Topological loops

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by

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INTRODUCTION

The purpose of this thesis is to present some of the ideas presented by L. Pontrjagin in chapter three of his book <u>Topological Groups</u>, but relaxing the conditions necessary for a group and considering topological loops.

The thesis is divided into three main parts: the first introducing and defining the topological loop along with listing a few of its properties. The second part deals with the subloop, normal subloop and the factor loop. The last part defines isomorphism, automorphism, and homomorphism and gives several results concerning these concepts.

This introduction would not be complete unless mention was made of the help received; first from Dr. Sigmund Hudson, who was kind enough to send me an abstract of his thesis on topological loops; second to Dr. K. H. Hofman, whose article on topological loops was used; and lastly, but perhaps most important, to Dr. Donald Schmidt, my thesis advisor.

PART I - CONCEPT OF A TOPOLOGICAL LOOP

Definition 1. A loop, L, is a set with a binary operation $(x,y) \rightarrow xy$ on $L \times L$ into L satisfying:

a) there is an element $e \in L$ such that $ex = xe = x$, for all $x \in L$.

b) for any x and y in L, there are z_1 and z_2, both unique, so that $xz_1 = y$ and $z_2 x = y$ (we denote z_2 by $yx^{(-1)}$ and z_1 by $x^{(-1)}y$).

Definition 2. A topological loop, L, is a loop as defined in definition 1 which has a topology making it a T_1 space, and so that

$$(x,y) \rightarrow xy$$
$$(x,y) \rightarrow x^{(-1)}y$$
$$(x,y) \rightarrow yx^{(-1)}$$

are all continuous functions of $L \times L$ into L.

Definition 3. The mapping $R_a: x \rightarrow xa$ for all $x \in L$ and a fixed a in L is called a right translation; likewise, $L_a: x \rightarrow ax$ is called a left translation.

In order to obtain certain results in topological loops it is necessary the right and left translations be homeomorphisms on L onto L. There is an example listed in Hofmann (2) of a topological loop in which the right translations are not homeomorphisms.

We next define an I - loop and show that in an I - loop the translations are homeomorphisms. (2)

Definition 4. An I - loop is a loop in which both L_a and R_a are one-to-one, onto mappings for all a in the loop; also, there is a mapping $x \rightarrow x^{-1}$ such that:

$$R_x^{-1} = R_{x^{-1}} \qquad \text{and} \qquad L_x^{-1} = L_{x^{-1}}$$

for all $x \in L$. x^{-1} is called the inverse element of x.

Lemma 1. In an I - loop, $a^{-1}(ax) = x$ and $(xa)a^{-1} = x$ for a given $a \in L$ and for all $x \in L$.

Proof: $a^{-1}(ax) = R_{a^{-1}}(R_a(x)) = i(x) = x$, where i is the identity mapping, since $R_{a^{-1}} = R_a^{-1}$.

$(xa)a^{-1} = L_{a^{-1}}(L_a(x)) = i(x) = x$, where i is the identity mapping, since $L_{a^{-1}} = L_a^{-1}$.

Hence in an I - loop the solution to $xz_1 = y$ is $z_1 = x^{-1}y$, and to $z_2x = y$ is $z_2 = yx^{-1}$.

Theorem 1. In a topological I - loop, L, the mappings R_a and L_a are homeomorphisms.

Proof: To prove this we must show that R_a: $x \rightarrow ax$ is continuous, and its inverse is continuous (we must show the same for L_a, also).

Consider the mapping, f, which takes the loop L homeomorphically onto the subspace $a \times L$ of $L \times L$.

$$f: x \rightarrow (a,x)$$

for a given $a \in L$ and for all $x \in L$. Also, consider the mapping, g, such that

$$g: (a,x) \rightarrow ax$$

for all $x \in L$ and a given $a \in L$. By the definition of a topological loop g is a continuous mapping. Since

$$R_a = gf: x \rightarrow ax,$$

and the composition of two continuous mappings is continuous, R_a is a continuous mapping.

Since R_a is continuous and one-to-one R_a^{-1} exists. By the definition of an I - loop, $R_a^{-1} = R_{a^{-1}}$, and hence we have that $R_a^{-1}: x \rightarrow a^{-1}x$ is continuous, because $R_{a^{-1}}$ is continuous from above.

The proof to show that L_a is a homeomorphism follows the same pattern.

Theorem 2. The mapping $x \rightarrow x^{-1}$, for any $x \in L$, is a homeomorphism in an I - loop.

Proof: Consider the subspace $L \times e$ of $L \times L$, and let h be the projection mapping on $L \times L$ to L, restricted to $L \times e$. h is a homeomorphism and hence

$$h^{-1}: L \rightarrow L \times e$$

is a homeomorphism: i.e.

$$h^{-1}: x \rightarrow (x,e).$$

Let k be the mapping

$$(x,y) \rightarrow x^{-1}y$$

of $L \times L$ to L be restricted to $L \times e$, then $k: (x,e) \rightarrow x^{-1}$ is continuous since $(x,y) \rightarrow x^{-1}y$ is continuous. The mapping kh^{-1} on L into L is continuous since it is the composition of two continuous mappings. Hence,

$$kh^{-1}(x) = k(h^{-1}(x)) = k(x,e) = x^{-1}$$

or

$$kh^{-1}: x \rightarrow x^{-1}$$

is a continuous mapping.

The mapping is one-to-one, hence the continuity of the inverse mapping follows immediately.

The continuity of these mappings can be restated as follows in terms of neighborhoods:

If x and y are in L, then for every neighborhood W of the element xy there exist neighborhoods U and V of the elements x and y such that $UV \subset W$.

If x is in L, then for every neighborhood V of the element x^{-1} there exists a neighborhood U of the element x such that $U^{-1} \subset V$.

Theorem 3. If L is an I - loop and a topological space such that the mappings $(x,y) \rightarrow xy$ and $x \rightarrow x^{-1}$ are continuous then L is a topological I - loop.

Proof: In otherwords we want to show that the continuity of $(x,y) \rightarrow xy$ and $x \rightarrow x^{-1}$ implies the continuity of the mappings $(x,y) \rightarrow x^{-1}y$ and $(x,y) \rightarrow yx^{-1}$.

To do this we shall use the properties of neighborhoods.

Since $(x,y) \rightarrow xy$ is continuous there exists a neighborhood W of xy such that for U a neighborhood of x and V a neighborhood of y $UV \subset W$. By the continuity of $x \rightarrow x^{-1}$, hence the continuity of $x^{-1} \rightarrow x$, there exists a neighborhood U of x such that for S a neighborhood of x^{-1} $S^{-1} \subset U$. Now, $W \supset UV \supset S^{-1}V$, which implies that $(x,y) \rightarrow x^{-1}y$ is continuous.

Similarly, $(x,y) \rightarrow yx^{-1}$ is continuous. Hence in a topological I - loop the continuity of the three mappings can be replaced by the continuity of the two mappings:

$$(x,y) \rightarrow xy$$

$$x \rightarrow x^{-1}.$$

Next we will show that the continuity of the two mappings $(x,y) \rightarrow xy$ and $x \rightarrow x^{-1}$ can be replaced by the continuity of the mapping $(x,y) \rightarrow xy^{-1}$, ie. if x and y are contained in L, then for every neighborhood W of xy^{-1} there exist neighborhoods U and V of the elements x and y such that $UV^{-1} \subset W$. Consider the elements x and y in L, and the neighborhood W of the element xy^{-1}. Since x and y^{-1} are in L and $(x,y) \rightarrow xy^{-1}$ is continuous, there also exist neighborhoods of x and y^{-1}, U and S respectively, such that $US \subset W$. Since $y \rightarrow y^{-1}$ is continuous, there exists a neighborhood V of y such that $V^{-1} \subset S$. $UV^{-1} \subset US$, and since $US \subset W$ this does imply that $UV^{-1} \subset W$.

We need also to show that if $(x,y) \rightarrow xy^{-1}$ is continuous, then $(x,y) \rightarrow xy$ and $x \rightarrow x^{-1}$ are continuous. The mapping $x \rightarrow x^{-1}$ is continuous since it is the composite of two continuous mappings as follows:

$$x \rightarrow (e,x) \rightarrow ex^{-1} = x^{-1}$$

Let W be a neighborhood of xy. The mapping $(x,y^{-1}) \rightarrow x(y^{-1})^{-1} = xy$ is continuous. Hence, there exist neighborhoods U and V^{-1} of x and y^{-1} respectively such that $UV^{-1} \subset W$. Since from above $y \rightarrow y^{-1}$ is continuous there exists a neighborhood V of y such that $V^{-1} \subset V'$. Hence $UV^{-1} \subset UV' \subset W$ and the mapping $(x,y) \rightarrow xy$ is continuous.

Definition 5. A topological space L is homogeneous if and only if for any two elements p and q in L there exists a homeomorphism h of L into itself such that $f(p) = q$.

Theorem 4. A topological I - loop is homogeneous.

Proof: Let p and $q \in L$. Consider the mapping $L_{qp^{-1}}$ which is a homeomorphic mapping on L into L.

$$L_{qp^{-1}}: x \rightarrow (qp^{-1})x$$

for all $x \in L$. Let $x = p \in L$. Then

$$L_{qp^{-1}}: p \rightarrow (qp^{-1})p = q$$

Hence, a topological I - loop is homogeneous.

From the property of homogeneity it will be sufficient to state and verify the local properties of a single element only of the topological loop. Hence, we may verify the property at e, and it will hold over the entire space.

Consider, for example, the property of local compactness. Let U be a neighborhood of e such that $\overline{U}$ is compact. Let $p \in L$, where L is a topological I - loop with e as its identity. By the homogeneity there exists a homeomorphism f on L to L such that $f(e) = p$. Hence $f(U)$ is a neighborhood of p. Since f is a homeomorphism $f(\overline{U}) = \overline{f(U)}$, and since continuity preserves compactness $\overline{f(U)}$ is compact. Hence, L is locally compact.

Definition 6. A topological space L is called regular if for every neighborhood U of an arbitrary point a there exists a neighborhood V of the same point such that $\overline{V} \subset U$.

Theorem 5. Every topological I - loop is a regular topological space.

Proof: We may establish the regularity of the space L by considering the neighborhoods of the identity, e. Let U be a neighborhood of e. Since $ee^{-1} = e$ it follows that there exists a neighborhood V of the identity such that $VV^{-1} \subset U$. To show that $\overline{V} \subset U$ let $p \in \overline{V}$. Then every neighborhood of the point p intersects V. pV is a neighborhood of p hence there exists in V a point b such that

$$pb = a \in V.$$

But then

$$(pb)b^{-1} = ab^{-1}$$

$$p = ab^{-1} \in VV^{-1} \subset U.$$

Hence, $\overline{V} \subset U$.

Theorem 6. Let L be a topological loop and let U and V be two of its compact subsets, then UV is also compact.

Proof: Consider the product $L \times L$. For (a,b), (c,d) contained in $L \times L$ define

$$(a,b)\,(c,d) = (ac,bd).$$

$L \times L$ is a topological loop under this operation with the product topology.

Let U and V be compact subsets of L. Let U' be the homeomorphic image of U in the first factor and V' the homeomorphic image of V in the second factor. Then U', V' are each compact and $U' \times V'$ is compact. Define a mapping

$$f\colon U' \times V' \to UV$$

by

$$(u,v) \to uv.$$

This mapping is continuous by the definition of a topological loop, and hence UV is compact.

PART II - SUBLOOP, NORMAL SUBLOOP, FACTOR LOOP

Definition 7. H is a subloop of the loop L if and only if $H \subset L$, and if H is again a loop.

Definition 8. Let L be a loop and let N be a subloop of L, then N is a normal subloop if and only if there exists a homomorphism f of L such that N is the kernel of f.

A second way of defining a normal subloop, which can be proven to be equivalent to the above definition (see (2)), is the following:

Let L be a loop and let N be a subloop of L, then N is a normal subloop of L if and only if:

a) $aN = Na$

b) $a(bN) = (ab)N$

c) $a(Nb) = (aN)b$

d) $N(ab) = (Na)b$

for all $a, b \in L$.

Definition 9. Let H be a subloop of L and also a subset of the topological space L. We say H is a topological subloop of L if H is a closed subset of L.

Theorem 7. If H is a subloop of a topological loop L, then H is itself a topological loop.

Proof: In order to show this we must show that the loop operations are continuous. Consider $a, b \in H$, b^{-1} is also contained in H, so $ab^{-1} = c \in H$. Every neighborhood W' of the element c in the relative topology can be obtained as follows:

$$W' = H \cap W,$$

where W is a neighborhood of the element c in the loop L. Since L is a topological loop there exist neighborhoods U and V, of a and b respectively, in L such that $UV^{-1} \subset W$. Now

$$U' = H \cap U$$

and $$V' = H \cap V$$

are also neighborhoods of the elements a and b respectively in the space H. So we have

$$U'V'^{-1} \subset W$$

and $$U'V'^{-1} \subset H.$$

Hence

$$U'V'^{-1} \subset W \cap H \subset W'$$

so the operations are continuous.

Theorem 8. If H is an open subset of L and a subloop of L, then L is closed.

Proof: To prove this first note that the space

$$(L - H)H = \bigcup \{xH \,/\, x \in L - H\},$$

which is the union of the open sets xH, is open. $(L - H)H$ has empty intersections with H, for suppose there exists an $x \in H \cap (L - H)H$. Then $x \in H$ and

$x \in (L - H)H$. Hence $x = lh$, $l \in L - H$, $h \in H$. Since $x \in H$, $lh = h'$ for some $h' \in H$. But, this implies that $l = h'h^{-1} \in H$ which contradicts the fact that $l \in L - H$.

Now,

$$(L - H)H \cup H \supset (L - H) \cup H = L.$$

So H is the complement of an open set, and is therefore closed.

Theorem 9. Let H be a subloop of a topological loop L. If L is compact, or locally compact, then H is respectively compact, or locally compact.

Proof: If L is compact, since H is a closed subset of L, then H is also compact. If L is locally compact and if $a \in H$, then there exists a neighborhood U of a in L such that $\overline{U}$ is compact. $H \cap U = U'$ is a neighborhood of the element a in the topological space H. Since H is closed in L, $\overline{U'} \subset H$, and $\overline{U'}* = \overline{U'} \cap H = \overline{U'}$. Also, $\overline{U'} \subset \overline{U}$, and so $\overline{U'}$ is compact, as a closed subset of a compact set. Hence, H is locally compact.

Definition 10. Let L be a topological loop and H a subloop of L. Let $L/H = \{Ha \,/\, a \in L\}$ be the totality of all right cosets of the subloop H. Let Σ be a basis for the topology on L and let $U \in \Sigma$. Denote by U^* the set of all cosets of the form Hx, $x \in U$. For the basis for a topology Σ^* of L/H we take the totality of all sets of the form U^*, where U is an arbitrary element of Σ. The topological space L/H is then called the space of right cosets of the subloop H in the loop L. (Likewise we can define the space of left cosets.)

To prove that Σ^*, as defined above, is a basis for a T_1 - topology on L/H we must show the following two conditions are met:

a) for any $A, B \in L/H$ there exists a U^* of the basis Σ^* which is such that $A \in U^*$, but $B \notin U^*$.

b) for any two sets U^* and V^* of the basis Σ^* which contain the point $A \in L/H$, there exists a $W^* \subset \Sigma^*$ such that $A \in W^* \subset U^* \cap V^*$.

To show that condition a) is met let A and B be two distinct cosets and let $a \in A$. Since $B = Hb$ is a closed set and $a \notin B$ there must exist a neighborhood U of a which does not intersect B. Then the set U^* of all the cosets of the form Hx, where $x \in U$, forms a neighborhood of the coset A which does not contain B.

To show that condition b) is met let U^* and V^* be two neighborhoods of a coset A, and let $a \in A$.

$$U^* = \{Hx \ / \ x \in U, \ \ U \subset \Sigma\}$$

$$V^* = \{ Hy \,/\, y \in V, \quad V \subset \Sigma \}$$

HU and HV are open sets such that $HU \subset L$ and $HV \subset L$ and both contain a. There exists a neighborhood W of the element a, which is contained in both open sets HU and HV. Let

$$W^* = \{ Hz \,/\, z \in W, \quad W \subset \Sigma \}$$

It follows that W^* is a neighborhood of the coset A, where $A \subset U^* \cap V^*$.

Definition 11. The mapping f of a topological space R into a topological space R' will be called open if for every open set $U \subset R$, $f(U) \subset R'$ is open.

The mapping f is open if and only if for every $a \in L$ and every neighborhood V of a there exists a neighborhood V' of the point $f(a) = a'$ such that $V' \subset f(V)$.

Theorem 10. Let L be a topological loop, H one of its subloops and L/H the space of cosets. Associate with every element $x \in L$ the element $X = f(x)$ of the space L/H which is the coset containing the element x. The mapping f (called the natural mapping) of the topological space L on the space L/H is a continuous open mapping.

Proof: Let us assume L/H is the space of right cosets (the same results will hold for the space of left cosets). Let a L, and $A = Ha$, i.e. $f(a) = A$. Let U^* L/H be some neighborhood of A. U^* is composed of all cosets of the form Hx, where x U and U is a neighborhood of a in the space L. HU L is open and a HU. Hence there exists a neighborhood V of a such that V HU. It follows that $f(V)$ U^*, so f is continuous.

Let a L and $A = Ha = f(a)$. Let U be some neighborhood of a. Hx / x U $= U^*$, and U^* is a neighborhood of the element A. We have $f(U) = U^*$, hence U^* $f(U)$. Therefore f is open.

Definition 12. Let L be a topological loop and N a normal subloop. Consider the set L/N of cosets. L/N is called the factor loop of the topological loop L by the normal subloop N. In L/N the solution to the equation

$$(Na)(Nx) = Nb$$

is the element Nx in L/N such that x is the solution of $ax = b$ in L. In an I - loop this element is $x = a^{1}b$. Hence

$$(Na)(Na^{1}b) = Nb,$$

or in general

$$(Nx)(Ny) = Nxy.$$

Similarly for the equation $(Nx)(Na) = Nb$.

To show that L/N is an I - loop we need to show that each $X \in L/N$ has a unique inverse. Let $X = Nx$, then consider $(Nx)(Nx^{-1})$. By the above result this gives

$$N(xx^{-1}) = Ne = N.$$

Hence $X^{-1} = Nx^{-1}$.

We also need to show that R_A and L_A, $A \in L/N$ are one-to-one and onto. $R_A: X \rightarrow XA$, $A = Na$, and $A^{-1} = Na^{-1}$, where $A \in L/N$, $a \in L$ and N is a normal subloop of L.

$$\begin{aligned} X_1 A &= X_2 A \\ (X_1A)A^{-1} &= (X_2A)A^{-1} \\ (Nx_1Na)Na^{-1} &= (Nx_2Na)Na^{-1} \\ (Nx_1a)Na^{-1} &= (Nx_2a)Na^{-1} \\ N[(x_1a)a^{-1}] &= N[(x_1a)a^{-1}] \\ Nx_1 &= Nx_2 \\ X_1 &= X_2 \end{aligned}$$

If $Y \in L/N$, $Y = Ny$, $y \in L$, then

$$Y = N(ya^{-1})Na = N[(ya^{-1})a] = Ny.$$

To prove the continuity of the loop operations in L/N we let A and B be two elements of L/N, $C = AB^{-1}$, and W^* be a neighborhood of C. W^* is composed of all the cosets of the form Nz, where $z \in W$ and W is a neighborhood in L. Since $C \in W^*$ there exists a $c \in W$ such that $C = Nc$. We have

$$\begin{aligned} C &= AB^{-1} \\ CB &= (AB^{-1})B = A \end{aligned}$$

so there exists a $b \in B$ such that $a = cb$, with $a \in A$. The loop operations in L are continuous so there exist neighborhoods U and V of a and b respectively such that $UV^{-1} \subset W$. Let U^* be the neighborhood of the element A which is composed of all cosets of the form Nx, $x \in U$ and V^* be the neighborhood of the element B. So

$$(Nx)(Ny)^{-1} = (Nx)(y^{-1}N^{-1}) = (Nx)(y^{-1}N) = (Nx)(Ny^{-1}) = Nxy^{-1} \subset W^*$$

hence $U^*V^{*-1} \subset W^*$, so the loop operations are continuous in L/N.

Theorem 11. If a topological loop is compact or locally compact then each of its factor groups, L/N, is correspondingly compact or locally compact.

Proof: Using the natural map $f: L \rightarrow L/N$, this map is continuous, and hence it preserves compactness.

Suppose L is locally compact, then there exists a neighborhood U of e such that $\overline{U}$ is compact. Let f be the natural mapping of L on L/N. Since f is an open mapping it follows $f(U) = U^*$ is an open set in L/N. Since f is continuous $f(\overline{U})$ is compact, furthermore, $N \in U^*$ since N is the kernel of the homomorphism f. Because of the regularity of L/N there exists a neighborhood V^* of the element N such that $\overline{V^*} \subset U^* \subset f(\overline{U})$. Since $\overline{V^*}$ is a closed subset of the compact set $f(\overline{U})$, it is also compact. Hence, L/N is locally compact.

Every topological loop has two trivial normal subloops: $\{e\}$ and the whole loop L.

Definition 13. A topological loop L is called simple only if each of its normal subloops is either discrete or coincides with L (a topological loop L is called discrete if it contains no limit points).

Theorem 12. Let L be a topological I - loop and H a subloop of the abstract loop L. Then $\bar{H}$ is a subloop of the topological loop L.

Proof: Let $a \in \bar{H}$, $b \in \bar{H}$. We shall prove that $ab^{-1} \in \bar{H}$. Let W be a neighborhood of the element ab^{-1}. Then there exist neighborhoods U and V of the elements a and b such that $UV^{-1} \subset W$. Since $a \in \bar{H}$ and $b \in \bar{H}$ there exist elements x and y of H such that $x \in U$ and $y \in V$; but then $xy^{-1} \in H$, and at the same time $xy^{-1} \in W$. Hence an arbitrary neighborhood W of the element ab^{-1} intersects $\bar{H}$, and $ab^{-1} \in \bar{H}$. Accordingly $\bar{H}$ is a subloop of the abstract loop L. But since $\bar{H}$ is closed in the space L, it follows that $\bar{H}$ is a subloop of the topological loop L.

PART III - ISOMORPHISM, AUTOMORPHISM, HOMOMORPHISM

Definition 14. A mapping f of a topological loop L on a topological loop L' is called isomorphic if:

a) f is an isomorphic mapping of the abstract loop L on the abstract loop L'.

b) f is a homeomorphic mapping of the topological space L on the topological space L'.

Two topological loops are called isomorphic if there exists an isomorphic mapping of one loop on the other.

Definition 15. An isomorphic mapping of a topological loop L into itself is called an automorphism of the loop L.

It is interesting to note that the set of all automorphisms of a topological loop L forms an abstract group. (3)

Definition 16. A mapping of a topological loop L into a topological loop L* is called homomorphic if:

a) g is a homomorphic mapping of the abstract loop L into the abstract loop L*.

b) g is a continuous mapping of the topological space L into the topological space L*.

Definition 17. A homomorphic mapping g of a topological loop L into a topological loop L^* is called open if g is an open mapping or the topological space L into the topological space L^*.

It is the open homomorphism which gives a natural generalization of the concept of a homomorphism in the theory of abstract loops.

Theorem 13. Let L and L^* be two topological loops, and let g be a homomorphic mapping of the loop L into the abstract loop L^*. In order that the mapping g should be continuous, or open, it is sufficient to show that it is continuous or open at the identity. To be continuous it is sufficient that for every neighborhood U^* of the identity e^* in the loop L^* there exists a neighborhood U of the identity $e \in L$ such that $g(U) \subset U^*$, and for g to be open, that for every neighborhood V of the identity e there exists a neighborhood V^* of the identity e^* such that $g(V) \supset V^*$.

Proof: Suppose the condition for continuity is fulfilled. Let $a \in G$, $g(a) = a^*$ and let U^* be an arbitrary neighborhood of the element a^*. Then U^*a^{*-1} is a neighborhood of e^* and hence from the continuity condition there exists a U' of the identity e such that $g(U') \subset U^*a^{*-1}$.

Since $U = U'a$ is a neighborhood of the element a we have

$$g(U) = g(U')g(a) = (U^*a^{*-1})a^* = U^*.$$

Hence the mapping is continuous.

Analogously, it follows, using the second condition, that g is open.

Let L be a topological loop, N one of its normal subloops, and L/N the corresponding factor loop. Associate with every $x \in L$ the coset $X \in N$ such that $x \in X$ and $g(x) = X$. Then the mapping g of the topological loop L on the topological loop L/N is an open homomorphic mapping. This mapping is called the natural homomorphic mapping of the loop L on the loop L/N. g is homomorphic since the mapping of the abstract loop L on the abstract loop L/N is homomorphic, and by theorem 10 the mapping was shown to be continuous.

Theorem 14. Let L and L^* be two topological loops, let g be an open homomorphic mapping of the loop L on the loop L^* and let N be the kernel of the homomorphism g. Then N is a normal subloop of the loop L, and the topological loop L^* is isomorphic with the loop L/N.

Proof: N is a normal subloop of the abstract loop L. Furthermore, because N is the complete inverse image of the element e^* under the continuous mapping g, N is a closed subset of the topological space L. Hence N is a normal subloop of the topological loop L.

Let x^* be an element of the loop L^* and X the totality of all the elements of L which go into x^* under the mapping g. X is a coset of N in the loop L. Suppose $f(x^*) = X$. $f: L^* \rightarrow L/N$ such that f is isomorphic. Next we show that f is a homeomorphic mapping of the space L^* on the space L/N. To do this it is sufficient to show that f is a bicontinuous mapping (it is obviously one-to-one since f is an isomorphic mapping between the abstract loops).

Let $a^* \subset L^*$, and $f(a^*) = A$. Let U^* be a neighborhood of the element A in L/N. U^* is composed of all cosets of the form Nx, where $x \in U$, and U is a fixed neighborhood in the space L. Let $a \in U$ such that $A = Na$. Since g is open and $g(a) = a^*$, there exists a neighborhood V^* of the element a^* such that $g(U) \supset V^*$. Hence $f(V^*) \subset U^*$ (by theorem 13) since if, $x^* \in V^*$, then there exists $x \in U$ such that $g(x) = x^*$. Hence $f(x^*) = Nx \in U^*$ and the mapping f is continuous.

Let f^{-1} denote the inverse mapping of f. Let $A = Na \in L/N$ and $f^{-1}(A) = a^*$. Let U^* be a neighborhood of a^*. Since g is a continuous mapping and $g(a) = a^*$, it follows that there exists a neighborhood V of the element a such that $g(V) \subset U^*$. Let V^* be the neighborhood of the element A which is composed of all cosets of the form Nx, $x \in V$. Since $g(V) \subset U^*$, it follows that $f^{-1}(V^*) \subset U^*$. Hence f^{-1} is a continuous mapping.

Thus f is an isomorphic mapping of the topological loop L^* on the topological loop L/N.

Theorem 15. If L and L^* are two locally compact topological loops satisfing the second axiom of countability, then every homomorphic mapping g of the loop L on the loop L^* is open.

Proof: Let $W \subset G$, be open. To show: $g(W)$ contains an open set.

Since L is both locally compact and regular, there exists an open set V such that its closure, $\overline{V}$, is compact and $\overline{V} \subset W$. The set of all the open sets of the form Vx covers the whole space L, and since the space L satisfies the second axiom of countability it follows that from this covering we can select a countable covering. Hence, there exists a countable sequence of points a_n, $n = 1,2,\ldots$ such that the system of open sets $\{Va_n\}$, $n = 1,2,\ldots$, covers the space L. Suppose that $g(\overline{V}a_n) = F_n$.

Since $\overline{V}a_n$ is compact, F_n is also compact. Since G^* is regular and satisfies the second axiom of countability, F_n is closed. The system of sets, F_n, $n = 1,2,\ldots$, covers the space G^*.

To show that among the sets F_n there exists at least one containing an open set, suppose the opposite to be true. Let V^* be an open set of G^* such that its closure $\overline{V}^*$ is compact. Since F_1 does not contain an open set there exists a point $b_1 \in V^*$ not belonging to F_1. There exists further a neighborhood V_1 of the point b_1 such that $\overline{V}_1 \subset V^*$, and $\overline{V}_1$ does not intersect F_1. In the open set V_1 we can also find a point b_2 not belonging to F_2, and a neighborhood V_2 such that $\overline{V}_2 \subset V_1$ and $\overline{V}_2$ does not intersect F_2. Continuing this process we construct a sequence of open sets V_n, $n = 1,2,\ldots$, such that $\overline{V}_{n+1} \subset V_n$, V_n is compact and does not intersect F_n. $\bigcap \overline{V}_n \neq \phi$, $n = 1,2,\ldots$, i.e. there exists a b such that $b \notin F_n$ for any n. This however is impossible, since the system F_n, $n = 1, 2,\ldots$, covers the space L^*. Hence one of the sets F_n, say F_k, contains an open set. But then $g(\overline{V}) = F_k g(a_k^{-1})$ also contains an open set. Since $\overline{V} \subset W$, $g(W)$ contains an open set.

Let W be a neighborhood of the identity

$e \in L$. There exists a neighborhood W of the identity such that $WW^{-1} \subset U$. From what has just been proved $g(W)$ contains an open set W^*. Let $q \in W^*$ and let p be a point of W such that $g(p) = q$. Then Wp^{-1} is a neighborhood of the point e which is contained in U (since $p \in W$, $Wp^{-1} \subset WW \subset U$). Also, W^*q^{-1} is a neighborhood of the identity $e^* \in L^*$. Since $g(W) \supset W^*$, it follows $g(Wp^{-1}) \supset W^*q^{-1}$ and therefore $g(W) \supset W^*q^{-1}$. Hence the mapping g is open.

It is interesting to note that if an open homomorphic mapping g of a topological loop L on a topological loop L^* has a kernel conatining only the identity, then this mapping is isomorphic.

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