A THEORY ON NORMALITY OF FINITE QUASIGROUPS

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PREFACE

This thesis is a discussion of normality theory for finite quasigroups. It is designed to develop this theory so that finite quasigroups with certain properties may be characterized. Most of this material is found in the Mathematics journals in one form or another.

Some other approaches to the study of normality of quasigroups are: equivalence relations defined by homomorphims, the kernel of a homomorphic mapping of a quasigroup into a loop, and congruence relations based on lattice theory.

The first definition of left-normality used in this paper is due to D. C. Murdock [7], and the second definition is taken from G. N. Garrison [4].

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A THEORY ON NORMALITY OF FINITE QUASIGROUPS

Definition 1.1. $\langle G,*\rangle$ is a quasigroup if and only if:

(1) G is a non-empty set

(2) $*$ is a binary operation on G

(3) For any two elements, a and b of G, there exist unique elements x and y of G, such that $a*x = b$ and $y*a = b$.

We will abbreviate $\langle G,*\rangle$ by G and $a*b$ by ab unless we wish to emphasize the operation.

Assumption 1.2. G will be assumed to be a finite quasigroup unless otherwise stated. Many of the theorems hold for non-finite quasigroups, but adjustments must be made in the proofs.

Definition 1.3. $\langle H,*\rangle$ is a subquasigroup of a quasigroup $\langle G,*\rangle$ if and only if:

(1) $H \subseteq G$

(2) $\langle H,*\rangle$ is a quasigroup.

Definition 1.4. Let G be a quasigroup, and let A and B be subsets of G. Then $AB = \{ab \mid a \in A, b \in B\}$. If $A = \{a\}$, then $AB = aB$.

We will define a quasigroup in a manner somewhat different from Definition 1.1, although the two definitions are equivalent.

Definition 1.5. Let G be a finite set consisting of n elements $a_1, a_2, \cdots, a_n$. Then G is a quasigroup if and only if:

(1) Each of the n^2 products $a_i a_j$ designates one and only one element of G

(2) If $a_k a_i = a_k a_j$ and $a_i a_k = a_j a_k$ then $i = j$ for every $k = 1, 2, \cdots, n$. Every a_m must be in G ($m = i, j$, or k).

Definition 1.6. The order of a quasigroup G is the number of elements in the set G, and the order of a subset H of G is the number of elements in H.

We will now define ℓ-normality for finite quasigroups.

Definition 1.7. Let H be a subquasigroup of G. Then H will be called ℓ-normal if and only if

$$(aH)(bH) = (ab)H$$

for every a and b in G. We will require that

$$(ah)H = aH$$

for every a in G and h in H.

Some elementary theorems will be needed in the development of this thesis, and these theorems on subsets and subquasigroups of quasigroups will precede the work on normality.

Theorem 1.8. Let H be a subset of G; then, if H is of order m, aH and Ha are of order m for any a in G.

Proof: Assume H is of order m and a is an element of G. Then $aH = \{ah \mid a \in G, h \in H\}$ and $Ha = \{ha \mid h \in H, a \in G\}$. Let $H = h_1, h_2, \cdots, h_m$. Now $aH = ah_1, ah_2, \cdots, ah_m$ and $Ha = h_1a, h_2a, \cdots, h_ma$. Hence, each of the cosets aH and Ha contain m distinct elements which implies that aH and Ha are of order m.

Theorem 1.9. Let H and K be subsets of G; then, if H is of order m and K is of order n, then HK (orKH) is of order at least as great as the larger of m and n.

Proof: Assume H is of order m and K is of order n. Then HK = $h_1k_1, h_2k_1, \cdots, h_mk_1, h_1k_2, h_2k_2, \cdots, h_mk_2, \cdots, h_1k_n, h_2k_n, \cdots, h_mk_n$. Assume that $m \geq n$. Then the elements $h_1k_1, h_2k_1, \cdots, h_mk_1$ must be distinct since G is a quasigroup. Hence HK contains at least m distinct elements. Similarly, if $n \geq m$, then HK contains at least n distinct elements. It is obvious this theorem also holds for KH.

Theorem 1.10. Let H and K be subsets of G, then $a(H \cap K) = aH \cap aK$ and $(H \cap K)a = Ha \cap Ka$.

Proof: Let b be any element of $H \cap K$. Then ab is an element of aH, and ab is an element of aK which implies

$$a(H \cap K) \subseteq aH \cap aK.$$

Let b be any element of both aH and aK. Then b = ah for some h in H, and b = ak for some k in K. Now, Definition 1.1 (3) implies h = k. Hence, we see that b = am where m is in $H \cap K$ which implies that b is in $a(H \cap K)$. Therefore $aH \cap aK \subseteq a(H \cap K)$. Hence $a(H \cap K) = aH \cap aK$. Similarly, $(H \cap K)a = Ha \cap Ka$.

Theorem 1.11. Let H and K be subsets of G; then, if Ha = Ka or aH = aK, we have that H = K. (a is any element of G).

Proof: Let h be an element of H, then we have that ha = ka for some k in K. By Definition 1.5, we obtain h = k which implies $H \subseteq K$. Similarly, $K \subseteq H$. It is now obvious that aH = aK, and hence H = K.

Theorem 1.12. Let H, K, and L be subsets of G such that H, K, and L all have the same order. If HK = L, then Hk = L and hK = L for every k in K and h in H.

Proof: From Theorem 1.8, we have that Hk is of the same order as H and hence of the same order as L. Now all of the elements of Hk are contained in HK which implies they are in L also. Since every Hk is in HK and HK = L we have that Hk = L. Similarly, hK = L.

Theorem 1.13. Let H, K, M, and L all be subsets of G such that they have the same order. If HL = M = KL or LH = M = LK, then H = K.

Proof: Let a be in L. Then Ha = M = Ka by Theorem 1.12. Since Ha = Ka we see that H = K by Theorem 1.11. Similarly, if LH = M = LK, then H = K.

Theorem 1.14. Let H be a subset of G, then H is a subquasigroup of G, if and only if, HH = H.

Proof: Suppose H is a subquasigroup of G. Then H is closed under *, that is, $h_1 * h_2$ is in H for every h_1 and h_2 in H which implies HH = H. Conversely, suppose HH = H. Then $h_1 * h_2 = h$ for every h_1 and h_2 in H. Since h is in H, we have closure. Since $h_k h_i = h_k h_j$ is impossible if $i \neq j$ for every element of G (hence of H) we see that Definition 1.5 is satisfied. Therefore, H is a suquasigroup of G if and only if HH = H.

Example 1.15. If H is any subquasigroup of G such that H is not ℓ-normal in G, then the order of H need not divide the order of G. Let <G,*> be defined by the following table:

*	1	2	3	4	5
1	1	2	4	5	3
2	2	1	5	3	4
3	5	4	3	2	1
4	3	5	1	4	2
5	4	3	2	1	5

Let $H = \{1,2\}$. $\langle H,*\rangle$ is a subquasigroup of order 2, but G is of order 5.

Theorem 1.16. Let H and K be subquasigroups of G, then $H \cap K$ is a subquasigroup or $H \cap K = \emptyset$.

Proof: Let H and K be subquasigroups of G such that $H \cap K = D \neq \emptyset$. Then $d_1 d_2$ is in H and in K for every d_i in D which implies $d_1 d_2$ is in D for every d_1 and d_2 in D. Hence $DD = D$, and D is a subquasigroup by Theorem 1.14. $H \cap K$ may be empty, for in Example 1.15, we have that $H = \{1,2\}$ and $K = \{5\}$ and $H \cap K = \emptyset$.

We now turn our attention to ℓ-normal subquasigroups. The previous theorems on subquasigroups are true if the subquasigroups are ℓ-normal, and the proofs would require very little alteration.

Theorem 1.17. Let H be a ℓ-normal subquasigroup of G, then the cosets aH and bH are identical or $aH \cap bH = \emptyset$ (that is, there exists left coset expansions with respect to any ℓ-normal subquasigroup H of G).

Proof: Assume that $aH \neq bH$. We have two possible cases.

Case 1. If $aH \cap bH = \emptyset$ the proof is complete.

Case 2. If $aH \cap bH = M \neq \emptyset$, then aH and bH both contain M. Assume that $M \neq aH$ and $M \neq bH$. Now $M(aH) \subseteq (aH)(aH)$, hence $M(aH) \subseteq (aa)H$ which implies that $M(aH) \subseteq a_1H$ where $a_1 = aa$. The order of $M(aH)$ cannot be less than the order of aH, and the order of $M(aH)$ cannot be greater than the order of a_1H since $M(aH) \subseteq a_1H$. Since aH and a_1H have the same order we see that $M(aH) =$

(aH)(aH). Similarly, M(aH) = (aH)(bH). Therefore aH = bH.

Definition 1.18. Let G be a quasigroup. For every a in G there exists a unique element e_a in G such that $a*e_a = a$. We shall call e_a the right unit of a.

If H and K are ℓ-normal subquasigroups of G, then $H \cap K$ is a ℓ-normal subquasigroup. The following theorem will prove that $H \cap K$ contains all the right units of G.

Theorem 1.19. Let H be a ℓ-normal subquasigroup of G, then H contains the right unit of every element of G.

Proof: Suppose H is ℓ-normal in G. Then by Definition 1.7, aH = (ah)H. This implies that ah is in (ah)H. The right unit of ah must be in H since H is a quasigroup, that is, the equation (ah)x = ah has a unique solution for x in H. Since a was chosen arbitrarily H must contain all right units of G.

Theorem 1.20. If H is a ℓ-normal subquasigroup of a quasigroup G, then there exists a finite number of disjoint cosets with respect to $H(H_1, H_2, \cdots, H_n)$ such that $G = H_1 \cup H_2 \cup \cdots \cup H_n$.

Proof: Since G is finite and there exists left coset expansions with respect to H, we see that aH = bH or $aH \cap bH = \phi$ by Theorem 1.17 for every a and b in G. This set of disjoint cosets, call them $H_1, H_2, \cdots, H_n$ comprise the entire set G. Therefore $G = H_1 \cup H_2 \cup \cdots \cup H_n$.

Corollary 1.21. If H is a ℓ-normal subquasigroup of G, then every element a in G is an element of some coset bH.

Proof: The disjoint cosets of Theorem 1.20 are of the form a_1H, $a_2H, \cdots, a_nH$. Now $G = a_1H \quad \cdots \quad a_nH$ which implies every element of G is in some a_iH $(i = 1,2, \cdots, n)$. Therefore, every element of G is in a coset bH.

Definition 1.22. If H is a ℓ-normal subquasigroup of G then G/H will denote the set of all cosets of H.

The set of all cosets, G/H, contains all of the elements of G by Theorem 1.20, and Corollary 1.21.

Theorem 1.23. Let H be a ℓ-normal subquasigroup of G such that (aH)(bH) = (aH)(cH), then bH = cH. Also, if (bH)(aH) = (cH)(aH), then bH = cH.

Proof: Suppose (aH)(bH) = (aH)(cH). Then (aH)(bH) = (ab)H = (aH)(cH) = (ac)H since H is ℓ-normal. H is finite which implies (aH)(bH), (ab)H, (ac)H, and (aH)(cH) all have the same order. Hence, by Theorem 1.13, we see that bH = cH. The last part of the theorem is proven in a similar manner.

Theorem 1.24. The set of cosets G/H forms a quasigroup with product of cosets as the product of Definition 1.5.

Proof: Since H is ℓ-normal in G the product of two cosets is a coset and Definition 1.5 (1) is satisfied. From Theorem 1.23 it follows that Definition 1.5 (2) is satisfied. Therefore G/H is a quasigroup.

The quasigroup G/H is called the quotient quasigroup, the elements of G/H being cosets. It is easily shown that G/H satisfies Definition 1.1. G/H is closed under the product of Definition 1.5, and the equations AX = B and YA = B are uniquely solvable in G/H. For, consider AX = B. Now aH = A and bH = B, and there exists an x in G such that ax = b. Hence, AX = (aH)(xH) = (ax)H = bH = B. Similarly, there exists a coset Y in G/H such that YA = B. Therefore, G/H is a quasigroup.

Theorem 1.25. Let a be an element in G, then b(aH) is equal to some coset xH if H is a ℓ-normal subquasigroup of G.

Proof: By Corollary 1.21, b is in some coset yH. Then Theorem 1.13 implies that b(aH) = (yH)(aH) = (ya)H = xH for some x in G. Therefore, b(aH) is equal to xH for some x in G.

Theorem 1.26. Let H be a ℓ-normal subquasigroup of G, then H is a single coset.

Proof: Let h be any element of H. Consider the coset hH. Now hH is equal to H since H is a subquasigroup. Therefore hH = H, and H is a single coset.

Theorem 1.27. Let H be a ℓ-normal subquasigroup of G, then aH = bH if and only if a and b are in cH for some c in G.

Proof: Assume that a is in cH and b is in dH. Then aH = (cH)H and bH = (dH)H by Theorem 1.26. By Theorem 1.23, (cH)H = (dH)H which implies cH = dH. Hence a and b are in cH. Conversely, let a be in cH and b be in cH. Then aH = (cH)H = bH which completes the proof.

We will now show that every quasigroup G with a ℓ-normal subquasigroup H has right coset decomposition as well as left coset decomposition.

Theorem 1.28. Let H be a ℓ-normal subquasigroup of G, then Ha may be written in the form dH for some d in G.

Proof: From Corollary 1.21, we have that a is in bH for some b in G. Hence $Ha \subseteq H(bH) = (hH)(bH) = (hb)H$ for each h in H. Since Ha and (hb)H have k elements if H is of order k, it follows that $Ha = (hb)H = dH$.

Theorem 1.29. (Converse of Theorem 1.28) Let H be a ℓ-normal subquasigroup of G, then dH may be written in the form Ha for some a in G.

Proof: Let H be of order k. Consider the equation $dH = HX$ in G/H. We can solve uniquely for the coset X since G/H is a quasigroup. Let a be any element of X. Then by Theorem 1.12, $dH = Ha$.

From Theorem 1.29, we see that right coset decomposition exists whenever left coset decomposition exists, and conversely.

The next two theorems concerning ℓ-normal subquasigroups will be familiar theorems of group theory.

Theorem 1.30. Let H be a ℓ-normal subquasigroup of G, then the order of H is a factor of the order of G.

Proof: Let H be of order m and G of order n $(m \leq n)$. Then each coset $a_i H$ has m elements by Theorem 1.8. Then by Theorem 1.17, G consists of k distinct cosets, each consisting of m elements. Hence there exist k times m elements in G. Therefore the order of H divides the order of G.

Corollary 1.31. Let H be a ℓ-normal subquasigroup of G. If G has order n and H has order m $(m \leq n)$, then the quotient quasigroup G/H has order n/m.

Proof: Let H be of order m and G be of order n. From Theorem 1. 30, we have that k times m equals n, and G/H is the set of cosets $a_1H, a_2H, \cdots, a_kH$ so that G/H consists of k elements (cosets). Now k = n/m and G/H has order n/m.

Theorem 1.32. Let H be a ℓ-normal subquasigroup of G, then a(bH) = (ab)H for every a and b in G.

Proof: Since a is in aH,

$$a(bH) \subseteq (aH)(bH) = (ab)H.$$

The order of a(bH) must be the same as the order of (ab)H which implies a(bH) = (ab)H.

Definition 1.33. Let H and K be ℓ-normal subquasigroups of G. Then $H \cup K$ is the minimal subquasigroup containing both H and K if and only if:

(1) $H \subseteq H \cup K$

(2) $K \subseteq H \cup K$

(3) If L is a subquasigroup of G such that $H \subseteq L$ and $K \subseteq L$, then $H \cup K \subseteq L$.

Theorem 1.34. Let H and K be ℓ-normal subquasigroups of G such that H and K contain all right units of G; then $H \cup K$ contain all products hk, where h is in H and k is in K.

Proof: Let h be any element of H and k be any element of K. Then h is in $H \cup K$ and k is in $H \cup K$ which implies hk is in $H \cup K$ since $H \cup K$ is a subquasigroup.

Definition 1.35. Let H and K be subquasigroups of G. Then H and K are permutable if and only if HK = KH.

Theorem 1.36. Let H and K be permutable ℓ-normal subquasigroups of G, then $H \cup K = HK = KH$ is a subquasigroup of G.

Proof: HK = KH since H and K are permutable. By Definition 1.4 $HK = \{hk \mid h \in H, k \in K\}$. Since K contains all right units of G (hence of H), $h_i k_j$ is in HK and KH for every i and j equal to 1, 2, $\cdots$, n if n is the order of HK. Now $h_1 k_1$ is in HK and KH, $h_2 k_2$ is in HK and KH, $h_1 k_1 = k_1' h_1'$, and $h_2 k_2 = k_2' h_2'$. We must show that $(h_1 k_1)(h_2 k_2) = (k_1' h_1')(k_2' h_2')$ for every h in H and k in K. Since H and K are of finite order,

$$(h_1 k_1)(h_2 k_2) \subseteq (h_1 K)(h_2 K) = (h_1 h_2)K = HK = KH = (k_1' k_2')H = (k_1' H)(k_2' H).$$

Hence $(h_1 k_1)(h_2 k_2)$ is equal to $(k_1' h_1')(k_2' h_2')$ for some h_1' and h_2' in H. Similarly $(k_1 h_1)(k_2 h_2) = (h_1' k_1')(h_2' k_2')$. Therefore, the product operation is closed in HK. Unique solutions exist to the equations ax = b and ya = b for every a and b in G (hence in HK). Therefore, HK is a quasigroup by Definition 1.1. Then Definition 1.33 implies that $H \cup K = HK = KH$.

Corollary 1.37. Let H and K be permutable ℓ-normal subquasigroups of G such that $H \cap K \neq \emptyset$, then $H \cup K$ consists of only those elements of the form hk(kh).

Proof: The Corollary is obvious from Theorem 1.36.

Before our discussion of homomorphism theory we will give an example of a finite quasigroup with a ℓ-normal subquasigroup.

Example 1.38. $\langle G, * \rangle$ is a quasigroup with the operation * defined by the following table:

*	1	2	3	4	5	6	7	8	9
1	1	3	2	5	6	4	9	7	8
2	3	2	1	6	4	5	7	8	9
3	2	1	3	4	5	6	8	9	7
4	6	4	5	7	8	9	2	3	1
5	4	5	6	8	9	7	3	1	2
6	5	6	4	9	7	8	1	2	3
7	8	9	7	3	1	2	6	5	4
8	9	7	8	1	2	3	4	6	5
9	7	8	9	2	3	1	5	4	6

If $H = \{1, 2, 3\}$, then $\langle H,*\rangle$ is a subquasigroup of $\langle G,*\rangle$, and H is the set of all right units of G. It is easily verified that H is a ℓ-normal subquasigroup of G.

G/H, the set of all cosets of H, consists of the sets $\{1, 2, 3\}$, $\{4, 5, 6\}$, and $\{7, 8, 9\}$.

Definition 1.39. Let G and G' be quasigroups. If there exists a mapping f such that $f:G \longrightarrow G'$ and f preserves the relations and operations of F such that for each relation and operation in G there is a corresponding relation or operation in G', then f is a homomorphism of G into G'. If the mapping f is onto G' in a 1-1 correspondence, then f is an isomorphism.

We may say if $f:G \longrightarrow G'$ such that $f(ab) = f(a)f(b)$, then f is a homomorphism of G into G'.

Theorem 1.40. Let G be a quasigroup and H be a ℓ-normal subquasigroup of G; then G is homomorphic to G/H.

Proof: Since H is ℓ-normal, we have that $(aH)(bH) = (ab)H$. Now we define a mapping f such that $f(a) = aH$, for every a in G. Then $f:G \longrightarrow G/H$ since G/H is the set of all cosets of the form aH. We have that $f(a)f(b) = (aH)(bH) = (ab)H$. Now, f is a function, for aH and bH are identical or $aH \cap bH = \phi$ by Theorem 1.17. Since $(ab)H = f(ab)$, Definition 1.14 is satisfied and G is homomorphic to G/H which is a quasigroup by Theorem 1.24.

A logical question which arises concerning ℓ-normal subquasigroups is the characterization of quasigroups which possess ℓ-normal subquasigroups. We may answer this question if we consider the homomorphic image of any quasigroup with a ℓ-normal subquasigroup.

Definition 1.41. Let G be a set of elements $a_1, a_2, \cdots a_n$ such that there exists a function $f: G \longrightarrow G'$. Then G' is a homomorphic image of G if and only if:

(1) Every a in G maps into one and only one a' in G'

(2) Every a' in G' has a pre-image in G

(3) If $a \longrightarrow a'$ and $b \longrightarrow b'$, then $ab \longrightarrow a'b'$.

Theorem 1.42. The homomorphic image G' of G is a quasigroup whenever Definition 1.41 is satisfied.

Proof: Let H_i be the set of elements in G which are mapped to the a_i''s in G' for every $i = 1, 2, \cdots, n$. Now $H_i \cap H_j = \phi$ if $i \neq j$ so that if $a_i'a_j' = a_k'$ in G' then $H_iH_j \subseteq H_k$ by Definition 1.41 (3). Each H_i is of the same order by Theorem 1.8 which implies $H_iH_j = H_k$. We see that G' is a quasigroup for every a in H_i maps into only one a' in G'. Suppose $a_ia_j = a_ia_k$ in G $(j \neq k)$, then in the quasigroup G we would have $H_iH_j = H_iH_k$ where $H_j \cap H_k = \phi$ which contridicts Theorem 1.13. Hence Definitions 1.39 and 1.41 are satisfied. Therefore G' is a quasigroup.

The quotient quasigroup G/H is an example of the homomorphic image

G' of G in Theorem 1.42 whenever H is a ℓ-normal subquasigroup.

We are now prepared to answer the question of characterization.

Theorem 1.43. Let G be a finite quasigroup, then there exists a proper ℓ-normal subquasigroup H of G if and only if there exists a non-trivial homomorphic image G' of G containing an element a' such that

$$(a_i'a')(a_j'a') = (a_i'a_j')a'$$

for every i and j such that $a \longrightarrow a'$ if a is in H.

Proof: From Theorem 1.40 we see that G/H is a homomorphic image of G. Now $G/H = G'$ such that if x is in aH and y is in bH, (aH) (bH) = (ab)H in G which implies

$$[(aH)H][(bH)H] = [(aH)(bH)]H \text{ in } G/H.$$

Hence the image $G' = G/H$ contains the a' (which is equal to H) with the property $(a_i'a')(a_j'a') = (a_i'a_j')a'$. Conversely, let H, H_i, and H_j denote the cosets in G which correspond to a', a_i', and a_j' respectively. Then by Theorem 1.42

$$(H_iH)(H_jH) = (H_iH_j)\ H$$

for every i and j. Let x and y be any elements of G. Suppose x is in H_i and y is in H_j. By Definition 1.41 (3) we see that xy is in H_iH_j. Now Theorem 1.12 implies that (xH)(yH) is equal to (xy)H which proves H is ℓ-normal.

Theorem 1.43 gives us a method of identifying quasigroups which contain proper ℓ-normal subquasigroups. To the author's knowledge, there is no way to characterize quasigroups containing ℓ-normal subquasigroups without an investigation of its homomorphic image. If Theorem 1.43 is altered so that $(aa_i')(aa_j') = a'(a_i'a_j')$, then a r-normality would exist, that is, $(Ha)(Hb) = H(ab)$ for every a and b in G. If the quasigroup G is abelian, then Theorem 1.43 would imply the existence of a r-normality.

A loop is a quasigroup containing a unique identity. The quotient quasigroup G/H is a right-loop, that is, G/H contains a unique right identity. If the quasigroup G/H is abelian, then G/H becomes a full loop.

If we alter the definition of ℓ-normality a new result may be obtained concerning normality of cosets.

Definition 1.44. A subquasigroup H of G will be called ℓ_1-normal if and only if

$$(aH)(bH) = cH$$

where a,b, and c are elements of G. We will require that $(ah)H = aH$ for every a in G.

This definition does not require that $ab = c$. The quasigroups which have ℓ-normal subquasigroups form a proper subset of the quasigroups with ℓ_1-normal subquasigroups. It is easily seen that Definition 1.7 is a special case of Definition 1.44.

Definition 1.45. Let H be a ℓ_1-normal subquasigroup of G. Then the coset aH is ℓ_1-normal if and only if

$$[b(aH)][c(aH)] = d(aH)$$

where a, b, c and d are elements of G. We could define aH to be ℓ-normal if and only if

$$[b(aH)][c(aH)] = (bc)(aH).$$

Theorem 1.46. Let H be a ℓ_1-normal subquasigroup of G, then the coset aH is a ℓ_1-normal coset for every a in G.

Proof: Every element in G is in some coset xH and b(aH) is equal to some coset yH. Hence the product [b(aH)][c(aH)] may be written in the form (pH)(qH) for some p and q in G. Since H is ℓ_1-normal, (pH)(qH) = rH for some r in G. Consider the equation X(aH) = rH in G/H. Assume X is non-empty and let d be an element of X. Then rH = d(aH) since the order of X(aH) is equal to the order of rH. Hence,

$$[b(aH)][c(aH)] = d(aH)$$

which implies that aH is ℓ_1-normal.

If we restrict H to be a ℓ-normal subquasigroup of G, then aH is not necessarily a ℓ-normal coset. If H is ℓ-normal in G, then aH is not ℓ-normal in Example 1.38. However, H is also ℓ_1-normal in Example 1.38, and aH is ℓ_1-normal for every a in G.

Definition 1.47. Let H be a subquasigroup of G. Then H is a r_1-normal subquasigroup of G if and only if

$$(Ha)(Hb) = Hc \text{ and } H(ha) = Ha$$

where a, b, and c are elements of G and h is in H.

This leads us to our final theorem on finite quasigroups.

Theorem 1.48. Let H be a subquasigroup of G, then H is a ℓ_1-normal subquasigroup of G if and only if H is a r_1-normal subquasigroup of G.

Proof: Since H is ℓ_1-normal, (aH)(bH) = cH. Every cH is equal to

(Hp)(Hq) for some p and q since these cosets must be of equal order. Hence,

$$(aH)(bH) = cH = Ha = (Hp)(Hq)$$

which implies H is r_1-normal. Similarly, H is ℓ_1-normal if H is r_1-normal.

This theorem does not depend on the commutative law but is based on the fact that right coset decomposition exists if and only if we have left coset decomposition. Since the quasigroup $G = a_1H \cup a_2H \cup \cdots \cup a_kH = Hb_1 \cup \cdots \cup Hb_k$ we see that every Hb_j is equal to some a_iH.

Certain problems arise if G is a non-finite quasigroup. We are not permitted to rely on the order properties which were essential in the proofs of many of our previous theorems. Our final theorem will illustrate one of the problems of infinite quasigroup theory when approached from the ℓ-normality point of view.

Definition 1.49. Let G be a non-finite set. Then $\langle G,*\rangle$ is a left-quasigroup if and only if:

(1) * is a binary operation on G

(2) For any two elements a and b in G, there exists a unique element x in G such that $a*x = b$.

Similarly G is a right-quasigroup if and only if * is a binary operation on G, and for any two elements a and b in G there exists a unique element y in G such that $y*a = b$. Then a quasigroup is both a right and left quasigroup.

In our final theorem, we will assume that $(ab)H = a(bH)$ for every a and b in G.

Theorem 1.50. Let H be a ℓ-normal subquasigroup of G where G is non-finite, then G/H is a left quasigroup.

Proof: Since H is ℓ-normal in G,

$$(aH)(bH) = (ab)H \text{ which implies } G/H \text{ is closed.}$$

The left cancellation law holds, for consider

$$(aH)(bH) = (aH)(cH)$$
$$(ab)H = (ac)H$$
$$a(bH) = a(cH)$$
$$a(bh_1) = a(ch_2)$$
$$bh_1 = ch_2$$

(that is, for every h_1 in H there exists an h_2 in H such that $bh_1 = ch_2$ and vice-versa)

Hence bH = cH which shows the left cancellation law holds. Consider the equation AX = B in G/H. A solution to this equation exists, for consider

aH and bH in G/H.

There exists a x in G such that ax = b which implies (aH)(xH) = (ax)H = bH. Letting A = aH, B = bH and X = xH we see a solution to the equation AX = B does exist. This solution is unique since the existence of AX = B and $AX_1 = B$ implies $X_1 = X$ from the left cancellation law. Therefore, G/H is a left quasigroup.

With the properties given above we have no method of proof which will guarantee the existence of unique solutions to the equation YA = B.

Some interesting questions arose during the writing of this paper which provide for further study in this area. However, the results obtained give us a clear picture of the limitations on general quasigroup theory when approached from a normality point of view. The following questions are left open in this thesis:

(1) What is a necessary and sufficient condition for aH to be ℓ-normal for every coset in G/H if H is ℓ-normal?

(2) If H is a ℓ-normal subquasigroup of G, what are the conditions required for H to be r-normal?

(3) What are the necessary conditions for aH to be equal to Ha for every a in G? (assuming H is ℓ-normal)

(4) Could the definition of normality be changed to produce extensive results in general homomorphism theory for quasigroups?

(5) If we define H to be the kernel of a homomorphism which maps G into G/H, will more general results be obtained?

(6) What conditions are necessary for G/H to be a quasigroup if G is non-finite?

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