

A Unified Approach to Stabilize an Arbitrary Order Discrete or Continuous Time Transfer Function with Time Delay Using a PI Controller

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Abstract. The object in this paper is to find the stability regions of an arbitrary order discrete or continuous time transfer function with time delay. The stability bounds of the proportional integral (PI) controller are found in terms of the proportional (K_p) and integral gain (K_i). The delta operator is used to describe the controllers because it provides not only numerical properties superior to the discrete time shift operator, but also converges to the continuous time derivative operator as the sampling period approaches a zero.

The advantage of this method is that designers can find the stability boundaries when only the frequency response and not the parameters of the plant transfer function are known. A unified approach allows us to use the same procedure for finding the discrete time and continuous time stability regions. If the plant transfer function is known, the stability regions can be found analytically. Regions where phase and gain margin specifications are met can also be found.

1. INTRODUCTION

There has been a significant amount of research concerning the stability boundaries for PI (proportional integral) control of systems with time delay. Most of these methods depend on earlier developed theorems and the fact that the plant parameters are known. However, in [1], a new method, which does not involve complex mathematical derivations, is used to solve the problem of stabilizing an arbitrary order transfer function when only the numerical frequency response of the plant transfer function is known. Typically, the digital control systems are described in a different framework than those used for continuous time control systems. In [2] and [3], it is shown that the continuous and discrete cases can be understood under a common framework through the use of delta operators. In this paper, delta operators are used to obtain unified stability boundaries for PI control of arbitrary order transfer functions with time delay.

2. STABILITY BOUNDS

A general single-input single-output continuous-time plant transfer function with time delay τ_d , $G_p(s) = G(s)e^{-\tau_d s}$, will be considered. If the output of plant is sampled with a zero-order hold input, the equivalent delta domain transfer function $G_p(\gamma)$ can be found

from $G_p(\gamma) = \frac{\Delta\gamma}{1+\Delta\gamma} T \left[L^{-1} \left\{ \frac{1}{s} G(s) \right\} \right]$, where Δ is the

sampling period, $\gamma = \begin{cases} s, & \Delta = 0 \\ \frac{e^{s\Delta} - 1}{\Delta}, & \Delta \neq 0 \end{cases}$, and T is the

generalized transform defined in [2].

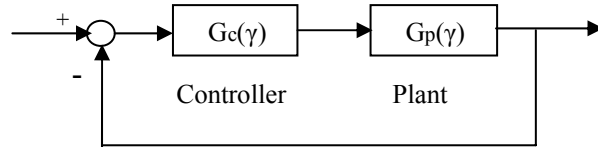


Fig. 1. Basic system with unity feedback

A standard unity feedback control system is shown in Fig. 1, where a PI controller of the form

$$G_c(\gamma) = K_p + K_i \frac{1 + \Delta\gamma}{\gamma}$$

is used for control. The goal in this paper is to find the regions in the K_p and K_i plane that make the closed-loop characteristic polynomial $\Delta(\gamma)$ of the system in Fig. 1 Hurwitz stable. The characteristic equation is written as

$$\Delta(\gamma) = 1 + G_c(\gamma)G_p(\gamma).$$

The frequency response can be found by letting $\gamma = \beta \equiv (e^{j\omega\Delta} - 1) / \Delta$. The plant

frequency response, $G_p(\beta)$, can be written as

$$G_p(\beta) = R_p(\omega) + jI_p(\omega).$$

Expanding the frequency response equation, $\Delta(\beta)$, and setting the real and imaginary parts equal to zero results in

$$\begin{bmatrix} \frac{\sin(\omega\Delta)R_p(\omega)}{\Delta} & X_{12} \\ \frac{\sin(\omega\Delta)I_p(\omega)}{\Delta} & X_{22} \end{bmatrix} \begin{bmatrix} K_p \\ K_i \end{bmatrix} = \begin{bmatrix} -\sin(\omega\Delta) \\ \Delta \\ 0 \end{bmatrix}, \quad (1)$$

where

$$X_{12} = \frac{\sin(\omega\Delta)}{2} R_p(\omega) + \frac{(1 + \cos(\omega\Delta))}{2} I_p(\omega) \quad (2)$$

$$X_{22} = \frac{\sin(\omega\Delta)}{2} I_p(\omega) - \frac{(1 + \cos(\omega\Delta))}{2} R_p(\omega) \quad (3)$$

The stability boundaries for the PI controllers can be expressed by the solutions of the above equations.

For $\Delta \neq 0$ and $\omega \neq 0$, the solution to (1) is given by

$$K_i(\omega) = -\frac{2 \sin(\omega\Delta) I_p(\omega)}{\Delta (1 + \cos(\omega\Delta)) \left((I_p(\omega))^2 + (R_p(\omega))^2 \right)} \quad (4)$$

$$K_p(\omega) = -\frac{\Delta}{2} K_i(\omega) - \frac{R_p(\omega)}{\left((I_p(\omega))^2 + (R_p(\omega))^2 \right)}. \quad (5)$$

At $\omega = 0$, the solution to (1) is given by

$$\begin{bmatrix} 0 & I_p(0) \\ 0 & R_p(0) \end{bmatrix} \begin{bmatrix} K_p \\ K_i \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}. \quad (6)$$

From (6) it can be seen that K_p is arbitrary and the value for K_i must be zero, unless $R_p(0) = I_p(0) = 0$.

By setting $\Delta \rightarrow 0$ in (4) and (5), we obtain the continuous-time stability boundaries

$$K_i(\omega) = -\frac{\omega I_p(\omega)}{|G_p(j\omega)|^2} \quad \text{and} \quad K_p(\omega) = -\frac{R_p(\omega)}{|G_p(j\omega)|^2},$$

which are equivalent with those found in [1].

Example. A continuous system with a time delay is given by

$$G_p(s) = \frac{-0.5s + 1}{(s + 1)(2s + 1)} e^{-0.6s}. \quad (7)$$

Fig. 2 shows the stability boundaries for a discrete-time PI controller with $\Delta = 0.5$ and continuous-time PI controller. Plot of the closed-loop step responses are shown in Fig. 3 for discrete time and continuous time PI controllers with $K_p = 1.5$ and $K_i = 0.5$.

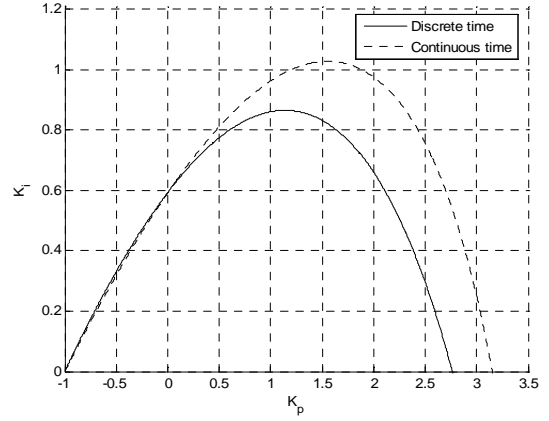


Fig. 2 Stability boundaries for Example

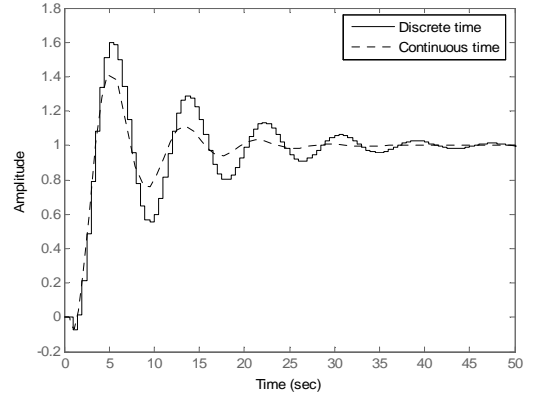


Fig. 3 Step response of compensated system

3. CONCLUSION

By using the delta operator, we can easily find the unified stability boundaries of PI controllers from the frequency response of a plant without dealing with complex mathematical derivations. It is shown that the continuous and discrete cases can be understood under a common framework through the use of the delta operator. If the parameters of the plant are known, this method can be used to find the stability boundary analytically. We can also apply this procedure to find stability regions that meet specific gain and phase margin requirements.

Reference

- [1] S. Sujoldzic and J. M. Watkins, "Stabilization of an arbitrary order transfer function with time delay using PI and PD controllers," Proceedings of the American Control Conference, Minneapolis, MN., Jun 2006.
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