed for testing these hypotheses when the variances are assumed to be known. If the variances are assumed to be not known, the classical F-tests have been develpoed for this case.

1.1 Univariate order restriction

In some applications, a researcher may believe a priori that the ordering of the means is known. The likelihood ratio test for homogeneity of normal means with order restricted alternatives have been developed and studied since the second half of the previous century. Bartholomew (1959) considered a simple ordering over the normal means

$$H_0 : \mu_1 = \mu_2 = ... = \mu_q \text{ versus } H_a : \mu_1 \leq \mu_2 \leq ... \leq \mu_q$$

Since then, diffrent types of order restrictions have been imposed on the normal means, some of them are

$$H_a \; : \; \mu_1 \geq \mu_2 \geq ... \geq \mu_q \text{ (monotone decreasing)}$$

$$H_a \; : \; \mu_1 \leq \mu_j \text{ for } j = 2, 3, ..., q \text{ (simple tree ordering)}$$

$$H_a \; : \; \mu_1 \leq \mu_2 \leq ... \leq \mu_j \geq \mu_{j+1} \geq \mu_{j+2} \geq ... \geq \mu_q \text{ (umberlla ordering)}$$

These types of problems gave rise to the topic of Order Restricted Staistical Inference. Therefore, a general order restriction alternative hypothesis has been considered. This was done using closed convex cones in Hilbert spaces to define a general order resticted alternative hypothesis. Such problem has been studied extensively. The test developed here is a generalization of the χ^2-test and is refered to as the $\overline{\chi}^2$-test. Properties of this test, such as consistency and unbiasedness as well as monotonicity of its power function, have been investigated. Mukerjee et al (1986) established the unbiasedness for this test. Robertson et al (1988) and Silvapulle et al (2005) summerized the results of the work done in this field.

1.2 Multivariate order restriction

Sasabuchi et al (1983) presented a multivariate generalization of the order restricted statistical inference analysis, with known covariances, discussed by Bartholomew (1959,1961) , Barlow et al (1972) and Robertson et al (1988). $\leq$, as well as $\geq$ and $=$, are orders of real numbers that are reflexive ($x \leq x$ for all $x \in R$), transitive ($x \leq y$ and $y \leq z \Rightarrow x \leq z$), and preservable under multiplication with non-negative numbers, ($x \leq y \Rightarrow \alpha x \leq \alpha y$ for all $\alpha \geq 0$), under addition ($x \leq y$ and $u \leq v \Rightarrow x + u \leq y + v$) and under limit ($x_n \leq x$, $y_n \leq y$, $x_n \to x$ and $y_n \to y \Rightarrow x \leq y$). Such relations extended to R^p can be utilized to describe various prior knowledge about parameter vectors in statistical inferences including the ones considered by Bartholomew (1961) and by Sasabuchi et al. (1983) where one parameter vector is componentwise less than or equal to the other one; the ones studied by Hu et al (2012) where the columns of a parameter matrix are constrained by synchronized orders; and the ones explored by Pincus (1975), by Conaway et al. (1990, 1991) where one parameter vector is from the other one in an approximate given direction. In this paper we show that a relation of vectors in R^p possesses the five properties mentioned above if and only if it is induced from a closed convex cone in a way pointed out by Cohen et al (1996). The cone-induced multivariate orders contain diversified relations and have greater potential in restricted statistical inferences.

The research on multivariate order restricted vectors is not as extensively explored as that on univariate variables. The study on restricted univariate parameters has many results and examples in Barlow et al. (1972), in Robertson et al. (1988) and in Silvapulle et al (2005).

For multivariate order restrictions many researchers assume that the order is componentwise univariate orders such as in Nüesch (1966), in Wang et al (1998), in Sasabuchi et al. (2003) and in Sasabuchi (2007).

The framework in this paper assumes that the order is induced from a closed convex cone without any specifics on particular structures of the cones. For the model of q normal populations this paper derives the restricted maximum likelihood estimators (RMLEs) for the mean vectors under multivariate order restrictions, develops the likelihood ratio tests (LRTs) for the hypotheses about the restricted mean vectors, and establishes the unbiasedness for the LRTs. Both the null and alternative hypotheses are generally described in terms of closed convex cones. The statistical procedures are generally described through the projections onto those closed convex cones.

CHAPTER 2

PRELIMINARIES

2.1 Order in R^p and order restriction cone in $R^{p\times q}$

In this section, cone-induced multivariate-order for vectors in R^p is defined; also, sufficient and necessary conditions for such orders are established. These orders will be used to define order resrictions on matrices in $R^{p\times q}$. The section begins with some definitions and lemmas.

Definition 1. **(a)** *Let A be a set of elements (may or may not be a set in a linear space), and $\ll$ be a binary relation for the elements in A such that $\ll$ is reflexive, $x \ll x$ for all $x \in A$; and $\ll$ is transitive, if $x \ll y$ and $y \ll z$, then $x \ll z$. Then $\ll$ is called a quasi order in A.*

(b) *Let C be a set in a linear space V. Set C is a cone if $x \in C$ implies $\alpha x \in C$ for all $\alpha \geq 0$.*

(c) *Let C be a set in a linear space V. Set C is a convex set if $x, y \in C$ imply $\alpha x + (1-\alpha)y \in C$, for all $\alpha \in (0,1)$.*

(d) *Let C be a set in a linear space V with norm defined. Set C is closed if $x^{(n)} \in C$ and $x^{(n)} \to x$ imply $x \in C$.*

Lemma 2.1.1. **(a)** *C is a convex cone if and only if $x, y \in C$ imply $\alpha x + \beta y \in C$, for all α, $\beta \geq 0$.*

(b) *The intersection of two closed convex cones is also a closed convex cone.*

(c) *Let C be a convex cone in a linear space V. Define $x \ll y$, or equivalently $y \gg x$, if $x, y \in V$ and $y - x \in C$. Then $\ll$ is a quasi order and is preservable under linear combinations with non-negative coefficients ($x \ll y$ and $u \ll v \Rightarrow \alpha x + \beta u \ll \alpha y + \beta v$ for all $\alpha, \beta \geq 0$). This $\ll$ is called the order induced from C.*

(d) *Let $\ll$ be a quasi order induced from a closed convex cone C in a linear space V. Then $\ll$ is preserved under limit ($x_n \ll y_n$, $x_n \to x$ and $y_n \to y \Rightarrow x \ll y$).*

The proof of Lemma 2.1.1. is directly based on the definitions in Definition 1, and is hence skipped.

Theorem 2.1.1. *Let $\ll$ be a relation for vectors in R^p.*

(a) *$\ll$ is a quasi order preservable under linear combinations with non-negative coefficients if and only if it is induced from a convex cone.*

(b) *$\ll$ is a quasi order preservable under linear combinations with non-negative coefficients and under limit if and only if it is induced from a closed convex cone.*

Proof : (a) The "if part" is part (c) of Lemma 2.1.1. We now show the "only if" part.

Suppose quasi order $\ll$ is preservable under linear combinations with non-negative coefficients. Let $C = \{x \in R^p : x \gg 0\}$. From (a) of Lemma 2.1.1. one can see that C is a convex cone. Clearly, $\ll$ is induced from C.

(b) The "if part" is part (d) of Lemma 2.1.1. We now show the "only if" part. Suppose quasi order $\ll$ is preservable under linear combinations with non-negative coefficients and preservable under limit. Let $C = \{x \in R^p : x \gg 0\}$. One can check that C is a closed convex cone. Clearly, $\ll$ is induced from this cone. □

Suppose $\ll$ is an order for vectors in R^p induced from a closed convex cone C in R^p. For vectors $x_1, ..., x_q$ where $x_i \in R^p$, the restriction $x_i \ll x_j$ for $(i, j) \in H$ where H is a specified subset of $\{1, ..., q\} \times \{1, ..., q\}$, is called an order restriction on $x_1, ..., x_q$, or on matrix $X = (x_1, ..., x_q) \in R^{p \times q}$. A matrix that satisfies this order restriction is called an isotonic matrix. With underlying H, let $C_{p\times q} \subset R^{p\times q}$ be the collection of all isotonic matrices,

$$C_{p\times q} = \{X = (x_1, ..., x_q) \in R^{p\times q} : x_i \ll x_j, (i, j) \in H\} \tag{2.1}$$

The theorem below identifies $C_{p\times q}$ as a closed convex cone in $R^{p\times q}$.

Theorem 2.1.2. *Let $C_{p\times q}$ be as defined in (2.1). Then $C_{p\times q}$ is a closed convex cone in $R^{p\times q}$.*

Proof : Suppose $A = (A_1, ..., A_q)$ and $B = (B_1, ..., B_q)$ are both in $C_{p\times q}$. Then $\alpha A + \beta B = (\alpha A_1 + \beta B_1, ..., \alpha A_q + \beta B_q)$ with $\alpha \geq 0$ and $\beta \geq 0$. For $(i, j) \in H$,

$A_i \ll A_j$ and $B_i \ll B_j$ since $A \in C_{p\times q}$ and $B \in C_{p\times q}$. But $\ll$ is preservable under linear combinations with non-negative coefficients. So, $\alpha A_i + \beta B_i \ll \alpha A_j + \beta B_j$ i.e. $\alpha A + \beta B \in C_{p\times q}$. By (a) of Lemma 2.1.1. $C_{p\times q}$ is a convex cone.

Note that $R^{p\times q}$ is a Hilbert space in which there is a norm induced from an inner product such that the convergence of sequences in $R^{p\times q}$ is component wise. Now, suppose $A^{[n]} = (A_1^{[n]}, ..., A_q^{[n]}) \in C_{p\times q}$ and $A^{[n]} \to A = (A_1, .., A_q)$. Then for $(i, j) \in H$, $A_i^{[n]} \ll A_j^{[n]}$, $A_i^{[n]} \to A_i$ and $A_j^{[n]} \to A_j$. Consequently $A_i \ll A_j$, i.e., $A \in C_{p\times q}$. Thus $C_{p\times q}$ is closed. $\square$

2.2 Projections onto closed convex cones

Projections play an important role in maximization and minimization in statistical inference under order restrictions. With reference to Rudin (1987), we present the definition of projection below.

Definition 2. *Let A be a closed convex set and x be a vector in Hilbert space $\mathcal{H}$ equipped with inner product $\langle \cdot, \cdot \rangle$ and induced norm $\|\cdot\|$. x^* is called the minimum distance projection of x onto A if $x^* \in A$ and $\|x - x^*\| = inf\{\|x - y\| : y \in A\}$. This projection is denoted by $x^* = \pi(x|A)$.*

It is a well known fact in the geometry of Hilbert space that $x^* = \pi(x|A)$, the projection of x onto closed convex set A in Hilbert space $\mathcal{H}$, exists and is unique. Moreover, see Lemma 1.1 of E. H. Zarantonello (1971),

$$x^* = \pi(x|A) \Leftrightarrow x^* \in A, \text{ and } \langle x - x^*, x^* - y\rangle \geq 0 \text{ for all } y \in A. \tag{2.2}$$

The next two lemmas give the sufficient and necessary conditions for determining the projections onto closed convex cones and closed linear subspaces, two special cases of closed convex sets. These two types of sets are encountered in statistical inferences under order restrictions.

Lemma 2.2.1. *The projection of x onto closed convex cone C, a special closed convex set, $\pi(x|C)$ exists, is unique, and*

$$x^* = \pi(x|C) \Leftrightarrow x^* \in C, \langle x - x^*, x^*\rangle = 0 \text{ and } \langle x - x^*, y\rangle \leq 0 \text{ for all } y \in C.$$

Proof : The proof of the "if part" is trivial. We now show the "only if" part. Suppose $x^* = \pi(x|C)$. Then $x^* \in C$ is clearly true. Note that $0 \in C$. So, $\langle x - x^*, x^*\rangle = \langle x - x^*, x^* - 0\rangle \geq 0$ by (2.2). But, $x^* \in C$ implies $2x^* \in C$. Thus, $\langle x - x^*, x^*\rangle = -\langle x - x^*, x^* - 2x^*\rangle \leq 0$ by (2.2). So $\langle x - x^*, x^*\rangle = 0$. With $y \in C$, by (2.2) again, $\langle x - x^*, y\rangle = -\langle x - x^*, x^* - y\rangle \leq 0$. □

Lemma 2.2.2. *The projection of x onto a closed linear subspace L, a special closed convex cone, $\pi(x|L)$ exists, is unique, and*

$$x^* = \pi(x|L) \Leftrightarrow x^* \in L, \text{ and } \langle x - x^*, y\rangle = 0 \text{ for all } y \in L.$$

Proof : The proof of the "if part" is trivial. We now show the "only if" part. Suppose $x^* = \pi(x|L)$. Then $x^* \in L$ is clearly true. Suppose $y \in L$. Then $-y \in L$ also. Thus, by Lemma 2.2.1., $\langle x - x^*, \pm y\rangle \leq 0$ since L is a closed convex cone. Hence, $\langle x - x^*, y\rangle = 0$. □

The following lemma presents some relations for the projections onto closed linear subspace and closed convex cone when the closed linear subspace is contained in a closed convex cone. These relations will be utilized to write differnt forms of the test statistic for the hypothesis test under consideration. The test will be presented in chapter 3.

Lemma 2.2.3. *If L is a closed linear subspace, C is a closed convex cone, and $L \subset C$, then*

(a) $\|x - \pi(x|L)\|^2 - \|x - \pi(x|C)\|^2 = \|\pi(x|L) - \pi(x|C)\|^2$ *for all x.*

(b) *If $y \in L$, then $\pi(x \pm y|L) = \pi(x|L) \pm y$ for all x.*

(c) *If $y \in L$, then $\pi(x \pm y|C) = \pi(x|C) \pm y$ for all x.*

(d) *If $y \in L$, then $\|\pi(x \pm y|L) - \pi(x \pm y|C)\|^2 = \|\pi(x|L) - \pi(x|C)\|^2$ for all x.*

Proof : (a) $\langle x - \pi(x|C), \pi(x|C) - \pi(x|L)\rangle = \langle x - \pi(x|C), \pi(x|C)\rangle - \langle x - \pi(x|C), \pi(x|L)\rangle$.

The first inner product on the right, by Lemma 2.2.1., is 0. The second inner product, by Lemma 2.2.1., is also 0 since $\pi(x|L) \in L \Rightarrow \pm\pi(x|L) \in L \subset C$ which implies $\langle x - \pi(x|C), \pm\pi(x|L)\rangle \leq 0$. Thus $[x - \pi(x|C)] \perp [\pi(x|C) - \pi(x|L)]$ and so by Pythagorean Theorem, $\|x - \pi(x|L)\|^2 = \|x - \pi(x|C)\|^2 + \|\pi(x|C) - \pi(x|L)\|^2$.

(b) If $y \in L,$ then $\pi(x|L) \pm y \in L$. For $z \in L$,

$\langle x \pm y - [\pi(x|L) \pm y], z\rangle = \langle x - \pi(x|L), z\rangle = 0$. So, by Lemma 2.2.2.,

$\pi(x|L) \pm y = \pi(x \pm y|L)$.

(c) If $y \in L \subset C,$ then $\pi(x|C) \pm y \in C$. So

$\langle (x \pm y) - [\pi(x|C) \pm y], \pi(x|C) \pm y\rangle = \langle x - \pi(x|C), \pi(x|C) \pm y\rangle = 0$. For $z \in C$,

$\langle (x \pm y) - [\pi(x|C) \pm y], z\rangle = \langle x - \pi(x|C), z\rangle \leq 0$

So, by Lemma 2.2.1., $\pi(x|C) \pm y = \pi(x \pm y|C)$.

(d) Applying parts (b) and (c), we see that if $y \in L$, then

$$\begin{aligned} \|\pi(x \pm y|L) - \pi(x \pm y|C)\|^2 &= \|[\pi(x|L) \pm y] - [\pi(x|C) \pm y]\|^2 \\ &= \|\pi(x|L) - \pi(x|C)\|^2. \ \square \end{aligned}$$

CHAPTER 3
THE HOMOGENEITY TEST UNDER MULTIVARIATE ORDER RESTRICTIONS

This chapter presents the homogeneity test for mean vectors of multi-variate normal populations under multi-variate order restrictions. It also presents the corresponding likelihood function, likelihood ratio and likelihood ratio test statistic.

3.1 The homogeneity test

Let $N(\mu_i, \Sigma)$, $i = 1, ..., q$, be q independent normal populations with known Σ and unknown $\mu_i \in R^p$. Assume that $\mu_i \ll \mu_j$ for $(i, j) \in H$ where $H \subset \{1, ..., q\} \times \{1, ..., q\}$ is a given set. We need to test $\mu_1 = \cdots = \mu_q$. Let

$$L_{p\times q} = \{\mu = (\mu_1, ..., \mu_q) \in R^{p\times q} :\ \mu_1 = \cdots = \mu_q\} \tag{3.1}$$

and $C_{p\times q}$ as defined in(2.1). Then $L_{p\times q}$ is a closed linear subspace in $R^{p\times q}$, and $L_{p\times q} \subset C_{p\times q}$. We consider test on

$$H_0 : \mu \in L_{p\times q} \text{ versus } H_a : \mu \in C_{p\times q} - L_{p\times q} \tag{3.2}$$

3.2 The maximized likelihood function

Let $X_{i1}, ..., X_{in_i}$, $i = 1, ..., q$, be a random sample from $N(\mu_i, \Sigma)$ with sample mean $\overline{X}_i \in R^p$.

Then, the likelihood function of μ can be expressed as

$$\begin{aligned} L(\mu) &= \textstyle\prod_{i=1}^{q} \prod_{j=1}^{n_i} (2\pi)^{-p/2} |\Sigma|^{-1/2} \exp[(-1/2)(x_{ij} - \mu_i)' \Sigma^{-1} (x_{ij} - \mu_i)] \\ &= (2\pi)^{-np/2} |\Sigma|^{-n/2} \exp[(-1/2) \textstyle\sum_{i=1}^{q} \sum_{j=1}^{n_i} (x_{ij} - \mu_i)' \Sigma^{-1} (x_{ij} - \mu_i)], \end{aligned}$$

where $n = n_1 + \cdots + n_q$.

Let $c = (2\pi)^{-np/2} |\Sigma|^{-n/2}$ and $d = \exp[(-1/2) \sum_{i=1}^{q} \sum_{j=1}^{n_i} (x_{ij} - \overline{X}_i)' \Sigma^{-1} (x_{ij} - \overline{X}_i)]$. Note that

$$\begin{aligned} \textstyle\sum_{i=1}^{q} \sum_{j=1}^{n_i} (x_{ij} - \overline{X}_i)' \Sigma^{-1} (\overline{X}_i - \mu_i) &= \textstyle\sum_{i=1}^{q} [\sum_{j=1}^{n_i} (x_{ij} - \overline{X}_i)'] \Sigma^{-1} (\overline{X}_i - \mu_i) \\ &= \textstyle\sum_{i=1}^{q} \ 0 \ \Sigma^{-1} (\overline{X}_i - \mu_i) = 0 \\ \text{and } \textstyle\sum_{i=1}^{q} \sum_{j=1}^{n_i} (\overline{X}_i - \mu_i)' \Sigma^{-1} (x_{ij} - \overline{X}_i) &= [\textstyle\sum_{i=1}^{q} \sum_{j=1}^{n_i} (x_{ij} - \overline{X}_i)' \Sigma^{-1} (\overline{X}_i - \mu_i)]' = 0' = 0. \end{aligned}$$

Then,

$$\begin{aligned} L(\mu) &= c \cdot \exp[(-1/2) \textstyle\sum_{i=1}^{q} \sum_{j=1}^{n_i} (x_{ij} - \overline{X}_i + \overline{X}_i - \mu_i)' \Sigma^{-1} (x_{ij} - \overline{X}_i + \overline{X}_i - \mu_i)] \\ &= c \cdot d \cdot \exp[(-1/2) \textstyle\sum_{i=1}^{q} n_i (\overline{X}_i - \mu_i)' \Sigma^{-1} (\overline{X}_i - \mu_i)]. \end{aligned}$$

For $A = (A_1, ..., A_q)$ and $B = (B_1, ..., B_q)$ in $R^{p \times q}$ define

$$\langle A, B \rangle_{p \times q} = \sum_{j=1}^{q} n_j A_j' \Sigma^{-1} B_j. \tag{3.3}$$

Then $\langle A, B \rangle_{p \times q}$ is an inner product in $R^{p \times q}$, i.e.,

(i) $\langle A, A\rangle_{p\times q} \geq 0$ for all $A \in R^{p\times q}$. $\langle A, A\rangle_{p\times q} = 0 \Leftrightarrow A = 0$

(ii) $\langle A, B\rangle_{p\times q} = \langle B, A\rangle_{p\times q}$

(iii) $\langle \alpha A + \beta B, C\rangle_{p\times q} = \alpha\langle A, C\rangle_{p\times q} + \beta\langle B, C\rangle_{p\times q}$.

Denote the norm induced from the inner product defined in (3.3) by $\|\cdot\|_{p\times q}$. With $\overline{X} = (\overline{X}_1, ..., \overline{X}_q) \in R^{p\times q}$, the likelihood function is finally expressed as

$$L(\mu) = c \cdot d \cdot \exp[(-1/2)\|\overline{X} - \mu\|^2_{p\times q}]$$

Since $L(\mu)$ is an increasing function of $\exp[(-1/2)\|\overline{X} - \mu\|^2_{p\times q}]$, it is a decreasing function of $\|\overline{X} - \mu\|^2_{p\times q}$. Therefore, $L(\mu)$ is maximized when $\|\overline{X} - \mu\|_{p\times q}$ is minimized.

Let D be a closed convex set in $R^{p\times q}$. By Definition 2, when $\mu \in D$, $L(\mu)$ is maximized at $\widehat{\mu} = \pi(\overline{X}|D)$ with

$$L(\widehat{\mu}) = c \cdot d \cdot \exp[(-1/2)\|\overline{X} - \pi(\overline{X}|D)\|^2_{p\times q}]$$

3.3 The likelihood ratio test statistic

For the homogeneity test in (3.2), where $L_{p\times q}$ and $C_{p\times q}$ are as defined in (2.1) and (3.1), respectively, the likelihood ratio is

$$\begin{aligned} \Lambda &= \frac{\max[L(\mu) \colon \mu \in C_{p\times q}]}{\max[L(\mu) \colon \mu \in L_{p\times q}]} = \frac{\exp[(-1/2)\,\|\overline{X} - \pi(\overline{X}|C_{p\times q})\|^2_{p\times q}]}{\exp[(-1/2)\,\|\overline{X} - \pi(\overline{X}|L_{p\times q})\|^2_{p\times q}]} \\ &= \exp\{(1/2)\,[\|\overline{X} - \pi(\overline{X}|L_{p\times q})\|^2_{p\times q} - \|\overline{X} - \pi(\overline{X}|C_{p\times q})\|^2_{p\times q}]\}. \end{aligned}$$

By (a) of Lemma 2.2.3., we see that

$$\Lambda = \exp[(1/2)\, \|\pi(\overline{X}|L_{p\times q}) - \pi(\overline{X}|C_{p\times q})\|^2_{p\times q}]$$

which is clearly an increasing function of

$$T(\overline{X}) = \|\pi(\overline{X}|L_{p\times q}) - \pi(\overline{X}|C_{p\times q})\|^2_{p\times q} \tag{3.4}$$

Thus, this $T(\overline{X})$ is a LRT statistic and H_0 is rejected for the large values of $T(\overline{X})$.

CHAPTER 4

PROPERTIES OF THE TEST STATISTIC OF THE HOMOGENEITY TEST

The first section in this chapter defines polar cones and lists some of their properties. These properties are utilized to express the homogeneity test statistic in multiple forms in the second section.

4.1 Polar cones

Suppose C is a closed convex cone in a Hilbert space equipped with inner product $\langle \cdot, \cdot \rangle$ and its induced norm $\| \cdot \|$. Define

$$C^{(p)} = \{x : \langle x, y \rangle \leq 0 \text{ for all } y \in C\} \tag{4.1}$$

Suppose x_1 and x_2 are in $C^{(p)}$. $\langle \alpha x_1 + \beta x_2, y \rangle = \alpha \langle x_1, y \rangle + \beta \langle x_2, y \rangle \leq 0$, for all α, $\beta \geq 0$ and $y \in C$. Thus $\alpha x_1 + \beta x_2 \in C^{(p)}$. So by (a) of Lemma 2.1.1. $C^{(p)}$ is a convex cone.

Suppose $x_n \in C^{(p)}$ and $x_n \to x$ as $n \to \infty$. With $y \in C$,

$|\langle x_n, y \rangle - \langle x, y \rangle| = |\langle x_n - x, y \rangle| \leq \|x_n - x\| \cdot \|y\| \to 0$, as $n \to \infty$. Thus, $\langle x_n, y \rangle \to \langle x, y \rangle$. But, $\langle x_n, y \rangle \leq 0$ since $y \in C$. So, $\langle x, y \rangle \leq 0$. Hence, $x \in C^{(p)}$. By (d) of Definition 1, $C^{(p)}$ is a closed set. The closed convex cone $C^{(p)}$ in (4.1) is called the polar cone, or the dual cone of C.

Some of the properties of dual cones are presented in the following lemma.

Lemma 4.1.1. *Let C, C_1 and C_2 be closed convex cones, and $C_1 \subset C_2$. Then*

(a) $C_2^{(p)} \subset C_1^{(p)}$.

(b) $\pi(x|C^{(p)}) = x - \pi(x|C)$.

(c) $\|\pi(x|C^{(p)})\|^2 = \|x\|^2 - \|\pi(x|C)\|^2$.

(d) $\|\pi(x - y|C)\| \geq \|\pi(x|C)\|$, *for all* $y \in C^{(p)}$.

Proof : (a) Let $x \in C_2^{(p)}$. For $y \in C_1$, $\langle x, y\rangle \leq 0$ since $y \in C_1 \subset C_2$. Thus $x \in C_1^{(p)}$. Conclusion $C_2^{(p)} \subset C_1^{(p)}$ follows.

(b) By Lemma 2.2.1. we need to show $x - \pi(x|C) \in C^{(p)}$; $\langle x - [x - \pi(x|C)], x - \pi(x|C)\rangle = 0$; and $\langle x - [x - \pi(x|C)], y\rangle \leq 0$ for all $y \in C^{(p)}$. Suppose $y \in C$. By Lemma 2.2.1. $\langle x - \pi(x|C), y\rangle \leq 0$. But, by (4.1), $x - \pi(x|C) \in C^{(p)}$.

Secondly, $\langle x - [x - \pi(x|C)], x - \pi(x|C)\rangle = \langle \pi(x|C), x - \pi(x|C)\rangle = 0$;

Finally, for $y \in C^{(p)}$, $\langle x - [x - \pi(x|C)], y\rangle = \langle \pi(x|C), y\rangle \leq 0$, by (4.1). Thus, (b) holds.

(c) Note that $x = [x - \pi(x|C)] + \pi(x|C)$ where $\langle x - \pi(x|C), \pi(x|C)\rangle = 0$. By Pythagorean Theorem,

$$\|x\|^2 = \|x - \pi(x|C)\|^2 + \|\pi(x|C)\|^2, \text{ i.e., } \|x\|^2 = \|\pi(x|C^{(p)})\|^2 + \|\pi(x|C)\|^2.$$

(d) For $y \in C^{(p)}$, by (b) and the definition of projection,

$$\begin{aligned} \|\pi(x - y|C)\| &= \|x - y - \pi(x - y|C^{(p)})\| = \|x - [y + \pi(x - y|C^{(p)})]\| \\ &\geq \|x - \pi(x|C^{(p)})\| = \|\pi(x|C)\|. \ \square \end{aligned}$$

Definition 3. *For linear subspace L, $L^{\perp} = \{x : \langle x, y\rangle = 0$ for all $y \in L\}$ is called the orthogonal complement of L.*

Clearly, $L^{\perp}$ is also a linear subspace since for x_1, $x_2 \in L^{\perp}$, with α, $\beta \in R$ and $y \in L$, $\langle \alpha x_1 + \beta x_2, y\rangle = \alpha\langle x_1, y\rangle + \beta\langle x_2, y\rangle = 0$, i.e., $\alpha x_1 + \beta x_2 \in L^{\perp}$.

Lemma 4.1.2. *Let L be a closed linear subspace, a special closed convex cone, and $L \subset C$, where C is a closed convex cone. Then*

(a) $L^{(p)} = L^{\perp}$.

(b) $\|\pi(x \pm y|L)\| \geq \|\pi(x|L)\|$, *for all $y \in C^{(p)}$.*

Proof : (a) By the definition $L^{\perp} \subset L^{(p)}$. We show $L^{(p)} \subset L^{\perp}$. Let $x \in L^{(p)}$. For $y \in L$, $\pm y \in L$. By (4.1), $\langle x, \pm y\rangle \leq 0$. Consequently, $\langle x, y\rangle = 0$. Thus, $x \in L^{\perp}$.

(b) Note that $y \in C^{(p)} \subset L^{\perp}$ implies that $\pm y \in L^{\perp}$. Hence, by (a) of this Lemma and (b) of Lemma 4.1.1.,

$$\begin{aligned} \|\pi(x \pm y|L)\| &= \|x \pm y - \pi(x \pm y|L^{\perp})\| = \|x - [\mp y + \pi(x \pm y|L^{\perp})]\| \\ &\geq \|x - \pi(x|L^{\perp})\| = \|\pi(x|L)\|. \ \square \end{aligned}$$

4.2 Different forms of the test statistic

For the test on the hypothesis in (3.2), a Likelihood ratio test statistic $T(\overline{X})$ has been derived in Chapter 3. This test statistic has several equivalent forms. In other words, $T(X) = \|\pi(X|L_{p\times q}) - \pi(X|C_{p\times q})\|^2_{p\times q}$ with $X \in R^{p\times q}$ has several equivalent expressions. These expressions are summerized in Lemma 4.2.1. below, and will be used in Chapter 5 to establish the unbiasedness of the test.

Lemma 4.2.1. $T(X) = \|\pi(X|L_{p\times q}) - \pi(X|C_{p\times q})\|^2_{p\times q}$ *has the following equivalent forms*

(a) $T(X) = \|\pi(X|L^{\perp}_{p\times q}) - \pi(X|C^{(p)}_{p\times q})\|^2_{p\times q}$

(b) $T(X) = \|\pi(X|C_{p\times q} \cap L^{\perp}_{p\times q})\|^2_{p\times q}$

(c) $T(X) = \|X - \pi(X|(C_{p\times q} \cap L^{\perp}_{p\times q})^{(p)})\|^2_{p\times q}$

(d) $T(X) = \|\pi(X \pm Y|L_{p\times q}) - \pi(X \pm Y|C_{p\times q})\|^2_{p\times q}$ *for* $Y \in L_{p\times q}$

Proof : (a) By (b) of Lemma 4.1.1. and (a) of Lemma 4.1.2. $\pi(X|C_{p\times q}) = X - \pi(X|C^{(p)}_{p\times q})$ and $\pi(X|L_{p\times q}) = X - \pi(X|L^{\perp}_{p\times q})$. Therefore,

$$\begin{aligned} T(X) &= \|\pi(X|L_{p\times q}) - \pi(X|C_{p\times q})\|^2_{p\times q} \\ &= \|\pi(X|L_{p\times q}) - X + X - \pi(X|C_{p\times q})\|^2_{p\times q} \\ &= \| - \pi(X|L^{\perp}_{p\times q}) + \pi(X|C^{(p)}_{p\times q})\|^2_{p\times q} = \|\pi(X|L^{\perp}_{p\times q}) - \pi(X|C^{(p)}_{p\times q})\|^2_{p\times q} \end{aligned}$$

(b) Note that by (b) of Lemma 2.1.1. $C_{p\times q} \cap L^{\perp}_{p\times q}$ is a closed convex cone. We show $\pi(X|C_{p\times q})-\pi(X|L_{p\times q}) = \pi(X|C_{p\times q}\cap L^{\perp}_{p\times q})$. First, $\pi(X|C_{p\times q})-\pi(X|L_{p\times q}) \in C_{p\times q}$ and for $Y \in L_{p\times q}$, $\langle \pi(X|C_{p\times q}) - \pi(X|L_{p\times q}), Y\rangle_{p\times q}$

$$\begin{aligned} &= \langle \pi(X|C_{p\times q}) - X + X - \pi(X|L_{p\times q}), Y\rangle_{p\times q} \\ &= \langle \pi(X|C_{p\times q}) - X, Y\rangle_{p\times q} + \langle X - \pi(X|L_{p\times q}), Y\rangle_{p\times q} \\ &= 0 + 0 = 0. \end{aligned}$$

So, $\pi(X|C_{p\times q}) - \pi(X|L_{p\times q}) \in L^{\perp}_{p\times q}$. Thus $\pi(X|C_{p\times q}) - \pi(X|L_{p\times q}) \in C_{p\times q} \cap L^{\perp}_{p\times q}$.

Secondly, $\langle X - [\pi(X|C_{p\times q}) - \pi(X|L_{p\times q})], \pi(X|C_{p\times q}) - \pi(X|L_{p\times q})\rangle_{p\times q}$

$$\begin{aligned} &= \langle X - \pi(X|C_{p\times q}), \pi(X|C_{p\times q}) - \pi(X|L_{p\times q})\rangle_{p\times q} \\ &\quad + \langle \pi(X|L_{p\times q}), \pi(X|C_{p\times q}) - \pi(X|L_{p\times q})\rangle_{p\times q} \\ &= 0 + \langle \pi(X|L_{p\times q}), \pi(X|C_{p\times q}) - \pi(X|L_{p\times q})\rangle_{p\times q} \\ &= \langle \pi(X|L_{p\times q}), \pi(X|C_{p\times q}) - X + X - \pi(X|L_{p\times q})\rangle_{p\times q} \\ &= \langle \pi(X|L_{p\times q}), \pi(X|C_{p\times q}) - X\rangle_{p\times q} + \langle \pi(X|L_{p\times q}), X - \pi(X|L_{p\times q})\rangle_{p\times q} \\ &= 0 + 0 = 0. \end{aligned}$$

Finally, $Y \in C_{p\times q} \cap L^{\perp}_{p\times q}$ implies that

$$
\begin{aligned}
& \langle X - [\pi(X|C_{p\times q}) - \pi(X|L_{p\times q})], Y\rangle_{p\times q} \\
= & \langle X - \pi(X|C_{p\times q}), Y\rangle_{p\times q} + \langle \pi(X|L_{p\times q}), Y\rangle_{p\times q} \\
= & \langle X - \pi(X|C_{p\times q}), Y\rangle_{p\times q} \leq 0;
\end{aligned}
$$

Thus by Lemma 2.2.1. $\pi(X|C_{p\times q}) - \pi(X|L_{p\times q}) = \pi(X|C_{p\times q} \cap L^{\perp}_{p\times q})$. Consequently, $T(X) = \|\pi(X|C_{p\times q} \cap L^{\perp}_{p\times q})\|^2_{p\times q}$.

(c) By (b) of this Lemma and (b) of Lemma 4.1.1. ,

$$T(X) = \|\pi(X|C_{p\times q} \cap L^{\perp}_{p\times q})\|^2_{p\times q} = \|X - \pi(X|(C_{p\times q} \cap L^{\perp}_{p\times q})^{(p)})\|^2_{p\times q}.$$

(d) Applying (d) of Lemma 2.2.3. gives (d). □

CHAPTER 5
THE PROBLEM OF UNBIASEDNESS

This chapter presents the problem of unbiasedness of the homogeneity test. In the first section, the definition of unbiased test is presented. The second section is devoted for developing a sufficient condition for the unbiasedness.

5.1 The definition of unbiased test

Because the distribution of $T(\overline{X})$ depends on μ, $P[T(\overline{X}) > t]$ is a function of μ that is denoted as $P[T(\overline{X}) > t|\mu]$. Let t_α be the α-level critical value, i.e., $\max[P(T(\overline{X}) > t_\alpha|\mu) : \mu \in L_{p\times q}] = \alpha$. Then the LRT that rejects $H_0 : \mu \in L_{p\times q}$ when $T(\overline{X}) > t_\alpha$ has significance level α. This test is called an α-level test. The α-level test is unbiased if

$$P[T(\overline{X}) > t_\alpha|\mu = \mu_*] \leq P[T(\overline{X}) > t_\alpha|\mu = \mu^*] \tag{5.1}$$

for all $\mu_* \in L_{p\times q}$ and all $\mu^* \in C_{p\times q}$. The rest of this dissertation is devoted for the establishment of the unbiasedness of the LRT.

5.2 Sufficient condition for the unbiasedness

This section develops a sufficient condition for the unbiasedness of the homogeneity test. The sufficient condition is presented in the following theorem. The theorem is preceded by a lemma. The lemma is the mere tool to prove the theorem.

Lemma 5.2.1. (a) $P[T(\overline{X}) > t_\alpha | \mu = \mu^*] = P[T(\overline{X}) > t_\alpha | \mu = \pi(\mu^* | L^\perp_{p\times q})]$.

(b) *If* $\mu_* \in L_{p\times q}$, *then* $P[T(\overline{X}) > t_\alpha | \mu = \mu_*] = P[T(\overline{X}) > t_\alpha | \mu = 0]$.

(c) *If* $\mu^* \in C_{p\times q}$, *then* $\pi(\mu^* | L^\perp_{p\times q}) \in C_{p\times q} \cap L^\perp_{p\times q}$.

Proof : (a) $P[T(\overline{X}) > t_\alpha | \mu = \mu^*] = P[T(\overline{X} - \pi(\mu^* | L_{p\times q})) > t_\alpha | \mu = \mu^*]$, by (d) of Lemma 4.2.1. since, by the definition of projection, $\pi(\mu^* | L_{p\times q}) \in L_{p\times q}$. But, the distribution of $\overline{X} - \pi(\mu^* | L_{p\times q})$ when $\mu = \mu^*$ and the distribution of $\overline{X}$ when $\mu = \mu^* - \pi(\mu^* | L_{p\times q}) = \pi(\mu^* | L^\perp_{p\times q})$ are equal. Hence,

$$P[T(\overline{X} - \pi(\mu^* | L_{p\times q})) > t_\alpha | \mu = \mu^*] = P[T(\overline{X}) > t_\alpha | \mu = \pi(\mu^* | L^\perp_{p\times q})].$$

(b) Suppose $\mu_* \in L_{p\times q}$. Then $\pi(\mu_* | L^\perp_{p\times q}) = \mu_* - \pi(\mu_* | L_{p\times q}) = \mu_* - \mu_* = 0$. By (a) of this Lemma,

$$P[T(\overline{X}) > t_\alpha | \mu = \mu_*] = P[T(\overline{X}) > t_\alpha | \mu = \pi(\mu_* | L^\perp_{p\times q})] = P[T(\overline{X}) > t_\alpha | \mu = 0].$$

(c) It suffices to show $\pi(\mu^* | L^\perp_{p\times q}) \in C_{p\times q}$ for $\mu^* \in C_{p\times q}$ since, by the definition of projection, $\pi(\mu^* | L^\perp_{p\times q}) \in L^\perp_{p\times q}$. By (b) of Lemma 4.1.1. and (a) of Lemma 4.1.2.

one may write, $\pi(\mu^*|L_{p\times q}^{\perp}) = \mu^* - \pi(\mu^*|L_{p\times q}) = \mu^* + [-\pi(\mu^*|L_{p\times q})]$. Both μ^* and $-\pi(\mu^*|L_{p\times q})$ are in $C_{p\times q}$. So is $\pi(\mu^*|L_{p\times q}^{\perp})$ by (a) of Lemma 2.1.1. Conclusion follows. □

The next theorem presents a sufficient condition for the unbiasedness of the homogeneity test.

Theorem 5.2.1. *The homogeneity test is unbiased if*

$$P[T(\overline{X}) > t_\alpha|\mu = 0] \le P[T(\overline{X}) > t_\alpha|\mu = \nu] \text{ for all } 0 \ne \nu \in C_{p\times q} \cap L_{p\times q}^{\perp}.$$

Proof : Recall the inequality in (5.1), the homogeneity test is unbiased if $P[T(\overline{X}) > t_\alpha|\mu = \mu_*] \le P[T(\overline{X}) > t_\alpha|\mu = \mu^*]$ for all $\mu_* \in L_{p\times q}$ and all $\mu^* \in C_{p\times q}$. By (a) and (b) of Lemma 5.2.1., the inequality above in (5.1) becomes

$$P[T(\overline{X}) > t_\alpha|\mu = 0] \le P[T(\overline{X}) > t_\alpha|\mu = \pi(\mu^*|L_{p\times q}^{\perp})]$$

for all $\mu^* \in C_{p\times q}$. By (c) of Lemma 5.2.1., $\pi(\mu^*|L_{p\times q}^{\perp}) \in C_{p\times q} \cap L_{p\times q}^{\perp}$. Thus, $P[T(\overline{X}) > t_\alpha|\mu = 0] \le P[T(\overline{X}) > t_\alpha|\mu = \nu]$ for all $0 \ne \nu \in C_{p\times q} \cap L_{p\times q}^{\perp}$ implies $P[T(\overline{X}) > t_\alpha|\mu = 0] \le P[T(\overline{X}) > t_\alpha|\mu = \pi(\mu^*|L_{p\times q}^{\perp}]$ for all $\mu^* \in C_{p\times q}$. □

CHAPTER 6

THE ESTABLISHMENT OF THE UNBIASEDNESS

6.1 The partition of the acceptance region

6.1.1 The acceptance region

By Theorem 5.2.1 the homogeneity test under the consideration is unbiased if $P[T(\overline{X}) > t_\alpha | \mu = 0] \leq P[T(\overline{X}) > t_\alpha | \mu = \nu]$ for all $0 \neq \nu \in C_{p\times q} \cap L^\perp_{p\times q}$ which is equivalent to $P[T(\overline{X}) \leq t_\alpha | \mu = 0] \geq P[T(\overline{X}) \leq t_\alpha | \mu = \nu]$ for all $0 \neq \nu \in C_{p\times q} \cap L^\perp_{p\times q}$. Define

$$A = \{X \in R^{p\times q} : T(X) \leq t_\alpha\} \tag{6.1}$$

Here A is called the acceptance region since the H_0 is accepted if the observed $\overline{X}$ is in A. The inequality, as a sufficient condition for the unbiasedness, becomes

$$P(\overline{X} \in A | \mu = 0) \geq P(\overline{X} \in A | \mu = \nu) \text{ for all } 0 \neq \nu \in C_{p\times q} \cap L^\perp_{p\times q} \tag{6.2}$$

Lemma 6.1.1. *If A in (6.1) is partitioned into A_1, A_2 and A_3 such that*

$$\begin{aligned} P(\overline{X} \in A_1 | \mu = 0) &= P(\overline{X} \in A_2 | \mu = \nu) \\ P(\overline{X} \in A_2 | \mu = 0) &= P(\overline{X} \in A_1 | \mu = \nu) \\ P(\overline{X} \in A_3 | \mu = 0) &\geq P(\overline{X} \in A_3 | \mu = \nu) \end{aligned}$$

for all $0 \neq \nu \in C_{p\times q} \cap L^{\perp}_{p\times q}$*, then the likelihood ratio test under the consideration is unbiased.*

Proof : Under the condition of this Lemma (6.2) is true. Hence, the test is unbiased. $\square$

6.1.2 Two subsets of the acceptance region

Let $A_1 = \{X \in A : \langle X - \nu/2, \nu\rangle_{p\times q} > 0\}$. Then A_1 is a subset of A. For $X \in R^{p\times q}$ define a transformation

$$f(X) = X - \frac{2\,\langle X - \nu/2,\, \nu\rangle_{p\times q}}{\|\nu\|^2_{p\times q}}\nu \tag{6.3}$$

where $0 \neq \nu \in C_{p\times q} \cap L^{\perp}_{p\times q}$. Clearly, if $Y = f(X)$, then

$$\|Y\|^2_{p\times q} = \|X\|^2_{p\times q} - 2\langle X - \nu/2,\, \nu\rangle_{p\times q} \tag{6.4}$$

$$\begin{aligned}
\text{since} \qquad \|Y\|^2_{p\times q} &= \|X - \tfrac{2\,\langle X-\nu/2,\,\nu\rangle_{p\times q}}{\|\nu\|^2_{p\times q}}\nu\|^2_{p\times q} \\
&= \|X\|^2_{p\times q} - \tfrac{4\,\langle X-\nu/2,\,\nu\rangle_{p\times q}}{\|\nu\|^2_{p\times q}}\langle X,\, \nu\rangle_{p\times q} + \|\tfrac{2\,\langle X-\nu/2,\,\nu\rangle_{p\times q}}{\|\nu\|^2_{p\times q}}\nu\|^2_{p\times q} \\
&= \|X\|^2_{p\times q} + \tfrac{4\,\langle X-\nu/2,\,\nu\rangle_{p\times q}}{\|\nu\|^2_{p\times q}}\langle -X,\, \nu\rangle_{p\times q} + \tfrac{4\,\langle X-\nu/2,\,\nu\rangle_{p\times q}}{\|\nu\|^2_{p\times q}}\langle X - \nu/2,\, \nu\rangle_{p\times q} \\
&= \|X\|^2_{p\times q} + \tfrac{4\,\langle X-\nu/2,\,\nu\rangle_{p\times q}}{\|\nu\|^2_{p\times q}}\langle -X + X - \nu/2,\, \nu\rangle_{p\times q} \\
&= \|X\|^2_{p\times q} - 2\langle X - \nu/2,\, \nu\rangle_{p\times q}\,.
\end{aligned}$$

Lemma 6.1.2. *With* A_1 *and* $f(X)$ *defined above, let* $A_2 = f(A_1)$*. Then, both* A_1 *and* A_2 *are subsets of* A*.*

Proof : It is known that A_1 is a subset of A. We now show that A_2 is also a subset of A. Suppose $Y = f(X) \in A_2$ where $X \in A_1 \subset A$. Then $\langle X - \nu/2, \nu\rangle_{p\times q} > 0$.

To show $Y \in A$ we need to show $T(Y) \leq t_\alpha$. By (6.4),

$\|Y\|^2_{p\times q} = \|X\|^2_{p\times q} - 2\,\langle X - \nu/2,\, \nu\rangle_{p\times q} < \|X\|^2_{p\times q}$. By (b) and (c) of Lemma 4.1.1. and

(b) of Lemma 4.2.1.

$$\begin{aligned} T(X) &= \|\pi(X|C_{p\times q} \cap L^{\perp}_{p\times q})\|^2_{p\times q} = \|X\|^2_{p\times q} - \|\pi(X|(C_{p\times q} \cap L^{\perp}_{p\times q})^{(p)})\|^2_{p\times q} \\ &= \|X\|^2_{p\times q} - \|X - \pi(X|C_{p\times q} \cap L^{\perp}_{p\times q})\|^2_{p\times q} \\ \text{and } T(Y) &= \|Y\|^2_{p\times q} - \|Y - \pi(Y|C_{p\times q} \cap L^{\perp}_{p\times q})\|^2_{p\times q}. \end{aligned}$$

$$\begin{aligned} \text{But, } \|Y - \pi(Y|C_{p\times q} \cap L^{\perp}_{p\times q})\|^2_{p\times q} &= \|X - \tfrac{2\,\langle X-\nu/2,\nu\rangle_{p\times q}}{\|\nu\|^2_{p\times q}}\nu - \pi(Y|C_{p\times q} \cap L^{\perp}_{p\times q})\|^2_{p\times q} \\ &= \|X - [\tfrac{2\,\langle X-\nu/2,\nu\rangle_{p\times q}}{\|\nu\|^2_{p\times q}}\nu + \pi(Y|C_{p\times q} \cap L^{\perp}_{p\times q})]\|^2_{p\times q} \\ &\geq \|X - \pi(X|C_{p\times q} \cap L^{\perp}_{p\times q})]\|^2_{p\times q}. \end{aligned}$$

$$\begin{aligned} \text{Thus, } T(Y) &= \|Y\|^2_{p\times q} - \|Y - \pi(Y|C_{p\times q} \cap L^{\perp}_{p\times q})\|^2_{p\times q} \\ &\leq \|X\|^2_{p\times q} - \|X - \pi(x|C_{p\times q} \cap L^{\perp}_{p\times q})\|^2_{p\times q} = T(X) \leq t_\alpha. \end{aligned}$$

So, $Y \in A$. $\square$

6.1.3 Partition of the acceptance region

Define $A_3 = A \cap (A_1 \cup A_2)^c$. Then $A = A_1 \cup A_2 \cup A_3$ where $A_1 \cap A_3 = \emptyset$ and $A_2 \cap A_3 = \emptyset$.

If $A_1 \cap A_2 = \emptyset$, then A is partitioned into A_1, A_2 and A_3. We establish $A_1 \cap A_2 = \emptyset$ below.

Suppose $Y = f(X) \in A_2$ where $X \in A_1 \subset A$. We show

$$\langle Y - \nu/2,\, \nu\rangle_{p\times q} = -\langle X - \nu/2,\, \nu\rangle_{p\times q} < 0 \tag{6.5}$$

so that $Y \notin A_1$ and consequently $A_1 \cap A_2 = \emptyset$.

$$\begin{aligned}
\langle Y - \nu/2,\, \nu \rangle_{p\times q} &= \langle X - \tfrac{2\,\langle X - \nu/2,\, \nu \rangle_{p\times q}}{\|\nu\|^2_{p\times q}} \nu - \nu/2,\, \nu \rangle_{p\times q} \\
&= \langle X, \nu \rangle_{p\times q} - \tfrac{2\,\langle X - \nu/2,\, \nu \rangle_{p\times q}}{\|\nu\|^2_{p\times q}} \langle \nu,\, \nu \rangle_{p\times q} - \langle \nu/2,\, \nu \rangle_{p\times q} \\
&= \langle X, \nu \rangle_{p\times q} - 2\,\langle X - \nu/2,\, \nu \rangle_{p\times q} - \langle \nu/2,\, \nu \rangle_{p\times q} \\
&= \langle X - \upsilon/2, \nu \rangle_{p\times q} - 2\,\langle X - \nu/2,\, \nu \rangle_{p\times q} \\
&= -\langle X - \nu/2,\, \nu \rangle_{p\times q} < 0.
\end{aligned}$$

Therefore, it is concluded that A_1, A_2 and A_3 partition A.

6.2 Two equalities

In this section we establish the two equations in Lemma 6.1.1.

$$\begin{aligned}
P(\overline{X} \in A_1 | \mu = 0) &= P(\overline{X} \in A_2 | \mu = \nu) \\
P(\overline{X} \in A_2 | \mu = 0) &= P(\overline{X} \in A_1 | \mu = \nu)
\end{aligned} \tag{6.6}$$

for $0 \neq \nu \in C_{p\times q} \cap L^{\perp}_{p\times q}$.

6.2.1 Two equivalent equations

Let $Y = f(X)$ be the transformation in (6.3). We claim that

$$f[f(X)] = X \;\text{ for all }\; X \in R^{p\times q}$$

Let $Y = f(X)$. Then $Y = X - \frac{2\langle X-\nu/2, \nu\rangle_{p\times q}}{\|\nu\|^2_{p\times q}}\nu$. By (6.3) and (6.5),

$$\begin{aligned} f[f(X)] &= f(Y) = Y - \frac{2\langle Y-\nu/2, \nu\rangle_{p\times q}}{\|\nu\|^2_{p\times q}}\nu \\ &= X - \frac{2\langle X-\nu/2, \nu\rangle_{p\times q}}{\|\nu\|^2_{p\times q}}\nu - \frac{2\langle Y-\nu/2, \nu\rangle_{p\times q}}{\|\nu\|^2_{p\times q}}\nu \\ &= X - \frac{2\langle X-\nu/2, \nu\rangle_{p\times q}}{\|\nu\|^2_{p\times q}}\nu + \frac{2\langle X-\nu/2, \nu\rangle_{p\times q}}{\|\nu\|^2_{p\times q}}\nu \\ &= X. \end{aligned}$$

The claim is established. Based on this claim, the transformation in (6.3) is a 1-1 transformation. With this result, the next lemma gives two equivalent equations to those in Lemma 6.1.1.

Lemma 6.2.1. *The two equations in Lemma 6.1.1. are equivalent to* $P[f(\overline{X}) \in A_2|\mu = 0] = P(\overline{X} \in A_2|\mu = \nu)$ *and* $P[f(\overline{X}) \in A_1|\mu = 0] = P(\overline{X} \in A_1|\mu = \nu)$ *for all* $0 \neq \nu \in C_{p\times q} \cap L^{\perp}_{p\times q}$.

Proof : The two equations in Lemma 6.1.1. are $P(\overline{X} \in A_1|\mu = 0) = P(\overline{X} \in A_2|\mu = \nu)$ and $P(\overline{X} \in A_2|\mu = 0) = P(\overline{X} \in A_1|\mu = \nu)$ for $0 \neq \nu \in C_{p\times q} \cap L^{\perp}_{p\times q}$. But with the 1-1 transformation in (6.3),

$$P(\overline{X} \in A_1|\mu = 0) = P[f(\overline{X}) \in f(A_1)|\mu = 0] = P[f(\overline{X}) \in A_2|\mu = 0],$$

$$P(\overline{X} \in A_2|\mu = 0) = P[f(\overline{X}) \in f(A_2)|\mu = 0] = P[f(\overline{X}) \in A_1|\mu = 0].$$

Conclusion follows. $\square$

6.2.2 Distributions of $\overline{X}$ and $f(\overline{X})$

The distribution of $\overline{X} = (\overline{X}_1, ..., \overline{X}_q)$ at $\mu = (\mu_1, ..., \mu_q)$ where $\overline{X}_i \sim N\left(\mu_i, \frac{1}{n_i}\Sigma\right)$, $i = 1, ..., q$, are independent is characterized by that of $\text{vec}(\overline{X})$ at $\text{vec}(\mu)$. Clearly with $N = \text{diag}(n_1, ..., n_q)$

$$\text{vec}(\overline{X}) \sim N\left(\text{vec}(\mu),\, N^{-1} \otimes \Sigma\right) \tag{6.7}$$

Rewrite (6.3) as

$$Y = f(X) = X - \frac{2\langle X,\, \nu\rangle_{p\times q}}{\|\nu\|^2_{p\times q}}\nu + \nu$$

where $\langle X,\, \nu\rangle_{p\times q} = \sum_j n_j X_j' \Sigma^{-1}\nu_j = \sum_j X_j' (\frac{\Sigma}{n_j})^{-1}\nu_j = (X_1', ..., X_q')\, diag(n_1\Sigma^{-1}, ..., n_q\Sigma^{-1})\, (\nu_1', ..., \nu_q')'$ using (3.3). Hence,

$$\langle X,\, \nu\rangle_{p\times q} = [\text{vec}(X)]'(N \otimes \Sigma^{-1})\text{vec}(\nu) = [\text{vec}(\nu)]'(N \otimes \Sigma^{-1})\text{vec}(X) \tag{6.8}$$

Therefore,

$$\begin{aligned}
\text{vec}[f(X)] &= \text{vec}(X) - \tfrac{2}{\|\nu\|^2_{p\times q}}\, [\text{vec}(\nu)]'(N \otimes \Sigma^{-1})\text{vec}(X)\text{vec}(\nu) + \text{vec}(\nu) \\
&= \text{vec}(X) - \tfrac{2}{\|\nu\|^2_{p\times q}}\, \text{vec}(\nu)[\text{vec}(\nu)]'(N \otimes \Sigma^{-1})\text{vec}(X) + \text{vec}(\nu) \\
&= [I_{pq} - \tfrac{2}{\|\nu\|^2_{p\times q}}\, \text{vec}(\nu)[\text{vec}(\nu)]'(N \otimes \Sigma^{-1})]\text{vec}(X) + \text{vec}(\nu) \\
&= D \cdot \text{vec}(X) + \text{vec}(\nu)
\end{aligned}$$

where

$$D = I_{pq} - \frac{2}{\|\nu\|^2_{p\times q}}\, \text{vec}(\nu)[\text{vec}(\nu)]'(N \otimes \Sigma^{-1}) \tag{6.9}$$

Hence

$$\text{vec}[f(\overline{X})] = D \cdot \text{vec}(\overline{X}) + \text{vec}(\nu) \sim N\left(D\,\text{vec}(\mu) + \text{vec}(\nu),\ D(N^{-1} \otimes \Sigma)D'\right) \tag{6.10}$$

Expressions (6.7) and (6.10) characterize the distributions of $\overline{X}$ and $f(\overline{X})$ at μ.

6.2.3 An equation in distributions

From (6.7), the distribution of $\overline{X}$ at $\mu = \nu$ is rendered by

$$[\text{vec}(\overline{X})|\mu = \nu] \sim N(\text{vec}(\nu),\ N^{-1} \otimes \Sigma)$$

From (6.10), the distribution of $f(\overline{X})$ at $\mu = 0$ is characterized

$$[\text{vec}(f(\overline{X}))|\mu = 0] \sim N(\text{vec}(\nu),\ D(N^{-1} \otimes \Sigma)D')$$

In this subsection we establish the equality of $\text{vec}[f(\overline{X})]|\mu = 0$ and $\text{vec}(\overline{X})|\mu = \nu]$ in distribution.

Lemma 6.2.2. *For all $0 \neq \nu \in C_{p\times q} \cap L^{\perp}_{p\times q}$, $vec[f(\overline{X})]|\mu = 0$ and $vec(\overline{X})|\mu = \nu$ are equal in distribution.*

Proof : Based on the comparison of the two distributions, we need to show

$D(N^{-1} \otimes \Sigma)D' = N^{-1} \otimes \Sigma$ where D is given in (6.9). Note that

$$
\begin{aligned}
D' &= [I_{pq} - \tfrac{2}{\|V\|^2_{p\times q}}\,[vec(V)]\,[vec(V)]'\,(N \otimes \Sigma^{-1})]' \\
&= I_{pq} - \tfrac{2}{\|V\|^2_{p\times q}}\,(N \otimes \Sigma^{-1})'\,[vec(V)]\,[vec(V)]' \\
&= I_{pq} - \tfrac{2}{\|V\|^2_{p\times q}}\,(N \otimes \Sigma^{-1})\,[vec(V)]\,[vec(V)]' \\
\text{and } D(N^{-1} \otimes \Sigma) &= [I_{pq} - \tfrac{2}{\|V\|^2_{p\times q}}\,[vec(V)]\,[vec(V)]'\,(N \otimes \Sigma^{-1})]\,(N^{-1} \otimes \Sigma) \\
&= (N^{-1} \otimes \Sigma) - \tfrac{2}{\|V\|^2_{p\times q}}\,[vec(V)]\,[vec(V)]'\,(N \otimes \Sigma^{-1})(N^{-1} \otimes \Sigma) \\
&= (N^{-1} \otimes \Sigma) - \tfrac{2}{\|V\|^2_{p\times q}}\,[vec(V)]\,[vec(V)]'\,(I_q \otimes I_p) \\
&= (N^{-1} \otimes \Sigma) - \tfrac{2}{\|V\|^2_{p\times q}}\,[vec(V)]\,[vec(V)]'.
\end{aligned}
$$

So, $\quad D(N^{-1} \otimes \Sigma)D'$

$$
\begin{aligned}
= \quad & [(N^{-1} \otimes \Sigma) - \tfrac{2\,vec(V)[vec(V)]'}{\|V\|^2_{p\times q}}][I - \tfrac{2\,(N \otimes \Sigma^{-1})\,vec(V)[vec(V)]'}{\|V\|^2_{p\times q}}] \\
= \quad & (N^{-1} \otimes \Sigma) - \tfrac{2\,vec(V)[vec(V)]'}{\|V\|^2_{p\times q}} - \tfrac{2\,(N^{-1}N \otimes \Sigma\Sigma^{-1})\,vec(V)[vec(V)]'}{\|V\|^2_{p\times q}} + \tfrac{4\;vec(V)\,\langle V,V\,\rangle_{p\times q}\,[vec(V)]'}{\|V\|^4_{p\times q}} \\
= \quad & (N^{-1} \otimes \Sigma) - \tfrac{2}{\|V\|^2_{p\times q}}\,vec(V)[vec(V)]' - \tfrac{2}{\|V\|^2_{p\times q}}\,I_{pq}\,vec(V)[vec(V)]' + \tfrac{4}{\|V\|^2_{p\times q}}\,vec(V)[vec(V)]' \\
= \quad & N^{-1} \otimes \Sigma.
\end{aligned}
$$

Conclusion follows. □

Based on the results of this lemma $f(\overline{X})|\mu = 0$ and $\overline{X}|\mu = \nu$ are equal in distributions. Consequently, the two equations in Lemma 6.2.1. hold. In turn, the two equations in Lemma 6.1.1. are true.

6.3 One inequality

In this section we establish the inequality in Lemma 6.1.1.

$$P(\overline{X} \in A_3 | \mu = 0) \geq P(\overline{X} \in A_3 | \mu = \nu)$$

But first we show a lemma on the behavior of the probability density function of $\overline{X}$ on A_3, with $0 \neq \nu \in C_{p\times q} \cap L^{\perp}_{p\times q}$.

Lemma 6.3.1. *Let $g_\mu(X)$ with $X \in R^{p\times q}$ be the probability density function of $\overline{X}$ given $E(\overline{X}) = \mu$. Then $g_0(X) \geq g_\nu(X)$ for all $X \in A_3$, with $0 \neq \nu \in C_{p\times q} \cap L^{\perp}_{p\times q}$ in (6.3).*

Proof : From (6.7), the distribution of $\overline{X}$ at μ is rendered by

$$[\text{vec}(\overline{X})|\mu] \sim N(\text{vec}(\mu),\ N^{-1} \otimes \Sigma)$$

By (6.8),

$$\begin{aligned} g_\mu(X) &= (2\pi)^{-pq/2}|N^{-1} \otimes \Sigma|^{-1/2} \exp[\tfrac{-1}{2}(\text{vec}(X-\mu))'(N^{-1} \otimes \Sigma)^{-1}(\text{vec}(X-\mu))] \\ &= (2\pi)^{-pq/2}|N^{-1} \otimes \Sigma|^{-1/2} \exp(-\tfrac{1}{2}\|X-\mu\|^2_{p\times q}). \end{aligned}$$

Thus,

$$\begin{aligned} g_0(X) &= (2\pi)^{-pq/2}|N^{-1} \otimes \Sigma|^{-1/2} \exp(-\tfrac{1}{2}\|X\|^2_{p\times q}) \text{ and} \\ g_\nu(X) &= (2\pi)^{-pq/2}|N^{-1} \otimes \Sigma|^{-1/2} \exp(-\tfrac{1}{2}\|X-\nu\|^2_{p\times q}). \end{aligned}$$

Clearly, $g_0(X) \geq g_\nu(X)$ for all $X \in A_3$, if and only if $\|X\|^2_{p\times q} \leq \|X - \nu\|^2_{p\times q}$ for all $X \in A_3$.

Suppose $X \in A_3$. Then $X \notin A_1$. Thus, $\langle X - \nu/2,\, \nu\rangle_{p\times q} \leq 0$. Therefore,

$$
\begin{aligned}
\|X\|^2_{p\times q} &= \|X - \nu/2 + \nu/2\|^2_{p\times q} \\
&= \|X - \nu/2\|^2_{p\times q} + \|\nu/2\|^2_{p\times q} + \langle X - \nu/2,\, \nu\rangle_{p\times q} \\
&\leq \|X - \nu/2\|^2_{p\times q} + \|\nu/2\|^2_{p\times q} - \langle X - \nu/2,\, \nu\rangle_{p\times q} \\
&= \|X - \nu/2 - \nu/2\|^2_{p\times q} \\
&= \|X - \nu\|^2_{p\times q}.
\end{aligned}
$$

The lemma is established. □

With the result in Lemma 6.3.1. $P(\overline{X} \in A_3|\mu = 0) \geq P(\overline{X} \in A_3|\mu = \nu)$ for all $0 \neq \nu \in C_{p\times q} \cap L^{\perp}_{p\times q}$. So the inequality in Lemma 6.1.1. holds. By now the conditions in Lemma 6.1.1. are all true. Hence, the likelihood ratio test on the hypothesis in (3.2) is unbiased.

CHAPTER 7
CONCLUSION

In this dissertation, the problem of testing the homogeneity of several normal mean vectors under multivariate order restriction is considered. To formulate the problem, let $N(\mu_i, \Sigma)$, $i = 1, ..., q$, be q independent normal populations with known Σ and unknown $\mu_i \in R^p$. Assume that $\mu_i \ll \mu_j$ for $(i, j) \in H$ where $H \subset \{1, ..., q\} \times \{1, ..., q\}$ is a given set. We need to test $\mu_1 = \cdots = \mu_q$. With $\mu = (\mu_1, ..., \mu_q) \in R^{p\times q}$, let

$$L_{p\times q} = \{\mu : \ \mu_1 = \cdots = \mu_q\}$$

and $C_{p\times q}$ as defined in(2.1). Then $L_{p\times q}$ is a closed linear subspace in $R^{p\times q}$, and $L_{p\times q} \subset C_{p\times q}$. We consider test on

$$H_0 : \mu \in L_{p\times q} \text{ versus } H_a : \mu \in C_{p\times q} - L_{p\times q}$$

For the model of q normal populations, this dissertation derives the restricted maximum likelihood estimators (RMLEs) for the mean vectors under multivariate order restrictions, develops the likelihood ratio tests (LRTs) for the hypotheses about the restricted mean vectors. The likelihood ratio test statistic and its properties are generally described through the projections onto closed convex cones from which the multivariate orders are induced.

In hypothesis testing, two types of wrong decisions can occur. The first type of wrong decision occurs if the null hypothesis is rejected when it is true. This is referred to as Type I error. The probablity of Type I error is referred to as the size or the signifigance level of the test. The second type of wrong decision occurs if the alternative hypothesis is rejected when it is true. This is referred to as Type II error. It is desired, of course, to minimize the probablities of these two types of error. In general, this is not possible. Since minimizing or decreasing the probablity of one error results in maximizing or increasing the probability of the other error. Type I error is often considered the worse of the two errors. Hence, an upper bound is usually chosen for the size of the test. Then, the probablity of Type II error is to be minimized. This is equivalent to maximizing the power of the test. The power of the test is the probability of accepting the alternative hypothesis when it is true. It is always desired that the size of the test is not greater than the power of the test. A hypothesis test having this property is called **unbiased test**, as defined in Hog et al (2005).

This work establishes the unbiasedness of the likelihood ratio test considered in chapter 3 of this dissertation. The work assumed that the covariance matrix Σ is known. Mukerjee et al (1986) established the unbiasedness for the univariate counterpart (when $p = 1$) of the considered test.

To continue the work in this direction, one may consider the case when the covariance matrix Σ is unknown. It is worth mentioning here that Hu et al (1994) established the unbiasedness for the univariate counterpart of the test under consideration when the covariance matrix Σ is unknown (in this case, the covariance matrix is a scalar and is usually denoted by σ^2).

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