Optimal design of decentralized large-scale systems with low sensitivity to non-linearity

OPTIMAL DESIGN OF DECENTRALIZED LARGE-SCALE SYSTEMS WITH LOW SENSITIVITY TO NON-LINEARITY

A Dissertation by

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DEDICATION

I dedicate my dissertation work to my family and many friends. A special feeling of gratitude to my loving parents.

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ABSTRACT

The optimal design of complex systems with low sensitivity to nonlinear subsystems is an important problem in engineering and science. When talking about non-linearity, linearization of non-linear control systems is an important technique used in control engineering to analyze the behavior of complex nonlinear systems. The process involves approximating a nonlinear system around an operating point by a linear system that can be analyzed using conventional linear control theory. This technique is particularly useful in control system design since it allows the use of well-established techniques such as state feedback, and linear quadratic regulator (LQR) to design controllers for nonlinear systems. Linearization is a powerful tool that can help researchers understand the behavior of nonlinear systems and design effective control strategies to achieve desired performance objectives.

This paper proposes the single perturbation method as a model reduction technique to reduce the order of the system. The singular perturbation approach identifies the slow and fast dynamics of a complex system and develops reduced-order models that represent the system's core characteristics. The reduced-order models are then used to design optimal controllers for the complex system that are insensitive to the nonlinear subsystems.

Game theory is a branch of mathematics that looks at how people make strategic decisions when the results of their choices depend on the decisions made by others. It can be used in a lot of different ways in control systems, especially in the design of systems with more than one person making decisions and interacting with each other. Game theory may be used to control systems to evaluate agent behavior and create controller designs that maximize system performance. The reduced order model was optimized using Nash strategy of game theory.

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x	Slow subsystem states variable in interconnected systems.
z_i	Fast subsystem i states variable in interconnected systems.
u_i	system input i.
y_i	system output i.
F	Feedback gain.
ϵ	Singular Perturbation, small, real-valued, positive parameter.
$\bar{x}, \bar{z}, \bar{u},$	Quasi-steady state system variables.
x_s, z_s, u_s	Reduced order slow model variables.
x_f, z_f, u_f	Reduced order fast model variables.
"A'"	Denotes the derivative of A with respect to "μ" evaluated at "μ_0".
P, K	Lyapunov and Riccati equations variables.
$E\{.\}$	Expected value.
$tr[A]$	The trace of the matrix A.
$\dot{x}$	First derivative of x with respect to time.
$\frac{\partial y}{\partial x}$	Partial derivative of y with respect to x.
μ	Non-linearity.
σ	Low Sensitivity.
ζ_i	Linearized subsystem i states variables in interconnected systems.
α_i	Linearized system input i.

CHAPTER 1

INTRODUCTION

1.1 Background

The foundation of our society today is built on control theory, from cruise control to intelligent robotics. Control theory's goal is to push a system, often known as the plant, to produce the desired output while maintaining the desired performance. It is feasible and crucial for system performance to create a controller utilizing control theory techniques that is optimum, stabilizing, and resilient. Large-scale systems include a wide range of actual physical systems, including power networks, wireless telecommunications networks, and computer networks. All large-scale systems require control in order to function. However, due to dimensionality, restrictions on the information structure, uncertainty in the information exchanges, and delays for long-distance data transfer, large-scale systems are frequently too massive and the problems are too difficult to create and evaluate. When studying, designing, and putting control techniques and algorithms into practice, these complexities result in significant challenges [1][2].

When several interconnected subsystems are present, a system is said to be large-scale. Large-scale systems are frequently characterized by a multidimensional, complex structure, as well as multi-state and multi-input variables. A system is deemed huge from the standpoint of the control system if it is larger than a single controller structure. Large-scale systems are specifically controlled using a multi-controller structure. A control designer must create effective control schemes that can be executed with the least amount of money and resources because the control problem may grow exponentially with an increase in the complexity of the large-scale system. A control designer's responsibility is to assure stability, robustness, and dependable operation of a large-scale system in the event of a subsystem(s) failure. It is required to divide large-scale systems

into a number of smaller interconnected subsystems in order to handle the complexity in the analysis and synthesis activities for these systems. Decentralized control is crucial in this situation to make analysis and control synthesis for large-scale systems simpler.

1.2 Control System

A control system can be defined as a system that controls or manages other systems to obtain a specific outcome. There are various classifications of control system, such as; linear or nonlinear system, and continuous or discrete time system. Also, we have different system structures like centralized or decentralized system.

1.3 Decentralized System

A decentralized control system contains a main system and one or more subsystems controlling the main system. In this research, we will consider the system structure to be a decentralized large scale system with non-linearity in the continuous time.

Large-scale systems commonly incorporate decentralized control. A decentralized system is made up of numerous separate subsystems, each of which functions independently and makes decisions based on its measurements. Each subsystem also has its own input, output, and control unit. Decentralization is accomplished by breaking down a large-scale system into smaller subsystems, resolving each subsystem independently, and then merging the results to solve the original system as a whole. In control systems, this can be accomplished by decomposing the analysis and synthesis of the entire system into uncoupled subsystems, where there is no information transmission between various local controllers. As a result, a decentralized controller comprised of a number of autonomous controllers that work together to control the entire system. Each connected subsystem that is under decentralized management is in charge of carrying out a certain system duty. The control unit for each individual subsystem in decentralized control schemes is created in such a way that it can only access a small, localized collection of information. A decentralized control system's core feature is its lack of reliance on information from other

subsystems, which completely eliminates the issue of distorted information sharing. Decentralized control's dependability objective is to stabilize the system by building a control unit for each control channel such that it can withstand failures [3].

1.4 Non-Linearity

In control system, a part of the system can be linear while the other part is nonlinear. For example, the main system can be linear but the subsystem is nonlinear or the opposite. This is usually a problem to deal with in analysis and design. So, what needs to be done, in order to deal with that, is linearizing the system. Another issue here is that we will lose accuracy after linearizing the model because the state variables in the linearized model have a different nature from the state variables in the nonlinear system since they do not represent the actual value of $x(t)$ but represent how far the value is from a certain value.

Linearizing a nonlinear model is by expanding it around a particular point, we call it an equilibrium point. The equilibrium point is where the system reaches a steady state. If it is linear, then we have one equilibrium point which will be 0 at infinity(∞).

For a nonlinear system, however; it can have more than one equilibrium point. So, we consider the system to be linear around a particular equilibrium point, and for that equilibrium point, we have the linearized model. This linearized model is an approximate representation of the actual nonlinear system even if the model changes a little bit, the linearized model will still represent the system well enough. That approximation means that there is some uncertainty because the linearization process gets into the picture some uncertain parameters. We assume that this uncertain parameter has a certain value, but it is not equal to that value all the time. Therefore, we say that the non-linearity is an uncertain parameter μ, which is evaluated at the nominal value μ_0.

1.5 Singular Perturbation

The first essential stage in performing system analysis and control design is mathematical modeling of a physical system. Modeling should achieve a balance between being straightforward and accurate with sufficient in-depth dynamics. Ordinary differential equations of high order are necessary for the accurate representation of many physical systems. A model's dynamic order and stiffness are frequently strengthened by the presence of some parasitic factors, such as tiny time constants, masses, resistances, and capacitances [2][4]. Time scales are created by stiffness, which is characterized by the coexistence of slow and fast phenomena. Singularly perturbed systems, which are a subset of more general time-scale systems, are high-order systems in which the suppression of a minor parameter leads to a reduction in dimension [5]. The singular perturbation theory often decouples the slow dynamics, which are the dominating features of a two-time-scale (TTS) system, from the fast dynamics, which are only present for a brief period of time. Determining if the system equations are of the singly perturbed type is crucial for the control designer. Singular perturbation (SP) theory may be applied if just a small number of the system states' temporal derivatives are being multiplied by small coefficients. Singularly perturbed systems (SPSs) are based on a tiny, SP real-valued parameter and have a time-scale breakdown, reduced order (slow), and fast subsystems. Such SPSs frequently involve the interaction of two different speed groups of eigenvalues. As a result, its eigenvalues ratio is sufficiently low. Determining how states can be divided into slow and rapid states requires careful investigation and understanding [6]. Designing distinct slow and fast controllers for each subsystem before combining them to create a composite controller for the original full-order system is suggested by the decomposition of TTS systems into separate slow and fast subsystems. When the rapid control law is not required, just the reduced (slow) control law is used [7]. So it is possible to research and analyze the performance of a system using a reduced-order (slow) subsystem or a composite system, and then apply the findings to a real full-order system. The control

designer can easily generate a rough solution of a lower-order system than the original by applying SP theory. In addition to lowering system order, the separation of time scales also gets rid of stiffness issues. In several areas of applied mathematics, including as electrical power systems, robotics, aerospace systems, and nuclear reactors, areas of application for SP and time-scale approaches appear to be expanding. For more information, see [4] and the references therein. By assuming that the fast modes are infinitely fast when deriving the slow subsystem and that the slow modes are constant during a fast transient when deriving the fast subsystem, the characteristics of a singularly perturbed system enable the separation of the actual full-order system into the reduced (slow) and fast subsystems [8].

1.6 Literature Survey

In the early 1960s, singular perturbation methods were initially developed as a tool to simplify the computing of optimal trajectories. In the control literature, Sannuti and Kokotovic [9] developed SP as a model-reduction methodology for finite-dimensional dynamic continuous-time systems that disregard high frequency parasitics in the late 1960s. Since then, SP theory has received a great deal of interest, and fast growth in the further development of these control systems has happened, as evidenced by surveys of the scientific literature [4, 7, 10]. A 2002-to-2012 survey by Zhang et al. [11] covered six stability, estimation, optimum and other control problems, as well as applications of singularly perturbed systems. It is important to note that SP theory has its origins in fluid dynamics and has widespread applications in the field of aerospace systems, as demonstrated by the review of SP applications on aerospace systems [12]. In the design of optimum control for large-scale systems, the need for order reduction associated with the SP approach is most pronounced. Saksena et al. [10] have conducted a comprehensive analysis of the composite control design based on separate control law designs for slow and rapid subsystems. The so-called reduced technique [6] is a prevalent way for handling SPSs. Without knowledge of the tiny perturbation parameter, Chow and Kokotovic [8] built a near-optimal state regulator for slow and fast mode systems. Haddad and

Kokotovic [13] utilized SP to solve the linear quadratic Gaussian (LQG) problem for stochastic systems exhibiting both fast and slow dynamics. Khalil and Gajic [14] introduced a new technique for decomposing and approximating LQG estimation and control problems for SPSs. The Kalman-Bucy filter was separated into slow- and fast-mode filters via decoupling transformation. Shen et al. [15] designed a composite feedback controller for discrete systems subject to stochastic jump parameters using an SP method. Oloomi et al. [16] derived an output feedback regulator for discrete temporal systems that are singularly affected. Using the sequential Galerkin approximation (SGA) method, Kim et al. [17] provided algorithms for finding composite control laws for singularly perturbed bilinear systems. Yao [18] has developed a method of iteration for designing sub-optimal static output feedback computer control for decentralized SPSs. Zhou and Challoo [19] discovered a nearly optimum solution for a singularly perturbed large-scale interconnected discrete-time system with undetermined parameters. Corless et al. [20] have investigated and constructed controllers for uncertain linear time-varying systems utilizing singular perturbation techniques. Stepanyan and Nguyen [21] devised a controller for a plant with slow actuators utilizing singular perturbation techniques with tracking error guaranteed to be constrained to a minimal parameter in the dynamics of the actuator.

CHAPTER 2

INTRODUCTORY BACKGROUND ON CONTROL THEORY

2.1 Control Theory and Linear Systems

Control theory deals with the regulation of dynamic systems that have distinct inputs and outputs. The primary objective is to determine the optimal system inputs that lead to the desired output outcome. This chapter provides an overview of the fundamental concepts and principles of control theory, which are essential for designing control systems.

2.2 Problem Formulation

Given a system model and a function representing the cost of operation, find the input that would result in the minimum value of the cost. The main specification is to minimize cost.

$$\begin{aligned} \dot{x} &= Ax + Bu \\ y &= Cx \end{aligned} \tag{1}$$

where $x(t) \in R^n$ is the state vector, $u(t) \in R^m$ is the control vector, and $y(t) \in R^n$ is the output vector. The notations $A \in R^{n\times n}$, $B \in R^{n\times m}$,$C \in R^{p\times m}$,are constant matrices of appropriate dimensions.

2.3 Linear-Quadratic Regulator (LQR)

Performance index function "J", defined in terms of u, x, t_0, and t_f, t_0 and t_f are the initial and final times.

This design is for a linear system model and a quadratic cost function in the form

$$J = \frac{1}{2}\, x^T(t_f)\; V\; x(t_f) + \frac{1}{2}\int_{t_0}^{t_f} \left(x^T(t)\; Q\; x(t) + u^T(t)\; R\; u(t)\right) dt \tag{2}$$

where

$x : n \times 1$,

$u : n \times 1$,

$Q : n \times n$,

$R : m \times m$,

$V : n \times n$,

J : scalar.

2.4 Optimal Control Theory

The objective is to minimize the cost function J, defined in equation 2) by determining the optimal values of $u(t)$, $x(t)$, and $x(t_f)$, while ensuring that the model equation, given by equation 1 is satisfied. This can be referred to as a conditional optimization problem.

To do this, we propose a new function that combines the cost function and the model equation into a single formula. By minimizing this function, we can automatically satisfy the model equation while also minimizing the cost function.

$$\bar{J} = \frac{1}{2}\ x^T(t_f)\ V\ x(t_f) + \int_0^{t_f} \left[\frac{1}{2}(x^T\ Q\ x + u^T\ R\ u) + (P^T(Ax + Bu - \dot{x}))\right] dt \tag{3}$$

Where P is a vector with the same dimension as x. It is called Lagrange Multiplier. So, we have

$$\bar{J} = \frac{1}{2}x^T(t_f)Vx(t_f) + \int_0^{t_f} \left[\frac{1}{2}x^TQx + \frac{1}{2}u^TRu + P^TAx + P^TBu - P^T\dot{x}\right] dt \tag{4}$$

We apply integration by parts to the last term.

$$\int_0^{t_f} P^T\dot{x}dt = P^T(t_f)x(t_f) - P^T(0)X(o) - \int_0^{t_f} x^T\dot{P}dt \tag{5}$$

Using 5 in 4, we get

$$\bar{J} = \bar{J}_1 + \bar{J}_2 + \bar{J}_3 \tag{6}$$

such that

$$\bar{J}_1 = \frac{1}{2}x^T(t_f)Vx(t_f) - P^T(t_f)x(t_f) + P^T(0)x(o) \tag{7}$$

$$\bar{J}_2 = \int_0^{t_f} \left[\frac{1}{2}x^T Qx + P^T Ax + x^T \dot{P}\right] dt \tag{8}$$

$$\bar{J}_3 = \int_0^{t_f} \left[\frac{1}{2}u^T Ru + P^T Bu\right] dt \tag{9}$$

We note that 7, 8, and 9 are uncoupled with respect to $x(t_f)$, x, and u, respectively. So, to minimize $\bar{J}$, we need to minimize $\bar{J}_1$, $\bar{J}_2$, and $\bar{J}_3$ independently. The results are $x^*(t_f)$, $u^*(t)$, and $x^*(t)$ in terms of $P(t)$

$$0 = \frac{\partial \bar{J}_1}{\partial x(t_f)}$$

$$0 = Vx(t_f) - P(t_f) \tag{10}$$

$$0 = \frac{\partial \bar{J}_2}{\partial x}$$

$$0 = Qx - A^T P + \dot{P} \tag{11}$$

$$0 = \frac{\partial \bar{J}_3}{\partial u}$$

$$0 = Ru - B^T P \tag{12}$$

Equation 11 is linear. So, the relationship between P and X can be expressed as

$$P(t) = K(t)x(t) \tag{13}$$

$K(t)$ is a matrix with dimension $n \times n$.

Note:

$$P(t_f) = K(t_f)x(t_f) \tag{14}$$

Furthermore, we have

$$\begin{aligned} \dot{P} &= \dot{K}x + K\dot{x} \\ &= \dot{K}x + K[Ax + Bu] \end{aligned} \tag{15}$$

Now, from 12, we have

$$u = -R^{-1}B^T P \tag{16}$$

And, by using 13, we have

$$u = -R^{-1}B^T K x \tag{17}$$

Now, we use 17 in 15.

$$\dot{P} = \dot{K}x + K[Ax + Bu] = \dot{K}x + KAx - KBR^{-1}B^T Kx \tag{18}$$

But, we also have, from 11, the following

$$\dot{P} = -Qx - A^T P = -Qx - A^T Kx \tag{19}$$

Finally, from 18 and 19, we have

$$\dot{K}x + KAx - KBR^{-1}B^T Kx = -Qx - A^T Kx \tag{20}$$

This must be satisfied for all x. Therefore, we must have

$$-\dot{K} = KA + A^T K + Q - KBR^{-1}B^T K \tag{21}$$

From 10 and 14, we have

$$K(t_f) = V \tag{22}$$

Equation 21 is a matrix differential equation. We solve it to obtain K, using the boundary condition 22.

Equation 21 is Known as **Riccati Equation**.

2.5 LQ Problem

The LQ problem, which refers to a linear system that is subject to the quadratic cost function, can be viewed as a particular instance of the general optimization process. The system is:

$$\dot{x} = Ax + Bu \tag{23}$$

and the cost function is quadratic:

$$J = \frac{1}{2}\, x_f^T\, V\, x_f + \frac{1}{2}\int_0^{t_f} \left(x^T\, Q\, x + u^T\, R\, u\right) dt \tag{24}$$

Then, the Hamiltonian

$$\mathcal{H}(u^*) = \frac{1}{2}x^T\, Q\, x + \frac{1}{2}u^T\, R\, u + J_x^{*T}\{Ax + Bu\} \tag{25}$$

The partial derivative is to be taken with respect to u, for minimization

$$0 = \frac{\partial \mathcal{H}}{\partial u} = Ru + B^T J_x^* \tag{26}$$

yields to the optimal control input

$$u^* = -R^{-1}B^T J_x^* \tag{27}$$

and the optimal Hamiltonian $\mathcal{H}^*$

$$\mathcal{H}^* = \mathcal{H}(u^*) = \frac{1}{2}x^T Q x - \frac{1}{2}J_x^{*T} B R^{-1} B^T J_x^* + J_x^{*T} A x \tag{28}$$

For a linear system and a quadratic cost function, it is expected that

$$\begin{aligned} J_x^* &= K(t)x(t) \\ J(t) &= \frac{1}{2}x^T(t)K(t)x(t) \end{aligned} \tag{29}$$

where $K(t)$ is an unknown symmetric non-singular matrix. Then,

$$0 = J_t^* + \mathcal{H}(u^*) \tag{30}$$

which can be written as

$$0 = \frac{1}{2}x^T Kx + \frac{1}{2}x^T Qx - \frac{1}{2}x^T KBR^{-1}B^T Kx + x^T KAx \tag{31}$$

or

$$-\dot{K} = Q + KBR^{-1}B^T K + KA + A^T K \tag{32}$$

This is the Riccati Equation. The optimal value for feedback gain is to be found using the solution K and thus the control input u and optimal cost. The representation 32 is called the Algebraic Riccati Equation (ARE), which has the form

$$0 = Q + KBR^{-1}B^T K + KA + A^T K \tag{33}$$

2.6 Output Feedback

The Output feedback method involves using only the information from the output signal, without estimating the system states, to compute the feedback gain. As a result, the feedback gain will only relate to the output signal's properties and will not be connected to the state vector x. So, consider the system

$$\dot{x} = Ax + Bu \tag{34}$$

$$y = Cx \tag{35}$$

In case the state is not available, the controller will base its decisions solely on the output signal and will take the following structure

$$u = -Fy \tag{36}$$

And the quadratic performance index

$$J = \frac{1}{2}\int_t^\infty (x^T(\tau)Qx(\tau) + u^T(\tau)Ru(\tau))d\tau \tag{37}$$

Now, using 35 and 36 in 34, we get

$$\dot{x} = Ax + B(-FCx) = (A - FC)x = A_c x \tag{38}$$

And the performance index will be expressed as

$$J = \frac{1}{2}\int_t^\infty (x^T(\tau)Q_c x(\tau))d\tau \tag{39}$$

where

$$Q_c = Q + C^T F^T RFC$$

minimizing J can be found by selecting a positive semi-definite P so that

$$J = \frac{1}{2}x^T(0)Px(0) = \frac{1}{2}tr\{P\Sigma\} \tag{40}$$

where

$$0 = A_c^T P + PA_c + Q_c \tag{41}$$

and

$$\Sigma = x(0)x^T(0) \tag{42}$$

Note that 41 is a condition for 40 and will be noted as κ.

$$\kappa = A_c^T P + PA_c + Q_c \tag{43}$$

Then, the performance index in 40 can be minimized by introducing the Lagrangian multiplier Z to the Hamiltonian $\mathcal{H}$ in 44 satisfying 43

$$\mathcal{H} = tr\{P\Sigma\} + tr\{\kappa Z\} \tag{44}$$

taking the partial derivative of $\mathcal{H}$ with respect to P, z, and F and then set it equal to zero to get optimization

$$0 = \frac{\partial \mathcal{H}}{\partial Z} = Q_c + A_c^T P + P A_c \tag{45}$$

$$0 = \frac{\partial \mathcal{H}}{\partial P} = \Sigma + A_c Z + Z A_c^T \tag{46}$$

$$0 = \frac{\partial \mathcal{H}}{\partial F} = 2RFCZC^T - 2B^T PZC^T \implies F = R^{-1}B^T PZC^T(CZC^T)^{-1} \tag{47}$$

These equations can be solved by assuming CZC^T is non-singular and R is positive definite. To begin, select an initial value for F. Then, use equations 45 to find P and 46 to find Z, by solving the Lyapunov equations. Next, use the values of P and Z to solve equation 47 for F. Repeat this process by updating the values of A_c and Q_c with the calculated F until convergence is achieved.

2.7 State Feedback Optimal

State feedback is considered to be a special case of output feedback with $C = I$. For the system 34 we will have the feedback gain:

$$u = -Fx \tag{48}$$

and substitute this gain into the performance index function 37, we get

$$J = \frac{1}{2}\int_t^\infty (x^T(\tau)Qx(\tau) + u^T(\tau)Ru(\tau))d\tau \tag{49}$$

CHAPTER 3

PROBLEM STATEMENT

3.1 Overview

In a decentralized control system, the main system is controlled by one or more subsystems. For decentralized systems, controller synthesis contains a local controller for each subsystem, which operates the main system properly. Figure 2.1 shows that the subsystems are not interconnected; rather, they operate independently.

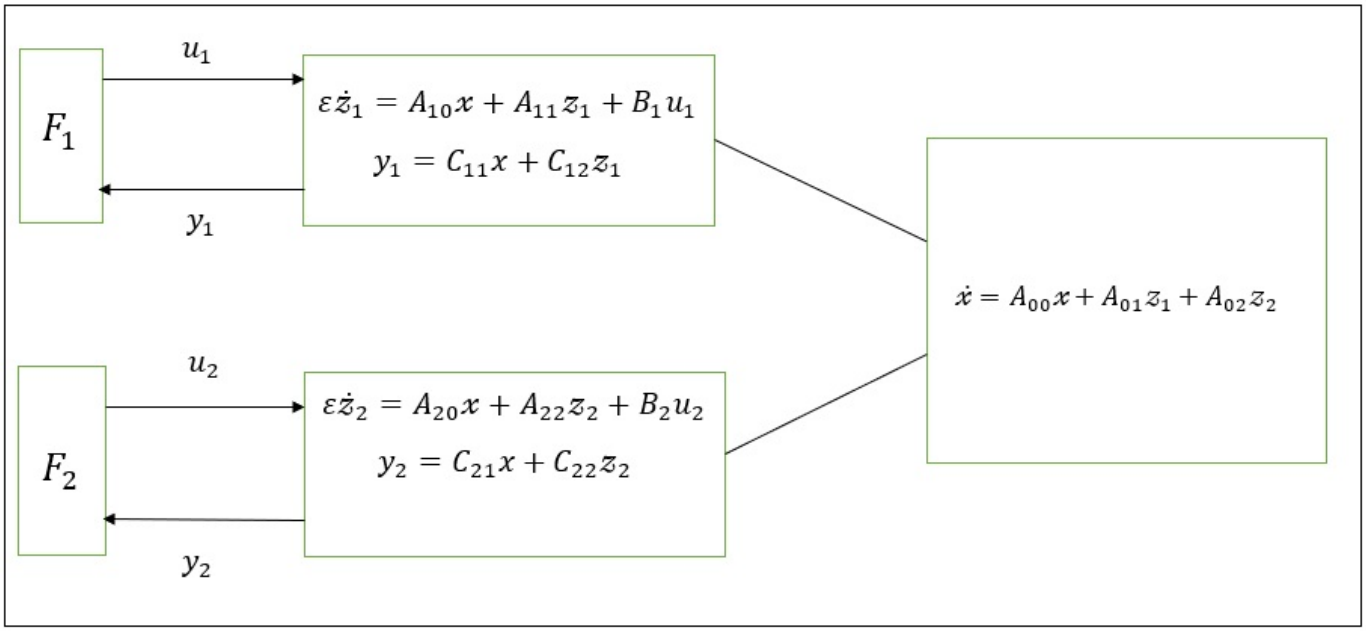

Figure 1: Decentralized system setup with two subsystems.

Through the utilization of output feedback gain, the controllers F_1 and F_2 are able to exert control over the respective subsystems z_1 and z_2, which, in turn, exert control over the primary system x. After the gains F_1 and F_2 have been calculated, it is possible to identify the controller for the composite system. The response of the slow system is expected to be the same for the entirety of the singular perturbation, which allows us to

define the fast dynamics as the reaction of the fast converging subsystems. When the responses of the quick systems converge to zero, we have reached the slow state, also known as the steady state dynamic. This occurs when the reaction of the slow system is the only one that has any impact on the system. The system that is being investigated is a decentralized singularly perturbed linear time-invariant system.

3.2 Conditions and Limitations

Since there is uncertainty in the system, the solution may not exist for some values of "S". So, "S" has to be small enough to have a solution. Furthermore, limitations exist if the singular perturbation $\epsilon < 1$.

3.3 Mathematical Model

3.3.1 Model without Sensitivity

Consider the linear time-invariant decentralized singularly-perturbed model.

$$\dot{x} = A_{00}x + A_{01}z_1 + A_{02}z_2 \tag{50a}$$

$$\epsilon\dot{z_1} = A_{10}x + A_{11}z_1 + B_1u_1 \tag{50b}$$

$$\epsilon\dot{z_2} = A_{20}x + A_{22}z_2 + B_2u_2 \tag{50c}$$

$$y_1 = C_{11}x + C_{12}z_1 \tag{50d}$$

$$y_2 = C_{21}x + C_{22}z_2 \tag{50e}$$

where $x \in R^n$ is the state variable of the main system $\dot{x}$, $z_1 \in R^{m_1}$, and $z_2 \in R^{m_2}$ are the state variables of the subsystems $\dot{z}_1$ and $\dot{z}_2$ respectively. Inputs $u_1 \in R^{r_1}$ and $u_2 \in R^{r_2}$ are the control input vectors of subsystems $\dot{z}_1$ and $\dot{z}_2$, and outputs $y_1 \in R^{p_1}$ and $y_2 \in R^{p_2}$ are the control output of subsystems $\dot{z}_1$ and $\dot{z}_2$.
$A_{00}, A_{01}, A_{02}, A_{10}, A_{11}, A_{20}, A_{22}, B_1, B_2, C_{11}, C_{12}, C_{21}$ and C_{22} are constant matrices with appropriate matching dimensions.It is assumed that the singular perturbation parameter

$\epsilon < 1$ is a small positive number.

The performance index in equations (50) can be calculated by

$$J_1(t) = \frac{1}{2}\int_0^\infty (x^T(\tau)Q_1x(\tau)+z_1^T(\tau)Q_{z_1}z_1(\tau)+z_2^T(\tau)Q_{z_2}z_2(\tau)+u_1^T(\tau)R_1u_1(\tau)+u_2^T(\tau)R_2u_2(\tau))dt \tag{51a}$$

$$J_2(t) = \frac{1}{2}\int_0^\infty (x^T(\tau)Q_2x(\tau)+z_1^T(\tau)Q_{z_1}z_1(\tau)+z_2^T(\tau)Q_{z_2}z_2(\tau)+u_1^T(\tau)R_3u_1(\tau)+u_2^T(\tau)R_4u_2(\tau))dt \tag{51b}$$

where $Q_1, Q_2, Q_{z_1}, Q_{z_2} \geq 0$ and $R_1, R_2, R_3, R_4 > 0$.

This system is linear and has no source of uncertainty. But, what if we have non-linearity as a source of uncertainty?

3.3.2 Model with Sensitivity

Now, consider the same system with σ_1 and σ_2 which represent the sensitivity to nonlinear subsystems z_1 and z_2.

$$\begin{bmatrix} \dot{x} \\ \epsilon\dot{z_1} \\ \epsilon\dot{z_2} \\ \dot{\sigma_1} \\ \dot{\sigma_2} \end{bmatrix} = \begin{bmatrix} A_{00} & A_{01} & A_{02} & 0 & 0 \\ A_{10} & A_{11} & 0 & 0 & 0 \\ A_{20} & 0 & A_{22} & 0 & 0 \\ 0 & A'_{11}|_{\mu_0} & 0 & A_{11}|_{\mu_0} & 0 \\ 0 & 0 & A'_{22}|_{\mu_0} & 0 & A_{22}|_{\mu_0} \end{bmatrix} \begin{bmatrix} x \\ z_1 \\ z_2 \\ \sigma_1 \\ \sigma_2 \end{bmatrix} + \begin{bmatrix} 0 & 0 \\ B_1 & 0 \\ 0 & B_2 \\ 0 & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} u_1 \\ u_2 \end{bmatrix}$$

$$\dot{x} = A_{00}x + A_{01}z_1 + A_{02}z_2 \tag{52a}$$

$$\epsilon\dot{z_1} = A_{10}x + A_{11}z_1 + B_1u_1 \tag{52b}$$

$$\epsilon\dot{z_2} = A_{20}x + A_{22}z_2 + B_2u_2 \tag{52c}$$

$$\epsilon\dot{\sigma_1} = A'_{11}|_{\mu_0}z_1 + A_{11}|_{\mu_0}\sigma_1 \tag{52d}$$

$$\epsilon\dot{\sigma_2} = A'_{22}|_{\mu_0}z_2 + A_{22}|_{\mu_0}\sigma_2 \tag{52e}$$

$$y_1 = C_{11}x + C_{12}z_1 \tag{52f}$$

$$y_2 = C_{21}x + C_{22}z_2 \tag{52g}$$

where $\sigma_1 \in R^{m_1}$ and $\sigma_2 \in R^{m_2}$ are the sensitivity such that σ_1 and σ_2 are defined as the derivatives of z_1 and z_2, respectively, with respect to μ evaluated at μ_0 which is the nominal value of μ.

$\sigma_1 = \frac{\partial z_1}{\partial \mu}|_{\mu_0}$ and $\sigma_2 = \frac{\partial z_2}{\partial \mu}|_{\mu_0}$; also, μ is a parameter in A_{11} and A_{22}. and μ_0 is the nominal value of μ. (μ_0 is given by the manufacturer).

The performance index in equations (51a) and (51b) can be calculated by

$$\begin{aligned} J_1(t) = \frac{1}{2}\int_0^\infty (x^T(\tau)Q_1x(\tau) + z_1^T(\tau)Q_{z_1}z_1(\tau) + z_2^T(\tau)Q_{z_2}z_2(\tau) \\ + u_1^T(\tau)R_1u_1(\tau) + u_2^T(\tau)R_2u_2(\tau) + \sigma_1^T S_1\sigma_1 + \sigma_2^T S_2\sigma_2)dt \end{aligned} \tag{53}$$

$$\begin{aligned} J_2(t) = \frac{1}{2}\int_0^\infty (x^T(\tau)Q_2x(\tau) + z_1^T(\tau)Q_{z_1}z_1(\tau) + z_2^T(\tau)Q_{z_2}z_2(\tau) \\ + u_1^T(\tau)R_3u_1(\tau) + u_2^T(\tau)R_4u_2(\tau) + \sigma_1^T S_1\sigma_1 + \sigma_2^T S_2\sigma_2)dt \end{aligned} \tag{54}$$

where $S_1, S_2, Q_1, Q_2, Q_{z_1}, Q_{z_2} \geq 0$ and $R_1, R_2, R_3, R_4 > 0$.

So, we have a system with sensitivity to nonlinear subsystems. Our goal is to linearize this system and then we optimize it to get the best feedback.

CHAPTER 4

APPROACH

4.1 Mathematical Tools

Calculus of variations is used in this research, which is also known as Pontryagin's maximum principle.In optimal control theory, Pontryagin's maximum principle is used to determine the optimal control for transitioning a dynamical system from one state to another, particularly in the presence of restrictions for the state or input controls. It claims that any optimal control must solve the so-called Hamiltonian system, which is a two-point boundary value problem plus a maximum condition of the control Hamiltonian, coupled with the optimal state trajectory. These necessary conditions become sufficient under specific goal and constraint function convexity conditions. Also, singular perturbation will be used to reduce a large-scale system.

4.2 Linearization

Linearizing a nonlinear model goes through some steps [22].

Let

$\dot{z}_1 = f_1(x, z_1, u_1)$ and $\dot{z}_2 = f_2(x, z_2, u_2)$.

$x_e, z_{1_e}, z_{2_e}, u_{1_e}$, and u_{2_e} are equilibrium points.

Here,

$$f(x_e, z_{1_e}, u_{1_e}) = f(x_e, z_{2_e}, u_{2_e}) = 0$$

Now, define $\Delta x = x - x_e$, $\Delta z_1 = z_1 - z_{1_e}$, $\Delta z_2 = z_2 - z_{2_e}$, $\Delta u_1 = u_1 - u_{1_e}$, and $\Delta u_2 = u_2 - u_{2_e}$.

So,

$x = x_e + \Delta x$, $z_1 = z_{1_e} + \Delta z_1$, $z_2 = z_{2_e} + \Delta z_2$, $u_1 = u_{1_e} + \Delta u_1$, and $u_2 = u_{2_e} + \Delta u_2$.

Note that, since $\dot{x}$ is linear, this means the equilibrium point $x_e = 0$.

$$\Rightarrow x = \Delta x$$

Thus,

$$f_i(x, z_i, u_i) = f(x_e + \Delta x, z_{i_e} + \Delta z_i, u_{i_e} + \Delta u_i) \quad i = 1, 2$$

$$= f(x_e, z_{i_e}, u_{i_e}) + \frac{\partial f_i}{\partial x}|_{x_e, z_{i_e}, u_{i_e}} \Delta x + \frac{\partial f_i}{\partial z_i}|_{x_e, z_{i_e}, u_{i_e}} \Delta z_i + \frac{\partial f_i}{\partial u_i}|_{x_e, z_{i_e}, u_{i_e}} \Delta u_i + \textit{ higher order terms}$$

Note that,

$$f(x_e, z_{i_e}, u_{i_e}) = 0$$

The higher order terms can be represented as μ. Hence,

$$f_i(x, z_i, u_i) = \frac{\partial f_i}{\partial x}|_{x_e, z_{i_e}, u_{i_e}} \Delta x + \frac{\partial f_i}{\partial z_i}|_{x_e, z_{i_e}, u_{i_e}} \Delta z_i + \frac{\partial f_i}{\partial u_i}|_{x_e, z_{i_e}, u_{i_e}} \Delta u_i + \mu$$

Where μ =higher order terms as we mentioned earlier, and the nominal value of $\mu_0 = 0$, and then, this μ will be incorporated as follows:

$$\dot{\zeta}_i = \bar{A}_{i0}(\mu)x + \bar{A}_{ii}(\mu)\zeta_i + \bar{B}_i(\mu)\alpha_i$$

Where,

$\bar{A}_{i0} = \frac{\partial f_i}{\partial x}|_{x_e, z_{i_e}, u_{i_e}}$

$\bar{A}_{ii} = \frac{\partial f_i}{\partial z_i}|_{x_e, z_{i_e}, u_{i_e}}$

$\bar{B}_i = \frac{\partial f_i}{\partial u_i}|_{x_e, z_{i_e}, u_{i_e}}$

So, in the linearized model, the state variables ζ_i represent the variation between z_i and the equilibrium points z_{i_e} respectively. Also, the input α_i represent the variation between the input u_i and the equilibrium points u_{i_e} respectively.

Also, note that: $\sigma_1 = \frac{\partial \zeta_1}{\partial \mu}|_{\mu_0}$ and $\sigma_2 = \frac{\partial \zeta_2}{\partial \mu}|_{\mu_0}$

Therefore, the new linearized model will be as the following :

$$\dot{x} = A_{00}x + A_{01}\zeta_1 + A_{02}\zeta_2 \tag{55a}$$

$$\epsilon\dot{\zeta_1} = \bar{A}_{10}x + \bar{A}_{11}\zeta_1 + \bar{B}_1\alpha_1 \tag{55b}$$

$$\epsilon\dot{\zeta_2} = \bar{A}_{20}x + \bar{A}_{22}\zeta_2 + \bar{B}_2\alpha_2 \tag{55c}$$

$$\epsilon\dot{\sigma_1} = \bar{A}'_{11}|_{\mu_0}\zeta_1 + \bar{A}_{11}|_{\mu_0}\sigma_1 \tag{55d}$$

$$\epsilon\dot{\sigma_2} = \bar{A}'_{22}|_{\mu_0}\zeta_2 + A_{22}|_{\mu_0}\sigma_2 \tag{55e}$$

$$y_1 = C_{11}x + C_{12}\zeta_1 \tag{55f}$$

$$y_2 = C_{21}x + C_{22}\zeta_2 \tag{55g}$$

4.3 Optimal Control For Non-Linearity

In order to optimize the model (55), we first need to convert the system by using a technique known as singular perturbation (SP)[1].

Preliminary Results of:

4.3.1 Slow Subsystem

First, we set $\epsilon = 0$, then $\epsilon\dot{\zeta_i} = 0$, it signifies that only the slow portion dynamics have an impact on the system. Now,let $\bar{x}, \bar{\zeta_i}$, and $\bar{\alpha}_i$ represent the system's quasi-steady state variables. The system's slow variables can be set as follows:

$$x_s = \bar{x}, \quad y_{i_s} = \bar{y}_i, \quad \alpha_{i_s} = \bar{\alpha}_i$$

Then, the system (55) will be reduced to the following model

$$\dot{\bar{x}} = A_{00}\bar{x} + A_{01}\bar{\zeta_1} + A_{02}\bar{\zeta_2} \tag{56a}$$

$$0 = \bar{A}_{10}\bar{x} + \bar{A}_{11}\bar{\zeta_1} + \bar{B}_1\bar{\alpha}_1 \tag{56b}$$

$$0 = \bar{A}_{20}\bar{x} + \bar{A}_{22}\bar{\zeta_2} + \bar{B}_2\bar{\alpha}_2 \tag{56c}$$

Now, from (56b) and (56c), we get the following:

$$\bar{\zeta_1} = -[\bar{A}_{11}]^{-1}(\bar{A}_{10}\bar{x} + \bar{B}_1\bar{\alpha}_1) \tag{57a}$$

$$\bar{\zeta_2} = -[\bar{A}_{22}]^{-1}(\bar{A}_{20}\bar{x} + \bar{B}_2\bar{\alpha}_2) \tag{57b}$$

Next, by substituting $\bar{\zeta_1}$ and $\bar{\zeta_2}$ in $\dot{\bar{x}}$, (56a) will be as the following

$$\dot{\bar{x}} = A_0\bar{x} + B_{01}\bar{\alpha}_1 + B_{02}\bar{\alpha}_2 \tag{58}$$

Where,

$$A_0 = A_{00} - A_{01}[\bar{A}_{11}]^{-1}\bar{A}_{10} - A_{02}[\bar{A}_{22}]^{-1}\bar{A}_{20}$$
$$B_{01} = -A_{01}[\bar{A}_{11}]^{-1}\bar{B}_1$$
$$B_{02} = -A_{02}[\bar{A}_{22}]^{-1}\bar{B}_2$$

The quadratic performance index for slow subsystem is

$$J_s(t) = \frac{1}{2}\int_0^\infty (x_s^T Q_x x_s + \bar{\zeta_1}^T Q_{\zeta_1}\bar{\zeta_1} + \bar{\zeta_2}^T Q_{\zeta_2}\bar{\zeta_2} + \alpha_{1_s}^T R_1 \alpha_{1_s} + \alpha_{2_s}^T R_2 \alpha_{2_s})dt \tag{59}$$

Now, by inserting (57a) and (57b)in (59), we will have the following:

$$\begin{aligned} J_s(t) = \frac{1}{2}\int_0^\infty (x_s^T Q_x x_s + [-A_{11}^{-1}(A_{10}\bar{x} + B_1\bar{\alpha}_1]^T Q_{\zeta_1}[-A_{11}^{-1}(A_{10}\bar{x} + B_1\bar{\alpha}_1] \\ +[-A_{22}^{-1}(A_{20}\bar{x} + B_2\bar{\alpha}_2]^T Q_{\zeta_2}[-A_{22}^{-1}(A_{20}\bar{x} + B_2\bar{\alpha}_2] + \alpha_{1_s}^T R_1 \alpha_{1_s} + \alpha_{2_s}^T R_2 \alpha_{2_s})dt \end{aligned} \tag{60}$$

Next, we can use same structure described in [23] to rewrite (60)

$$J_s(t) = \frac{1}{2}\int_0^\infty (x_s^T Q_0 x_s + 2x_s^T M_1 \alpha_{1_s} + 2x_s^T M_2 \alpha_{2_s} + \alpha_{1_s}^T R_{01} \alpha_{1_s} + \alpha_{2_s}^T R_{02} \alpha_{2_s})dt \tag{61}$$

Where,

$$Q_0 = Q_x + A_{10}^T A_{11}^{-T} Q_{\zeta_1} A_{11}^{-1} A_{10} + A_{20}^T A_{22}^{-T} Q_{\zeta_2} A_{22}^{-1} A_{20}$$
$$M_1 = A_{10}^T A_{11}^{-T} Q_{\zeta_1} A_{11}^{-1} B_1$$

$M_2 = A_{20}^T A_{22}^{-T} Q_{\zeta_2} A_{22}^{-1} B_2$

$R_{01} = R_1 + B_1^T A_{11}^{-T} Q_{\zeta_1} A_{11}^{-1} B_1$

$R_{02} = R_2 + B_2^T A_{22}^{-T} Q_{\zeta_2} A_{22}^{-1} B_2$

To eliminate $(2x_s^T M_1 \alpha_{1_s})$ and $(2x_s^T M_2 \alpha_{2_s})$,

let

$$\alpha_{1_s} = \alpha_{01_s} + F_{01} x_s$$

$$\alpha_{2_s} = \alpha_{02_s} + F_{02} x_s$$

The preliminary feedback input is

$$F_{01} = -R_{01}^{-1} M_1^T$$

$$F_{02} = -R_{02}^{-1} M_2^T$$

By applying this feedback gain, we will have

$$\begin{aligned} \alpha_{1_s} &= \alpha_{01_s} - R_{01}^{-1} M_1^T x_s \\ \alpha_{2_s} &= \alpha_{02_s} - R_{02}^{-1} M_2^T x_s \end{aligned} \tag{62}$$

Now, we can rewrite (61) to be as the following

$$J_s(t) = \frac{1}{2} \int_0^\infty (x_s^T Q_s x_s + \alpha_{1_s}^T R_{01} \alpha_{1_s} + \alpha_{2_s}^T R_{02} \alpha_{2_s}) dt \tag{63}$$

Where,

$Q_s = Q_0 + 2M_1 F_{01} + 2M_2 F_{02}$

The optimal feedback control of the system will be as

$$\begin{aligned} \alpha_{1_s}^*(t) &= F_{01}^* x_s(t) \\ \alpha_{2_s}^*(t) &= F_{02}^* x_s(t) \end{aligned} \tag{64}$$

Where,

$$\begin{aligned} F_{01}^{*} &= -R_{01}^{-1}(M_1^T + B_{01}^T K_s) \\ F_{02}^{*} &= -R_{02}^{-1}(M_2^T + B_{02}^T K_s) \end{aligned} \tag{65}$$

K_s is the solution to the following algebraic Riccati equation:

$$0 = K_s A_s + A_s^T K_s - K_s B_{01} R_{01}^{-1} B_{01}^T K_s - K_s B_{02} R_{02}^{-1} B_{02}^T K_s + Q_s \tag{66}$$

Where,

$$A_s = A_0 - B_{01} R_{01}^{-1} M_1^T - B_{02} R_{02}^{-1} M_2^T$$

4.3.2 Fast Subsystem

First, we consider the fast dynamics. We define the fast variables as follows: $\zeta_{1_f} = \zeta_1 - \bar{\zeta_1},\ \zeta_{2_f} = \zeta_2 - \bar{\zeta_2},\ \alpha_{1_f} = \alpha_1 - \bar{\alpha}_1,\ \alpha_{2_f} = \alpha_2 - \bar{\alpha}_2,\ y_{1_f} = y_1 - \bar{y}_1,$ and $y_{2_f} = y_2 - \bar{y}_2$. Also, we assume $x = \bar{x}$ =constant, $\dot{\bar{\zeta_1}} = 0$, and $\dot{\bar{\zeta_2}} = 0$. The slow part of each variable is assumed to be constant during the transient period.

Thus, we have the two subsystems

$$\begin{aligned} \epsilon\dot{\zeta_1} &= \bar{A}_{10} x + \bar{A}_{11}\zeta_1 + \bar{B}_1\alpha_1 \\ \epsilon\dot{\zeta_2} &= \bar{A}_{20} x + \bar{A}_{22}\zeta_2 + \bar{B}_2\alpha_2 \end{aligned} \tag{67}$$

And the slow parts, i.e. when $\epsilon = 0$:

$$\begin{aligned} 0 &= \bar{A}_{10}\bar{x} + \bar{A}_{11}\bar{\zeta_1} + \bar{B}_1\bar{\alpha}_1 \\ 0 &= \bar{A}_{20}\bar{x} + \bar{A}_{22}\bar{\zeta_2} + \bar{B}_2\bar{\alpha}_2 \end{aligned} \tag{68}$$

By subtracting (68) from (67), we will have the following fast subsystem:

$$\begin{aligned} \epsilon\dot{\zeta_1} &= \bar{A}_{10}(x - \bar{x}) + \bar{A}_{11}(\zeta_1 - \bar{\zeta_1}) + \bar{B}_1(\alpha_1 - \bar{\alpha}_1) \\ \epsilon\dot{\zeta_2} &= \bar{A}_{20}(x - \bar{x}) + \bar{A}_{22}(\zeta_2 - \bar{\zeta_2}) + \bar{B}_2(\alpha_2 - \bar{\alpha}_2) \end{aligned} \tag{69}$$

But we assumed $x = \bar{x}$, $\dot{\bar{\zeta}}_1 = 0$, and $\dot{\bar{\zeta}}_2 = 0$. We can rewrite (69) to be as

$$\epsilon(\dot{\zeta}_1 - \dot{\bar{\zeta}}_1) = \epsilon\dot{\zeta}_{1_f} = \bar{A}_{11}\zeta_{1_f} + \bar{B}_1\alpha_{1_f}$$

$$\epsilon(\dot{\zeta}_2 - \dot{\bar{\zeta}}_2) = \epsilon\dot{\zeta}_{2_f} = \bar{A}_{22}\zeta_{2_f} + \bar{B}_2\alpha_{2_f}$$

Furthermore, for the output, we will have

$$y_{1_f} = y_1 - \bar{y}_1 = C_{12}\zeta_{1_f}$$

$$y_{2_f} = y_2 - \bar{y}_2 = C_{22}\zeta_{2_f}$$

Also, we have the gain feedback as:

$\alpha_{1_f} = -F_{03}y_{1_f} = -F_{03}C_{12}\zeta_{1_f}$ *and* $\alpha_{2_f} = -F_{04}y_{2_f} = -F_{04}C_{22}\zeta_{2_f}$

By inserting the gain feedback in our fast subsystem, it will be

$$\dot{\zeta}_{1_f} = \frac{1}{\epsilon}[\bar{A}_{11}\zeta_{1_f} - \bar{B}_1F_{03}C_{12}\zeta_{1_f}] = \frac{1}{\epsilon}[\bar{A}_{11} - \bar{B}_1F_{03}C_{12}]\zeta_{1_f} = \hat{A}_{1_f}\zeta_{1_f} \tag{70}$$

$$\dot{\zeta}_{2_f} = \frac{1}{\epsilon}[\bar{A}_{22}\zeta_{2_f} - \bar{B}_2F_{04}C_{22}\zeta_{2_f}] = \frac{1}{\epsilon}[\bar{A}_{22} - \bar{B}_2F_{04}C_{22}]\zeta_{2_f} = \hat{A}_{2_f}\zeta_{2_f} \tag{71}$$

$$\dot{\sigma}_{1_f} = \hat{A}'_{1_f}\zeta_{1_f} + \hat{A}_{1_f}\sigma_{1_f} \tag{72}$$

$$\dot{\sigma}_{2_f} = \hat{A}'_{2_f}\zeta_{2_f} + \hat{A}_{2_f}\sigma_{2_f} \tag{73}$$

Where

$\hat{A}_{1_f} = \frac{1}{\epsilon}(\bar{A}_{11} - \bar{B}_1F_{03}C_{12})$ and $\hat{A}_{2_f} = \frac{1}{\epsilon}(\bar{A}_{22} - \bar{B}_2F_{04}C_{22})$

We consider the quadratic Performance index which includes σ_{1_f} and σ_{2_f} in order to minimize the following cost

$$J_f(t) = \frac{1}{2}\int_0^\infty (\zeta_{1_f}^T\hat{Q}_1\zeta_{1_f} + \zeta_{2_f}^T\hat{Q}_2\zeta_{2_f} + \sigma_{1_f}^T S_{1_f}\sigma_{1_f} + \sigma_{2_f}^T S_{2_f}\sigma_{2_f})dt \tag{74}$$

where

$\hat{Q}_1 = Q_{\zeta_1} + C_{12}^T F_{03}^T R_1 F_{03} C_{12}$

$\hat{Q}_2 = Q_{\zeta_2} + C_{22}^T F_{04}^T R_2 F_{04} C_{22}$

Also, we have

$$\zeta_{1_f} = \phi_1(t)\zeta_{1_f}(0)$$

$$\zeta_{2_f} = \phi_2(t)\zeta_{2_f}(0)$$

$$\sigma_{1_f} = \int_0^t \phi_3(t-\tau)\hat{A}'_{1_f}\phi_1(\tau)\zeta_{1_f}(0)d\tau$$

$$\sigma_{2_f} = \int_0^t \phi_4(t-\tau)\hat{A}'_{2_f}\phi_2(\tau)\zeta_{2_f}(0)d\tau$$

where

$\phi_1(t) = e^{(\hat{A}_{1_f})t}$, $\phi_2(t) = e^{(\hat{A}_{2_f})t}$, $\phi_3(t-\tau) = e^{(\hat{A}_{1_f})(t-\tau)}$, $\phi_4(t-\tau) = e^{(\hat{A}_{2_f})(t-\tau)}$

and $\zeta_{i_f}(0)$, $i = 1, 2$, are initial values.

We form the Hamiltonian as

$$\begin{aligned}\mathcal{H} = \frac{1}{2}(\zeta_{1_f}^T \hat{Q}_1 \zeta_{1_f} + \zeta_{2_f}^T \hat{Q}_2 \zeta_{2_f} + \sigma_{1_f}^T S_{1_f}\sigma_{1_f} + \sigma_{2_f}^T S_{2_f}\sigma_{2_f}) + \lambda_{\zeta_{1_f}}^T \hat{A}_{1_f}\zeta_{1_f} \\ +\lambda_{\zeta_{2_f}}^T \hat{A}_{2_f}\zeta_{2_f} + \lambda_{\sigma_{1_f}}^T(\hat{A}'_{1_f}\zeta_{1_f} + \hat{A}_{1_f}\sigma_{1_f}) + \lambda_{\sigma_{2_f}}^T(\hat{A}'_{2_f}\zeta_{2_f} + \hat{A}_{2_f}\sigma_{2_f})\end{aligned} \tag{75}$$

Applying the minimum principle [24], We get

$$\begin{aligned}\dot{\lambda}_{\zeta_{1_f}} &= -\frac{\partial\mathcal{H}}{\partial\zeta_{1_f}} = -[\hat{Q}_1\zeta_{1_f} + \hat{A}_{1_f}^T\lambda_{\zeta_{1_f}} + \hat{A}_{1_f}'^T\lambda_{\sigma_{1_f}}] \\ \dot{\lambda}_{\zeta_{2_f}} &= -\frac{\partial\mathcal{H}}{\partial\zeta_{2_f}} = -[\hat{Q}_2\zeta_{2_f} + \hat{A}_{2_f}^T\lambda_{\zeta_{2_f}} + \hat{A}_{2_f}'^T\lambda_{\sigma_{2_f}}] \\ \dot{\lambda}_{\sigma_{1_f}} &= -\frac{\partial\mathcal{H}}{\partial\sigma_{1_f}} = -[S_{1_f}\sigma_{1_f} + \hat{A}_{1_f}^T\lambda_{\sigma_{1_f}}] \\ \dot{\lambda}_{\sigma_{2_f}} &= -\frac{\partial\mathcal{H}}{\partial\sigma_{2_f}} = -[S_{2_f}\sigma_{2_f} + \hat{A}_{2_f}^T\lambda_{\sigma_{2_f}}]\end{aligned} \tag{76}$$

Now, Let

$$\lambda_{\zeta_{1_f}} = K_{11}\zeta_{1_f} + K_{13}\sigma_{1_f}$$

$$\lambda_{\zeta_{2_f}} = K_{22}\zeta_{2_f} + K_{24}\sigma_{2_f}$$

$$\lambda_{\sigma_{1_f}} = K_{31}\zeta_{1_f} + K_{33}\sigma_{1_f}$$

$$\lambda_{\sigma_{2_f}} = K_{42}\zeta_{2_f} + K_{44}\sigma_{2_f}$$

K_{ij} can be obtained by solving the following equations, $(i, j = 1, 2, 3, 4)$

$$K_{11}\hat{A}_{1_f} + \hat{A}_{1_f}^T K_{11} = -\hat{Q}_1 - K_{13}\hat{A}'_{1_f} - \hat{A}_{1_f}^{'T} K_{31} \tag{77a}$$

$$K_{13}\hat{A}_{1_f} + \hat{A}_{1_f}^T K_{13} = -\hat{A}_{1_f}^{'T} K_{33} \tag{77b}$$

$$K_{22}\hat{A}_{2_f} + \hat{A}_{2_f}^T K_{22} = -\hat{Q}_2 - K_{24}\hat{A}'_{2_f} - \hat{A}_{2_f}^{'T} K_{42} \tag{77c}$$

$$K_{24}\hat{A}_{2_f} + \hat{A}_{2_f}^T K_{24} = -\hat{A}_{2_f}^{'T} K_{44} \tag{77d}$$

$$K_{31}\hat{A}_{1_f} + \hat{A}_{1_f}^T K_{31} = -K_{33}\hat{A}'_{1_f} \tag{77e}$$

$$K_{33}\hat{A}_{1_f} + \hat{A}_{1_f}^T K_{33} = -S_{1_f} \tag{77f}$$

$$K_{42}\hat{A}_{2_f} + \hat{A}_{2_f}^T K_{42} = -K_{44}\hat{A}'_{2_f} \tag{77g}$$

$$K_{44}\hat{A}_{2_f} + \hat{A}_{2_f}^T K_{44} = -S_{2_f} \tag{77h}$$

The Hamiltonian is rewritten as follows:

$$\begin{aligned}
\mathcal{H} = tr[&\frac{1}{2}(Q_{\zeta_1} + C_{12}^T F_{03}^T R_1 F_{03} C_{12})\zeta_{1_f}\zeta_{1_f}^T + \frac{1}{2}(Q_{\zeta_2} + C_{22}^T F_{04}^T R_2 F_{04} C_{22})\zeta_{2_f}\zeta_{2_f}^T \\
&+ \frac{1}{2}S_{1_f}\sigma_{1_f}\sigma_{1_f}^T + \frac{1}{2}S_{2_f}\sigma_{2_f}\sigma_{2_f}^T \\
&+ \hat{A}_{1_f}\zeta_{1_f}\lambda_{\zeta_{1_f}}^T + \hat{A}_{2_f}\zeta_{2_f}\lambda_{\zeta_{2_f}}^T \\
&+ (\hat{A}'_{1_f}\zeta_{1_f} + \hat{A}_{1_f}\sigma_{1_f})\lambda_{\sigma_{1_f}}^T + (\hat{A}'_{2_f}\zeta_{2_f} + \hat{A}_{2_f}\sigma_{2_f})\lambda_{\sigma_{2_f}}^T]
\end{aligned} \tag{78}$$

The necessary condition of minimizing J_f given by;

$$0 = E\int_0^\infty \frac{\partial \mathcal{H}}{\partial F_{0i}} dt \quad , \quad i = 3, 4 \tag{79}$$

will result the following equations:

$$\begin{aligned} F_{03}^{*} &= \frac{1}{\epsilon} R_1^{-1} \bar{B}_1^T [K_{11}E + K_{13}G_1 + K_{31}G_1^T + K_{33}G_2] C_{12}^T (C_{12}EC_{12}^T)^{-1} \\ F_{04}^{*} &= \frac{1}{\epsilon} R_2^{-1} \bar{B}_2^T [K_{22}N + K_{24}T_1 + K_{42}T_1^T + K_{44}T_2] C_{22}^T (C_{22}NC_{22}^T)^{-1} \end{aligned} \tag{80}$$

where E, G_1, G_2, N, T_1, and T_2 are obtained by solving the following equations:

$$\hat{A}_{1_f} E + E\hat{A}_{1_f}^T = -\Omega_1 \tag{81a}$$

$$\hat{A}_{1_f} G_1 + G_1 \hat{A}_{1_f}^T = -\hat{A}'_{1_f} E \tag{81b}$$

$$\hat{A}_{1_f} G_2 + G_2 \hat{A}_{1_f}^T = -G_1 \hat{A}_{1_f}'^T - \hat{A}'_{1_f} G_1^T \tag{81c}$$

$$\hat{A}_{2_f} N + N\hat{A}_{2_f}^T = -\Omega_2 \tag{81d}$$

$$\hat{A}_{2_f} T_1 + T_1 \hat{A}_{2_f}^T = -\hat{A}'_{2_f} N \tag{81e}$$

$$\hat{A}_{2_f} T_2 + T_2 \hat{A}_{2_f}^T = -T_1 \hat{A}_{2_f}'^T - \hat{A}'_{2_f} T_1^T \tag{81f}$$

Where,

$$\Omega_1 = E[\zeta_{1_f}(0)\zeta_{1_f}^T(0)]$$

$$\Omega_2 = E[\zeta_{2_f}(0)\zeta_{2_f}^T(0)]$$

4.4 Composite Control

The composite optimal control solution for the main system (55) is

$$\begin{aligned} \alpha_1^{*} &= \alpha_{1_s}^{*} + \alpha_{1_f}^{*} = F_{01}^{*} x_s - F_{03}^{*} C_{12} \zeta_{1_f} \\ \alpha_2^{*} &= \alpha_{2_s}^{*} + \alpha_{2_f}^{*} = F_{02}^{*} x_s - F_{04}^{*} C_{22} \zeta_{2_f} \end{aligned} \tag{82}$$

But we have

$$\bar{x} = x_s \cong x$$

$$\zeta_{1_f} = \zeta_1 - \zeta_{1_s}$$

$$\zeta_{2_f} = \zeta_2 - \zeta_{2_s}$$

Where

$$\zeta_{1_s} = \bar{\zeta_1} = -\bar{A}_{11}^{-1}(\bar{A}_{10}x_s + \bar{B}_1 u_{1_s}^*) = -\bar{A}_{11}^{-1}\bar{A}_{10}x - \bar{A}_{11}^{-1}\bar{B}_1 F_{01}^* x$$

$$\zeta_{2_s} = \bar{\zeta_2} = -\bar{A}_{22}^{-1}(\bar{A}_{20}x_s + \bar{B}_2 u_{2_s}^*) = -\bar{A}_{22}^{-1}\bar{A}_{20}x - \bar{A}_{22}^{-1}\bar{B}_2 F_{02}^* x$$

Now,

$$\begin{aligned}\alpha_1^* &= F_{01}^* x - F_{03}^* C_{12}(\zeta_1 - \zeta_{1_s}) = F_{01}^* x - F_{03}^* C_{12}\zeta_1 + F_{03}^* C_{12}\zeta_{1_s} \\ &= F_{01}^* x - F_{03}^* C_{12}\zeta_1 + F_{03}^* C_{12}(-\bar{A}_{11}^{-1}\bar{A}_{10}x - \bar{A}_{11}^{-1}\bar{B}_1 F_{01}^* x) \\ &= [F_{01}^* x - F_{03}^* C_{12}\bar{A}_{11}^{-1}\bar{B}_1 F_{01}^* x] - F_{03}^* C_{12}\bar{A}_{11}^{-1}\bar{A}_{10}x - F_{03}^* C_{12}\zeta_1\end{aligned} \tag{83}$$

Similarly,

$$\alpha_2^* = [F_{02}^* x - F_{04}^* C_{22}\bar{A}_{22}^{-1}\bar{B}_2] - F_{04}^* C_{22}\bar{A}_{22}^{-1}\bar{A}_{20}x - F_{04}^* C_{22}\zeta_2 \tag{84}$$

Therefore,

$$\begin{aligned}\alpha_1^* &= F_{05}^* x - F_{03}^* C_{12}\zeta_1 \\ \alpha_2^* &= F_{06}^* x - F_{04}^* C_{22}\zeta_2\end{aligned} \tag{85}$$

Where,

$$F_{05}^* = [F_{01}^* - F_{03}^* C_{12}\bar{A}_{11}^{-1}\bar{B}_1 F_{01}^*] - F_{03}^* C_{12}\bar{A}_{11}^{-1}\bar{A}_{10}$$

$$F_{06}^* = [F_{02}^* - F_{04}^* C_{22}\bar{A}_{22}^{-1}\bar{B}_2 F_{02}^*] - F_{04}^* C_{22}\bar{A}_{22}^{-1}\bar{A}_{20}$$

4.5 Numerical Example 1

Consider the following system

$$\dot{x} = \begin{bmatrix} -2 & 0 \\ 0 & -3 \end{bmatrix} x(t) + \begin{bmatrix} 3 & -1 \\ -1 & -2 \end{bmatrix} \zeta_1(t) + \begin{bmatrix} -3 & 2 \\ -4 & -1 \end{bmatrix} \zeta_2(t) \tag{86}$$

$$\epsilon\dot{\zeta_1} = \begin{bmatrix} -1 & 2 \\ -4 & 3 \end{bmatrix} x(t) + \begin{bmatrix} -5 & 0 \\ 2 & -6+2\mu \end{bmatrix} \zeta_1(t) + \begin{bmatrix} -2 \\ 2 \end{bmatrix} \alpha_1(t) \tag{87}$$

$$\epsilon\dot{\zeta}_2 = \begin{bmatrix} -1 & 5 \\ -3 & 2 \end{bmatrix} x(t) + \begin{bmatrix} -6 & 1 \\ 1 & -3+\mu \end{bmatrix} \zeta_2(t) + \begin{bmatrix} -1 \\ 1 \end{bmatrix} \alpha_2(t) \tag{88}$$

$$\epsilon\dot{\sigma}_{1_f} = \begin{bmatrix} 0 & 0 \\ 0 & 2 \end{bmatrix} \zeta_{1_f} + \left(\begin{bmatrix} -5 & 0 \\ 2 & -6 \end{bmatrix} - \begin{bmatrix} -2 \\ 2 \end{bmatrix} F_{03} \begin{bmatrix} 2 & 1 \\ 1 & 3 \end{bmatrix} \right) \sigma_{1_f} \tag{89}$$

$$\epsilon\dot{\sigma}_{2_f} = \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} \zeta_{2_f} + \left(\begin{bmatrix} -6 & 1 \\ 1 & -3 \end{bmatrix} - \begin{bmatrix} -1 \\ 1 \end{bmatrix} F_{04} \begin{bmatrix} 1 & 3 \\ 2 & 1 \end{bmatrix} \right) \sigma_{2_f} \tag{90}$$

$$y_1 = \begin{bmatrix} x \\ \zeta_1 \end{bmatrix} \tag{91}$$

$$y_2 = \begin{bmatrix} x \\ \zeta_2 \end{bmatrix} \tag{92}$$

With $\epsilon = 0.1$, $R_1 = R_2 = 1$, and $Q_x = Q_{z_1} = Q_{z_2} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = \Omega_1 = \Omega_2$.

Solution: Two cases are considered in solving this example. The first case is when S_{1_f} and S_{2_f} are set equal to $0 \times I$, which means there is no sensitivity in the system. The solution in this case does exist and it is guaranteed. The second case is when $S_{1_f} = S_{2_f} = 40000 \times I$, and the solution exists in this case.

When $S_{1_f} = S_{2_f} = 0 \times I$. The optimal solutions for the composite control are

$$\alpha_1^* = \begin{bmatrix} 0.4049 & -0.0906 \end{bmatrix} x - \begin{bmatrix} -0.1302 & 0.0867 \end{bmatrix} C_{12}\zeta_1$$

$$\alpha_2^* = \begin{bmatrix} -0.0783 & -0.0229 \end{bmatrix} x - \begin{bmatrix} 0.0695 & -0.0641 \end{bmatrix} C_{22}\zeta_2$$

and

$$\alpha_{1_s} y^* = \begin{bmatrix} 0.4249 & -0.0944 \end{bmatrix} x_s$$

$$\alpha_{2_s}^* = \begin{bmatrix} 0.0070 & -0.1009 \end{bmatrix} x_s$$

With

$$K_s = \begin{bmatrix} 0.3166 & -0.1047 \\ -0.1047 & 0.1880 \end{bmatrix}$$

.

$$K_{11} = \begin{bmatrix} 0.0010 & 0.0002 \\ 0.0002 & 0.0008 \end{bmatrix}, K_{22} = \begin{bmatrix} 0.0009 & 0.0003 \\ 0.0003 & 0.0017 \end{bmatrix}$$

$$K_{13} = K_{24} = K_{31} = K_{33} = K_{42} = K_{44} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$$

$$E = 10^{-3} \begin{bmatrix} 0.9451 & 0.2109 \\ 0.2109 & 0.8777 \end{bmatrix}, G_1 = 10^{-4} \begin{bmatrix} 0.0011 & 0.0036 \\ 0.0399 & 0.1483 \end{bmatrix}, G_2 = 10^{-6} \begin{bmatrix} 0.0008 & 0.0171 \\ 0.0171 & 0.4802 \end{bmatrix}$$

$$N = \begin{bmatrix} 0.0009 & 0.0003 \\ 0.0003 & 0.0017 \end{bmatrix}, T_1 = 10^{-4} \begin{bmatrix} 0.0095 & 0.0363 \\ 0.0695 & 0.2974 \end{bmatrix}, T_2 = 10^{-3} \begin{bmatrix} 0.0044 & 0.0266 \\ 0.0226 & 0.1531 \end{bmatrix}$$

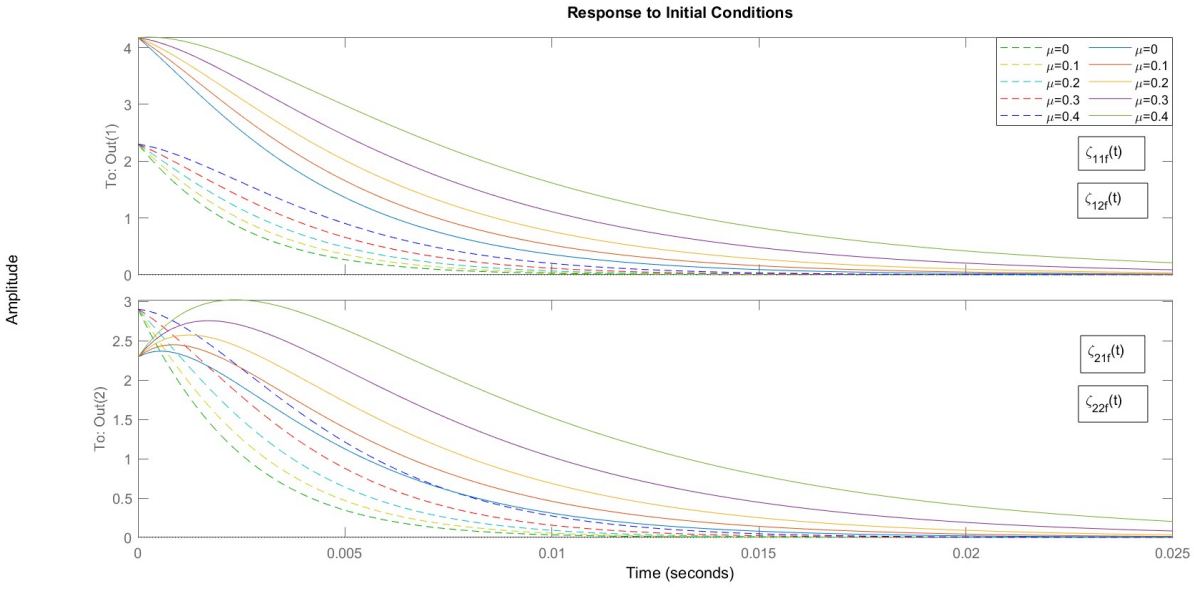

Figure 2: Systems Responses To Initial Condition When $S_{1_f} = 0 \times I$ and $S_{2_f} = 0 \times I$

When $S_{1_f} = S_{2_f} = 40000 \times I$. The optimal solutions for the composite control are

$$\alpha_1^* = \begin{bmatrix} -0.9509 & 1.2270 \end{bmatrix} x - \begin{bmatrix} -0.2808 & 0.7627 \end{bmatrix} C_{12}\zeta_1$$

$$\alpha_2^* = \begin{bmatrix} -2.1556 & 2.2095 \end{bmatrix} x - \begin{bmatrix} 1.1354 & -0.5256 \end{bmatrix} C_{22}\zeta_2$$

and

$$\alpha_{1s}^* = \begin{bmatrix} 0.4249 & -0.0944 \end{bmatrix} x_s$$

$$\alpha_{2_s}^* = \begin{bmatrix} 0.0070 & -0.1009 \end{bmatrix} x_s$$

With

$$K_s = \begin{bmatrix} 0.3166 & -0.1047 \\ -0.1047 & 0.1880 \end{bmatrix}$$

.

$$K_{11} = \begin{bmatrix} 0.0021 & 0.0027 \\ 0.0027 & 0.0102 \end{bmatrix}, K_{13} = \begin{bmatrix} 0.0571 & 0.0519 \\ 0.2766 & 0.3310 \end{bmatrix}$$

$$K_{31} = \begin{bmatrix} 0.0571 & 0.2766 \\ 0.0519 & 0.3310 \end{bmatrix}, K_{33} = \begin{bmatrix} 49.2064 & 16.4222 \\ 16.4222 & 26.5542 \end{bmatrix}$$

$$K_{22} = \begin{bmatrix} 0.0014 & 0.0024 \\ 0.0024 & 0.0175 \end{bmatrix}, K_{24} = \begin{bmatrix} 0.0059 & 0.0230 \\ 0.1433 & 0.5637 \end{bmatrix}$$

$$K_{42} = \begin{bmatrix} 0.0059 & 0.1433 \\ 0.0230 & 0.5637 \end{bmatrix}, K_{44} = \begin{bmatrix} 38.0135 & 12.2069 \\ 12.2069 & 49.0865 \end{bmatrix}$$

$$E = \begin{bmatrix} 0.0013 & 0.0003 \\ 0.0003 & 0.0005 \end{bmatrix}, G_1 = 10^{-5} \begin{bmatrix} 0.3557 & 0.2070 \\ 0.6146 & 0.6154 \end{bmatrix}, G_2 = 10^{-6} \begin{bmatrix} 0.0624 & 0.0721 \\ 0.0721 & 0.1349 \end{bmatrix}$$

$$N = \begin{bmatrix} 0.0011 & 0.0003 \\ 0.0003 & 0.0011 \end{bmatrix}, T_1 = 10^{-4} \begin{bmatrix} 0.0253 & 0.0328 \\ 0.0658 & 0.1171 \end{bmatrix}, T_2 = 10^{-6} \begin{bmatrix} 0.0797 & 0.1716 \\ 0.1392 & 0.3816 \end{bmatrix}$$

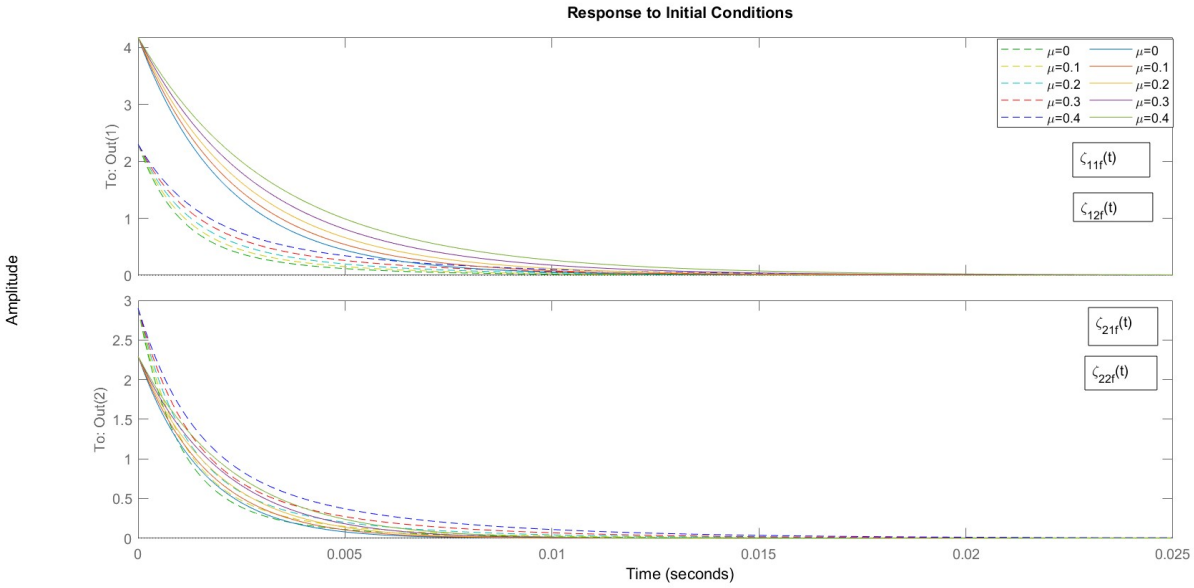

Figure 3: Systems Responses To Initial Condition When $S_{1_f} = S_{2_f} = 40000 \times I$

As shown in Figures (2) and (3) the sensitivity has been reduced, considering different values of μ_0. The values considered for μ_0 are 0, 0.1, 0.2, 0.3, and 0.4.

4.6 Numerical Example 2

Consider the following system

$$\dot{x} = \begin{bmatrix} -2 & 0 & 1 & 0 \\ 0 & -3 & 0 & 0 \\ 0 & 0 & -4 & 0 \\ -4 & 0 & 0 & -1 \end{bmatrix} x(t) + \begin{bmatrix} 1 & 0 & 0 & -1 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 2 & 0 & 0 & 1 \end{bmatrix} \zeta_1(t) + \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ -1 & 0 & 0 & 0 \\ 0 & 0 & -2 & 0 \end{bmatrix} \zeta_2(t) \qquad (93)$$

$$\epsilon\dot{\zeta}_1 = \begin{bmatrix} 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} x(t) + \begin{bmatrix} -1+3\mu & 0 & 0 & -1 \\ 4\mu & -1 & 0 & -1 \\ 0 & 0 & -1 & -2+2\mu \\ 1 & 0 & 0 & -4+2\mu \end{bmatrix} \zeta_1(t) + \begin{bmatrix} -1 \\ 0 \\ 0 \\ 1 \end{bmatrix} \alpha_1(t) \qquad (94)$$

$$\epsilon\dot{\zeta}_2 = \begin{bmatrix} 1 & 0 & 0 & 0 \\ -1 & 0 & -1 & 0 \\ 0 & 0 & 0 & 2 \\ -1 & 0 & 0 & 0 \end{bmatrix} x(t) + \begin{bmatrix} -1+\mu & 0 & -1+2\mu & 0 \\ 0 & -2 & 0 & \mu \\ 0 & -2+2\mu & -3 & -3 \\ -3 & 0 & 0 & -4+\mu \end{bmatrix} \zeta_2(t) + \begin{bmatrix} -1 \\ 0 \\ -1 \\ -1 \end{bmatrix} \alpha_2(t) \tag{95}$$

$$\epsilon\dot{\sigma}_{1_f} = \begin{bmatrix} 3 & 0 & 0 & 0 \\ 4 & 0 & 0 & 0 \\ 0 & 0 & 0 & 2 \\ 0 & 0 & 0 & 2 \end{bmatrix} \zeta_{1_f} + \left(\begin{bmatrix} -1 & 0 & 0 & -1 \\ 0 & -1 & 0 & -1 \\ 0 & 0 & -1 & -2 \\ 1 & 0 & 0 & -4 \end{bmatrix} - \begin{bmatrix} -1 \\ 0 \\ 0 \\ 1 \end{bmatrix} F_{03} \begin{bmatrix} 1 & 0 & 1 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 1 & 1 \\ 3 & 0 & 0 & 1 \end{bmatrix} \right) \sigma_{1_f} \tag{96}$$

$$\epsilon\dot{\sigma}_{2_f} = \begin{bmatrix} 1 & 0 & 2 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 2 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \zeta_{2_f} + \left(\begin{bmatrix} -1 & 0 & -1 & 0 \\ 0 & -2 & 0 & 0 \\ 0 & -2 & -3 & -3 \\ -3 & 0 & 0 & -4 \end{bmatrix} - \begin{bmatrix} -1 \\ 0 \\ -1 \\ -1 \end{bmatrix} F_{03} \begin{bmatrix} 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 1 \\ -1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 2 \end{bmatrix} \right) \sigma_{2_f} \tag{97}$$

$$y_1 = \begin{bmatrix} x \\ \zeta_1 \end{bmatrix} \tag{98}$$

$$y_2 = \begin{bmatrix} x \\ \zeta_2 \end{bmatrix} \tag{99}$$

With $\epsilon = 0.1$, $R_1 = R_2 = 1$, and

$$Q_x = Q_{\zeta_1} = Q_{\zeta_2} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} = \Omega_1 = \Omega_2..$$

Solution: In this example, also two cases are considered. The first case is when $S = 0 \times I$. The second case is when $S = 100 \times I$.

When $S_{1_f} = S_{2_f} = 0 \times I$. The optimal solutions for the composite control are

$$\alpha_1^* = \begin{bmatrix} 0.5742 & -0.1585 & 0.3434 & 0.3448 \end{bmatrix} x - \begin{bmatrix} -0.2798 & -0.0363 & 0.2435 & -0.0448 \end{bmatrix} C_{12}\zeta_1$$

$$\alpha_2^* = \begin{bmatrix} 0.6965 & 0.0982 & -0.0963 & -0.3003 \end{bmatrix} x - \begin{bmatrix} -0.4613 & -0.3836 & -0.0293 & 0.3988 \end{bmatrix} C_{22}\zeta_2$$

and

$$\alpha_{1_s}^* = \begin{bmatrix} 0.5865 & -0.0864 & 0.4747 & 0.1314 \end{bmatrix} x_s$$

$$\alpha_{2_s}^* = \begin{bmatrix} 0.6559 & 0.0796 & -0.1286 & -0.3569 \end{bmatrix} x_s$$

with

$$K_s = \begin{bmatrix} 1.5718 & -0.0461 & 0.2322 & -0.5994 \\ -0.0461 & 0.4963 & -0.1159 & -0.0634 \\ 0.2322 & -0.1159 & 0.3927 & -0.0414 \\ -0.5994 & -0.0634 & -0.0414 & 0.5311 \end{bmatrix}$$

$$K_{11} = \begin{bmatrix} 0.0044 & -0.0005 & -0.0010 & 0.0002 \\ -0.0005 & 0.0050 & -0.0000 & -0.0009 \\ -0.0010 & -0.0000 & 0.0050 & -0.0018 \\ 0.0002 & -0.0009 & -0.0018 & 0.0022 \end{bmatrix}, K_{22} = \begin{bmatrix} 0.0058 & -0.0000 & -0.0009 & -0.0006 \\ -0.0000 & 0.0033 & -0.0008 & 0.0006 \\ -0.0009 & -0.0008 & 0.0020 & -0.0008 \\ -0.0006 & 0.0006 & -0.0008 & 0.0018 \end{bmatrix}$$

$$K_{13} = K_{24} = K_{31} = K_{33} = K_{42} = K_{44} = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

$$G_1 = 10^{-3} \begin{bmatrix} 0.2883 & -0.0557 & -0.1113 & 0.0933 \\ 0.4534 & -0.1086 & -0.2173 & 0.1546 \\ -0.0299 & -0.0001 & -0.0001 & 0.0206 \\ 0.0893 & -0.0394 & -0.0787 & 0.0637 \end{bmatrix}, G_2 = 10^{-3} \begin{bmatrix} 0.0487 & 0.0833 & -0.0107 & 0.0192 \\ 0.0833 & 0.1471 & -0.0231 & 0.0343 \\ -0.0107 & -0.0231 & 0.0110 & -0.0034 \\ 0.0192 & 0.0343 & -0.0034 & 0.0097 \end{bmatrix}$$

$$E = \begin{bmatrix} 0.0032 & -0.0003 & -0.0005 & 0.0006 \\ -0.0003 & 0.0053 & 0.0006 & -0.0003 \\ -0.0005 & 0.0006 & 0.0062 & -0.0006 \\ 0.0006 & -0.0003 & -0.0006 & 0.0014 \end{bmatrix}$$

$$T_1 = 10^{-3}\begin{bmatrix} 0.0364 & -0.0710 & 0.2311 & -0.0902 \\ -0.0404 & -0.0038 & -0.0616 & 0.0716 \\ 0.0313 & 0.0844 & 0.0677 & -0.0716 \\ -0.0270 & 0.0373 & -0.1752 & 0.0870 \end{bmatrix}, T_2 = 10^{-4}\begin{bmatrix} 0.1623 & -0.0150 & 0.0925 & -0.1833 \\ -0.0581 & -0.0225 & -0.0150 & 0.0711 \\ 0.0592 & 0.0322 & 0.0934 & -0.1166 \\ -0.1172 & 0.0023 & -0.0934 & 0.1605 \end{bmatrix}$$

$$N = \begin{bmatrix} 0.0035 & 0.0003 & 0.0001 & -0.0020 \\ 0.0003 & 0.0025 & -0.0009 & -0.0002 \\ 0.0001 & -0.0009 & 0.0034 & -0.0012 \\ -0.0020 & -0.0002 & -0.0012 & 0.0029 \end{bmatrix}$$

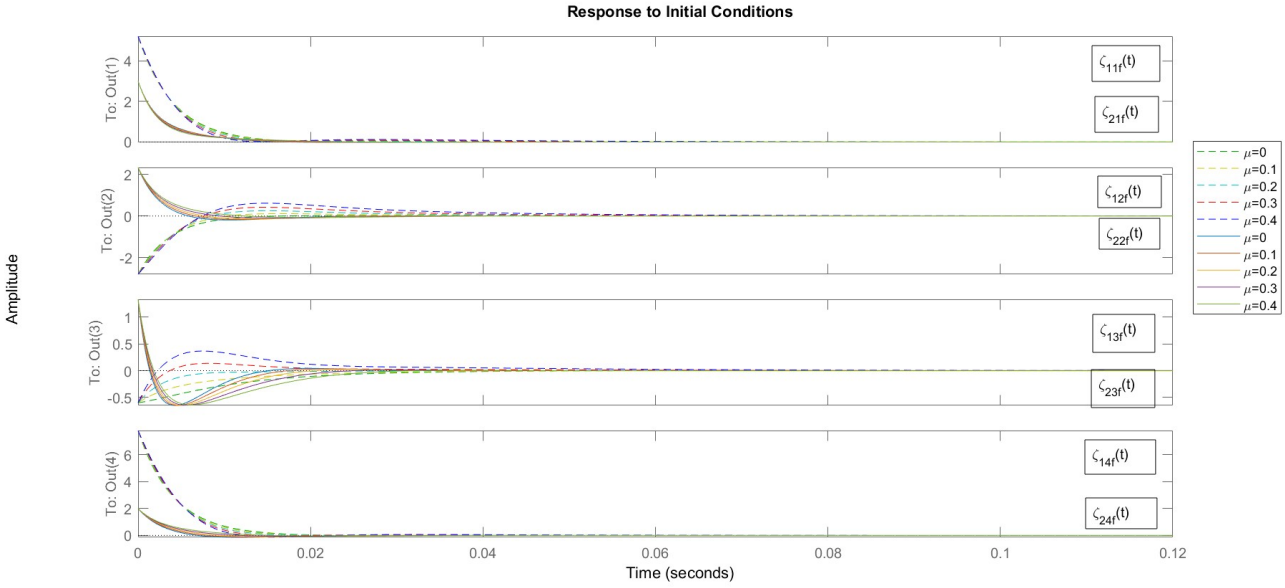

Figure 4: Systems Responses To Initial Condition When $S_{1_f} = 0 \times I$ and $S_{2_f} = 0 \times I$

When $S_{1_f} = S_{2_f} = 100 \times I$. The optimal solutions for the composite control are

$$\alpha_1^* = \begin{bmatrix} 0.2436 & -0.3007 & -0.0802 & 0.8123 \end{bmatrix} x - \begin{bmatrix} -0.6087 & -0.0518 & 0.6587 & -0.4234 \end{bmatrix} C_{12}\zeta_1$$

$$\alpha_2^* = \begin{bmatrix} 0.5091 & 0.1196 & -0.1700 & -0.2484 \end{bmatrix} x - \begin{bmatrix} -0.9350 & -0.5903 & -0.1413 & 0.8317 \end{bmatrix} C_{22}\zeta_2$$

and

$$\alpha_{1_s}^* = \begin{bmatrix} 0.5865 & -0.0864 & 0.4747 & 0.1314 \end{bmatrix} x_s$$

$$\alpha_{2_s}^* = \begin{bmatrix} 0.6559 & 0.0796 & -0.1286 & -0.3569 \end{bmatrix} x_s$$

with

$$K_s = \begin{bmatrix} 1.5718 & -0.0461 & 0.2322 & -0.5994 \\ -0.0461 & 0.4963 & -0.1159 & -0.0634 \\ 0.2322 & -0.1159 & 0.3927 & -0.0414 \\ -0.5994 & -0.0634 & -0.0414 & 0.5311 \end{bmatrix}$$

.

$$K_{11} = \begin{bmatrix} 0.0129 & -0.0007 & -0.0013 & -0.0011 \\ -0.0007 & 0.0050 & -0.0000 & -0.0008 \\ -0.0013 & -0.0000 & 0.0050 & -0.0017 \\ -0.0011 & -0.0008 & -0.0017 & 0.0030 \end{bmatrix}, K_{13} = \begin{bmatrix} 0.0054 & 0.0397 & -0.0007 & -0.0068 \\ 0.0000 & -0.0013 & 0.0003 & 0.0002 \\ 0.0000 & -0.0000 & 0.0000 & 0.0000 \\ -0.0016 & -0.0088 & 0.0126 & -0.0005 \end{bmatrix}$$

$$K_{22} = \begin{bmatrix} 0.0088 & -0.0009 & -0.0002 & -0.0015 \\ -0.0009 & 0.0047 & -0.0012 & 0.0008 \\ -0.0002 & -0.0012 & 0.0027 & -0.0012 \\ -0.0015 & 0.0008 & -0.0012 & 0.0023 \end{bmatrix}, K_{24} = \begin{bmatrix} 0.0200 & -0.0009 & -0.0062 & 0.0007 \\ -0.0048 & -0.0097 & 0.0097 & -0.0072 \\ 0.0143 & 0.0025 & -0.0039 & -0.0001 \\ -0.0096 & 0.0049 & 0.0009 & 0.0027 \end{bmatrix}$$

$$K_{31} = \begin{bmatrix} 0.0054 & 0.0000 & 0.0000 & -0.0016 \\ 0.0397 & -0.0013 & -0.0000 & -0.0088 \\ -0.0007 & 0.0003 & 0.0000 & 0.0126 \\ -0.0068 & 0.0002 & 0.0000 & -0.0005 \end{bmatrix}, K_{33} = \begin{bmatrix} 0.2712 & -0.0649 & -0.1278 & 0.0975 \\ -0.0649 & 0.4990 & -0.0012 & -0.0844 \\ -0.1278 & -0.0012 & 0.4999 & -0.1721 \\ 0.0975 & -0.0844 & -0.1721 & 0.2016 \end{bmatrix}$$

$$K_{42} = \begin{bmatrix} 0.0200 & -0.0048 & 0.0143 & -0.0096 \\ -0.0009 & -0.0097 & 0.0025 & 0.0049 \\ -0.0062 & 0.0097 & -0.0039 & 0.0009 \\ 0.0007 & -0.0072 & -0.0001 & 0.0027 \end{bmatrix}, K_{44} = \begin{bmatrix} 0.4442 & 0.0110 & -0.0756 & -0.0622 \\ 0.0110 & 0.3260 & -0.0761 & 0.0641 \\ -0.0756 & -0.0761 & 0.1899 & -0.0726 \\ -0.0622 & 0.0641 & -0.0726 & 0.1810 \end{bmatrix}$$

$$G_1 = 10^{-3}\begin{bmatrix} 0.0725 & -0.0145 & -0.0200 & 0.0423 \\ 0.1381 & -0.0621 & -0.1000 & 0.1005 \\ -0.0003 & -0.0150 & -0.0366 & 0.0331 \\ 0.0377 & -0.0325 & -0.0607 & 0.0631 \end{bmatrix}, G_2 = 10^{-4}\begin{bmatrix} 0.0610 & 0.1377 & -0.0036 & 0.0457 \\ 0.1377 & 0.4175 & -0.0404 & 0.1350 \\ -0.0036 & -0.0404 & 0.0451 & 0.0105 \\ 0.0457 & 0.1350 & 0.0105 & 0.0625 \end{bmatrix}$$

$$E = \begin{bmatrix} 0.0016 & -0.0001 & -0.0001 & 0.0005 \\ -0.0001 & 0.0053 & 0.0006 & -0.0003 \\ -0.0001 & 0.0006 & 0.0063 & -0.0006 \\ 0.0005 & -0.0003 & -0.0006 & 0.0015 \end{bmatrix}$$

$$T_1 = 10^{-3}\begin{bmatrix} -0.0019 & -0.0594 & 0.1510 & -0.0507 \\ -0.0276 & -0.0035 & -0.0507 & 0.0700 \\ 0.0236 & 0.0888 & 0.0266 & -0.0550 \\ -0.0006 & 0.0337 & -0.1339 & 0.0706 \end{bmatrix}, T_2 = 10^{-5}\begin{bmatrix} 0.6029 & -0.3736 & 0.3901 & -0.9366 \\ -0.3185 & -0.1007 & -0.0645 & 0.5624 \\ -0.0023 & 0.0665 & 0.5306 & -0.4955 \\ -0.3636 & 0.3510 & -0.5403 & 0.9591 \end{bmatrix}$$

$$N = \begin{bmatrix} 0.0030 & 0.0004 & -0.0004 & -0.0017 \\ 0.0004 & 0.0025 & -0.0008 & -0.0001 \\ -0.0004 & -0.0008 & 0.0030 & -0.0009 \\ -0.0017 & -0.0001 & -0.0009 & 0.0030 \end{bmatrix}$$

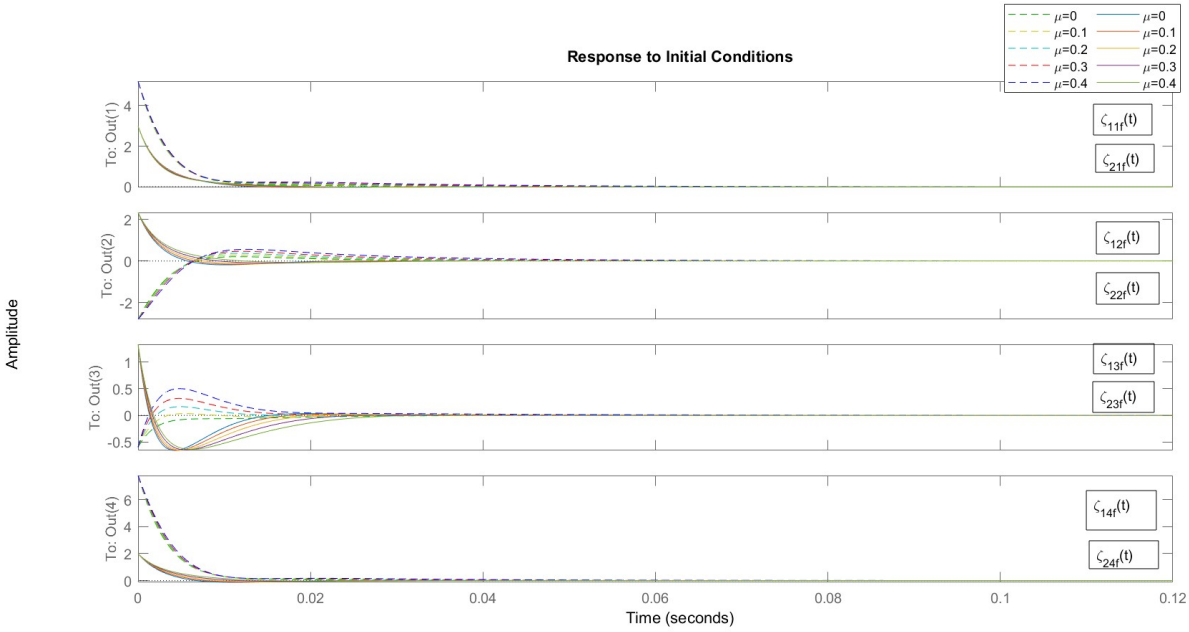

Figure 5: Systems Responses To Initial Condition When $S_{1_f} = S_{2_f} = 100 \times I$

CHAPTER 5

GAME THEORY

5.1 Introduction to Game Theory

Game theory is a technique of strategic decision-making. It focuses on optimizing costs among multiple players. It takes into account the decisions of other players, whether they are cooperative or non-cooperative, in determining an individual player's choice. This approach is applicable in various fields and not limited to economics. In control systems, decision-making is a crucial aspect of controller design. When there are multiple controllers with mutual impact, game theory can be used to analyze their interactions. Each controller is considered a "player" in this scenario, and they can be either cooperating or competing to improve the overall performance of the system.

5.1.1 Game theory in control systems

A non-cooperative game is comprised of three fundamental components, including the number of players, a set of strategies, and utility functions or performance index. Extensive research has been conducted on various game-theoretical strategies and techniques to address non-cooperative games. Among these, the Nash equilibrium is a well-known solution concept used to minimize the cost function and achieve a steady-state equilibrium. It characterizes the state where a player's performance cannot be improved given the fixed actions of other players [25].

In this research, the Nash equilibrium solution is utilized to optimize the defined system. A game can be modeled to minimize the objective function $J(x, u_1, u_2)$ to obtain the optimal $J^*(x, u_1^*, u_2^*)$.

such that

$$J_1^*(x, u_1^*, u_2^*) \leq J_1(x, u_1, u_2^*) \tag{100}$$

$$J_2^*(x, u_1^*, u_2^*) \leq J_2(x, u_1^*, u_2) \tag{101}$$

So, we suppose that we have the system

$$\dot{x} = Ax + B_1 u_1 + B_2 u_2 \tag{102}$$

$$\begin{aligned} y_1 &= C_1 x \\ y_2 &= C_2 x \end{aligned} \tag{103}$$

with $x(0) = x_0$ and output feedback input,

$$\begin{aligned} u_1 &= -F_1 C_1 x \\ u_2 &= -F_2 C_2 x \end{aligned} \tag{104}$$

For each controller input, the given performance index function

$$J_1 = \frac{1}{2} \int_0^\infty (x^T Q_x x + u_1^T R_{11} u_1 + u_2^T R_{12} u_2) dt \tag{105}$$

where $R_{12} \geq 0$, $Q_x \geq 0$, $R_{11} > 0$, (C_1 and C_2) have full row rank. Then

$$\dot{x} = (A - B_1 C_1 F_1 - B_2 C_2 F_2) x = A_c x \tag{106}$$

so,

$$J_1 = \frac{1}{2} \int_0^\infty x^T (Q_x + C_1^T F_1^T R_{11} F_1 C_1 + C_2^T F_2^T R_{12} F_2 C_2) x dt = \int_0^\infty x^T Q_c x dt \tag{107}$$

The solution for this LQR problem was already found previously,

$$J_1 = \frac{1}{2} tr(P_1 \Sigma) \tag{108}$$

where $\Sigma = x(0)x^T(0)$, and $P_1 \geq 0$ is the solution of κ_1 defined as

$$\kappa_1 \equiv A_c^T P_1 + P_1 A_c + Q_x + C_1^T F_1^T R_{11} F_1 C_1 + C_2^T F_2^T R_{12} F_2 C_2 = 0 \tag{109}$$

Introducing the symmetric undetermined Z_1 as a co-state variable for input 1 into the following Hamiltonian function

$$\mathcal{H}_1 = tr(P_1\Sigma) + tr(\kappa_1 Z_1) \tag{110}$$

Taking partial derivatives of $\mathcal{H}$ with respect to the variables Z_1, P_1, and F_1 and set equal to zero, $\mathcal{H}$ will be minimized,

$$\frac{\partial \mathcal{H}}{\partial Z_1} = \kappa_1^T = A_c^T P_1 + P_1 A_c + Q_x + C_1^T F_1^T R_{11} F_1 C_1 + C_2^T F_2^T R_{12} F_2 C_2 \equiv 0 \tag{111}$$

$$\frac{\partial \mathcal{H}}{\partial P_1} = A_c Z_1 + Z_1 A_c^T + \Sigma \equiv 0 \tag{112}$$

$$\frac{\partial \mathcal{H}}{\partial F_1} = -B_1^T P_1 Z_1 C_1^T + R_{11} F_1 Z_1 C_1^T \equiv 0 \tag{113}$$

where P_1, $R's$, and Q_x are symmetric.

Subsystem 1 will assume that the feedback gain of the other player's decision (F_2) remains constant. Through an iterative process of solving these equations, the Nash equilibrium can be determined.

5.2 Decentralized Singularly Perturbed Model

Let's consider the following linearized decentralized singularly-perturbed model :

$$\dot{x} = A_{00}x + A_{01}\zeta_1 + A_{02}\zeta_2 \tag{114a}$$

$$\epsilon\dot{\zeta}_1 = \bar{A}_{10}x + \bar{A}_{11}\zeta_1 + \bar{B}_1\alpha_1 \tag{114b}$$

$$\epsilon\dot{\zeta}_2 = \bar{A}_{20}x + \bar{A}_{22}\zeta_2 + \bar{B}_2\alpha_2 \tag{114c}$$

$$\dot{\sigma}_1 = \bar{A}'_{11}|_{\mu_0}\zeta_1 + \bar{A}_{11}|_{\mu_0}\sigma_1 \tag{114d}$$

$$\dot{\sigma}_2 = \bar{A}'_{22}|_{\mu_0}\zeta_2 + A_{22}|_{\mu_0}\sigma_2 \tag{114e}$$

$$y_1 = C_{11}x + C_{12}\zeta_1 \tag{114f}$$

$$y_2 = C_{21}x + C_{22}\zeta_2 \tag{114g}$$

Where,

$\bar{A}_{i0} = \frac{\partial f_i}{\partial x}|_{x_e, z_{i_e}, u_{i_e}}$

$\bar{A}_{ii} = \frac{\partial f_i}{\partial z_i}|_{x_e, z_{i_e}, u_{i_e}}$

$\bar{B}_i = \frac{\partial f_i}{\partial u_i}|_{x_e, z_{i_e}, u_{i_e}}$

The quasi-steady state model of the singularly perturbed decentralized large scale is obtained as

$$\dot{x}_s = A_0 x_s + B_{01}\alpha_{1_s} + B_{02}\alpha_{2_s} \tag{115a}$$

$$y_{1_s} = C_{01}x_s + D_{01}\alpha_{1_s} \tag{115b}$$

$$y_{2_s} = C_{02}x_s + D_{02}\alpha_{2_s} \tag{115c}$$

Where,

$A_0 = A_{00} - A_{01}[\bar{A}_{11}]^{-1}\bar{A}_{10} - A_{02}[\bar{A}_{22}]^{-1}\bar{A}_{20}$

$B_{01} = -A_{01}[\bar{A}_{11}]^{-1}\bar{B}_1$

$B_{02} = -A_{02}[\bar{A}_{22}]^{-1}\bar{B}_2$

$C_{01} = C_{11} - C_{12}A_{11}^{-1}\bar{A}_{10}$

$D_{01} = -C_{12}[\bar{A}_{11}^{-1}]\bar{B}_1$

$C_{02} = C_{21} - C_{22}A_{22}^{-1}\bar{A}_{20}$

$D_{02} = -C_{22}[\bar{A}_{22}]^{-1}\bar{B}_2$

Note,

$$\bar{\zeta}_1 = -[\bar{A}_{11}]^{-1}(\bar{A}_{10}\bar{x} + \bar{B}_1\bar{\alpha}_1) \tag{116a}$$

$$\bar{\zeta}_2 = -[\bar{A}_{22}]^{-1}(\bar{A}_{20}\bar{x} + \bar{B}_2\bar{\alpha}_2) \tag{116b}$$

Input $\bar{\alpha}$ for each subsystem can be written in terms of $\bar{x}$ as

$$\bar{\alpha}_1^* = -F_1\bar{y}_1 = -F_1C_{01}\bar{x} - F_1D_{01}\bar{\alpha}_1 = -(I_1 + F_1D_{01})^{-1}F_1C_{01}\bar{x} = -\hat{F}_1C_{01}\bar{x} \tag{117a}$$

$$\bar{\alpha}_2^* = -F_2\bar{y}_2 = -F_2C_{02}\bar{x} - F_2D_{02}\bar{\alpha}_2 = -(I_2 + F_2D_{02})^{-1}F_2C_{02}\bar{x} = -\hat{F}_2C_{02}\bar{x} \tag{117b}$$

Where,

$\hat{F}_1 = (I_1 + F_1 D_{01})^{-1} F_1$

$\hat{F}_2 = (I_2 + F_2 D_{02})^{-1} F_2$

Now, (115a) can be rewritten as

$$\dot{\bar{x}} = A_0 \bar{x} + B_{01}(-\hat{F}_1 C_{01} \bar{x}) + B_{02}(-\hat{F}_2 C_{02} \bar{x}) = (A_0 - B_{01}\hat{F}_1 C_{01} - B_{02}\hat{F}_2 C_{02})\bar{x} = A_r \bar{x} \quad (118)$$

Now, the uncertainty parameter μ has been incorporated in (118). Therefore, $\bar{\sigma}$ can be defined as the change of $\bar{x}$ and can be represented as $\bar{\sigma} = \frac{\partial \bar{x}}{\partial \mu}$. Hence,

$$\dot{\bar{\sigma}} = A'_r \bar{x} + A_r \bar{\sigma} \quad (119)$$

Where A'_r is the derivative of A_r with respect to μ.

Thus, the reduced model is obtained as

$$\dot{\bar{x}} = A_r \bar{x} \quad (120a)$$

$$\dot{\bar{\sigma}} = A'_r \bar{x} + A_r \bar{\sigma} \quad (120b)$$

$$\bar{y}_1 = C_{01} \bar{x} + D_{01} \bar{\alpha}_1 \quad (120c)$$

$$\bar{y}_2 = C_{02} \bar{x} + D_{02} \bar{\alpha}_2 \quad (120d)$$

The performance index functions are

$$J_1(t) = \frac{1}{2} \int_0^\infty (\bar{x}^T Q_1 \bar{x} + \bar{\alpha}_1^T R_1 \bar{\alpha}_1 + \bar{\alpha}_2^T R_2 \bar{\alpha}_2 + \bar{\sigma}^T S \bar{\sigma}) dt \quad (121)$$

$$J_2(t) = \frac{1}{2} \int_0^\infty (\bar{x}^T Q_2 \bar{x} + \bar{\alpha}_1^T R_3 \bar{\alpha}_1 + \bar{\alpha}_2^T R_4 \bar{\alpha}_2 + \bar{\sigma}^T S \bar{\sigma}) dt \quad (122)$$

Now, by inserting (117) in (121) and (122), the performance index can be rewritten as:

$$J_1(t) = \frac{1}{2} \int_0^\infty (\bar{x}^T Q_1 \bar{x} + [-\hat{F}_1 C_{01} \bar{x}]^T R_1 [-\hat{F}_1 C_{01} \bar{x}] + [-\hat{F}_2 C_{02} \bar{x}]^T R_2 [-\hat{F}_2 C_{02} \bar{x}] + \bar{\sigma}^T S \bar{\sigma}) dt \quad (123)$$

$$J_2(t) = \frac{1}{2}\int_0^\infty (\bar{x}^T Q_2 \bar{x} + [-\hat{F}_1 C_{01}\bar{x}]^T R_3[-\hat{F}_1 C_{01}\bar{x}] + [-\hat{F}_2 C_{02}\bar{x}]^T R_4[-\hat{F}_2 C_{02}\bar{x}] + \bar{\sigma}^T S\bar{\sigma})dt \tag{124}$$

So,

$$J_1(t) = \frac{1}{2}\int_0^\infty (\bar{x}^T[Q_1 + C_{01}^T \hat{F}_1^T R_1 \hat{F}_1 C_{01} + C_{02}^T \hat{F}_2^T R_2 \hat{F}_2 C_{02}]\bar{x} + \bar{\sigma}^T S\bar{\sigma})dt \tag{125}$$

$$J_2(t) = \frac{1}{2}\int_0^\infty (\bar{x}^T[Q_2 + C_{01}^T \hat{F}_1^T R_3 \hat{F}_1 C_{01} + C_{02}^T \hat{F}_2^T R_4 \hat{F}_2 C_{02}]\bar{x} + \bar{\sigma}^T S\bar{\sigma})dt \tag{126}$$

Thus,

$$J_1(t) = \frac{1}{2}\int_0^\infty (\bar{x}^T \hat{Q}_1 \bar{x} + \bar{\sigma}^T S\bar{\sigma})dt \tag{127}$$

$$J_2(t) = \frac{1}{2}\int_0^\infty (\bar{x}^T \hat{Q}_2 \bar{x} + \bar{\sigma}^T S\bar{\sigma})dt \tag{128}$$

Where,

$\hat{Q}_1 = Q_1 + C_{01}^T \hat{F}_1^T R_1 \hat{F}_1 C_{01} + C_{02}^T \hat{F}_2^T R_2 \hat{F}_2 C_{02}$

$\hat{Q}_2 = Q_2 + C_{01}^T \hat{F}_1^T R_3 \hat{F}_1 C_{01} + C_{02}^T \hat{F}_2^T R_4 \hat{F}_2 C_{02}$

To minimize J_1, introducing the Hamiltonian:

$$\mathcal{H}_1 = \frac{1}{2}(\bar{x}^T \hat{Q}_1 \bar{x} + \bar{\sigma}^T S\bar{\sigma}) + \lambda_{\bar{x}}^T[A_r\bar{x}] + \lambda_{\bar{\sigma}}^T[A_r'\bar{x} + A_r\bar{\sigma}] \tag{129}$$

Now, by applying the minimum principle, We get

$$\begin{aligned} \dot{\lambda}_{\bar{x}} &= -\frac{\partial \mathcal{H}_1}{\partial \bar{x}} = -[\hat{Q}_1\bar{x} + A_r^T\lambda_{\bar{x}} + A_r'^T\lambda_{\bar{\sigma}}] \\ \dot{\lambda}_{\bar{\sigma}} &= -\frac{\partial \mathcal{H}_1}{\partial \bar{\sigma}} = -[S\bar{\sigma} + A_r^T\lambda_{\bar{\sigma}}] \end{aligned} \tag{130}$$

Now, Let

$$\lambda_{\bar{x}} = K_{11}\bar{x} + K_{12}\bar{\sigma}$$

$$\lambda_{\bar{\sigma}} = K_{21}\bar{x} + K_{22}\bar{\sigma}$$

K_{ij} can be obtained by solving the following equations, $(i, j = 1, 2)$

$$K_{11}A_r + A_r^T K_{11} = -\hat{Q}_1 - K_{12}A_r' - [A_r']^T K_{21} \tag{131a}$$

$$K_{12}A_r + A_r^T K_{12} = -[A_r']^T K_{22} \tag{131b}$$

$$K_{21}A_r + A_r^T K_{21} = -K_{22}A_r' \tag{131c}$$

$$K_{22}A_r + A_r^T K_{22} = -S \tag{131d}$$

The Hamiltonian is rewritten as follows:

$$\mathcal{H}_1 = tr\{[\frac{1}{2}(Q_1 + C_{01}^T \hat{F}_1^T R_1 \hat{F}_1 C_{01} + C_{02}^T \hat{F}_2^T R_2 \hat{F}_2 C_{02})\bar{x}\bar{x}^T + \frac{1}{2}S\bar{\sigma}\bar{\sigma}^T + A_r\bar{x}\lambda_{\bar{x}}^T + A_r\bar{\sigma}\lambda_{\bar{\sigma}}^T + A_r'\bar{x}\lambda_{\bar{\sigma}}^T\} \tag{132}$$

The necessary condition of minimizing J_1 (127) given by;

$$0 = E\int_0^\infty \frac{\partial \mathcal{H}_1}{\partial \hat{F}_1} dt \quad . \tag{133}$$

will result the following equation:

$$\hat{F}_1 = R_1^{-1}[B_{01}^T(K_{11}M + K_{12}G_1 + K_{21}G_1^T + K_{22}G_2)C_{01}^T + (B_{01}')^T(K_{21}M + K_{22}G_1)(C_{01})^T](C_{01}MC_{01}^T)^{-1} \tag{134}$$

where M, G_1 and G_2 are obtained by solving the following equations:

$$A_r M + M A_r^T = -\Omega \tag{135}$$

$$A_r G_1 + G_1 A_r^T = -A_r' M \tag{136}$$

$$A_r G_2 + G_2 A_r^T = -G_1(A_r')^T - A_r' G_1^T \tag{137}$$

where

$$\Omega = E\{x_0 x_0^T\}$$

Now, to minimize J_2 (128), the same process is applied to the Hamiltonian (138)

$$\mathcal{H}_2 = \frac{1}{2}(\bar{x}^T\hat{Q}_2\bar{x} + \bar{\sigma}^T S\bar{\sigma}) + \lambda_{\bar{x}}^T[A_r\bar{x}] + \lambda_{\bar{\sigma}}^T[A_r'\bar{x} + A_r\bar{\sigma}] \tag{138}$$

P_{ij} can be obtained by solving the following equations, $(i, j = 1, 2)$

$$P_{11}A_r + A_r^T K_{11} = -\hat{Q}_2 - P_{12}A_r' - [A_r']^T P_{21} \tag{139a}$$

$$P_{12}A_r + A_r^T P_{12} = -[A_r']^T P_{22} \tag{139b}$$

$$P_{21}A_r + A_r^T P_{21} = -P_{22}A_r' \tag{139c}$$

$$P_{22}A_r + A_r^T P_{22} = -S \tag{139d}$$

The Hamiltonian is rewritten as follows:

$$\mathcal{H}_2 = tr\{[\frac{1}{2}(Q_2 + C_{01}^T \hat{F}_1^T R_3 \hat{F}_1 C_{01} + C_{02}^T \hat{F}_2^T R_4 \hat{F}_2 C_{02})\bar{x}\bar{x}^T + \frac{1}{2}S\bar{\sigma}\bar{\sigma}^T + A_r\bar{x}\lambda_{\bar{x}}^T + A_r\bar{\sigma}\lambda_{\bar{\sigma}}^T + A_r'\bar{x}\lambda_{\bar{\sigma}}^T\} \tag{140}$$

The necessary condition of minimizing J_1 (128) given by;

$$0 = E\int_0^\infty \frac{\partial \mathcal{H}_2}{\partial \hat{F}_2} dt \quad . \tag{141}$$

And then:

$$\hat{F}_2 = R_3^{-1}[B_{02}^T(P_{11}N + P_{12}T_1 + P_{21}T_1^T + P_{22}T_2)C_{02}^T + (B_{02}')^T(P_{21}N + P_{22}T_1)(C_{02})^T](C_{02}NC_{02}^T)^{-1} \tag{142}$$

where N, T_1 and T_2 are obtained by solving the following equations:

$$A_r N + N A_r^T = -\Omega \tag{143}$$

$$A_r T_1 + T_1 A_r^T = -A_r' N \tag{144}$$

$$A_r T_2 + T_2 A_r^T = -T_1(A_r')^T - A_r' T_1^T \tag{145}$$

where

$$\Omega = E\{x_0 x_0^T\}$$

5.3 Nash Equilibrium

In game theory, the controllers must be acquainted in order to minimize while taking each other into account. First, controller 1 begins without any knowledge of Controller 2's feedback gain. Controller 1 will first estimate the feedback gain F_2 and then use equations (147) and (148) to determine the feedback gain F_1 using the optimal cost function of the singularly perturbed system given by (146).

$$J_1(t) = \frac{1}{2}\int_0^\infty (\bar{x}^T \hat{Q}_1 \bar{x} + \bar{\sigma}^T S \bar{\sigma}) dt \tag{146}$$

and the optimal feedback gain

$$\hat{F}_1 = R_1^{-1}[B_{01}^T(K_{11}M + K_{12}G_1 + K_{21}G_1^T + K_{22}G_2)C_{01}^T + (B_{01}')^T(K_{21}M + K_{22}G_1)(C_{01})^T](C_{01}MC_{01}^T)^{-1} \tag{147}$$

where M, G_1 and G_2 are obtained by solving the following equations:

$$A_r M + M A_r^T = -\Omega \tag{148a}$$

$$A_r G_1 + G_1 A_r^T = -A_r' M \tag{148b}$$

$$A_r G_2 + G_2 A_r^T = -G_1(A_r')^T - A_r' G_1^T \tag{148c}$$

where

$$\Omega = E\{x_0 x_0^T\}$$

K_{ij} can be obtained by solving the following equations, $(i, j = 1, 2)$

$$K_{11}A_r + A_r^T K_{11} = -\hat{Q}_1 - K_{12}A_r' - [A_r']^T K_{21} \tag{149a}$$

$$K_{12}A_r + A_r^T K_{12} = -[A_r']^T K_{22} \tag{149b}$$

$$K_{21}A_r + A_r^T K_{21} = -K_{22}A_r' \tag{149c}$$

$$K_{22}A_r + A_r^T K_{22} = -S \tag{149d}$$

Using the results of the previous iteration, and following the same minimization process, the feedback gain F_2 for controller 2 will be obtained.

$$J_2(t) = \frac{1}{2}\int_0^\infty (\bar{x}^T \hat{Q}_2 \bar{x} + \bar{\sigma}^T S \bar{\sigma}) dt \tag{150}$$

With optimal feedback,

$$\hat{F}_2 = R_4^{-1}[B_{02}^T(P_{11}N + P_{12}T_1 + P_{21}T_1^T + P_{22}T_2)C_{02}^T + (B_{02}')^T(P_{21}N + P_{22}T_1)(C_{02})^T](C_{02}NC_{02}^T)^{-1} \tag{151}$$

where N, T_1 and T_2 are obtained by solving the following equations:

$$A_r N + N A_r^T = -\Omega \tag{152}$$

$$A_r T_1 + T_1 A_r^T = -A_r' N \tag{153}$$

$$A_r T_2 + T_2 A_r^T = -T_1(A_r')^T - A_r' T_1^T \tag{154}$$

And

$$\Omega = E\{x_0 x_0^T\}$$

Then the two calculated feedback gains F_1 and F_2 that are now known can be used to recalculate F_1 and F_2 with estimates that are better than the initial guess for F_2. Until an equilibrium that reflects the Nash solution to this minimization problem is reached, this method can be repeated.

5.4 Numerical Example 3

We have the following model

$$\dot{x} = \begin{bmatrix} -2 & 0 \\ 0 & -3 \end{bmatrix} x(t) + \begin{bmatrix} 1 & -1 \\ -1 & -2 \end{bmatrix} \zeta_1(t) + \begin{bmatrix} -3 & 2 \\ -4 & -1 \end{bmatrix} \zeta_2(t) \tag{155}$$

$$\epsilon\dot{\zeta}_1 = \begin{bmatrix} -1 & 2 \\ -4 & 3 \end{bmatrix} x(t) + \begin{bmatrix} -5 & 4 \\ -3 & -1+\mu \end{bmatrix} \zeta_1(t) + \begin{bmatrix} 2 \\ -2 \end{bmatrix} \alpha_1(t) \tag{156}$$

$$\epsilon\dot{\zeta}_2 = \begin{bmatrix} -1 & 5 \\ -3 & 2 \end{bmatrix} x(t) + \begin{bmatrix} -2 & -1 \\ -2 & -7+\mu \end{bmatrix} \zeta_2(t) + \begin{bmatrix} -1 \\ 1 \end{bmatrix} \alpha_2(t) \tag{157}$$

With $\epsilon = 0.1$.

Consider only the reduced model obtained from section (5.2), given by:

$$\dot{\bar{x}} = A_r \bar{x} \tag{158}$$

$$\dot{\bar{\sigma}} = A_r' \bar{x} + A_r \bar{\sigma} \tag{159}$$

$$\bar{y}_1 = C_{01} \bar{x} + D_{01} \bar{\alpha}_1 \tag{160a}$$

$$\bar{y}_2 = C_{02} \bar{x} + D_{02} \bar{\alpha}_2 \tag{160b}$$

$$\bar{\alpha}_1^* = -\hat{F}_1 C_{01} \bar{x} \tag{161a}$$

$$\bar{\alpha}_2^* = -\hat{F}_2 C_{02} \bar{x} \tag{161b}$$

Where,

$$A_0 = \begin{pmatrix} -\frac{7\,\mu^2 - 129\,\mu + 340}{\omega_1} & -\frac{71\,\mu^2 - 776\,\mu + 1827}{\omega_1} \\ \frac{11\,\mu^2 - 141\,\mu + 476}{\omega_2} & -\frac{67\,\mu^2 - 670\,\mu + 1569}{\omega_2} \end{pmatrix}$$

such that

$$\omega_1 = 2\,(5\,\mu - 17)\,(\mu - 6)$$

$$\omega_2 = (5\,\mu - 17)\,(\mu - 6)$$

$$B_{01} = \begin{pmatrix} \frac{2\,(\mu-5)}{5\,\mu-17} \\ -\frac{2\,(\mu+19)}{5\,\mu-17} \end{pmatrix},$$

$$B_{02} = \begin{pmatrix} \frac{3\,\mu-32}{2\,(\mu-6)} \\ \frac{2\,(\mu-7)}{\mu-6} \end{pmatrix}$$

Setting μ equal to the nominal value $\mu_0 = 0$ gives:

$$A_r = \left(\begin{bmatrix} -1.6667 & -8.9559 \\ 4.6667 & -15.3824 \end{bmatrix} - \begin{bmatrix} 0.5882 \\ 2.2353 \end{bmatrix} \hat{F}_1 \begin{bmatrix} 0 & 0.8235 \\ -1.0000 & 1.5294 \end{bmatrix} - \begin{bmatrix} 2.6667 \\ 2.3333 \end{bmatrix} \hat{F}_2 \begin{bmatrix} 0.6667 & 2.7500 \\ -0.3333 & 0.5000 \end{bmatrix} \right) \tag{162}$$

$$A_r' = \left(\begin{bmatrix} -0.1356 & -0.3228 \\ 0.7679 & -0.5193 \end{bmatrix} - \begin{bmatrix} 0.0553 \\ 0.7750 \end{bmatrix} \hat{F}_1 \begin{bmatrix} 0 & 0.8235 \\ -1.0000 & 1.5294 \end{bmatrix} - \begin{bmatrix} 0.1944 \\ 0.0555 \end{bmatrix} \hat{F}_2 \begin{bmatrix} 0.6667 & 2.7500 \\ -0.3333 & 0.5000 \end{bmatrix} \right)$$

(163)

Followed by the performance index functions for the reduced model as

$$J_1(t) = \frac{1}{2} \int_0^\infty (\bar{x}^T Q_1 \bar{x} + \bar{\alpha}_1^T R_1 \bar{\alpha}_1 + \bar{\alpha}_2^T R_2 \bar{\alpha}_2 + \bar{\sigma}^T S \bar{\sigma}) dt \tag{164}$$

$$J_2(t) = \frac{1}{2} \int_0^\infty (\bar{x}^T Q_2 \bar{x} + \bar{\alpha}_1^T R_3 \bar{\alpha}_1 + \bar{\alpha}_2^T R_4 \bar{\alpha}_2 + \bar{\sigma}^T S \bar{\sigma}) dt \tag{165}$$

with

$$Q_1 = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \text{ and } Q_2 = \begin{bmatrix} 0.2 & 0 \\ 0 & 0.2 \end{bmatrix} \text{ and } R_1 = R_2 = R_3 = R_4 = 1,$$

$$C_{11} = C_{12} = C_{21} = C_{22} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = \Omega.$$

Solution

The cases in solving this example are $(S = 0 \times I)$ and $(S = 5 \times I)$. The solution exists for each case, and the sensitivity has been reduced.

When $S = 0 \times I$, we have

$$\bar{\alpha}_1^* = \begin{bmatrix} 0.0578 & 0.0415 \end{bmatrix} C_{01}\bar{x}$$

$$\bar{\alpha}_2^* = \begin{bmatrix} 0.0197 & -0.1063 \end{bmatrix} C_{02}\bar{x}$$

with

$$K_{11} = \begin{bmatrix} 0.1388 & -0.0551 \\ -0.0551 & 0.0642 \end{bmatrix}$$

$$K_{12} = K_{21} = K_{22} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$$

$$M = \begin{bmatrix} 0.1632 & 0.0234 \\ 0.0234 & 0.0389 \end{bmatrix}, G_1 = \begin{bmatrix} -0.0100 & -0.0036 \\ 0.0040 & -0.0001 \end{bmatrix}, G_2 = 10^{-3} \begin{bmatrix} 0.0000 & -0.3026 \\ 0.3026 & -0.0000 \end{bmatrix}$$

$$P_{11} = \begin{bmatrix} 0.0281 & -0.0113 \\ -0.0113 & 0.0133 \end{bmatrix}$$

$$P_{12} = P_{21} = P_{22} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$$

$$N = \begin{bmatrix} 0.1632 & 0.0234 \\ 0.0234 & 0.0389 \end{bmatrix}, T_1 = \begin{bmatrix} -0.0100 & -0.0036 \\ 0.0040 & -0.0001 \end{bmatrix}, T_2 = 10^{-3} \begin{bmatrix} 0.0000 & -0.3026 \\ 0.3026 & -0.0000 \end{bmatrix}$$

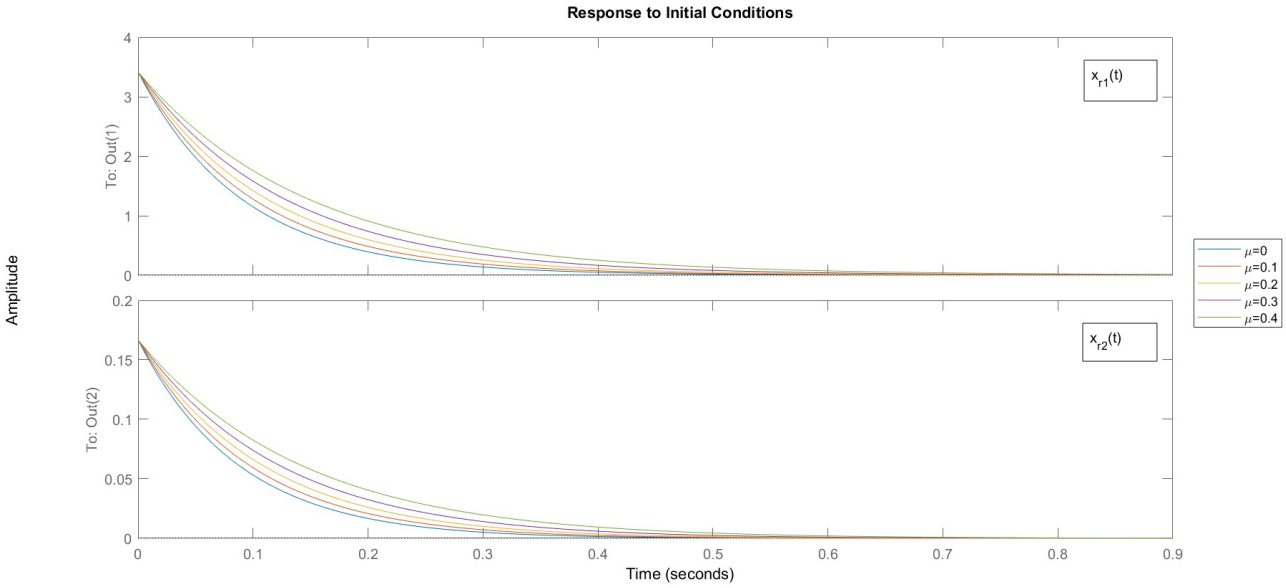

Figure 6: Systems Responses To Initial Condition When $S = 0 \times I$

When $S = 5 \times I$, we have

$$\bar{\alpha}_1^* = \begin{bmatrix} 0.0308 & 0.0093 \end{bmatrix} C_{01}\bar{x}$$

$$\bar{\alpha}_2^* = \begin{bmatrix} 0.0194 & -0.3620 \end{bmatrix} C_{02}\bar{x}$$

with

$$K_{11} = \begin{bmatrix} 0.1427 & -0.0570 \\ -0.0570 & 0.0663 \end{bmatrix}, K_{12} = \begin{bmatrix} -0.0337 & 0.0297 \\ 0.0097 & -0.0142 \end{bmatrix}$$

$$K_{21} = \begin{bmatrix} -0.0337 & 0.0097 \\ 0.0297 & -0.0142 \end{bmatrix}, K_{22} = \begin{bmatrix} 0.6755 & -0.2599 \\ -0.2599 & 0.3127 \end{bmatrix}$$

$$M = \begin{bmatrix} 0.1587 & 0.0208 \\ 0.0208 & 0.0390 \end{bmatrix}, G_1 = \begin{bmatrix} -0.0095 & -0.0033 \\ 0.0040 & -0.0001 \end{bmatrix}, G_2 = 10^{-3} \begin{bmatrix} -0.0000 & -0.2853 \\ 0.2853 & 0 \end{bmatrix}$$

$$P_{11} = \begin{bmatrix} 0.0347 & -0.0154 \\ -0.0154 & 0.0162 \end{bmatrix}, P_{12} = \begin{bmatrix} -0.0337 & 0.0297 \\ 0.0097 & -0.0142 \end{bmatrix}$$

$$P_{21} = \begin{bmatrix} -0.0337 & 0.0097 \\ 0.0297 & -0.0142 \end{bmatrix}, P_{22} = \begin{bmatrix} 0.6755 & -0.2599 \\ -0.2599 & 0.3127 \end{bmatrix}$$

$$N = \begin{bmatrix} 0.1587 & 0.0208 \\ 0.0208 & 0.0390 \end{bmatrix}, T_1 = \begin{bmatrix} -0.0095 & -0.0033 \\ 0.0040 & -0.0001 \end{bmatrix}, T_2 = 10^{-3} \begin{bmatrix} 0 & -0.2853 \\ 0.2853 & 0 \end{bmatrix}$$

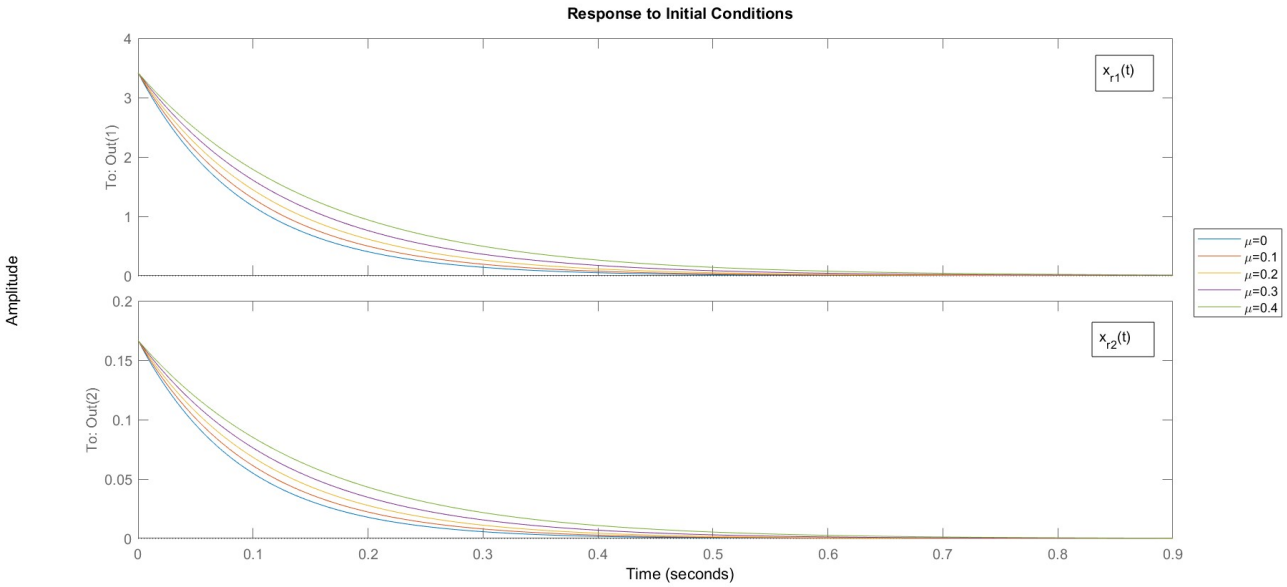

Figure 7: Systems Responses To Initial Condition When $S = 5 \times I$

5.5 Numerical Example 4

We have the following model

$$\dot{x} = \begin{bmatrix} -4 & 0 & 0 & -3 \\ -3 & -1 & -1 & 0 \\ 2 & -1 & -2 & 1 \\ 5 & 1 & 1 & -1 \end{bmatrix} x(t) + \begin{bmatrix} 1 & -1 & -1 & 0 \\ -1 & -1 & -3 & 0 \\ 0 & 0 & 1 & 0 \\ -7 & 0 & -1 & 1 \end{bmatrix} \zeta_1(t) + \begin{bmatrix} 0 & -2 & 0 & 0 \\ 0 & 0 & 1 & -1 \\ 0 & -2 & -1 & -1 \\ 0 & -1 & -2 & 0 \end{bmatrix} \zeta_2(t) \quad (166)$$

$$\epsilon\dot{\zeta}_1 = \begin{bmatrix} 2 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 5 & -4 & -2 & -6 \\ -1 & 4 & -2 & -5 \end{bmatrix} x(t) + \begin{bmatrix} -1 & -2 & -1 & 0 \\ -1 & -1 & 0 & 0 \\ 0 & -1 & 1 & -1 \\ -2 & -1 & -3 & -1+\mu \end{bmatrix} \zeta_1(t) + \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \end{bmatrix} \alpha_1(t) \tag{167}$$

$$\epsilon\dot{\zeta}_2 = \begin{bmatrix} -3 & 0 & -2 & 0 \\ 0 & -3 & -1 & -1 \\ 2 & 1 & -3 & -1 \\ 0 & -1 & 0 & -3 \end{bmatrix} x(t) + \begin{bmatrix} -1 & -1 & 0 & -1 \\ -1 & -2 & 0 & 0 \\ -1 & 0 & -2 & 0 \\ 0 & 0 & 0 & -2+\mu \end{bmatrix} \zeta_2(t) + \begin{bmatrix} 0 \\ 0 \\ 1 \\ 0 \end{bmatrix} \alpha_2(t) \tag{168}$$

Consider applying the same reduced model in Numerical Example 3 (5.4) to this example with the given values. The reduced model is

$$\dot{\bar{x}} = A_r \bar{x} \tag{169}$$

$$\dot{\bar{\sigma}} = A_r' \bar{x} + A_r \bar{\sigma} \tag{170}$$

$$A_0 = \begin{pmatrix} -\frac{31\,\mu-86}{2\,(\mu-3)} & \frac{2\,(2\,\mu-5)^2}{\omega_1} & \omega_2 & \frac{3\left(3\,\mu^2-13\,\mu+18\right)}{\omega_4} \\ \omega_3 & -\frac{2\,(4\,\mu-9)}{\mu-3} & -\omega_3 & -\frac{19\,\mu-3}{2\,(\mu-3)} \\ -\frac{19\,\mu-50}{2\,(\mu-3)} & \frac{2\left(4\,\mu^2-21\,\mu+28\right)}{\omega_1} & \omega_2 & \frac{13\,\mu^2-71\,\mu+114}{\omega_4} \\ \frac{20\,\mu-39}{\mu-3} & -\frac{10\,\mu^2-35\,\mu+27}{\omega_1} & -\frac{5\,\mu+6}{\mu-3} & -\frac{32\,\mu^2-97\,\mu+57}{\omega_1} \end{pmatrix}$$

where

$$\omega_1 = (\mu-2)\,(\mu-3)$$

$$\omega_2 = -\frac{5\,\mu-22}{2\,(\mu-3)}$$

$$\omega_3 = \frac{11\,\mu-12}{2\,(\mu-3)}$$

$$\omega_4 = 2\,(\mu-2)\,(\mu-3)$$

$B_{01} = \begin{pmatrix} -\frac{1}{2\,(\mu-3)} \\ \frac{3}{2\,(\mu-3)} \\ -\frac{1}{2\,(\mu-3)} \\ \frac{3}{\mu-3} \end{pmatrix}$, $B_{02} = \begin{pmatrix} 0 \\ 0.5 \\ -0.5 \\ -1.0 \end{pmatrix}$. After differentiating with respect to μ and Setting $\mu = 0$,

$$A_0 = \begin{pmatrix} -14.3333 & 8.3333 & -3.6667 & 4.5 \\ 2.0 & -6.0 & -2.0 & -0.5 \\ -8.3333 & 9.3333 & -3.6667 & 9.5 \\ 13.0 & -4.5 & 2.0 & -9.5 \end{pmatrix} \quad B_{01} = \begin{pmatrix} 0.1667 \\ -0.5 \\ 0.1667 \\ -1.0 \end{pmatrix}$$

$$B_{02} = \begin{pmatrix} 0 \\ 0.5 \\ -0.5 \\ -1.0 \end{pmatrix}$$

$$C_{01} = \begin{pmatrix} -1.3333 & 1.3333 & -0.6667 & 1.5000 \\ 2.3333 & -0.3333 & 0.6667 & -0.5000 \\ -0.3333 & 1.3333 & 1.3333 & -0.5000 \\ 2.3333 & -1.3333 & -2.3333 & -5.0000 \end{pmatrix}$$

$$C_{02} = \begin{pmatrix} -5.0000 & 4.0000 & -3.0000 & 4.0000 \\ 3.0000 & -2.5000 & 1.0000 & -2.5000 \\ 4.0000 & -1.5000 & 1.0000 & -2.5000 \\ 0 & -0.5000 & 0 & -0.5000 \end{pmatrix}$$

$$A_0' = \begin{pmatrix} 0.38889 & 0.2778 & -0.3889 & 0.5 \\ -1.1667 & 0.6667 & 1.1667 & 3.0 \\ 0.3889 & 0.7778 & -0.3889 & 2.0 \\ -2.3333 & 2.0833 & 2.3333 & 8.25 \end{pmatrix}, B_{01}' = \begin{pmatrix} 0.0556 \\ -0.1667 \\ 0.0556 \\ -0.3333 \end{pmatrix}, B_{02}' \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}.$$

With $\epsilon = 0.1$, $R_1 = R_2 = R_3 = R_4 = 1$,

$$Q_1 = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}, \text{ and } Q_2 = \begin{bmatrix} 0.2 & 0 & 0 & 0 \\ 0 & 0.2 & 0 & 0 \\ 0 & 0 & 0.2 & 0 \\ 0 & 0 & 0 & 0.2 \end{bmatrix}, \Omega = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

Solution: The considered cases in this example are three cases. The first case is when $S = 0 \times I$. The second case is when $S = 20 \times I$.

When $S = 0 \times I$, we have

$$\bar{\alpha}_1^* = \begin{bmatrix} -0.1263 & -0.1528 & -0.0297 & 0.0254 \end{bmatrix} C_{01}\bar{x}$$

$$\bar{\alpha}_2^* = \begin{bmatrix} 0.0207 & -0.0170 & 0.0370 & 0.1581 \end{bmatrix} C_{02}\bar{x}$$

with

$$K_{11} = \begin{bmatrix} 0.1501 & 0.0943 & -0.0344 & 0.0910 \\ 0.0943 & 0.2097 & 0.0523 & 0.1105 \\ -0.0344 & 0.0523 & 0.2030 & 0.1185 \\ 0.0910 & 0.1105 & 0.1185 & 0.2049 \end{bmatrix}$$

$$K_{12} = K_{21} = K_{22} = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

$$M = \begin{bmatrix} 0.1085 & 0.0700 & -0.0567 & 0.0576 \\ 0.0700 & 0.1149 & -0.0344 & 0.0106 \\ -0.0567 & -0.0344 & 0.3803 & 0.0784 \\ 0.0576 & 0.0106 & 0.0784 & 0.1367 \end{bmatrix}$$

$$G_1 = \begin{bmatrix} 0.0442 & 0.0361 & -0.0117 & 0.0363 \\ 0.0334 & 0.0277 & 0.0061 & 0.0346 \\ -0.1016 & -0.1182 & 0.3980 & 0.0990 \\ -0.0016 & -0.0210 & 0.2232 & 0.1104 \end{bmatrix}, G_2 = \begin{bmatrix} -0.0000 & 0.0023 & -0.1127 & -0.0440 \\ -0.0023 & -0.0000 & -0.1435 & -0.0633 \\ 0.1127 & 0.1435 & -0.0000 & 0.1602 \\ 0.0440 & 0.0633 & -0.1602 & 0.0000 \end{bmatrix}$$

$$P_{11} = \begin{bmatrix} 0.0357 & 0.0250 & -0.0073 & 0.0228 \\ 0.0250 & 0.0491 & 0.0110 & 0.0281 \\ -0.0073 & 0.0110 & 0.0435 & 0.0260 \\ 0.0228 & 0.0281 & 0.0260 & 0.0471 \end{bmatrix}$$

$$P_{12} = P_{21} = P_{22} = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

$$N = \begin{bmatrix} 0.1085 & 0.0700 & -0.0567 & 0.0576 \\ 0.0700 & 0.1149 & -0.0344 & 0.0106 \\ -0.0567 & -0.0344 & 0.3803 & 0.0784 \\ 0.0576 & 0.0106 & 0.0784 & 0.1367 \end{bmatrix}$$

$$T_1 = \begin{bmatrix} 0.0442 & 0.0361 & -0.0117 & 0.0363 \\ 0.0334 & 0.0277 & 0.0061 & 0.0346 \\ -0.1016 & -0.1182 & 0.3980 & 0.0990 \\ -0.0016 & -0.0210 & 0.2232 & 0.1104 \end{bmatrix}, T_2 = \begin{bmatrix} 0.0000 & 0.0023 & -0.1127 & -0.0440 \\ -0.0023 & 0.0000 & -0.1435 & -0.0633 \\ 0.1127 & 0.1435 & 0.0000 & 0.1602 \\ 0.0440 & 0.0633 & -0.1602 & -0.0000 \end{bmatrix}$$

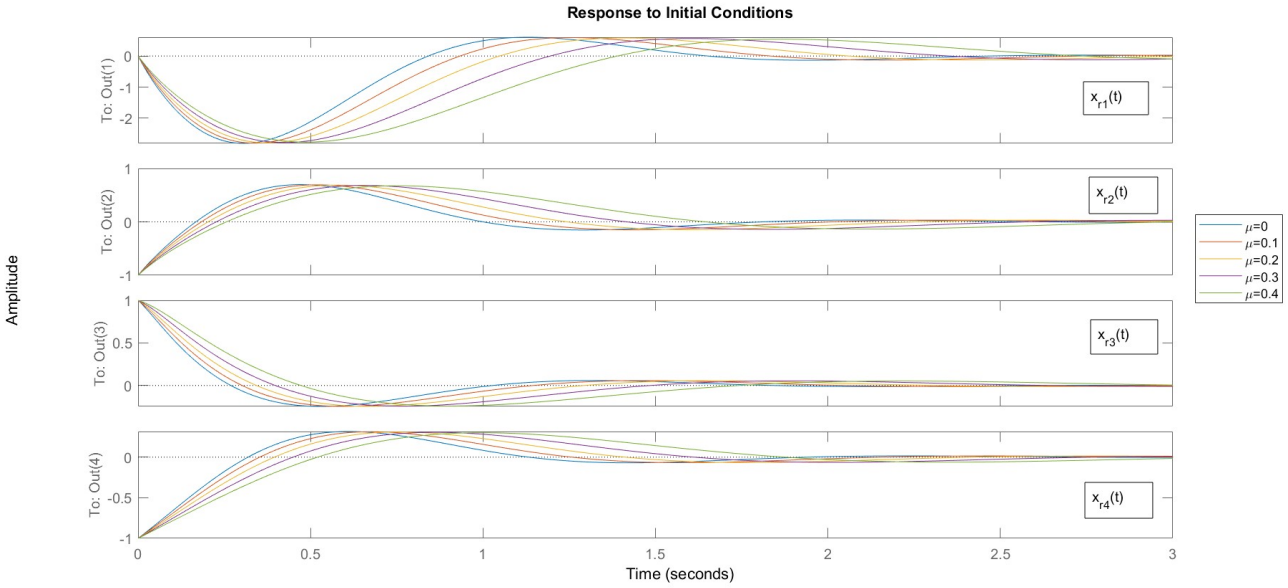

Figure 8: Systems Responses To Initial Condition When $S = 0 \times I$

When $S = 20 \times I$. we have

$$\bar{\alpha}_1^* = \begin{bmatrix} -0.0507 & -1.4791 & 1.0926 & 0.7909 \end{bmatrix} C_{01}\bar{x}$$

$$\bar{\alpha}_2^* = \begin{bmatrix} -1.1260 & -10.1207 & 5.9874 & 21.9703 \end{bmatrix} C_{02}\bar{x}$$

with

$$K_{11} = \begin{bmatrix} 0.8884 & 0.2475 & 0.0150 & 0.9657 \\ 0.2475 & 0.6568 & 0.0299 & 0.2735 \\ 0.0150 & 0.0299 & 0.4652 & 0.4747 \\ 0.9657 & 0.2735 & 0.4747 & 1.9194 \end{bmatrix}, K_{12} = \begin{bmatrix} -0.0625 & 0.1661 & 0.1976 & 0.0589 \\ 0.0384 & 0.6294 & 0.2647 & 0.2146 \\ 0.0776 & 0.1720 & -0.0372 & 0.0582 \\ -0.0183 & 0.8159 & 0.5538 & 0.3739 \end{bmatrix}$$

$$K_{21} = \begin{bmatrix} -0.0625 & 0.0384 & 0.0776 & -0.0183 \\ 0.1661 & 0.6294 & 0.1720 & 0.8159 \\ 0.1976 & 0.2647 & -0.0372 & 0.5538 \\ 0.0589 & 0.2146 & 0.0582 & 0.3739 \end{bmatrix}, K_{22} = \begin{bmatrix} 1.5408 & 0.0773 & -1.3027 & 0.0517 \\ 0.0773 & 3.1566 & 1.0276 & 0.5508 \\ -1.3027 & 1.0276 & 3.1568 & 0.7088 \\ 0.0517 & 0.5508 & 0.7088 & 0.8497 \end{bmatrix}$$

$$M = \begin{bmatrix} 0.0895 & 0.0530 & -0.0394 & 0.0360 \\ 0.0530 & 0.0947 & 0.0046 & 0.0148 \\ -0.0394 & 0.0046 & 0.2061 & -0.0117 \\ 0.0360 & 0.0148 & -0.0117 & 0.0449 \end{bmatrix}$$

$$G_1 = \begin{bmatrix} 0.0021 & 0.0025 & 0.0038 & 0.0022 \\ -0.0066 & -0.0019 & 0.0183 & -0.0010 \\ 0.0039 & 0.0057 & 0.0382 & 0.0078 \\ 0.0026 & 0.0066 & 0.0229 & 0.0068 \end{bmatrix}, G_2 = \begin{bmatrix} 0.0000 & 0.0002 & -0.0047 & -0.0001 \\ -0.0002 & -0.0000 & -0.0077 & -0.0005 \\ 0.0047 & 0.0077 & 0 & 0.0062 \\ 0.0001 & 0.0005 & -0.0062 & 0.0000 \end{bmatrix}$$

$$P_{11} = \begin{bmatrix} 0.8268 & 0.2444 & 0.0672 & 0.9636 \\ 0.2444 & 0.5305 & -0.0112 & 0.2514 \\ 0.0672 & -0.0112 & 0.3389 & 0.4464 \\ 0.9636 & 0.2514 & 0.4464 & 1.8854 \end{bmatrix}, P_{12} = \begin{bmatrix} -0.0625 & 0.1661 & 0.1976 & 0.0589 \\ 0.0384 & 0.6294 & 0.2647 & 0.2146 \\ 0.0776 & 0.1720 & -0.0372 & 0.0582 \\ -0.0183 & 0.8159 & 0.5538 & 0.3739 \end{bmatrix}$$

$$P_{21} = \begin{bmatrix} -0.0625 & 0.0384 & 0.0776 & -0.0183 \\ 0.1661 & 0.6294 & 0.1720 & 0.8159 \\ 0.1976 & 0.2647 & -0.0372 & 0.5538 \\ 0.0589 & 0.2146 & 0.0582 & 0.3739 \end{bmatrix}, P_{22} = \begin{bmatrix} 1.5408 & 0.0773 & -1.3027 & 0.0517 \\ 0.0773 & 3.1566 & 1.0276 & 0.5508 \\ -1.3027 & 1.0276 & 3.1568 & 0.7088 \\ 0.0517 & 0.5508 & 0.7088 & 0.8496 \end{bmatrix}$$

$$N = \begin{bmatrix} 0.0895 & 0.0530 & -0.0394 & 0.0360 \\ 0.0530 & 0.0947 & 0.0046 & 0.0148 \\ -0.0394 & 0.0046 & 0.2061 & -0.0117 \\ 0.0360 & 0.0148 & -0.0117 & 0.0449 \end{bmatrix}$$

$$T_1 = \begin{bmatrix} 0.0021 & 0.0025 & 0.0038 & 0.0022 \\ -0.0066 & -0.0019 & 0.0183 & -0.0010 \\ 0.0039 & 0.0057 & 0.0382 & 0.0078 \\ 0.0026 & 0.0066 & 0.0229 & 0.0068 \end{bmatrix}, T_2 = \begin{bmatrix} 0.0000 & 0.0002 & -0.0047 & -0.0001 \\ -0.0002 & 0.0000 & -0.0077 & -0.0005 \\ 0.0047 & 0.0077 & 0 & 0.0062 \\ 0.0001 & 0.0005 & -0.0062 & 0.0000 \end{bmatrix}$$

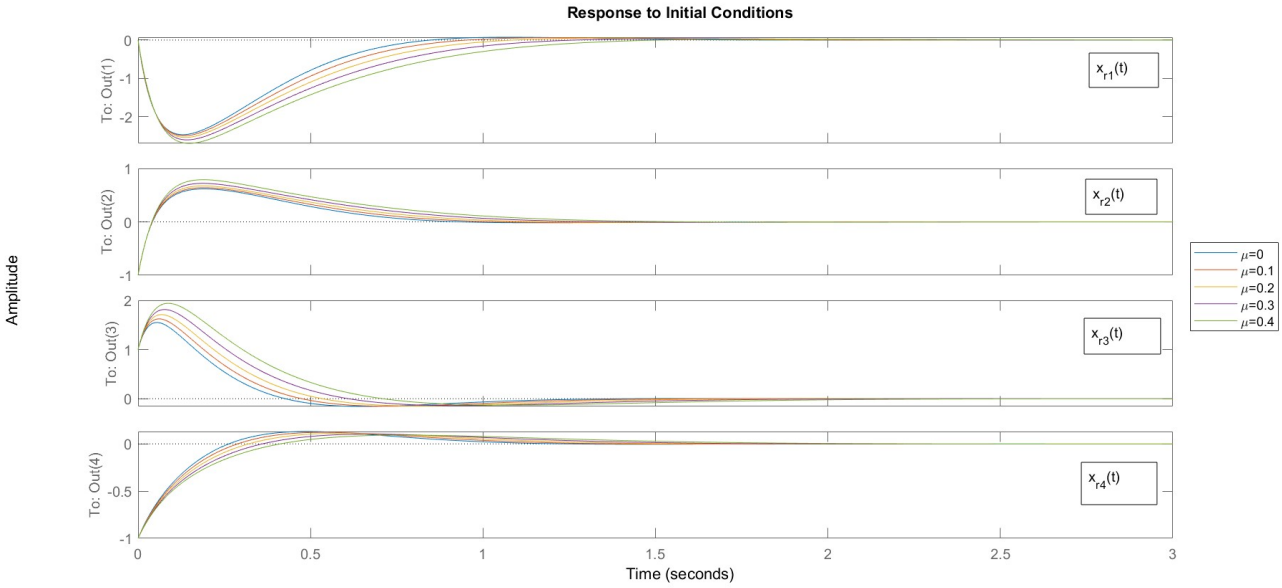

Figure 9: Systems Responses To Initial Condition When $S = 20 \times I$

CHAPTER 6

CONCLUSION AND FUTURE WORK

6.1 Conclusion

In conclusion, the design of complex systems with low sensitivity to nonlinear subsystems is an essential problem in engineering and science. The linearization of non-linear control systems is a useful technique in control system design as it allows the use of conventional linear control theory to design controllers for nonlinear systems. This research proposes the singular perturbation method as a model reduction technique to develop reduced-order models that represent the system's core characteristics. The reduced-order models can be used to design optimal controllers for the complex system that are insensitive to the nonlinear subsystems.

The use of game theory in control systems has also proven to be a valuable tool, especially in systems with multiple agents making decisions and interacting with each other. In this research, the reduced order model was optimized using the Nash strategy of game theory to maximize system performance.

The results of this study demonstrate the effectiveness of the singular perturbation method as a model reduction technique and the use of game theory in designing optimal controllers for complex systems. This research contributes to the understanding of nonlinear systems' behavior and the design of effective control strategies to achieve desired performance objectives. Further research can be done to explore the potential of other model reduction techniques and game theory strategies in optimizing control systems for complex nonlinear systems.

6.2 Future Work

There are several areas for future work on the optimal design of complex systems with low sensitivity to nonlinear subsystems. One potential area of research is to investigate the application of other model reduction techniques, such as aggregation technique, in designing reduced-order models for nonlinear systems. A comparative study of the effectiveness of different model reduction techniques in controlling complex systems would be a valuable contribution to the field.

Another area of future work in the optimal design of complex systems with low sensitivity to nonlinear subsystems is the impact of the sampling period on the performance of the designed controllers. The sampling period is the time interval between two consecutive measurements or updates of the control system. In practical applications, the sampling period is finite, and it can have a significant impact on the performance of the control system. The impact of the sampling period on the performance of complex nonlinear control systems is an important area of future work that can lead to significant advancements in control theory and practice.

Additionally, it would be interesting to explore the application of game theory in control systems beyond the Nash strategy. Other game theory strategies, such as cooperative game theory or evolutionary game theory, could be used to design optimal controllers for complex systems with multiple agents making decisions and interacting with each other.

Overall, the future work in the optimal design of complex systems with low sensitivity to nonlinear subsystems is vast and varied. Further research in these areas could lead to significant advancements in engineering and science, with practical applications.

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