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Pursuit Evasion Games using Collision Cones

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This paper presents a preliminary study into the use of collision cones in analyzing pursuit evasion games between two objects of arbitrary shapes. A two-player pursuit evasion game is considered, wherein a single pursuer tries to capture a single evader who, in turn, tries to avoid capture. The collision cone is defined as the set of instantaneous velocities that cause one player to lie on a collision course with the other. The evader applies an acceleration to drive the relative velocity vector out of the collision cone, while the pursuer applies an acceleration to drive the relative velocity vector into the collision cone. Two different matrix games are considered, and saddle points (defined by the scenarios wherein the relative velocity vector is aligned with the boundary of the collision cone at the instant of closest approach) are determined.

I. Introduction

In a pursuit evasion game, one or more pursuers try to capture one or more evaders, who try to avoid being captured [1]. In some pursuit evasion games, capture is said to occur when the distance between the pursuer and the evader becomes less than some pre-specified threshold, while in other games capture is said to occur when the pursuers see or surround the evader. Pursuit evasion games have applications in warfare, motion planning problems in robotics, network security, modeling animal behavior, etc. These games are also used to study collision avoidance in dynamic environments, in order to determine the worst-case behavior of the obstacle(s) and then check if the collision avoidance algorithm can guarantee collision avoidance even in the presence of this worst-case behavior. Examples of pursuit evasion differential games include the homicidal chauffeur game [2],[3], the game of two cars [4], the lion and the man, the lady in the lake [5], etc.

While the literature on pursuit evasion games largely assumes the shapes of the objects to be circles, or in some cases ellipses, there can be many scenarios wherein analysis of pursuit evasion games for a larger class of shapes can be of interest. For instance, consider robots moving in a cluttered environment. As the robots move along their trajectories, at the times when they come in close proximity, the relative shapes of the robots play an important role in the determination of collision avoidance trajectories. In such cases, the circle approximation can be overly conservative, and this is specially true when the robots have elongated and/or non-convex shapes. In such cases, using the exact shapes of the objects for collision avoidance can enable closer maneuvers to be performed, which may not be possible with the circle approximation.

As a second example, consider an environment that is not cluttered and a circular approximation to the robot shape is adopted. Then, the effects of uncertainties in the position and velocity of the object(s) can be modeled by enlarging the object from its original shape to a new shape that accounts for these uncertainties. From a strict mathematical standpoint, the shape of this enlarged object will depend on the type of the underlying probability distribution that models these uncertainties. The shape of this enlarged object is not necessarily circular and could be entirely arbitrary. As a third example, consider two vehicle swarms. The shape of the boundary of each swarm of vehicles can, in general, be arbitrary. Then, the problem of a non-cooperative merger of the two swarms can be viewed as a pursuit evasion game between two (virtual) objects of arbitrary shapes. This would allow one to evaluate whether any guidance laws designed to enable this merger, are robust to worst-case disturbances that attempt to disrupt this merger. As a fourth example,

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consider a pursuer missile carrying a selectively aimable warhead [6],[7], which is known to have a conical directed blast zone. The target may be modelled as an evader trying to avoid this non-circular zone of the pursuer. There are thus several scenarios wherein collision dynamics of objects involving arbitrary shapes are of interest.

In this paper, we analyze pursuit-evasion games for robots with arbitrary shapes, using the collision cone approach [8]. The collision cone approach is a one-step look-ahead approach that takes both the shape of the obstacles, as well as their instantaneous velocities into account, in the determination of collision avoidance trajectories. The collision cone approach has some similarities with the velocity obstacle approach in that both approaches generate cones of forbidden velocities/headings, which the robots should avoid. However, while the velocity obstacle approach [11] as well as its many extensions [12] have been developed for circular robots, the collision cone approach has been developed for objects with arbitrary shapes. Exact analytical expressions governing the collision conditions for objects with entirely arbitrary shapes (both convex and non-convex) moving on a plane have been developed in [8], while analytical expressions for collision conditions for quadric surfaces moving in 3-D space have been developed in [9],[10]. The great benefit of such analytical expressions is that they lay a foundation for development of analytical expressions for collision avoidance laws. The benefit of such analytical collision avoidance laws is that they can lead to reduced computational load, which is particularly important in robotic platforms with limited computational resources.

The collision cone approach of [8] has been extensively employed in the literature, for example, in driver assistance systems [13]-[15], safe trajectories for aircraft [16],[18], multi-vehicle collision avoidance problem [19], and investigation of collision avoidance in biological systems such as fish schools [20].

This paper formulates two pursuit evasion games for vehicles with arbitrary shapes. In both games, the objects are assumed to have double integrator dynamics, and the control inputs are the accelerations of the vehicles. The first game plays the acceleration magnitudes of the two vehicles against each other, while keeping the directions of these acceleration vectors normal to the velocity vector of the respective vehicles. The second game plays the directions of the acceleration vectors against each other, while the magnitudes of the accelerations are determined by a guidance law that uses feedback information from the instantaneous collision cone between the two vehicles. The two games are each formulated both as games of kind, as well as games of degree, and the occurrence of saddle points is investigated.

The rest of this paper is organized as follows. In Section II, we briefly review the collision cone concept and discuss collision avoidance strategies based on this concept. In Section III, we illustrate how a pursuit evasion game can be connected to collision cone concepts. In Sections IV and V, we formulate two pursuit evasion games based on the collision cone, and also present numerical results for these two games. Finally, Section VI presents the conclusions.

II. Collision Cone

In this section, we first review the collision cone concept presented in [8], and subsequently formulate two pursuit evasion games based on this concept. The collision cone method is a local, reactive method suitable for online avoidance of obstacles in *a priori* unknown environments. It is defined as the cone of velocity headings that will cause two objects to be on a collision course. Consider an engagement between two arbitrary shaped objects A and B (these objects may be convex or non-convex) as shown in Figure 1. Here, A and B are moving with speeds V_A and V_B, acting at angles α_A and α_B, respectively. The kinematics of this engagement is characterized by the equations governing the Line-of-Sight (LOS), which is defined as the line P_1P_2 joining the centers (suitably defined, for instance, a point located at or near the geometric centroid of the object) of the two objects. These equations are as follows:

$$\begin{bmatrix} \dot{\hat{r}} \\ \dot{\hat{\theta}} \\ \dot{\hat{V}}_\theta \\ \dot{\hat{V}}_r \end{bmatrix} = \begin{bmatrix} \hat{V}_r \\ \frac{\hat{V}_\theta}{\hat{r}} \\ \frac{\hat{V}_\theta \hat{V}_r}{\hat{r}} \\ \frac{\hat{V}_\theta^2}{\hat{r}} \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ -\sin(\delta_A - \hat{\theta}) \\ -\cos(\delta_A - \hat{\theta}) \end{bmatrix} a_A + \begin{bmatrix} 0 \\ 0 \\ \sin(\delta_B - \hat{\theta}) \\ \cos(\delta_B - \hat{\theta}) \end{bmatrix} a_B \quad (1)$$

Here, $\hat{r}$ and $\hat{\theta}$ represent, respectively, the range and the bearing angle of P_1P_2; $\hat{V}_\theta$ and $\hat{V}_r$ represent, respectively, the relative velocity components (of B with respect to A) that are normal to, and along, P_1P_2; a_A and a_B represent, respectively, the magnitudes of the accelerations of A and B, while δ_A and δ_B represent the angles at which these accelerations are applied.

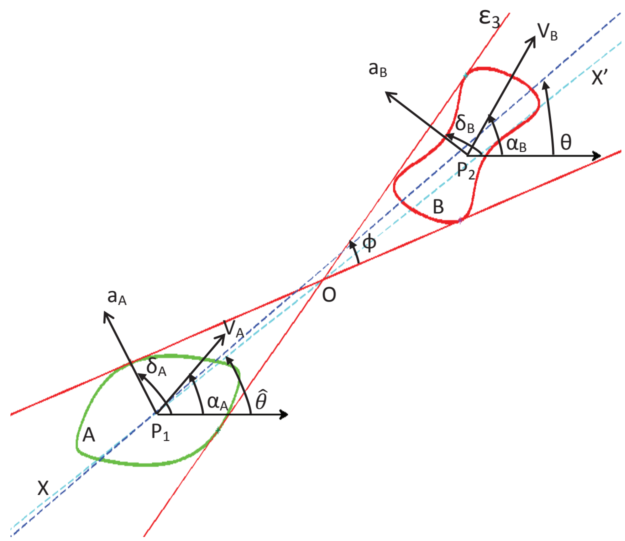

Figure 1. Engagement geometry between arbitrary objects A and B

The objects A and B are on a collision course if their relative velocities belong to a specific set. As was demonstrated in [8], this set of relative velocities can be encapsulated in a scalar quantity y, which is defined as follows:

$$y = \left(\frac{V_\theta^2}{V_\theta^2 + V_r^2} \right) - \sin^2 \frac{\phi}{2} \tag{2}$$

The collision cone is then defined as the region in (V_θ, V_r) space for which $y < 0$, $V_r < 0$ is satisfied. The condition $y < 0, V_r < 0$ physically means that the relative velocity vector lies inside the collision cone, while the condition $y = 0, V_r < 0$ physically means that the relative velocity vector is aligned with the boundary of the collision cone. In order to define the individual terms that contribute to the equation for y in (2), we construct a cone ϵ_3 (shown in Figure 1), which is the smallest cone that can be formed that completely contains A and B, and is such that A and B lie on opposite sides of the vertex of the cone. In (2), ϕ represents the vertex angle of this cone, while V_r and V_θ represent the relative velocity components of the angular bisector of this cone. Here, V_r is the relative velocity component along this bisector and V_θ is the corresponding component normal to this bisector. Finally, θ represents the angle made by this angular bisector with a reference line. From (2), it is evident that the definition of y is such that $-1 \leq y \leq 1$ always holds.

In general, V_r is distinct from $\hat{V}_r$, and V_θ is distinct from $\hat{V}_\theta$. In the special case when A and B are both circular objects, then the angular bisector of ϵ_3 coincides with P_1P_2, that is, $\hat{\theta} = \theta$, and thus, $V_r = \hat{V}_r$, $V_\theta = \hat{V}_\theta$. If A and B move with constant velocities, then $y < 0, V_r < 0$ are both necessary and sufficient conditions for collision to occur [8]. If A and/or B move with varying velocities, then a necessary condition for collision to occur is that at the instant of closest approach, $y < 0$ holds. The conditions (a) $y < 0, V_r > 0$ for all future time, and (b) $y > 0$ for all future time, are each sufficient conditions for collision avoidance.

The collision cone can be represented in physical space as the instantaneous cone of headings of A that will cause it to be on a collision course with B. It can also be equivalently represented in the (V_θ, V_r) relative velocity space (shown in Figure 2) as the instantaneous cone of relative velocities that will cause A and B to lie on a collision course. Note that Equation (2) is an expression for the collision cone when $\phi \leq \pi$ (as in Figure 1). When $\phi > \pi$, the collision cone is given by an appropriate function of the complement of Equation (2), as described in [8].

III. Problem Formulation

We now incorporate the collision cone approach in the context of a two player pursuit evasion game. The pursuer seeks to cause a collision by applying an acceleration so as to drive the relative velocity vector deep into the collision cone, while the evader seeks to avoid collision by applying an acceleration that will drive

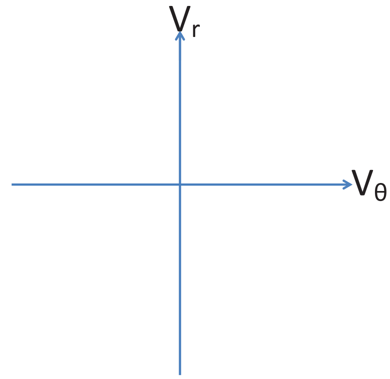

Figure 2. Representation of the collision cone in relative velocity space

the relative velocity vector out of the collision cone. The game is played on a plane, and it is assumed that both the players have bounded acceleration, and these acceleration bounds are identical for both the players.

The state and output equations for the 2-player game are as follows:

$$\dot{x} = f(x) + g_A(x)a_A + g_B(x)a_B \quad (3)$$

$$y = h(x) \quad (4)$$

where, the state vector x is defined as $x = \begin{bmatrix} \hat{r} & \hat{\theta} & \hat{V}_\theta & \hat{V}_r & V_A & V_B & \alpha_A & \alpha_B \end{bmatrix}^T$. The state equations for the first four states are as given in (1), while those for the remaining four states are as follows.

$$\begin{aligned} \dot{V}_A &= a_A \cos(\delta_A - \alpha_A) \\ \dot{V}_B &= a_B \cos(\delta_B - \alpha_B) \\ \dot{\alpha}_A &= \frac{a_A}{V_A} \sin(\delta_A - \alpha_A) \\ \dot{\alpha}_B &= \frac{a_B}{V_B} \sin(\delta_B - \alpha_B) \end{aligned} \quad (5)$$

The output equation for y is given in (2).

Here, a_A and a_B are the accelerations of the two players, with A being the pursuer and B being the evader. These accelerations are applied at angles δ_A and δ_B, respectively, with the angles being measured relative to an inertial horizontal reference. In the context of the collision cone, B applies an acceleration so as to bring the relative velocity vector out of the collision cone, while A applies an acceleration so as to bring the relative velocity vector inside the collision cone. From a mathematical perspective, B attempts to get the quantity $y(t)$ to a positive value (which corresponds to the physical scenario of the relative velocity vector being outside the collision cone), while A attempts to get the quantity $y(t)$ to a negative value. Let t_m represent the time at which V_r becomes zero, that is, $V_r(t_m) = 0$. Two games are considered.

At one level, the two games are qualitative (that is, games of kind) in that the game ends if the pursuer captures the evader within a specified time duration, or the evader avoids the pursuer for that time duration. Since the quantity $y(t)$ defines the instantaneous depth of the relative velocity vector inside (or outside) the collision cone, the games are also quantitative (that is, game of degree) in the following sense. (a) In the scenario that the pursuer captures the evader (that is, $y(t_m) < 0$, $V_r(t_m) < 0$), the larger the value of $|y(t_m)|$, the deeper the impact of the pursuer into the evader, while the smaller the value of $|y(t_m)|$, the shallower the impact. (b) In the scenario that the evader manages to avoid capture, then the smaller the value of $|y(t_m)|$, the smaller the miss-distance and thus the narrower the escape by the evader, while the larger the value of $|y(t_m)|$, the larger the miss-distance, and the more comfortable the evader's escape. Game 1 is open-loop in the sense that neither the pursuer nor the evader use any feedback of the other agent's states, while Game 2 is closed-loop in the sense that both the pursuer and the evader use continuous feedback of the value of $y(t)$, and incorporate this value in their respective guidance laws.

IV. Pursuit Evasion Game 1

In the first game (Game 1), both the players A and B apply their individual accelerations a_A and a_B normal to the respective velocity vectors. Thus, the maneuver is one of pure heading change by both the players, while maintaining constant speed. This makes the trajectories of both the players circular, with the radius of each player's trajectory depending on the magnitude of the applied acceleration. These acceleration magnitudes a_A and a_B are discretized to lie between -2 m/sec^2 to 3.5 m/sec^2 for A, and between -3 m/sec^2 and 3 m/sec^2 for B. A matrix game is played on the kinematic ODEs (1) using these discretized controls. A collision has occurred if, at the time instant when $V_r = 0$ (which corresponds to the instant of closest approach), we have $y < 0$.

In all the cases shown below, the initial conditions are identical. The initial co-ordinates of the center of the pursuer are $(0, 0)$, while that of the center of the pursuer are $(15, 0)$. The initial speed of the pursuer is 7 m/sec, while that of the evader is 6 m/sec. The initial heading of the pursuer is 45^0, while that of the evader is 120^0. For these initial conditions, the initial value of y, that is, $y(0)$ is negative, and $V_r(0)$ is also negative, thus indicating that the initial speeds and headings of the pursuer and evader are such that they are on a collision course. The shapes of the pursuer and the evader are represented by the following equation, and are shown in Figure 3.

$$\begin{aligned} x_i &= R_i \cos\phi \\ y_i &= R_i(\sin\phi - 0.75\sin^3\phi), \quad \phi \in [0, 2\pi] \end{aligned} \tag{6}$$

In the above equation, (x_i, y_i) represent a local co-ordinate frame. When the heading angle of the vehicle is at an angle α_i, as shown in Fig 3, the shape of each object assumes the following expression in a global (X, Y) co-ordinate frame:

$$\begin{aligned} X_i &= X_{ci} + x_i\cos\alpha_i - y_i\sin\alpha_i \\ Y_i &= Y_{ci} + x_i\sin\alpha_i + y_i\cos\alpha_i \end{aligned} \tag{7}$$

where, (X_{ci}, Y_{ci}) represent the coordinates of the center of the ith object in the (X, Y) frame. Fig 3 shows

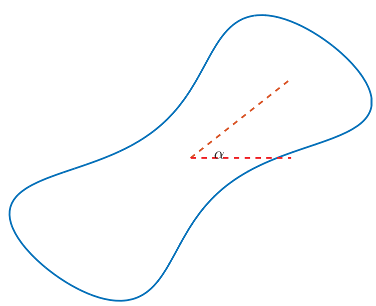

Figure 3. Shape of objects A and B

that the shape of the object is not omni-directional, and furthermore, as the heading angle of the object changes, the orientation of the object changes as well, and this influences the relative geometry between the pursuer A and evader B.

In the results of Figures 4-6, the pursuer applies a constant acceleration of 1 m/sec^2, while the evader applies a constant acceleration of -3 m/sec^2. Figure 6 shows the trajectories of the two objects, and as can be seen from the figure they come very close to each other, but do not collide. The instant at which the closest approach occurs, is when $V_r = 0$, and this is seen from Figure 4 to be at around $t = 2.1$ sec. Figure 5 shows that y, which was initially negative, becomes positive at the instant of closest approach, which makes sense because the two objects did not collide in this case. Since the vehicles apply their accelerations normal to their respective velocity vectors, their speeds remain constant throughout the engagement. The plots of their velocity heading angles are shown in Figure 5.

In Figures 7,8,9, the pursuer applies a constant acceleration of -1 m/sec^2, while the evader's acceleration is -3 m/sec^2. As can be seen from Figure 8, the value of y decreases from its initial negative value, and the two agents eventually collide (as seen in Figure 9). In other words, this means that the net effect of their

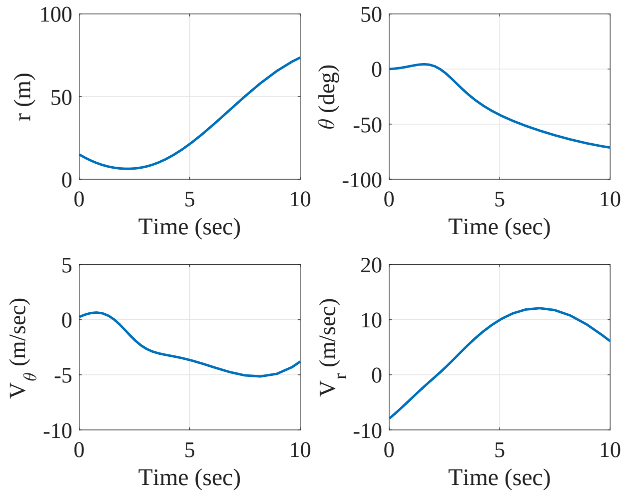

Figure 4. State Time Histories for Game 1, with accelerations $a_A = 1\ m/sec^2, a_B = -3$ m/sec^2

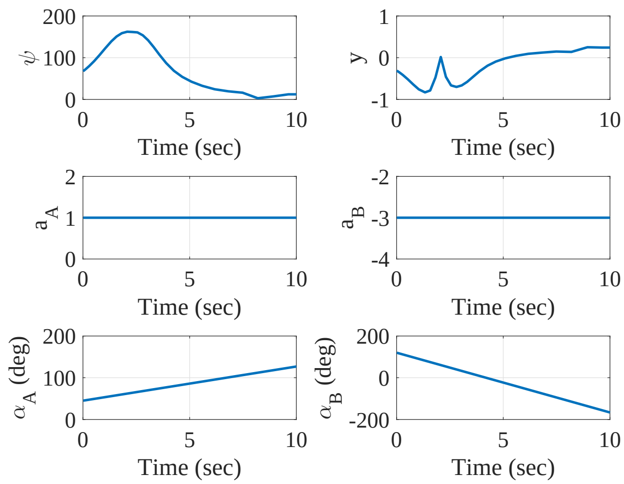

Figure 5. Time Histories of ϕ, y, acceleration inputs, and velocity heading angles for Game 1, with $a_A = 1\ m/sec^2, a_B =$ -3 m/sec^2

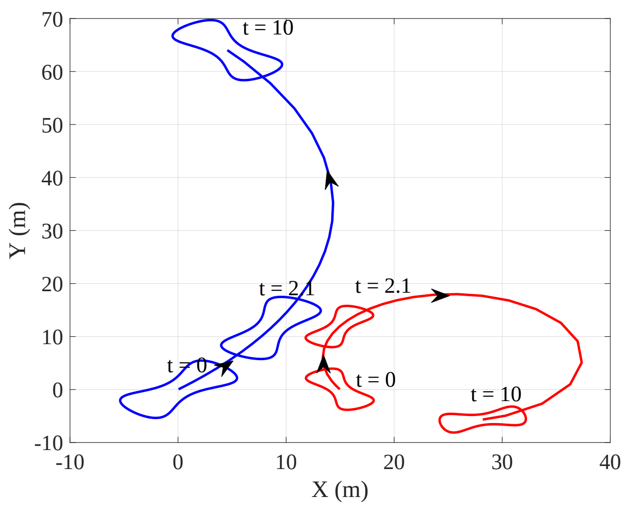

Figure 6. XY Trajectories of the Pursuer and Evader for Game 1, with $a_A = 1$ m/sec$^2, \mathrm{a_B} = -3$ m/sec^2

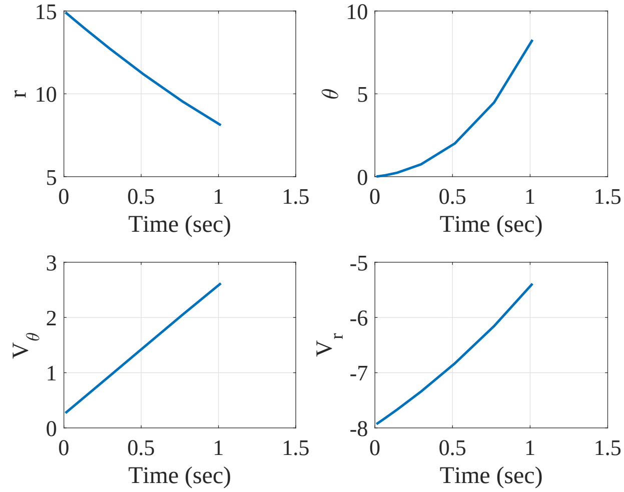

Figure 7. State Time Histories for Game 1, with accelerations $a_A = -1$ m/sec^2, $a_B = -3$ m/sec^2

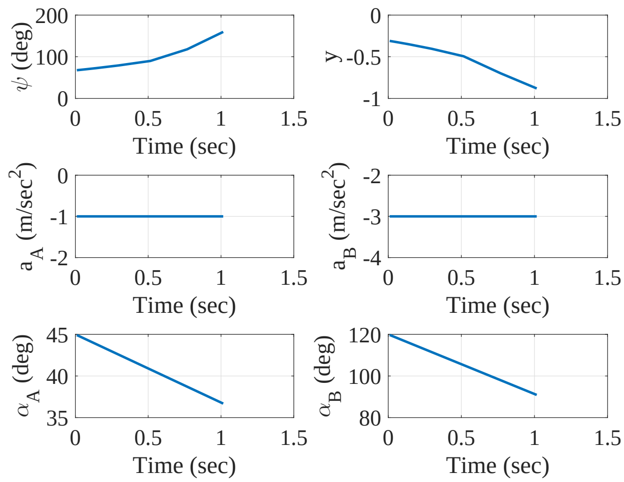

Figure 8. Time Histories of ϕ, y, acceleration inputs, and velocity heading angles for Game 1, with $a_A = -1$ m/sec^2, $a_B = -3$ m/sec^2

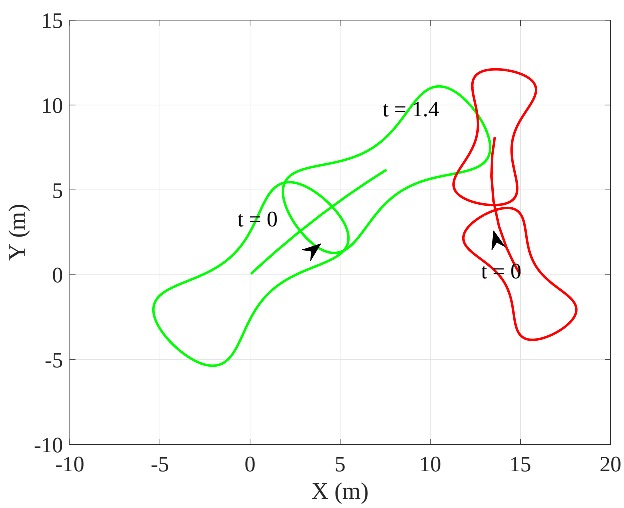

Figure 9. XY Trajectories of the Pursuer and Evader for Game 1, with $a_A = -1$ m/sec^2, $a_B = -3$ m/sec^2

accelerations is to cause the relative velocity vector to go deeper inside the collision cone. The influence of the applied accelerations by the pursuer and evader on their velocity heading angles are seen in Figure 8.

In Figure 10, the evader applies a constant acceleration of 3 $\mathrm{m/sec^2}$, while the pursuer applies a constant acceleration of -2 $\mathrm{m/sec^2}$. Accordingly, the evader traces a counterclockwise trajectory, while the pursuer's trajectory is clockwise. At the instant of closest approach shown in the figure, the pursuer and the evader are well separated, and there is no collision. The effect of the applied accelerations thus has been to drive the relative velocity vector out of the collision cone (or, in other words, drive y from its initial negative value to a positive value at time $t = t_m$).

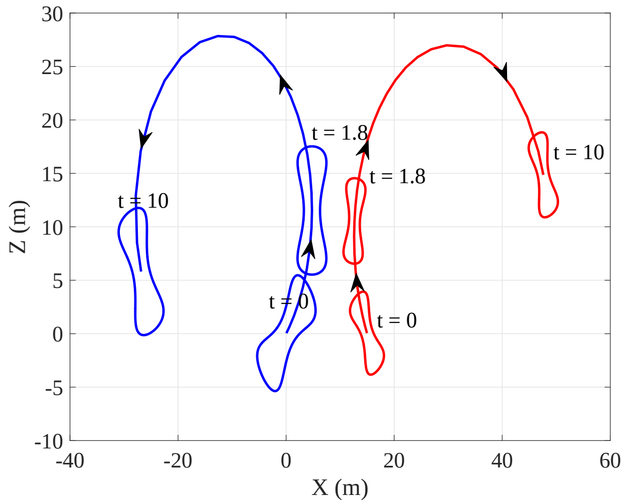

Figure 10. ***XY*** **Trajectories of the Pursuer and Evader for Game 1, with** $a_A = 3$ $\mathrm{m/sec^2}, a_B = -2$ $\mathrm{m/sec^2}$

Figures 4-10 are thus the simulation results for a few specific cases of acceleration combinations. Defining the value of $y(t_m)$ (which is y at the instant when $V_r = 0$ occurs) to be the payoff, the results of the matrix game for multiple (a_A, a_B) combinations are shown in Table 1. All the entries correspond to the same initial condition discussed earlier.

Table 1. Results for Matrix Game 1: Game of Degree

$a_A \rightarrow$ / $a_B \downarrow$	-2	-1	0	1	2	3	3.5
-3	-0.14	-0.88	0.45	0.01	0.15	0.03	0.51
-2	-0.75	-0.79	-0.79	-0.73	0.04	0.57	0.13
-1	-0.74	-0.82	-0.52	-0.56	-0.76	-0.15	0.08
0	-0.75	-0.68	-0.50	-0.81	-0.39	-0.73	-0.73
1	-0.54	-0.49	-0.48	-0.55	-0.63	-0.64	-0.68
2	-0.57	-0.48	-0.64	-0.65	-0.43	-0.58	-0.61
3	-0.46	-0.45	-0.65	-0.59	-0.44	-0.51	-0.53

If $y(t_m) = 0$ occurs, then this means that at the instant of closest approach, the relative velocity vector is aligned with the boundary of the collision cone, and the two objects just graze each other. While the entries in Table 1 correspond to those of a game of degree, we can extract a game of kind matrix from Table 1. We do so by replacing negative values of $y(t_m)$ (which indicate collision) by -1, positive values of $y(t_m)$ (which indicate avoidance) by $+1$, and values of $y(t_m)$ that belong to a small interval of $[-0.01, 0.01]$ by 0. By doing so, Table 1, assumes the form shown in Table 2.

If we now focus our attention exclusively on the set of columns that range from $a_A = 0$ to $a_A = 3.5$ $\mathrm{m\,sec^2}$, we see that the zero entry (corresponding to $a_A = 1$ $\mathrm{m/sec^2}$, $a_B = -3$ $\mathrm{m\,sec^2}$) corresponds to a local saddle point for this game.

Table 2. Results for Matrix Game 1: Game of Kind

$a_A \rightarrow$ $a_B \downarrow$	-2	-1	0	1	2	3	3.5
-3	-1	-1	+1	0	+1	+1	+1
-2	-1	-1	-1	-1	+1	+1	+1
-1	-1	-1	-1	-1	-1	-1	+1
0	-1	-1	-1	-1	-1	-1	-1
1	-1	-1	-1	-1	-1	-1	-1
2	-1	-1	-1	-1	-1	-1	-1
3	-1	-1	-1	-1	-1	-1	-1

V. Pursuit Evasion Game 2

In the second game (Game 2), again the pursuer A tries to make $y < 0$, while the evader B tries to make $y > 0$. In this game, both A and B use guidance laws for the magnitudes of a_A and a_B (based on the current values of y), to achieve the above objective. These guidance laws are obtained using a nonlinear dynamic inversion technique. For player A, the acceleration law is:

$$a_A = \frac{\begin{bmatrix} -K_A(y-w)(V_r^2+V_\theta^2)^2 \\ +2(V_r^2+V_\theta^2)V_\theta^2\text{cosec}^2(\frac{\phi}{2})(V_r\dot{\theta}/V_\theta+\cot(\frac{\phi}{2})\frac{\dot{\phi}}{2}) \end{bmatrix}}{\left[2V_rV_\theta\text{cosec}^2(\frac{\phi}{2})(-V_r\sin(\delta_A-\theta)+V_\theta\cos(\delta_A-\theta))\right]} \tag{8}$$

and this seeks to make $y = w$, where $w < 0$ represents the negative reference value chosen by the pursuer A. A similar acceleration law exists for the evader B with the difference that the reference value for the evader is positive (that is, $w > 0$). When a_B is zero (that is, B moves with constant velocity), the above acceleration law (8) guarantees that A will achieve $y = w < 0$, and the dynamics of y will evolve according to the equation $\dot{y} + y = K_A w$, where K_A represents the acceleration gain of A. Similarly, when a_A is zero (that is, A moves with constant velocity), the guidance law guarantees that B will achieve $y = w > 0$, and the dynamics of y will evolve according to $\dot{y} + y = K_B w$, where K_B represents the acceleration gain of B.

An examination of (8) shows that the magnitude of the acceleration is also a function of the angle δ_A, which represents the angle at which a_A is applied. When $\delta_A = \alpha + \frac{\pi}{2}$, the above acceleration is applied normal to the velocity vector of A (that is, the influence of the applied acceleration is to cause a pure heading change in the velocity vector of A), while when $\delta_A = \alpha_A$, the above acceleration is applied along the velocity vector of A (that is, the influence of the applied acceleration is to cause a pure speed change in A). For other values of δ_A, the effect of the applied acceleration is to cause a combination of speed and heading change in A. A corresponding set of statements can be made for the angle δ_B.

This game is played between the angles δ_A and δ_B at which a_A and a_B are applied. These angles belong to the range $[0, \frac{\pi}{2}]$, relative to the line normal to the velocity vector of each vehicle. In other words, $\delta_A = \alpha_A + \frac{\pi}{2} + \eta_A$, and $\delta_B = \alpha_B + \frac{\pi}{2} + \eta_B$, where $\eta_A \in [0, \frac{\pi}{2}]$, and $\eta_B \in [0, \frac{\pi}{2}]$. The initial conditions are taken as identical to those of Game 1.

Figures 11-13 represent the scenario when $\eta_A = 0$, $\eta_B = \frac{5\pi}{12}$, which means that $\delta_A = \alpha_A + \frac{\pi}{2}$, and $\delta_B = \alpha_B + \frac{5\pi}{12} + \frac{\pi}{2}$. The time of closest approach occurs when $V_r = 0$ and from Figure 11, it is seen that this occurs at around 1.4 seconds. At this time, $y(0)$ is very nearly zero, indicating that the evader has just barely grazed the pursuer. The accelerations of both the pursuer and the evader hit their saturation limits (± 10 m/sec^2) for a short while at the initial portion of the trajectory, but at the time when the two objects graze each other, the acceleration magnitudes are as obtained from the guidance law.

In Figures 14-16, we have $\eta_A = \frac{5\pi}{18}$ and $\eta_B = \frac{\pi}{2}$. In this case, the evader managed to increase y from its initial value (in other words, it could increase the miss-distance from what it originally would have been), and could successfully avoid the collision. At the instant when $V_r = 0$ (around 6 seconds), y is around 0.3, which is a positive number, and thus represents a case of no collision.

In Figures 17-19, we have $\eta_A = \frac{\pi}{2}$ and $\eta_B = 0$. In this case, the evader managed to increase y from its initial value (in other words, it could reduce the miss-distance from its original value) but could not quite

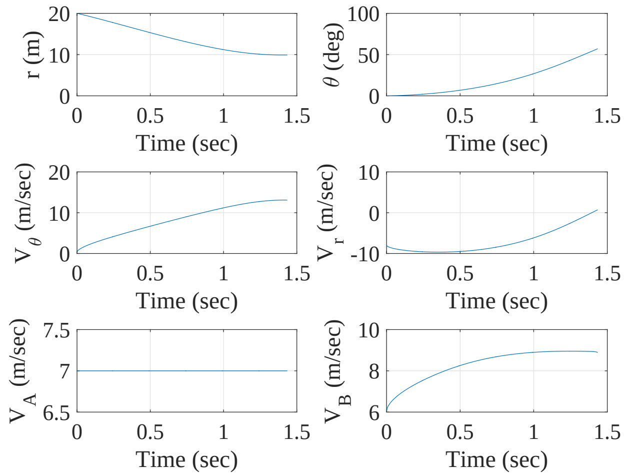

Figure 11. State Time Histories for Game 2, with $\eta_A = 0, \eta_B = \frac{5\pi}{12}$

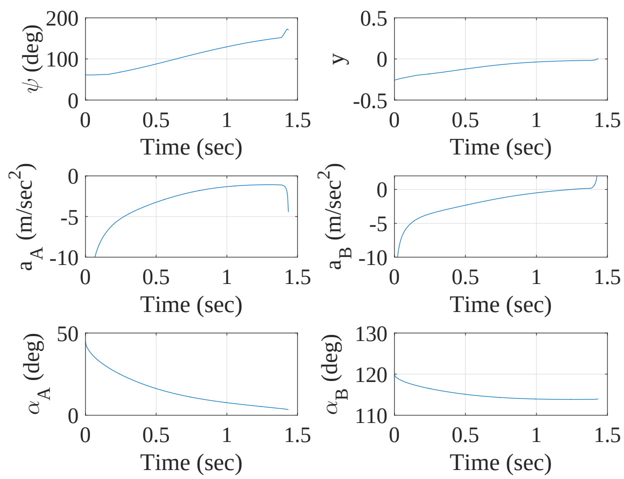

Figure 12. Time Histories of ϕ, y, acceleration inputs, and velocity heading angles for Game 2, with $\eta_A = 0, \eta_B = \frac{5\pi}{12}$

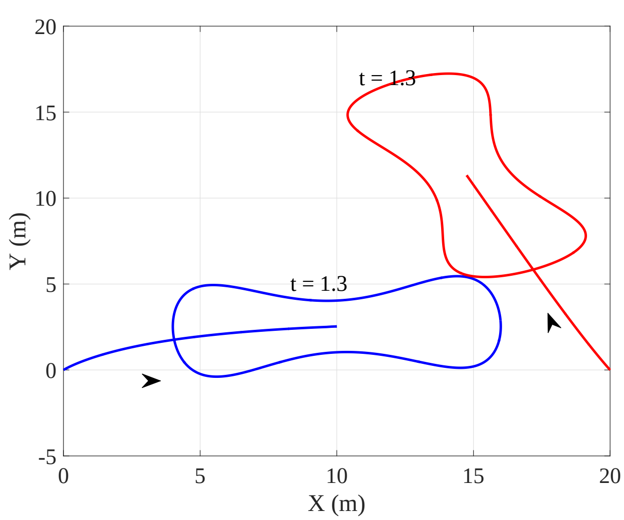

Figure 13. XY Trajectories of the Pursuer and Evader for Game 2, with $\eta_A = 0, \eta_B = \frac{5\pi}{12}$

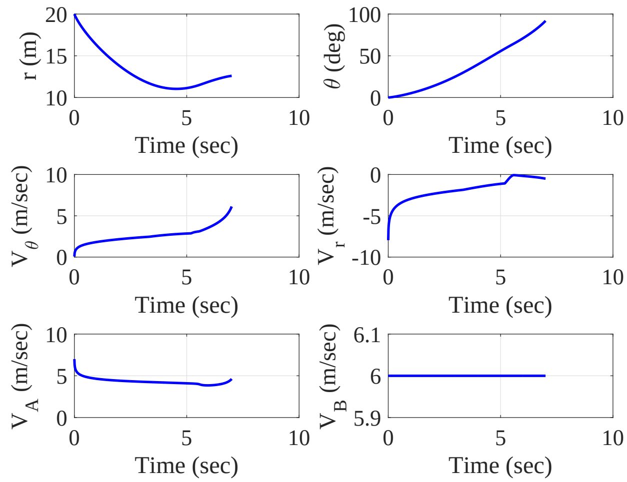

Figure 14. State Time Histories for Game 2, with $\eta_A = \frac{5\pi}{18}, \eta_B = 0$

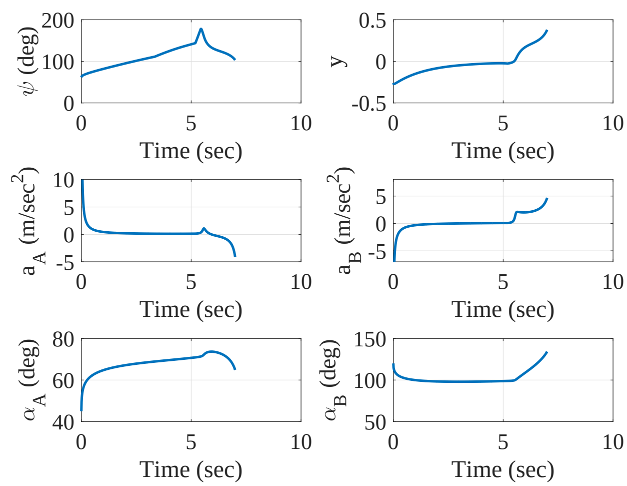

Figure 15. Time Histories of ϕ**,** y**, acceleration inputs, and velocity heading angles for Game 2, with** $\eta_A = \frac{5\pi}{18}, \eta_B = 0$

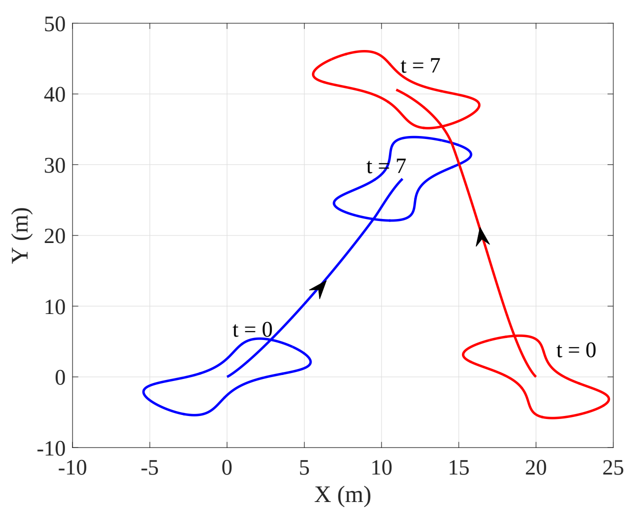

Figure 16. XY **Trajectories of the Pursuer and Evader for Game 2, with** $\eta_A = \frac{5\pi}{18}, \eta_B = 0$

avoid the collision. At the instant when $V_r = 0$ (around 3.3 seconds), y is around -0.1, which represents a collision, and this collision is evident from Figure 19.

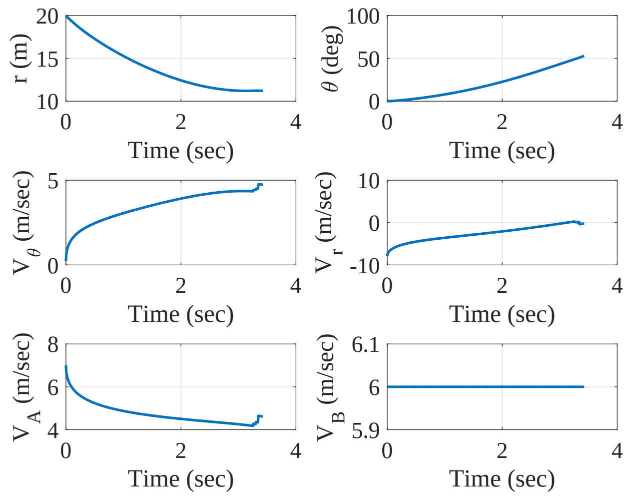

Figure 17. State Time Histories for Game 2, with $\eta_A = \frac{\pi}{2}, \eta_B = 0$

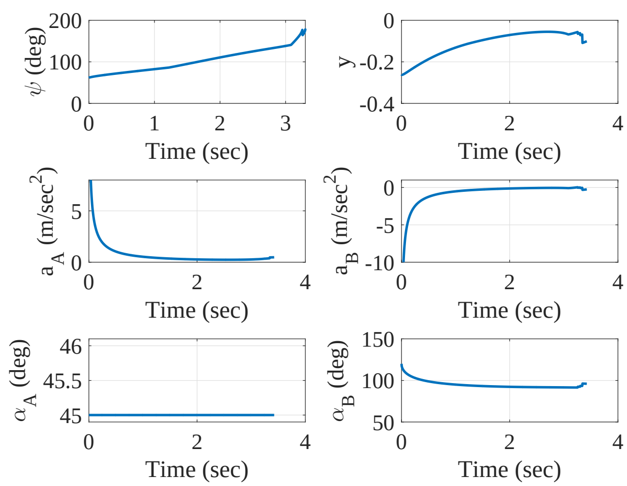

Figure 18. Time Histories of ϕ, y, acceleration inputs, and velocity heading angles for Game 2, with $\eta_A = \frac{\pi}{2}, \eta_B = 0$

Defining $y(t_m)$ to represent the payoff, the results of the matrix game for different combinations of (ϕ_A, ϕ_B) chosen by the players are given in Table 3. Table 3 thus represents a game of degree. Similar to what was done in Game 1, we can extract a game of kind matrix from Table 3, and do so by replacing negative values of $y(t_m)$ (which indicate collision) by -1, positive values of $y(t_m)$ (which indicate avoidance) by $+1$, and values of $y(t_m)$ that belong to a small interval of $[-0.006, 0.006]$ by 0. By doing so, Table 3, assumes the form shown in Table 4.

If we now focus our attention exclusively on the last two columns and the bottom four rows of Table 4 (these correspond to $\eta_A = \frac{5\pi}{36}$ to $\eta_A = 0$, and $\eta_B = \frac{5\pi}{12}$ to $\eta_B = \frac{5\pi}{36}$), we see that the zero entry (corresponding to $\eta_A = 0$, $\eta_B = \frac{5\pi}{12}$) corresponds to a local saddle point for this game.

VI. Conclusions

In this paper, a preliminary investigation into the incorporation of the collision cone in a pursuit evasion game, is conducted. The shapes of the objects in the pursuit evasion game are arbitrary. The study considers two players with the pursuer applying an acceleration that attempts to bring the relative velocity vector into the interior of the collision cone, and the evader applying an acceleration that attempts to bring the relative

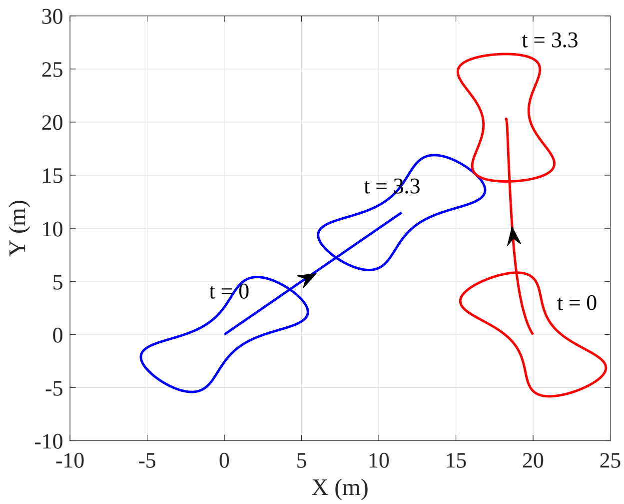

Figure 19. XY **Trajectories of the Pursuer and Evader for Game 2, with** $\eta_A = \frac{\pi}{2}, \eta_B = 0$

Table 3. Results for Matrix Game 2: Game of Degree

$\eta_A \rightarrow$ $\eta_B \downarrow$	$\frac{\pi}{2}$	$\frac{5\pi}{12}$	$\frac{5\pi}{18}$	$\frac{5\pi}{36}$	0
$\frac{\pi}{2}$	-0.05	-0.06	-0.04	0.01	0.03
$\frac{5\pi}{12}$	-0.05	-0.06	-0.03	0.02	0.006
$\frac{5\pi}{18}$	-0.05	-0.05	-0.03	0.14	-0.03
$\frac{5\pi}{36}$	-0.05	-0.06	-0.02	-0.03	-0.06
0	-0.10	-0.06	0.3	-0.07	-0.09

Table 4. Results for Matrix Game 2: Game of Kind

$\eta_A \rightarrow$ $\eta_B \downarrow$	$\frac{\pi}{2}$	$\frac{5\pi}{12}$	$\frac{5\pi}{18}$	$\frac{5\pi}{36}$	0
$\frac{\pi}{2}$	-1	-1	-1	+1	+1
$\frac{5\pi}{12}$	-1	-1	-1	+1	0
$\frac{5\pi}{18}$	-1	-1	-1	+1	-1
$\frac{5\pi}{36}$	-1	-1	-1	-1	-1
0	-1	-1	+1	-1	-1

velocity vector out of the collision cone. Two types of matrix games are considered. In the first game, the acceleration magnitudes of the two players are the (constant) controls and each player applies its acceleration normal to its velocity vector. In the second game, the acceleration magnitudes vary according to a dynamic inversion guidance law designed based on the collision cone, and the (constant) controls are the angles at which these accelerations are applied relative to the velocity vector of each player. The first game is an open loop game, while the second game is a closed loop game in the sense that both players use continuous feedback of the instantaneous collision cone between the players. These investigations demonstrate several interesting facets illustrating the presence of (local) saddle points in the game, and call for further investigations using the tools of differential games and optimal control theory.

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