

**DERIVATION AND REGULARIZATION OF CORRECTIONS TO
PROPAGATORS AND VERTICES FOR CUBICALLY INTERACTING
SCALAR FIELD THEORIES**

A Thesis by

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The following faculty members have examined the final copy of this thesis for form and content, and recommend that it be accepted in partial fulfillment of the requirement for the degree of Master of Science with a major in Mathematics.

Thomas DeLillo, Committee Chair

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ABSTRACT

This thesis, a mainly expository work, attempts to give all the calculational details for some popular models in scalar quantum field theory that serve a mostly pedagogical purpose. Starting with a real scalar field with a given Lagrangian in classical field theory, standard quantization results given in most introductory QFT texts are derived with as many explicit calculations (which are often left to the reader) as possible. After presenting this development, we gather specific calculations that model scattering and decay processes. In many texts, the author(s) simply gives one or two calculations as an example. Here we organize several different calculations in a single work.

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PREFACE

The idea for this masters thesis originated out of Tom DeLillo's 2009 Quantum Field Theory course. He suggested that a good possible topic for the preliminary calculations and formalism for scalar quantum field theory. Complete details for most theories are often scattered over several sources, therefore one objective of this thesis is to lay out all the calculational details of a popular model, or "toy" theory, for students of physics or mathematics as preparation for more complicated physical theories (like Quantum Electrodynamics) including spin or charge. These details are also laid out as reference for a more rigorous mathematical development of QFT; see eg. [1][2]. These calculations when combined into a manuscript could also be used as a guide for a student embarking on a course in QFT or even be used as a set of notes accompanying the standard texts of the field.

With this in mind, the reader is assumed to be familiar with introductory quantum mechanics at the level of [3]. Also some basic special relativity is required. Though often presented in a tensor notation we forego this method in a general sense. Four-vectors, however, will be denoted in the typical sense: $x_\mu = (x_0, -x_1, -x_2, -x_3)$ and $x^\mu = (x_0, x_1, x_2, x_3)$ which are given by the metric signature $(1, -1, -1, -1)$. In the introductory development we make use of the Einstein summation convention $x^\mu x_\mu = x_0^2 - x_1^2 - x_2^2 - x_3^2$.

TABLE OF CONTENTS

Chapter	Page
1 INTRODUCTION	1
2 CLASSICAL FIELD THEORY	3
2.1 Lagrangian Mechanics; the Euler Lagrange Equations	3
2.2 The Euler-Lagrange Equations for Fields	5
2.3 Hamiltonian Mechanics	6
2.4 The Real Scalar Field	7
2.5 A Scalar Field Theory with Interactions	8
3 QUANTIZATION OF THE CLASSICAL KLEIN-GORDON FIELD	10
3.1 Obtaining the Hamiltonian and Momentum	13
3.2 Constructing the Fock Space and Evaluating the Energy and Momentum of its Elements	15
3.3 Obtaining a Time-Dependant Form of ϕ	20
3.4 Quantizing the ABB and ABC-Theory	22
3.5 Calculating $\langle 0 a_{\mathbf{p}}, \phi(x) 0\rangle$	23
3.6 Derivation of the Feynman Propagator	25
4 PERTURBATION THEORY	29
4.1 Deriving the perturbation expansion	29
4.2 Expressing $\phi(x)$ in Terms of $\phi_I(x)$	30
5 TREE-LEVEL AMPLITUDES IN ABC AND ABB THEORY	32
5.1 AB-Scattering in ABC Theory	32
5.1.1 The Zeroth Order Term in Expansion of g	32
5.1.2 The First Order Term in Expansion of g	33
5.1.3 The Second-Order Term in Expansion of g	35
5.2 $C \rightarrow A + B$ Decay	43
5.3 $A \rightarrow B + B$ Decay	47
6 DIMENSIONAL REGULARIZATION	52
6.1 One-loop Correction to the A-Propagator in ABB Theory	52
6.2 One-loop Correction to the Vertex in ABC Theory	55
6.3 Fourth-Order Self Energy Diagram: Nested Divergence	56
6.4 Fourth-Order Self-Energy Diagram: Overlapping Divergence	59
6.4.1 Calculating the Boundary Contribution of the $d\eta$ Integral	62
6.5 Calculating the integral contribution to the $d\eta$ Integral	64

TABLE OF CONTENTS (continued)

Chapter	Page
7 CONCLUSION AND FUTURE WORK	68
REFERENCES	71

LIST OF FIGURES

Figure	Page
5.1 Zeroth-order term in the power series expansion of g	33
5.2 C-Exchange in AB-scattering. The q_A refers to the p'_A in the discussion and similarly for q_B	37
5.3 AB Annihilation for ABC Theory. q_A corresponds to p'_A and similarly for q_B . .	41
5.4 A C-particle decaying into an A and a B-particle	43
5.5 An A-particle decaying into 2 B-particles	47
6.1 Self-Energy of The A-Particle in ABB Theory	53
6.2 One-loop correction to the vertex of ABC theory	55
6.3 Two-loop correction to the A-propagator: nested divergence	57
6.4 Two-loop correction to the propagator: overlapping divergence	59

CHAPTER 1

INTRODUCTION

Quantum field theory is the attempt to combine quantum mechanics with special relativity. In quantum mechanics, the (non-relativistic) Schrodinger equation which is second order in space and first order in time gives rise to solutions which are interpreted as having positive energy. Unfortunately when we try to put forward this same interpretation in the relativistic case (an equation both second order in space and time) we obtain “negative energy” solutions.

To ameliorate this issue we extend relativistic quantum mechanics to relativistic quantum field theory. ϕ is then interpreted not as a wavefunctions for a certain particle but as a field for infinitely many particles. Chapter 2 introduces field theory in the classical sense in preparation for quantization of the fields which takes place in Chapter 3. For this introductory material I follow mainly Peskin and Schroeder [4], although some material (especially the derivation of the Euler-Lagrange equation) is contained in Maggiore [5].

After the introductory development and a brief explanation of perturbation theory (in Chapter 4), I present some basic interacting theories which are called the ABC and ABB-theories. The Feynman rules for ABC are stated in [6] and [7] and derived in [8]. The Feynman rules for the ABB theory are stated in [9], and to our knowledge have not been derived from a Lagrangian. We state this Lagrangian and derive the Feynman rule for a vertex.

Chapter 5 is devoted to calculating scattering amplitudes, or S-matrix elements. In the laboratory any n number of particles collide with one another and turn into m (possibly different) particles. The entries of the S-matrix $\langle n|S|m\rangle$ can be written in a perturbative expansion of some interaction parameter g and may be interpreted as the probability amplitude for the initial state to evolve into the particular final state. These probabilities are

physically measured by decay rates Γ or scattering cross-sections σ both of which will not be considered here.

This process of calculating amplitudes through evaluation of S-matrix elements works fine until higher-order contributions to the expansion are calculated. It turns out that almost all higher order calculations yield infinite results. We must then reinterpret what we mean by the parameters of the theory (mass, coupling, and field.) This is the subject of Chapter 6. Through a process known as *dimensional regularization* we are able to isolate the singular part of the integral and recover the finite part, discarding infinitesimal contributions. Covariant regularization is another method for isolating these infinite parts of the integrals (see [10]). It yields the same finite results (with different singular parts), but it will not be considered here.

CHAPTER 2

CLASSICAL FIELD THEORY

Before we are able to introduce quantum field theory, we need to first review the basics of classical mechanics. [11] formulates mechanics in a number of useful ways including Poisson brackets and Hamilton-Jacobi theory (both of which are useful in the transition from classical to quantum mechanics.) Of most importance here, however, will be Lagrangian and Hamiltonian mechanics. We will first introduce the Lagrangian then define the Hamiltonian from it. In future chapters we will rely mainly on the Hamiltonian formalism.

2.1 Lagrangian Mechanics; the Euler Lagrange Equations

In classical mechanics, a system may be described by N generalized coordinates which themselves depend on time, a non-measurable quantity which we define as a real-valued, continuous parameter of the system. These are constructed as an attempt to capture the measurable degrees of freedom of the system. The following derivations are following [5]. We denote these coordinates by $q_i = q_i(t), i = 1, \dots, N$. How these coordinates interact with one another is captured by the Lagrangian. The Lagrangian L of this system is defined by $L(q, \dot{q})$, a function of these generalized coordinates and their time derivatives which will be typically denoted as $\dot{q} \equiv \frac{dq}{dt}$. As a system evolves from an initial configuration to a final one, we may define the action S of each possible path $q(t)$ by

$$S = \int_{q(t)} L(q, \dot{q}) dt. \tag{2.1}$$

The cornerstone of this classical theory is *Hamilton's Variational Principle*, which states that the true path $q(t)$ that the system takes is the one for which the action S is stationary. This is made mathematically formal by the methods of variational calculus and

setting the *variation of S* δS equal to 0.

$$\begin{aligned} 0 = \delta S &= \delta \int_{t_{in}}^{t_f} d^A x \mathcal{L}(q, \dot{q}) dt \\ &= \int_{t_{in}}^{t_f} \delta \mathcal{L}(q, \dot{q}) dt \\ &= \int_{t_{in}}^{t_f} \left[L(q + \delta q, \dot{q} + \delta \dot{q}) - L(q, \dot{q}) \right] dt. \end{aligned}$$

Now the variation of the Lagrangian is

$$\delta L = \sum_i \frac{\partial L}{\partial q_i} \delta q_i + \frac{\partial L}{\partial \dot{q}_i} \delta \dot{q}_i, \quad (2.2)$$

so noting that $\delta \dot{q}_i \equiv \frac{d}{dt}(\delta q_i)$ we insert (2.2) into the formula above and integrate by parts to obtain

$$0 = \int_{t_{in}}^{t_f} \left[\sum_i \frac{\partial L}{\partial q_i} \delta q_i + \frac{\partial L}{\partial \dot{q}_i} \delta \dot{q}_i + O((\delta q_i)^2, (\delta \dot{q}_i)^2) \right] dt \quad (2.3)$$

$$= \int_{t_{in}}^{t_f} \sum_i \left[\frac{\partial L}{\partial q_i} - \frac{d}{dt} \left(\frac{\partial L}{\partial \dot{q}_i} \right) \right] \delta q_i dt + \frac{\partial L}{\partial \dot{q}_i} \delta q_i \Big|_{t_{in}}^{t_f}. \quad (2.4)$$

The boundary vanishes since $\delta q(t_{in}) = \delta q(t_f) = 0$, and we are left with

$$0 = \int_{t_{in}}^{t_f} \sum_i \left[\frac{\partial L}{\partial q_i} - \frac{d}{dt} \left(\frac{\partial L}{\partial \dot{q}_i} \right) \right] \delta q_i dt. \quad (2.5)$$

Since the $\delta q_i(t)$'s were arbitrary, we obtain at once the Euler-Lagrange equation

$$\boxed{\frac{\partial L}{\partial q_i} - \frac{d}{dt} \left(\frac{\partial L}{\partial \dot{q}_i} \right) = 0.} \quad (2.6)$$

As a motivation for (2.6) consider the mechanical energy E_{mech} for a simple harmonic oscillator

$$E_{mech} = \frac{1}{2} m v^2 + \frac{1}{2} k x^2. \quad (2.7)$$

(2.7) “looks like” Hooke’s Law if one takes the derivative of PE with respect to x and the derivative of KE with respect to v (followed by a time derivative)

$$\frac{\partial(PE)}{\partial x} = kx \qquad \frac{\partial(KE)}{\partial v} = mv \Rightarrow \frac{d}{dt} \left(\frac{\partial(KE)}{\partial v} \right) = ma.$$

The way to actually recover Hooke's Law, $F = -kx$ where $k > 0$ is the spring constant, is to define the Lagrangian $L = T - V$ (where $T = KE$ and $V = PE$) for the oscillator by

$$L = \frac{1}{2}m\dot{x}^2 - \frac{1}{2}kx^2 \quad (2.8)$$

for the generalized coordinate is $q = x$. Then we can apply (2.7) to obtain our result

$$0 = \frac{\partial L}{\partial q_i} - \frac{d}{dt} \left(\frac{\partial L}{\partial \dot{q}_i} \right) = -kx - m\ddot{x}. \quad (2.9)$$

2.2 The Euler-Lagrange Equations for Fields

We now make the transition from a system with a finite number of degrees of freedom to one with infinitely many. Consider a flexible string fixed at two endpoints on the x-axis. Each point on the string can be thought of as being some distance away from the axis connecting the endpoints. Thus the string has infinitely many degrees of freedom. By thinking of the infinitely many degrees of freedom as a continuous variable, we create the concept of a field $\phi(x)$ where $x = x_\mu = (x_0, \mathbf{-x})$. Then for a system described by this field, the corresponding Lagrangian takes the form

$$L = \int d^3x \mathcal{L}(\phi, \partial_\mu \phi) \quad (2.10)$$

where \mathcal{L} is called the *Lagrangian density*. Since we are considering special relativity we will be using the four-vector notation throughout the text. In this case $\partial_\mu = (\frac{\partial}{\partial t}, -\nabla)$. Also, in keeping with custom \mathcal{L} will be referred to simply as the Lagrangian. The action of a field is then defined by

$$S = \int dt L = \int d^4x \mathcal{L}(\phi, \partial_\mu \phi). \quad (2.11)$$

This is an integral over all space-time. For the boundary conditions, instead of setting $\delta q(t_{in}) = \delta q(t_f) = 0$ like in the case of finitely many degrees of freedom, we assume that $\phi, \partial_\mu \phi \rightarrow 0$ as $x^\mu \rightarrow \infty$ sufficiently fast so that the boundary terms may be neglected and the spacetime integrals are finite [5].

Once again the action S must be stationary for the actual (physical) field ϕ

$$0 = \delta S = \delta \int d^4x \mathcal{L}(\phi, \partial_\mu \phi) \quad (2.12)$$

$$= \int d^4x \delta \mathcal{L}(\phi, \partial_\mu \phi) \quad (2.13)$$

$$= \int d^4x \left[\frac{\partial \mathcal{L}}{\partial \phi} \delta \phi + \frac{\partial \mathcal{L}}{\partial (\partial_\mu \phi)} \delta (\partial_\mu \phi) \right]. \quad (2.14)$$

We set $\delta(\partial_\mu \phi) = \partial_\mu(\delta \phi)$ and integrate by parts using a form of the divergence theorem for a finite volume of spacetime and taking the limit to all spacetime [5] to obtain

$$0 = \lim_{V \rightarrow \infty} \int_V d^4x \left[\frac{\partial \mathcal{L}}{\partial \phi} + \partial_\mu \left(\frac{\partial \mathcal{L}}{\partial (\partial_\mu \phi)} \right) \right] \delta(\partial_\mu \phi) + \int_{\partial V} dA \frac{\partial \mathcal{L}}{\partial (\partial_\mu \phi)} \delta \phi \cdot n_\mu. \quad (2.15)$$

Once again the boundary term goes vanishes as the volume approaches all of spacetime and we are left with

$$\int d^4x \left[\partial_\mu \left(\frac{\partial \mathcal{L}}{\partial (\partial_\mu \phi)} \right) - \frac{\partial \mathcal{L}}{\partial \phi} \right] \delta \phi = 0. \quad (2.16)$$

Since the $\delta \phi$ is arbitrary, we finally obtain the Euler-Lagrange equations for fields

$$\boxed{\partial_\mu \left(\frac{\partial \mathcal{L}}{\partial (\partial_\mu \phi)} \right) - \frac{\partial \mathcal{L}}{\partial \phi} = 0}. \quad (2.17)$$

This is a 2nd-order PDE in our case.

2.3 Hamiltonian Mechanics

Next we pass to the Hamiltonian formalism. Note above that L was defined in terms of q and \dot{q} : $L = L(q, \dot{q})$. An alternative method to classical mechanics would be to define the conjugate momentum of a corresponding generalized coordinate by $p \equiv \frac{\partial L}{\partial \dot{q}}$. The Hamiltonian for a given system is then defined from its Lagrangian by $H \equiv p\dot{q} - L$ and is a function of the generalized coordinates and their corresponding conjugate momenta $H = H(p, q)$. The Euler-Lagrange equations are replaced by *Hamilton's equations*

$$-\dot{p}_i = \frac{\partial H}{\partial q_i} \quad \dot{q}_i = \frac{\partial H}{\partial p_i}. \quad (2.18)$$

The same process is followed for fields by defining the *conjugate momentum* by

$$p(\mathbf{x}) \equiv \frac{\partial L}{\partial \dot{\phi}} = \frac{\partial}{\partial \dot{\phi}(\mathbf{x})} \int \mathcal{L}(\phi(\mathbf{y}), \dot{\phi}(\mathbf{y})) d^3y. \quad (2.19)$$

Inverting the derivative with the integral we are able to obtain the definition for the *conjugate momentum density* $\pi(x)$ by

$$\pi(x) \equiv \frac{\partial \mathcal{L}}{\partial \dot{\phi}}. \quad (2.20)$$

The Hamiltonian H and corresponding Hamiltonian density \mathcal{H} is then defined by

$$H = \int d^3x (\pi(x)\dot{\phi}(x) - \mathcal{L}) = \int d^3x \mathcal{H}. \quad (2.21)$$

It is worth noting that in this thesis we use the Lagrangian to define the Hamiltonian and stay with the Hamiltonian formalism. It appears that the Hamiltonian is well suited for perturbation theory, whereas the Lagrangian formalism is best used in the path-integral formulation [5][4].

2.4 The Real Scalar Field

In this discussion we will only consider real scalar fields $\phi(x)$. From the Lagrangian

$$\mathcal{L} = \frac{1}{2}\dot{\phi}^2 - \frac{1}{2}(\nabla\phi)^2 - \frac{1}{2}m^2\phi^2 = \frac{1}{2}(\partial_\mu\phi)^2 - \frac{1}{2}m^2\phi^2. \quad (2.22)$$

We can plug this into (2.17) to obtain the classical Klein-Gordon equation

$$\left(\frac{\partial^2}{\partial t^2} - \nabla^2 + m^2\right)\phi = (\partial_\mu\partial^\mu + m^2)\phi = 0. \quad (2.23)$$

Note the summation notation on the right-hand side. For a general 4-vector $x_\mu x^\mu = t^2 - \mathbf{x}^2$.

In this case $\partial_\mu\partial^\mu = \frac{\partial^2}{\partial t^2} - \nabla^2$. From the Lagrangian we can determine the Hamiltonian.

Noting that

$$\pi(x) = \frac{\partial \mathcal{L}}{\partial \dot{\phi}(x)} = \frac{\partial}{\partial \dot{\phi}(x)} \left(\frac{1}{2}\dot{\phi}^2 - \frac{1}{2}(\nabla\phi)^2 + \frac{1}{2}m^2\phi^2 \right) = \dot{\phi}(x), \quad (2.24)$$

we use the definition of \mathcal{H} to obtain

$$\mathcal{H} = \pi\dot{\phi} - \mathcal{L} \quad (2.25)$$

$$= \pi^2 - \frac{1}{2}\dot{\phi}^2 + \frac{1}{2}(\nabla\phi)^2 + \frac{1}{2}m^2\phi^2 \quad (2.26)$$

$$= \frac{1}{2}\pi^2 + (\nabla\phi)^2 + \frac{1}{2}m^2\phi^2 \quad (2.27)$$

with the corresponding total Hamiltonian being

$$H = \int \mathcal{H} d^3x = \int \left(\frac{1}{2} \pi^2 + \frac{1}{2} (\nabla \phi)^2 + \frac{1}{2} m^2 \phi^2 \right) d^3x. \quad (2.28)$$

Each of the terms in \mathcal{H} has a specific interpretation ([4], p. 17) as the energy cost for:

$$\begin{aligned} \frac{1}{2} \pi^2 &\Rightarrow \text{moving in time} \\ (\nabla \phi)^2 &\Rightarrow \text{shearing in space} \\ \frac{1}{2} m^2 \phi^2 &\Rightarrow \text{existence of the field.} \end{aligned}$$

2.5 A Scalar Field Theory with Interactions

Consider a physical theory describing 3 scalar particles as described in [6] and [8]. It is referred to as the ABC-theory. We will call them A, B, and C particles and the corresponding fields will be denoted ϕ_A , ϕ_B , and ϕ_C , respectively. If they were non-interacting particles, the Lagrangian of the system would be written as

$$\mathcal{L}_{ABC} = \frac{1}{2} (\partial_\mu \phi_A)^2 - \frac{1}{2} m_A^2 \phi_A^2 + \frac{1}{2} (\partial_\mu \phi_B)^2 - \frac{1}{2} m_B^2 \phi_B^2 + \frac{1}{2} (\partial_\mu \phi_C)^2 - \frac{1}{2} m_C^2 \phi_C^2 \quad (2.29)$$

which contains 3 identical copies of (2.22). And if we apply the Euler-Lagrange equations to this we obtain 3 identical KG-equations.

$$(\partial_\mu \partial^\mu + m_A^2) \phi_A = 0 \quad (2.30)$$

$$(\partial_\mu \partial^\mu + m_B^2) \phi_B = 0 \quad (2.31)$$

$$(\partial_\mu \partial^\mu + m_C^2) \phi_C = 0. \quad (2.32)$$

Now suppose we introduce an interaction between the particles by adding a term to (2.29) which is a product of the three fields at a point in spacetime $\mathcal{L}_{int} = -g \phi_A(x) \phi_B(x) \phi_C(x)$. With this simple interaction term the total Lagrangian becomes

$$\mathcal{L}_{ABC} = \frac{1}{2} (\partial_\mu \phi_A)^2 - \frac{1}{2} m_A^2 \phi_A^2 + \frac{1}{2} (\partial_\mu \phi_B)^2 - \frac{1}{2} m_B^2 \phi_B^2 + \frac{1}{2} (\partial_\mu \phi_C)^2 - \frac{1}{2} m_C^2 \phi_C^2 - g \phi_A \phi_B \phi_C. \quad (2.33)$$

An application of the Euler-Lagrange equations gives

$$(\partial_\mu \partial^\mu + m_A^2)\phi_A = -g\phi_B\phi_C \quad (2.34)$$

$$(\partial_\mu \partial^\mu + m_B^2)\phi_B = -g\phi_A\phi_C \quad (2.35)$$

$$(\partial_\mu \partial^\mu + m_C^2)\phi_C = -g\phi_A\phi_B. \quad (2.36)$$

We will also consider another physical theory: one describing only 2 particles, an A and a B. We will call it the ABB-Theory, and it is discussed in [9]. The full Lagrangian is a bit simpler than the ABC

$$\mathcal{L}_{ABB} = \frac{1}{2}(\partial_\mu \phi_A)^2 - \frac{1}{2}m_A^2\phi_A^2 + \frac{1}{2}(\partial_\mu \phi_B)^2 - \frac{1}{2}m_B^2\phi_B^2 - \frac{1}{2}g\phi_A\phi_B^2. \quad (2.37)$$

The Euler-Lagrange Equations for this Lagrangian give 2 equations of motion with different interacting terms

$$(\partial_\mu \partial^\mu + m_A^2)\phi_A = -g\phi_B^2 \quad (2.38)$$

$$(\partial_\mu \partial^\mu + m_B^2)\phi_B = -g\phi_A\phi_B. \quad (2.39)$$

The equations of motion for ABC-Theory and ABB-Theory are systems of PDE's, and a mathematical analysis to study existence and uniqueness of solutions could be undertaken. We will not pursue this here. Instead, we will quantize the free Klein-Gordon equation to obtain non-interacting states and rely on a perturbative expansion of $e^{i \int d^4x H_{int}}$ to compute probabilities of one free state evolving to another *through* the interaction.

CHAPTER 3

QUANTIZATION OF THE CLASSICAL KLEIN-GORDON FIELD

Now that we have the classical field equation, we quantize the field by promoting $\phi(\mathbf{x})$ and $\pi(\mathbf{x})$ to operators (acting on a Hilbert space) and imposing commutation relations between them. Recall that the commutator between two operators is defined by

$$[A, B] := AB - BA. \quad (3.1)$$

The commutation relations which will be imposed on ϕ and π are

$$[\phi(\mathbf{x}, t), \pi(\mathbf{y}, t)] = i\delta^3(\mathbf{x} - \mathbf{y}) \quad (3.2)$$

$$[\phi(\mathbf{x}, t), \phi(\mathbf{y}, t)] = [\pi(\mathbf{x}, t), \pi(\mathbf{y}, t)] = 0. \quad (3.3)$$

Note that the commutation relations above are for the field operators in the Heisenberg picture, where they depend on time. In fact, these are *equal-time* commutation relations. Were we in the Schrodinger picture the fields, and hence the commutator, would be fixed for all time, since only the states depend on time in the Schrodinger picture.

To “solve” this quantum field theory, we first must find an expression for the Klein-Gordon equation in momentum space. We do this by first expanding $\phi(x)$ itself as an integral over momentum space

$$\phi(\mathbf{x}, t) = \int \frac{d^3p}{(2\pi)^3} e^{i\mathbf{p}\cdot\mathbf{x}} \phi(\mathbf{p}, t). \quad (3.4)$$

Now the Klein-Gordon equation in position space may be written

$$\left(\frac{\partial}{\partial t^2} - \nabla^2 + m^2 \right) \phi(\mathbf{x}) = 0, \quad (3.5)$$

so we can replace the field $\phi(\mathbf{x})$ in (3.5) with its momentum space expression (3.4) and write the KG-equation itself in momentum space by the following calculation

$$\begin{aligned} 0 &= \left(\frac{\partial}{\partial t^2} - \nabla^2 + m^2 \right) \phi(\mathbf{x}) \\ &= \left(\frac{\partial}{\partial t^2} - \nabla^2 + m^2 \right) \int \frac{d^3 p}{(2\pi)^3} e^{i\mathbf{p}\cdot\mathbf{x}} \phi(\mathbf{p}, t) \\ &= \int \frac{d^3 p}{(2\pi)^3} \left(\frac{\partial}{\partial t^2} - \nabla^2 + m^2 \right) e^{i\mathbf{p}\cdot\mathbf{x}} \phi(\mathbf{p}, t). \end{aligned}$$

And since $\nabla^2 e^{i\mathbf{p}\cdot\mathbf{x}} = (i\mathbf{p} \cdot i\mathbf{p}) e^{i\mathbf{p}\cdot\mathbf{x}} = -|\mathbf{p}|^2 e^{i\mathbf{p}\cdot\mathbf{x}}$ we can plug this into our expression to obtain

$$0 = \int \frac{d^3 p}{(2\pi)^3} \left(\frac{\partial}{\partial t^2} + |\mathbf{p}|^2 + m^2 \right) e^{i\mathbf{p}\cdot\mathbf{x}} \phi(\mathbf{p}, t).$$

This of course yields our result, the Klein-Gordon equation in momentum space

$$\left(\frac{\partial}{\partial t^2} + |\mathbf{p}|^2 + m^2 \right) \phi(\mathbf{p}, t) = 0. \quad (3.6)$$

Since ϕ is real we must have solutions of the form

$$\phi(\mathbf{p}, t) \propto a_{\mathbf{p}} e^{i\omega_{\mathbf{p}} t} \quad \text{or} \quad \phi(\mathbf{p}, t) \propto a_{\mathbf{p}}^\dagger e^{-i\omega_{\mathbf{p}} t}$$

where the frequency $\omega_{\mathbf{p}}$ is given by

$$\omega_{\mathbf{p}} = \sqrt{|\mathbf{p}|^2 + m^2}. \quad (3.7)$$

This takes the same form as the quantum harmonic oscillator (see [3] for a discussion of the quantum harmonic oscillator.) In this way we may think of each mode of the Klein-Gordon field as a harmonic oscillator with its own raising and lowering operators ($a_{\mathbf{p}}^\dagger$ and $a_{\mathbf{p}}$, respectively). For the field $\phi(\mathbf{x})$ and its corresponding conjugate momentum density $\pi(\mathbf{x})$ we have the following mode expansions

$$\phi(\mathbf{x}) = \int \frac{d^3 p}{(2\pi)^3} \frac{1}{\sqrt{2\omega_{\mathbf{p}}}} \left(a_{\mathbf{p}} e^{i\mathbf{p}\cdot\mathbf{x}} + a_{\mathbf{p}}^\dagger e^{-i\mathbf{p}\cdot\mathbf{x}} \right) \quad (3.8)$$

$$\pi(\mathbf{x}) = \int \frac{d^3 p}{(2\pi)^3} (-i) \sqrt{\frac{\omega_{\mathbf{p}}}{2}} \left(a_{\mathbf{p}} e^{i\mathbf{p}\cdot\mathbf{x}} - a_{\mathbf{p}}^\dagger e^{-i\mathbf{p}\cdot\mathbf{x}} \right), \quad (3.9)$$

and if we allow \mathbf{p} to take on negative values we obtain a more useful form for ϕ and π

$$\phi(\mathbf{x}) = \int \frac{d^3p}{(2\pi)^3} \frac{1}{\sqrt{2\omega_{\mathbf{p}}}} (a_{\mathbf{p}} + a_{-\mathbf{p}}^\dagger) e^{i\mathbf{p}\cdot\mathbf{x}} \quad (3.10)$$

$$\pi(\mathbf{x}) = \int \frac{d^3p}{(2\pi)^3} (-i) \sqrt{\frac{\omega_{\mathbf{p}}}{2}} (a_{\mathbf{p}} - a_{-\mathbf{p}}^\dagger) e^{i\mathbf{p}\cdot\mathbf{x}}. \quad (3.11)$$

The initial commutation relations for ϕ and π are actually equivalent to the commutation relation between $a_{\mathbf{p}}$ and $a_{\mathbf{q}}^\dagger$

$$[a_{\mathbf{p}}, a_{\mathbf{q}}^\dagger] = (2\pi)^3 \delta^3(\mathbf{p} - \mathbf{q}) \quad (3.12)$$

as is shown by the following calculation:

$$\begin{aligned} [\phi(\mathbf{x}), \pi(\mathbf{y})] &= \phi(\mathbf{x})\pi(\mathbf{y}) - \pi(\mathbf{y})\phi(\mathbf{x}) \\ &= \int \frac{d^3p}{(2\pi)^3} \frac{1}{\sqrt{2\omega_{\mathbf{p}}}} (a_{\mathbf{p}} + a_{-\mathbf{p}}^\dagger) e^{i(\mathbf{p}\cdot\mathbf{x})} \int \frac{d^3q}{(2\pi)^3} (-i) \sqrt{\frac{\omega_{\mathbf{q}}}{2}} (a_{\mathbf{q}} - a_{-\mathbf{q}}^\dagger) e^{i(\mathbf{q}\cdot\mathbf{y})} \\ &\quad - \int \frac{d^3q}{(2\pi)^3} (-i) \sqrt{\frac{\omega_{\mathbf{q}}}{2}} (a_{\mathbf{q}} - a_{-\mathbf{q}}^\dagger) e^{i(\mathbf{q}\cdot\mathbf{y})} \int \frac{d^3p}{(2\pi)^3} \frac{1}{\sqrt{2\omega_{\mathbf{p}}}} (a_{\mathbf{p}} + a_{-\mathbf{p}}^\dagger) e^{i(\mathbf{p}\cdot\mathbf{x})} \\ &= \int \frac{d^3pd^3q}{(2\pi)^6} \left(\frac{-i}{2}\right) \sqrt{\frac{\omega_{\mathbf{q}}}{\omega_{\mathbf{p}}}} [(a_{\mathbf{p}} + a_{-\mathbf{p}}^\dagger)(a_{\mathbf{q}} - a_{-\mathbf{q}}^\dagger)] e^{i(\mathbf{p}\cdot\mathbf{x} + \mathbf{q}\cdot\mathbf{y})} \\ &\quad - \int \frac{d^3pd^3q}{(2\pi)^6} \left(\frac{-i}{2}\right) \sqrt{\frac{\omega_{\mathbf{q}}}{\omega_{\mathbf{p}}}} [(a_{\mathbf{q}} - a_{-\mathbf{q}}^\dagger)(a_{\mathbf{p}} + a_{-\mathbf{p}}^\dagger)] e^{i(\mathbf{p}\cdot\mathbf{x} + \mathbf{q}\cdot\mathbf{y})} \\ &= \int \frac{d^3pd^3q}{(2\pi)^6} \frac{-i}{2} \sqrt{\frac{\omega_{\mathbf{q}}}{\omega_{\mathbf{p}}}} [(a_{\mathbf{p}} + a_{-\mathbf{p}}^\dagger)(a_{\mathbf{q}} - a_{-\mathbf{q}}^\dagger) - (a_{\mathbf{q}} - a_{-\mathbf{q}}^\dagger)(a_{\mathbf{p}} + a_{-\mathbf{p}}^\dagger)] e^{i(\mathbf{p}\cdot\mathbf{x} + \mathbf{q}\cdot\mathbf{y})}. \end{aligned}$$

The product of creation and annihilation operators in the integrand may be simplified to a sum of commutators:

$$\begin{aligned} (a_{\mathbf{p}} + a_{-\mathbf{p}}^\dagger)(a_{\mathbf{q}} - a_{-\mathbf{q}}^\dagger) - (a_{\mathbf{q}} - a_{-\mathbf{q}}^\dagger)(a_{\mathbf{p}} + a_{-\mathbf{p}}^\dagger) &= [a_{\mathbf{p}}, a_{\mathbf{q}}] + [a_{-\mathbf{q}}, a_{\mathbf{p}}] + [a_{-\mathbf{p}}^\dagger, a_{\mathbf{q}}] + [a_{-\mathbf{q}}^\dagger, a_{-\mathbf{p}}^\dagger] \\ &= 0 - (2\pi)^3 \delta^3(\mathbf{p} + \mathbf{q}) - (2\pi)^3 \delta^3(\mathbf{q} + \mathbf{p}) + 0 \\ &= -2(2\pi)^3 \delta^3(\mathbf{p} + \mathbf{q}). \end{aligned}$$

Inserting this result into the above expression we obtain the desired result

$$[\phi(\mathbf{x}), \pi(\mathbf{y})] = \int \frac{d^3pd^3q}{(2\pi)^6} \left(\frac{-i}{2}\right) \sqrt{\frac{\omega_{\mathbf{q}}}{\omega_{\mathbf{p}}}} [-2(2\pi)^3 \delta^3(\mathbf{p} + \mathbf{q})] e^{i(\mathbf{p}\cdot\mathbf{x} + \mathbf{q}\cdot\mathbf{y})} \quad (3.13)$$

$$= \int \frac{d^3p}{(2\pi)^3} (i) e^{i\mathbf{p}\cdot(\mathbf{x}-\mathbf{y})} \quad (3.14)$$

$$= i\delta^3(\mathbf{x} - \mathbf{y}). \quad (3.15)$$

3.1 Obtaining the Hamiltonian and Momentum

Next we calculate the Hamiltonian of the system. Recall that

$$H = \int \mathcal{H} d^3x = \int \left(\frac{1}{2}\pi^2 + \frac{1}{2}(\nabla\phi)^2 + \frac{1}{2}m^2\phi^2 \right) d^3x, \quad (3.16)$$

so to express H in terms of a and a^\dagger we first calculate the individual terms of \mathcal{H} . For the first term we have

$$\pi^2 = \int \frac{d^3p}{(2\pi)^3} (-i) \sqrt{\frac{\omega_{\mathbf{p}}}{2}} (a_{\mathbf{p}} - a_{-\mathbf{p}}^\dagger) e^{i\mathbf{p}\cdot\mathbf{x}} \int \frac{d^3q}{(2\pi)^3} (-i) \sqrt{\frac{\omega_{\mathbf{q}}}{2}} (a_{\mathbf{q}} - a_{-\mathbf{q}}^\dagger) e^{i\mathbf{q}\cdot\mathbf{x}} \quad (3.17)$$

$$= \int \frac{d^3p d^3q}{(2\pi)^6} \frac{-\sqrt{\omega_{\mathbf{p}}\omega_{\mathbf{q}}}}{2} (a_{\mathbf{p}} - a_{-\mathbf{p}}^\dagger)(a_{\mathbf{q}} - a_{-\mathbf{q}}^\dagger) e^{i(\mathbf{p}+\mathbf{q})\cdot\mathbf{x}}. \quad (3.18)$$

For the second term we have

$$\nabla\phi = \nabla \int \frac{d^3p}{(2\pi)^3} \frac{1}{\sqrt{2\omega_{\mathbf{p}}}} (a_{\mathbf{p}} + a_{-\mathbf{p}}^\dagger) e^{i\mathbf{p}\cdot\mathbf{x}} \quad (3.19)$$

$$= \int \frac{d^3p}{(2\pi)^3} \frac{1}{\sqrt{2\omega_{\mathbf{p}}}} (a_{\mathbf{p}} + a_{-\mathbf{p}}^\dagger) [\nabla e^{i\mathbf{p}\cdot\mathbf{x}}] \quad (3.20)$$

$$= \int \frac{d^3p}{(2\pi)^3} \frac{1}{\sqrt{2\omega_{\mathbf{p}}}} (a_{\mathbf{p}} + a_{-\mathbf{p}}^\dagger) (i\mathbf{p}) e^{i\mathbf{p}\cdot\mathbf{x}} \quad (3.21)$$

which, when squared, becomes

$$(\nabla\phi)^2 = \int \frac{d^3p}{(2\pi)^3} \frac{1}{\sqrt{2\omega_{\mathbf{p}}}} (a_{\mathbf{p}} + a_{-\mathbf{p}}^\dagger) (i\mathbf{p}) e^{i\mathbf{p}\cdot\mathbf{x}} \int \frac{d^3q}{(2\pi)^3} \frac{1}{\sqrt{2\omega_{\mathbf{q}}}} (a_{\mathbf{q}} + a_{-\mathbf{q}}^\dagger) (i\mathbf{q}) e^{i\mathbf{q}\cdot\mathbf{x}} \quad (3.22)$$

$$= \int \frac{d^3p d^3q}{(2\pi)^6} \frac{1}{\sqrt{2\omega_{\mathbf{p}}\omega_{\mathbf{q}}}} (a_{\mathbf{p}} + a_{-\mathbf{p}}^\dagger)(a_{\mathbf{q}} + a_{-\mathbf{q}}^\dagger) (-\mathbf{p}\cdot\mathbf{q}) e^{i(\mathbf{p}+\mathbf{q})\cdot\mathbf{x}}. \quad (3.23)$$

The last term takes the same form as $(\nabla\phi)^2$ but with no $-(\mathbf{p}\cdot\mathbf{q})$

$$\phi^2 = \int \frac{d^3p d^3q}{(2\pi)^6} \frac{1}{\sqrt{2\omega_{\mathbf{p}}\omega_{\mathbf{q}}}} (a_{\mathbf{p}} + a_{-\mathbf{p}}^\dagger)(a_{\mathbf{q}} + a_{-\mathbf{q}}^\dagger) e^{i(\mathbf{p}+\mathbf{q})\cdot\mathbf{x}}. \quad (3.24)$$

Now that we have the form for each of the terms of \mathcal{H} in terms of creation and annihilation operators we can finally integrate to obtain a form for H (cf. [4], eq. 2.31).

$$H = \int d^3x \int \frac{d^3p d^3q}{(2\pi)^6} e^{i(\mathbf{p}+\mathbf{q})\cdot\mathbf{x}} \times \left(\frac{-\sqrt{\omega_{\mathbf{p}}\omega_{\mathbf{q}}}}{4} (a_{\mathbf{p}} - a_{-\mathbf{p}}^\dagger)(a_{\mathbf{q}} - a_{-\mathbf{q}}^\dagger) + \frac{-\mathbf{p}\cdot\mathbf{q} + m^2}{4\sqrt{\omega_{\mathbf{p}}\omega_{\mathbf{q}}}} (a_{\mathbf{p}} + a_{-\mathbf{p}}^\dagger)(a_{\mathbf{q}} + a_{-\mathbf{q}}^\dagger) \right).$$

We first evaluate the d^3x integral to obtain a $\delta^3(\mathbf{p} + \mathbf{q})$ which will enforce $\mathbf{q} = -\mathbf{p}$ upon integration over \mathbf{q} .

$$H = \int \frac{d^3p}{(2\pi)^3} \left[\frac{-\sqrt{\omega_{\mathbf{p}}\omega_{-\mathbf{p}}}}{4} (a_{\mathbf{p}} - a_{-\mathbf{p}}^\dagger)(a_{-\mathbf{p}} - a_{\mathbf{p}}^\dagger) + \frac{-\mathbf{p} \cdot (-\mathbf{p}) + m^2}{4\sqrt{\omega_{\mathbf{p}}\omega_{-\mathbf{p}}}} (a_{\mathbf{p}} + a_{\mathbf{p}}^\dagger)(a_{-\mathbf{p}} + a_{\mathbf{p}}^\dagger) \right] \quad (3.25)$$

$$= \int \frac{d^3p}{(2\pi)^3} \left(\frac{\omega_{\mathbf{p}}}{4} \right) \left(-a_{\mathbf{p}}a_{-\mathbf{p}} + a_{\mathbf{p}}a_{\mathbf{p}}^\dagger + a_{-\mathbf{p}}^\dagger a_{-\mathbf{p}} - a_{-\mathbf{p}}^\dagger a_{\mathbf{p}}^\dagger + a_{\mathbf{p}}a_{-\mathbf{p}} + a_{\mathbf{p}}a_{\mathbf{p}}^\dagger + a_{-\mathbf{p}}^\dagger a_{-\mathbf{p}} + a_{-\mathbf{p}}^\dagger a_{\mathbf{p}}^\dagger \right) \quad (3.26)$$

$$= \int \frac{d^3p}{(2\pi)^3} \left(\frac{\omega_{\mathbf{p}}}{4} \right) \left(2a_{\mathbf{p}}a_{\mathbf{p}}^\dagger + 2a_{-\mathbf{p}}^\dagger a_{-\mathbf{p}} \right) \quad (3.27)$$

$$= \int \frac{d^3p}{(2\pi)^3} \left(\frac{\omega_{\mathbf{p}}}{4} \right) \left(2a_{\mathbf{p}}a_{\mathbf{p}}^\dagger + 2a_{-\mathbf{p}}^\dagger a_{-\mathbf{p}} + 2a_{\mathbf{p}}^\dagger a_{\mathbf{p}} - 2a_{\mathbf{p}}^\dagger a_{\mathbf{p}} \right) \quad (3.28)$$

$$= \int \frac{d^3p}{(2\pi)^3} \left(\frac{\omega_{\mathbf{p}}}{4} \right) \left(4a_{\mathbf{p}}^\dagger a_{\mathbf{p}} + 2[a_{\mathbf{p}}, a_{\mathbf{p}}^\dagger] \right) \quad (3.29)$$

$$= \int \frac{d^3p}{(2\pi)^3} \omega_{\mathbf{p}} \left(a_{\mathbf{p}}^\dagger a_{\mathbf{p}} + \frac{1}{2}[a_{\mathbf{p}}, a_{\mathbf{p}}^\dagger] \right). \quad (3.30)$$

The $[a_{\mathbf{p}}, a_{\mathbf{p}}^\dagger]$ term is the *zero-point energy*, a c-number which is immeasurable. It can be eliminated if we redefine the Hamiltonian with *normal ordering*. For any product of creation and annihilation operators $a_{\mathbf{p}}$ and $a_{\mathbf{p}}^\dagger$ the normal ordering operation takes all the $a_{\mathbf{p}}^\dagger$'s and puts them on the left side of the product leaving all the $a_{\mathbf{p}}$'s on the right side (standard commutation rules apply.) The operation is denoted with a pair of colons on either side of the product. As an example $: a_{\mathbf{p}}a_{\mathbf{p}}^\dagger := a_{\mathbf{p}}^\dagger a_{\mathbf{p}}$. Hence if we define our Hamiltonian density by

$$\mathcal{H} = : \frac{1}{2}\pi^2 + \frac{1}{2}(\nabla\phi)^2 + \frac{1}{2}m^2\phi^2 : . \quad (3.31)$$

then we obtain the result we need

$$H = \int \frac{d^3p}{(2\pi)^3} \omega_{\mathbf{p}} a_{\mathbf{p}}^\dagger a_{\mathbf{p}}. \quad (3.32)$$

Next we follow a similar process to express the (total) momentum operator in terms of creation and annihilation operators (cf. [4], eq. 2.33). From the momentum-energy tensor ([5], sec. 3.2.1) the total momentum operator is defined as

$$\mathbf{P} = - \int d^3x \pi(\mathbf{x}) \nabla \phi(\mathbf{x}). \quad (3.33)$$

Substituting our expressions for π and $\nabla\phi$ we obtain

$$\begin{aligned}
\mathbf{P} &= - \int d^3x \int \frac{d^3p}{(2\pi)^3} (-i) \sqrt{\frac{\omega_{\mathbf{p}}}{2}} (a_{\mathbf{p}} - a_{-\mathbf{p}}^\dagger) e^{i\mathbf{p}\cdot\mathbf{x}} \int \frac{d^3q}{(2\pi)^3} \frac{1}{\sqrt{2\omega_{\mathbf{q}}}} (a_{\mathbf{q}} + a_{-\mathbf{q}}^\dagger) (i\mathbf{p}) e^{i\mathbf{q}\cdot\mathbf{x}} \\
&= - \int \frac{d^3p d^3q}{(2\pi)^6} \left(\frac{1}{2}\right) \sqrt{\frac{\omega_{\mathbf{p}}}{\omega_{\mathbf{q}}}} (a_{\mathbf{p}} - a_{-\mathbf{p}}^\dagger) (a_{\mathbf{q}} + a_{-\mathbf{q}}^\dagger) \mathbf{q} \int d^3x e^{i(\mathbf{p}+\mathbf{q})\cdot\mathbf{x}} \\
&= - \int \frac{d^3p d^3q}{(2\pi)^6} \left(\frac{1}{2}\right) \sqrt{\frac{\omega_{\mathbf{p}}}{\omega_{\mathbf{q}}}} (a_{\mathbf{p}} - a_{-\mathbf{p}}^\dagger) (a_{\mathbf{q}} + a_{-\mathbf{q}}^\dagger) \mathbf{q} (2\pi)^3 \delta^3(\mathbf{p} + \mathbf{q}),
\end{aligned}$$

and upon integration over q the δ -function enforces $\mathbf{q} = -\mathbf{p}$ so we are left with

$$\begin{aligned}
\mathbf{P} &= - \int \frac{d^3p}{(2\pi)^3} \left(\frac{1}{2}\right) \sqrt{\frac{\omega_{\mathbf{p}}}{\omega_{-\mathbf{p}}}} (a_{\mathbf{p}} - a_{-\mathbf{p}}^\dagger) (a_{-\mathbf{p}} + a_{\mathbf{p}}^\dagger) (-\mathbf{p}) \\
&= \int \frac{d^3p}{(2\pi)^3} \frac{1}{2} \mathbf{P} (a_{\mathbf{p}} a_{-\mathbf{p}} + a_{\mathbf{p}} a_{\mathbf{p}}^\dagger - a_{-\mathbf{p}}^\dagger a_{-\mathbf{p}} - a_{-\mathbf{p}}^\dagger a_{\mathbf{p}}^\dagger).
\end{aligned}$$

The first and last terms vanish since they are odd in \mathbf{p} , and if we redefine \mathbf{P} by a normal ordering of the integrand as was the case with \mathcal{H} we are left with our result (cf. [4], eq. 2.33)

$$\mathbf{P} = \int \frac{d^3p}{(2\pi)^3} \mathbf{P} a_{\mathbf{p}}^\dagger a_{\mathbf{p}}. \tag{3.34}$$

3.2 Constructing the Fock Space and Evaluating the Energy and Momentum of its Elements

Now that we have our Hamiltonian H and momentum \mathbf{P} operators expressed in terms of $a_{\mathbf{p}}$ and $a_{\mathbf{p}}^\dagger$ we proceed to construct the mathematical space on which these operators will act. The resulting elements of this space will be interpreted as free, non-interacting multi-particle states.

Definition 1. *The vacuum state, denoted by $|0\rangle$, is the state that is destroyed when acted upon by the annihilation operator. Formally, for all \mathbf{p} we have*

$$a_{\mathbf{p}}|0\rangle = 0. \tag{3.35}$$

This state is normalized by $\langle 0|0\rangle = 1$. Upon hermitian conjugation of (3.35) we clearly also have $\langle 0|a_{\mathbf{p}}^\dagger = 0$. The elements of the space which will carry physical significance are those obtained when the creation operator acts on $|0\rangle$.

Definition 2. A one-particle state $|\mathbf{p}\rangle$ is defined by

$$|\mathbf{p}\rangle = \sqrt{2E_{\mathbf{p}}} a_{\mathbf{p}}^{\dagger} |0\rangle. \quad (3.36)$$

We may interpret $|\mathbf{p}\rangle$ as a single relativistic particle propagating through space with momentum \mathbf{p} . We can define an inner product on the space by

$$\begin{aligned} \langle \mathbf{p} | \mathbf{q} \rangle &= \langle 0 | a_{\mathbf{p}} (2E_{\mathbf{p}})^{1/2} (2E_{\mathbf{q}})^{1/2} a_{\mathbf{q}}^{\dagger} |0\rangle \\ &= (2E_{\mathbf{p}})^{1/2} (2E_{\mathbf{q}})^{1/2} \langle 0 | a_{\mathbf{p}} a_{\mathbf{q}}^{\dagger} - a_{\mathbf{q}}^{\dagger} a_{\mathbf{p}} |0\rangle \\ &= (2E_{\mathbf{p}})^{1/2} (2E_{\mathbf{q}})^{1/2} \langle 0 | [a_{\mathbf{p}}, a_{\mathbf{q}}^{\dagger}] |0\rangle \\ &= (2E_{\mathbf{p}})^{1/2} (2E_{\mathbf{q}})^{1/2} \langle 0 | (2\pi)^3 \delta^3(\mathbf{p} - \mathbf{q}) |0\rangle \\ &= 2E_{\mathbf{p}} (2\pi)^3 \delta^3(\mathbf{p} - \mathbf{q}). \end{aligned}$$

From this we can define the notion of a multi-particle state by ([5], eq. 4.9)

$$|\mathbf{p}_1, \mathbf{p}_2, \dots, \mathbf{p}_n\rangle \equiv (2E_{\mathbf{p}_1})^{1/2} (2E_{\mathbf{p}_2})^{1/2} \dots (2E_{\mathbf{p}_n})^{1/2} a_{\mathbf{p}_1}^{\dagger} a_{\mathbf{p}_2}^{\dagger} \dots a_{\mathbf{p}_n}^{\dagger} |0\rangle. \quad (3.37)$$

Now that we have the elements of our space constructed we proceed to operate on them with our energy and momentum operators, H and \mathbf{P} . First note that, by definition of the vacuum state, $H|0\rangle = 0$ and $\mathbf{P}|0\rangle = 0$

$$H|0\rangle = \int \frac{d^3p}{(2\pi)^3} E_{\mathbf{p}} a_{\mathbf{p}}^{\dagger} a_{\mathbf{p}} |0\rangle = 0 \quad (3.38)$$

$$\mathbf{P}|0\rangle = \int \frac{d^3p}{(2\pi)^3} \mathbf{p} a_{\mathbf{p}}^{\dagger} a_{\mathbf{p}} |0\rangle = 0. \quad (3.39)$$

For a one-particle state we compute the energy and momentum as follows. First from H and

(3.36) we have

$$\begin{aligned}
H|\mathbf{p}\rangle &= \int \frac{d^3q}{(2\pi)^3} E_{\mathbf{q}} a_{\mathbf{q}}^\dagger a_{\mathbf{q}} \sqrt{2E_{\mathbf{p}}} a_{\mathbf{p}}^\dagger |0\rangle \\
&= \sqrt{2E_{\mathbf{p}}} \int \frac{d^3q}{(2\pi)^3} E_{\mathbf{q}} a_{\mathbf{q}}^\dagger (a_{\mathbf{q}} a_{\mathbf{p}}^\dagger - a_{\mathbf{p}}^\dagger a_{\mathbf{q}}) |0\rangle \\
&= \sqrt{2E_{\mathbf{p}}} \int \frac{d^3q}{(2\pi)^3} E_{\mathbf{q}} a_{\mathbf{q}}^\dagger [a_{\mathbf{q}}, a_{\mathbf{p}}^\dagger] |0\rangle \\
&= \sqrt{2E_{\mathbf{p}}} \int \frac{d^3q}{(2\pi)^3} E_{\mathbf{q}} a_{\mathbf{q}}^\dagger (2\pi)^3 \delta^3(\mathbf{q} - \mathbf{p}) |0\rangle \\
&= E_{\mathbf{p}} \sqrt{2E_{\mathbf{p}}} a_{\mathbf{p}}^\dagger |0\rangle \\
&= E_{\mathbf{p}} |\mathbf{p}\rangle.
\end{aligned}$$

Similarly for momentum we have

$$\begin{aligned}
\mathbf{P}|\mathbf{p}\rangle &= \int \frac{d^3q}{(2\pi)^3} \mathbf{q} a_{\mathbf{q}}^\dagger a_{\mathbf{q}} \sqrt{2E_{\mathbf{p}}} a_{\mathbf{p}}^\dagger |0\rangle \\
&= \sqrt{2E_{\mathbf{p}}} \int \frac{d^3q}{(2\pi)^3} \mathbf{q} a_{\mathbf{q}}^\dagger (a_{\mathbf{q}} a_{\mathbf{p}}^\dagger - a_{\mathbf{p}}^\dagger a_{\mathbf{q}}) |0\rangle \\
&= \sqrt{2E_{\mathbf{p}}} \int \frac{d^3q}{(2\pi)^3} \mathbf{q} a_{\mathbf{q}}^\dagger [a_{\mathbf{q}}, a_{\mathbf{p}}^\dagger] |0\rangle \\
&= \sqrt{2E_{\mathbf{p}}} \int \frac{d^3q}{(2\pi)^3} \mathbf{q} a_{\mathbf{q}}^\dagger (2\pi)^3 \delta^3(\mathbf{q} - \mathbf{p}) |0\rangle \\
&= \mathbf{p} \sqrt{2E_{\mathbf{p}}} a_{\mathbf{p}}^\dagger |0\rangle \\
&= \mathbf{p} |\mathbf{p}\rangle.
\end{aligned}$$

We can also compute the energy and momentum for a two-particle state $|\mathbf{p}_1, \mathbf{p}_2\rangle$.

$$\begin{aligned}
H|\mathbf{p}_1, \mathbf{p}_2\rangle &= \int \frac{d^3q}{(2\pi)^3} E_{\mathbf{q}} a_{\mathbf{q}}^\dagger a_{\mathbf{q}} \left[(2E_1)^{1/2} (2E_2)^{1/2} a_{\mathbf{p}_1}^\dagger a_{\mathbf{p}_2}^\dagger |0\rangle \right] \\
&= (2E_1)^{1/2} (2E_2)^{1/2} \int \frac{d^3q}{(2\pi)^3} E_{\mathbf{q}} a_{\mathbf{q}}^\dagger [a_{\mathbf{q}}, a_{\mathbf{p}_1}^\dagger a_{\mathbf{p}_2}^\dagger] |0\rangle.
\end{aligned}$$

Note that $[A, BC] = [A, B]C + B[A, C]$ so that the commutator above reduces to

$$(2E_1)^{1/2} (2E_2)^{1/2} \int \frac{d^3q}{(2\pi)^3} E_{\mathbf{q}} a_{\mathbf{q}}^\dagger \left([a_{\mathbf{q}}, a_{\mathbf{p}_1}^\dagger] a_{\mathbf{p}_2}^\dagger + a_{\mathbf{p}_1}^\dagger [a_{\mathbf{q}}, a_{\mathbf{p}_2}^\dagger] \right) |0\rangle. \quad (3.40)$$

Once we have integrated out the δ -functions we are left with

$$(2E_1)^{1/2} (2E_2)^{1/2} \left(E_1 a_{\mathbf{p}_1}^\dagger a_{\mathbf{p}_2}^\dagger + E_2 a_{\mathbf{p}_2}^\dagger a_{\mathbf{p}_1}^\dagger \right) |0\rangle. \quad (3.41)$$

For our result we obtain $H|\mathbf{p}_1, \mathbf{p}_2\rangle = (E_1 + E_2)|\mathbf{p}_1, \mathbf{p}_2\rangle$. As expected, the Hamiltonian acting on a two particle state gives us the sum of the energies of the particles. A similar result holds for \mathbf{P} .

To compute the energy or momentum for a more general state we first need the following lemma for the commutator of a product of operators. The basic idea is that the commutator “works its way through” the product with different terms in a sum. For example, $[A, B_1B_2B_3B_4]$ simplifies to

$$[A, B_1B_2B_3B_4] = [A, B_1]B_2B_3B_4 + B_1[A, B_2]B_3B_4 + B_1B_2[A, B_3]B_4 + B_1B_2B_3[A, B_4].$$

Lemma 1. *Let A, B_1, B_2, \dots, B_n be arbitrary operators. Define $B_0 = B_{n+1} = \mathbf{1}$. Then for $n \geq 2$*

$$[A, B_1B_2 \cdots B_n] = \sum_{j=1}^n \left(\prod_{i=0}^{j-1} B_i[A, B_j] \prod_{k=j+1}^{n+1} B_k \right). \quad (3.42)$$

Proof. The proof is by induction. For $n = 2$ we have

$$[A, B_1B_2] = AB_1B_2 - B_1B_2A = AB_1B_2 - B_1AB_2 + B_1AB_2 - B_1B_2A = [A, B_1]B_2 + B_1[A, B_2]. \quad (3.43)$$

If we assume by inductive hypothesis that (3.42) holds for $[A, B_1B_2 \cdots B_n]$, then we can compute the commutator for $n + 1$

$$\begin{aligned} [A, B_1B_2 \cdots B_nB_{n+1}] &= \left[A, \left(\prod_{i=1}^n B_i \right) B_{n+1} \right] \\ &= A \left(\prod_{i=1}^n B_i \right) B_{n+1} - \left(\prod_{i=1}^n B_i \right) B_{n+1} A \\ &= A \left(\prod_{i=1}^n B_i \right) B_{n+1} - \left(\prod_{i=1}^n B_i \right) A B_{n+1} + \left(\prod_{i=1}^n B_i \right) A B_{n+1} - \left(\prod_{i=1}^n B_i \right) B_{n+1} A \\ &= \left[A, \prod_{i=1}^n B_i \right] B_{n+1} + \prod_{i=1}^n B_i [A, B_{n+1}] \\ &= \sum_{j=1}^n \left(\prod_{i=0}^{j-1} B_i [A, B_j] \prod_{k=j+1}^{n+1} B_k \right) B_{n+1} + \prod_{l=0}^n B_l [A, B_{n+1}] B_{n+2} \\ &= \sum_{j=1}^{n+1} \left(\prod_{i=0}^{j-1} B_i [A, B_j] \prod_{k=j+1}^{n+2} B_k \right). \end{aligned}$$

Thus the result holds for the case of $n + 1$ and our proof is complete. \square

Using this result we can compute the energy and momentum of a general multi-particle state:

Proposition 1.

$$H|\mathbf{p}_1, \mathbf{p}_2, \dots, \mathbf{p}_n\rangle = (E_{\mathbf{p}_1} + E_{\mathbf{p}_2} + \dots + E_{\mathbf{p}_n})|\mathbf{p}_1, \mathbf{p}_2, \dots, \mathbf{p}_n\rangle. \quad (3.44)$$

Proof.

$$\begin{aligned} H|\mathbf{p}_1, \mathbf{p}_2, \dots, \mathbf{p}_n\rangle &= \int \frac{d^3p}{(2\pi)^3} E_{\mathbf{p}} a_{\mathbf{p}}^\dagger a_{\mathbf{p}} \prod_{i=1}^n (2E_{\mathbf{p}_i})^{1/2} a_{\mathbf{p}_i}^\dagger |0\rangle \\ &= \left(\prod_{i=1}^n (2E_{\mathbf{p}_i})^{1/2} \right) \int \frac{d^3p}{(2\pi)^3} E_{\mathbf{p}} a_{\mathbf{p}}^\dagger \left(a_{\mathbf{p}} \prod_{j=1}^n a_{\mathbf{p}_j}^\dagger - \prod_{j=1}^n a_{\mathbf{p}_j}^\dagger a_{\mathbf{p}} \right) |0\rangle \\ &= \left(\prod_{i=1}^n (2E_{\mathbf{p}_i})^{1/2} \right) \int \frac{d^3p}{(2\pi)^3} E_{\mathbf{p}} a_{\mathbf{p}}^\dagger \left[a_{\mathbf{p}}, \prod_{j=1}^n a_{\mathbf{p}_j}^\dagger \right] |0\rangle. \end{aligned}$$

After applying the result of (3.42) we reduce the commutator to a sum of individual commutators of the form $[a_{\mathbf{p}}, a_{\mathbf{p}_i}^\dagger]$. From this we evaluate these commutators as a series of δ -functions and proceed to integrate them:

$$\begin{aligned} H|\mathbf{p}_1, \mathbf{p}_2, \dots, \mathbf{p}_n\rangle &= \left(\prod_{i=1}^n (2E_{\mathbf{p}_i})^{1/2} \right) \sum_{j=1}^n \int \frac{d^3p}{(2\pi)^3} E_{\mathbf{p}} a_{\mathbf{p}}^\dagger (2\pi)^3 \delta^3(\mathbf{p} - \mathbf{p}_j) \prod_{i \neq j} a_{\mathbf{p}_i}^\dagger |0\rangle \\ &= \left(\prod_{i=1}^n (2E_{\mathbf{p}_i})^{1/2} \right) \sum_{j=1}^n E_{\mathbf{p}_j} \prod_{i=1}^n a_{\mathbf{p}_i}^\dagger |0\rangle \\ &= \sum_{j=1}^n E_{\mathbf{p}_j} \prod_{i=1}^n (2E_{\mathbf{p}_i})^{1/2} a_{\mathbf{p}_i}^\dagger |0\rangle \\ &= (E_{\mathbf{p}_1} + E_{\mathbf{p}_2} + \dots + E_{\mathbf{p}_n})|\mathbf{p}_1, \mathbf{p}_2, \dots, \mathbf{p}_n\rangle, \end{aligned}$$

where in the second line we have used the fact that all creation operators commute with one another. Thus we have arrived at our result. \square

A similar result which we state without proof holds for the spatial momentum of a multi-particle state:

$$\mathbf{P}|\mathbf{p}_1, \mathbf{p}_2, \dots, \mathbf{p}_n\rangle = (\mathbf{p}_1 + \mathbf{p}_2 + \dots + \mathbf{p}_n)|\mathbf{p}_1, \mathbf{p}_2, \dots, \mathbf{p}_n\rangle. \quad (3.45)$$

We have started from the classical Klein-Gordon equation as derived from the invariant action and imposed canonical commutation relations between $\phi(\mathbf{x}, t)$ and $\pi(\mathbf{y}, t)$ at equal times. Switching to a momentum space representation, the classical KG-field was quantized by interpreting each mode of the field as a quantum harmonic oscillator with the associated creation and annihilation operators ($a_{\mathbf{p}}$ and $a_{\mathbf{p}}^\dagger$.) We then expressed the Hamiltonian H and momentum \mathbf{P} operators in terms of $a_{\mathbf{p}}$ and $a_{\mathbf{p}}^\dagger$. After constructing the Fock space by defining an element (with appropriate relativistic normalization) to be a product of creation operators acting on the initial “vacuum” element $|0\rangle$, we operated on these states with H and \mathbf{P} so that the appropriate interpretation of an arbitrary element of the Fock space as a multiparticle state with a definite energy and momentum may be applied. In this way we say that the (free) quantum field theory has been solved.

3.3 Obtaining a Time-Dependant Form of ϕ

An important physical quantity that will arise is the amplitude of a particle to propagate from spacetime points x to y . This process will be represented by

$$\langle 0|\phi(x)\phi(y)|0\rangle. \tag{3.46}$$

This is a *vacuum expectation value* or vev of a product of two fields. These vev’s are in effect what are actually measured in the laboratory. Mathematically, it can be shown by The Reconstruction Theorem [2] that the field can be recovered (or reconstructed) from the vacuum expectation values. In order to calculate this, the first task is to convert our KG-field

$$\phi(\mathbf{x}) = \int \frac{d^3x}{(2\pi)^3} \frac{1}{\sqrt{2E_{\mathbf{p}}}} \left(a_{\mathbf{p}} e^{i\mathbf{p}\cdot\mathbf{x}} + a_{\mathbf{p}}^\dagger e^{-i\mathbf{p}\cdot\mathbf{x}} \right) \tag{3.47}$$

into a form which contains x and p , not just the spatial parts \mathbf{x} and \mathbf{p} (four-vectors x containing temporal and spatial components are denoted in the usual way as $x = x_\mu = (x_0, -x_1, -x_2, -x_3)$.) The following discussion fills in the calculational details of the last paragraph of p. 25 in Section 2.4 of [4]. To compute the time-dependance of $\phi(\mathbf{x})$ we switch to the Heisenberg picture by defining the time-dependant field operator $\phi(x)$ in terms of the

time-independent $\phi(\mathbf{x})$

$$\phi(x) \equiv \phi(t, \mathbf{x}) = e^{iHt} \phi(\mathbf{x}) e^{-iHt} \quad (3.48)$$

$$= \int \frac{d^3p}{(2\pi)^3} \frac{1}{\sqrt{2E_{\mathbf{p}}}} \left(e^{iHt} a_{\mathbf{p}} e^{-iHt} e^{i\mathbf{p}\cdot\mathbf{x}} + e^{iHt} a_{\mathbf{p}}^\dagger e^{-iHt} e^{-i\mathbf{p}\cdot\mathbf{x}} \right). \quad (3.49)$$

Since promoting $\phi(\mathbf{x})$ to an operator essentially meant promoting $a_{\mathbf{p}}$ and $a_{\mathbf{p}}^\dagger$ to operators, the problem of determining the time-dependence of $\phi(\mathbf{x})$ ultimately rests on the time-dependence of $a_{\mathbf{p}}$ and $a_{\mathbf{p}}^\dagger$. To determine this we first evaluate the commutator $[H, a_{\mathbf{p}}]$ by the following calculation:

$$\begin{aligned} [H, a_{\mathbf{p}}] &= \left[\int \frac{d^3q}{(2\pi)^3} E_{\mathbf{q}} a_{\mathbf{q}}^\dagger a_{\mathbf{q}}, a_{\mathbf{p}} \right] \\ &= \int \frac{d^3q}{(2\pi)^3} E_{\mathbf{q}} [a_{\mathbf{q}}^\dagger a_{\mathbf{q}}, a_{\mathbf{p}}] \\ &= - \int \frac{d^3q}{(2\pi)^3} E_{\mathbf{q}} (2\pi)^3 \delta^3(\mathbf{p} - \mathbf{q}) a_{\mathbf{q}} \\ &= -E_{\mathbf{p}} a_{\mathbf{p}}. \end{aligned}$$

By a similar calculation $[H, a_{\mathbf{p}}^\dagger] = E_{\mathbf{p}} a_{\mathbf{p}}^\dagger$ so that we have the following two identities

$$H a_{\mathbf{p}} = a_{\mathbf{p}} (H - E_{\mathbf{p}}) \quad H a_{\mathbf{p}}^\dagger = a_{\mathbf{p}}^\dagger (H + E_{\mathbf{p}}). \quad (3.50)$$

To find the quantity $H^2 a_{\mathbf{p}}$ we compute $[H^2, a_{\mathbf{p}}]$ using the commutator $[H, a_{\mathbf{p}}]$ and obtain

$$\begin{aligned} [H^2, a_{\mathbf{p}}] &= H^2 a_{\mathbf{p}} - a_{\mathbf{p}} H^2 \\ &= H^2 a_{\mathbf{p}} - H a_{\mathbf{p}} H + H a_{\mathbf{p}} H - a_{\mathbf{p}} H^2 \\ &= H[H, a_{\mathbf{p}}] + [H, a_{\mathbf{p}}]H \end{aligned}$$

from this we can deduce the expression for $H^2 a_{\mathbf{p}}$ by letting

$$\begin{aligned} [H^2, a_{\mathbf{p}}] &= H[H, a_{\mathbf{p}}] + [H, a_{\mathbf{p}}]H \\ H^2 a_{\mathbf{p}} &= a_{\mathbf{p}} H^2 - H(E_{\mathbf{p}} a_{\mathbf{p}}) - (E_{\mathbf{p}} a_{\mathbf{p}})H \\ &= a_{\mathbf{p}} H^2 - E_{\mathbf{p}} a_{\mathbf{p}} (H - E_{\mathbf{p}}) - (E_{\mathbf{p}} a_{\mathbf{p}})H \\ &= a_{\mathbf{p}} H^2 - a_{\mathbf{p}} (2E_{\mathbf{p}} H) + a_{\mathbf{p}} E_{\mathbf{p}}^2 \\ &= a_{\mathbf{p}} (H - E_{\mathbf{p}})^2. \end{aligned}$$

Similarly for $a_{\mathbf{p}}^\dagger$ we have $H^2 a_{\mathbf{p}}^\dagger = a_{\mathbf{p}}^\dagger (H + E_{\mathbf{p}})^2$. Clearly the general terms take the form

$$H^n a_{\mathbf{p}} = a_{\mathbf{p}} (H - E_{\mathbf{p}})^n \quad H^n a_{\mathbf{p}}^\dagger = a_{\mathbf{p}}^\dagger (H + E_{\mathbf{p}})^n. \quad (3.51)$$

Using this general form we can then compute $e^{iHt} a_{\mathbf{p}} e^{-iHt}$ by expanding out the exponential on the left-hand side and applying (3.51);

$$\begin{aligned} e^{iHt} a_{\mathbf{p}} e^{-iHt} &= \left(1 + (iHt) + \frac{1}{2}(iHt)^2 + \dots \right) a_{\mathbf{p}} e^{-iHt} \\ &= \left(a_{\mathbf{p}} + iH a_{\mathbf{p}} t + \frac{1}{2}(-i)^2 H^2 a_{\mathbf{p}} t^2 + \dots \right) e^{-iHt} \\ &= \left(a_{\mathbf{p}} + i a_{\mathbf{p}} (H - E_{\mathbf{p}}) + \frac{1}{2}(-i)^2 a_{\mathbf{p}} (H - E_{\mathbf{p}})^2 t^2 + \dots \right) e^{-iHt} \\ &= a_{\mathbf{p}} e^{i(H - E_{\mathbf{p}})t} e^{-iHt} \\ &= a_{\mathbf{p}} e^{-iE_{\mathbf{p}}t}. \end{aligned}$$

And with a similar calculation for $a_{\mathbf{p}}^\dagger$ we obtain the time evolution of our creation and annihilation operators $a_{\mathbf{p}}$ and $a_{\mathbf{p}}^\dagger$ ([4], eq. 2.46)

$$e^{iHt} a_{\mathbf{p}} e^{-iHt} = a_{\mathbf{p}} e^{-iE_{\mathbf{p}}t} \quad e^{iHt} a_{\mathbf{p}}^\dagger e^{-iHt} = a_{\mathbf{p}}^\dagger e^{iE_{\mathbf{p}}t} \quad (3.52)$$

Plugging in our results of (3.52) into $\phi(x)$ we obtain its explicit form

$$\phi(x) = \int \frac{d^3p}{(2\pi)^3} \frac{1}{\sqrt{2E_{\mathbf{p}}}} \left(a_{\mathbf{p}} e^{-ip \cdot x} + a_{\mathbf{p}}^\dagger e^{ip \cdot x} \right). \quad (3.53)$$

Note also that the time evolution of the conjugate momentum field is defined in a similar manner by $\pi(x) = e^{iHt} \pi(\mathbf{x}) e^{-iHt}$ so that the same relation $\pi(x) = \partial^0 \phi(x)$ holds.

3.4 Quantizing the ABB and ABC-Theory

All of the previous development has been for only one real scalar field. The calculations in Chapter 5 will be for scalar field theories involving more than one particle. So for the ABC theory ([6], [7], [8]) we define the A-field, B-field, and C-field from the following commutation relations

$$[\phi_i(\mathbf{x}), \pi_j(\mathbf{y})] = i\delta^3(\mathbf{x} - \mathbf{y})\delta_{ij} \quad \text{for } i, j = A, B, C \quad (3.54)$$

$$[\phi_i(\mathbf{x}), \phi_j(\mathbf{y})] = [\pi_i(\mathbf{x}), \pi_j(\mathbf{y})] = 0, \quad (3.55)$$

where δ_{ij} is the Kronecker delta. The field operators are defined in an analagous way:

$$\phi_A(x) = \int \frac{d^3p}{(2\pi)^3} \frac{1}{\sqrt{2E_{\mathbf{p}}}} \left(a_{\mathbf{p}} e^{-ip \cdot x} + a_{\mathbf{p}}^\dagger e^{ip \cdot x} \right) \quad (3.56)$$

$$\phi_B(x) = \int \frac{d^3p}{(2\pi)^3} \frac{1}{\sqrt{2E_{\mathbf{p}}}} \left(b_{\mathbf{p}} e^{-ip \cdot x} + b_{\mathbf{p}}^\dagger e^{ip \cdot x} \right) \quad (3.57)$$

$$\phi_C(x) = \int \frac{d^3p}{(2\pi)^3} \frac{1}{\sqrt{2E_{\mathbf{p}}}} \left(c_{\mathbf{p}} e^{-ip \cdot x} + c_{\mathbf{p}}^\dagger e^{ip \cdot x} \right). \quad (3.58)$$

The Hamiltonian and momentum are defined similarly to (3.32) and (3.34) by

$$H = \int \frac{d^3p}{(2\pi)^3} \left(E_{\mathbf{p}A} a_{\mathbf{p}}^\dagger a_{\mathbf{p}} + E_{\mathbf{p}B} b_{\mathbf{p}}^\dagger b_{\mathbf{p}} + E_{\mathbf{p}C} c_{\mathbf{p}}^\dagger c_{\mathbf{p}} \right) \quad (3.59)$$

$$\mathbf{P} = \int \frac{d^3p}{(2\pi)^3} \left(\mathbf{p}A a_{\mathbf{p}}^\dagger a_{\mathbf{p}} + \mathbf{p}B b_{\mathbf{p}}^\dagger b_{\mathbf{p}} + \mathbf{p}C c_{\mathbf{p}}^\dagger c_{\mathbf{p}} \right), \quad (3.60)$$

and a general multi-particle state of l A-particles, m B-particles, and n C-particles can be written

$$\begin{aligned} & |\mathbf{p}_{A_1}, \dots, \mathbf{p}_{A_l}, \mathbf{p}_{B_1}, \dots, \mathbf{p}_{B_m}, \mathbf{p}_{C_1}, \dots, \mathbf{p}_{C_n} \rangle \\ &= (2E_{A_1})^{1/2} \dots (2E_{A_l})^{1/2} (2E_{B_1})^{1/2} \dots (2E_{B_m})^{1/2} (2E_{C_1})^{1/2} \dots (2E_{C_n})^{1/2} \\ & \quad \times a_{\mathbf{p}_1}^\dagger \dots a_{\mathbf{p}_l}^\dagger b_{\mathbf{p}_1}^\dagger \dots b_{\mathbf{p}_m}^\dagger c_{\mathbf{p}_1}^\dagger \dots c_{\mathbf{p}_n}^\dagger. \end{aligned}$$

All these results hold for the ABB-theory ([9]) as well with the obvious removal of the the C-particle and corresponding field/creation and annihilation operators. Note also that this is a framework for *non-interacting* field theories. Once we introduce an interaction between the fields we may not undertake the same quantization procedure. A perturbative approach must be used to obtain a probability amplitude for a multi-particle state to evolve to another as a power series expansion in g , the coupling between the fields (the strength of the interaction.)

3.5 Calculating $\langle 0|a_{\mathbf{p}}, \phi(x)|0 \rangle$

We will need the following result

$$\langle 0|a_{\mathbf{p}}\phi(x)|0 \rangle = \frac{1}{\sqrt{2E_{\mathbf{p}}}} e^{ip \cdot x}, \quad (3.61)$$

which we compute in detail below. First we express $\phi(x)$ in its mode expansion and take the integral out of the vev

$$\begin{aligned}\langle 0|a_{\mathbf{p}}\phi(x)|0\rangle &= \langle 0|a_{\mathbf{p}}\int\frac{d^3q}{(2\pi)^3}\frac{1}{\sqrt{2E_{\mathbf{q}}}}\left(a_{\mathbf{q}}e^{-iq\cdot x}+a_{\mathbf{q}}^\dagger e^{iq\cdot x}\right)|0\rangle \\ &= \int\frac{d^3q}{(2\pi)^3}\frac{1}{\sqrt{2E_{\mathbf{q}}}}\langle 0|a_{\mathbf{p}}\left(a_{\mathbf{q}}e^{-iq\cdot x}+a_{\mathbf{q}}^\dagger e^{iq\cdot x}\right)|0\rangle.\end{aligned}$$

Then forming two vev's while taking out the exponentials we obtain a sum of vev's of products of two creation and annihilation operators

$$\int\frac{d^3q}{(2\pi)^3}\frac{1}{\sqrt{2E_{\mathbf{q}}}}\left(\langle 0|a_{\mathbf{p}}a_{\mathbf{q}}|0\rangle e^{-iq\cdot x}+\langle 0|a_{\mathbf{q}}a_{\mathbf{p}}^\dagger|0\rangle e^{iq\cdot x}\right). \quad (3.62)$$

Note that the $a_{\mathbf{q}}|0\rangle$ will cause the first term to vanish by definition of operators acting on the vacuum state. The second term contains a vev seen in previous calculations which can be reduced to the commutator $[a_{\mathbf{p}}, a_{\mathbf{q}}^\dagger]$. Finishing the steps,

$$\int\frac{d^3q}{(2\pi)^3}\frac{1}{\sqrt{2E_{\mathbf{q}}}}\langle 0|[a_{\mathbf{p}}, a_{\mathbf{q}}^\dagger]|0\rangle e^{iq\cdot x}=\int\frac{d^3q}{(2\pi)^3}\frac{1}{\sqrt{2E_{\mathbf{q}}}}(2\pi)^3\delta^3(\mathbf{p}-\mathbf{q})e^{iq\cdot x}=\frac{1}{\sqrt{2E_{\mathbf{p}}}}e^{ip\cdot x}. \quad (3.63)$$

and applying the rule for integrating with δ -functions, we obtain our desired result (where $p\cdot x=E_{\mathbf{p}}t-\mathbf{p}\cdot\mathbf{x}$).

A similar result holds for $\phi(x)$ and a^\dagger

$$\langle 0|\phi(x)a_{\mathbf{p}}^\dagger|0\rangle=\frac{1}{\sqrt{2E_{\mathbf{p}}}}e^{-ip\cdot x}. \quad (3.64)$$

The calculation is similar to the one above, so we perform it without explanation. Refer to the above discussion for calculation details.

$$\begin{aligned}\langle 0|\phi(x), a_{\mathbf{p}}^\dagger|0\rangle &= \langle 0|\int\frac{d^4q}{(2\pi)^3}\frac{1}{\sqrt{2E_{\mathbf{q}}}}\left(a_{\mathbf{q}}e^{-iq\cdot x}+a_{\mathbf{q}}^\dagger e^{iq\cdot x}\right)a_{\mathbf{p}}^\dagger|0\rangle \\ &= \int\frac{d^4q}{(2\pi)^3}\frac{1}{\sqrt{2E_{\mathbf{q}}}}\left(\langle 0|a_{\mathbf{q}}a_{\mathbf{p}}^\dagger|0\rangle e^{-iq\cdot x}+\langle 0|a_{\mathbf{q}}^\dagger a_{\mathbf{p}}^\dagger|0\rangle e^{iq\cdot x}\right) \\ &= \int\frac{d^4q}{(2\pi)^3}\frac{1}{\sqrt{2E_{\mathbf{q}}}}\langle 0|[a_{\mathbf{q}}, a_{\mathbf{p}}^\dagger]|0\rangle e^{-iq\cdot x} \\ &= \int\frac{d^4q}{(2\pi)^3}\frac{1}{\sqrt{2E_{\mathbf{q}}}}(2\pi)^3\delta^3(\mathbf{q}-\mathbf{p})e^{-iq\cdot x} \\ &= \frac{1}{\sqrt{2E_{\mathbf{p}}}}e^{-ip\cdot x}.\end{aligned}$$

3.6 Derivation of the Feynman Propagator

The next reasonable calculation would be

$$\langle 0|\phi(x)\phi(y)|0\rangle, \quad (3.65)$$

and the following derivation of this quantity is found in [8], sec. 6.3.2. We first expand both of the field operators into their mode expansions and take the integrals out of the vev:

$$\begin{aligned} \langle 0|\phi(x)\phi(y)|0\rangle &= \langle 0|\int \frac{d^3p}{(2\pi)^3} \frac{1}{\sqrt{2E_{\mathbf{p}}}} \left(a_{\mathbf{p}}e^{-ip\cdot x} + a_{\mathbf{p}}^\dagger e^{ip\cdot x} \right) \\ &\quad \times \int \frac{d^3q}{(2\pi)^3} \frac{1}{\sqrt{2E_{\mathbf{q}}}} \left(a_{\mathbf{q}}e^{-iq\cdot y} + a_{\mathbf{q}}^\dagger e^{iq\cdot y} \right) |0\rangle \\ &= \iint \frac{d^3p}{(2\pi)^3} \frac{d^3q}{(2\pi)^3} \frac{1}{\sqrt{2E_{\mathbf{p}}2E_{\mathbf{q}}}} \langle 0|\left(a_{\mathbf{p}}e^{-ip\cdot x} + a_{\mathbf{p}}^\dagger e^{ip\cdot x} \right) \left(a_{\mathbf{q}}e^{-iq\cdot y} + a_{\mathbf{q}}^\dagger e^{iq\cdot y} \right) |0\rangle. \end{aligned}$$

The product in the vev is of the form $\langle 0|(a_{\mathbf{p}} + a_{\mathbf{p}}^\dagger)(a_{\mathbf{q}} + a_{\mathbf{q}}^\dagger)|0\rangle$ and can be reduced to $(2\pi)^3\delta^3(\mathbf{p} - \mathbf{q})$ in the following way:

$$\begin{aligned} \langle 0|\left(a_{\mathbf{p}}e^{-ip\cdot x} + a_{\mathbf{p}}^\dagger e^{ip\cdot x} \right) \left(a_{\mathbf{q}}e^{-iq\cdot y} + a_{\mathbf{q}}^\dagger e^{iq\cdot y} \right) |0\rangle \\ = \langle 0|a_{\mathbf{p}}a_{\mathbf{q}}|0\rangle e^{-iq\cdot x} e^{-iq\cdot y} + \langle 0|a_{\mathbf{p}}a_{\mathbf{q}}^\dagger|0\rangle e^{-ip\cdot x} e^{iq\cdot y} \\ + \langle 0|a_{\mathbf{p}}^\dagger a_{\mathbf{q}}|0\rangle e^{ip\cdot x} e^{iq\cdot y} + \langle 0|a_{\mathbf{p}}^\dagger a_{\mathbf{q}}^\dagger|0\rangle e^{ip\cdot x} e^{iq\cdot y} \quad (3.66) \end{aligned}$$

Every term except for the second on the right-hand side vanishes by the rules $a|0\rangle = 0$ or $0 = \langle 0|a^\dagger$. The second term itself can be reduced in the usual way to the commutation relation between the two operators:

$$\langle 0|a_{\mathbf{p}}a_{\mathbf{q}}^\dagger|0\rangle = (2\pi)^3\delta^3(\mathbf{p} - \mathbf{q}).$$

It then follows that the vev has been reduced to $(2\pi)^3\delta^3(\mathbf{p} - \mathbf{q})e^{-ip\cdot x}e^{iq\cdot y}$, so the amplitude becomes

$$\begin{aligned} \langle 0|\phi(x)\phi(y)|0\rangle &= \iint \frac{d^3p}{(2\pi)^3} \frac{d^3q}{(2\pi)^3} \frac{1}{\sqrt{2E_{\mathbf{p}}2E_{\mathbf{q}}}} \\ &\quad \times (2\pi)^3\delta^3(\mathbf{p} - \mathbf{q})e^{-ip\cdot x}e^{iq\cdot y}. \end{aligned} \quad (3.67)$$

We may now choose to evaluate either integral using the $\delta^3(\mathbf{p} - \mathbf{q})$, and here the d^3q integration is performed. We are left with

$$\langle 0|\phi(x)\phi(y)|0\rangle = \int \frac{d^3p}{(2\pi)^3} \frac{1}{2E_{\mathbf{p}}} e^{-ip(x-y)}. \quad (3.68)$$

Also, to this point we had been ignoring the time ordering of these two fields

$$\begin{aligned} \langle 0|\phi(x)\phi(y)|0\rangle &= \int \frac{d^3p}{(2\pi)^3} \frac{1}{2E_{\mathbf{p}}} e^{-ip(x-y)} \\ &= \int \frac{d^3p}{(2\pi)^3} \frac{1}{2\omega_{\mathbf{p}}} \\ &\quad \left[\theta(t_1 - t_2) e^{-i\omega_{\mathbf{p}}(t_1 - t_2)} e^{-i\mathbf{p}\cdot(x-y)} \right. \\ &\quad \left. + \theta(t_2 - t_1) e^{-i\omega_{\mathbf{p}}(t_2 - t_1)} e^{-i\mathbf{p}\cdot(y-x)} \right], \end{aligned} \quad (3.69)$$

where we have replaced the E_k with the ω_k notation and separated the exponentials into their respective temporal and spatial coordinates. This form is guaranteed to be covariant by the left-hand side, but the right-hand side does not appear to be so, since the integral is over 3-space and would leave only a function of t which would seem to be non-covariant. However, the key to expressing this integral into a more convincing form is to turn it into a four-fold integral. We do this by introducing an integral representation of $\theta(t)$:

$$\theta(t) = i \int_{-\infty}^{\infty} \frac{dz}{2\pi} \frac{e^{-izt}}{z + i\epsilon}. \quad (3.70)$$

Multiplying $\theta(t)$ by $e^{-i\omega t}$ we get

$$\theta(t)e^{-i\omega t} = i \int_{-\infty}^{\infty} \frac{dz}{2\pi} \frac{e^{-i(z+\omega)t}}{z + i\epsilon}. \quad (3.71)$$

And by changing integration variables from z to $z + \omega$, the integral becomes

$$\theta(t)e^{-i\omega t} = i \int_{-\infty}^{\infty} \frac{dz}{2\pi} \frac{e^{-izt}}{z - (\omega_{\mathbf{p}} - i\epsilon)}. \quad (3.72)$$

To turn the vev into a four-fold integral, we plug this into our original integral

$$\begin{aligned} \langle 0|T(\phi(x)\phi(y))|0\rangle &= i \int \frac{d^3p}{(2\pi)^4\omega_{\mathbf{p}}} \left[\int_{-\infty}^{\infty} dz \frac{e^{-iz(t_1-t_2)+i\mathbf{p}\cdot(\mathbf{x}-\mathbf{y})}}{z - (\omega_{\mathbf{p}} + i\epsilon)} \right. \\ &\quad \left. + \int_{-\infty}^{\infty} dz \frac{e^{-iz(t_2-t_1)+i\mathbf{p}\cdot(\mathbf{y}-\mathbf{x})}}{z - (\omega_{\mathbf{p}} + i\epsilon)} \right], \end{aligned}$$

and setting $z = p_0$ so that it's the first component of the four-momentum

$$= \int \frac{d^4p}{(2\pi)^4} \frac{i}{2\omega_{\mathbf{p}}} \left[\frac{e^{-ip\cdot(x-y)}}{p_0 - (\omega_{\mathbf{p}} - i\epsilon)} + \frac{e^{ip\cdot(x-y)}}{p_0 + (\omega_{\mathbf{p}} - i\epsilon)} \right]. \quad (3.73)$$

Finally, by changing the integration variable in the second integral from p to $-p$, we obtain a common denominator so the exponential may be taken out and we get

$$\langle 0|T\{\phi(x)\phi(y)\}|0\rangle = \int \frac{d^4p}{(2\pi)^4} e^{ip\cdot(x-y)} \frac{i}{(p_0)^2 - (\omega_{\mathbf{p}} - i\epsilon)^2}. \quad (3.74)$$

The last step is to simplify the denominator of the fraction in the integrand by noting recalling (3.7) so that

$$\begin{aligned} p_0^2 - (\omega_{\mathbf{p}} - i\epsilon)^2 &= p_0^2 - \omega_{\mathbf{p}}^2 + 2\omega_{\mathbf{p}}i\epsilon - (i\epsilon)^2 \\ &= p_0^2 - \omega_{\mathbf{p}}^2 + 2\omega_{\mathbf{p}}i\epsilon \\ &= p_0^2 - \omega_{\mathbf{p}}^2 + i\epsilon \\ &= p_0^2 - \mathbf{p}^2 - m^2 + i\epsilon, \end{aligned}$$

where we dropped the term of order $\mathcal{O}(\epsilon^2)$ in the second step, absorbed the 2 and $\omega_{\mathbf{p}}$ (in which we assumed $\omega_{\mathbf{p}} > 0$). The final expression for the propagator is expressed as follows:

$$\boxed{\langle 0|T(\phi(x)\phi(y))|0\rangle = \int \frac{d^4p}{(2\pi)^4} e^{ip\cdot(x-y)} \frac{i}{p_0^2 - \mathbf{p}^2 - m^2 + i\epsilon}}. \quad (3.75)$$

This quantity is known as the *Feynman Propagator* and in the literature it is commonly denoted as $D(x-y)$ or $\Delta(x-y)$. This is obviously an expression in position space, and one can clearly see that the expression in momentum space is just

$$D(p) = \frac{i}{p^2 - m^2 + i\epsilon}. \quad (3.76)$$

In either representation it has the interpretation of a particle propagating from spacetime points x to y . Mathematically, it is the Green's function of the Klein-Gordon equation in momentum space.

CHAPTER 4

PERTURBATION THEORY

We have previously described free, non-interacting scalar multi-particle states as elements $|\mathbf{p}_1, \mathbf{p}_2, \dots, \mathbf{p}_n\rangle$ of the constructed Fock space. We are now in a position to describe how states evolve from one to another. Therefore we attempt to solve quantum field theories of the form

$$H = H_0 + H_{int}, \quad (4.1)$$

where H_0 is the Hamiltonian corresponding to a (classical) free scalar field theory and H_{int} an interacting Hamiltonian with a corresponding density \mathcal{H}_{int} defined in the usual way by

$$H_{int} = \int d^3x \mathcal{H}_{int} = - \int d^3x \mathcal{L}_{int}. \quad (4.2)$$

Note that $\mathcal{H}_{int} = -\mathcal{L}_{int}$.

4.1 Deriving the perturbation expansion

We must resort instead to a perturbative approach which will allow us to use our interacting fields and multiparticle states in terms of the free fields above and the multiparticle states in the product Fock space. Following [4] (sec. 4.2), we will derive an expression for the two-point correlation function in ϕ^3 theory, a theory describing one scalar field with a cubic interaction

$$\mathcal{L}_{int} = -\frac{\lambda}{3!}\phi^3. \quad (4.3)$$

Denote $|\Omega\rangle$ as the ground state of the interacting theory ($|0\rangle$ is still the ground state of the free theory). We want to compute the quantity

$$\langle\Omega|T\phi(x)\phi(y)|\Omega\rangle, \quad (4.4)$$

where ϕ in this case is the field corresponding to the interacting theory. The goal here is to express both $|\Omega\rangle$ and ϕ , the interaction theory ground state and field, in terms of $|0\rangle$ and $\phi_I(x)$, the free theory ground state and field.

4.2 Expressing $\phi(x)$ in Terms of $\phi_I(x)$

First we note that in the Heisenberg picture the full Heisenberg field evolves in time according to

$$\phi(x) = e^{-iHt} \phi(\mathbf{x}) e^{iHt}. \quad (4.5)$$

Now at any time t_0 the field may be written in terms of its creation and annihilation operators

$$\phi(t_0, \mathbf{x}) = \int \frac{d^3p}{(2\pi)^3} \frac{1}{\sqrt{2E_{\mathbf{p}}}} \left(a_{\mathbf{p}} e^{i\mathbf{p}\cdot\mathbf{x}} + a_{\mathbf{p}}^\dagger e^{-i\mathbf{p}\cdot\mathbf{x}} \right), \quad (4.6)$$

then for any subsequent time t the field evolves according to

$$\phi(t, \mathbf{x}) = e^{-iH(t-t_0)} \phi(t_0, \mathbf{x}) e^{iH(t-t_0)}. \quad (4.7)$$

If there was no interaction term, no coupling between the fields, then the above evolution would be for a free field. It is commonly referred to as $\phi_I(x)$, the interaction picture field

$$\phi_I(x) = \int \frac{d^3p}{(2\pi)^3} \frac{1}{\sqrt{2E_{\mathbf{p}}}} \left(a_{\mathbf{p}} e^{ip\cdot x} + a_{\mathbf{p}}^\dagger e^{-ip\cdot x} \right), \quad (4.8)$$

where $x^0 = t - t_0$. We can then express the full Hamiltonian field $\phi(x)$ from this interaction picture field by introducing the time-evolution operator $U(t, t_0)$ defined by

$$U(t, t_0) \equiv e^{iH_0 t} e^{-iH(t-t_0)}. \quad (4.9)$$

Then $\phi(x)$ can be expressed as

$$\begin{aligned} \phi(x) &= U^\dagger(t, t_0) \phi_I(x) U(t, t_0) \\ &= e^{iH(t-t_0)} e^{-iH_0 t} \phi_I(x) e^{iH_0 t} e^{-iH(t-t_0)} \\ &= e^{iH(t-t_0)} e^{-iH_0 t} \left(e^{iH_0 t} \phi(t_0, \mathbf{x}) e^{-iH_0 t} \right) e^{iH_0 t} e^{-iH(t-t_0)} \\ &= e^{iH(t-t_0)} \phi(t_0, \mathbf{x}) e^{-iH(t-t_0)}. \end{aligned}$$

Now $U(t, t_0)$ satisfies the Schrodinger equation

$$i \frac{\partial}{\partial t} U(t, t_0) = H_I(t) U(t, t_0), \quad (4.10)$$

where $H_I(t) = e^{iH_0(t-t_0)}(H_{int})e^{-iH_0(t-t_0)}$. Clearly the solution to (4.10) must be some form of exponential, and the true solution takes the form of a power series in $H_I(t)$

$$U(t, t_0) = 1 + (-i) \int_{t_0}^t dt_1 H_I(t_1) + (-i)^2 \int_{t_0}^t dt_1 \int_{t_0}^{t_1} dt_2 H_I(t_1) H_I(t_2) \quad (4.11)$$

$$+ (-i)^3 \int_{t_0}^t dt_1 \int_{t_0}^{t_1} dt_2 \int_{t_0}^{t_2} dt_3 H_I(t_1) H_I(t_2) H_I(t_3) + \dots \quad (4.12)$$

Note that for each integral in the sum we have

$$\int_{t_0}^t dt_1 \int_{t_0}^{t_1} dt_2 \dots \int_{t_0}^{t_{n-1}} dt_n H(t_1) \dots H(t_n) = \frac{1}{n!} \int_{t_0}^t dt_1 \dots dt_n T\{H_I(t_1) \dots H_I(t_n)\}, \quad (4.13)$$

thus we are left with

$$U(t, t_0) \equiv T\left\{ \exp\left[-i \int_{t_0}^t dt' H_I(t') \right] \right\}. \quad (4.14)$$

and we arrive at our result of expressing the full Hamiltonian field $\phi(x)$ in terms of the free field $\phi_I(x)$. Combining this result with the expression of $|\Omega\rangle$ in terms of $|0\rangle$ by evolving $|0\rangle$ with the time-evolution operator $U(T, t_0)$ (see [4] section 4.2)

$$|\Omega\rangle = \lim_{T \rightarrow \infty(1-i\epsilon)} \frac{U(T, t_0)|0\rangle}{e^{-iE_0(t_0-T)}\langle\Omega|0\rangle} \quad (4.15)$$

for which $\langle\Omega|0\rangle$ was assumed nonzero and E_0 is the dominant term in the evolution as $T \rightarrow \infty$; we are left with our final result ([4], eq. 4.31)

$$\langle\Omega|T\{\phi(x)\phi(y)\}|\Omega\rangle = \lim_{t \rightarrow \infty(1-i\epsilon)} \frac{\langle 0|T\left\{ \phi_I(x)\phi_I(y) \exp\left[-i \int_{-t}^t dt' H_I(t') \right] \right\}|0\rangle}{\langle 0|T\left\{ \exp\left[-i \int_{-t}^t dt' H_I(t') \right] \right\}|0\rangle}. \quad (4.16)$$

From this expression we can calculate *S-matrix elements*, or scattering amplitudes, for different processes. Given some initial state $|i\rangle$ and a final state $|f\rangle$, the probability of a state evolving after some period of time to the final state is given by

$$\langle f|S|i\rangle, \quad (4.17)$$

where the S-operator is given by the Dyson expansion (see eq. 6.42 of [8])

$$S = \sum_0^\infty \frac{(-i)^n}{n!} \int d^4x_1 d^4x_2 \dots d^4x_n H_I(x_1) H_I(x_2) \dots H_I(x_n). \quad (4.18)$$

CHAPTER 5

TREE-LEVEL AMPLITUDES IN ABC AND ABB THEORY

From the Dyson expansion ([6], [7], [9], [8]) we can immediately start calculating S-matrix elements for different theories.

5.1 AB-Scattering in ABC Theory

Consider two incoming A and B particles with momentum \mathbf{p}_A and \mathbf{p}_B in ABC-theory. The two-particle state representing this is the incoming “ket”

$$|\mathbf{p}_A, \mathbf{p}_B\rangle = (2E_{\mathbf{p}_A})^{1/2}(2E_{\mathbf{p}_B})^{1/2}a_{\mathbf{p}}^\dagger b_{\mathbf{p}}^\dagger|0\rangle. \quad (5.1)$$

Upon an interaction we are left two outgoing A and B particles with momentum \mathbf{q}_A and \mathbf{q}_B which are represented by the outgoing “bra”

$$\langle\mathbf{q}_A, \mathbf{q}_B| = \langle 0|a_{\mathbf{q}}b_{\mathbf{q}}(2E_{\mathbf{q}_A})^{1/2}(2E_{\mathbf{q}_B})^{1/2}. \quad (5.2)$$

We calculate up to and including second order the terms in the perturbative expansion of the S-matrix element $\langle\mathbf{q}_A, \mathbf{q}_B|S||\mathbf{p}_A, \mathbf{p}_B\rangle$ in the ABC theory ($H_I = g\phi_A\phi_B\phi_C$).

5.1.1 The Zeroth Order Term in Expansion of g

We first calculate the 0-th order term $\langle\mathbf{q}_A, \mathbf{q}_B|I||\mathbf{p}_A, \mathbf{p}_B\rangle$. Performing the calculation we get

$$\langle\mathbf{q}_A, \mathbf{q}_B|I||\mathbf{p}_A, \mathbf{p}_B\rangle = (16E_{\mathbf{q}_A}E_{\mathbf{q}_B}E_{\mathbf{p}_B}E_{\mathbf{p}_A})^{1/2}\langle 0|a_{\mathbf{q}}b_{\mathbf{q}}a_{\mathbf{p}}^\dagger b_{\mathbf{p}}^\dagger|0\rangle.$$

We interchange the $b_{\mathbf{q}}$ and $a_{\mathbf{p}}^\dagger$ and subtract a term with the two a operators flipped to obtain a commutator in the vev

$$\langle 0|a_{\mathbf{q}}a_{\mathbf{p}}^\dagger b_{\mathbf{q}}b_{\mathbf{p}}^\dagger|0\rangle = \langle 0|[a_{\mathbf{q}}, a_{\mathbf{p}}^\dagger]b_{\mathbf{q}}b_{\mathbf{p}}^\dagger|0\rangle. \quad (5.3)$$

This commutator can be taken out of the vev and will enforce $\mathbf{q}_A = \mathbf{p}_A$. We repeat the process for the b operators to obtain another commutator which enforces $\mathbf{q}_B = \mathbf{p}_B$. The

δ -functions also enforce conservation of energy of each particle from the conservation of momentum as follows:

$$E_{\mathbf{q}_A} = \sqrt{(\mathbf{q}_A)^2 + m_A^2} = \sqrt{(\mathbf{p}_A)^2 + m_A^2} = E_{\mathbf{p}_A} \quad (5.4)$$

and similarly for E_B . The resulting 0-th order term is

$$(2E_A)^{1/2}(2E_B)^{1/2}\delta^3(\mathbf{q}_A - \mathbf{p}_A)\delta^3(\mathbf{q}_B - \mathbf{p}_B). \quad (5.5)$$

This calculation shows that the 0-th order term in the expansion is merely the two particles passing by each other not interacting as shown in the following figure.

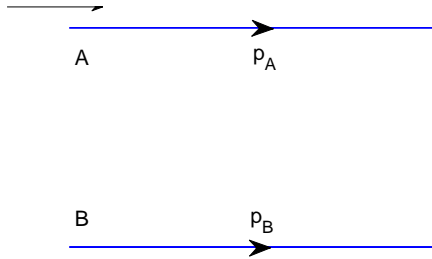


Figure 5.1: Zeroth-order term in the power series expansion of g

5.1.2 The First Order Term in Expansion of g

Next we proceed to calculate the first order term in the series expansion. It is the first term to include the actual interaction, and it is of the form

$$-ig \int d^4x \langle \mathbf{q}_A, \mathbf{q}_B | \phi_A \phi_B \phi_C | \mathbf{p}_A, \mathbf{p}_B \rangle. \quad (5.6)$$

Recall that each ϕ_i is an operator in the interaction picture; thus each ϕ_i is expressed by a standard mode expansion of free creation and annihilation operators:

$$\phi(x) = \int \frac{d^3q}{(2\pi)^3} \frac{1}{\sqrt{2E_{\mathbf{q}}}} \left(a_{\mathbf{q}} e^{-iq \cdot x} + a_{\mathbf{q}}^\dagger e^{iq \cdot x} \right). \quad (5.7)$$

We expand the first term in the amplitude as follows:

$$\begin{aligned}
-ig \int d^4x \langle \mathbf{q}_A, \mathbf{q}_B | \phi_A \phi_B \phi_C | \mathbf{p}_A, \mathbf{p}_B \rangle &= -ig \int d^4x \langle 0 | a_{\mathbf{q}} b_{\mathbf{q}} (2E_{\mathbf{q}_A})^{1/2} (2E_{\mathbf{q}_B})^{1/2} \\
&\int \frac{d^3k_A}{(2\pi)^3} \frac{1}{\sqrt{2E_{\mathbf{k}_A}}} \left(a_{\mathbf{k}} e^{-ik_A \cdot x} + a_{\mathbf{k}}^\dagger e^{ik_A \cdot x} \right) \int \frac{d^3k_B}{(2\pi)^3} \frac{1}{\sqrt{2E_{\mathbf{k}_B}}} \left(b_{\mathbf{k}} e^{-ik_B \cdot x} + b_{\mathbf{k}}^\dagger e^{ik_B \cdot x} \right) \\
&\int \frac{d^3k_C}{(2\pi)^3} \frac{1}{\sqrt{2E_{\mathbf{k}_C}}} \left(c_{\mathbf{k}} e^{-ik_C \cdot x} + c_{\mathbf{k}}^\dagger e^{ik_C \cdot x} \right) (2E_{\mathbf{p}_A})^{1/2} (2E_{\mathbf{p}_B})^{1/2} a_{\mathbf{p}}^\dagger b_{\mathbf{p}}^\dagger | 0 \rangle.
\end{aligned}$$

We rearrange the terms and combine the integrals so that only the creation and annihilation operators (with associated exponentials where appropriate) are present within the vacuum expectation

$$\begin{aligned}
-ig \int d^4x \frac{d^3k_A}{(2\pi)^3} \frac{d^3k_B}{(2\pi)^3} \frac{d^3k_C}{(2\pi)^3} \left(\frac{2E_{\mathbf{q}_A} 2E_{\mathbf{q}_B} 2E_{\mathbf{p}_A} 2E_{\mathbf{p}_B}}{2E_{\mathbf{k}_A} 2E_{\mathbf{k}_B} 2E_{\mathbf{k}_C}} \right)^{1/2} \\
\langle 0 | a_{\mathbf{q}} b_{\mathbf{q}} (a_{\mathbf{k}} e^{-ik_A \cdot x} + a_{\mathbf{k}}^\dagger e^{ik_A \cdot x}) (b_{\mathbf{k}} e^{-ik_B \cdot x} + b_{\mathbf{k}}^\dagger e^{ik_B \cdot x}) (c_{\mathbf{k}} e^{-ik_C \cdot x} + c_{\mathbf{k}}^\dagger e^{ik_C \cdot x}) a_{\mathbf{p}}^\dagger b_{\mathbf{p}}^\dagger | 0 \rangle.
\end{aligned}$$

Suppressing the associated exponentials, the vacuum expectation value (vev) takes the following form:

$$\langle 0 | a_{\mathbf{q}} b_{\mathbf{q}} (a_{\mathbf{k}} + a_{\mathbf{k}}^\dagger) (b_{\mathbf{k}} + b_{\mathbf{k}}^\dagger) (c_{\mathbf{k}} + c_{\mathbf{k}}^\dagger) a_{\mathbf{p}}^\dagger b_{\mathbf{p}}^\dagger | 0 \rangle. \quad (5.8)$$

Notice that by the commutation relations imposed at the outset (3 noninteracting free fields) we can move either the $c_{\mathbf{k}}$ or the $c_{\mathbf{k}}^\dagger$ to the appropriate $\langle 0 |$ or $| 0 \rangle$ and the entire expectation vanishes;

$$\begin{aligned}
&\langle 0 | a_{\mathbf{q}} b_{\mathbf{q}} (a_{\mathbf{k}} + a_{\mathbf{k}}^\dagger) (b_{\mathbf{k}} + b_{\mathbf{k}}^\dagger) c_{\mathbf{k}} a_{\mathbf{p}}^\dagger b_{\mathbf{p}}^\dagger | 0 \rangle + \langle 0 | a_{\mathbf{q}} b_{\mathbf{q}} (a_{\mathbf{k}} + a_{\mathbf{k}}^\dagger) (b_{\mathbf{k}} + b_{\mathbf{k}}^\dagger) c_{\mathbf{k}}^\dagger a_{\mathbf{p}}^\dagger b_{\mathbf{p}}^\dagger | 0 \rangle \\
&= \langle 0 | a_{\mathbf{q}} b_{\mathbf{q}} (a_{\mathbf{k}} + a_{\mathbf{k}}^\dagger) (b_{\mathbf{k}} + b_{\mathbf{k}}^\dagger) a_{\mathbf{p}}^\dagger b_{\mathbf{p}}^\dagger c_{\mathbf{k}} | 0 \rangle + \langle 0 | c_{\mathbf{k}}^\dagger a_{\mathbf{q}} b_{\mathbf{q}} (a_{\mathbf{k}} + a_{\mathbf{k}}^\dagger) (b_{\mathbf{k}} + b_{\mathbf{k}}^\dagger) c_{\mathbf{k}}^\dagger a_{\mathbf{p}}^\dagger b_{\mathbf{p}}^\dagger | 0 \rangle \\
&= 0.
\end{aligned}$$

Since the vacuum expectation value vanishes, the entire integral goes to 0, thus the first-order term in the expansion is zero. Note that there is no Feynman diagram for first order in this case.

5.1.3 The Second-Order Term in Expansion of g

Next we calculate the second-order term in the perturbative expansion. It takes the form

$$\frac{(-ig)^2}{2} \iint d^4x d^4y \langle \mathbf{q}_A, \mathbf{q}_B | T \{ \phi_{Ax} \phi_{Bx} \phi_{Cx} \phi_{Ay} \phi_{By} \phi_{Cy} \} | \mathbf{p}_A, \mathbf{p}_B \rangle. \quad (5.9)$$

Rather than expressing each field into its respective mode expansion of free creation and annihilation operators and rearranging terms to cancel members of the sum of products of operators we defer to some algebraic techniques to simplify the product. The discussion follows [8], but the identity below is a special case of the lemma in Chapter 2.

Consider the vacuum expectation $\langle 0 | ABCD | 0 \rangle$ where A , B , C , and D are arbitrary creation operators, annihilation operators, or linear combinations of the two. If $A = a + a^\dagger$, then

$$\begin{aligned} \langle 0 | ABCD | 0 \rangle &= \langle 0 | aBCD | 0 \rangle + \langle 0 | a^\dagger BCD | 0 \rangle \\ &= \langle 0 | aBCD | 0 \rangle - \langle 0 | BCDa | 0 \rangle + \langle 0 | a^\dagger BCD | 0 \rangle \\ &= \langle 0 | aBCD - BCDa | 0 \rangle + 0 \\ &= \langle 0 | [a, BCD] | 0 \rangle. \end{aligned}$$

Using the simple identity

$$[a, BCD] = [a, B]CD + B[a, C]D + BC[a, D] \quad (5.10)$$

given by the proof

$$\begin{aligned} [a, BCD] &= aBCD - BCDa \\ &= aBCD - BaCD + BaCD - BCDa \\ &= (aB - Ba)CD + BaCD - BCaD + BCaD - BCDa \\ &= [a, B]CD + B(aC - Ca)D + BC(aD - Da) \\ &= [a, B]CD + B[a, C]D + BC[a, D], \end{aligned}$$

the vev becomes

$$\langle 0|[a, BCD]|0\rangle = [a, B]\langle 0|CD|0\rangle + [a, C]\langle 0|BD|0\rangle + [a, D]\langle 0|BC|0\rangle. \quad (5.11)$$

Note that for each of the commutators above which are in the form $[a, X]$, the following substitution may be made (since $\langle 0|0\rangle$)

$$[a, X] = \langle 0|[a, X]|0\rangle = \langle 0|aX|0\rangle - \langle 0|Xa|0\rangle = \langle 0|aX|0\rangle. \quad (5.12)$$

Note also that $[X, a^\dagger] = \langle 0|Xa^\dagger|0\rangle$. Plugging (5.12) into the vev we get

$$\begin{aligned} \langle 0|ABCD|0\rangle &= \langle 0|ab|0\rangle\langle 0|CD|0\rangle + \langle 0|aC|0\rangle\langle 0|BD|0\rangle \\ &\quad + \langle 0|aD|0\rangle\langle 0|BC|0\rangle. \end{aligned}$$

Thus the vev of a product of multiple creation and annihilation operators has been reduced to sums of products of vev's of only two. Typically most of these terms will vanish by the rules set forth in the beginning for creation and annihilation operators acting on vacuum states.

This result holds much more generally to time ordered products of fields. It is known as Wick's Theorem and is used in advanced treatments on QFT (e.g., [4], [5]) as a device to reduce a large product of fields, say in an interaction term, vev's of commutators which ultimately become Feynman propagators. The matrix element takes the form

$$(-ig)^2 \iint x_1 x_2 \langle \mathbf{p}'_A, \mathbf{p}'_B | T \{ \phi_A(x_1) \phi_B(x_1) \phi_C(x_1) \phi_A(x_2) \phi_B(x_2) \phi_C(x_2) \} | \mathbf{p}_A, \mathbf{p}_B \rangle. \quad (5.13)$$

There are two diagrams which correspond to a nonzero amplitude; these will be discussed in the following sections.

u-Channel Process: $u = (p_A - p'_B)^2$

The first way that we say the particles can interact is by exchanging a C-particle as indicated in the figure. An A and a B-particle are accelerated to relativistic energies, and upon a short-range interaction, they exchange a virtual C-particle. This is known as a

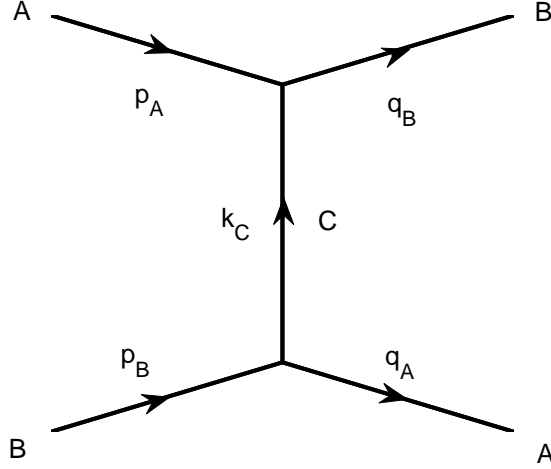


Figure 5.2: C-Exchange in AB-scattering. The q_A refers to the p'_A in the discussion and similarly for q_B .

u-channel process, and the integral describing the process is

$$\begin{aligned}
 (-ig)^2 \iint dx_1 dx_2 \langle 0 | \phi_A(x_2) a_{\mathbf{p}}^\dagger | 0 \rangle \langle 0 | \phi_B(x_1) b_{\mathbf{p}}^\dagger | 0 \rangle \\
 \langle 0 | \phi_C(x_1) \phi_C(x_2) | 0 \rangle \langle 0 | a_{\mathbf{p}'} \phi_A(x_1) | 0 \rangle \langle 0 | b_{\mathbf{p}'} \phi_B(x_2) | 0 \rangle \sqrt{16E'_A E'_B E_A E_B}.
 \end{aligned} \tag{5.14}$$

From the properties calculated in the previous section we can replace the vev's with the corresponding exponential or propagating term as follows:

$$\begin{aligned}
 (-ig)^2 \iint dx_1 dx_2 \frac{1}{2E_A} e^{-ip_A \cdot x_1} \frac{1}{2E_B} e^{-ip_B \cdot x_2} \frac{1}{2E'_A} e^{ip'_A \cdot x_2} \frac{1}{2E'_B} e^{ip'_B \cdot x_1} \\
 \int \frac{d^4 k}{(2\pi)^4} e^{-ik \cdot (x_1 - x_2)} \sqrt{16E'_A E'_B E_A E_B}.
 \end{aligned} \tag{5.15}$$

Cancelling the normalization factors and gathering all the exponentials not in the propagator term we get

$$(-ig)^2 \iint d^4 x_1 d^4 x_2 e^{i(p'_B - p_A) \cdot x_1} e^{i(p'_A - p_B) \cdot x_2} \int \frac{d^4 k}{(2\pi)^4} e^{-ik \cdot (x_1 - x_2)}. \tag{5.16}$$

We will now evaluate the $d^4 x_1$ and $d^4 x_2$ integrals. Notice that the propagator term is a function of the difference $x_1 - x_2$ of the coordinates. We expect this to be so, because the physics of the situation should be independant of the choice of the coordinate system [8]. To

show this explicitly in the calculation we change the variable of the space integrations from x_1 and x_2 to x and X which are defined coordinate-wise by

$$x = x_1 - x_2 \qquad X = \frac{x_1 + x_2}{2}. \qquad (5.17)$$

It is useful at this point to make a remark. Were the integration variables one-dimensional over Euclidean space, the change of variables formula would take the form

$$\iint dx_1 dx_2 f(x_1, x_2) = \iint dx dX f(x_1(x, X)x_2(x, X)) \left| \frac{\partial(x_1, x_2)}{\partial(x, X)} \right|, \qquad (5.18)$$

where $\left| \frac{\partial(x_1, x_2)}{\partial(x, X)} \right|$ is the determinant of the Jacobian matrix in the change of variables formula (see [12], sec. 6.2, p. 377). The simple calculation would show that this is unity:

$$\frac{\partial(x_1, x_2)}{\partial(x, X)} = \begin{vmatrix} \frac{\partial x_1}{\partial x} & \frac{\partial x_1}{\partial X} \\ \frac{\partial x_2}{\partial x} & \frac{\partial x_2}{\partial X} \end{vmatrix} = \begin{vmatrix} \frac{1}{2} & 1 \\ -1 & 1 \end{vmatrix} = 1. \qquad (5.19)$$

However we are changing two integration variables in a 4-dimensional Minkowski space, so we are actually changing from a set of 8 one-dimensional variables to another set of 8. The actual Jacobian in the change of variables looks like

$$\begin{bmatrix} \frac{\partial x_1^0}{\partial x^0} & \frac{\partial x_1^0}{\partial x^1} & \frac{\partial x_1^0}{\partial x^2} & \frac{\partial x_1^0}{\partial x^3} & \frac{\partial x_1^0}{\partial X^0} & \frac{\partial x_1^0}{\partial X^1} & \frac{\partial x_1^0}{\partial X^2} & \frac{\partial x_1^0}{\partial X^3} \\ \frac{\partial x_1^1}{\partial x^0} & \frac{\partial x_1^1}{\partial x^1} & \frac{\partial x_1^1}{\partial x^2} & \frac{\partial x_1^1}{\partial x^3} & \frac{\partial x_1^1}{\partial X^0} & \frac{\partial x_1^1}{\partial X^1} & \frac{\partial x_1^1}{\partial X^2} & \frac{\partial x_1^1}{\partial X^3} \\ \frac{\partial x_1^2}{\partial x^0} & \frac{\partial x_1^2}{\partial x^1} & \frac{\partial x_1^2}{\partial x^2} & \frac{\partial x_1^2}{\partial x^3} & \frac{\partial x_1^2}{\partial X^0} & \frac{\partial x_1^2}{\partial X^1} & \frac{\partial x_1^2}{\partial X^2} & \frac{\partial x_1^2}{\partial X^3} \\ \frac{\partial x_1^3}{\partial x^0} & \frac{\partial x_1^3}{\partial x^1} & \frac{\partial x_1^3}{\partial x^2} & \frac{\partial x_1^3}{\partial x^3} & \frac{\partial x_1^3}{\partial X^0} & \frac{\partial x_1^3}{\partial X^1} & \frac{\partial x_1^3}{\partial X^2} & \frac{\partial x_1^3}{\partial X^3} \\ \frac{\partial x_2^0}{\partial x^0} & \frac{\partial x_2^0}{\partial x^1} & \frac{\partial x_2^0}{\partial x^2} & \frac{\partial x_2^0}{\partial x^3} & \frac{\partial x_2^0}{\partial X^0} & \frac{\partial x_2^0}{\partial X^1} & \frac{\partial x_2^0}{\partial X^2} & \frac{\partial x_2^0}{\partial X^3} \\ \frac{\partial x_2^1}{\partial x^0} & \frac{\partial x_2^1}{\partial x^1} & \frac{\partial x_2^1}{\partial x^2} & \frac{\partial x_2^1}{\partial x^3} & \frac{\partial x_2^1}{\partial X^0} & \frac{\partial x_2^1}{\partial X^1} & \frac{\partial x_2^1}{\partial X^2} & \frac{\partial x_2^1}{\partial X^3} \\ \frac{\partial x_2^2}{\partial x^0} & \frac{\partial x_2^2}{\partial x^1} & \frac{\partial x_2^2}{\partial x^2} & \frac{\partial x_2^2}{\partial x^3} & \frac{\partial x_2^2}{\partial X^0} & \frac{\partial x_2^2}{\partial X^1} & \frac{\partial x_2^2}{\partial X^2} & \frac{\partial x_2^2}{\partial X^3} \\ \frac{\partial x_2^3}{\partial x^0} & \frac{\partial x_2^3}{\partial x^1} & \frac{\partial x_2^3}{\partial x^2} & \frac{\partial x_2^3}{\partial x^3} & \frac{\partial x_2^3}{\partial X^0} & \frac{\partial x_2^3}{\partial X^1} & \frac{\partial x_2^3}{\partial X^2} & \frac{\partial x_2^3}{\partial X^3} \end{bmatrix}. \qquad (5.20)$$

The corresponding determinant is then calculated by taking each partial derivative for each

coordinate:

$$\begin{vmatrix} \frac{1}{2} & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & \frac{1}{2} & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & \frac{1}{2} & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & \frac{1}{2} & 0 & 0 & 0 & 1 \\ -\frac{1}{2} & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & -\frac{1}{2} & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & -\frac{1}{2} & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -\frac{1}{2} & 0 & 0 & 0 & 1 \end{vmatrix} = 1. \quad (5.21)$$

Once again, we obtain unity for the proportionality between the $dx_1 dx_2$ variables and the $dx dX$ variables.

Also we need to solve (5.17) for x_1 and x_2 which can be done by the simple substitution method of linear algebra:

$$x_1 = \frac{1}{2}x + X \quad x_2 = \frac{-1}{2}x + X. \quad (5.22)$$

With these two we can finally make the change of variables in (5.16) to get

$$\begin{aligned} & \left(\frac{-ig}{2}\right)^2 \iint d^4x d^4X e^{i(p'_B - p_A) \cdot (\frac{x}{2} + X)} e^{i(p'_A - p_B) \cdot (\frac{-x}{2} + X)} \\ & \int \frac{d^4k_C}{(2\pi)^4} e^{-ik \cdot x} \frac{i}{k_C^2 - m_C^2 + i\epsilon}. \end{aligned} \quad (5.23)$$

Multiplying terms in the exponential and rearranging them to calculate the dX integral first we obtain

$$\begin{aligned} & \left(\frac{-ig}{2}\right)^2 \int d^4X e^{i(p'_B - p_A + p'_A - p_B) \cdot X} \int d^4x e^{i(p'_B - p_A - p'_A + p_B) \cdot \frac{x}{2}} \\ & \times \int \frac{d^4k_C}{(2\pi)^4} e^{-ik \cdot x} \frac{i}{k_C^2 - m_C^2 + i\epsilon}. \end{aligned} \quad (5.24)$$

The dX integral is clearly now calculated to be a δ -function. We can use the even function property of the δ -function to switch the signs on the p_i 's inside:

$$\begin{aligned} & \left(\frac{-ig}{2}\right)^2 (2\pi)^4 \delta^4(p_A + p_B - p'_A - p'_B) \\ & \times \int d^4x e^{i(p'_B - p_A - p'_A + p_B) \cdot \frac{x}{2}} \\ & \times \int \frac{d^4k}{(2\pi)^4} e^{-ik \cdot x} \frac{i}{k_C^2 - m_C^2 + i\epsilon}. \end{aligned} \quad (5.25)$$

Notice that the δ -function implies that conservation of momentum: $p_A + p_B = p'_A + p'_B$. We can then define the *four-momentum transfer* q by $q = p_B - p'_A$. By the δ -function, this also enforces $q = p'_B - p_A$ so that the term in the exponential of the dx integral becomes

$$i(p'_B - p_A - p'_A + p_B) \cdot \frac{x}{2} = i(q + q) \cdot \frac{x}{2} = i(q \cdot x). \quad (5.26)$$

Plugging this new expression in and switching the integrals (with appropriate continuity assumptions), we get

$$\begin{aligned} & \left(\frac{-ig}{2}\right)^2 (2\pi)^4 \delta^4(p_A + p_B - p'_A - p'_B) \\ & \times \int \frac{d^4k}{(2\pi)^4} \left[\int d^4x e^{i(q-k_C) \cdot x} \right] \frac{i}{k_C^2 - m_C^2 + i\epsilon}. \end{aligned} \quad (5.27)$$

The bracketed integral is of course $(2\pi)^4 \delta^4(q - k_C)$ which when used in the dk_C integration singles out q as a particular value of k_C which leaves us with the final result (the $(2\pi)^4$'s obviously cancel)

$$\left(\frac{-ig}{2}\right)^2 (2\pi)^4 \delta^4(p_A + p_B - p'_A - p'_B) \frac{i}{q^2 - m_C^2 + i\epsilon}. \quad (5.28)$$

This entire process can be repeated by switching the x_1 and x_2 in the original expression of the matrix element. The exact same result as (5.28) would be obtained and added to it, cancelling the 2's. Doing this we obtain as our final result

$$\boxed{\mathcal{M} = (-ig)^2 (2\pi)^4 \delta^4(p_A + p_B - p'_A - p'_B) \frac{i}{(p_B - p'_A)^2 - m_C^2 + i\epsilon}}. \quad (5.29)$$

Once again this is a *u-channel* process (see [8], sec. 6.3, eq. 6.104.) Any time a tree-level scattering process is considered, it will occur through different channels. The second possible channel is discussed next.

s-Channel Process: $s = (p_A + p_B)^2$

The second non-zero contribution to the scattering amplitude (5.13) is shown in the following figure. In this process the incoming particles interact through a temporary annihilation reproduction of one another through a virtual C-particle. For the expression to

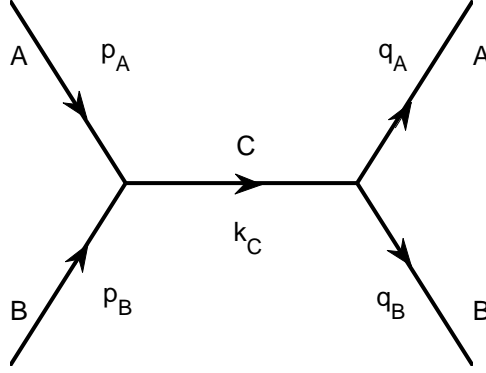


Figure 5.3: AB Annihilation for ABC Theory. q_A corresponds to p'_A and similarly for q_B .

be computed we have

$$\begin{aligned}
& \left(\frac{-ig}{2}\right)^2 \iint d^4x_1 d^4x_2 \sqrt{16E'_A E'_B E_A E_B} \\
& \quad \langle 0 | a_{\mathbf{p}'} \phi_A(x_2) | 0 \rangle \langle 0 | b_{\mathbf{p}'} \phi_B(x_2) | 0 \rangle \\
& \quad \langle 0 | \phi_C(x_1) \phi_C(x_2) | 0 \rangle \\
& \quad \langle 0 | \phi_A(x_1) a_{\mathbf{p}}^\dagger | 0 \rangle \langle 0 | \phi_B(x_2) b_{\mathbf{p}}^\dagger | 0 \rangle.
\end{aligned} \tag{5.30}$$

We calculate this integral in a similar manner as the u-channel process. Reorganizing terms and substituting values we get

$$\begin{aligned}
& \left(\frac{-ig}{2}\right)^2 \iint d^4x_1 d^4x_2 \sqrt{16E'_A E'_B E_A E_B} \frac{1}{\sqrt{2E'_A}} e^{ip'_A \cdot x_2} \frac{1}{\sqrt{2E'_B}} e^{ip'_B \cdot x_2} \\
& \quad \int \frac{d^4k_C}{(2\pi)^4} e^{-ik \cdot x} \frac{i}{k_C^2 - m_C^2 + i\epsilon} \frac{1}{\sqrt{2E_A}} e^{-ip_A \cdot x_1} \frac{1}{\sqrt{2E_B}} e^{-ip_B \cdot x_1}.
\end{aligned} \tag{5.31}$$

Cancelling the normalizations and gathering like terms we further obtain

$$\left(\frac{-ig}{2}\right)^2 \iint d^4x_1 d^4x_2 e^{-i(p_A + p_B) \cdot x_1} e^{i(p'_A + p'_B) \cdot x_2} \int \frac{d^4k_C}{(2\pi)^4} e^{-ik \cdot x} \frac{i}{k_C^2 - m_C^2 + i\epsilon}. \tag{5.32}$$

Once again we make the change of variables (with the same Jacobian)

$$\begin{aligned}
x &= x_1 - x_2 & x_1 &= X + x/2 \\
X &= (x_1 + x_2)/2 & x_2 &= X - x/2.
\end{aligned}$$

Upon substitution, the exponentials in the $x_1 x_2$ integral become

$$\begin{aligned} e^{-i(p_A+p_B)\cdot(X+x/2)} e^{i(p'_A+p'_B)\cdot(X-x/2)} &= e^{-i(p_A+p_B)\cdot X} e^{-i(p_A+p_B)\cdot x/2} e^{i(p'_A+p'_B)\cdot X} e^{-i(p'_A+p'_B)\cdot x/2} \\ &= e^{i(p'_A+p'_B-p_A-p_B)\cdot X} e^{-i(p_A+p_B+p'_A+p'_B)\cdot x/2}. \end{aligned}$$

We may thus rewrite (5.32) as

$$\begin{aligned} \left(\frac{-ig}{2}\right)^2 \int d^4 X e^{i(p'_A+p'_B-p_A-p_B)\cdot X} \int d^4 x e^{-i(p_A+p_B+p'_A+p'_B)\cdot x/2} \\ \int \frac{d^4 k_C}{(2\pi)^4} e^{-ik\cdot x} \frac{i}{k_C^2 - m_C^2 + i\epsilon}. \end{aligned} \quad (5.33)$$

The dX integral becomes a δ -function

$$\int d^4 X e^{i(p'_A+p'_B-p_A-p_B)\cdot X} = (2\pi)^4 \delta^4(p_A + p_B - p'_A - p'_B),$$

and the δ -function enforces $p'_A + p'_B = p_A + p_B$ which cancels the 2 in the exponential of the dx integral. Combining this and interchanging the order of the dx and dk integrals we obtain

$$\left(\frac{-ig}{2}\right)^2 (2\pi)^4 \delta^4(p_A + p_B - p'_A - p'_B) \int \frac{d^4 k_C}{(2\pi)^4} \left[\int d^4 x e^{i(p_A+p_B-k_C)\cdot x} \right] \frac{i}{k_C^2 - m_C^2 + i\epsilon}. \quad (5.34)$$

The dx integral is another δ -function

$$\int d^4 x e^{i(p_A+p_B-k_C)\cdot x} = (2\pi)^4 \delta^4(p_A + p_B - k_C), \quad (5.35)$$

so with this result we can calculate the final integral

$$\left(\frac{-ig}{2}\right)^2 (2\pi)^4 \delta^4(p_A + p_B - p'_A - p'_B) \int \frac{d^4 k_C}{(2\pi)^4} (2\pi)^4 \delta^4(p_A + p_B - k_C) \frac{i}{k_C^2 - m_C^2 + i\epsilon} \quad (5.36)$$

which gives our final result (note that we obtain an identical result by interchanging the x_1 and x_2 values which cancels the two in front)

$$(-ig)^2 (2\pi)^4 \delta^4(p_A + p_B - p'_A - p'_B) \frac{i}{(p_A + p_B)^2 - m_C^2 + i\epsilon}. \quad (5.37)$$

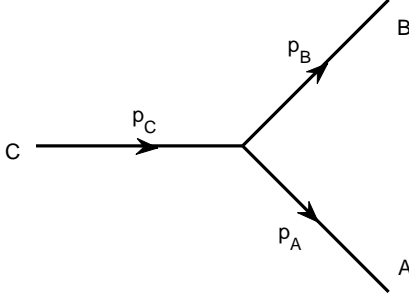


Figure 5.4: A C-particle decaying into an A and a B-particle

5.2 C → A + B Decay

The next process we calculate is C-decay into two particles: an A and a B. The corresponding figure below illustrates the process. The amplitude we seek to calculate is

$$(-ig)\langle \mathbf{p}_B, \mathbf{p}_A | \int d^4x \phi_A \phi_B \phi_C | \mathbf{p}_C \rangle. \quad (5.38)$$

We expand this term using the mode expansion of fields and the momentum states constructed from the vacuum,

$$\begin{aligned} & (-ig)\langle \mathbf{p}_B, \mathbf{p}_A | \int d^4x \phi_A \phi_B \phi_C | \mathbf{p}_C \rangle \quad (5.39) \\ &= -ig \langle 0 | a_{\mathbf{p}} b_{\mathbf{p}} \sqrt{2E_A} \sqrt{2E_B} \int d^4x \int \frac{d^3k_A}{(2\pi)^3} \frac{1}{\sqrt{2E_{k_A}}} (a_{\mathbf{k}} e^{-ik_A \cdot x} + a_{\mathbf{k}}^\dagger e^{ik_A \cdot x}) \\ & \quad \times \int \frac{d^3k_B}{(2\pi)^3} \frac{1}{\sqrt{2E_{k_B}}} (b_{\mathbf{k}} e^{-ik_B \cdot x} + b_{\mathbf{k}}^\dagger e^{ik_B \cdot x}) \int \frac{d^3k_C}{(2\pi)^3} \frac{1}{\sqrt{2E_{k_C}}} (c_{\mathbf{k}} e^{-ik_C \cdot x} + c_{\mathbf{k}}^\dagger e^{ik_C \cdot x}) \\ & \quad \times \sqrt{2E_C} c_{\mathbf{p}}^\dagger | 0 \rangle. \quad (5.40) \end{aligned}$$

Rearranging terms so that only the creation and annihilation operators (and associated exponentials) are contained within the vev we obtain:

$$\begin{aligned} & -ig \int d^4x \frac{d^3k_A}{(2\pi)^3} \frac{d^3k_B}{(2\pi)^3} \frac{d^3k_C}{(2\pi)^3} \sqrt{\frac{2E_A 2E_B 2E_C}{2E_{k_A} 2E_{k_B} 2E_{k_C}}} \\ & \quad \langle 0 | a_{\mathbf{p}} b_{\mathbf{p}} (a_{\mathbf{k}} e^{-ik_A \cdot x} + a_{\mathbf{k}}^\dagger e^{ik_A \cdot x}) (b_{\mathbf{k}} e^{-ik_B \cdot x} + b_{\mathbf{k}}^\dagger e^{ik_B \cdot x}) \\ & \quad (c_{\mathbf{k}} e^{-ik_C \cdot x} + c_{\mathbf{k}}^\dagger e^{ik_C \cdot x}) c_{\mathbf{p}}^\dagger | 0 \rangle. \end{aligned}$$

Suppressing the exponentials, the vev in the integral above takes the following form

$$\langle 0|a_{\mathbf{p}}b_{\mathbf{p}}(a_{\mathbf{k}} + a_{\mathbf{k}}^\dagger)(b_{\mathbf{k}} + b_{\mathbf{k}}^\dagger)(c_{\mathbf{k}} + c_{\mathbf{k}}^\dagger)c_{\mathbf{p}}^\dagger|0\rangle.$$

We systematically reduce this sum of 8 products of 6 creation/annihilation operators to a single product of operators which will in turn be used with commutation relations to transform the vev to a product of δ -functions. To begin the process we first eliminate the $c_{\mathbf{k}}^\dagger$ by moving it to the left

$$\begin{aligned} & \langle 0|a_{\mathbf{p}}b_{\mathbf{p}}(a_{\mathbf{k}} + a_{\mathbf{k}}^\dagger)(b_{\mathbf{k}} + b_{\mathbf{k}}^\dagger)(c_{\mathbf{k}} + c_{\mathbf{k}}^\dagger)c_{\mathbf{p}}^\dagger|0\rangle \\ &= \langle 0|a_{\mathbf{p}}b_{\mathbf{p}}(a_{\mathbf{k}} + a_{\mathbf{k}}^\dagger)(b_{\mathbf{k}} + b_{\mathbf{k}}^\dagger)c_{\mathbf{k}}c_{\mathbf{p}}^\dagger|0\rangle + \langle 0|a_{\mathbf{p}}b_{\mathbf{p}}(a_{\mathbf{k}} + a_{\mathbf{k}}^\dagger)(b_{\mathbf{k}} + b_{\mathbf{k}}^\dagger)c_{\mathbf{k}}^\dagger c_{\mathbf{p}}^\dagger|0\rangle \\ &= \langle 0|a_{\mathbf{p}}b_{\mathbf{p}}(a_{\mathbf{k}} + a_{\mathbf{k}}^\dagger)(b_{\mathbf{k}} + b_{\mathbf{k}}^\dagger)c_{\mathbf{k}}c_{\mathbf{p}}^\dagger|0\rangle + \langle 0|c_{\mathbf{k}}^\dagger a_{\mathbf{p}}b_{\mathbf{p}}(a_{\mathbf{k}} + a_{\mathbf{k}}^\dagger)(b_{\mathbf{k}} + b_{\mathbf{k}}^\dagger)c_{\mathbf{p}}^\dagger|0\rangle \\ &= \langle 0|a_{\mathbf{p}}b_{\mathbf{p}}(a_{\mathbf{k}} + a_{\mathbf{k}}^\dagger)(b_{\mathbf{k}} + b_{\mathbf{k}}^\dagger)c_{\mathbf{k}}c_{\mathbf{p}}^\dagger|0\rangle + 0, \end{aligned}$$

then moving $b_{\mathbf{k}}$ to the right

$$\begin{aligned} &= \langle 0|a_{\mathbf{p}}b_{\mathbf{p}}(a_{\mathbf{k}} + a_{\mathbf{k}}^\dagger)b_{\mathbf{k}}c_{\mathbf{k}}c_{\mathbf{p}}^\dagger|0\rangle + \langle 0|a_{\mathbf{p}}b_{\mathbf{p}}(a_{\mathbf{k}} + a_{\mathbf{k}}^\dagger)b_{\mathbf{k}}^\dagger c_{\mathbf{k}}c_{\mathbf{p}}^\dagger|0\rangle \\ &= \langle 0|a_{\mathbf{p}}b_{\mathbf{p}}(a_{\mathbf{k}} + a_{\mathbf{k}}^\dagger)c_{\mathbf{k}}c_{\mathbf{p}}^\dagger b_{\mathbf{k}}|0\rangle + \langle 0|a_{\mathbf{p}}b_{\mathbf{p}}(a_{\mathbf{k}} + a_{\mathbf{k}}^\dagger)b_{\mathbf{k}}^\dagger c_{\mathbf{k}}c_{\mathbf{p}}^\dagger|0\rangle \\ &= 0 + \langle 0|a_{\mathbf{p}}b_{\mathbf{p}}(a_{\mathbf{k}} + a_{\mathbf{k}}^\dagger)b_{\mathbf{k}}^\dagger c_{\mathbf{k}}c_{\mathbf{p}}^\dagger|0\rangle, \end{aligned}$$

and finally moving $a_{\mathbf{k}}$ to the right in the same way

$$\begin{aligned} &= \langle 0|a_{\mathbf{p}}b_{\mathbf{p}}a_{\mathbf{k}}b_{\mathbf{k}}^\dagger c_{\mathbf{k}}c_{\mathbf{p}}^\dagger|0\rangle + \langle 0|a_{\mathbf{p}}b_{\mathbf{p}}a_{\mathbf{k}}^\dagger b_{\mathbf{k}}^\dagger c_{\mathbf{k}}c_{\mathbf{p}}^\dagger|0\rangle \\ &= \langle 0|a_{\mathbf{p}}b_{\mathbf{p}}b_{\mathbf{k}}^\dagger c_{\mathbf{k}}c_{\mathbf{p}}^\dagger a_{\mathbf{k}}|0\rangle + \langle 0|a_{\mathbf{p}}b_{\mathbf{p}}a_{\mathbf{k}}^\dagger b_{\mathbf{k}}^\dagger c_{\mathbf{k}}c_{\mathbf{p}}^\dagger|0\rangle \\ &= 0 + \langle 0|a_{\mathbf{p}}b_{\mathbf{p}}a_{\mathbf{k}}^\dagger b_{\mathbf{k}}^\dagger c_{\mathbf{k}}c_{\mathbf{p}}^\dagger|0\rangle. \end{aligned}$$

We obtain a simplification of the original vev

$$\langle 0|a_{\mathbf{p}}b_{\mathbf{p}}(a_{\mathbf{k}} + a_{\mathbf{k}}^\dagger)(b_{\mathbf{k}} + b_{\mathbf{k}}^\dagger)(c_{\mathbf{k}} + c_{\mathbf{k}}^\dagger)c_{\mathbf{p}}^\dagger|0\rangle = \langle 0|a_{\mathbf{p}}b_{\mathbf{p}}a_{\mathbf{k}}^\dagger b_{\mathbf{k}}^\dagger c_{\mathbf{k}}c_{\mathbf{p}}^\dagger|0\rangle. \quad (5.41)$$

From this reduction we then use each product aa^\dagger to turn the vev into a product of δ -functions using the commutation relations given by (3.12). The calculation proceeds as

follows. First we do the commutator for A;

$$\begin{aligned}
\langle 0|a_{\mathbf{p}}b_{\mathbf{p}}a_{\mathbf{k}}^{\dagger}b_{\mathbf{k}}^{\dagger}c_{\mathbf{k}}c_{\mathbf{p}}^{\dagger}|0\rangle &= \langle 0|a_{\mathbf{p}}a_{\mathbf{k}}^{\dagger}b_{\mathbf{p}}b_{\mathbf{k}}^{\dagger}c_{\mathbf{k}}c_{\mathbf{p}}^{\dagger}|0\rangle \\
&= \langle 0|(a_{\mathbf{p}}a_{\mathbf{k}}^{\dagger} - a_{\mathbf{k}}^{\dagger}a_{\mathbf{p}})b_{\mathbf{p}}b_{\mathbf{k}}^{\dagger}c_{\mathbf{k}}c_{\mathbf{p}}^{\dagger}|0\rangle \\
&= \langle 0|[a_{\mathbf{p}}, a_{\mathbf{k}}^{\dagger}]b_{\mathbf{p}}b_{\mathbf{k}}^{\dagger}c_{\mathbf{k}}c_{\mathbf{p}}^{\dagger}|0\rangle \\
&= (2\pi)^3\delta^3(\mathbf{p}_A - \mathbf{k}_A)\langle 0|b_{\mathbf{p}}b_{\mathbf{k}}^{\dagger}c_{\mathbf{k}}c_{\mathbf{p}}^{\dagger}|0\rangle.
\end{aligned}$$

Then we perform the commutator for B;

$$\begin{aligned}
&= (2\pi)^3\delta^3(\mathbf{p}_A - \mathbf{k}_A)\langle 0|(b_{\mathbf{p}}b_{\mathbf{k}}^{\dagger} - b_{\mathbf{k}}^{\dagger}b_{\mathbf{p}})c_{\mathbf{k}}c_{\mathbf{p}}^{\dagger}|0\rangle \\
&= (2\pi)^3\delta^3(\mathbf{p}_A - \mathbf{k}_A)\langle 0|[b_{\mathbf{p}}, b_{\mathbf{k}}^{\dagger}]c_{\mathbf{k}}c_{\mathbf{p}}^{\dagger}|0\rangle \\
&= (2\pi)^3\delta^3(\mathbf{p}_A - \mathbf{k}_A)(2\pi)^3\delta^3(\mathbf{p}_B - \mathbf{k}_B)\langle 0|c_{\mathbf{k}}c_{\mathbf{p}}^{\dagger}|0\rangle.
\end{aligned}$$

Finally we perform the commutator for C;

$$\begin{aligned}
&= (2\pi)^3\delta^3(\mathbf{p}_A - \mathbf{k}_A)(2\pi)^3\delta^3(\mathbf{p}_B - \mathbf{k}_B)\langle 0|c_{\mathbf{k}}c_{\mathbf{p}}^{\dagger} - c_{\mathbf{p}}^{\dagger}c_{\mathbf{k}}|0\rangle \\
&= (2\pi)^3\delta^3(\mathbf{p}_A - \mathbf{k}_A)(2\pi)^3\delta^3(\mathbf{p}_B - \mathbf{k}_B)\langle 0|[c_{\mathbf{k}}, c_{\mathbf{p}}^{\dagger}]|0\rangle \\
&= (2\pi)^3\delta^3(\mathbf{p}_A - \mathbf{k}_A)(2\pi)^3\delta^3(\mathbf{p}_B - \mathbf{k}_B)(2\pi)^3\delta^3(\mathbf{k}_C - \mathbf{p}_C)\langle 0|0\rangle \\
&= (2\pi)^3\delta^3(\mathbf{p}_A - \mathbf{k}_A)(2\pi)^3\delta^3(\mathbf{p}_B - \mathbf{k}_B)(2\pi)^3\delta^3(\mathbf{k}_C - \mathbf{p}_C).
\end{aligned}$$

Thus the vev of the product of operators has been reduced to a product of δ -functions

$$\langle 0|a_{\mathbf{p}}b_{\mathbf{p}}a_{\mathbf{k}}^{\dagger}b_{\mathbf{k}}^{\dagger}c_{\mathbf{k}}c_{\mathbf{p}}^{\dagger}|0\rangle = (2\pi)^3\delta^3(\mathbf{p}_A - \mathbf{k}_A)(2\pi)^3\delta^3(\mathbf{p}_B - \mathbf{k}_B)(2\pi)^3\delta^3(\mathbf{k}_C - \mathbf{p}_C). \quad (5.42)$$

We can now take this expression for the vev and plug it back into our integral, remembering to include the appropriate exponentials: $e^{ik_A \cdot x}$, $e^{ik_B \cdot x}$, and $e^{-ik_C \cdot x}$ for $a_{\mathbf{k}}^{\dagger}$, $b_{\mathbf{k}}^{\dagger}$, and $c_{\mathbf{k}}$, respectively.

$$\begin{aligned}
\mathcal{M} &= (-ig) \int d^4x \frac{d^3k_A}{(2\pi)^3} \frac{d^3k_B}{(2\pi)^3} \frac{d^3k_C}{(2\pi)^3} \sqrt{\frac{2E_A 2E_B 2E_C}{2E_{\mathbf{k}_A} 2E_{\mathbf{k}_B} 2E_{\mathbf{k}_C}}} \\
&\quad \times (2\pi)^3\delta^3(\mathbf{p}_A - \mathbf{k}_A)(2\pi)^3\delta^3(\mathbf{p}_B - \mathbf{k}_B)(2\pi)^3\delta^3(\mathbf{k}_C - \mathbf{p}_C)e^{i(k_A+k_B-k_C)\cdot x}.
\end{aligned}$$

After the simplification of the vev we are in a position to actually calculate the integral. Using the property of δ -functions on the integrals over k-space

$$\int \frac{d^3k}{(2\pi)^3} f(\mathbf{k}) \delta^3(\mathbf{p} - \mathbf{k}) = f(\mathbf{p}) \quad (5.43)$$

and then taking the Fourier Transform of the resulting exponential in 4-space we obtain after rearranging terms

$$\begin{aligned} \mathcal{M} &= (-ig) \int d^4x \int \frac{d^3k_A}{(2\pi)^3} \sqrt{\frac{2E_A}{2E_{\mathbf{k}_A}}} e^{ik_A \cdot x} (2\pi)^3 \delta^3(\mathbf{p}_A - \mathbf{k}_A) \\ &\quad \times \int \frac{d^3k_B}{(2\pi)^3} \sqrt{\frac{2E_B}{2E_{\mathbf{k}_B}}} e^{ik_B \cdot x} (2\pi)^3 \delta^3(\mathbf{p}_B - \mathbf{k}_B) \int \frac{d^3k_C}{(2\pi)^3} \sqrt{\frac{2E_C}{2E_{\mathbf{k}_C}}} e^{-ik_C \cdot x} (2\pi)^3 \delta^3(\mathbf{k}_C - \mathbf{p}_C) \\ &= (-ig) \int d^4x e^{ip_A \cdot x} e^{ip_B \cdot x} e^{-ip_C \cdot x} \\ &= (-ig) \int d^4x e^{i(p_A + p_B - p_C) \cdot x} \\ &= (-ig) (2\pi)^4 \delta^4(p_A + p_B - p_C). \end{aligned}$$

We have at last obtained the amplitude for C-decay into an A and a B particle

$$\boxed{\mathcal{M} = (-ig) (2\pi)^4 \delta^4(p_A + p_B - p_C)}. \quad (5.44)$$

At this point it is worthwhile to make a remark. Although this is a scattering amplitude for a C-particle into an A and a B, what we in effect have just done is derive a Feynman rule of ABC theory. For any diagram that should occur in a process described by an ABC interaction, every vertex will look like the one in Figure 5.4. We can simply write $(-ig)$ for each vertex instead of repeating this entire calculation. Every Feynman diagram with vertices will include this factor as many times as necessary.

Another remark is that $(-ig)$ is part of the *invariant amplitude*. Different texts give different arguments for why the $(2\pi)^4 \delta^4(p_A + p_B - p_C)$ can be removed. [4] solves the problem formally using wave packets, but [13] uses the quicker method of letting the particles interact in a finite volume for a finite time so that an element of the S-matrix may ultimately written as

$$S = -2\pi i \delta^4(p_A + p_B - p_C) \mathcal{M}. \quad (5.45)$$

5.3 A → B + B Decay

We now calculate the decay amplitude for the process of an A-particle decaying into two B particles in ABB-theory. The calculation is similar to $C \rightarrow A + B$ in ABC-theory, but a couple new features arise due to the indistinguishability of the two resulting particles. We will show that we recover the same amplitude as for C-decay (interpreting the g as the coupling for ABB theory). The figure has similar structure to the ABC-vertex and is shown below.

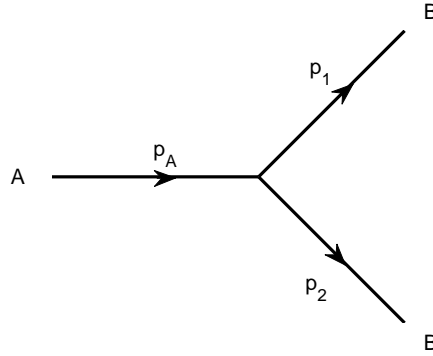


Figure 5.5: An A-particle decaying into 2 B-particles

We proceed as before by first writing the S-matrix element out with all the terms explicitly stated. For this calculation we will denote the operators and fields corresponding to the B-particles with a 1 or 2 subscript. The A-particle is denoted as usual.

$$\begin{aligned}
 \mathcal{M} &= \frac{(-ig)}{2} \langle \mathbf{p}_1, \mathbf{p}_2 | \int d^4x \phi_1(x) \phi_2(x) \phi_A(x) | \mathbf{p}_A \rangle \\
 &= \frac{(-ig)}{2} \int d^4x \langle 0 | b_{\mathbf{p}_1} b_{\mathbf{p}_2} \sqrt{2E_1} \sqrt{2E_2} \int \frac{d^3k_1}{(2\pi)^3} \frac{1}{\sqrt{2E_{\mathbf{k}_1}}} (b_1 e^{-ik_1 \cdot x} + b_1^\dagger e^{ik_1 \cdot x}) \\
 &\quad \times \int \frac{d^3k_2}{(2\pi)^3} \frac{1}{\sqrt{2E_{\mathbf{k}_2}}} (b_2 e^{-ik_2 \cdot x} + b_2^\dagger e^{ik_2 \cdot x}) \int \frac{d^3k_A}{(2\pi)^3} \frac{1}{\sqrt{2E_{\mathbf{k}_A}}} (a_{\mathbf{k}} e^{-ik_A \cdot x} + a_{\mathbf{k}}^\dagger e^{ik_A \cdot x}) \\
 &\quad \times \sqrt{2E_A} a_{\mathbf{p}}^\dagger | 0 \rangle.
 \end{aligned}$$

From this expansion we rearrange terms so that only the creation and annihilation operators

(with appropriate exponentials) are left within the vev;

$$\mathcal{M} = \frac{-ig}{2} \int d^4x \frac{d^3k_1}{(2\pi)^3} \frac{d^3k_2}{(2\pi)^3} \frac{d^3k_A}{(2\pi)^3} \sqrt{\frac{2E_1 2E_2 2E_a}{2E_{\mathbf{k}_1} 2E_{\mathbf{k}_2} 2E_{\mathbf{k}_A}}} \\ \times \langle 0 | b_{\mathbf{p}_1} b_{\mathbf{p}_2} (b_1 e^{-ik_1 \cdot x} + b_1^\dagger e^{ik_1 \cdot x}) (b_2 e^{-ik_2 \cdot x} + b_2^\dagger e^{ik_2 \cdot x}) (a_{\mathbf{k}} e^{-ik_A \cdot x} + a_{\mathbf{k}}^\dagger e^{ik_A \cdot x}) a_{\mathbf{p}}^\dagger | 0 \rangle.$$

Suppressing the exponentials, the vev takes the following form:

$$\langle 0 | b_{\mathbf{p}_1} b_{\mathbf{p}_2} (b_1 + b_1^\dagger) (b_2 + b_2^\dagger) (a_{\mathbf{k}} + a_{\mathbf{k}}^\dagger) a_{\mathbf{p}}^\dagger | 0 \rangle. \quad (5.46)$$

Once again we are tasked with reducing this vev to a product of δ -functions. We follow the same manipulations to simplify it. First we distribute the $a_{\mathbf{k}}$ and $a_{\mathbf{k}}^\dagger$;

$$\begin{aligned} & \langle 0 | b_{\mathbf{p}_1} b_{\mathbf{p}_2} (b_1 + b_1^\dagger) (b_2 + b_2^\dagger) (a_{\mathbf{k}} + a_{\mathbf{k}}^\dagger) a_{\mathbf{p}}^\dagger | 0 \rangle \\ &= \langle 0 | b_{\mathbf{p}_1} b_{\mathbf{p}_2} (b_1 + b_1^\dagger) (b_2 + b_2^\dagger) a_{\mathbf{k}} a_{\mathbf{p}}^\dagger | 0 \rangle + \langle 0 | b_{\mathbf{p}_1} b_{\mathbf{p}_2} (b_1 + b_1^\dagger) (b_2 + b_2^\dagger) a_{\mathbf{k}}^\dagger a_{\mathbf{p}}^\dagger | 0 \rangle \\ &= \langle 0 | b_{\mathbf{p}_1} b_{\mathbf{p}_2} (b_1 + b_1^\dagger) (b_2 + b_2^\dagger) a_{\mathbf{k}} a_{\mathbf{p}}^\dagger | 0 \rangle + \langle 0 | a_{\mathbf{k}}^\dagger b_{\mathbf{p}_1} b_{\mathbf{p}_2} (b_1 + b_1^\dagger) (b_2 + b_2^\dagger) a_{\mathbf{p}}^\dagger | 0 \rangle \\ &= \langle 0 | b_{\mathbf{p}_1} b_{\mathbf{p}_2} (b_1 + b_1^\dagger) (b_2 + b_2^\dagger) a_{\mathbf{k}} a_{\mathbf{p}}^\dagger | 0 \rangle + 0. \end{aligned}$$

Then we distribute the $b_{\mathbf{k}_2}$ and $b_{\mathbf{k}_2}^\dagger$;

$$\begin{aligned} &= \langle 0 | b_{\mathbf{p}_1} b_{\mathbf{p}_2} (b_1 + b_1^\dagger) b_2 a_{\mathbf{k}} a_{\mathbf{p}}^\dagger | 0 \rangle + \langle 0 | b_{\mathbf{p}_1} b_{\mathbf{p}_2} (b_1 + b_1^\dagger) b_2^\dagger a_{\mathbf{k}} a_{\mathbf{p}}^\dagger | 0 \rangle \\ &= \langle 0 | b_{\mathbf{p}_1} b_{\mathbf{p}_2} (b_1 + b_1^\dagger) a_{\mathbf{k}} a_{\mathbf{p}}^\dagger b_2 | 0 \rangle + \langle 0 | b_{\mathbf{p}_1} b_{\mathbf{p}_2} (b_1 + b_1^\dagger) b_2^\dagger a_{\mathbf{k}} a_{\mathbf{p}}^\dagger | 0 \rangle \\ &= 0 + \langle 0 | b_{\mathbf{p}_1} b_{\mathbf{p}_2} (b_1 + b_1^\dagger) b_2^\dagger a_{\mathbf{k}} a_{\mathbf{p}}^\dagger | 0 \rangle. \end{aligned}$$

We have reduced the vev to the following simpler form

$$\langle 0 | b_{\mathbf{p}_1} b_{\mathbf{p}_2} (b_1 + b_1^\dagger) b_2^\dagger a_{\mathbf{k}} a_{\mathbf{p}}^\dagger | 0 \rangle. \quad (5.47)$$

We can eliminate the product $a_{\mathbf{k}} a_{\mathbf{p}}^\dagger$ just as we did in the C-decay of ABC theory: simply subtract a vev with the term $a_{\mathbf{p}}^\dagger a_{\mathbf{k}}$ to produce a commutator. It proceeds as follows;

$$\begin{aligned} \langle 0 | b_{\mathbf{p}_1} b_{\mathbf{p}_2} (b_1 + b_1^\dagger) b_2^\dagger a_{\mathbf{k}} a_{\mathbf{p}}^\dagger | 0 \rangle &= \langle 0 | b_{\mathbf{p}_1} b_{\mathbf{p}_2} (b_1 + b_1^\dagger) b_2^\dagger (a_{\mathbf{k}} a_{\mathbf{p}}^\dagger - a_{\mathbf{p}}^\dagger a_{\mathbf{k}}) | 0 \rangle \\ &= \langle 0 | b_{\mathbf{p}_1} b_{\mathbf{p}_2} (b_1 + b_1^\dagger) b_2^\dagger [a_{\mathbf{k}}, a_{\mathbf{p}}^\dagger] | 0 \rangle \\ &= (2\pi)^3 \delta^3(\mathbf{p}_A - \mathbf{k}_A) \langle 0 | b_{\mathbf{p}_1} b_{\mathbf{p}_2} (b_1 + b_1^\dagger) b_2^\dagger | 0 \rangle. \end{aligned} \quad (5.48)$$

To deal with this reduced vev,

$$\langle 0|b_{\mathbf{p}_1}b_{\mathbf{p}_2}(b_1 + b_1^\dagger)b_2^\dagger|0\rangle = \langle 0|b_{\mathbf{p}_1}b_{\mathbf{p}_2}b_1b_2^\dagger|0\rangle + \langle 0|b_{\mathbf{p}_1}b_{\mathbf{p}_2}b_1^\dagger b_2^\dagger|0\rangle, \quad (5.49)$$

we first insert the relation $b_1b_2^\dagger = (2\pi)^3\delta^3(\mathbf{k}_1 - \mathbf{k}_2) + b_2^\dagger b_1$ into the first vev on the right hand side to get

$$\begin{aligned} \langle 0|b_{\mathbf{p}_1}b_{\mathbf{p}_2}b_1b_2^\dagger|0\rangle &= \langle 0|b_{\mathbf{p}_1}b_{\mathbf{p}_2}((2\pi)^3\delta^3(\mathbf{k}_1 - \mathbf{k}_2) + b_2^\dagger b_1)|0\rangle \\ &= (2\pi)^3\delta^3(\mathbf{k}_1 - \mathbf{k}_2)\langle 0|b_{\mathbf{p}_1}b_{\mathbf{p}_2}|0\rangle + \langle 0|b_{\mathbf{p}_1}b_{\mathbf{p}_2}b_2^\dagger b_1|0\rangle \\ &= 0. \end{aligned}$$

For the second term on the right-hand side of we apply the same method twice to ultimately obtain a product of δ -functions; first we do the 2 middle terms $b_{\mathbf{p}_2}$ and b_1^\dagger

$$\begin{aligned} \langle 0|b_{\mathbf{p}_1}b_{\mathbf{p}_2}b_1^\dagger b_2^\dagger|0\rangle &= \langle 0|b_{\mathbf{p}_1}((2\pi)^3\delta^3(\mathbf{p}_2 - \mathbf{k}_1) + b_1^\dagger b_{\mathbf{p}_2})b_2^\dagger|0\rangle \\ &= (2\pi)^3\delta^3(\mathbf{p}_2 - \mathbf{k}_1)\langle 0|b_{\mathbf{p}_1}b_2^\dagger|0\rangle + \langle 0|b_{\mathbf{p}_1}b_1^\dagger b_{\mathbf{p}_2}b_2^\dagger|0\rangle. \end{aligned}$$

Doing the same thing for $b_{\mathbf{p}_1}b_2^\dagger$ in the first term,

$$\begin{aligned} \langle 0|b_{\mathbf{p}_1}b_{\mathbf{p}_2}b_1^\dagger b_2^\dagger|0\rangle &= (2\pi)^3\delta^3(\mathbf{p}_2 - \mathbf{k}_1)\langle 0|(2\pi)^3\delta^3(\mathbf{p}_1 - \mathbf{k}_2) + b_2^\dagger b_{\mathbf{p}_1}|0\rangle + \langle 0|b_{\mathbf{p}_1}b_1^\dagger b_{\mathbf{p}_2}b_2^\dagger|0\rangle \\ &= (2\pi)^3\delta^3(\mathbf{p}_2 - \mathbf{k}_1)(2\pi)^3\delta^3(\mathbf{p}_1 - \mathbf{k}_2) + \langle 0|b_{\mathbf{p}_1}b_1^\dagger b_{\mathbf{p}_2}b_2^\dagger|0\rangle. \end{aligned}$$

For the term $\langle 0|b_{\mathbf{p}_1}b_1^\dagger b_{\mathbf{p}_2}b_2^\dagger|0\rangle$ we use a similar technique as in the case of reducing the aa^\dagger to reduce it to another product of δ -functions

$$\begin{aligned} \langle 0|b_{\mathbf{p}_1}b_1^\dagger b_{\mathbf{p}_2}b_2^\dagger|0\rangle &= \langle 0|(b_{\mathbf{p}_1}b_1^\dagger - b_1^\dagger b_{\mathbf{p}_1})b_{\mathbf{p}_2}b_2^\dagger|0\rangle \\ &= \langle 0|[b_{\mathbf{p}_1}, b_1^\dagger]b_{\mathbf{p}_2}b_2^\dagger|0\rangle \\ &= (2\pi)^3\delta^3(\mathbf{p}_1 - \mathbf{k}_1)\langle 0|b_{\mathbf{p}_2}b_2^\dagger|0\rangle, \end{aligned}$$

and doing the same thing for $b_{\mathbf{p}_2}b_2^\dagger$,

$$\begin{aligned} &= (2\pi)^3\delta^3(\mathbf{p}_1 - \mathbf{k}_1)\langle 0|b_{\mathbf{p}_2}b_2^\dagger - b_2^\dagger b_{\mathbf{p}_2}|0\rangle \\ &= (2\pi)^3\delta^3(\mathbf{p}_1 - \mathbf{k}_1)(2\pi)^3\delta^3(\mathbf{p}_2 - \mathbf{k}_2). \end{aligned}$$

Thus our initial vev has been reduce to the following product of δ -functions:

$$(2\pi)^3 \delta^3(\mathbf{p}_2 - \mathbf{k}_1)(2\pi)^3 \delta^3(\mathbf{p}_1 - \mathbf{k}_2) + (2\pi)^3 \delta^3(\mathbf{p}_1 - \mathbf{k}_1)(2\pi)^3 \delta^3(\mathbf{p}_2 - \mathbf{k}_2).$$

Notice that there are two products instead of one as was the case in calculating $C \rightarrow A + B$ in the ABC theory. Since the ABB theory contains 2 indistinguishable particles, we need to add a $\frac{1}{2}$ to the interaction term which is why we started with $\mathcal{H}_I = \frac{1}{2}g\phi_A\phi_B^2$. If all the particles were indistinguishable (say for a self-interacting ϕ^3 theory) we would need a factor of $\frac{1}{3!}$ in front of our interaction term.

We now plug this expression for the vev back into our original integral calculation of decay amplitude

$$\begin{aligned} \mathcal{M} &= \frac{-ig}{2} \int d^4x \frac{d^3k_1}{(2\pi)^3} \frac{d^3k_2}{(2\pi)^3} \frac{d^3k_A}{(2\pi)^3} \sqrt{\frac{2E_1 2E_2 2E_a}{2E_{\mathbf{k}_1} 2E_{\mathbf{k}_2} 2E_{\mathbf{k}_a}}} e^{i(k_1+k_2+k_A)\cdot x} \\ &\quad \times \left((2\pi)^3 \delta^3(\mathbf{p}_2 - \mathbf{k}_1)(2\pi)^3 \delta^3(\mathbf{p}_1 - \mathbf{k}_2) + (2\pi)^3 \delta^3(\mathbf{p}_1 - \mathbf{k}_1)(2\pi)^3 \delta^3(\mathbf{p}_2 - \mathbf{k}_2) \right) \\ &\quad \times (2\pi)^3 \delta^3(\mathbf{k}_A - \mathbf{p}_A). \end{aligned}$$

The integral over k_A may be calculated in a manner similar to those calculated in the decay amplitude in ABC-theory, using the property of δ -functions:

$$\int \frac{d^3k_A}{(2\pi)^3} \sqrt{\frac{2E_A}{2E_{\mathbf{k}_A}}} e^{-ik_A\cdot x} (2\pi)^3 \delta(\mathbf{p}_A - \mathbf{k}_A) = e^{-ip_A\cdot x}. \quad (5.50)$$

Plugging this in we are left with a sum of a product of coupled integrals over k_1 and k_2 , respectively:

$$\begin{aligned} \mathcal{M} &= \frac{-ig}{2} \int d^4x e^{-ip_A\cdot x} \\ &\quad \times \left[\int \frac{d^3k_1}{(2\pi)^3} \sqrt{\frac{2E_1}{2E_{\mathbf{k}_1}}} e^{ik_1\cdot x} (2\pi)^3 \delta^3(\mathbf{p}_2 - \mathbf{k}_1) \int \frac{d^3k_2}{(2\pi)^3} \sqrt{\frac{2E_2}{2E_{\mathbf{k}_2}}} e^{ik_2\cdot x} (2\pi)^3 \delta^3(\mathbf{p}_1 - \mathbf{k}_2) \right] \\ &\quad + \left[\int \frac{d^3k_1}{(2\pi)^3} \sqrt{\frac{2E_2}{2E_{\mathbf{k}_2}}} e^{ik_1\cdot x} (2\pi)^3 \delta^3(\mathbf{p}_1 - \mathbf{k}_1) \int \frac{d^3k_2}{(2\pi)^3} \sqrt{\frac{2E_1}{2E_{\mathbf{k}_1}}} e^{ik_2\cdot x} (2\pi)^3 \delta^3(\mathbf{p}_2 - \mathbf{k}_2) \right]. \end{aligned} \quad (5.51)$$

Once more we apply the rules of δ -functions to obtain

$$\mathcal{M} = \frac{-ig}{2} \int d^4x e^{-ip_A\cdot x} \left\{ \frac{E_1}{E_2} e^{ip_2\cdot x} \frac{E_2}{E_1} e^{ip_1\cdot x} + e^{ip_1\cdot x} e^{ip_2\cdot x} \right\}. \quad (5.52)$$

Combining the exponentials and cancelling the E_1 and E_2

$$\mathcal{M} = \frac{-ig}{2} \int d^4x 2e^{i(p_1+p_2-p_A)\cdot x}. \quad (5.53)$$

Notice that we get an extra factor of 2 in the integrand which is why the $\frac{1}{2}$ was included in the interaction term of H , $\mathcal{H}_{int} = \frac{1}{2}g\phi_A\phi_B^2$. Using the Fourier Transform of the exponential, we obtain as our final result

$$\boxed{\mathcal{M} = -ig(2\pi)^4\delta^4(p_1 + p_2 - p_A)}. \quad (5.54)$$

CHAPTER 6

DIMENSIONAL REGULARIZATION

The previous calculations have been for *tree-level diagrams*, ones with no internal loops. When we calculate higher-order contributions to the perturbative expansion, these internal lines will cause the resulting integrals to diverge. In order to make physical sense of these infinite values, we introduce a process called *dimensional regularization*, a process originally found in [14].

The basic idea is as follows. First we perform the calculation in D “continuous” dimensions. Note that we are in no way attempting to give a physical interpretation to the idea of a continuous dimension. It is simply a mathematical device we use to isolate the divergence in the integral. After an application of Feynman’s Trick to the integrand, we simplify the denominator and perform a shift in the integration variable which results in a radial integral. After integrating out the angular dependence and evaluating the remaining radial integral, we then let $D = N - \epsilon$ where N is the number of (physical) dimensions being considered and expand everything in a power series of ϵ . We will obtain a singular part, a finite part, and a vanishing contribution.

Ultimately, the singular part of the integral will be removed through a process known as *renormalization*. See [10], vol. II of [13], part II of [4], Ch. 9 of [16], and Ch’s. 12 and 16 of [15] for introductory treatments. By redefining \mathcal{L} to include a “counterterm” which has a renormalization constant equal to the singular term, we can derive a new Feynman rule which will cancel out the singular part. Thus we are left with a finite result for the higher order contribution to the scattering amplitude.

6.1 One-loop Correction to the A-Propagator in ABB Theory

Consider now the “self-energy” correction to the propagator in a theory given by a cubic interaction (ϕ^3 , ABC, or ABB) as shown in Figure 6.1. Physically, this diagram is interpreted as an A-particle propagating in space, splitting into two “virtual” particles which

then recombine. We perform the 2nd-order contribution in a continuous D dimensions in order to isolate the singularity. The integral corresponding to the one-loop correction to the A-propagator in ABB theory is given by

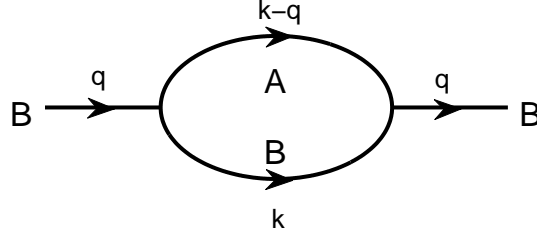


Figure 6.1: Self-Energy of The A-Particle in ABB Theory

$$\Sigma(q^2) = i \frac{g^2}{2} \frac{1}{4} \int \frac{d^D k}{(2\pi)^D} \frac{1}{(k^2 - m_B^2)((k - q)^2 - m_A^2)}. \quad (6.1)$$

The integrand in (6.1) may be expressed as an integral via Feynmann's trick

$$\frac{1}{AB} = \int_0^1 dx \frac{1}{[Ax + B(1 - x)]^2}, \quad (6.2)$$

so that (6.1) simplifies after inserting this result:

$$\begin{aligned} \Sigma(q^2) &= i \frac{g^2}{8} \int \frac{d^D k}{(2\pi)^D} \int_0^1 dx \frac{1}{[(k^2 - m_B^2)(1 - x) + ((k - q)^2 - m_A^2)x]^2} \\ &= i \frac{g^2}{8} \int \frac{d^D k}{(2\pi)^D} \int_0^1 dx \frac{1}{[k^2 - 2k \cdot qx + q^2x - m_B^2(1 - x) - m_A^2x]^2}. \end{aligned}$$

Next we complete the square of the terms containing k by adding and subtracting a q^2x^2 term. We then perform a change of variables $k' = k - qx$. The dx and $d^D k'$ integrals are iterated integrals and can be flipped so that we are left with

$$\Sigma(q^2) = i \frac{g^2}{8} \int_0^1 dx \int \frac{d^D k'}{(2\pi)^D} \frac{1}{[(k')^2 - q^2x(1 - x) - m_B^2(1 - x) - m_A^2x]^2}. \quad (6.3)$$

The $d^D k'$ integral is then given by

$$\int \frac{d^D k}{(2\pi)^D} \frac{1}{[X^2 - k^2]^a} = \frac{i}{(4\pi)^{D/2}} \frac{\Gamma(a - D/2)}{\Gamma(a)} \left(\frac{1}{X^2}\right)^{a-D/2}, \quad (6.4)$$

so we obtain for our particular case ($X^2 = q^2 x(1-x) - m_B^2(1-x) - m_A^2 x$ and $a = 2$)

$$\Sigma(q^2) = -\frac{g^2}{8(4\pi)^{D/2}} \frac{\Gamma(2 - D/2)}{\Gamma(2)} \int_0^1 dx \frac{1}{[q^2 x(1-x) - M^2]^{2-D/2}}. \quad (6.5)$$

$M^2 \equiv m_B^2(1-x) + m_A^2 x$ is a term that will change depending on the propagator being computed in the theory being considered. Notice that upon a change of variables $y = 1-x$ we recover the same integral

$$\int_0^1 dx \frac{1}{q^2 x(1-x) - m_B^2(1-x) - m_A^2 x} = \int_0^1 dy \frac{1}{q^2 y(1-y) - m_B^2 y - m_A^2(1-y)}, \quad (6.6)$$

so that the choice of parameterization is arbitrary. Letting $D = 4 - \epsilon$ we will ultimately expand (6.5) in a power series of ϵ :

$$\Sigma(q^2) = \frac{g}{8} \frac{g}{(4\pi)^2} \Gamma\left(\frac{\epsilon}{2}\right) \int_0^1 dx \left[\frac{4\pi\mu^2}{q^2 x(1-x) - M^2} \right]^{\epsilon/2}$$

where we made the shift $g \rightarrow g\mu^{\epsilon/2}$ to keep the integrand unitless. Now the series expansion for $\Gamma(z)$ is well known for any complex z ,

$$\Gamma(z) = \frac{1}{z} - \gamma + \mathcal{O}(z), \quad (6.7)$$

For the integrand in (6.1) we use the identity

$$z^\epsilon = e^{\epsilon \log(z)} = 1 + \epsilon \log(z) + \frac{1}{2}(\epsilon \log(z))^2 + \dots, \quad (6.8)$$

so that upon insertion of these series expansions, (6.1) becomes

$$\Sigma q^2 = \frac{1}{8} \left(\frac{-g}{4\pi}\right)^2 \left(\frac{2}{\epsilon} - \gamma + \mathcal{O}(\epsilon)\right) \int_0^1 d \left[1 - \frac{\epsilon}{2} \log\left(\frac{q^2 x(1-x) - M^2}{4\pi\mu^2}\right) + \mathcal{O}(\epsilon^2) \right].$$

Our final result for the one-loop correction to the B-propagator in ABB theory is

$$\Sigma(q^2) = -\frac{1}{8} \left(\frac{g\mu}{4\pi}\right)^2 \left[\frac{2}{\epsilon} - \gamma - \int_0^1 dx \log\left(\frac{q^2 x(1-x) - M^2}{4\pi\mu^2}\right) + \mathcal{O}(\epsilon) \right] \quad (6.9)$$

Once we introduce a new Feynman rule for self-interactions in the ABB theory (by the renormalization process discussed above), the offending singular term in the expansion above will vanish and we will be left with a finite result.

6.2 One-loop Correction to the Vertex in ABC Theory

We have previously calculated the first order contribution to the vertex of ABC theory, namely the amplitude corresponding to the vertex itself. If we were to calculate the 2nd-order contribution

$$\frac{1}{2} \int d^4x_1 d^4x_2 \langle 0 | c \phi_{Ax} \phi_{Bx} \phi_{Cx} \phi_{Ay} \phi_{By} \phi_{Cy} a^\dagger b^\dagger | 0 \rangle, \quad (6.10)$$

the vev vanishes since there are an odd number of fields inside it. So we continue to the third-order contribution which is shown in the following diagram: If we write the third-order

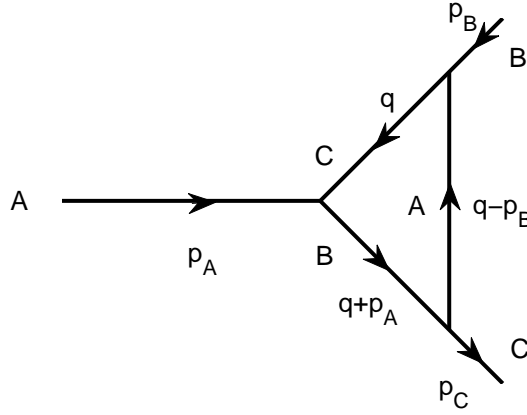


Figure 6.2: One-loop correction to the vertex of ABC theory

contribution to this process we obtain (neglecting normalizations)

$$\frac{(-ig)^3}{3!} \int d^4x_1 d^4x_2 d^4x_3 \langle 0 | ab \phi_{Ax} \phi_{Bx} \phi_{Cx} \phi_{Ay} \phi_{By} \phi_{Cy} \phi_{Az} \phi_{Bz} \phi_{Cz} c^\dagger | 0 \rangle \quad (6.11)$$

After proper manipulations which amount to applying the Feynman Rules for the theory we are left with the remaining integral for our amplitude

$$(-ig)^3 \int \frac{d^D q}{(2\pi)^D} \frac{1}{[(q + p_A)^2 - m_B^2][(q - p_B)^2 - m_A^2](q^2 - m_C^2)}. \quad (6.12)$$

Using Feynman's trick we obtain for the integrand

$$\int_0^1 dx dy \frac{1}{\left[[(q - p_B)^2 - m_A^2]x + [(q + p_A)^2 - m_B^2]y + (q^2 - m_C^2)(1 - x - y) \right]^3}. \quad (6.13)$$

To simplify the denominator we expand everything out and complete the square then perform a change of variables $q' = q - (p_B x - p_A y)$

$$\left[q^2 - 2q \cdot (p_B x - p_A y) + p_B^2 x + p_A^2 - m_A^2 x - m_B^2 y - m_C^2 (1 - x - y) \right]^3 \quad (6.14)$$

$$= \left[(q - (p_B x - p_A y))^2 - p_B^2 x(1 - x) - p_A^2 y(1 - y) - M^2 \right]^2 \quad (6.15)$$

$$= [(q')^2 - P^2 - M^2]^3, \quad (6.16)$$

so that 6.12 becomes

$$(-ig)^3 \int_0^1 dx dy \int \frac{d^D q'}{(2\pi)^D} \frac{1}{[(q')^2 - P^2 - M^2]^3}. \quad (6.17)$$

We can now evaluate the $d^D q'$ integral by methods of [16] (see Ch. 8) and get

$$\frac{(-g)^3}{(4\pi)^{D/2}} \int_0^1 dx dy \frac{\Gamma(3 - D/2)}{\Gamma(3)} \frac{1}{[P^2 + M^2]^{3-D/2}}. \quad (6.18)$$

(6.18) clearly has a form similar to the same integral done for the self-energy of the B-propagator in ABB theory. The only difference here is the $\Gamma(3 - D/2)$ instead of a $\Gamma(2 - D/2)$. This means that this integral will be finite in 4-dimensional spacetime. But we will be left with a singular term in 6 dimensions. Letting $D = 6 - \epsilon$ we obtain

$$, -\frac{1}{2} \left(\frac{g\mu^{\epsilon/6}}{4\pi} \right)^3 \Gamma\left(\frac{\epsilon}{2}\right) \int_0^1 dx dy \quad (6.19)$$

so that the power series expansion of the one-loop correction to the vertex becomes

$$\mathcal{M} = -\frac{1}{2} \left(\frac{g\mu^{\epsilon/6}}{4\pi} \right)^3 \left[\frac{2}{\epsilon} - \gamma - \log\left(\frac{P^2 + M^2}{4\pi\mu^2}\right) + \mathcal{O}(\epsilon) \right]. \quad (6.20)$$

We obtain a result similar to the one-loop correction to the propagator, but in 6 spacetime-dimensions.

6.3 Fourth-Order Self Energy Diagram: Nested Divergence

Next we calculate the fourth-order contribution to the self-energy propagator in a cubic interaction. We will calculate this for the A-propagator in ABC theory, and we will show that different diagrams will give different resulting contributions to the finite part which is recovered after renormalization.

Consider the propagator correction corresponding to an A-particle splitting into a B and C particle. The virtual B particle then splits itself into virtual A and C particles before recombining into a B particle which then recombines with the original virtual C particle. Since there are two loops we will be integrating over two independent momentum variables.

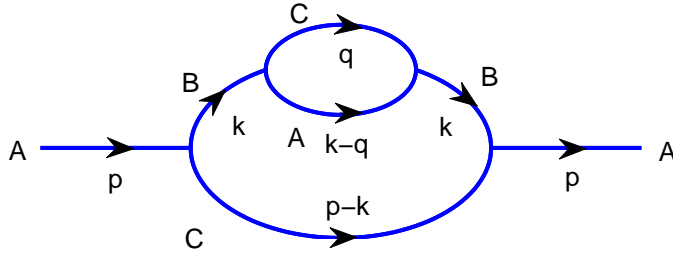


Figure 6.3: Two-loop correction to the A-propagator: nested divergence

Let the incoming momentum of the A-particle be denoted by q . By four-momentum conservation enforced by the Feynman rules, the momentum of the outgoing virtual B-particle will be denoted as k ; for the C-particle, $k + q$. For the nested loop the momentum of the virtual A is denoted p and the virtual B, $p + k$. The resulting integral is

$$g^4 \int \frac{d^D k}{(2\pi)^D} \frac{d^D p}{(2\pi)^D} \frac{1}{[(p - k)^2 - m_C^2](k^2 - m_B^2)(q^2 - m_C^2)[(q - k)^2 - m_A^2]}. \quad (6.21)$$

One can clearly see that the $d^D p$ can be evaluated first and it is the same calculation as that of the one-loop correction to the self-energy. We thus replace it with the computed values

expressed in terms of D (not yet in a power series of ϵ):

$$\begin{aligned}
\mathcal{M} &= g^4 \int \frac{d^D k}{(2\pi)^D} \frac{1}{[(p-k)^2 - m_C^2](k^2 - m_B^2)^2} \int \frac{d^D q}{(2\pi)^D} \frac{1}{(q^2 - m_C^2)[(q-k)^2 - m_A^2]} \\
&= g^4 \int \frac{d^D k}{(2\pi)^D} \frac{1}{[(p-k)^2 - m_C^2](k^2 - m_B^2)^2} \frac{1}{(4\pi)^{D/2}} \frac{\Gamma(2 - D/2)}{\Gamma(2)} \\
&\quad \times \int_0^1 dx \frac{1}{[k^2 x(1-x) - m_C^2(1-x) - m_A^2 x]^{2-D/2}} \\
&= g^4 \frac{\Gamma(2 - D/2)}{(4\pi)^{D/2}} \int_0^1 dx [x(1-x)]^{D/2-2} \\
&\quad \times \int \frac{d^D k}{(2\pi)^D} \frac{1}{\left[[(p-k)^2 - m_C^2](k^2 - m_B^2)^2 \left(k^2 - \frac{m_C^2}{x} - \frac{m_A^2}{1-x}\right)\right]^{2-D/2}}.
\end{aligned}$$

In the last line we collected the terms in the new integrand and factored out the $[x(1-x)]^{D/2-2}$. Using Feynman's Trick, we obtain for the new integrand

$$\frac{\Gamma(5 - D/2)}{\Gamma(2 - D/2)} \int_0^1 dy dz \frac{y^{1-D/2} z}{\left\{ \left(k^2 - \frac{m_C^2}{x} - \frac{m_A^2}{1-x}\right) y + (k^2 - m_B^2) z + [(p-k)^2 - m_C^2] (1-y-z) \right\}^{5-D/2}},$$

so that the original integral becomes

$$\begin{aligned}
\mathcal{M} &= g^4 \frac{\Gamma(5 - D/2)}{(4\pi)^{D/2}} \int_0^1 dx [x(1-x)]^{2-D/2} \int_0^1 dy y^{1-D/2} \int_0^1 dz z \\
&\quad \times \int \frac{d^D k}{(2\pi)^D} \frac{1}{\left\{ \left(k^2 - \frac{m_C^2}{x} - \frac{m_A^2}{1-x}\right) y + (k^2 - m_B^2) z + [(p-k)^2 - m_C^2] (1-y-z) \right\}^{5-D/2}}.
\end{aligned}$$

To evaluate the $d^D k$ integral we once again complete the square in k and then make a change of variables $k' = k - p(1-y-z)$ so that the resulting integral becomes

$$\int \frac{d^D k'}{(2\pi)^D} \frac{1}{\left[(k')^2 + p^2(1-y-z)(y+z) - M^2 \right]^{5-D/2}}, \quad (6.22)$$

where M^2 is used to collect the mass terms and will change depending on both the theory and the particular loop. It is given by

$$M^2 = m_C^2 \left(\frac{y}{x} \right) + m_A^2 \left(\frac{y}{1-x} \right) - m_B^2 z - m_C^2 (1-y-z). \quad (6.23)$$

The integral (6.22) can then be evaluated accordingly and we obtain

$$\frac{1}{(4\pi)^{D/2}} \frac{\Gamma(5 - D)}{\Gamma(5 - D/2)} \frac{1}{(p^2(1-y-z)(y+z) - M^2)^{5-D}}.$$

Inserting this result into our original integral we then obtain

$$\mathcal{M} = g^4 \frac{\Gamma(5-D)}{(4\pi)^D} \int_0^1 dx [x(1-x)]^{2-D/2} \int_0^1 dy y^{1-D/2} \int_0^1 dz z \frac{1}{(p^2(1-y-z)(y+z) - M^2)^{5-D}}. \quad (6.24)$$

From this we can conclude that the amplitude corresponding to this process is finite in 4 dimensions of spacetime. For if we insert $D = 4 - \epsilon$, the Γ -function would yield a finite expansion of $\Gamma(1 + \epsilon/2)$.

6.4 Fourth-Order Self-Energy Diagram: Overlapping Divergence

The other contribution to the self-energy at fourth-order is one where a particle splits into two virtual particles which then exchange another virtual particle before the original virtual particles recombine into the original particle as indicated by the following figure. We

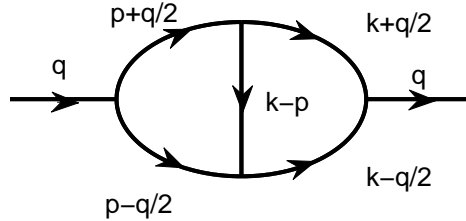


Figure 6.4: Two-loop correction to the propagator: overlapping divergence

are going to work in ϕ^3 theory. And in order to evaluate this type of integral, we will set $m = 0$. This discussion follows [15], pp. 541-546. The entire integral looks like

$$\Sigma_{ov} = (-ig)^4 \int \frac{d^D k}{(2\pi)^D} \frac{d^D q}{(2\pi)^D} \frac{1}{\left[(p - \frac{1}{2}q)^2 (p + \frac{1}{2}q)^2 (k - \frac{1}{2}q)^2 (k + \frac{1}{2}q)^2 (k - p)^2 \right]}. \quad (6.25)$$

We will evaluate the $d^D k$ integral first. Expressing the denominator as an integral of Feynman parameters we have

$$-2ig \int_0^1 dx \int_0^{1-x} dy \frac{d^D k}{(2\pi)^D} \frac{1}{\left[(k + \frac{1}{2}q)^2 x + (k - \frac{1}{2}q)^2 y + (k-p)^2 (1-x-y) \right]^3}. \quad (6.26)$$

The term which is cubed in the denominator of the integrand can be simplified to

$$k^2 - 2k \cdot q \left(\frac{1}{2}q(x-y) - p(1-x-y) \right) + \frac{1}{4}q^2(x+y) + p^2(1-x-y). \quad (6.27)$$

So we shift the k variable $k = k - \frac{1}{2}q(x-y) - p(1-x-y)$ to complete the square, and we're left with

$$-2ig^2 \int_0^1 dx \int_0^{1-x} dy \frac{d^D k}{(2\pi)^D} \frac{1}{[k^2 - X^2]^3}, \quad (6.28)$$

where we define X^2 from

$$X^2 = [p(1-x-y) - \frac{1}{2}q(x-y)]^2 - p^2(1-x-y) - \frac{1}{4}q^2(x+y). \quad (6.29)$$

If we make the following change of variables

$$\xi = x + y \qquad \xi\eta = \frac{1}{2}(x-y), \quad (6.30)$$

then (6.29) becomes

$$\begin{aligned} X^2 &= [p(1-\xi) - \xi\eta q]^2 - p^2(1-\xi) - \frac{1}{4}q^2\xi \\ &= p^2(1-\xi)\xi - 2p \cdot q(1-\xi)\xi\eta - q^2(1-\xi)\xi\eta^2 + q^2\eta^2(1-\xi)\xi + \xi^2\eta^2q^2 - \frac{1}{4}q^2\xi \\ &= -(1-\xi)\xi(p+q\eta)^2 - q^2\xi\left(\frac{1}{4} - \eta^2\right) \\ &\equiv (1-\xi)\xi Y^2. \end{aligned}$$

From this, Y^2 can be expressed as

$$Y^2 = q^2 \left(\frac{\frac{1}{4} - \eta^2}{1 - \xi} \right) - (p + q\eta)^2. \quad (6.31)$$

This change of variables from x, y to ξ, η imposes the following Jacobian

$$\left| \frac{\partial(x, y)}{\partial(\xi, \eta)} \right| = \begin{vmatrix} \frac{\partial x}{\partial \xi} & \frac{\partial x}{\partial \eta} \\ \frac{\partial y}{\partial \xi} & \frac{\partial y}{\partial \eta} \end{vmatrix} = \begin{vmatrix} \frac{1}{2} + \eta & \xi \\ \frac{1}{2} - \eta & -\xi \end{vmatrix} = \xi,$$

which also changes the integration of the Feynman parameters

$$\int_0^1 dx \int_0^1 dy \Rightarrow \int_0^1 \xi d\xi \int_{-1/2}^{1/2} d\eta.$$

Our $d^D k$ integral then becomes

$$-2ig^2 \int_0^1 \xi d\xi \int_{-1/2}^{1/2} d\eta \int \frac{d^D k}{(2\pi)^D} \frac{1}{[k^2 - X^2]^3} = \frac{2g^2}{(4\pi)^{D/2}} \frac{\Gamma(3 - D/2)}{\Gamma(3)} \int_0^1 \xi d\xi \int_{-1/2}^{1/2} d\eta \left(\frac{\mu^2}{X^2}\right)^{3-D/2}.$$

Setting $D = 6 - \epsilon$ we obtain our result for the $d^D k$ integral in terms of ϵ . Below is written the singular part $\Lambda_0(p, q)$ and the finite contribution $\Lambda_R(p, q)$ of the $d^D k$ integral:

$$\Lambda_R(p, q) = -\frac{g^2}{2(4\pi)^3} \int_0^1 \xi d\xi \int_{-1/2}^{1/2} d\eta \log\left(\frac{X^2}{\mu^2}\right) \quad (6.32)$$

$$\Lambda_0 = \frac{g^2}{(4\pi)^{D/2}} \Gamma\left(1 + \frac{\epsilon}{2}\right) \frac{1}{\epsilon}. \quad (6.33)$$

We use the integral for $\Lambda(p, q)$ for the full integral Σ_{ov} . Recalling that the momentums for the other two virtual particles are $(p \pm \frac{1}{2}q)$, our full integral reads

$$\Sigma_{ov} = i\frac{g^2}{2} \int \frac{d^D p}{(2\pi)^D} \frac{\Lambda(p, q)}{(p + \frac{1}{2}q)^2 (p - \frac{1}{2}q)^2} \quad (6.34)$$

$$= \frac{ig^4}{2(4\pi)^{D/2}} \Gamma\left(\frac{\epsilon}{2}\right) \int_0^1 d\xi \int_{-1/2}^{1/2} d\eta \int \frac{d^D p}{(2\pi)^D} \frac{\xi^{1-\epsilon/2} (1-\xi)^{-\epsilon/2} (\mu^2)^\epsilon}{(p + \frac{1}{2}q)^2 (p - \frac{1}{2}q)^2 (Y^2)^{-\epsilon/2}}, \quad (6.35)$$

where in the second line we combined the ξ from the Jacobian with the $[(1-\xi)\xi]^{-\epsilon/2}$ from the X^2 term leaving Y^2 .

Because of the overlapping divergence we must resort to improving the convergence of the $d^D p$ integral by following a process similar to the original one in [14]. It amounts to an integration by parts of

$$\int_{-1/2}^{1/2} d\eta \frac{1}{(p + \frac{1}{2}q)^2 (p - \frac{1}{2}q)^2 (Y^2)^{-\epsilon/2}}. \quad (6.36)$$

We first set

$$u = \frac{1}{(Y^2)^{\epsilon/2}} \quad dv = \frac{1}{(p - \frac{1}{2}q)^2 (p + \frac{1}{2}q)^2} \quad (6.37)$$

which gives the terms in the new integrand:

$$du = \frac{\epsilon}{2} \frac{1}{(Y^2)^{1+\epsilon/2}} \frac{\partial Y^2}{\partial \eta} \quad v = \frac{1}{(p - \frac{1}{2}q)^2 (p + \frac{1}{2}q)^2}.$$

The calculation of $\frac{\partial Y^2}{\partial \eta}$ is straightforward:

$$\frac{\partial Y^2}{\partial \eta} = -2p \cdot q - 2\eta q^2 + \frac{2\eta q^2}{1 - \xi}.$$

These results give us our integration by parts:

$$\begin{aligned} & \int_{-1/2}^{1/2} d\eta \frac{1}{(p - \frac{1}{2}q)^2 (p + \frac{1}{2}q)^2 (Y^2)^{\epsilon/2}} \\ &= \frac{1}{(Y^2)^{\epsilon/2}} \frac{1}{(p + \frac{1}{2}q)^2 (p + \frac{1}{2}q)^2} \Big|_{-1/2}^{1/2} - \frac{\epsilon}{2} \int_{-1/2}^{1/2} \eta d\eta \frac{-2p \cdot q - 2\eta q^2 + \frac{2\eta q^2}{1 - \xi}}{(p - \frac{1}{2}q)^2 (p + \frac{1}{2}q)^2 (Y^2)^{1+\epsilon/2}}. \end{aligned} \quad (6.38)$$

6.4.1 Calculating the Boundary Contribution of the $d\eta$ Integral

For the boundary term, setting $\eta = \pm \frac{1}{2}$ into (6.31) cancels the q^2 term leaving $(p \pm \frac{1}{2}q)^2$, and we are left with

$$\frac{1}{(Y^2)^{\epsilon/2}} \frac{1}{(p + \frac{1}{2}q)^2 (p + \frac{1}{2}q)^2} \Big|_{-1/2}^{1/2} = \frac{1}{2} \left[\frac{1}{(p + \frac{1}{2}q)^{1+\epsilon/2} (p - \frac{1}{2}q)^2} + \frac{1}{(p + \frac{1}{2}q)^2 (p - \frac{1}{2}q)^{1+\epsilon/2}} \right]. \quad (6.39)$$

The first part of Σ_{ov} from the boundary term can then be calculated:

$$\Sigma_{ov}^{(1)} = \frac{ig^4}{2(4\pi)^{D/2}} \Gamma\left(\frac{\epsilon}{2}\right) \int_0^1 \xi^{1-\epsilon/2} (1 - \xi)^{-\epsilon/2} \int \frac{d^D p}{(2\pi)^D} \frac{(\mu^2)^\epsilon}{[(p + \frac{1}{2}q)^2]^{1+\epsilon/2} (p - \frac{1}{2}q)^2}. \quad (6.40)$$

For the $d\xi$ integral note that $\xi^{1-\epsilon/2} (1 - \xi)^{-\epsilon/2} = \xi^{(2-\epsilon/2)-1} (1 - \xi)^{(1-\epsilon/2)-1}$ which is exactly the form we need for the definition of the B-function

$$B(\rho, \sigma) \equiv \int_0^1 dz z^{\rho-1} (1 - z)^{\sigma-1} = \frac{\Gamma(\rho)\Gamma(\sigma)}{\Gamma(\rho + \sigma)}. \quad (6.41)$$

To calculate the $d^D p$ integral we use Feynman parameters as usual. The denominator in the integrand is expressed as

$$\frac{1}{(p + \frac{1}{2}q)^{1+\epsilon/2} (p - \frac{1}{2}q)^2} = \frac{\Gamma((2 + \epsilon/2))}{\Gamma(1 + \epsilon/2)} \int_0^1 dx \frac{x^{\epsilon/2}}{\left[(p + \frac{1}{2}q)^2 x + (p - \frac{1}{2}q)^2 (1 - x) \right]^{2+\epsilon/2}} \quad (6.42)$$

which is the same expression for the other boundary term. The term in the denominator raised to the $2 + \epsilon/2$ power can be simplified to

$$\begin{aligned}
(p + \frac{1}{2}q)^2 x + (p - \frac{1}{2}q)^2 (1 - x) &= p^2 - p \cdot q(1 - 2x) + \frac{1}{4}q^2 \\
&= p^2 - 2p \cdot q \frac{1}{2}(1 - 2x) + q^2 \frac{1}{4}(1 - 2x)^2 - q^2 \frac{1}{4}(1 - 2x)^2 + \frac{1}{4}q^2 \\
&= \left(p - q \frac{1}{2}(1 - 2x)\right)^2 - \frac{1}{4} \left[q^2 - 4q^2 x + 4q^2 x^2\right] + \frac{1}{4}q^2 \\
&= p^2 + q^2 x(1 - x),
\end{aligned}$$

where in the last step we shifted $p \rightarrow \frac{1}{2}q(1 - 2x)$. Combining (6.42) and the ξ 's in (6.41) we obtain

$$\Sigma_{ov}^{(1)} = \frac{ig^4}{2(4\pi)^{D/2}} \Gamma\left(\frac{\epsilon}{2}\right) B\left(2 - \frac{\epsilon}{2}, 1 - \frac{\epsilon}{2}\right) \frac{\Gamma(2 + \epsilon/2)}{\Gamma(1 + \epsilon/2)} \int_0^1 dx \int \frac{d^D p}{(2\pi)^{D/2}} \frac{(\mu^2)^\epsilon x^{\epsilon/2}}{[p^2 + q^2 x(1 - x)]^{2 + \epsilon/2}}. \quad (6.43)$$

The $d^D p$ integral can finally be calculated and we obtain

$$\int \frac{d^D p}{(2\pi)^{D/2}} \frac{(\mu^2)^\epsilon}{[p^2 + q^2 x(1 - x)]^{2 + \epsilon/2}} = \frac{1}{(4\pi)^{D/2}} \frac{\Gamma(2 + \epsilon/2 - D/2)}{\Gamma(2 + \epsilon/2)} \frac{(\mu^2)^\epsilon}{[q^2 x(1 - x)]^{2 + \epsilon/2 - D/2}}. \quad (6.44)$$

So we plug this result into (6.43). The $(4\pi)^{D/2}$'s combine. Noting that $2 + \epsilon/2 - D/2 = \epsilon - 1$ when $D = 6 - \epsilon$, we obtain a factor of q^2 in the numerator. The $[x(x - 1)]^{\epsilon - 1}$ term in the denominator combines with the $x^{\epsilon/2}$ to give another B-function. Finally the $\Gamma(\frac{\epsilon}{2})$ cancels with the $\Gamma(1 + \frac{\epsilon}{2}) = \frac{\epsilon}{2}\Gamma(\frac{\epsilon}{2})$ in the denominator leaving a $\frac{2}{\epsilon}$. Collecting all these results we are left with

$$\Sigma_{ov}^{(1)} = q^2 \frac{g^4}{2(4\pi)^D} \left(\frac{2}{\epsilon}\right) B\left(2 - \frac{\epsilon}{2}, 1 - \frac{\epsilon}{2}\right) B\left(2 - \frac{\epsilon}{2}, 2 - \epsilon\right) \Gamma(\epsilon - 1) \left(\frac{\mu^2}{q^2}\right)^\epsilon. \quad (6.45)$$

Clearly the $\frac{2}{\epsilon}$ and $\Gamma(\epsilon - 1)$ terms will be the ones that contribute to the singularity at the end of an expansion of $\Sigma_{ov}^{(1)}$ in a series of ϵ . From [16] we know that all poles of the Γ -function (at the negative integers) are simple (see Appendix 8D of [16]):

$$\Gamma(-n + \epsilon) = \frac{(-1)^n}{n!} \left(\frac{1}{\epsilon} + \mathcal{O}(\epsilon^0)\right). \quad (6.46)$$

By using (6.46) and ignoring the ϵ terms in the B-functions we obtain an approximation for (6.45):

$$\Sigma_{ov} \approx q^2 \frac{g^4}{2(4\pi)^6} \left(\frac{2}{\epsilon}\right) B(2, 1) B(2, 2) \left(\frac{-1}{\epsilon}\right) \left(\frac{\mu^2}{q^2}\right)^\epsilon \quad (6.47)$$

$$= -q^2 \frac{g^4}{12(4\pi)^6} \left[\frac{1}{\epsilon^2} \left(\frac{\mu^2}{q^2}\right)^\epsilon \right] \quad (6.48)$$

$$\approx -q^2 \frac{g^4}{12(4\pi)^6} \left[\frac{1}{\epsilon^2} - \frac{1}{\epsilon} \log\left(\frac{q^2}{\mu^2}\right) + \dots \right]. \quad (6.49)$$

6.5 Calculating the integral contribution to the $d\eta$ Integral

We now turn our attention to calculating the 2nd term in (6.25) which is from the integral in (6.38). It takes the form

$$\begin{aligned} \Sigma_{ov}^{(2)} &= \frac{ig^4}{2(4\pi)^{D/2}} \Gamma\left(\frac{\epsilon}{2}\right) \int_0^1 d\xi \int \frac{d^D p}{(2\pi)^D} \xi^{1-\epsilon/2} (1-\epsilon)^{-\epsilon/2} \left(\frac{-\epsilon}{2}\right) \\ &\quad \int_{-1/2}^{1/2} \eta d\eta \frac{2p \cdot q + 2\eta q^2 - \frac{2\eta q^2}{1-\xi}}{(p + \frac{1}{2}q)^2 (p - \frac{1}{2}q)^2 (Y^2)^{1+\epsilon/2}} \\ &= -\frac{ig^4}{2(4\pi)^{D/2}} \Gamma\left(1 + \frac{\epsilon}{2}\right) \int_0^1 d\xi \int_{-1/2}^{1/2} \eta d\eta \xi^{1-\epsilon/2} (1-\epsilon)^{-\epsilon/2} \\ &\quad \int \frac{d^D p}{(2\pi)^D} \frac{(\mu^2)^\epsilon \left(2p \cdot q + 2\eta q^2 - \frac{2\eta q^2}{1-\xi}\right)}{(p + \frac{1}{2}q)^2 (p - \frac{1}{2}q)^2 (Y^2)^{1+\epsilon/2}}. \end{aligned}$$

If we make a shift in p by $p \rightarrow p - \eta q$ then the $d^D p$ integral becomes ($Y^2 \rightarrow p^2 + q^2 \frac{1/4 - \eta^2}{1-\xi}$)

$$I = \int \frac{d^D p}{(2\pi)^D} \frac{(\mu^2)^\epsilon \left(2p \cdot q - \frac{2\eta q^2}{1-\xi}\right)}{\left[p + \left(\frac{1}{2} - \eta\right)q\right]^2 \left[p - \left(\frac{1}{2} + \eta\right)q\right]^2 \left[p^2 + q^2 \frac{1/4 - \eta^2}{1-\xi}\right]^{1+\epsilon/2}}. \quad (6.50)$$

We apply Feynman's trick as usual. The denominator in the integrand of (6.50) can be written as

$$\begin{aligned} &\frac{1}{\left[p + \left(\frac{1}{2} - \eta\right)q\right]^2 \left[p - \left(\frac{1}{2} + \eta\right)q\right]^2 \left[p^2 + q^2 \frac{1/4 - \eta^2}{1-\xi}\right]^{1+\epsilon/2}} \\ &= \frac{\Gamma(3 + \epsilon/2)}{\Gamma(1 + \epsilon/2)} \int_0^1 dx \int_0^{1-x} dy (1-x-y)^{\epsilon/2} \\ &\quad \frac{1}{\left[\left(p + \left(\frac{1}{2} - \eta\right)q\right)^2 x + \left(p - \left(\frac{1}{2} + \eta\right)q\right)^2 y + \left(p^2 + q^2 \frac{1/4 - \eta^2}{1-\xi}\right)(1-x-y)\right]^{3+\epsilon/2}}. \end{aligned}$$

Some algebra is required to reduce the denominator to a more useful form, and we perform it in detail here. The same process applies to previous simplifications: we complete the square with the term linear in p and simplify the remaining terms.

$$\begin{aligned}
& \left(p + \left(\frac{1}{2} - \eta\right)q\right)^2 x + \left(p - \left(\frac{1}{2} + \eta\right)q\right)^2 y + \left(p^2 + q^2 \frac{1/4 - \eta^2}{1 - \xi}\right)(1 - x - y) \\
&= p^2 + 2p \cdot q \left[\left(\frac{1}{2} - \eta\right)x - \left(\frac{1}{2} + \eta\right)y\right] \\
&\quad + q^2 \left[\left(\frac{1}{2} - \eta\right)^2 x - \left(\frac{1}{2} + \eta\right)^2 y\right] + q^2 \left(\frac{1/4 - \eta^2}{1 - \xi}\right)(1 - x - y) \\
&= \left(p + q \left[\left(\frac{1}{2} - \eta\right)x - \left(\frac{1}{2} + \eta\right)y\right]\right)^2 - q^2 \left[\left(\frac{1}{2} - \eta\right)x - \left(\frac{1}{2} + \eta\right)y\right]^2 \\
&\quad + q^2 \left[\left(\frac{1}{2} - \eta\right)^2 x - \left(\frac{1}{2} + \eta\right)^2 y\right] + q^2 \left(\frac{1/4 - \eta^2}{1 - \xi}\right)(1 - x - y) \\
&= p^2 + Zq^2.
\end{aligned}$$

In the last line we shifted $p \rightarrow p + q\left[\left(\frac{1}{2} - \eta\right)x - \left(\frac{1}{2} + \eta\right)y\right]$ and defined Z by

$$\begin{aligned}
Z &= \left(\frac{1}{2} - \eta\right)x(1 - x) + \left(\frac{1}{2} + \eta\right)y(1 - y) + \left(\frac{1}{4} - \eta^2\right)2xy + \frac{1/4 - \eta^2}{1 - \xi}(1 - x - y) \\
&= \frac{1}{4}\left(x(1 - x) + y(1 - y) + 2xy\right) - \eta\left(x(1 - x) - y(1 - y)\right) \\
&\quad + \eta^2\left(x(1 - x) + y(1 - y) - 2xy\right) + \frac{1/4 - \eta^2}{1 - \xi}(1 - x - y)\left(\frac{1}{2} - \eta\right)x - \left(\frac{1}{2} + \eta\right)y.
\end{aligned}$$

Each of the first three terms can be simplified to recover the eq. (16.57) in [15]. For the $\frac{1}{4}$ term we have

$$\begin{aligned}
x(1 - x) + y(1 - y) + 2xy &= x + y - (x^2 + y^2 - 2xy) \\
&= x + y - (x - y)^2.
\end{aligned}$$

For the η term we have

$$\begin{aligned}
x(1 - x) - y(1 - y) &= x(1 - x) - xy - y(1 - y) + xy \\
&= (x - y)(1 - x - y),
\end{aligned}$$

and for the η^2 term we have

$$\begin{aligned}
x(1 - x) + y(1 - y) - 2xy &= x(1 - x) - xy + y(1 - y) - xy \\
&= (x + y)(1 - x - y).
\end{aligned}$$

Both the η and η^2 terms contain a $(1 - x - y)$ which can be factored out along with the one in the last term finally obtain eq. (16.57) of [15] (see p. 544):

$$\begin{aligned} Z &= \frac{1}{4}(x + y - (x - y)^2) + \left[\eta^2(x + y) - \eta(x - y) + \frac{1/4 - \eta^2}{1 - \xi} \right] (1 - x - y) \\ &= a_Z + \frac{b_Z}{1 - \xi}. \end{aligned}$$

where a_Z and b_Z are defined by

$$a_Z = \frac{1}{4}(x + y - (x - y)^2) + [\eta^2(x + y) - \eta(x - y)](1 - x - y) \quad (6.51)$$

$$b_Z = \left(\frac{1}{4} - \eta^2\right)(1 - x - y). \quad (6.52)$$

Now that we have the denominator of (6.50) in a more simplified form we can actually compute it. One small matter, however is that the numerator was also shifted:

$$2p \cdot q - \frac{2\eta q^2}{1 - \xi} \rightarrow (2q^2) \left[\left(\frac{1}{2} - \eta\right)x - \left(\frac{1}{2} + \eta\right)y \right].$$

Note that the $p \cdot q$ term vanishes since it is now odd in the $d^D p$ integral (by the p^2 now in the denominator.) The $d^D p$ integral can now be computed in the normal way:

$$\int \frac{d^D p}{(2\pi)^D} \frac{1}{[p^2 + Zq^2]^{3+\epsilon/2}} = i \frac{1}{(4\pi)^{D/2}} \frac{\Gamma(3 + \epsilon/2 - D/2)}{\Gamma(3 + \epsilon/2)} \left(\frac{\mu}{Zq^2}\right)^\epsilon. \quad (6.53)$$

We obtain at once the result for I :

$$I = \frac{-i\Gamma(\epsilon)(2q^2)}{(4\pi)^{D/2}\Gamma(1 + \epsilon/2)} \int_0^1 dx \int_0^{1-x} dy (1 - x - y)^{\epsilon/2} \left[\frac{1}{2}(x - y) - \eta(x + y) + \frac{\eta}{1 + \xi} \right] \left(\frac{\mu^2}{Zq^2}\right)^\epsilon. \quad (6.54)$$

Of all the Feynman parameters that have been generated through the course of this calculation we integrate $d\xi$ by parts. Note that

$$\left(\frac{\mu^2}{Zq^2}\right)^\epsilon = \left(\frac{\mu^2}{q^2}\right)^\epsilon Z^{-\epsilon} = \left(\frac{\mu^2}{q^2}\right)^\epsilon [(1 - \xi)a_Z + b_Z]^{-\epsilon} (1 - \xi)^\epsilon.$$

We can combine the $(1 - \xi)^\epsilon$ term with the original one in the $d\xi$ integral. Neglecting the first two terms in the square bracket of (6.54), we can also absorb the other $1 - \xi$ terms so that the $d\xi$ integral takes the form

$$\int_0^1 d\xi \xi^{1-\epsilon/2} [(1 - \xi)a_Z + b_Z]^{-\epsilon} (1 - \xi)^{\epsilon/2-1}. \quad (6.55)$$

To integrate by parts we take

$$u = \xi^{1-\epsilon/2}[(1-\xi)a_Z + b_Z]^{-\epsilon}$$

$$dv = (1-\xi)^{\epsilon/2-1}d\xi$$

which give v and du by

$$du = \left[(1 - \frac{\epsilon}{2}\xi^{-\epsilon/2}[(1-\xi)a_Z + b_Z]^{-\epsilon} + \epsilon\xi^{1-\epsilon/2}[(1-\xi)a_Z + b_Z]^{-(1+\epsilon)}a_Z \right] d\xi$$

$$v = -\frac{2}{\epsilon}(1-\xi)^{\epsilon/2}.$$

The boundary term vanishes and so will the 2nd term in du . We are left with

$$-\frac{2}{\epsilon} \int_0^1 d\xi (1-\xi)^{\epsilon/2} \left(1 - \frac{\epsilon}{2}\xi^{-\epsilon/2}[(1-\xi)a_Z + b_Z]^{-\epsilon}\right). \quad (6.56)$$

Taking $\epsilon \rightarrow 0$ for everything we are left with (keeping only the $\frac{1}{\epsilon}$) terms

$$\Sigma_{ov}^{(2)} = -q^2 \frac{g^4}{12(4\pi)^6} \left[\frac{1}{\epsilon^2} - \frac{1}{\epsilon} \log\left(\frac{q^2}{\mu^2}\right) + \dots \right]. \quad (6.57)$$

CHAPTER 7

CONCLUSION AND FUTURE WORK

In conclusion, we set out to give as many possible details in deriving some standard and some uncommon results in scalar quantum field theory. For a real scalar field ϕ we started with a Lagrangian (density.) We found the conjugate momentum density π then used it along with ϕ to determine our classical physical quantities of interest.

Next we promoted our fields to operators and imposed commutation relations between them. We expressed them as a Fourier integral of a linear combination of creation and annihilation operators. These creation and annihilation operators were defined to act on multi-particle states in the Fock space, creating or destroying particles at will. When the Hamiltonian and momentum operators were expressed in terms of $a_{\mathbf{p}}$ and $a_{\mathbf{p}}^\dagger$ we then showed that the energy and momentum of these states were precisely equal to the sum of Energy/momentum of each of the particles in the state.

We then introduced an interaction term in the Lagrangian. After expressing the new ground state $|\Omega\rangle$ in terms of the original ground state $|0\rangle$ of the free theory we calculated probability amplitudes (or scattering amplitude) for states to evolve to different states through the interaction. These scattering amplitudes are interpreted as Feynman diagrams. Unfortunately diagrams containing internal loops led to infinities, so we regularized the diagram to isolate and remove the singularity, leaving a finite contribution.

There are several ways this work could be extended. The first would be to continue with even higher-order calculations. One example would be the *box diagram* and its corresponding one and two-loop corrections. The calculations are straightforward, and no new techniques would be needed. It would also better highlight the aspects of renormalization.

Another interesting possibility, one that I have not yet seen, would be to study interactions of distinguishable particles with *quartic* interactions. ϕ^4 theory is a well-known,

commonly studied theory with similar calculations, yet there exist related theories such as

$$ABCD \qquad ABCC \qquad AABB \qquad ABBB. \qquad (7.1)$$

Deriving the Feynman rules, computing the symmetry factors of the diagrams, and determining renormalization constants are all potential options to pursue in this direction.

Yet another direction to extend this work, and perhaps a more useful one, would be to extend our discussion to *complex* scalar fields. Determined from the Lagrangian (cf. eq. (2.22))

$$\mathcal{L} = \partial_\mu \phi^* \partial^\mu \phi - m^2 \phi^* \phi. \qquad (7.2)$$

The dynamical variables ϕ and ϕ^* (with corresponding conjugate momenta π and π^*) can be shown to satisfy the Hamiltonian (see [4], Problem 2.2, p. 33)

$$\mathcal{H} = \pi^* \pi + (\nabla \phi^*) \cdot (\nabla \phi) + m^2 \phi^* \phi \qquad (7.3)$$

which is similar to \mathcal{H} for a real scalar field. The new quantity one can compute for a complex scalar field is the *charge* defined by

$$Q = \int d^3x \frac{i}{2} (\phi^* \pi^* - \phi \pi). \qquad (7.4)$$

Once interactions are introduced (after appropriate quantization) such as $\mathcal{H}_{int} = g \phi_A \phi_B \phi_B^*$, then we can begin interpreting creation and annihilation of *antiparticles*, particles corresponding to a certain particle with the same mass but opposite charge.

As a means to gain more physical insight into these theories, we could calculate the cross-sections and decay rates corresponding to the particular amplitudes. All loop diagrams, after appropriate renormalization, ultimately yield a finite contribution to the scattering amplitude. It is my understanding that in actually physical theories these higher-order terms give a correction to the measurable quantity (the mass, for instance.)

A final direction would be to discuss the renormalization of these theories. Depending on the dimension in which the divergent integrals are regularized, a theory may be super-renormalizable, renormalizable, or non-renormalizable ([7].) In $D = 4$ dimensions, ABC

is a super-renormalizable theory, meaning it only takes a finite number of counterterms to make terms of all orders finite. The quantum theory of gravity is an example of a non-renormalizable theory.

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