Integration of the laminar boundary layer equations for flow past wedges by Gauss's method.

INTEGRATION OF THE LAMINAR BOUNDARY LAYER EQUATIONS FOR FLOW PAST WEDGES BY GAUSS'S METHOD

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LIST OF SYMBOLS

A. Symbol	Definition
a	$f''(0) = a$ Second derivative of η at the wall
g	Acceleration due to gravity
$f(\eta)$ or f	Function of η
f (x)	Function of x in quadrature
f', f'', f'''	First, second and third total derivatives
H	Quadrature weight
h (x)	Function of x in quadrature
n	Order of quadrature
P	Pressure
u	Velocity component in the x direction
x	Coordinate in direction of surface
y	Coordinate perpendicular to surface
λ	Wedge angle parameter
μ	Coefficient of viscosity
ν	Kinematic viscosity
ρ	Density
$\tau_o(x)$	Shear stress at wall
ψ	Stream function
$\mathcal{D}$	Degree of Gauss's quadrature
η	Dimensionless quantity
y_1	Variable substituted for f''

B. Subscripts

0	At the wall
∞	Undisturbed air stream

C. Supercripts

k	Power in polynomial
n	Order of the power series of f (η)

INTRODUCTION

In many aeronautical problems, such as the flow past a body, the flow in rocket nozzles and the flow on the wings of high speed aircraft, friction effects between the solid body and liquid or gaseous flow are important factors. The problem of skin friction acting on a flat plate moving parallel to the surface through a fluid was of primary interest to ship builders. In 1872 William Froud clearly stated that frictional force must have its counterpart in the loss of momentum of the fluid which has passed along the surface of the plate. The theoretical analysis of the phenomenon based on the equation of motion of fluids started with Prandtl's paper presented in 1904 at Heidelberg. He showed that the small thickness of the boundary layer permits essential simplifications in the equations of motion of a viscous fluid so that the problem of frictional drag becomes accessible to mathematical analysis.

Skin friction effects take place between the body and the fluid in a thin layer (boundary layer) around the body. Knowing the local pressure gradient the skin friction drag may be calculated by integration of the

boundary layer equations for flow past wedges with different wedge angles.[1]

The subject of this investigation is to obtain the numerical value of the velocity gradient at the wall by integrating the boundary layer equation for flow past wedges with different wedge angles. The approach to the above problem is to use generalized Gauss's quadrature from first to fifth order successively. The application of this quadrature formula was recommended by William Squire.[2] Details about the quadrature formula are given in Appendix C.

The historical development and previous work pertaining to the solutions of the boundary layer problems are described in the first section. Derivation of equations of motion for wedge flows with different wedge angles and for the semi-infinite plane (a special case of the wedge) are presented in Appendix A. The general methods of solution for each flow case and the development of the power series of the solution are presented in Sections III and IV respectively. A convenient form of the generalized Gaussian quadrature is developed in Section V. This form enables the numerical value of the velocity gradient at the wall to be calculated for any arbitrary wedge angle.

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[1]See page 45 for List of References

SECTION I

HISTORICAL DEVELOPMENT AND PREVIOUS WORK ON THE SOLUTION OF BOUNDARY LAYER PROBLEMS

In the past sixty years a great number of scientific publications have been directed to the solutions of boundary layer equations. One important paper by Blasius appeared in 1908 for the case of the semi-infinite plane. He found that if the new variable was made proportional to $y/\sqrt{x}$, an ordinary differential equation resulted. He used the dimensionless quantity

$$= \frac{y}{2}\left[\frac{u_\infty}{\nu x}\right]^{1/2}, \qquad \psi = (\nu u_\infty x)^{1/2} f(\eta) \qquad \text{(I-1)}$$

and substituted η and ψ in equation of motion and the following equation resulted:[3]

$$f''' + ff'' = 0 \qquad \text{(I-2)}$$

Another mathematical approach to the boundary layer forming around a half-plane immersed in an incompressible viscous fluid was presented by Hermann Weyl in 1941.[4] He showed that the Blasius power series $f(\eta)$, about $\eta = 0$, has a convergence in the interval

$0 < n < 2.08$ but will not converge for $n > 3.11$.

The differential equation

$$f''' + ff'' = \lambda (1 - f'^2) \qquad \text{(I-3)}$$

describes the steady two-dimensional flow of a slightly viscous incompressible fluid past a wedge of arbitrary angle $-\pi\lambda$ and satisfying the boundary conditions

$$\begin{aligned} &f(\eta) = 0 \qquad f'(\eta) = 0 \qquad \text{at } \eta = 0 \\ &f''(\eta) = 0 \qquad\qquad\quad \text{at } \eta = \infty \end{aligned} \qquad \text{(I-4)}$$

This equation was first deduced by V. M. Falkner and S. W. Skan, and its solutions were later investigated in detail by D. R. Hartree.[5] Another solution by the steepest descent method was given in a somewhat different manner by Meksyn.[8] The existence of a solution for arbitrary λ, was given by H. Weyl but not its uniqueness.[6] The limits for the value of $f''(0)$ which characterized the solution of the problem (I-3) with boundary conditions (I-4) was given by W. A. Coppel[7] such that

$$-\frac{4}{3}\lambda < \left[f''(0) \right]^2 < -\frac{4}{3}\lambda + \frac{1}{3} \qquad \text{(I-5)}$$

SECTION II

ILLUSTRATION OF THE FINDING OF THE SOLUTION

In general, finding solutions of the boundary layer equations presents mathematical difficulties. This is primarily a consequence of their being non-linear. The following procedure was used to find the solution of the problem:

1. Substituting some quantities, the non-linear boundary layer equation for flow past wedges with different wedge angles is transformed into a non-homogeneous linear differential equation with known coefficients. The general solution of such equations is well known.

2. The functions in non-homogeneous linear equations are expressed as power series of an independent variable (η); they satisfy the corresponding boundary conditions at the wall. These power series are substituted into the non-linear differential equation and their coefficients are found as functions of a single coefficient (a_2).

3. Using integral relations and corresponding boundary conditions, the non-homogeneous linear differential equation was simplified and with some substitutions it is transformed into a new form which is applicable to a generalized Gauss's quadrature.

4. To evaluate the coefficient a_2 this quadrature was then solved numerically by using corresponding abscissa and weight coefficients for the right order. In order to decrease the calculation time and the possibility of errors, a solution by a digital computer program was used.

SECTION III

GENERAL METHOD OF SOLUTION FOR VELOCITY GRADIENT AT WALL

In this section the development of a general method of solution for obtaining the unknown quantity $f''(\eta)$ at the wall is presented. The methods of solution are developed for flow past wedges with different wedge angles and for a semi-infinite plane immersed in a flow stream with zero angle of incidence. A separate solution including additional terms for flow past a semi-infinite plane is presented. This is a required step because a simple substitution of $\lambda = 0$ into the power series solution for wedges causes a number of terms to vanish. If all terms were included for wedge angles not zero, more than fifteen additional terms would be involved. These additional terms would exceed the storage capacity of the IBM 1620 computer. To find $f''(0) = a$ the procedure as explained in Section II (see page 5) is followed.

Flow Past Wedges With Different Wedge Angles

The steady two-dimensional flow of a slightly viscous incompressible fluid past a wedge of angle $-\pi\lambda$

was described in Appendix A by the differential equation

$$f''' + ff'' = \lambda\ (1 - f'^2) \qquad \text{(A-9)}$$

with the boundary conditions

$$f(\eta) = 0 \qquad f'(\eta) = 0 \qquad \text{at } \eta = 0$$
$$f'(\eta) = 1 \qquad \text{at } \eta = \infty \qquad \text{(A-10)}$$

where

$$f' = \frac{df}{d\eta} = \left[\frac{1 + m}{2\nabla x\, \nu}\right] \frac{u}{u_\infty^{3/2}}$$ (velocity in the x direction)

$$f'' = \frac{d^2 f}{d\eta^2} = \frac{1}{\nu u_\infty^2} \left(\frac{du}{dy}\right)$$ (velocity gradient)

substituting quantities

$$f'' = y_1 \qquad \text{(III-1)}$$

$$f''' = y_1' \qquad \text{(III-1a)}$$

Equation (A-9) may be written as

$$y_1' + f\, y_1 = \lambda\ (1 - f'^2) \qquad \text{(III-2)}$$

This equation is a first order linear non-homogeneous differential equation with integrating factor

$$e^{\int f\, d\eta}$$

Integrating Equation (III-2) yields

$$y_1\, e^{\int f d\eta} = \int_0^\eta \lambda (1 - f'^2)\ e^{\int f d\eta}\, d\eta + C \qquad \text{(III-3)}$$

From this and the boundary conditions (A-10) it follows that

$$y(0) = C$$

Hence $C = a = f''(0)$ (III-4)

2. By inserting Equations (III-4) and (III-1) into the Equation (III-3)

$$f''(\eta) = e^{-\int f d\eta}\left[a + \int_0^{\eta} \lambda (1 - f'^2)\, e^{\int f d\eta}\, d\eta\right] \quad \text{(III-5)}$$

or denoting the function in parentheses by $\varphi(\eta)$

$$f''(\eta) = e^{-\int f d\eta}\, \varphi(\eta) \quad \text{(III-6)}$$

where

$$f(\eta) = \sum_{n=2}^{\infty} \frac{a_n \eta^n}{n!} \quad \text{(III-7)}$$

$$\varphi(\eta) = \sum_{n=0}^{\infty} \frac{b_n \eta^n}{n!} \quad \text{(III-8)}$$

The coefficients of the power series (III-7) and (III-8) as functions of the unknown parameter $f''(0) = a_2 = a$ are explained in Section IV.

3. To find $f''(0)$, integrate Equation (III-6) between zero and infinity

$$f'(\infty) - f'(0) = \int_0^{\infty} e^{-\int f d\eta}\, \varphi(\eta)\, d\eta \quad \text{(III-9)}$$

From this and the boundary conditions (A-10)

$$\int_0^\infty e^{-\int f d\eta} \varphi(\eta) \, d\eta = 1 \qquad \text{(III-10)}$$

The unknown quantity, f'' (0) = a, is inside the power series for f (η) and $\varphi(\eta)$. In order to obtain this unknown quantity, it is desired to evaluate the quantities

$$\int_0^\infty f(\eta) \, d\eta \quad , \quad \varphi(\eta)$$

by a power series about $\eta = 0$ (see Section V). To obtain a convenient form which is applicable for Gauss's quadrature let

$$e^{-\int f d\eta} = e^{-\sum_{n=3}^{\infty} \frac{\lambda^p a^r \eta^n}{n!}} \qquad \text{(III-11)}$$

$$\varphi(\eta) = \sum_{m=0}^{\infty} \frac{\lambda^g a^s \eta^m}{m!} \qquad \text{(III-12)}$$

where p, r, g, s are all constants given by Meksyn.[8]

Considering the series

$$e^{-\int f d\eta} = e^{-\frac{a\eta^3}{3!}} - \sum_{n=4}^{\infty} \lambda^p \frac{a^r \eta^n}{n!}$$

and using notations

$$x = \frac{a\eta^3}{3!} \qquad \text{(III-13)}$$

$$\eta = \left(\frac{3! \, x}{a}\right)^{1/3}$$

$$d\eta = \left(\frac{2}{9a}\right)^{1/3} x^{-2/3} \, dx$$

Equations (III-11) and (III-12) may be written in the forms

$$- e^{\int f d\eta} = e^{-x} \quad e^{h(x)} \tag{III-14}$$

$$\varphi(\eta) = \varphi(x) \tag{III-15}$$

By inserting these equations into Equation (III-10)

$$a^p \; C \int_0^\infty x^\beta \quad e^{-x} \quad e^{h(x)} \quad \varphi(x) \quad dx = 1 \tag{III-16}$$

where C, p, β are constants and h (x), φ(x) are functions of x. This equation is now in a form solvable by Gauss's quadrature.

4. For numerical calculations of this form, computer programs and table for abscissa and corresponding weight coefficients for the right order are presented in Appendix B.

Semi-Infinite Plane ($\lambda = 0$)

1. Starting with the general expression (A-16)

$$f''' + ff'' = 0$$

$$f = - \frac{f'''}{f''}$$

where

$$f' = \frac{df}{d\eta} = \frac{2u}{u_\infty} = \text{(velocity in the x direction)}$$

$$f'' = \left[\frac{\nu x}{u_\infty}\right]^{1/2} \frac{1}{u_\infty} \left(\frac{du}{dy}\right) \quad \text{(velocity gradient)} \tag{III-17}$$

and using substitution (III-1), Equation (III-17) becomes

$$f = \frac{y_1}{y_1} \tag{III-18}$$

By integration between 0 to η the following equation results

$$\int_0^{\eta} f d\eta = - \log \frac{f''(\eta)}{a}$$

or

$$f''(\eta) = a\, e^{-\int f d\eta} \qquad \text{(III-19)}$$

2. Integrating Equation (III-19) between the limits 0 and ∞ and using the corresponding boundary conditions of the (A-17)

$$a \int_0^{\infty} e^{-\int f d\eta} d\eta = 1 \qquad \text{(III-20)}$$

In order to evaluate the quantity

$$- \int_0^{\infty} f d\eta$$

and to obtain unknown quantity $f''(0) = a$, the same method is used as in the case of arbitrary wedge angle (see page 9).

SECTION IV

DEVELOPMENT OF THE POWER SERIES OF THE SOLUTION FOR BOUNDARY LAYER EQUATION

The present section considers some circumstances for which the power series of $f(\eta)$ may be applied, with sufficient accuracy, to the solution of Equations (III-10) and (III-20). For solutions of the above equations, the power series expansion of $f(\eta)$ at $\eta = 0$, which satisfies the corresponding boundary conditions, is derived in this section.

Power Series for the Solutions for Flow Past Wedges With a Different Wedge Angles

Solution of Equation (III-10) has two power series; namely,[8]

$$f(\eta) = \sum_{n=2}^{\infty} \frac{a_n \eta^n}{n!} \qquad \text{(III-7)}$$

$$\varphi(\eta) = \sum_{n=0}^{\infty} \frac{b_n \eta^n}{n!} \qquad \text{(III-8)}$$

Substituting Equation (III-7) in Equation (A-16) the coefficients of $f(\eta)$ which satisfy the corresponding boundary conditions may be determined in terms of $a_2 = a$ as

$$a_2 = a \qquad a_7 = -2(2 + 3\lambda)\lambda^2$$

$$a_3 = \lambda \qquad a_8 = (1 + 2\lambda)(11 + 10\lambda)a^3$$

$$a_5 = -(1 + 2\lambda)a^2 \qquad a_9 = 2(45 + 112\lambda + 66\lambda^2)\lambda a^2$$

$$a_6 = -2(2 + 3\lambda)\lambda a \qquad \text{(IV-1)}$$

Substituting Equations (III-8) and (III-7) in Equation (III-6) and expanding the right-hand side of Equation (III-6) in powers of η, and comparing the coefficients of the powers of η, the coefficients of $\phi(\eta)$ may be determined in terms of $a_2 = a$

$$b_0 = a \qquad b_4 = (1 - 6\lambda)\lambda a$$

$$b_1 = \lambda \qquad b_5 = (1 - 6\lambda)\lambda^2$$

$$b_3 = -2\lambda a^2 \qquad b_6 = -10(1 - 2\lambda)\lambda a^3 \qquad \text{(IV-2)}$$

Substituting relations (IV-1) and (IV-2) in Equations (III-8) and (III-7) respectively we find

$$f(\eta) = a\frac{\eta^2}{2!} + \frac{\lambda\eta^3}{3!} - a^2\frac{1 + 2\lambda}{5!}\eta^5 - a\frac{2(2 + 3\lambda)\lambda}{6!}\eta^6$$

$$- \frac{2(2 + 3\lambda)\lambda^2}{7!}\eta^7 + a^3\frac{(1 + 2\lambda)(11 + 10\lambda)}{8!}\eta^8$$

$$+ a^2\frac{2\lambda(45 + 113\lambda + 66\lambda^2)}{9!}\eta^9 \qquad \text{(IV-3)}$$

$$\varphi(\eta) = a + \lambda \eta - a^2 \frac{2\lambda}{3!} \eta^3 + a \frac{(1 - 6\lambda)\lambda}{4!} \eta^4$$

$$+ \frac{(1 - 6\lambda)\lambda^2}{5!} \eta^5 - a^3 \frac{10(1 - 2\lambda)\lambda}{6!} \eta^6 \qquad \text{(IV-4)}$$

Power Series for the Solution for a Semi-Infinite Plane

The power series near $\eta = 0$ is assumed to be of the form[10]

$$f(\eta) = a_0 + a_1 \eta + \frac{a_2}{2!} \eta^2 + - - - - - - - + \frac{a_n}{n!} \eta^n \qquad \text{(IV-5)}$$

Differentiating the above series with respect to η and substituting in Equation (A-16) we find

$$a_3 \eta^0 + a_4 \eta + \frac{a_5 + a_2^2}{2!} \eta^2 + - - - - - - + \frac{a_n \eta^n}{n!} = 0 \qquad \text{(IV-6)}$$

For the assumed series for $f(\eta)$ to be a solution of Equation (A-16) all coefficients of the various powers of η must vanish identically. Hence we have a_2, a_5, a_8 differ from zero. The coefficient a_2 remains undetermined, because the boundary condition at $\eta = \infty$ still remains to be satisfied. Introducing a_3, a_4, a_6, a_7, a_8 in terms of $a_2 = a$ the following series results:

$$f(\eta) = a \frac{\eta^2}{2!} - a^2 \frac{\eta^5}{5!} + \frac{11 a^3 \eta^8}{8!} - \frac{375 a^4 \eta^{11}}{11!}$$

$$+ \frac{27897 a^5 \eta^{14}}{14!} - \frac{3817173 a^6 \eta^{17}}{17!} + \frac{3294299710 a^7 \eta^{20}}{20!} \qquad \text{(IV-7)}$$

SECTION V

CONVENIENT FORM FOR GENERALIZED GAUSS'S QUADRATURE

The purpose of this section is to obtain the form for the boundary layer equation which is applicable to the generalized Gauss's quadrature. To approach the above problem, the power series Equation (IV-3), (IV-4) and (IV-7) which represent corresponding solutions respectively are placed in Equations (III-6) and (III-20) respectively. By inserting notations (III-13) the following equations applicable for generalized Gauss's quadrature formula were obtained:

$$c_o\, a^{c_1} = \int_o^\infty x^{-2/3}\, e^{-x}\, e^{h(x)}\, \varphi(x)\, dx \qquad \text{(V-1)}$$

$$d_o\, a^{-2/3} = \int_o^\infty x^{-2/3}\, e^{-x}\, e^{d_1 x^2 + d_2 x^3 + \cdots + d_n x^n}\, dx \qquad \text{(V-2)}$$

where c_0, c_1, d_0, d_1 - - -d_n are all constants, n is the number of the terms, and h (x), φ(x) are functions of x.

Flow Past Wedges With Different Wedge Angles

Integrating Equation (IV-3) and multiplying by (-1) on both sides we obtain

$$-f\,d\eta = a_1\eta^3 + a_2\eta^4 + a_3\eta^6 + a_4\eta^7 + a_5\eta^8 + a_6\eta^9 + a_7\eta^{10} + a_8\eta^{11} \qquad \text{(V-3)}$$

where

$$a_1 = -\frac{a}{3!} \qquad a_6 = -\frac{(1+2\lambda)(11+10\lambda)a^3}{9!}$$

$$a_2 = -\frac{\lambda}{4!} \qquad a_7 = -\frac{2\lambda a^2(45+113\lambda+66\lambda^2)}{10!}$$

$$a_3 = \frac{a^2(1+2\lambda)}{6!} \qquad a_8 = -\frac{16\,a\,\lambda^2(2+3\lambda)(8+7\lambda)}{11!}$$

$$a_4 = \frac{2\lambda a(2+3\lambda)}{7!}$$

$$a_5 = \frac{2\lambda^2(2+3\lambda)}{8!}$$

Let $\frac{a\eta^3}{3!} = x$ $\qquad \eta = \left[\frac{x\,3!}{a}\right]^{1/3}$

then $d\eta = \left[\frac{2}{9a}\right]^{1/3} x^{-2/3}\,dx \qquad \text{(V-4)}$

Substituting the above relations in Equations (V-3) and (IV-4) and substituting them in Equation (III-10) the following equation results:

$$\left[\frac{9}{2}\right]^{1/3} a^{-1/3} = \int_0^\infty x^{-2/3} e^{-x} e^{h(x)} \varphi(x)\,dx \qquad \text{(V-5)}$$

where

$$h(x) = b_1 x^{4/3} + b_2 x^2 + b_3 x^{7/3} + b_4 x^{8/3} + b_5 x^3 + b_6 x^{10/3} \qquad \text{(V-6)}$$

$$\varphi(x) = c_1 + c_2 x^{1/3} + c_3 x + c_4 x^{4/3} + c_5 x^{5/3} + c_6 x^2 \qquad \text{(V-7)}$$

and

$$b_1 = -\frac{\lambda}{24} 6^{4/3} a^{-4/3}$$

$$b_2 = \frac{1 + 2\lambda}{20} \qquad \text{(V-8)}$$

$$b_3 = \frac{2\lambda(2 + 3\lambda)}{5040} 6^{7/3} a^{-4/3}$$

$$b_4 = \frac{2\lambda^2 (2 + 3\lambda)}{40320} 6^{8/3} a^{-8/3}$$

$$b_5 = -\frac{(1 + 2\lambda)(11 + 10\lambda) 216}{362880}$$

$$b_6 = -\frac{2(45 + 113\lambda + 66\lambda^2)\lambda 6^{10/3}}{3 \cdot 6288\ 106} a^{-4/3}$$

$$c_1 = a$$

$$c_2 = \lambda 6^{1/3} a^{-1/3}$$

$$c_3 = -2\lambda a$$

$$c_4 = \frac{(1 - 6\lambda)\lambda}{4} 6^{4/3} a^{-1/3} \qquad \text{(V-3)}$$

$$c_5 = \frac{(1 - 6\lambda)\lambda^2}{120} 6^{5/3} a^{-5/3}$$

$$c_6 = -\frac{10(1 - 2\lambda)\lambda}{20}$$

Semi-Infinite Plane

Integrating Equation (IV-7) and multiplying by -1 on both sides we obtain

$$- \int f d\eta = d_1 \eta^3 + d_2 \eta^6 + d_3 \eta^9 + d_4 \eta^{12} + d_5 \eta^{15} + d_6 \eta^{18} + d_7 \eta^{21} \qquad \text{(V-10)}$$

where

$$d_1 = - \frac{a}{3!} \qquad d_5 = - \frac{27897\ a^5}{15!}$$

$$d_2 = \frac{a^2}{6!} \qquad d_6 = \frac{3817173\ a^6}{18!} \qquad \text{(V-11)}$$

$$d_3 = - \frac{11\ a^3}{9!} \qquad d_7 = - \frac{3294299710\ a^7}{21!}$$

$$d_4 = \frac{375\ a^4}{12!}$$

introducing substitutions (V-4) in Equation (V-10) and substituting it in Equation (III-19) the following equation results:

$$\left[\frac{9}{2}\right]^{1/3} a^{-2/3} = \int_0^\infty x^{-2/3} e^{-x} e^{f_1 x^2 + f_2 x^3 + - - - + f_6 x^7} dx \qquad \text{(V-12)}$$

where

$$f_1 = \frac{1}{20} \qquad f_5 = \frac{3817173\ (46654)}{18!}$$

$$f_2 = - \frac{11\ (216)}{9!} \qquad f_6 = - \frac{3294299710\ (279936)}{21!}$$

$$f_3 = \frac{375\ (1296)}{12!}$$

$$f_4 = - \frac{7776\ (27897)}{15!}$$

SECTION VI

NUMERICAL RESULTS

The equations

$$A(a)^{L} = \sum_{j=1}^{5} H_j \, e^{b(a)^{w} x^{z} + c(a)^{p} x^{q} + d(a)^{r} x^{s} + e(a)^{t} x^{v} + f(a)^{i} x^{n} + g(a)^{k} x^{m}}$$

$$A(a)^{L} = \sum_{J=1}^{5} H_j \, e^{b(a)^{w} x^{z} + c(a)^{p} x^{q} + d(a)^{r} x^{s} + e(a)^{t} x^{v} + f(a)^{i} x^{n} + g(a)^{k} x^{m} \left[b'(a)^{w'} x^{z'} + c(a)^{p'} x^{q'} + d(a)^{r'} x^{s'} + e(a)^{t'} x^{v'} + f(a)^{i'} x^{n'} + g(a)^{k'} x^{m'} \right]}$$

were programmed for solution on IBM 1620 digital computer. The flow chart for numerical quadrature was given in Reference 9, page 248. The numerical results with exact values taken from some references are presented in tabular form on the following pages. The computer program is presented in Appendix B and average computer running time for one example was approximately forty minutes. In this thesis numerical results were obtained for three values of λ.

- $\lambda = 0$ refers to flat plate
- $\lambda = 1$ refers to flat plate normal to the flow (180° wedge)
- $\lambda = 0.25$ refers to 45° wedge immersed in the flow stream

The limits of f'' (0) for these wedge angles were calculated using Equation (I-5) and results are tabulated on page 24.

Semi-Infinite Plane

The exact value (obtained by exact solution*) of a = 0.4696 for semi-infinite plane was taken from Reference 8, page 41.

Results of Present Method

Order	Terms	f'' (0) = a	Absolute Error
1	4	0.4799	0.0103
1	6	0.4799	0.0103
2	4	0.4703	0.0007
	6	0.4706	0.0010
3	4	0.4711	0.0005
	6	0.4742	0.0046
4	4	0.4707	0.0011
	6	0.4721	0.0025
5	4	0.4709	0.0013
	6	0.4722	0.0026

*A solution is considered exact when it is complete solution of the boundary layer equations, irrespective of whether it is obtained analytically or by numerical methods.

Previous Results

Previous results are converted to the same variable as L. Howarth used (see Appendix D).

References	η	Results $f''(0)$	Multiplication Factor	Comparable Results
L. Howarth[14]	$y\left[\frac{u}{\nu x}\right]^{1/2}$	0.332	1	0.332
D. Meksyn[8]	$y\left[\frac{u}{2\nu x}\right]^{1/2}$	0.4696	$\frac{1}{\sqrt{2}}$	0.333
William Squire[2]	$y\left[\frac{u}{2\nu x}\right]^{1/2}$	0.480	$\frac{1}{\sqrt{2}}$	0.340
Present Method 1st order	$y\left[\frac{u}{2\nu x}\right]^{1/2}$	0.479	$\frac{1}{\sqrt{2}}$	0.339
Present Method 4th order	$y\left[\frac{u}{2\nu x}\right]^{1/2}$	0.472	$\frac{1}{\sqrt{2}}$	0.334

Flow Past Wedges With Different Wedge Angles

The ranges of the values for f'' (0) = a for corresponding wedge angles (see Figure 6) were taken from Reference 7.

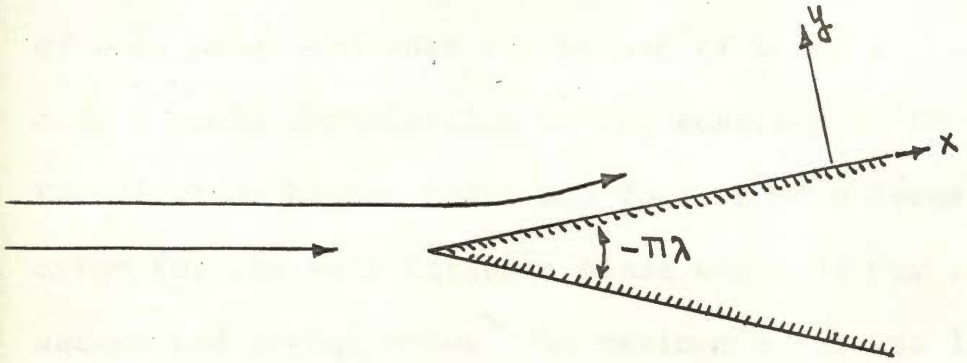

Figure 6

Results of Present Method

Wedge Angle (Degrees)	$-\lambda$	Order	Terms	f'' (0) = a	Ranges[7]
0	0	1	6	0.480	$0 < a < 0.577$
45	0.25	1	6	0.674	$0.577 < a < 0.816$
180	1	1	6	1.269	$1.155 < a < 1.29$

Comparison of Results

$-\lambda$	Hartree, D. R.[5]	Present Method	Absolute Error
0	0.469	0.480	0.011
0.25	0.731*	0.674	0.057
1	1.2326	1.269	0.036

*Interpolated from Hartree's result.[5]

DISCUSSION OF RESULTS

Applicability of generalized Gauss's quadrature to boundary layer problems is relatively new. The results of this paper indicate the number of terms in the quadrature made a small contribution to the accuracy of the results. For first or higher order and four or more terms the maximum error for the semi-infinite plane was 1.03 per cent. For second and higher orders the maximum error was less than .5 per cent using four and six terms. The solutions with six terms actually show a greater error than for four terms, indicating either a loss of accuracy or a tendency to converge to a different answer than that given by Meksyn. The digital computer was capable of handling only eight significant digits, and, therefore, rounding errors did appear in fifth and sixth terms of the series.

The results for flow past wedges with different wedge angles are also satisfactory, but it was not possible to carry out calculations with higher order because the uniqueness of the solution for arbitrary wedge angles has not been proven. One solution is easily obtained by using a first order of quadrature. No general relationship was found between the order of the quadrature and the number of terms required in the power series expansion.* The

*See Section V for definition of the term in quadrature.

error which is associated with the first order solution was expected. In this case abcissae and weights are assigned to a small portion of the power series where this power series is more accurate because of terms in it. This is why an addition of two more terms (six terms) did not make any significant change in the result for the first order. Because there are no values of the abcissae and weights assigned to the remainder of the curve, the accuracy of the power series is not affected. The formulas which were derived by Gauss for finding the value of an indefinite integral are in general approximate and thus there is an inherent error associated with them. This error is dependent upon the theory of Legendre polynomials. Time did not permit an evaluation of this error.

Advantages of the Gauss method are ease of application to the computer and the adaptibility to many other flow cases which have the same degree of quadrature ($\beta = -2/3$); for example, boundary layers with suction.*

The disadvantages are the difficulty in determining the order of the quadrature formula and number of terms in the power series needed for accurate results.

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*See Appendix C for β in quadrature formula.

CONCLUSIONS

The following conclusions can be drawn from the foregoing investigation:

1. In order to get a better understanding of the application of the generalized Gauss's quadrature for boundary layer problems, more extensive work is needed for different flow cases. This concept which shows promise is to develop the criteria for number of terms in the power series and the number of order in the quadrature formula.

2. It seems that more accurate results will be obtained if more terms (seven terms or more) and higher order is used. This would require the use of a digital computer which has more storage capacity than the IBM 1620.

3. For the flat plate solutions of arbitrary order (first to fifth) and term (four terms or more) used, maximum error does not exceed one per cent of the correct value. When a higher degree of accuracy is required for fixed number of terms, higher orders must be used.

4. Using more than two terms in the power series does not effect the result if the first order is used, and consequently the per cent error is greatest for this case.

5. From a method consideration, when two or three initial boundary conditions are not known in the non-linear high order differential equation, this is the one of the best methods for investigation. This method gives a concept which shows promise for more complex boundary layer problems in all branches of engineering.

APPENDIX A

DERIVATION OF EQUATION OF MOTION FOR FLOW PAST WEDGES WITH DIFFERENT WEDGE ANGLES AND SEMI-INFINITE PLANE

$$u \frac{\partial u}{\partial x} + u \frac{\partial u}{\partial y} = - \frac{\partial p}{\partial x} \frac{1}{\rho} + \nu \frac{\partial^2 u}{\partial y^2} \qquad \text{(A-1)*}$$

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0 \qquad \text{(A-2)*}$$

The components u and v each apparently vary with x and y independently. However, because of the form of boundary layer equations, it is often possible (for some boundary conditions) to determine a new variable η which combines the effects of x and y and reduces the two partial differential equations to one ordinary differential equation. This means if η is properly chosen, as a combination of x and y, u may become a function only η. Such a variable is called a "similarity variable," and the velocity distributions at all points along the surface are similar when expressed in terms of η.

*See Reference 10.

Flow Past Wedges With Different Wedge Angles

In the case of two-dimensional motion the boundary layer equations are described by Equations (A-1) and (A-2) where u is the only velocity which varies in the x direction and it does not vary in the y direction, so

$$u_\infty \frac{\partial u_\infty}{\partial x} = -\frac{1}{\rho}\frac{\partial p}{\partial x} \qquad \text{(Euler's Equation)} \qquad \text{(A-3)}$$

where u_∞ is the velocity at the edge of the boundary layer. Equations (A-1) and (A-2) are then rewritten for two-dimensional bodies as follows:

$$u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} = u_\infty \frac{\partial u_\infty}{\partial x} + \nu/\rho \frac{\partial^2 u}{\partial y^2} \qquad \text{(A-4)}$$

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0 \qquad \text{(A-5)}$$

where x is the distance measured from the forward stagnation point of the body and y is the distance measured normal to the surface of the body (see Figure A).

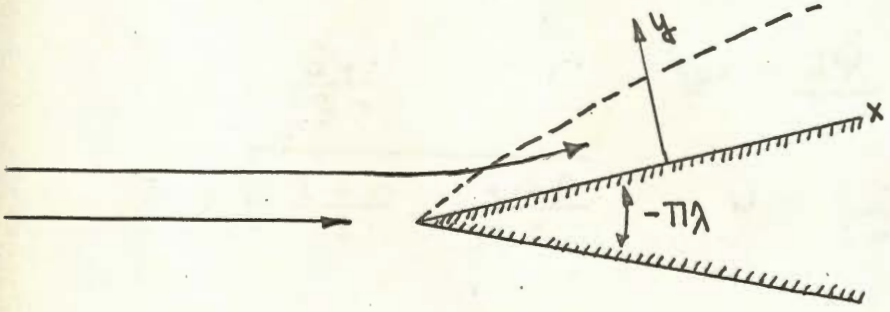

Figure A

In order to reduce Equation (A-4) to an ordinary non-linear differential equation, we may assume the following relation between u_∞ and x exist:

$$u_\infty = cx^m \tag{A-6}$$

where c is constant m is related to the wedge angle parameter λ as follows:

$$-m = \frac{\lambda}{2-\lambda} \tag{A-7}$$

$$\lambda = -\frac{2m}{m+1} \tag{A-8}$$

The existence of the similar solution when the velocity of the potential flow is proportional to a power of the length coordinate measured from the stagnation point Equation (A-6) was proved by Schlichting.[10] In order to satisfy the equation of continuity, Falkner and Skan[11] introduced the stream function ψ and the similarity variable such that

$$u = \frac{\partial \psi}{\partial y} \qquad v = -\frac{\partial \psi}{\partial x}$$

$$\eta = y\sqrt{\frac{1+m}{2}\,\frac{u_\infty \rho}{x}} \qquad \psi = \frac{2}{1+m}\,\frac{\nu u_\infty x}{\rho}\, f(\eta)$$

Equation (A-4) with corresponding boundary conditions

$$f''' + ff'' = \lambda (1 - f'^2) \qquad \text{(A-9)}$$

$$f(\eta) = 0 \qquad f'(\eta) = 0 \qquad \text{at} \quad \eta = 0$$
$$f'(\eta) = 1 \qquad \text{at} \quad \eta = \infty \qquad \text{(A-10)}$$

where $-\pi\lambda$ is the leading edge angle of the wedge.

Semi-Infinite Plane

Starting with complete momentum equations

$$\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} = -g_c\frac{\partial \Omega}{\partial x} - \frac{g_c}{\rho}\frac{\partial p}{\partial x} + \nu\left[\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2}\right]$$

$$\frac{\partial v}{\partial t} + u\frac{\partial v}{\partial x} + v\frac{\partial v}{\partial y} = -g_c\frac{\partial \Omega}{\partial y} - \frac{g_c}{\rho}\frac{\partial p}{\partial y} + \nu\left[\frac{\partial^2 v}{\partial x^2} + \frac{\partial^2 v}{\partial y^2}\right]$$

and assuming

1. Flow is steady
2. Boundary layer is thin compared to the distance measured from the leading edge.
3. $v \ll u$
4. As a result of (2), $\frac{\partial^2 u}{\partial x^2} \ll \frac{\partial^2 u}{\partial y^2}$
5. $\frac{\partial p}{\partial y} \backsimeq 0$
6. The field force Ω is neglected.

Equation (A-11) may be simplified as

$$u \frac{\partial u}{\partial x} + v \frac{\partial u}{\partial y} = - \frac{g_c}{\rho} \frac{\partial p}{\partial x} + \nu \frac{\partial^2 u}{\partial y^2} \qquad \text{(A-12)}$$

and continuity equation is

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0 \qquad \text{(A-2)}$$

Blasius[11] obtained exact* solution of Equations (A-12) and (A-2) for laminar boundary layer on a flat plate immersed in a fluid with uniform velocity. He defined the variable η by

$$\eta = \frac{y}{2} \left[\nu \frac{u_\infty}{x} \right]^{1/2} \qquad \psi = (\nu \, u_\infty \, x)^{1/2} \, f(\eta) \qquad \text{(A-13)}$$

Without showing the details Equation (A-13) may be used to obtain the following expressions as

$$\frac{\partial \psi}{\partial y} = \frac{u_\infty}{2} f' \qquad \text{(A-14)}$$

$$\frac{\partial^2 \psi}{\partial x \partial y} = - \frac{u_\infty}{4x} \eta f''$$

$$\frac{\partial \psi}{\partial x} = - \frac{1}{2} \left[\frac{\nu \, u_\infty}{x} \right]^{1/2} f + \frac{y \, u_\infty}{4x} f' \qquad \text{(A-15)}$$

*See page 22.

$$\frac{\partial^2 \psi}{\partial y^2} = u_\infty \; f'' \left[\frac{u_\infty}{\nu x} \right]^{1/2}$$

$$\frac{\partial^3 \psi}{\partial y^3} = \frac{u_\infty}{8} \; f''' \left[\frac{u_\infty}{\nu x} \right]$$

where f', f'', f''' are first, second and third total derivatives of f with respect to η. Substituting Equation (A-15) in Equation (A-12) gives

$$f''' + ff'' = 0 \qquad \text{(A-16)}$$

This is a non-linear ordinary differential equation of third order. The boundary conditions are

$$f(\eta) = 0 \qquad f'(\eta) = 0 \qquad \text{at} \quad \eta = 0$$
$$f'(\eta) = 1 \qquad \text{at} \quad \eta = \infty \qquad \text{(A-17)}$$

APPENDIX B

Tabulation of Weights and Abcissas

Example: Using Equation (A-7) and (A-8) in Appendix A for $\beta = -2/3$ and $n = 1$

$$L_1^{-2/3}(x) = 1/3 - x$$

$$x_1 = 0.3333$$

$$h_1 = \Gamma(1/3) = 2.6790$$

Order	X_j	H_j
1	0.3333	2.6790
2	0.1786	2.4594
	0.4880	0.1794
3	0.1220	2.3214
	1.6133	0.3485
	5.2645	0.0089
4	0.0927	2.1783
	1.2044	0.4662
	3.7213	0.0340
	8.3148	0.00037
5	0.0748	2.0629
	0.9633	0.5457
	2.9150	0.0676
	6.1884	0.0025
	11.5251	0.00001

Computer Program

Compiled Using UTO FORTRAN on an

IBM 1620 Digital Computer

```
    DIMENSION H(5),Y(5),PP(5),RR(5),SS(5),AA(5),BB(5),O(5)
    DIMENSION XX(5),YY(5)
    READ,A,B,C,D,E,Z
    READ,P,Q,R,S,T,U
    READ,W,ALP,DALP,TOL,V
    READ,PO,PM,P1,PK,PEP,PUS
    READ,QN,QP,QR,QQ,QS
 80 ACCEPT,N,ALP,DALP
 20 ALP=ALP+DALP
    PRINT,ALP
    F=A*(ALP**V)
    COW=).)
    DO 100 1=1,N
    IF(SENSE SWITCH 1)40,50
 40 ACCEPT,Y(1),H(1)
 50 G(1)=B*(Y(1)**Z)*(ALP**W)
    RR(1)=D*(Y(1)**S)*(ALP**R)
    PP(1)=C*(Y(1)**Q)*(ALP**P)
    SS(1)=E*(Y(1)**U)*(ALP**T)
    XX(1)=PEP*(Y(1)**PO)*(ALP**P1)
    YY(1)=PUS*(Y(1)**PM)*(ALP**PK)
    O(1)=G(1)+PP(1)+RR(1)+SS(1)+XX(1)+YY(1)
    EEE=ALP+QP*ALP*Y(1)+OR*(ALP)**(-.33333)*Y(1)**(1.33333)
    EEE=EEE+QQ*(ALP)**(-1.66667)*Y(1)**(1.66667)+OS*(ALP)*Y(1)**2.0
    EEE=EEE+QN*(ALP)**(-.33333333)*Y(1)**(.33333333)
    AA(1)=EXP(O(1))*(EEE)
    BB(1)=H(1)*AA(1)
100 COW=COW+BB(1)
    PRINT,COW,F
    IF(F-COW)130,200,20
130 ALP=ALP-DALP
    DALP=.1*DALP
    IF(DALP-TOL)200,200,20
200 PRINT,ALP
    GO TO 80
    END
```

APPENDIX C

NUMERICAL QUADRATURE

In numerical quadrature there are two types of problems such as

1. The evaluation of a definite integral
2. The evaluation of an indefinite integral as a function of the upper limit.

In this investigation the evaluation of an indefinite integral is of interest.

An approximation of the definite integral of a function f (x) is given by Gauss's quadrature for a function f (x) in the form

$$\int_a^b f(x)\,dx = \sum_{j=1}^{n} H_j\, f(X_j) \qquad \text{(C-1)}$$

where X_j's are abcissae and H_j's are corresponding weight coefficients. The principle is to obtain the best subdivision of the interval (a, b), the value of the function at these points, and coefficients to multiply by the functional values to yield the value of the definite integral. In geometric approach, the points X_j of coincidence of both curves are chosen so that the shaded area above the curve f_{2n-1} (x) cancels those below it within the interval (see Figure C-1).

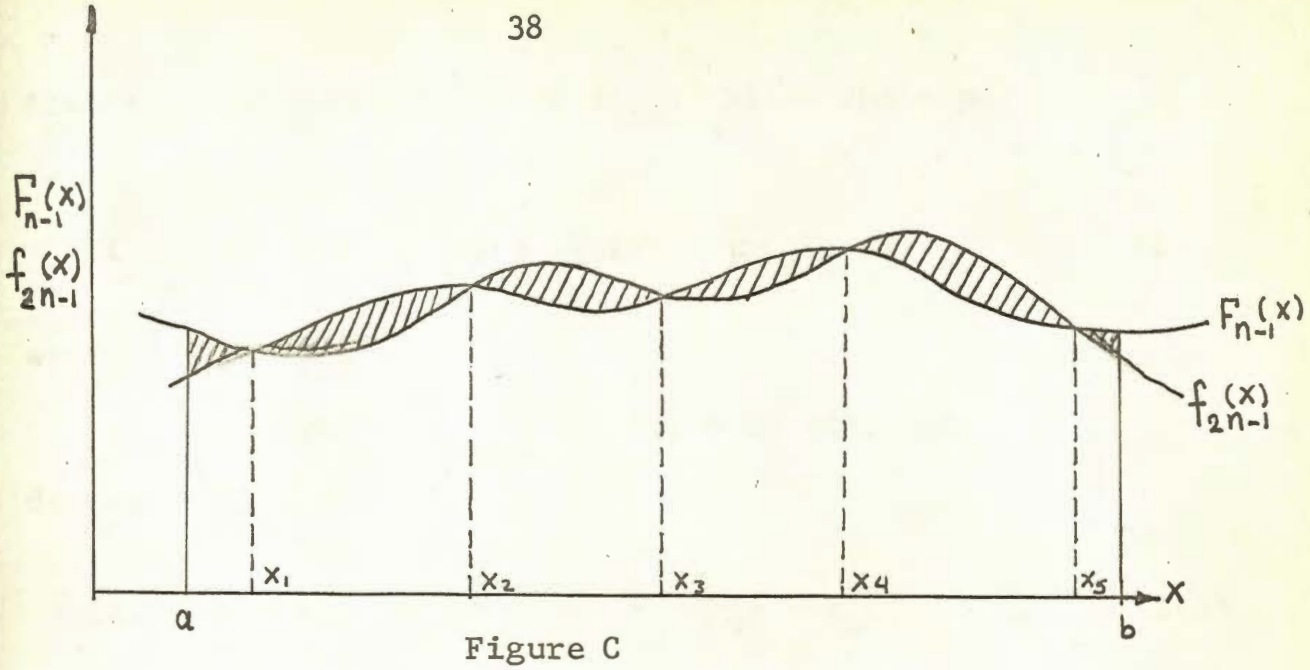

Figure C

The curve $F_{n-1}(x)$ represents a power polynomial of degree (n-1) and this curve passes through n points where these points are between the limit x = a, x = b. The curve $f_{2n-1}(x)$ represents another power polynomial of degree (2n-1) and the integral on the left-hand side of Equation (C-1) represented by the area between the x axis and the curve $f_{2n-1}(x)$, limited by the ordinates x = a and x = b, respectively. In order to have an exact solution for the problem is to specify the abcissae X_j's and the corresponding weight coefficients H_j's so that the right-hand side of Equation (C-1) represents the left-hand side exactly for as many powers x^k as possible (k = 0, 1, - - - -, n-1).

To determine the abcissae X_j's and the corresponding weights H_j's, assume x_1, x,- - - -, x_n are n points

that $F_{n-1}(x)$ coincides with $f_{2n-1}(x)$ at these points

Let

$$f_{2n-1}(x) = F_{n-1}(x) + \Pi(x) \quad G_{n-1}(x) \tag{C-2}$$

where

$F_{n-1}(x)$ and $G_{n-1}(x)$ are both polynomials of degree (n-1) and

$$\Pi(x) = (x - x_1)(x - x_2) - - - - -(x - x_n) \tag{C-3}$$

Integrating both sides of Equation (C-2) with respect to x between the limits a and b. The areas subtended by the curves f and F between the limits x = a, x = b, will be identical if $x_1, x_2 - - - -, x_n$ are chosen so that

$$(x - x_1)(x - x_2) - - -(x - x_n)\, G_{n-1}(x)\, dx = 0 \tag{C-4}$$

If Equation (C-4) holds for any polynomial $G_{n-1}(x)$ with arbitrary coefficients, it must certainly be fulfilled for each one of its constituent terms separately and this will be so if

$$(x - x_1)(x - x_2) - - -(x - x_n)\, x^k\, dx = 0 \tag{C-5}$$

where

$$k = 0, 1, 2, - - - (n - 1)$$

Then Equation (C-5) represents n independent conditions for a determination of n unknown constant X_j's.

In order to find the values of X_j we have to find the roots of a polynomial

$$P_n(x) = (x - x_1)(x - x_2) - - -(x - x_n)$$

which, as Equation (C-4) or (C-5) discloses, must be orthogonal to any polynomial in x of degree less than n. Determination of the coefficients of polynomial is given in Reference 9.

If the interval of integration is infinite such as the case where $\left[a, b \right] = \left[0, \infty \right]$ then

$$\int_0^\infty \omega(x) \; f(x) \; dx = \sum_{J=0}^{\infty} H_j \; f(X_j)$$

where $\omega(x)$ is the weight function. The weight function for generalized Gauss's quadrature is $x^{\beta} e^{-x}$ and it is an integrable function throughout the interval of integration. Corresponding quadrature formula used in this investigation is

$$\int_0^\infty x^{\beta} \; e^{-x} \; f(x) \; dx = \sum_{J=1}^{n} H_j \; f(X_j) \qquad \text{(C-6)}$$

where n and β are order and degree of the quadrature respectively. The abcissae X_j's (roots of the generalized

Laguerre polynomial $L_n^{\beta}(x)$ and corresponding weights H_j's may be found by the following relations:

$$L_n^{\beta}(x) = e^{x} x^{-\beta} \frac{d^n}{dx^n} \left[e^{-x} x^{\beta+n} \right]$$

or

$$L_n^{\beta}(x) = x^{-\beta} e^{x} \frac{d^n}{dx^n} \left[e^{-x} x^{\beta+n} \right]$$

$$H_j = \frac{n! \, \Gamma(n + \beta + 1)}{\left[L_{n+1}^{\beta}(X_j) \right]^2}$$

for $\beta = -2/3$ corresponding abcissae X_j's and weights H_j's are tabulated in Appendix B. The discussion about the numerical quadrature is taken from References 9, 12 and 13.

APPENDIX D

In case of the flat plate because of the transformation variable η, there are three forms of the boundary layer equations which appear in the literature. The following two forms are discussed in this thesis.

$$2 f_2''' + f_2 f_2'' = 0 \tag{D-1}$$

$$f_3''' + f_3 f_3'' = 0 \tag{D-2}$$

In this section the method is developed for converting one numerical value to another and this is used for semi-infinite plane.

Let

$$f_3''' + f_3 f_3'' = 0$$

where

$$f_3'(\eta_3) = 1 \qquad \text{for } \eta_3 = A$$

and

$$2f_2''' + f_2 f_2'' = 0$$

where

$$f_2'(\eta_2) = B \qquad \text{for } \eta_2 = C$$

substitute

$$f_2 = D\, f_3 \tag{D-3}$$

$$\eta_2 = E\, \eta_3 \tag{D-4}$$

Differentiate Equation (D-3)

$$f_2' = \frac{D}{E} f_3'$$

$$f_2'' = \frac{D}{E^2} f_3'' \qquad \text{(D-5)}$$

$$f_2''' = \frac{D}{E^3} f_3'''$$

Then substitute Equations (D-5) and (D-3) in (D-1)

$$2 \frac{D}{E^3} f_3''' + \frac{D^2}{E^2} f_3 f_3''' = 0 \qquad \text{(D-6)}$$

Equating Equation (D-6) to Equation (D-2)

$$2 \frac{D}{E^3} = 1 \qquad \frac{D^2}{E^2} = 1$$

$$2D = E^3 \qquad D^2 = E^2$$

$$2 = E^2 \qquad D = E$$

$$E = \sqrt{2} \qquad D = \sqrt{2}$$

Using Equation (D-5)

$$f_2'' = \frac{D}{E^2} f_3''$$

or

$$f_2''(0) = \frac{1}{E} f_3''(0)$$

$$f_2''(0) = \frac{1}{\sqrt{2}} f_3''(0)$$

An example application from page is as follows:

For $f_3''(0) = 0.4722$ (Present Method)

$$f_2''(0) = \frac{1}{1.414} = 0.4722$$

Present Method $f_2''(0) = 0.334$

L. Howarth $f_2''(0) = 0.33206$

Squire $f_2''(0) = 0.340$

Meksyn $f_2''(0) = 0.333$

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