ALMOST NON-NEGATIVE CURVATURE AND TORUS SYMMETRY IN LOW DIMENSIONS

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The following faculty members have examined the final copy of this thesis for form and content, and recommend that it be accepted in partial fulfillment of the requirement for the degree of Master of Science with a major in Mathematics.

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ABSTRACT

The classification of almost non-negatively curved, simply connected, closed manifolds is an open and difficult problem. The Symmetry Program suggests a means to approach such a classification. In this thesis we consider almost non-negatively curved n-manifolds, $4 \leq n \leq 6$, with torus symmetry. We obtain a classification up to equivariant diffeomorphism for $4 \leq n \leq 6$ when the action is of maximal symmetry rank. We also obtain a partial classification of the almost maximal symmetry rank case for $n = 5$, noting that a complete classification for $n = 4$ was obtained by Harvey and Searle.

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TABLE OF CONTENTS

LIST OF FIGURES

Figure Page

CHAPTER 1

Introduction

The classification of Riemannian manifolds with a lower sectional curvature bound is an open and difficult question. When there is a uniformly positive lower curvature bound, work due to Bonnet and Myers [37] tells us that the manifold M must be compact and have finite fundamental group. Work of Synge [48] tells us that, for a positively curved, compact manifold, if the dimension is even, M is either simply connected or has $\pi_1(M) \cong \mathbb{Z}_2$ and if the dimension is odd, M is orientable. It is then natural to restrict our attention to the class of closed, simply connected manifolds. Few examples of such manifolds are known, but all of them possess large symmetry. The Symmetry Program suggests a method to approach a classification with the guiding principle:

Classify positively and non-negatively curved manifolds with ”large” isometry groups.

Despite the fact that there are vastly more examples of closed, simply connected Riemannian manifolds that admit non-negative or almost non-negative sectional curvature, we know of no examples of manifolds admitting almost non-negative curvature but not admitting non-negative curvature, and likewise admitting non-negative curvature but not admitting positive curvature. Defining $\mathcal{M}_\kappa(n)$ to be the class of closed, simply connected Riemannian n-manifolds with sectional curvature greater than or equal to κ and $\mathcal{M}_{\sim 0}(n)$ to be the class of closed, simply connected, almost non-negatively curved Riemannian n-manifolds, we have

$$\mathcal{M}_1 \subset \mathcal{M}_0 \subset \mathcal{M}_{\sim 0},$$

but we do not know whether these inclusions are strict.

However, we can distinguish between these classes if we assume the existence of symmetries. Setting $\mathcal{M}_*^{C1}(n) = \{M \in \mathcal{M}_*(n) : M^n$ admits a cohomogeneity one isometric Lie group action$\}$, where a smooth action of a Lie group G on a manifold M is of *cohomogeneity*

n if the orbit space M/G is n-dimensional, we note that $\mathcal{M}^{C1}_{\sim 0}$ contains all cohomogeneity 1 manifolds by work of Schwachhöfer and Tuschmann [45]. We have

$$\mathcal{M}^{C1}_1 \subsetneqq \mathcal{M}^{C1}_0 \subsetneqq \mathcal{M}^{C1}_{\sim 0}$$

The proper inclusion $\mathcal{M}^{C1}_1 \subsetneqq \mathcal{M}^{C1}_0$ comes from classification work of Verdiani [49], Searle [46], and Grove, Wilking, and Ziller [20]. The proper inclusion $\mathcal{M}^{C1}_0 \subsetneqq \mathcal{M}^{C1}_{\sim 0}$ comes from work of He [24] and Grove, Verdiani, Wilking, and Ziller [17].

An outstanding conjecture of Bott illustrates the connection between the geometry of non-negatively curved manifolds and their topology:

Bott Conjecture. *A closed, simply connected, non-negatively curved Riemannian manifold is rationally elliptic.*

Here, a manifold is said to be *rationally elliptic* if its rational homotopy groups, viewed as a rational vector space, have finite dimension $\dim(\pi_*(M) \otimes \mathbb{Q}) < \infty$. This is equivalent to all but finitely many homotopy groups of M being finite [19]. Grove and Halperin have extended this conjecture to include almost non-negatively curved manifolds as well.

It is known that the isometry group of a closed Riemannian manifold is a compact Lie group by work of Myers and Steenrod [38], so it is natural to consider the rank of this group as a measure of symmetry. The *symmetry rank* of a Riemannian manifold M is defined to be the largest integer k such that T^k acts effectively by isometries on M. Grove and Searle [16] calculated the maximal symmetry rank for closed, positively curved n-manifolds to be $\lfloor \frac{n+1}{2} \rfloor$ and proved that those manifolds admitting a maximal symmetry rank torus action are S^n, $\mathbb{R}P^n$, $\mathbb{C}P^{n/2}$, or a lens space. For non-negative curvature, we have the following conjecture.

Maximal Symmetry Rank Conjecture. *[8] Let the torus T^k act isometrically and effectively on M^n, a closed, simply connected, non-negatively curved Riemannian manifold. Then the following hold:*

(1) $k \leq \lfloor \frac{2n}{3} \rfloor$.

(2) When $k = \lfloor \frac{2n}{3} \rfloor$, M^n is equivariantly diffeomorphic to Z/T^m with a linear T^k-action, where

$$Z = \prod_{i \leq r} S^{2n_i - 1} \times \prod_{i > r} S^{2n_i},$$

with $n_i \geq 2$, $r = 2\lfloor \frac{2n}{3} \rfloor - n$, $0 \leq m \leq 2n$ mod 3, and the T^m-action on Z is free and linear.

Galaz-García and Searle [11] and Escher and Searle [8] have confirmed the conjecture for dimensions ≤ 9. Note that a resolution of this conjecture would give a classification of closed, simply connected n-manifolds admitting a rank $\lfloor \frac{2n}{3} \rfloor$ torus action up to equivariant diffeomorphism based on the value of n mod 3.

$$\begin{aligned} M^{3m} &\simeq S^3 \times \cdots \times S^3 \\ M^{3m-1} &\simeq \begin{cases} S^5 \times S^3 \times \cdots \times S^3, \text{ or} \\ M^{3m}/T^1 \end{cases} \\ M^{3m-2} &\simeq \begin{cases} S^7 \times S^3 \times \cdots \times S^3, \text{ or} \\ S^3 \times S^4 \times S^3 \times \cdots \times S^3, \text{ or} \\ S^5 \times S^5 \times S^3 \times \cdots \times S^3, \text{ or} \\ M^{3m}/T^2 \text{ or } M^{3m-1}/T^2 \end{cases} \end{aligned}$$

Recent work of Harvey and Searle [22] in dimension 4 together with known results in dimensions 2 and 3 verify that this classification holds for closed, simply connected almost non-negatively curved manifolds in dimensions ≤ 4. It is natural to conjecture that the classification extends to higher dimensions, and in this thesis we show that it holds for $n = 5$ and 6.

Theorem A. *Let T^k act smoothly and effectively on a closed, smooth, simply connected n-manifold M, $4 \leq n \leq 6$. Then the following hold.*

(1) $k \leq \lfloor \frac{2n}{3} \rfloor = n - 2$.

(2) If $k = n-2$ and there exists a sequence $\{g_i\}_{i=1}^{\infty}$ of Riemannian metrics on M for which the T^k-action is isometric, and $\{(M, g_i)\}_{i=1}^{\infty}$ is almost non-negatively curved, then M admits a T^k-invariant metric of non-negative sectional curvature.

Using the classification from [8], this implies that such a manifold admitting a T^{n-2}-action will be equivariantly diffeomorphic to one of the following spaces with a linear T^{n-2}-action.

$$
\begin{aligned}
&M^4 \simeq \begin{cases} S^4, \\ S^2 \times S^2, \\ \mathbb{C}P^2, \\ \mathbb{C}P^2 \# \pm \mathbb{C}P^2, \end{cases} \\
&M^5 \simeq \begin{cases} S^5, \\ S^3 \times S^2, \\ S^3 \tilde{\times} S^2, \text{ the nontrivial } S^3\text{-bundle over } S^2, \text{ or} \end{cases} \\
&M^6 \simeq S^3 \times S^3.
\end{aligned}
$$

In the almost maximal symmetry rank case, that is, when $k = \lfloor \frac{2n}{3} \rfloor - 1$, we have the following result for $n = 5$.

Theorem B. *Let T^2 act smoothly and effectively on a closed, smooth, simply connected 5-manifold M. Let $\{g_n\}_{n=1}^{\infty}$ be a sequence of Riemannian metrics on M for which the T^2-action is isometric, and suppose that $\{(M, g_n)\}_{n=1}^{\infty}$ is almost non-negatively curved. Then there are at most 5 isolated orbits and if there are exactly 5, one of the following holds:*

a) All 5 are isolated circle orbits and there are no exceptional orbits; or

b) There is a unique isolated $T^2/(\mathbb{Z}_2 \times \mathbb{Z}_2)$ orbit.

Comparing to the case where $\sec(M) \geq 0$ [12], this results in 3 additional orbit spaces appearing when M is almost non-negatively curved. Restricting to the case where there are at most 4 isolated orbits, none of these configurations occur and we obtain the following result.

Theorem C. *Let T^2 act smoothly and effectively on a closed, smooth, simply connected 5-manifold M. Let $\{g_n\}_{n=1}^{\infty}$ be a sequence of Riemannian metrics on M for which the T^2 action is isometric and suppose that $\{M, g_n\}_{n=1}^{\infty}$ is almost non-negatively curved. Suppose that there are at most 4 isolated orbits. Then M is diffeomorphic to one of S^5, $S^3 \times S^2$, $S^3 \widetilde{\times} S^2$ (the nontrivial S^3-bundle over S^2), or the Wu manifold $SU(3)/SO(3)$.*

This thesis consists of four chapters. In Chapter 2, we discuss relevant notation and background material from Riemannian geometry, as well as the basics of isometric actions of compact Lie groups on Riemannian manifolds, and Alexandrov geometry, with a focus on orbit spaces of isometric group actions.

In Chapter 3 we prove Theorem A. We first apply results of Kim, McGavran, and Pak [27] and Ishida [25] to prove Part (1). For Part (2), theorems of Grove, Wilking, and Yeager [19] give us information about the orbit space. We then show that the action is locally standard, which allows us to apply a theorem of Dong, Escher, and Searle [7] to achieve the result.

In Chapter 4, we prove Theorems B and C. The arguments used extend those of Harvey and Searle [22] in dimension 4 for closed, simply connected manifolds with almost non-negative curvature and Galaz-García and Searle [12] in dimension 5 for closed, simply connected manifolds with non-negative curvature. The chapter is split into sections, where Section 4.1 proves bounds on the number of isolated circle orbits, Section 4.2 proves a lemma eliminating certain configurations when there are 5 isolated circle orbits, Sections 4.3 and 4.4 consider the case when the orbit space M^* is without boundary, and Section 4.5 considers the case where M^* has boundary.

CHAPTER 2

Preliminaries

In this chapter we introduce notation and define basic concepts that are used throughout this thesis.

2.1 Riemannian Manifolds

The fundamental objects of study in this thesis are Riemannian manifolds. To build up to a definition of the same, we start with smooth manifolds, which are spaces that are locally homeomorphic to $\mathbb{R}^n$, equipped with a differentiable structure. For further information on this topic, see, for example, Lee [31], [30] and do Carmo [6].

Definition 2.1.1 [Smooth manifold]. A *smooth manifold* of dimension n is a second-countable, Hausdorff topological space M along with a family of injective maps $\varphi_\alpha : U_\alpha \to M$ of open sets $U_\alpha \subset \mathbb{R}^n$ into M such that

(1) $\bigcup_\alpha \varphi_\alpha(U_\alpha) = M$;

(2) For any pair α, β with $\varphi_\alpha(U_\alpha) \cap \varphi_\beta(U_\beta) = W \neq \emptyset$, the sets $\varphi_\alpha^{-1}(W)$ and $\varphi_\beta^{-1}(W)$ are open in $\mathbb{R}^n$ and $\varphi_\beta^{-1} \circ \varphi_\alpha = \varphi_{\alpha,\beta}$ is a differentiable map; and

(3) The family $\{(U_\alpha, \varphi_\alpha)\}$ is maximal with regard to conditions (1) and (2), that is, there exists no family $\{(U'_\alpha, \varphi'_\alpha)\}$ containing $\{(U_\alpha, \varphi_\alpha)\}$ as a proper subset.

The pairs (U, φ) are called *coordinate charts* or *charts*, the collection $\{(U_\alpha, \varphi_\alpha)\}_{\alpha \in A}$ is called a *differentiable structure* or an *atlas*, and the maps $\varphi_{\alpha,\beta}$ are called *transition maps*. A trivial example of a smooth manifold is given by $\mathbb{R}^n$ with the atlas containing $\{(\mathbb{R}^n, \mathrm{Id}_{\mathbb{R}^n})\}$.

With this definition in place, we now provide a few relevant examples.

Example 2.1.2. *The n-sphere S^n is the set $\{x \in \mathbb{R}^{n+1} : |x| = 1\}$. S^n is a manifold with the subspace topology inherited from $\mathbb{R}^{n+1}$. For an odd-dimensional sphere, we may also use the* *definition* $S^{2n+1} = \{z \in \mathbb{C}^{n+1} : |z| = 1\}$, *where the sphere is a manifold with the subspace topology from the ambient space* $\mathbb{C}^{n+1}$.

Example 2.1.3. *Complex projective space* $\mathbb{C}P^n := (\mathbb{C}^n \setminus \{0\})/\sim$, *where* $z \sim w$ *if* $z = \lambda w$ *for some non-zero* $\lambda \in \mathbb{C}$. $\mathbb{C}P^n$ *with the quotient topology from* $\mathbb{C}^n \setminus \{0\}$ *induced by this equivalence is a* $2n$*-dimensional manifold.*

The differentiable structure allows us to define differentiability for functions between smooth manifolds.

Definition 2.1.4 [Smooth map]. Let M^m and N^n be smooth manifolds with differentiable structures $\{(U_\alpha, \varphi_\alpha)\}$ and $\{(V_\beta, \psi_\beta)\}$, respectively. A map $f : M \to N$ is *smooth* at $p \in M$ if there exist charts (U, φ) and (V, ψ) on M and N, respectively, such that $p \in \varphi(U)$, $f(\varphi(U)) \in \psi(V)$, and the map $\psi^{-1} \circ f \circ \varphi : U \to V$ is smooth in the standard Euclidean sense. The map f is called *smooth* if it is smooth at all points $p \in M$.

We may add to the structure of a smooth manifold the concept of a Riemannian metric, which is a smoothly varying inner product on the tangent space of a manifold. In $\mathbb{R}^n$, the tangent space is often identified with $\mathbb{R}^n$ itself, but for general manifolds we must define it more abstractly. We begin with the concept of the tangent space.

Definition 2.1.5 [Tangent space]. Let M be a smooth manifold, and let $\alpha : (-\varepsilon, \varepsilon) \subset \mathbb{R} \to M$ be a smooth function. Let $\mathcal{D}$ be the set of smooth maps $f : M \to \mathbb{R}$. Then the *tangent vector to the curve* α at $t = 0$ is a function $\alpha'(0) : \mathcal{D} \to \mathbb{R}$ defined by

$$\alpha'(0)f = \left.\frac{d(f \circ \alpha)}{dt}\right|_{t=0}.$$

A *tangent vector at* p is the tangent vector of some smooth curve $\alpha : (-\varepsilon, \varepsilon) \to M$ with $\alpha(0) = p$. We denote by T_pM the set of all tangent vectors of M at p, called the *tangent space* of M at p.

At each point $p \in M^n$, the tangent space T_pM admits the structure of an n-dimensional real vector space. If $p = \varphi(x_0)$ for some chart (U, φ) on M, $x_0 \in U$, then the set $\{\gamma_i'(0)\}$ defines a basis for this vector space, where

$$\gamma_i(t) = \varphi(x_0 + te_i),$$

with $\{e_i\}_{i=1}^n$ being the standard basis on $\mathbb{R}^n$. Since U is open, φ is defined in a neighborhood of x_0 and thus γ_i is defined in a neighborhood of 0.

Taking the disjoint union of the tangent space at each point of a smooth manifold M, we may construct another smooth manifold, called the tangent bundle of M.

Definition 2.1.6 [Tangent bundle]. Let M^n be a smooth manifold with a differentiable structure $\{(U_\alpha, \varphi_\alpha)\}$. Define the set $TM = \{(p, v) : p \in M, v \in T_pM\}$. For each α, set $(x_1^\alpha, \ldots, x_n^\alpha)$ to be the coordinates in U_α and $\{\gamma_1^\alpha, \ldots, \gamma_n^\alpha\}$ the corresponding bases of the tangent spaces. Define a map

$$\psi_\alpha : U_\alpha \times \mathbb{R}^n \to TM$$

by

$$\psi_\alpha(x_1^\alpha, \ldots, x_n^\alpha, u_1, \ldots, u_n) = \left(\varphi_\alpha(x_1^\alpha, \ldots, x_n^\alpha), \sum_{i=1}^n u_i \gamma_i^\alpha \right).$$

With the differentiable structure $\{(U_\alpha \times \mathbb{R}^n, \psi_\alpha)\}$, TM is a smooth manifold of dimension $2n$, called the *tangent bundle* of M.

With the concept of a tangent space, we may now define the essential structure of a Riemannian manifold, the Riemannian metric.

Definition 2.1.7 [Riemannian metric]. A *Riemannian metric* on a smooth manifold M is a correspondence g which associates to each point $p \in M$ an inner product $\langle \ , \ \rangle_p$ on T_pM, which varies differentiably in the following sense: If (U, φ) is a chart with $p \in \varphi(U)$, $\varphi(x_1, \ldots, x_n) = q \in \varphi(U)$, and $\frac{\partial}{\partial x_i}(q) = d\varphi_q(0, \ldots, 1, \ldots, 0)$, then $\langle \frac{\partial}{\partial x_i}(q), \frac{\partial}{\partial x_j}(q) \rangle_q = g_{ij}(x_1, \ldots, x_n)$ is a differentiable function on U.

A *Riemannian manifold* is a pair (M^n, g), where M is a smooth manifold and g is a Riemannian metric on M. When the metric is clear from context, we will omit g and refer to M as a Riemannian manifold.

The strongest concept of equivalence for Riemannian manifolds is that of isometry.

Definition 2.1.8 [Isometry]. Let (M, g) and (N, h) be Riemannian manifolds. A diffeomorphism $F : M \to N$ is called an *isometry* if $\langle u, v\rangle_p = \langle df_p(u), df_p(v)\rangle_{f(p)}$ for all $p \in M$ and all $u, v \in T_pM$.

To introduce the notion of curvature, we must define the concepts of vector fields and connections on smooth manifolds.

Definition 2.1.9 [Vector field]. A *vector field* X on a smooth manifold M is a correspondence assigning to each point $p \in M$ a tangent vector $X(p) \in T_pM$. This is a map $M \to TM$, and X is called *smooth* if this map is smooth.

The set of all smooth vector fields on M is denoted $\mathfrak{X}(M)$.

Definition 2.1.10 [Connection]. A *connection* on a smooth manifold M is a map

$$\nabla : \mathfrak{X}(M) \times \mathfrak{X}(M) \to \mathfrak{X}(M),$$

written $(X, Y) \mapsto \nabla_X Y$, satisfying the following properties:

(i) $\nabla_X Y$ is linear over $C^\infty(M)$ in X: for f_1, $f_2 \in C^\infty(M)$ and X_1, $X_2 \in \mathfrak{X}(M)$,

$$\nabla_{f_1 X_1 + f_2 X_2} Y = f_1 \nabla_{X_1} Y + f_2 \nabla_{X_2} Y;$$

(ii) $\nabla_X Y$ is linear over $\mathbb{R}$ in Y: for $a_1, a_2 \in \mathbb{R}$ and $Y_1, Y_2 \in \mathfrak{X}(M)$,

$$\nabla_X (a_1 Y_1 + a_2 Y_2) = a_1 \nabla_X Y_1 + a_2 \nabla_X Y_2; \text{ and}$$

(iii) ∇ satisfies the following product rule: for $f \in C^\infty(M)$,

$$\nabla_X (fY) = f \nabla_X Y + (Xf)Y.$$

$\nabla_X Y$ is called the *covariant derivative* of Y in the direction X.

We say that a connection ∇ on M is *compatible with* g if it satisfies the following product rule for all $X, Y, Z, \in \mathfrak{X}(M)$:

$$\nabla_X \langle Y, Z\rangle = \langle \nabla_X Y, Z\rangle + \langle Y, \nabla_X Z\rangle.$$

In fact, every Riemannian manifold (M, g) admits a unique connection on M which is compatible with g, called the *Levi-Civita connection*. This fact is known as the *Fundamental Theorem of Riemannian Geometry*.

Definition 2.1.11 [Riemann curvature tensor]. The *Riemann curvature tensor* R of a Riemannian manifold (M, g) is a correspondence that associates to every pair $X, Y \in \mathfrak{X}(M)$ a mapping $R(X, Y) : \mathfrak{X}(M) \to \mathfrak{X}(M)$ defined by

$$R(X, Y)Z = \nabla_Y \nabla_X Z - \nabla_X \nabla_Y Z + \nabla_{[X,Y]} Z,$$

where ∇ is the Levi-Civita connection on (M, g).

Definition 2.1.12 [Sectional curvature]. The *sectional curvature* $K(\sigma_p)$ at a point $p \in M$ of a 2-plane $\sigma \subset T_pM$ is

$$\frac{\langle R(u, v)u, v\rangle}{\langle u, u\rangle\langle v, v\rangle - \langle u, v\rangle^2},$$

where $\{u, v\}$ is a basis for σ.

A Riemannian manifold (M, g) is said to have *positive sectional curvature* if $K(\sigma_p) > 0$ for all $p \in M$ and for all 2-planes $\sigma \subset T_pM$. We denote this as $\sec(M, g) > 0$. Similarly, (M, g) has *non-negative sectional curvature* if $\sec(M, g) \geq 0$.

Three simply connected manifolds of particular note occur when $\sec(M, g) = \kappa$ for some fixed $\kappa \in \mathbb{R}$, called space forms. We discuss them in the following examples.

Example 2.1.13. *Euclidean space $\mathbb{R}^n$ with the Riemannian metric of the standard dot product on $\mathbb{R}^n$ (identifying $T_p\mathbb{R}^n$ with $\mathbb{R}^n$) has constant sectional curvature 0.*

Example 2.1.14. *The n-sphere S^n, $n \geq 1$, as discussed in Example 2.1.2 with the Riemannian metric inherited from the standard metric in the ambient $\mathbb{R}^{n+1}$ is a Riemannian manifold with constant sectional curvature 1. We may rescale the metric by $g_\kappa = \frac{1}{\kappa} g$ to obtain a Riemannian manifold of constant curvature κ for any $\kappa > 0$.*

Example 2.1.15. *Hyperbolic space $\mathbb{H}^n$ is an n-dimensional Riemannian manifold of constant sectional curvature -1. As with S^n, we may rescale the metric to achieve constant curvature of any negative value.*

In this thesis, we consider manifolds of almost non-negative curvature. As the name suggests, this condition approximates but does not necessarily achieve non-negative curvature.

Definition 2.1.16 [Almost non-negative curvature]. A smooth manifold M is *almost non-negatively curved* if it admits a sequence of Riemannian metrics $\{g_n\}_{n=1}^{\infty}$ such that $\sec(M, g_n) \geq -\frac{1}{n^2}$ and $\operatorname{diam}(M, g_n) \leq D$ for some fixed $D > 0$ for all n.

We highlight some intricacies of this definition. First, the diameter condition prevents us from taking an arbitrary Riemannian manifold and scaling the metric to meet the curvature condition. Second, we may rescale our metrics $\{g_n\}$ and possibly pass to a subsequence so that the resulting sequence $\{g'_n\}$ is almost non-negatively curved and D is any desired positive value. For the remainder of this thesis, we assume that $D = 1$.

2.2 Isometric Lie Group Actions

In this thesis, we consider compact Lie groups acting on Riemannian manifolds by isometries. Good resources for additional information on this topic are, for example, Bredon [2] and the survey by Ziller [52].

Definition 2.2.1 [Lie group acting by isometries]. A Lie group G is said to *act by isometries* on a Riemannian manifold (M, h) if there exists a smooth map $\Theta : G \times M \to M$ such that

(1) $\Theta(g, \Theta(g', x)) = \Theta(gg', x)$ for all $g, g' \in G$ and for all $x \in M$;

(2) $\Theta(e, x) = x$ for all $x \in M$, where e is the identity of G; and

(3) The map $\theta_g : M \to M$ defined by $\theta_g(p) = \Theta(g, p)$ is an isometry for all g.

We may think of a Lie group acting by isometries as a Lie group homomorphism $\varphi : G \to \text{Isom}(M)$.

We now fix some notation for the remainder of the paper. The *orbit* of a point $x \in M$ is the set $Gx = \{gx : g \in G\}$. The *isotropy subgroup* of a point $x \in M$, also called the *stabilizer*, is the subgroup $G_x = \{g \in G : gx = x\}$. The *ineffective kernel* of the G-action is the subgroup $K = \bigcap_{x \in M} G_x$. We say that G acts *effectively* if the only element of G which fixes all of M is the identity. This is equivalent to $K = \{e\}$.

An orbit is said to be *principal* if its isotropy subgroup is minimal, that is, it does not contain a proper subgroup conjugate to another isotropy subgroup. An *exceptional orbit* has the same dimension as a principal orbit but does not have minimal isotropy. If an orbit is neither principal nor exceptional, that is, if it has smaller dimension than a principal orbit, it is called *singular*.

The group G acts *freely* if G_x is trivial for all $x \in M$. Similarly, G acts *semifreely* if $G_x = \{e\}$ or G for all $x \in M$, *almost freely* if G_x is finite for all $x \in M$, and *almost semifreely* if G_x is finite or all of G for all $x \in M$.

Using the information of a G-action on a manifold M, we may construct a quotient space known as the *orbit space*, denoted M/G or M^*. The elements of this space are the orbits Gx, and it inherits the quotient topology from the map $\pi : M \to M/G$ defined by $\pi(x) = Gx$. We say that M is of *cohomogeneity* n if it admits a G-action such that $\dim(M/G) = n$.

The dimension of M/G is constrained by the *fixed point set* of the G-action, $M^G = \{x \in M : gx = x \ \forall g \in G\}$, also written $\text{Fix}(M;G)$. Define dim(Fix(M;G)) = $\max\{\dim(N) : N \text{ is a component of } \text{Fix}(M;G)\}$. In particular, $\dim M/G \geq \dim(M^G) + 1$ for any non-trivial G-action. In light of this, we define the *fixed point cohomogeneity* $\text{cohomfix}(M;G) = \dim(M/G) - \dim(M^G) - 1$. The G-action on M is *fixed-point-homogeneous* if $\text{cohomfix}(M;G) = 0$.

One tool for analyzing the orbit space M/G when it is homeomorphic to an interval is known as the group diagram.

Definition 2.2.2 [Group diagram]. Let G act isometrically on a manifold M. If the orbit space M/G is homeomorphic to an interval, the *group diagram* is the quadruple (G, H, K_+, K_-), where H is the isotropy subgroup of principal orbits and K_+ and K_- are the isotropy subgroups corresponding to the endpoints $\overline{p}_+$ and $\overline{p}_-$, respectively.

We note that Mostert [36] shows that group diagrams are in one-to-one correspondence with cohomogeneity one manifolds (see Theorem 4).

An important tool to understand orbit spaces of smooth actions on manifolds is the Slice Theorem. One formulation of the same is as follows.

Slice Theorem. *[15] Let M be a smooth manifold, and let the group G act smoothly on M. Then for any $x \in M$, a sufficiently small tubular neighborhood $D(Gx)$ of Gx is equivariantly diffeomorphic to $G \times_{G_x} D_x^{\perp}$.*

2.3 Orbit Spaces

In this section, we consider orbit spaces of isometric actions of compact groups on closed manifolds with a lower curvature bound. First, we note a significant result which underpins many arguments in the area.

Kleiner Isotropy Lemma. *[28] Let G be a Lie group acting isometrically on an Alexandrov space X (to be defined in Section 2.4). If $c : [0, d] \to X$ is a minimal geodesic between the orbits $Gc(0)$ and $Gc(d)$, then for any $t \in (0, d)$, the isotropy subgroup $G_{c(t)}$ is a subgroup of $G_{c(0)}$ and of $G_{c(d)}$.*

The following standard results give us significant topological information about the orbit space M^n/T^k.

Theorem 2.3.1. *[2] Let G be a compact Lie group acting on a topological space X. If either G is connected or G has a nonempty fixed point set, then the orbit projection map $\pi : X \to X/G$ induces an onto map on fundamental groups.*

Lemma 2.3.2. *If Y is a simply connected 3-manifold with m boundary components, then Y is homeomorphic to S^3 with m copies of D^3 removed.*

One relevant example is the orbit space $X_{s,t}$.

Example 2.3.3. *Let S^1 act on S^3 by $e^{i\theta} \cdot (z_1, z_2) = (e^{is\theta} z_1, e^{it\theta} z_2)$. To ensure that the action is effective, we assume that $\gcd(s,t) = 1$. The orbit space $S^3/S^1 = X_{s,t}$ is useful in the analysis of isometric torus actions on manifolds.*

2.4 Alexandrov Spaces

The quotient spaces of isometric group actions on a manifold with a lower sectional curvature bound κ need not be manifolds. By work of Perelman and Petrunin [43], they belong to a larger class of spaces called Alexandrov spaces, which also includes Riemannian manifolds with $\sec \geq \kappa$ as a subset. See, for example, Burago, Burago, and Ivanov [3] and Plaut's survey [44] for more details on Alexandrov spaces.

Definition 2.4.1 [Length structure]. A *length structure* on a topological space X is a class A of admissible paths $\gamma : I \to X$, along with a map $L : A \to \mathbb{R}_+ \cup \{\infty\}$, satifying the following:

(1) The class A is closed under restrictions: if $\gamma : [a,b] \to X$ is an admissible path and $a \leq c \leq d \leq b$, then the restriction $\gamma|_{[c,d]}$ of γ to $[c,d]$ is also admissible.

(2) A is closed under concatenations of paths.

(3) A is closed under linear reparametrizations: for an admissible path $\gamma : [a,b] \to X$ and a homeomorphism $\varphi : [c,d] \to [a,b]$ of the form $\varphi(t) = \alpha t + \beta$, the composition $\gamma \circ \varphi(t) = \gamma(\varphi(t))$ is also an admissible path.

The value $L(\gamma)$ is called the *length* of γ.

Given a topological space X equipped with a length structure A, we define a distance function for $x, y \in X$ by

$$d_L(x,y) = \inf\{L(\gamma); \gamma : [a,b] \to X, \gamma \in A, \gamma(a) = x, \gamma(b) = y\}.$$

A metric space whose distance function can be realized as such a d_L is called a *length space*.

Definition 2.4.2 [Alexandrov space]. An *Alexandrov space* is a finite dimensional, locally compact, locally complete length space with a lower curvature bound κ, denoted curv $\geq \kappa$, in the triangle comparison sense.

The lower curvature bound in the triangle comparison sense is defined as follows. For every geodesic triangle $\triangle ABC$ in an Alexandrov space X, we construct a geodesic triangle $\triangle A'B'C'$ with the same side lengths in a 2-dimensional space of constant curvature κ, called a *model space*. Letting α, β, γ denote the angles in $\triangle ABC$ and α', β', γ' the angles in $\triangle A'B'C'$, we say that $\mathrm{curv}(X) \geq \kappa$ if

$$\alpha \geq \alpha',$$
$$\beta \geq \beta', \text{ and}$$
$$\gamma \geq \gamma'.$$

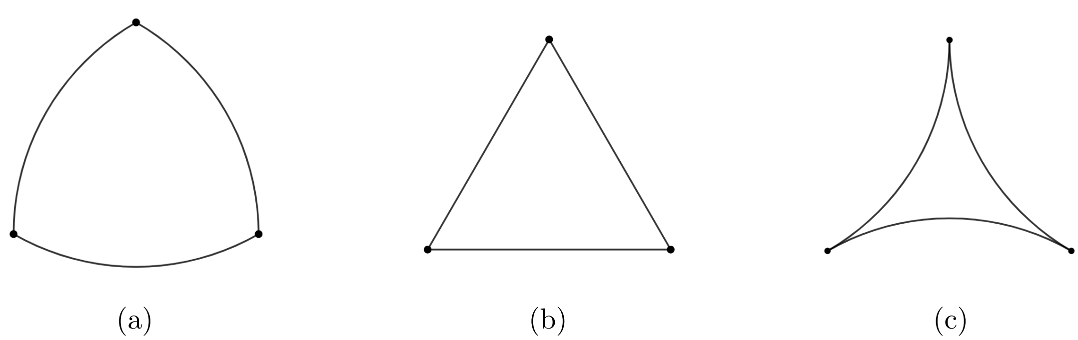


Figure 2.1: Visualization of geodesic triangles in constant positive (a), zero (b), and negative (c) curvature.

We may extend the concept of diameter of a metric space (X, d), denoted $\mathrm{diam}(X, d)$, defined to be the maximum distance between two points in a metric space, to more than two points. For an integer $q > 1$, the *q-extent* is a measure of the maximum average distance between a set of q points in X, defined by

$$\mathrm{xt}_q(X) = \binom{q}{2}^{-1} \sup_{x_1,\ldots,x_q \in X} \sum_{1 \leq i < j \leq q} \mathrm{dist}(x_i, x_j).$$

In this paper, we refer to $\mathrm{xt}_q(X)$ as the normalized q-extent. It is often of use to consider $\binom{q}{2}\mathrm{xt}_q(X)$, which we will refer to as the un-normalized q-extent. A set of points $\{x_i\}_{i=1}^q \subset X$ such that

$$\binom{q}{2}^{-1} \sum_{1\le i<j\le q} \mathrm{dist}(x_i, x_j) = \mathrm{xt}_q(X)$$

is called a *q-extender*.

Similarly to Riemannian manifolds, we may define the concept of almost non-negative curvature for Alexandrov spaces.

Definition 2.4.3 [Almost non-negatively curved Alexandrov space]. A sequence of Alexandrov spaces $\{(X, \mathrm{dist}_n)\}_{n=1}^{\infty}$ is *almost non-negatively curved* if there is a fixed $D > 0$ such that $\mathrm{diam}(X, \mathrm{dist}_n) \le D$ and $\mathrm{curv}(X, \mathrm{dist}_n) \ge -\frac{1}{n^2}$ for all n.

From [43], we know that if M is a Riemannian manifold with curvature bounded below by κ and G is a compact Lie group acting on M by isometries, then M/G is an Alexandrov space with $\mathrm{curv} \ge \kappa$. Since $\mathrm{diam}(M) \ge \mathrm{diam}(M/G)$, the quotient of an almost non-negatively curved manifold by a closed group of isometries yields an almost non-negatively curved Alexandrov space.

For an Alexandrov space, the analog of the unit tangent space of a smooth manifold is the so-called space of directions.

Definition 2.4.4 [Space of directions]. The *space of directions* at $p \in X$, $\Sigma_p X$, is the completion of the space of geodesic directions at p.

When X is the orbit space of an isometric G-action on a Riemannian manifold with $\sec \ge \kappa$, $\Sigma_{\overline{p}} X$ is isometric to $S_p^{\perp}/G_p$, where $S_p^{\perp}$ is the normal sphere to the orbit Gp at $p \in M$. This fact can be seen as a consequence of the Slice Theorem.

We distinguish between points in Alexandrov spaces by their spaces of directions. A point $p \in X^n$ is called *regular* if its space of directions Σ_p is isometric to a round sphere S^{n-1}. Otherwise, p is called *singular*. Among singular points, we distinguish those $p \in X$

for which $\text{diam}(\Sigma_p) \leq \pi/2$ and call them *extremal points*. An *extremal subset* is a subset of an Alexandrov space composed solely of extremal points.

We return to $X_{s,t}$ as defined in Example 2.3.3 in the context of Alexandrov spaces. Let T^{n-3} act on M^n, a closed manifold with $\sec \geq \kappa$, and assume $S^1 \leq T^{n-3}$ fixes an isolated point p. Then M/T^{n-3} is an Alexandrov space and $\Sigma_{\overline{p}} = S^3/S^1 = X_{s,t}$.

We have the following result from Kalka and Yang.

Lemma 2.4.5. *[26]* $\text{xt}_4(X_{s,t}) \leq \frac{\pi}{3}$ *and* $\text{xt}_5(X_{s,t}) \leq \frac{3\pi}{10}$. *Moreover, given 4 distinct points in* $X_{s,t}$ *with* $(|s|, |t|) \neq (1,1)$,

$$\sum_{1 \leq i < j \leq 4} \text{dist}(x_i, x_j) < 2\pi.$$

Thus the 4-extent of 4 distinct points in $X_{s,t}$ *is strictly less than* $\pi/3$.

Another relevant generalization from manifold theory to Alexandrov spaces is that of the First Variation of Arc Length.

Proposition 2.4.6 (First Variation of Arc Length)**.** *[3] Let* X *be an Alexandrov space, and let* $p, q, r \in X$. *Let* $\gamma : [0, \epsilon] \to X$ *be a shortest path with* $\gamma(0) = p$ *and* $\gamma(\epsilon) = r$. *Fix the shortest path* σ *between* p *and* q. *Define* $\alpha = \angle qpr$ *to be the angle between* γ *and* σ. *Then*

$$\text{dist}(q, \gamma(t)) \leq \text{dist}(p, q) - t\cos\alpha + o(t), \text{ as } t \to 0^+.$$

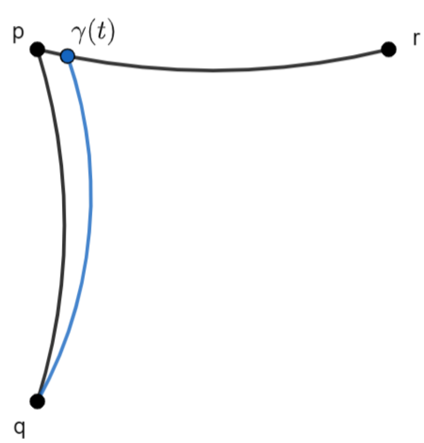


Figure 2.2: Visualization of the First Variation of Arc Length.

The following result concerning the derivative of $\mathrm{dist}(p, \gamma(t))$ from Plaut [44] is useful for many calculations.

Proposition 2.4.7. *[44] Let X be an Alexandrov space, and let $p, q \in X$ with $p \neq q$. Then for any shortest path γ with $\gamma(0) = p$, the right derivative of $\mathrm{dist}(q, \gamma(t))$ at $t = 0$ equals $-\cos(\alpha)$, where α is the angle between γ and the shortest path between q and p.*

It is possible to construct Alexandrov spaces from pre-existing examples and preserve the lower curvature bound. One such procedure involves gluing spaces together along their boundaries. The following Gluing Theorem summarizes work of Perelman [41], Petrunin [42], and Wörner [51].

Gluing Theorem ([41],[42],[51])**.** *Let X and Y be Alexandrov spaces of the same dimension, both with curv $\geq \kappa$. Suppose that $A \subset X$ and $B \subset Y$ are connected components of the boundaries of X and Y, respectively. If $f : A \to B$ is an isometry with respect to the intrinsic metrics on A and B, then $X \cup_f Y$ with the induced length metric is also an Alexandrov space of* $\mathrm{curv} \geq \kappa$.

Work by Liu and Shen [32] bounding the Betti numbers of an Alexandrov space was extended by Wong [50] to a bound on the number of boundary components. This upper bound $C(n, D, k)$ depends on the dimension n, the diameter D, and the lower curvature bound k. Combining this with the Gluing Theorem yields the following proposition.

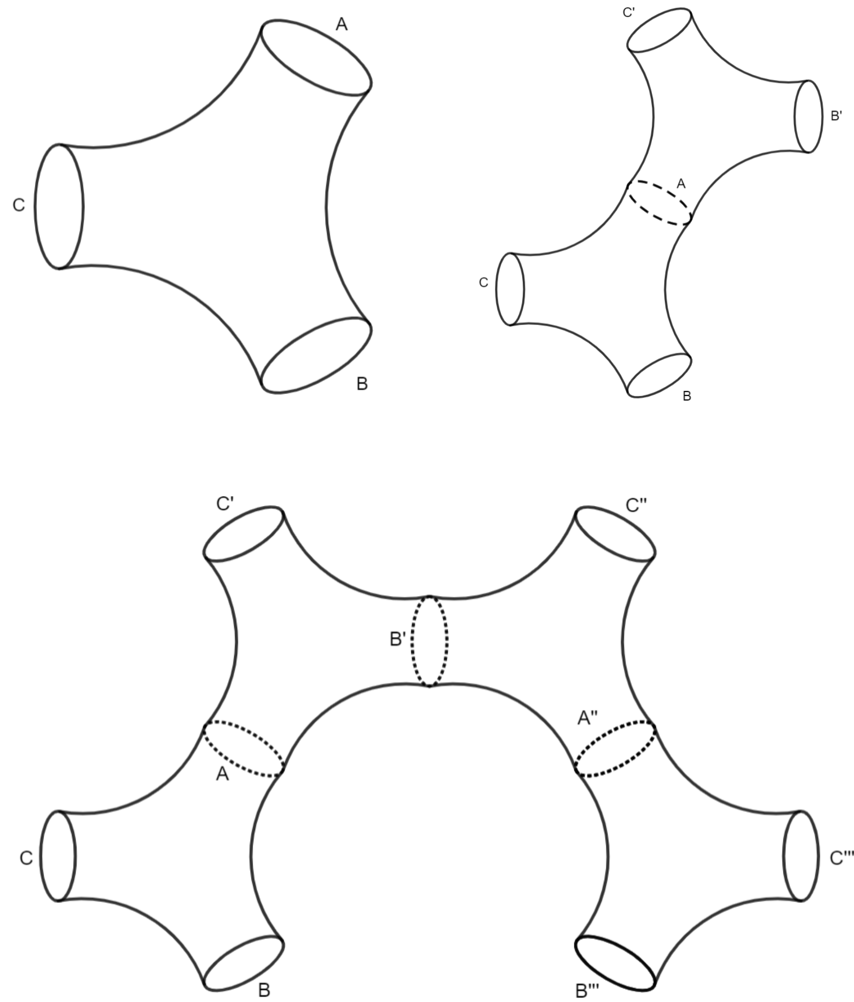


Figure 2.3: Visualization of the construction in Proposition 2.4.8.

Proposition 2.4.8. *[22] An almost non-negatively curved Alexandrov space has at most two boundary components.*

Proof. An almost non-negatively curved Alexandrov space X of dimension n has at most $C(n, 1, -1)$ boundary components. Suppose to derive a contradiction that X has three boundary components, call them A, B, C. By the Gluing Theorem, we may construct an almost non-negatively curved Alexandrov space by gluing two copies of X together along A, creating an almost non-negatively curved Alexandrov space X' with four boundary components B, B', C, C'. We now glue a copy of X to X' along the boundary components B' and B''. Repetition of this process, rescaling the metric if necessary, yields an almost non-negatively curved Alexandrov space with arbitrarily many boundary components, a contradiction to the upper bound $C(n, 1, -1)$. □

A second construction on Alexandrov spaces which preserves the lower curvature bound is that of taking a double branched cover under certain conditions. The following lemma due to Harvey and Searle [21] is a generalization of a result by Grove and Wilking [18].

Lemma 2.4.9. *[21] Let X be an Alexandrov space of* $\mathrm{curv} \geq k$ *which is homeomorphic to S^3. Let c be a simple closed curve in X which is an extremal subset. Then the double branched cover of X over c, $X_2(c)$, is an Alexandrov space of* $\mathrm{curv} \geq k$.

Recall the following definition.

Definition 2.4.10 [Defect]. The *defect* of a triangle $\triangle ABC$ in a metric space X is $\pi - (\alpha + \beta + \gamma)$.

In almost non-negative curvature, the defect of a geodesic triangle is very close to 0, as we see in the following lemma.

Lemma 2.4.11. *[22] Let X be an Alexandrov space with* $\mathrm{curv}(X) \geq -k^2$ *and* $\mathrm{diam}(X) \leq 1$. *Then the defect of any geodesic triangle $\triangle ABC$ in X is bounded above by a function $\mu(k) = O(k^2)$.*

Proof. By definition, for any geodesic triangle $\triangle ABC$ we have the inequality $\alpha + \beta + \gamma \geq \alpha' + \beta' + \gamma'$, where $\triangle A'B'C'$ is the comparison triangle in the hyperbolic plane of constant curvature $-k^2$. Thus the defect of $\triangle ABC$ is bounded above by the defect of $\triangle A'B'C'$. We also have that the defect of $\triangle A'B'C'$ is equal to its area multiplied by k^2 by the Gauss-Bonnet Theorem. From Bezdek [1], we know that the area of a polygon in the hyperbolic plane with a given perimeter is maximized by the area of a regular polygon with the same perimeter. Solving the hyperbolic law of cosines for α' with all side lengths equal to 1 and Gaussian curvature $-k^2$, we get

$$\alpha' = \arccos \frac{\cosh^2(k) - \cosh(k)}{\sinh^2(k)} = \arccos(u(k)).$$

Computing the Taylor expansion of the right hand side, we find

$$\alpha' = \arccos(u(k)) - \frac{1}{\sqrt{1-u^2(k)}} \frac{(2\cosh(k)-1)\sinh^2(k) - 2\cosh^3(k) + 2\cosh^2(k)}{\sinh^3(k)} k - O(k^2).$$

Taking the limit as $k \to 0^-$, the middle term vanishes and we find

$$\alpha' = \frac{\pi}{3} - O(k^2).$$

Similar inequalities hold for β' and γ', so

$$\begin{aligned} \pi - (\alpha + \beta + \gamma) &\leq \pi - (\alpha' + \beta' + \gamma') \\ &\leq O(k^2). \end{aligned}$$

□

2.5 Isometric Quotients of Almost Non-Negatively Curved Manifolds

In the previous section we proved that the orbit space of an almost non-negatively curved manifold can have at most two boundary components. This, along with the preceding topological facts and following lemma, allow us to classify the orbit space M^5/T^2 up to homeomorphism.

Lemma 2.5.1. *[2] Let G act on M by cohomogeneity 3, with $H_1(M;\mathbb{Z}_2) = 0$ and all orbits connected. Then M^* is a 3-manifold with or without boundary.*

Since the torus is connected and compact and the manifolds we consider are simply connected, this implies that the orbit spaces will also be simply connected. Thus M^* is a simply connected 3-manifold with at most two boundary components, and is therefore homeomorphic to S^3, D^3, or $S^2 \times I$.

The structure imposed by almost non-negative curvature allows us to control the size of the singular set in our orbit space.

Proposition 2.5.2. *[22] Let $\{(X, dist_n)\}$ be an almost non-negatively curved sequence of 3-dimensional Alexandrov spaces. Then for sufficiently large n, $(X, dist_n)$ can have at most five interior points with spaces of directions isometric to S^3/S^1.*

Proof. Let $S \subset X$ be the set of points with spaces of directions isometric to S^3/S^1, and suppose, to derive a contradiction, that $|S| = 6$. Set $S = \{p_i\}_{i=1}^6$. There are $\binom{6}{3} = 20$ distinct triangles with vertices in S. Since $\mathrm{curv}(X, \mathrm{dist}_n) \geq -\frac{1}{n^2}$, we have from Lemma 2.4.11 that each such triangle has $\alpha + \beta + \gamma \geq \pi - O(n^2)$. Noting by $\sum\limits_{i,j,k} \angle p_i p_j p_k$ the sum of all angles with $p_i, p_j, p_k \in S$, we have

$$\sum_{i,j,k} \angle p_i p_j p_k \geq 20(\pi - O(\tfrac{1}{n^2})).$$

We also have a bound on this sum based on the extents of the spaces of directions Σ_{p_j}. The 5-extent $\mathrm{xt}_5(\Sigma_{p_j}) \leq \frac{3\pi}{10}$, so the un-normalized 5-extent gives

$$\sum_{i,k} \angle p_i p_j p_k \leq 3\pi.$$

Summing this inequality over all p_j and combining with the previous displayed inequality yields

$$20(\pi - O(\tfrac{1}{n^2})) \leq 18\pi,$$

which does not hold for sufficiently large n, giving us the desired contradiction. □

CHAPTER 3

Maximal Symmetry Rank

In this chapter we prove Theorem A, which we restate for convenience.

Theorem A. *Let T^k act smoothly and effectively on a closed, smooth, simply connected n-manifold M, $4 \leq n \leq 6$. Then the following hold.*

(1) $k \leq \lfloor \frac{2n}{3} \rfloor = n-2$.

(2) If $k = n-2$ and there exists a sequence $\{g_i\}_{i=1}^{\infty}$ of Riemannian metrics on M for which the T^k-action is isometric, and $\{(M, g_i)\}_{i=1}^{\infty}$ is almost non-negatively curved, then M admits a T^k-invariant metric of non-negative sectional curvature.

Remark 3.0.1. The spaces that occur in Part (2) of Theorem A are equivariantly diffeomorphic to

$$M^4 \simeq \begin{cases} S^4, \\ S^2 \times S^2, \\ \mathbb{C}P^2, \\ \mathbb{C}P^2 \# \pm \mathbb{C}P^2, \end{cases}$$
$$M^5 \simeq \begin{cases} S^5, \\ S^3 \times S^2, \\ S^3 \tilde{\times} S^2, \text{ the nontrivial } S^3\text{-bundle over } S^2, \end{cases}$$
$$M^6 \simeq S^3 \times S^3,$$

with a linear T^k-action.

In order to prove Part (1) of Theorem A, we suppose that $k = n-2$. The following result due to Kim, McGavran, and Pak [27] gives significant information about the orbit space M^*.

Theorem 3.0.2. *[27] Let T^{n-2} act effectively on M^n, a closed, simply connected n-manifold, $n \geq 4$. Then both T^1 and T^2 subgroups of T^{n-2} must appear as isotropy subgroups of T^{n-2}*

and these are the only possible nontrivial isotropy subgroups. Hence the orbit space M^* is a disk D^2 with the boundary edges corresponding to orbits with T^1-isotropy, the boundary vertices corresponding to orbits with T^2-isotropy, and the interior points corresponding only to principal orbits. Moreover, for each orbit with T^2-isotropy, the T^1-isotropy subgroups corresponding to the two edges that emanate from this vertex must be subgroups of that T^2 isotropy subgroup.

This result is sufficient to show that the T^{n-2}-action is an isotropy-maximal action, as defined below.

Definition 3.0.3 [Isotropy-maximal action]. Let M^n be a connected manifold with an effective T^k-action. We call the T^k-action on M^n *isotropy-maximal* provided either of the following equivalent conditions hold:

(1) There is a point $x \in M$ such that the dimension of the isotropy subgroup is $n-k$, that is, $\dim(T^k_x) = n-k$; or

(2) There is a point $x \in M$ such that $\dim(T^k(x)) = 2k-n$, in which case the orbit $T^k(x)$ through $x \in M$ is called a *minimal orbit*.

From Theorem 3.0.2, we know that there are T^{n-2}/T^2-orbits, which are of dimension $n-4 = 2k-n$, with $k = n-2$, and are therefore minimal orbits. The following lemma of Ishida [25] provides enough information to verify that $n-2$ is the maximal symmetry rank under the hypotheses of Part (1) of Theorem A.

Lemma 3.0.4. *[25] Let M be a connected manifold and let T^k act effectively on M. If T^l is a subtorus of T^k such that the action restricted to T^l is isotropy-maximal, then $T^k = T^l$. In particular, $k = l$.*

Since the T^{n-2}-action is isotropy-maximal, no larger torus can act effectively on M^n. Thus, $\lfloor \frac{2n}{3} \rfloor$ is the maximal symmetry rank of a closed, simply connected n-manifold, $4 \leq$

$n \leq 6$, proving Part (1) of Theorem A. We note that this result is purely topological and independent of any curvature assumptions.

To prove Part (2), we require a deeper understanding of the orbit space M^*. We first recall a result from [25].

Lemma 3.0.5. *[25] Let M be a connected manifold with an isotropy-maximal T^k-action. Then each minimal orbit is isolated.*

We also need the following result from [27].

Theorem 3.0.6. *[27] Let T^{n-2} act effectively on M^n, a closed, simply connected n-manifold. Then all isotropy subgroups generate the whole T^{n-2}, and there are at least $(n-2)$ different circle isotropy subgroups.*

Combining the information from Theorem 3.0.2 and Lemma 3.0.5, the orbit space M^* is a 2-disk with at least $n-2$ sides by Theorem 3.0.6, where the vertices are the projections of singular orbits with T^2-isotropy and the edges are composed of projections of singular orbits with T^1-isotropy. We also have the following result from [19] which gives an upper bound for the number of orbits with T^2-isotropy.

Theorem 3.0.7. *[19] Let M be a closed, simply connected manifold and let G be a compact, connected Lie group. Suppose M admits a G-action such that M^* is homeomorphic to D^2. If M admits an almost non-negatively curved sequence of G-invariant metrics, then M^* has at most 4 edges.*

Thus the orbit space M^* is a 2-disk with at least $n-2$ edges and at most 4 edges, as shown in the following figure.

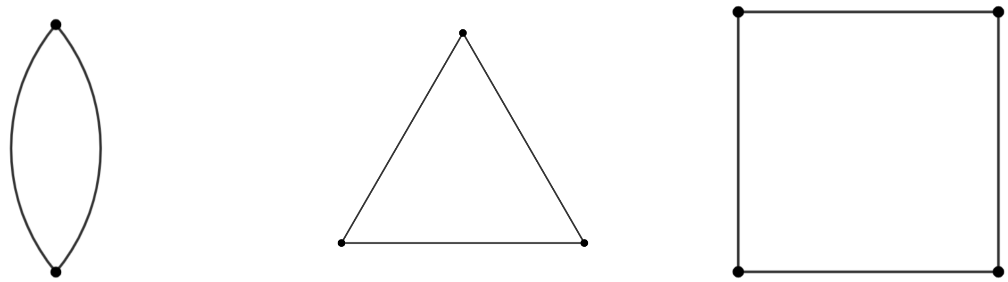


Figure 3.1: Possible orbit spaces for Theorem A.

Note that the lune occurs only when $n = 4$, the triangle when $n = 4$ or 5, and the square when $n = 4, 5$, or 6.

We gain additional topological information about M with the following result from [19].

Theorem 3.0.8. *[19] Any simply connected almost non-negatively curved manifold M of cohomogeneity at most two is rationally elliptic.*

In order to complete the proof of Theorem A, we must show that the torus action on M is locally standard, where a locally standard action is defined as follows.

Definition 3.0.9 [Locally standard action]. [8] A T^k-action on M^n is called *locally standard* if for each point $x \in M$, there is a neighborhood of x in M which is T^k-equivariantly diffeomorphic to

$$T^r \times W \times \mathbb{R}^m,$$

where $r = k - \dim(T^k_x)$, W is a faithful T^k_x-representation of real dimension $2\dim(T^k_x)$, and $T^k \cong T^r \times T^k_x$ acts trivially on $\mathbb{R}^m$, T^r acts trivially on W, and T^k_x acts trivially on T^r.

To prove that the torus actions are locally standard, we first note that since each orbit space in Figure 3.1 is contractible, we may decompose M as the union of disk bundles $D(F) \cup_E D(N)$ by the Slice Theorem. It is then sufficient to show that the T^{n-2}-action is locally standard on each of $D(F)$ and $D(N)$ and their common boundary E.

In order to proceed we first establish the following lemma.

Lemma 3.0.10. *A T^{m-1}-action on an m-manifold N^m with N/T^{m-1} homeomorphic to an interval and group diagram $(T^{n-2}, \{e\}, T^1, T^1)$ is locally standard.*

Proof. By the Slice Theorem, M decomposes as

$$D(T^{m-1}/T^1) \cup_{T^{m-1}/\{e\}} D(T^{m-1}/T^1) = D^2(T^{m-2}) \cup_{T^{m-1}} D^2(T^{m-2}).$$

The total spaces of the disk bundles $D^2(T^{m-2})$ are then $T^{m-2} \times D^2$, with $W = D^2$ admitting a free S^1-action. This provides the desired neighborhood of orbits corresponding to the endpoints of the interval. For interior points, the Slice Theorem gives a tubular neighborhood equivariantly diffeomorphic to $T^{m-1} \times D^1$, where $D^1 \cong \mathbb{R}^1$ is acted on trivially by T^{m-1}. $\square$

We now decompose M as a union of disk bundles according to whether M^* is a lune, a triangle, or a square. For the lune, we take $F = \pi^{-1}(\overline{p_1})$ and $N = \pi^{-1}(\overline{p_2})$, where $\overline{p_1}$ and $\overline{p_2}$ are the vertices. For the triangle, we take $N = \pi^{-1}(\overline{p})$ with $\overline{p}$ a vertex, and F the preimage of the closure of the side opposite to $\overline{p}$. For the square, we take N and F to be the preimages of the closures of two opposite sides. In all cases, E corresponds to the cohomogeneity 1 diagram $(T^{n-2}, \{e\}, T^1, T^1)$, which is locally standard by Lemma 3.0.10.

To show that the action is locally standard on F and N for each possible orbit space configuration, we consider two cases: Case 1, where F and/or N corresponds to the inverse image of a vertex, and Case 2, where F and/or N corresponds to the inverse image of an edge.

First, consider a T^{n-4} orbit, which projects to a vertex in the orbit space. This corresponds to N in the triangle case and F and N in the lune case. By the Slice Theorem we have a T^2-action on the normal S^3. We extend this to an action on the normal disk $D^4 \cong \mathbb{R}^4 \cong \mathbb{R}^2 \oplus \mathbb{R}^2$ by coning the action on the normal sphere. We then have a neighborhood equivariantly diffeomorphic to $T^{n-4} \times (\mathbb{R}^2 \oplus \mathbb{R}^2)$, where the isotropy subgroup T^2 acts on each $\mathbb{R}^2$ summand by rotations.

Now consider the preimage A of an edge of the orbit space. This corresponds to F in the triangle case and F and N in the square case. We have that $\pi(A)$ is homeomorphic to an interval with group diagram (T^{n-2}, T^1, T^2, T^2) and so A is a cohomogeneity one submanifold of M. By an argument similar to the proof of Lemma 3.0.10, the action of T^{n-2} on A is locally standard.

We may now apply the following result of Dong, Escher, and Searle [7].

Theorem 3.0.11. *[7] Let M^n be a closed, rationally elliptic n-manifold admitting a smooth, effective, locally standard, and isotropy-maximal T^k-action. Suppose all 4-dimensional faces of M^n/T^k are diffeomorphic to disks, after smoothing the corners. Then M^n is equivariantly diffeomorphic to a quotient of a free linear torus action of*

$$\mathcal{Z}^m = \prod_{i<r} S^{2n_i} \times \prod_{i \geq r} S^{2n_i - 1}, \;\; n_i \geq 2, \;\; \textit{where } n \leq m \leq 3n - 3k.$$

In particular, this tells us that M is equivariantly diffeomorphic to one of S^4, $S^2 \times S^2$, $\mathbb{C}P^2$, $\mathbb{C}P^2 \# \pm \mathbb{C}P^2$, S^5, $S^3 \times S^2$, $S^3 \widetilde{\times} S^2$ (the nontrivial S^3-bundle over S^2), or $S^3 \times S^3$, with a linear T^{n-2}-action. Each of these manifolds admits a T^{n-2}-invariant metric of non-negative curvature, completing the proof of Part (2).

CHAPTER 4

Almost Maximal Symmetry Rank

In this chapter, we consider the case of closed, simply connected, almost non-negatively curved 5-manifolds which admit an isometric torus action of almost maximal symmetry rank. In particular, we prove Theorems B and C, which we restate here for convenience.

Theorem B. *Let T^2 act smoothly and effectively on a closed, smooth, simply connected 5-manifold M. Let $\{g_n\}_{n=1}^{\infty}$ be a sequence of Riemannian metrics on M for which the T^2 action is isometric and suppose that $\{(M, g_n)\}_{n=1}^{\infty}$ is almost non-negatively curved. Then there are at most 5 isolated orbits and if there are exactly 5, one of the following holds:*

a) All 5 are isolated circle orbits and there are no exceptional orbits; or

b) There is a unique isolated $T^2/(\mathbb{Z}_2 \times \mathbb{Z}_2)$ orbit.

Theorem C. *Let T^2 act smoothly and effectively on a closed, smooth, simply connected 5-manifold M. Let $\{g_n\}_{n=1}^{\infty}$ be a sequence of Riemannian metrics on M for which the T^2 action is isometric and suppose that $\{M, g_n\}_{n=1}^{\infty}$ is almost non-negatively curved. Suppose that there are at most 4 isolated orbits. Then M is diffeomorphic to one of S^5, $S^3 \times S^2$, $S^3 \widetilde{\times} S^2$ (the nontrivial S^3-bundle over S^2), or the Wu manifold $SU(3)/SO(3)$.*

4.1 The Number of Isolated Circle Orbits

Our goal in this section is to obtain information about the orbit space of our action. In particular, we show that singular orbits must exist and provide bounds on the number of isolated singular orbits. We first recall the following lemma from Kobayashi [29].

Lemma 4.1.1. *[29] Let T^1 act isometrically and effectively on M^n, a closed orientable manifold. Then $\chi(M) = \chi(\mathrm{Fix}(M, T^1))$.*

Lemma 4.1.2. *[22] Let T^1 act isometrically and effectively on M^4, a closed, simply connected manifold. Then the fixed point set of this action is nonempty.*

Proof. Since M^4 is simply connected, $H_0(M;\mathbb{Z}) = \mathbb{Z}$ and $H_1(M;\mathbb{Z}) = 0$. We also have that $H_4(M;\mathbb{Z}) \cong H_0(M;\mathbb{Z})$ and $H_3(M;\mathbb{Z}) \cong H_1(M;\mathbb{Z})$ by Poincaré duality and the Universal Coefficients Theorem. Then $\chi(M) = 2 + \text{rank}(H_2(M;\mathbb{Z})) \geq 2$. We have that $\chi(M) = \chi(\text{Fix}(M, T^1))$ from Lemma 4.1.1. The Euler characteristic of the empty set is zero, and thus the fixed point set cannot be empty. □

We also need the following results from [2].

Theorem 4.1.3. *[2] Let M be a smooth, simply connected manifold admitting an action by a compact Lie group. If a principal orbit is connected (and hence all orbits are connected), then there are no special exceptional orbits, that is, the set of points belonging to exceptional orbits is of codimension greater than or equal to 2.*

Lemma 4.1.4. *[2] Let a compact group G act freely on a manifold M. Then the orbit space M/G is a manifold.*

The following is a straightforward result concerning the fixed point sets of a subgroup of an acting group.

Lemma 4.1.5. *Let G be an abelian group and H be a subgroup of G. Suppose G acts on a manifold M. Then the fixed point set of the H-sub-action on M, Fix$(M;H)$, is invariant under the G-action.*

Proof. Let $x \in \text{Fix}(M;H)$. Then for every $g \in G$ and every $h \in H$, $hgx = ghx = gx$. Thus, for every $x \in \text{Fix}(M;H)$, $Gx \subseteq \text{Fix}(M;H)$, so G acts invariantly on $\text{Fix}(M;H)$. □

To prove the following lemma, we use the concept of a group filtration, which is a decomposition of a group G into nested normal subgroups.

Definition 4.1.6 [Filtration]. A *filtration* of a group G is a set $\{G_i\}_{i=1}^n$ of normal subgroups of G such that $G_i \lhd G_{i+1}$ and $G_n = G$.

Lemma 4.1.7. *[12] Let T^n act on M^{n+3}, a closed, simply connected smooth manifold. Then some circle subgroup of T^n has nontrivial fixed point set.*

Proof. Suppose to derive a contradiction that T^n acts freely. Then for any choice of $T^{n-1} \leq T^n$, the orbit space M^{n+3}/T^{n-1} of the T^{n-1}-sub-action is a 4-manifold by Lemma 4.1.4. Thus, we have a free action of $T^1 \cong T^n/T^{n-1}$ on a 4-manifold $M^4 = M^{n+3}/T^{n-1}$, a contradiction to Lemma 4.1.2. Now suppose that the action is almost free, that is, all isotropy subgroups are finite. From [2] (see Theorem 10.5 in Chapter IV), we know that the torus action has finitely many isotropy types. The torus is abelian and thus there are finitely many finite isotropy subgroups. Let Γ be the subgroup of T^n generated by all the isotropy subgroups. Consider the action of T^n/Γ on M^{n+3}/Γ. In fact, we may consider the quotients

$$M \to M/\Gamma_1 \to \cdots \to M/\Gamma_k = M/\Gamma, \tag{4.1}$$

where $\Gamma = \Gamma_k \supset \cdots \supset \Gamma_1$ is a filtration with prime-order quotients Γ_i/Γ_{i-1}. This filtration exists since $\Gamma \subset T^n$ is abelian. Each Γ_i has nonempty fixed point set by assumption, so by Theorem 2.3.1 each quotient, including M/Γ, is a closed, simply connected topological space. The fixed point set of each Γ_i is invariant under the T^n-action by Lemma 4.1.5. Thus $\mathrm{Fix}(M; \Gamma_i)$ is at least n-dimensional and by Theorem 4.1.3 at most $n+1$-dimensional. The space of directions normal to the projection of a codimension 2 fixed point set in M^{n+3}/Γ is a sphere. For codimension 3, the isotropy subgroup will be a finite subgroup of $SO(3) \cap T^n$. Since $n \geq 2$, it must be a cyclic group of rotations or $\mathbb{Z}_2 \times \mathbb{Z}_2$. In either case, the quotient of the 2-sphere modulo the isotropy subgroup will again be a topological 2-sphere. Thus M^{n+3}/Γ is a topological manifold which is closed and simply connected. Since T^n/Γ acts freely on M^{n+3}/Γ and T^n/Γ is isomorphic to T^n, this implies that we have a free action of T^1 on a 4-manifold $(M^{n+3}/\Gamma)/T^{n-1}$. As shown earlier in this proof, this produces a contradiction. Thus T^n cannot act almost freely. This implies that there exists an isotropy subgroup of the T^n-action which has positive dimension. Such a group must contain a circle, and thus a circle subgroup has non-trivial fixed point set. □

With the existence of singular orbits verified, we establish a lower bound for the number of singular orbits in the following proposition.

Proposition 4.1.8. *[12] Let T^n act on M^{n+3}, a closed, simply connected, smooth manifold. Suppose that M^* is homeomorphic to S^3 and that there are exactly two orbit types: principal orbits T^n and isolated singular orbits T^{n-1}, that is, the isotropy subgroups are either trivial or isomorphic to T^1. Then there are at least $n+1$ isolated singular orbits T^{n-1}.*

Proof. Let M_0 denote the manifold with boundary obtained by removing a small open tubular neighborhood around each isolated singular orbit T^{n-1}. Let M_0^* denote the quotient space M_0/T^n, and let $r+1$ be the number of isolated singular orbits T^{n-1}. By a standard transversality argument [31] we know that

$$\pi_1(M_0) \cong \pi_1(M) = \{e\}, \text{ and}$$

$$\pi_2(M_0) \cong \pi_2(M).$$

Since we assume that there is no non-trivial isotropy subgroup of finite order we obtain a fibration

$$T^n \to M_0 \to M_0^*.$$

This gives rise to a long exact sequence in homotopy

$$0 \to \pi_2(M_0) \to \pi_2(M_0^*) \to \pi_1(T^n) \to \pi_1(M_0) \to \pi_1(M_0^*) \to 0,$$

noting that $\pi_i(T^n) \cong \pi_i(\mathbb{R}^n)$ for all $i \geq 2$. Defining E to be the closure of $M^* \setminus M_0^*$, we now apply the Mayer-Vietoris sequence to $M^* = M_0^* \cup E$ to obtain

$$\cdots \to H_3(M_0^*) \oplus H_3(E) \to H_3(M^*) \to H_2(M_0^* \cap E) \to H_2(M_0^*) \oplus H_2(E) \to H_2(M^*) \to \cdots$$

Since $\pi_1(M) \cong \pi_1(M_0) = 0$, it follows that $\pi_1(M_0^*) = 0$ by the long exact sequence in homotopy. We also have that M^* is a 3-sphere, so $H_i(M^*) \cong \mathbb{Z}$ for $i = 0, 3$ and $H_i(M^*) = 0$ for $i \neq 0, 3$. The set E is a disjoint union of closed 3-disks, so $H_i(E) = 0$ for all $i > 0$. Thus the Mayer-Vietoris sequence becomes

$$\cdots \to H_3(M_0^*) \to \mathbb{Z} \to H_2(M_0^* \cap E) \to H_2(M_0^*) \to 0 \to \cdots$$

The subspace $M_0^* \cap E$ is a disjoint union of 2-spheres, one for each isolated T^{n-1} orbit, so $H_2(M_0^* \cap E) \cong \mathbb{Z}^{r+1}$. The set M_0^* is $S^3 \setminus \bigsqcup_{i=1}^{r+1} D_i^3$, so it deformation retracts onto $\bigvee_{i=1}^{r} S_i^2$, the wedge sum of r 2-spheres. Thus $H_3(M_0^*) = 0$ and $H_2(M_0^*) \cong \mathbb{Z}^r$. Since M_0^* is simply connected it follows from the Hurewicz isomorphism [23] that $\pi_2(M_0^*) \cong H_2(M_0^*) \cong \mathbb{Z}^r$ and the above exact sequence in homotopy becomes

$$0 \to \pi_2(M_0) \to \mathbb{Z}^r \to \mathbb{Z}^n \to 0.$$

We conclude that $n \leq r$ and thus there are at least $n+1$ isolated singular orbits. □

Combining this proposition with Proposition 2.5.2 yields that, when M^* is homeomorphic to S^3, there are at least 3 and at most 5 isolated circle orbits.

4.2 The Case of Five Circle Orbits

In this section, we prove Proposition 4.2.9, which is necessary to prove Part (A) of Theorem B. We first recall some necessary definitions and then recall the proof of a technical lemma from [22] (Lemma 4.2.8 here) needed to complete the proof of Proposition 4.2.9.

Definition 4.2.1 [Locally Lipschitz function]. A function $f : \mathbb{R} \to \mathbb{R}$ is called *locally Lipschitz* if for every $x_0 \in \mathbb{R}$, there exist positive constants c_{x_0} and δ_{x_0} such that if $|x_0 - x| < \delta_{x_0}$, then $|f(x_0) - f(x)| < c_{x_0}|x_0 - x|$.

Definition 4.2.2 [Concave function]. A function $f : \mathbb{R} \to \mathbb{R}$ is called *concave* on an interval $[a, b]$ if for every $\alpha \in [0, 1]$ and every $x_0, x_1 \in [a, b]$,

$$f((1-\alpha)x_0 + \alpha x_1) \geq (1-\alpha)f(x_0) + \alpha f(x_1).$$

If f is twice differentiable in $[a, b]$, this is equivalent to saying that $f''(x) \leq 0$ for all $x \in [a, b]$.

Concavity is a very strong and useful condition. In particular, if f is a once-differentiable concave function on $[a, b]$, then $f'_+(x_0) \geq f'_-(x_1)$ for all x_0 and x_1 such that $a \leq x_0 \leq x_1 \leq b$. However, functions of interest such as the distance function along a geodesic in an Alexandrov space are not necessarily concave, but by precomposing with specific functions they satisfy the more flexible condition of λ-concavity.

Definition 4.2.3 **[λ-concave].** A locally Lipschitz function $f : \mathbb{R} \to \mathbb{R}$ is λ-*concave* if $\phi(t) = f(t) - \lambda t^2/2$ is a concave function. If f is twice differentiable, we have $f^{''} \leq \lambda$.

In particular, we note that

$$\phi^{'}_{+}(t_0) \geq \phi^{'}_{-}(t_1) \tag{4.2}$$

for $t_0 < t_1$ since ϕ is concave, where $\phi^{'}_{-}$ and $\phi^{'}_{+}$ are the left and right derivatives, respectively. We may extend this definition to length spaces and hence Alexandrov spaces as follows: A function $f : X \to \mathbb{R}$ on a length space X is λ-concave if its restriction to every unit speed geodesic is λ-concave.

In an Alexandrov space with curv $\geq -k^2$, the function $f_k = \rho_k \circ \operatorname{dist}(p, \cdot)$ with

$$\rho_k(x) = \frac{1}{k^2}(\cosh(kx) - 1)$$

is $(1 + k^2 f_k)$-concave. To see this, fix $p_1, p_2, p_3 \in X$, let γ be a geodesic from p_2 to p_3, and let $\alpha = \angle p_1 p_2 p_3$. Then

$$\begin{aligned}
f(t) &= \frac{1}{k^2}(\cosh(k \operatorname{dist}(p_1, \gamma(t))) - 1) \\
f^{'}(t) &= \frac{1}{k}\sinh(k \operatorname{dist}(p_1, \gamma(t)))\frac{d}{dt}[\operatorname{dist}(p_1, \gamma(t))] \\
&= \frac{1}{k}\sinh(k \operatorname{dist}(p_1, \gamma(t)))(-\cos(\alpha)) && \text{by Proposition 2.4.7} \\
f^{''}(t) &= \cosh(k \operatorname{dist}(p_1, \gamma(t)))\cos^2(\alpha) \\
&\leq \cosh(k \operatorname{dist}(p_1, \gamma(t))) \\
&= 1 + k^2 f(t).
\end{aligned}$$

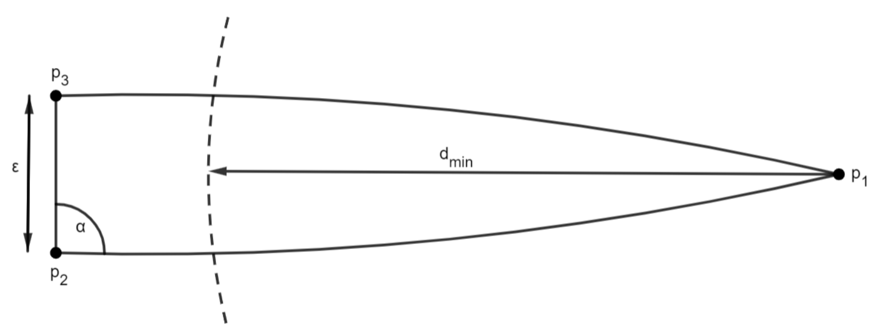


Figure 4.1: Thin triangle in Lemma 4.2.4

Lemma 4.2.4. *[22] Let X be an Alexandrov space with* curv $\geq -k^2$ *and fix* $d_{min} > 0$. *Let $p_1, p_2, p_3 \in X$ be three distinct points with p_2 and p_3 extremal,* $\operatorname{dist}(p_2, p_3) = \epsilon$ *and* $\operatorname{dist}(p_1, p_i) \geq d_{min}$ *for $i \in \{2, 3\}$. Then*

$$\frac{\pi}{2} \geq \angle p_1 p_2 p_3 \geq \frac{\pi}{2} - g(d_{min}, \epsilon, k),$$

where

$$g(d_{min}, \epsilon, k) = \epsilon \left(\frac{1}{d_{min}} + O(k^2) \right) + O(\epsilon^3). \tag{4.3}$$

Proof. Since p_2 is chosen to be extremal, $\operatorname{diam}(\Sigma_{p_2}) \leq \pi/2$, and thus $\angle p_1 p_2 p_3 \leq \pi/2$. We proceed to demonstrate the lower bound.

As shown previously, the function f_k defined above satisfies $f_k^{''} \leq 1 + k^2 f_k$. Let $\gamma : [0, \epsilon] \to X$ be an arclength-parametrized geodesic with $\gamma(0) = p_2$ and $\gamma(\epsilon) = p_3$. Define $f = f_k \circ \gamma = \rho_k \circ \operatorname{dist}(p_1, \gamma(t))$ to be the restriction of f_k to the geodesic. Choose $R > 0$ so that $f \leq R$ on $[0, \epsilon]$, which we may do because f is continuous and therefore must achieve a maximum on its compact domain. It now follows that $f^{''} \leq 1 + Rk^2$ on the geodesic; in other words, $\phi(t) = f(t) - (1 + Rk^2)\frac{t^2}{2}$ is concave.

Let $\alpha = \angle p_1 p_2 p_3$. Recall from Proposition 2.4.7 that the right derivative

$$\lim_{t \to 0^+} \frac{d}{dt}[\operatorname{dist}(p_1, \gamma(t))] = -\cos(\alpha).$$

Then we have

$$\begin{aligned}
f_+^{'}(0) &= \lim_{t \to 0^+} \frac{f(t) - f(0)}{t} \\
&= \lim_{t \to 0^+} \frac{(\cosh(k \operatorname{dist}(p_1, \gamma(t))) - 1) - (\cosh(k \operatorname{dist}(p_1, p_2)) - 1)}{k^2 t} \\
&= \lim_{t \to 0^+} \frac{\sinh(k \operatorname{dist}(p_1, \gamma(t)))}{k} \cdot \frac{d}{dt}[\operatorname{dist}(p_1, \gamma(t))] && \text{by L'Hopital's Rule} \\
&= \frac{\sinh(k \operatorname{dist}(p_1, p_2))}{k}(-\cos(\alpha)) \\
&\leq -\frac{\sinh(k d_{min})}{k} \cos(\alpha). && \text{since } \cos(\alpha) > 0
\end{aligned}$$

Note that $\phi'_+(0) = f'_+(0) = \lim_{t\to 0^+} f'(t)$. At the other end of γ, since p_3 is an extremal point, we have $\angle p_1p_3p_2 = \beta \le \pi/2$. Similarly as with $f'_+(0)$,

$$\begin{aligned} f'_-(\epsilon) &= \lim_{t\to 0^-} \frac{f(\epsilon+t) - f(\epsilon)}{t} \\ &= \lim_{t\to 0^-} \frac{(\cosh(k\,\mathrm{dist}(p_1,\gamma(\epsilon+t))) - 1) - (\cosh(k\,\mathrm{dist}(p_1,p_3)) - 1)}{k^2 t} \\ &= \lim_{t\to 0^-} \frac{\sinh(k\,\mathrm{dist}(p_1,\gamma(\epsilon+t)))}{k} \cdot \frac{d}{dt}[\mathrm{dist}(p_1,\gamma(\epsilon+t))] \qquad \text{by L'Hopital's Rule} \\ &= \lim_{t\to 0^+} \frac{\sinh(k\,\mathrm{dist}(p_1,\overline{\gamma}(t)))}{k} \left(-\frac{d}{dt}[\mathrm{dist}(p_1,\overline{\gamma}(t))]\right) \\ &= \frac{\sinh(k\,\mathrm{dist}(p_1,p_3))}{k}\cos(\beta) \end{aligned}$$

where $\overline{\gamma}$ is the path given by reversing γ. So $f'_-(\epsilon) \ge 0$ and hence $\phi'_-(\epsilon) \ge -(1+Rk^2)\epsilon$.

Now, by Equation 4.2, $\phi'_-(\epsilon) \le \phi'_+(0)$ so that

$$-(1+Rk^2)\epsilon \le -\frac{\sinh(kd_{min})}{k}\cos\alpha.$$

Rearranging the last inequality to solve for $\cos(\alpha)$ and taking a Taylor expansion of $\sinh(x)$, we conclude that

$$\cos\alpha \le \frac{k(1+Rk^2)}{\sinh(kd_{min})}\epsilon = \epsilon\left(\frac{1}{d_{min}} + O(k^2)\right).$$

It follows from the Taylor expansion of $\arccos(x)$ that

$$\begin{aligned} \angle p_1p_2p_3 &= \alpha \\ &= \arccos(\cos(\alpha)) \\ &= \arccos\left(\epsilon\left(\frac{1}{d_{min}} + O(k^2)\right)\right) \\ &\ge \frac{\pi}{2} - \epsilon\left(\frac{1}{d_{min}} + O(k^2)\right) - O(\epsilon^3), \end{aligned}$$

as required. □

In the next sublemma, we consider spaces $X_{s,t}$ with $(|s|,|t|) \ne (1,1)$. In this case, a singular point and its antipodal point are the unique pair realizing the diameter $\pi/2$, so we gain information about pairs of points v_0, v_1 with $\mathrm{dist}(v_0,v_1)$ nearly $\pi/2$.

Sublemma 4.2.5. *[22] Let $p, q \in X_{s,t}$ such that* $\operatorname{dist}(p,q) = \pi/2$. *Suppose that p is a singular point in $X_{s,t}$. For any $\epsilon > 0$ there is a $\delta > 0$ such that if $v_0, v_1 \in X_{s,t}$ with* $\operatorname{dist}(v_0, v_1) \geq \pi/2 - \delta$, *the following statements hold for some $i \in \{0,1\}$, where we read $i+1$ modulo 2:*

(1) $|\operatorname{dist}(v_i, p) - \operatorname{dist}(v_{i+1}, q)| \leq \delta$;

(2) $\operatorname{dist}(v_{i+1}, q) \leq (1+\epsilon)\delta$; *and*

(3) $\operatorname{dist}(v_i, p) \leq (2+\epsilon)\delta$.

Proof. Note that from the triangle inequality we have

$$\begin{aligned} \operatorname{dist}(v_i, v_{i+1}) &\leq \operatorname{dist}(v_i, p) + \operatorname{dist}(p, v_{i+1}) \\ &= \operatorname{dist}(v_i, p) + \frac{\pi}{2} - \operatorname{dist}(v_{i+1}, q), \end{aligned}$$

since

$$\operatorname{dist}(p, v_{i+1}) + \operatorname{dist}(v_{i+1}, q) = \pi/2. \tag{4.4}$$

We also have $\operatorname{dist}(v_i, v_{i+1}) \geq \pi/2 - \delta$ by hypothesis, so $\delta \geq \operatorname{dist}(v_i, p) - \operatorname{dist}(v_{i+1}, q)$ holds for $i = 0, 1$, and thus $|\operatorname{dist}(v_i, p) - dist(v_{i+1}, q)| \leq \delta$. This proves Part (1).

To prove Part (2), we note that $\operatorname{dist}(v_i, v_{i+1}) \geq \pi/2 - \delta$ by assumption. Consider the sequences $\{\delta_n\}_{n=1}^{\infty} = \{\frac{1}{n}\}_{n=1}^{\infty}$, $\{v_{i,n}\}_{n=1}^{\infty}$, and $\{v_{i+1,n}\}_{n=1}^{\infty}$. By compactness of $X_{s,t}$, there exists a convergent subsequence $\{v_{i,k_n}\} \xrightarrow[n\to\infty]{} L$. Since

$$\lim_{n\to\infty} \operatorname{dist}(v_{i,k_n}, v_{i+1,k_n}) \geq \lim_{n\to\infty} \left(\frac{\pi}{2} - \frac{1}{n}\right) = \frac{\pi}{2},$$

and p and q are the unique pair with $\operatorname{dist}(p,q) = \pi/2$, it follows that $L \in \{p, q\}$. By renumbering the v_i, assume $L = p$. Then $\lim\limits_{n\to\infty} v_{i+1,k_n} = q$. Thus there exists a function $C^{'}(\delta)$ with $\lim\limits_{\delta\to 0} C^{'}(\delta) = 0$ such that $\operatorname{dist}(v_i, p)$, $\operatorname{dist}(v_{i+1}, q) \leq C'(\delta)$, for some $i \in \{0,1\}$. It then follows from the first variation of arc length that

$$\operatorname{dist}(v_i, v_{i+1}) \leq \operatorname{dist}(v_{i+1}, p) - \cos(\beta) \operatorname{dist}(v_i, p) + C(\delta)(\operatorname{dist}(v_i, p))^2,$$

where $\beta = \angle v_{i+1} p v_i$, and $C(\delta) = C(C'(\delta))$, where $\lim_{\delta \to 0} C(\delta) = 0$. Then, since p is singular, $\operatorname{diam} \Sigma_p \leq \frac{\pi}{2}$, and it follows that $\beta \leq \pi/2$ so we have

$$\begin{aligned} \frac{\pi}{2} - \delta \leq \operatorname{dist}(v_i, v_{i+1}) &\leq \operatorname{dist}(v_{i+1}, p) + C(\delta)(\operatorname{dist}(v_i, p))^2 \\ &\leq \frac{\pi}{2} - \operatorname{dist}(v_{i+1}, q) + C(\delta)(\operatorname{dist}(v_i, p))^2 \qquad \text{by Equation 4.4} \\ &\leq \frac{\pi}{2} - \operatorname{dist}(v_{i+1}, q) + C(\delta)(\operatorname{dist}(v_{i+1}, q) + \delta)^2, \end{aligned}$$

by Part (1). Thus

$$\operatorname{dist}(v_{i+1}, q) \leq \delta + C(\delta)(\operatorname{dist}(v_{i+1}, q) + \delta)^2.$$

Then either

$$\operatorname{dist}(v_{i+1}, q) \leq \delta + 4C(\delta)(\operatorname{dist}(v_{i+1}, q))^2 \quad \text{or} \quad \operatorname{dist}(v_{i+1}, q) \leq \delta + 4C(\delta)\delta^2.$$

The first inequality holds if $\operatorname{dist}(v_{i+1}, q) \geq \delta$ and the second holds if $\operatorname{dist}(v_{i+1}, q) \leq \delta$. For the first inequality, taking a Taylor expansion of the right hand side and since $1 - 4C(\delta) \operatorname{dist}(v_{i+1}, q) > 0$ for small δ, we see that

$$\operatorname{dist}(v_{i+1}, q) \leq \delta(1 + 8C(\delta) \operatorname{dist}(v_{i+1}, q)) \leq \delta(1 + 4\pi C(\delta))$$

for small enough δ. For the second inequality, noting that $\delta < \pi$ trivially, the same inequality holds. We set $4\pi C(\delta) < \epsilon$, and so

$$\begin{aligned} \operatorname{dist}(v_{i+1}, q) &\leq \delta(1 + 4\pi C(\delta)) \\ &< \delta(1 + \epsilon), \end{aligned}$$

and we have proven Part (2).

To prove Part (3), we note that

$$\begin{aligned} \operatorname{dist}(v_i, p) &\leq \delta + \operatorname{dist}(v_{i+1}, q) && \text{by Part 1} \\ &\leq \delta + (1 + \epsilon)\delta && \text{by Part 2} \\ &= (2 + \epsilon)\delta, \end{aligned}$$

and the result follows. □

Of use in future proofs is the fact that if, for a given choice of ϵ, Sublemma 4.2.5 holds for some δ, then the result holds for all δ' with $0 < \delta' \leq \delta$.

For the following lemma, let $S \subset X$ be a set of 5 distinct points in the almost non-negatively curved Alexandrov space X with $\Sigma_{p_i} = X_{s_i,t_i}$ for each $p_i \in S$. Denote by v_{ij} the direction of a geodesic from p_i to p_j. The set S converges to some finite set, $S_\infty \subset X_\infty$, with $1 \leq |S_\infty| \leq 5$.

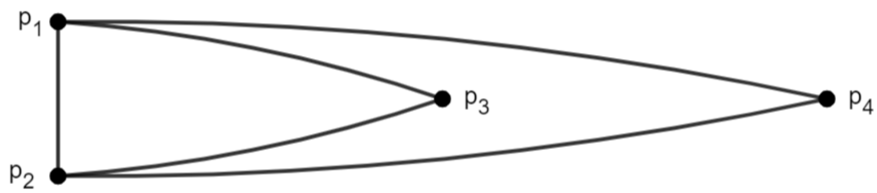


Figure 4.2: Configuration of two thin triangles in Sublemma 4.2.6.

Sublemma 4.2.6. *[22] Let $\{(X, \mathrm{dist}_n)\}$ be an almost non-negatively curved sequence of 3-dimensional Alexandrov spaces. Suppose that S is defined as above and that $|S_\infty| \leq 4$ in X_∞. Then there is a $\delta > 0$ such that for some k and for sufficiently large n,*

$$\mathrm{xt}_4(\{v_{kl} : k \neq l\}) \leq \frac{\pi}{3} - \delta.$$

Proof. We consider the cases where $|S_\infty| \leq 2$ and $|S_\infty| \geq 3$. We define a convergence map $\pi : S \to S_\infty$ so that $\pi(p_i) = \lim_{n\to\infty} p_i$. By passing to a subsequence of metrics, we may assume that $\mathrm{dist}_n(p_i, p_j) < 1/n$ for any $p_i, p_j \in \pi^{-1}(x)$, $x \in S$. Choose d so that $\mathrm{dist}_\infty(x, y) > 2d$ for each distinct pair $x, y \in S_\infty$. We also denote by dist the distance function on any space of directions Σ_{p_j}, and, for simplicity, we will omit the dependence of this space on n. Seeking a contradiction, suppose that for all k we have

$$\mathrm{xt}_4(\{v_{kl} : k \neq l\}) \to \frac{\pi}{3} \text{ as } n \to \infty.$$

Consider first the case in which $|S_\infty| \leq 2$. Then there is some $x \in S_\infty$ which is the limit of at least three points.

Suppose that exactly three points converge to x, so that $\pi^{-1}(x) = \{p_1, p_2, p_3\}$. We apply Lemma 4.2.4 to the three thin triangles given by (p_j, p_5, p_4) for $j = 1, 2, 3$, with $\epsilon < 1/n$ and $d_{min} = d$ as chosen above. Then, since $\angle p_j, p_5, p_4 = \text{dist}(v_{5j}, v_{54})$ in Σ_{p_5}, Equation 4.3 from Lemma 4.2.4 gives us that

$$\frac{\pi}{2} - \text{dist}(v_{5j}, v_{54}) \leq f\left(d, \frac{1}{n}, \frac{1}{n}\right) = \frac{1}{nd} + O\left(\frac{1}{n^3}\right) \text{ for } 1 \leq j \leq 3.$$

In the case where $(|s_5|, |t_5|) = (1, 1)$, then each v_{5j}, $1 \leq j \leq 3$, is close to the unique point $x \in \Sigma_{p_5}$ antipodal to v_{54}. Hence the v_{5j} are pairwise close to each other. In particular, it follows from the triangle inequality that

$$\text{dist}(v_{5i}, v_{5j}) \leq \text{dist}(v_{5i}, x) + \text{dist}(x, v_{5j}) \tag{4.5}$$

$$= (\pi/2 - \text{dist}(v_{5i}, v_{54})) + (\pi/2 - \text{dist}(v_{5j}, v_{54})) \tag{4.6}$$

$$\leq \frac{2}{nd} + O\left(\frac{1}{n^3}\right), \tag{4.7}$$

for $1 \leq i, j \leq 3$. Equality 4.6 follows from the fact that $X_{1,1}$ is a round sphere, so the v_{5i} each lie on a geodesic between x and v_{54}.

However, if $(|s_5|, |t_5|) \neq (1, 1)$, it follows from Parts (2) and (3) of Sublemma 4.2.5 that v_{54} is close to some $\xi \in \Sigma_{p_5}$, which is either a singular point or the unique point antipodal to a singular point. In this case, each v_{5j}, $1 \leq j \leq 3$, is close to the unique point ζ antipodal to ξ. We now apply Parts (2) and (3) of Sublemma 4.2.5, choosing $\epsilon = 1/2$. Since $\frac{\pi}{2} - \text{dist}(v_{5j}, v_{54}) \leq \frac{1}{nd} + O(\frac{1}{n^3})$, we set $\delta = \frac{1}{nd} + O(\frac{1}{n^3})$. Then

$$\begin{aligned} \text{dist}(v_{5i}, v_{5j}) &\leq \text{dist}(v_{5i}, \zeta) + \text{dist}(\zeta, v_{5j}) && \text{by the triangle inequality} \\ &\leq (4 + 2\epsilon)\delta && \text{by Sublemma 4.2.5} \\ &= \frac{5}{nd} + O\left(\frac{1}{n^3}\right), \end{aligned}$$

for $1 \leq i, j \leq 3$.

In both cases, we obtain that

$$\begin{aligned}\mathrm{xt}_4(\{v_{5l} : l \neq 5\}) &\leq \frac{1}{6}\left(3 \cdot \frac{\pi}{2} + 3 \cdot \left(\frac{5}{nd} + O\left(\frac{1}{n^3}\right)\right)\right) \\ &= \frac{1}{2}\left(\frac{\pi}{2} + \frac{5}{nd}\right) + O\left(\frac{1}{n^3}\right),\end{aligned}$$

which approaches $\pi/4$ as n approaches infinity, a contradiction.

Alternatively, suppose $\{p_1, p_2, p_3, p_4\} \subset \pi^{-1}(x)$. Let $y \in X_\infty$ such that $y \neq x$. Let $q_n \in (X, \mathrm{dist}_n)$ be such that $\lim_{n\to\infty} q_n = y$. We then apply Lemma 4.2.4 to the thin triangle given by (q_n, p_1, p_j) for $j = 2, 3, 4$, now with $d_{min} = d'$. We obtain at Σ_{p_1} that, as before,

$$\mathrm{dist}(v_{1i}, v_{1j}) \leq \frac{5}{nd'} + O\left(\frac{1}{n^3}\right) \text{ for } 2 \leq i, j \leq 4,$$

so that

$$\mathrm{xt}_4(\{v_{1l} : l \neq 1\}) \leq \frac{1}{2}\left(\frac{\pi}{2} + \frac{5}{nd'}\right) + O\left(\frac{1}{n^3}\right) \xrightarrow[n\to\infty]{} \frac{\pi}{4} < \frac{\pi}{3},$$

a contradiction.

We now consider the case $3 \leq |S_\infty| \leq 4$. Note first that, since these are all extremal points, each has a space of directions with diameter less than or equal to $\pi/2$. Thus no three are collinear in the sense that no shortest path between two points of S_∞ contains a third point of S_∞. If three of these points were collinear, then we would have $\angle p_i, p_j, p_k = \pi$ for some distinct $p_i, p_j, p_k \in S_\infty$, a contradiction since this angle corresponds to $\mathrm{dist}(v_{ji}, v_{jk})$ in Σ_{p_j}.

Suppose that $p_1, p_2 \in \pi^{-1}(x)$. Let $y = \pi(p_3)$ and $z = \pi(p_4)$. By renumbering, we may assume that x, y, and z are all distinct. Then applying Lemma 4.2.4 to (p_3, p_1, p_2) and (p_4, p_1, p_2) we obtain that, at Σ_{p_1},

$$\frac{\pi}{2} - \mathrm{dist}(v_{12}, v_{1j}) \leq \frac{1}{nd} + O\left(\frac{1}{n^3}\right) \text{ for } j = 3, 4.$$

Using a similar argument as in the case where $|S_\infty| \leq 2$, we deduce that

$$\mathrm{dist}(v_{13}, v_{14}) \leq \frac{5}{nd} + O\left(\frac{1}{n^3}\right) \xrightarrow[n\to\infty]{} 0.$$

It follows from the lower semi-continuity of angles in Alexandrov spaces (for example see Theorem 4.3.11 in [3]) that $\angle xyz = 0$, that is, the points x, y, and z are collinear; a contradiction. $\square$

Sublemma 4.2.7. *[22] Let $\{(X, dist_n)\}_{n=1}^{\infty}$, S, and S_∞ be as in Sublemma 4.2.6. Suppose that $|S_\infty|$=5 and that there is a $\overline{p_i} \in S$ such that $\Sigma_{\overline{p_i}}$ is isometric to X_{s_i,t_i} with $(|s_i|, |t_i|) \neq (1,1)$. Then there is a $\delta > 0$ such that for some k and for sufficiently large n,*

$$xt_4(\{v_{kl} : k \neq l\}) \leq \tfrac{\pi}{3} - \delta.$$

Proof. Without loss of generality, assume that $i = 5$. We suppose that $\mathrm{xt}_4(\{v_{kl} : k \neq l\}) = \pi/3$ to obtain a contradiction. By Lemma 2.4.5, the 4-extender in $\Sigma_{\overline{p_5}}$ is given by

$$\{w_1, w_2, w_3, w_4\} \subset \Sigma_{p_5} \text{ with } w_1 = w_2, w_3 = w_4, \text{ and } \operatorname{dist}(w_1, w_3) = \frac{\pi}{2}.$$

Then we may assume without loss of generality that $v_{51}, v_{52} \xrightarrow[n\to\infty]{} w_1$ and $v_{53}, v_{54} \xrightarrow[n\to\infty]{} w_3$. Hence, possibly passing to a subsequence,

$$\operatorname{dist}(v_{51}, v_{52}), \operatorname{dist}(v_{53}, v_{54}) < \frac{1}{n} \xrightarrow[n\to\infty]{} 0.$$

Thus the points in each of the two sets $\{\pi(p_5), \pi(p_1), \pi(p_2)\}$ and $\{\pi(p_5), \pi(p_3), \pi(p_4)\}$ are collinear, since $\operatorname{dist}(v_{51}, v_{52}) \to 0$ implies that $\angle \overline{p_1}, \overline{p_5}, \overline{p_2} = 0$. That is, the geodesic direction from $\overline{p_5}$ to $\overline{p_1}$ is the same as that from $\overline{p_5}$ to $\overline{p_2}$, so all three points must lie on the same geodesic. As noted in Sublemma 4.2.6 this contradicts the extremality of S_∞. $\square$

The previous two sublemmas combine to form the following statement.

Lemma 4.2.8. *[22] Let $\{(X, dist_n)\}_{n=1}^{\infty}$ be an almost non-negatively curved sequence of 3-dimensional Alexandrov spaces. Suppose that $S = \{p_i\}_{i=1}^{5}$ is a set of five distinct points in X with $\Sigma_{p_i} = X_{s_i,t_i}$ for each $p_i \in S$. Suppose there is a j such that $(|s_j|, |t_j|) \neq (1,1)$. Then there is a $\delta > 0$ such that for some k and for sufficiently large n,*

$$xt_4(\{v_{ij} : j \neq i\}) \leq \frac{\pi}{3} - \delta.$$

We may now prove the main proposition of this section.

Proposition 4.2.9. *[22] Let X be an almost non-negatively curved Alexandrov space with five points $\{\overline{p_i}\}_{i=1}^5$ whose spaces of directions are isometric to X_{s_i,t_i}. Then $(|s_i|, |t_i|) = (1,1)$ for all i.*

Proof. By Proposition 2.5.2, we know that there are at most 5 points $\overline{p}$ with $\Sigma_{\overline{p}} = X_{s,t}$. The points $\{\overline{p_i}\}_{i=1}^5$ define 10 triangles. Let $v_{jl} \in \Sigma_{\overline{p_j}}$ be the direction of a geodesic from $\overline{p_j}$ to $\overline{p_l}$. If we apply similar arguments to those of Proposition 2.5.2, we will find only that the inequality

$$10\left(\pi - \mu\left(\frac{1}{n}\right)\right) \leq 10\pi$$

must hold, and since $\mu(\frac{1}{n}) > 0$, this is always true.

The right hand side of this inequality stems from the fact that $\mathrm{xt}_4(\Sigma_{\overline{p_j}}) \leq \pi/3$. However, we can restrict our attention to calculating the potentially smaller value $\mathrm{xt}_4(\{v_{jl} : l \neq j\})$. By Sublemma 4.2.7, we have that if, for some i, $\Sigma_{\overline{p_i}}$ is isometric to some $X_{s,t}$ with $(|s|, |t|) \neq (1,1)$, the inequality

$$\mathrm{xt}_4(\{v_{ij} : j \neq i\}) \leq \frac{\pi}{3} - \delta$$

holds for some fixed $\delta > 0$ and sufficiently large n. In that case,

$$10(\pi - \mu(\frac{1}{n})) \leq 10\pi - \delta$$

holds, yielding a contradiction for sufficiently large n. It follows that $\Sigma_{\overline{p_i}}$ is isometric to $X_{1,1}$ for all i. □

4.3 M^* is Homeomorphic to S^3.

With the proposition proven, we return to direct consideration of the orbit space M^*. For the remainder of the chapter we separate the discussion based on the homeomorphism class of M^*.

When the orbit space is homeomorphic to S^3, we gain the following information about the orbits.

Proposition 4.3.1. *[12] Let T^2 act smoothly on M^5, a closed, simply connected smooth 5-manifold. If $M^* = S^3$, then the following hold.*

(1) The singular orbits of the action are T^1 and $T^1/\mathbb{Z}_k$, $k \in \mathbb{Z}^+$.

(2) The exceptional orbits are $T^2/\mathbb{Z}_k$, $k \geq 2$ and $T^2/(\mathbb{Z}_2 \times \mathbb{Z}_2)$.

(3) In all cases where there is finite cyclic isotropy, the corresponding fixed point set of finite cyclic isotropy is of dimension 3.

Proof. Since we have assumed that M^* is homeomorphic to S^3, there are no points with T^2 isotropy. If there were, the corresponding point in M^* would have a space of directions isometric to $S^3/T^2 = I$. Since this space of directions has boundary, the point would be a boundary point of M^*, a contradiction. Thus the isotropy subgroup corresponding to a singular orbit is a proper subgroup of T^2. Since the orbit is singular, the isotropy subgroup must have positive dimension, and therefore it contains a T^1. Thus the isotropy subgroup of a singular orbit is T^1 or $T^1 \times \mathbb{Z}_k$, proving Part (1). Observe that the normal sphere at any point of an exceptional orbit will be of dimension two. Thus the finite isotropy subgroup of an exceptional orbit must be a subgroup of $SO(3)$ and of T^2. Hence the only possible nontrivial finite isotropy subgroups are $\mathbb{Z}_k$, $k \geq 2$, and $\mathbb{Z}_2 \times \mathbb{Z}_2$. This proves Part (2).

Now we prove Part (3). We first consider the singular orbits, observing that if we have a singular orbit of the form $T^1/\mathbb{Z}_k$, then we have a $T^1 \times \mathbb{Z}_k$ action on the normal 3-sphere to any point of the orbit. In particular, there will be a finite cyclic subgroup of order k in $T^1 \times \mathbb{Z}_k$ fixing circles in this normal 3-sphere and therefore this orbit is contained in a fixed point set of finite isotropy of dimension 3. If the singular orbit is T^1, then the action of the circle on the normal S^3 is either free or almost free. In the latter case, a finite cyclic subgroup fixes a 3-dimensional submanifold which contains the singular orbit.

We now consider the exceptional orbits. For a $T^2/\mathbb{Z}_k$ orbit, $k \neq 2$, the $\mathbb{Z}_k$ action on S^2 is never free and thus this exceptional orbit will be contained in a 3-dimensional submanifold fixed by $\mathbb{Z}_k$, $k \neq 2$. It remains to show that for the exceptional orbit $T^2/\mathbb{Z}_2$, the $\mathbb{Z}_2$-isotropy subgroup also does not act freely on its normal S^2. This follows from the fact that the antipodal map, which reverses orientation, generates the only free $\mathbb{Z}_2$-action on S^2. This is not a subgroup of $SO(3)$ since there are no orientation-reversing involutions in $SO(3)$. Finally, we consider the exceptional orbit $T^2/(\mathbb{Z}_2 \times \mathbb{Z}_2)$. The action of the isotropy subgroup, $\mathbb{Z}_2 \times \mathbb{Z}_2$, on the normal S^2 produces a quotient space equal to the double right-angled spherical triangle with three vertices, each of which is fixed by a different $\mathbb{Z}_2$ subgroup of $\mathbb{Z}_2 \times \mathbb{Z}_2$. Each fixed vertex corresponds to a 3-dimensional submanifold fixed by the corresponding $\mathbb{Z}_2$ subgroup. For each $T^2/(\mathbb{Z}_2 \times \mathbb{Z}_2)$ orbit we will have exactly three such fixed point sets intersecting in this orbit. Thus, we conclude that the fixed point set of a finite cyclic group is always of dimension 3. □

The next proposition involves graph theory, so we note a few relevant definitions. For further information see Diestel [5].

Definition 4.3.2 [Graph]. A *graph* G is a pair (V, E) of disjoint sets, together with a map $\phi : E \to ([V]^2/\sim)$, where the equivalence relation $\sim$ is defined by $(x, y) \sim (y, x)$, where $(x, y) \in V \times V$. The set V is called the *vertices* of G, denoted $V(G)$, and E is called the *edges* of G, denoted $E(G)$. The map ϕ sends an edge $e \in E$ to its *ends*.

We note that in some references this definition describes an *undirected multigraph*, though we refer to it as a graph for the sake of brevity.

Definition 4.3.3 [Adjacent vertices]. Two vertices x and y of a graph G are called *adjacent* or *neighbors* if there exists an edge $e \in E(G)$ such that $\phi(e) = [(x, y)]$.

Definition 4.3.4 [Degree of a vertex]. The *degree* or *valency* of a vertex $v \in G$ is the number of edges at v, $|\{e \in E(G) : \phi(e) = [(v, v_1)] \text{ for some } v_1 \in V(G)\}|$.

Definition 4.3.5 [Cycle]. A *cycle* in a graph G is a collection of vertices $\{v_0, v_1, \ldots, v_{k-1}\}$ such that the v_i's are distinct and $\{[(v_0, v_1)], [v_1, v_2], \ldots, [(v_{k-1}, v_0)]\} \subset \phi(E(G))$. The graph-theoretic *length* of a cycle $\{v_0, v_1, \ldots, v_{k-1}\}$ is k.

With these definitions in place, we now prove that the subset of M^* corresponding to orbits with nontrivial isotropy constitutes a graph.

Proposition 4.3.6. *[12] Let T^2 act smoothly on M^5, a closed, simply connected smooth 5-manifold. If $M^* = S^3$, then the set of points with nontrivial isotropy corresponds to a graph G and the following hold.*

(1) The vertices of the graph correspond to isolated singular orbits or to isolated exceptional orbits with isotropy $\mathbb{Z}_2 \times \mathbb{Z}_2$.

(2) The graph must contain at least three vertices corresponding to isolated singular orbits.

(3) The vertices corresponding to isolated singular orbits have degree 0, 1, or 2.

(4) The vertices corresponding to isolated exceptional orbits with isotropy $\mathbb{Z}_2 \times \mathbb{Z}_2$ have degree 3.

(5) The edges of the graph correspond to points with finite, nontrivial, cyclic isotropy.

(6) Every edge must meet two different vertices.

(7) The points in the edges meeting an isolated exceptional orbit with isotropy $\mathbb{Z}_2 \times \mathbb{Z}_2$ have isotropy $\mathbb{Z}_2$.

(8) The preimage of the closure of an edge corresponds to a 3-dimensional manifold fixed by a nontrivial finite cyclic group admitting a T^2 action of cohomogeneity one.

Proof. We claim that the images of orbits of the form T^2/S^1, $T^2/(S^1 \times \mathbb{Z}_k)$, or $T^2/(\mathbb{Z}_2 \times \mathbb{Z}_2)$ in M^* correspond to $V(G)$ and that the images of the exceptional orbits of the form $T^2/\mathbb{Z}_k$ correspond to $E(G)$. We first consider the exceptional orbits $T^2/\mathbb{Z}_k$. By the Slice Theorem, the isotropy subgroup Z_k acts on the normal S^2. Since this $\mathbb{Z}_k$ is a subgroup of S^1, the action is orientation-preserving and therefore acts by rotations on S^2, fixing two antipodal points. The normal space to the orbit is the cone over S^2, so we see that $\mathbb{Z}_k$ fixes 1 dimension of the normal $\mathbb{R}^3$ to the orbit and hence $\dim(\mathrm{Fix}(M^5; \mathbb{Z}_k)) = 3$. Thus each $\mathbb{Z}_k$-orbit corresponds to a point on a geodesic in M^*. Recall that the components of $\mathrm{Fix}(M; \mathbb{Z}_k)$ are totally geodesic submanifolds N. Since $\dim(N) = 3$, N^3/T^2 is an interval in M^*. That is, T^2 acts by cohomogeneity 1 on N. By the classification of such actions in Mostert [36] and Neumann [39], the endpoints of I correspond to orbits of the form T^2/K, where $\mathbb{Z}_k \subseteq K$ and

$$K \cong \begin{cases} \mathbb{Z}_{2k} \\ S^1 \\ \mathbb{Z}_k \times \mathbb{Z}_2 \\ S^1 \times \mathbb{Z}_k \end{cases}$$

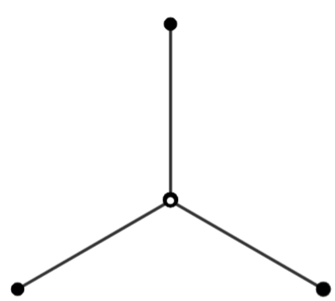
By Proposition 4.3.1, the case $K \cong \mathbb{Z}_{2k}$ may not occur, and if $K \cong \mathbb{Z}_k \times \mathbb{Z}_2$, then $k = 2$. Thus $V(G) = \{T^2/K : K \text{ is one of } \mathbb{Z}_2 \times \mathbb{Z}_2, S^1, \text{ or } \mathbb{Z}_k \times S^1\}$, $E(G) = \{T^2/\mathbb{Z}_k\}$, and $(V, E) = G$. This proves Parts (1), (5), and (8). Points $\overline{p}$ in M^* corresponding to isolated singular orbits have spaces of directions $X_{s,t}$, which have 0, 1, or 2 singular points. When they exist, these singular points in $X_{s,t}$ correspond to geodesic directions in $\Sigma_{\overline{p}}$ with nontrivial $\mathbb{Z}_s$- or $\mathbb{Z}_t$-isotropy, so these correspond to edges in M^* for which $\overline{p}$ is a vertex. This proves Part (3). Similarly, the space of directions $\Sigma_{\overline{q}} = S^2/(\mathbb{Z}_2 \times \mathbb{Z}_2)$ corresponding to $\overline{q} \in M^*$, the image of the orbit $T^2 q = T^2/(\mathbb{Z}_2 \times \mathbb{Z}_2)$, has three singular points. Thus there are 3 edges emanating from $\overline{q}$, proving Part (4). By Kleiner's isotropy lemma, any edge emanating from $\overline{q}$ has isotropy a subgroup of $\mathbb{Z}_2 \times \mathbb{Z}_2$. This isotropy is also finite cyclic, and therefore it must be $\mathbb{Z}_2$. This proves Part (7). Part (2) follows from Proposition 4.1.8. By Kleiner's isotropy lemma, the isotropy type on a cycle with one vertex and one edge must be constant, ruling out this configuration. Therefore, there cannot be cycles of graph-theoretic length 1 and, in particular, any edge must connect two different vertices, proving Part (6). □

We now define terms related to Proposition 4.3.6. An *arc* refers to the closure of an edge with finite cyclic isotropy in the set of orbits with nontrivial isotropy in M^*. We will refer to the graphs corresponding to nontrivial isotropy in M^* as *weighted graphs*, as they carry isotropy information. We will use black vertices to denote singular orbits and white vertices to denote exceptional orbits with isotropy $\mathbb{Z}_2 \times \mathbb{Z}_2$. Edges correspond to non-isolated exceptional orbits with nontrivial finite cyclic isotropy.

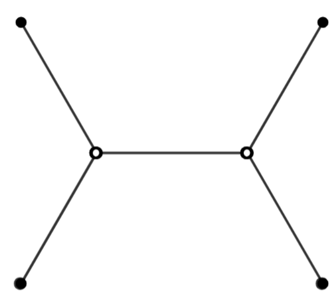
A geometric understanding of the space of directions $S^2/(\mathbb{Z}_2 \times \mathbb{Z}_2)$ corresponding to the image of a $T^2/(\mathbb{Z}_2 \times \mathbb{Z}_2)$ orbit in the orbit space is vital to many of the following proofs. In particular, its q-extent is necessary for certain calculations. We first recall a result from McGowan and Searle [35].

Proposition 4.3.7. *[35] For the $T^2/(\mathbb{Z}_2 \times \mathbb{Z}_2)$ orbits p represented by white vertices in the weighted graph, the spaces of directions $\Sigma_p = S^2/(\mathbb{Z}_2 \times \mathbb{Z}_2)$ have $\mathrm{xt}_4(\Sigma_p) = \frac{5\pi}{12}$ and $\mathrm{xt}_5(\Sigma_p) = \frac{2\pi}{5}$.*

There are two potential subgraphs of note which we will name. The first consists of a white vertex adjacent to three black vertices, called the weighted claw. The second consists of two white vertices and four black vertices, where each white vertex is adjacent to the other and to two black vertices, called the weighted tree.



(a)



(b)

Figure 4.3: The weighted claw (a) and weighted tree (b).

4.4 The Geometry of M^5/T^2

Throughout this section we assume that M is a closed, simply connected, almost non-negatively curved 5-manifold with an isometric and effective T^2-action such that $M^5/T^2 = S^3$. We now state and prove four lemmas which allow us to eliminate some of the possible graphs that can occur in M^5/T^2.

Lemma 4.4.1. *The complement of a cycle may contain at most 1 black vertex.*

Proof. Suppose to derive a contradiction that there is a cycle K whose complement contains 2 black vertices. Then by Lemma 2.4.9, the double branched cover B of M^* with branching set K is an almost non-negatively curved Alexandrov space. If the complement of K contains 3 black vertices, then B contains six points with spaces of directions $X_{s,t}$, a contradiction to Proposition 2.5.2. Thus K must contain at least one black vertex, call it p. Since p has degree 2, $\Sigma_p = X_{s,t}$ has $s \geq 2$, $t \geq 2$, and $(s,t) = 1$. From Grove and Wilking [18], the space of directions corresponding to $\pi^{-1}(p) \in B$ has $\Sigma_{\pi^{-1}(p)} = X_{s/2,t/2}$. Since $s \geq 2$ and $t \geq 2$, we maintain the inequality $\mathrm{xt}_q(X_{s/2,t/2}) \leq \mathrm{xt}_q(S^2(1/2))$. There are at least 5 points in B with spaces of directions $X_{s,t}$, hence each must be $X_{1,1}$ by Proposition 4.2.9. This contradicts that the point $\overline{p}$ has a space of directions $\Sigma_{\overline{p}} = X_{s,t}$ with $s \geq 2$ and $t \geq 2$. □

Lemma 4.4.2. *A black vertex is adjacent to at most one white vertex.*

Proof. Suppose to derive a contradiction that a black vertex, $\overline{p}$, is adjacent to two distinct white vertices. Recall that $\Sigma_{\overline{p}} = X_{s,t}$, where $(s,t) = 1$ and s and t correspond to the order of the isotropy subgroups of the orbits corresponding to the edges. Since the edges both have $\mathbb{Z}_2$ isotropy, we obtain a contradiction. □

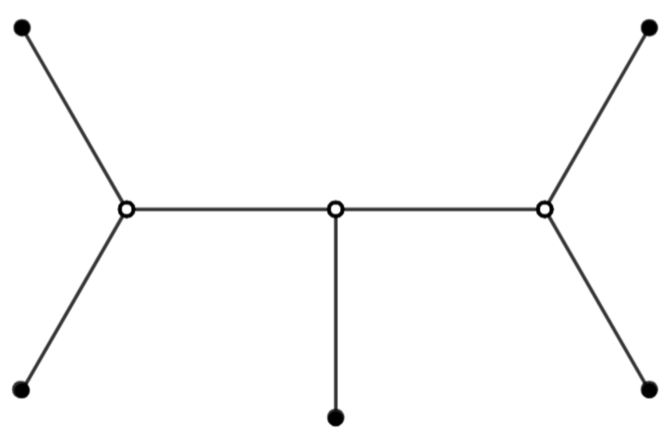


Figure 4.4: Configuration of three white vertices minimizing black vertices.

Lemma 4.4.3. *There are at most two white vertices.*

Proof. Suppose to derive a contradiction that there are at least three white vertices. Due to the restrictions from Lemmas 4.4.1 and 4.4.2, the graph configuration which minimizes the number of black vertices is a chain of three white vertices, each adjacent to one or two black vertices so that each have degree 3. However, this configuration has five black vertices, all of which have spaces of directions different from $X_{1,1}$, a contradiction to Proposition 4.2.9. □

We now set some notation for the following proposition. Let $\{\overline{p_i}\}_{i=1}^k$ be a set of points in the orbit space M^*. We denote by v_{ij} the point in $\Sigma_{\overline{p_i}}$ corresponding to the geodesic direction toward $\overline{p_j}$. We represent by α_{jik} the measure of $\angle \overline{p_j}\,\overline{p_i}\,\overline{p_k}$, which is equal to $\text{dist}(v_{ij}, v_{ik})$ in $\Sigma_{\overline{p_i}}$.

Proposition 4.4.4. *There is at most one white vertex.*

Proof. Suppose to derive a contradiction that there are at least two white vertices $\overline{p_1}$ and $\overline{p_2}$. Then by Lemma 4.4.3, these are the only white vertices. If $\overline{p_1}$ is not adjacent to $\overline{p_2}$, then we have two disjoint weighted claw subgraphs. These contain six black vertices, a contradiction to Proposition 2.5.2. Now suppose that $\overline{p_1}$ and $\overline{p_2}$ are adjacent. Then we have a weighted tree, otherwise the configuration would contradict Lemma 4.4.1. Call the four black vertices $\{\overline{p_i}\}_{i=3}^6$. Each $\overline{p_i}$ has positive degree, so the spaces of directions $\Sigma_{\overline{p_i}} = X_{s_i,t_i}$, $3 \leq i \leq 6$ have $(|s_i|, |t_i|) \neq (1, 1)$. We have $\binom{6}{3} = 20$ geodesic triangles constructed with vertices in the set $\{\overline{p_i}\}_{i=1}^6$. By Lemma 2.4.11, we know that the angle sum of each triangle is bounded below by $\pi - O(\frac{1}{n^2})$, where n is the index of the metric in the almost non-negatively curved sequence. Thus $\sum\limits_{i,j,k} \alpha_{ijk} \geq 20(\pi - O(\frac{1}{n^2}))$. We also have an upper bound based on $\text{xt}_5(\Sigma_{\overline{p_i}})$, since the distance in the space of directions corresponds to the angle between geodesics. Then

$$\sum_{i,j,k} \alpha_{ijk} \leq \sum_i \binom{5}{2} \mathrm{xt}_5(\Sigma_{\overline{p_i}}) \tag{4.8}$$

$$= \sum_{i=1}^{4} \binom{5}{2} \mathrm{xt}_5(X_{s_i,t_i}) + 2 \cdot \binom{5}{2} \mathrm{xt}_5(S^3/(\mathbb{Z}_2 \times \mathbb{Z}_2)) \tag{4.9}$$

$$\leq 4(10 \cdot \frac{3\pi}{10}) + 2(10 \cdot \frac{2\pi}{5}) \tag{4.10}$$

$$= 20\pi \tag{4.11}$$

Each geodesic direction v_{ij} is distinct, as $\mathrm{diam}(\Sigma_{\overline{p_i}}) = \frac{\pi}{2}$ for all i. Hence by Lemma 2.4.5, an extender is not achieved in X_{s_i,t_i}, $3 \leq i \leq 6$, so the inequality (4.8) is strict. Taking the limit as n goes to infinity, we achieve a contradiction. Note that none of the black orbits with S^1 isotropy can converge to each other. If they did, we would have a contradiction to Proposition 4.3.6, Lemma 4.4.1, or Lemma 4.4.2. □

Thus the only possible configuration containing a white vertex is the weighted claw, possibly with a fourth black vertex and/or an additional edge. In the following figures, we list all possible graphs.

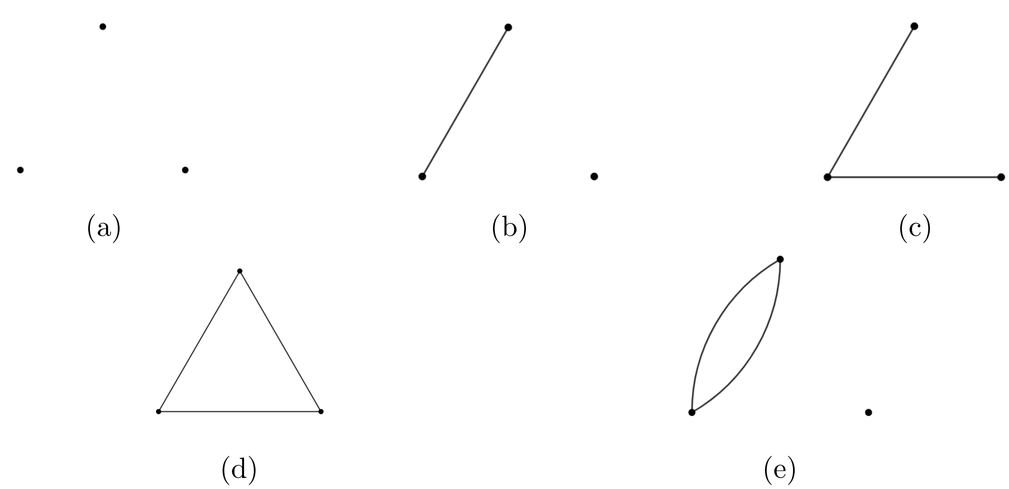


Figure 4.5: Possible graphs with three black vertices and no white vertices.

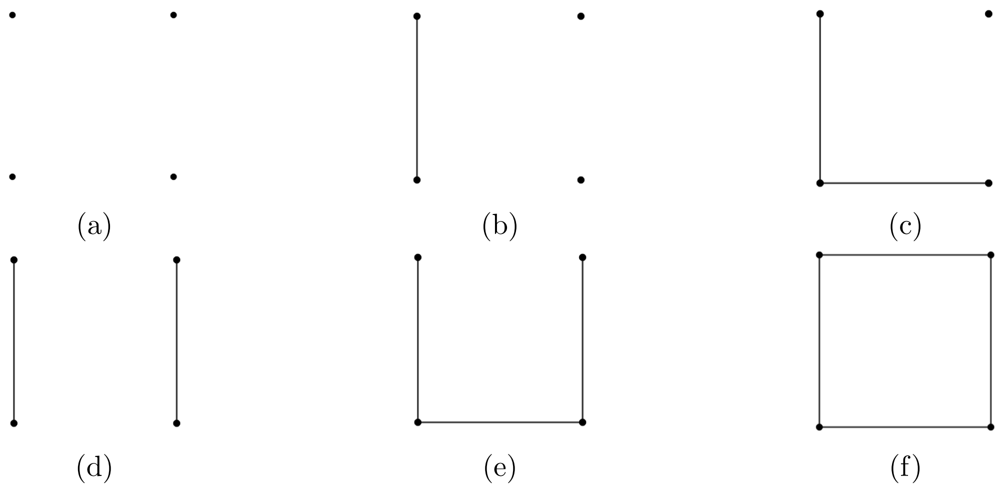


Figure 4.6: Possible graphs with four black vertices and no white vertices.

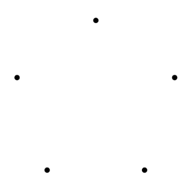


Figure 4.7: Possible graph with five black vertices.

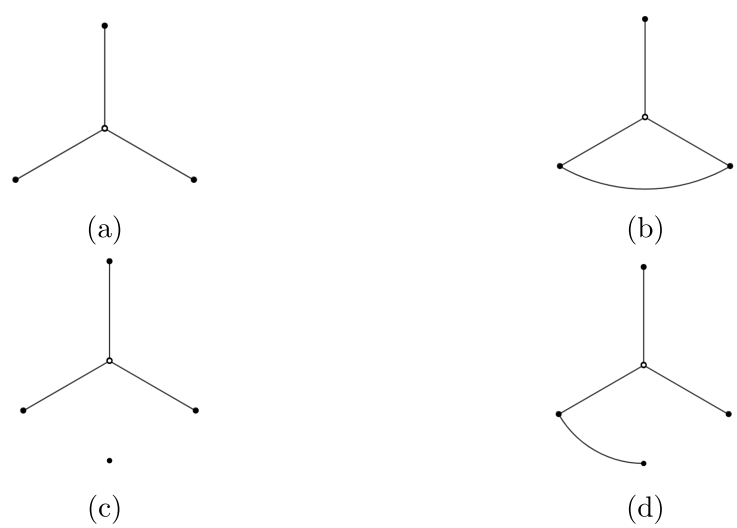


Figure 4.8: Possible graphs with one white vertex.

We note that a graph with four black vertices and a cycle containing three black vertices does not appear in Figure 4.6 by an argument similar to the proof of Lemma 4.4.1. This completes the proof of Theorem B when M^* is homeomorphic to S^3.

The graph shown in Figure 4.7 and the graphs (c) and (d) in Figure 4.8 are the only cases with 5 isolated orbits. The other graphs have at most 4 isolated orbits and appear in the classification of the non-negative curvature case in [12]. Thus these orbit spaces arise from actions equivariantly diffeomorphic to the actions described there. In particular, the orbit spaces (a)–(d) in Figure 4.5 correspond to $M = S^5$, (e) in Figure 4.5 corresponds to $M = S^5$ or the Wu manifold, each graph in Figure 4.6 corresponds to $M = S^3 \times S^2$ or $S^3 \widetilde{\times} S^2$, and (a) and (b) in Figure 4.8 correspond to the Wu manifold. Each of these manifolds are listed in Theorem C, verifying the theorem when M^* is homeomorphic to S^3.

4.5 M^* is Homeomorphic to D^3 or $S^2 \times I$

In this section, we assume that M^5 is a closed, simply connected, almost non-negatively curved manifold admitting an isometric and effective T^2-action such that M^5/T^2 is an Alexandrov space with boundary.

From Proposition 2.4.8 and Lemma 2.5.1, we know that M^* is a 3-manifold with at most two boundary components. Thus, since M^* has boundary, it is homeomorphic to D^3 or $S^2 \times I$ by Lemma 2.3.2. In the following two propositions we describe the isotropy of orbits corresponding to interior points of M^*.

Proposition 4.5.1. *If M^* is homeomorphic to $S^2 \times I$, then the isotropy subgroups corresponding to interior points of M^* are all trivial.*

Proof. Let $\overline{p} \in \mathrm{int}(M^*)$ correspond to the orbit Gp. From arguments similar to those in the proof of Proposition 4.3.1, we know that the isotropy subgroup G_p must be T^2, $S^1 \times \mathbb{Z}_k$ (possibly with $k = 1$), $\mathbb{Z}_k$, $\mathbb{Z}_2 \times \mathbb{Z}_2$, or $\{e\}$. As stated in the proof of Proposition 4.3.1, orbits with T^2-isotropy correspond to boundary points of M^*, so $G_p \ncong T^2$. If $G_p \cong S^1 \times \mathbb{Z}_k$, by the Slice Theorem the space of directions $\Sigma_{\overline{p}}$ is isometric to S^3/S^1. Using the Gluing Theorem, we may then construct an almost non-negatively curved Alexandrov space with more than 5 interior points with spaces of directions isometric to S^3/S^1, a contradiction to Proposition 2.5.2. If $G_p \cong \mathbb{Z}_2 \times \mathbb{Z}_2$, this implies the existence of at least 3 orbits with isotropy $S^1 \times \mathbb{Z}_k$, a contradiction. If $G_p \cong \mathbb{Z}_k$, similar arguments to the proof of Proposition 4.3.6 yield that $\overline{p}$ is contained in arc with endpoints which are the projection of orbits with $\mathbb{Z}_2 \times \mathbb{Z}_2$-, $S^1 \times \mathbb{Z}_k$-, or T^2-isotropy. Since no such orbit exists, this may not occur. Hence G_p must be trivial. □

Proposition 4.5.2. *If M^* is homeomorphic to D^3, interior points of M^* correspond to orbits with $S^1 \times \mathbb{Z}_k$-, $\mathbb{Z}_k$-, or trivial isotropy. There are at most two points in* $\mathrm{int}(M^*)$ *corresponding to singular orbits with isotropy containing a circle. Any interior points corresponding to orbits with $\mathbb{Z}_k$-isotropy must constitute an arc between the projections of two singular orbits.*

Proof. As stated in the proof of Proposition 4.5.1, an orbit Gp with $\pi(p) \in \mathrm{int}(M^*)$ may have G_p isomorphic to T^2, $S^1 \times \mathbb{Z}_k$, $\mathbb{Z}_2 \times \mathbb{Z}_2$, $\mathbb{Z}_k$, or the trivial group. From the proof of Proposition 4.3.1, we know that T^2-isotropy may not occur in the interior of M^*. Suppose to derive a contradiction that there are three points in $\mathrm{int}(M^*)$ corresponding to orbits with isotropy containing a circle. These points have spaces of directions isometric to S^3/S^1. Using the Gluing Theorem, we may then glue two copies of M^* together along their boundaries, constructing an almost non-negatively curved Alexandrov space with 6 interior points with spaces of directions isometric to S^3/S^1. This contradicts Proposition 2.5.2. The existence of an orbit with $\mathbb{Z}_2 \times \mathbb{Z}_2$-isotropy would imply the existence of at least 3 orbits with isotropy containing a circle, so this may not occur. As discussed in the proof of Proposition 4.3.6, the projection of an orbit with $\mathbb{Z}_k$-isotropy must be contained in an arc between projections of orbits with S^1- or $\mathbb{Z}_2 \times \mathbb{Z}_2$-isotropy. □

We note that in both propositions there are at most 2 isolated orbits, completing the proof of Theorem B.

We claim that there is a circle subgroup of T^2 acting fixed-point-homogeneously on M.

Lemma 4.5.3. *Let T^2 act on M^5 isometrically and effectively with $\partial M^* \neq \emptyset$. Then there exists some circle subgroup S^1 of T^2 such that the S^1-sub-action on M is fixed-point-homogeneous.*

Proof. Since the boundary of M^* is nonempty, then there exists T^2-isotropy corresponding to boundary points of the orbit space. Considering the isotropy representation of a T^2 fixed point, we see that T^2 acts on the normal $\mathbb{R}^5 = \mathbb{R}^2 \oplus \mathbb{R}^2 \oplus \mathbb{R}$ by rotations on the $\mathbb{R}^2$ summands. Considering a circle that fixes one of these summands, we have a codimension 2 fixed point set of this circle, which implies that it acts fixed-point-homogeneously. □

The remainder of the proof strategy for Theorem C is the following: we identify the structure of the singular set of the fixed-point-homogeneous S^1-sub-action, then verify that all such configurations appear in the proof of Theorem 3.5 in Galaz-García and Spindeler [13]. Note that they use C^* to denote the image in M^5/S^1 of the set at maximal distance from a boundary component F fixed by S^1. In particular, all singular orbits of the S^1-action are contained in $F \cup C$.

We split our consideration into cases based on the homeomorphism class of M^*, and further into subcases by the non-trivial isotropy of certain orbits. For clarity, we denote by S^1_a the circle acting fixed-point-homogeneously on M.

Case A: Assume that M^* is homeomorphic to $S^2 \times I$. Then at least one of the boundary components is fixed by S^1_a.

Case A.1: If both boundary components are fixed by S^1_a, this corresponds to the case where $\dim(C^*) = 3$ in [13].

Case A.2: If only one boundary component is fixed by S^1_a, then we split into subcases based on the intersection $S^1_a \cap S^1_b$, where S^1_b is the circle fixing the second boundary component.

Case A.2.a: If $S^1_a \cap S^1_b = \mathbb{Z}_k$, this corresponds to Case 2 in [13].

Case A.2.b: If $S^1_a \cap S^1_b = \{e\}$, this corresponds to Case 3.1 in [13].

Case B: Assume now that M^* is homeomorphic to D^3. There are three subcases based on the non-trivial isotropy corresponding to interior points of M^*.

Case B.1: If int(M^*) contains no non-trivial isotropy, this corresponds to Case 3.1 in [13].

Case B.2: If int(M^*) contains one point $\overline{p}$ with $S^1_b \times \mathbb{Z}_k$-isotropy, there are three subcases based on the intersection $(S^1_b \times \mathbb{Z}_k) \cap S^1_a$.

Case B.2.a: If $(S^1_b \times \mathbb{Z}_k) \cap S^1_a = S^1_a$, this corresponds to Case 1 in [13].

Case B.2.b: If $(S^1_b \times \mathbb{Z}_k) \cap S^1 = \mathbb{Z}_m$, this corresponds to Case 2 in [13].

Case B.2.c: $(S^1_b \times \mathbb{Z}_k) \cap S^1 = \{e\}$, this corresponds to Case 3.1 in [13].

Case B.3: If int(M^*) contains two points $\overline{p_1}$ and $\overline{p_2}$ with $S^1 \times \mathbb{Z}_{k_i}$-isotropy, $i \in \{1, 2\}$, there are 2 cases based on the group diagram (G, H, K_+, K_-) of I, where I is the geodesic interval with endpoints $\overline{p_1}$ and $\overline{p_2}$.

Case B.3.a: If we have $(T^2, \{e\}, S^1_1 \times \mathbb{Z}_{k_1}, S^1_2 \times \mathbb{Z}_{k_2})$ as the group diagram, there are two subcases.

Case B.3.a.i: If S^1_a is a subgroup of $S^1_1 \times \mathbb{Z}_{k_1}$ or $S^1_2 \times \mathbb{Z}_{k_2}$, this corresponds to Case 1 in [13].

Case B.3.a.ii: If S^1_a is not a subgroup of $S^1_1 \times \mathbb{Z}_{k_1}$ or $S^1_2 \times \mathbb{Z}_{k_2}$, this corresponds to Case 3.1 in [13].

Case B.3.b: If we have $(T^2, \mathbb{Z}_l, S^1_1 \times \mathbb{Z}_{k_1}, S^1_2 \times \mathbb{Z}_{k_2})$ as the group diagram, there are two subcases.

Case B.3.b.i: If $\mathbb{Z}_l \cap S^1_a = \{e\}$, we are in Case 2.3.1 above.

Case B.3.b.ii: If $\mathbb{Z}_l \cap S^1_a \neq \{e\}$, this corresponds to Case 2 in [13].

Since our orbit spaces are the same as those in [13], they arise from actions equivariantly diffeomorphic to the actions described there. In particular, when M^* is homeomorphic to $S^2 \times I$ or D^3, then M is diffeomorphic to S^5, $S^3 \times S^2$, or $S^3 \widetilde{\times} S^2$, the non-trivial S^3-bundle over S^2. This completes the proof of Theorem C.

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