SOME REMARKS ON CONSTRUCTIVE YUKAWA THEORY IN FOUR DIMENSIONS

A Dissertation by

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I have examined the final copy of this dissertation for form and content, and recommend that it be accepted in partial fulfillment of the requirement for the degree of Doctor of Philosophy with a major in Applied Mathematics.

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DEDICATION

To my dear wife Ruth for her support and to…..
Walt's

ABSTRACT

We have found an exact solution to the nonlinear Yukawa system in four dimensions, and used it to derive the time development of the system. Theorems are stated and proved regarding the essential self-adjointness of the operator solutions. Suggestions are given to complete the task of proving existence of the theory in the Wightman sense. An explicit statement of all new results obtained in this research is the content of Chapter 10.

PREFACE

"Quantum Field Theory (QFT) is quintessentially the algebra and analysis of infinite dimensional dynamical systems, as constrained by quantum phenomenology, causality and symmetry."[Baez 92] Within this definition, the subject of this thesis falls under the topic of *Constructive Quantum Field Theory* which is concerned with exact mathematical questions: existence, well-posedness and rigorous description of the formal apparatus of the subject.

In particular, we have found an exact solution to the nonlinear Yukawa system in four dimensions, Y4, and used it to derive the time development of the system. Theorems are stated and proved regarding the essential self-adjointness of the operator solutions. Suggestions are given to complete the task of proving existence of the theory in the Wightman sense. Standard theorems quoted for development of the subject are labeled by letters of the alphabet, and theorems original to the author are labeled by numbers. An explicit statement of all new results obtained in this research is the content of Chapter 10.

Y4 is the second simplest interaction term in the standard model Lagrangian for particle interactions. It is incorporated in the Higgs sector, the section wherein most of the unknown quantum phenomena in the universe resides. The simplest, the self interacting scalar field (or Φ^4 as it is labeled when considered by itself), appears to be trivial (i.e. a free field) [Chalm 97].

The point of view herein assumes that the reader has a working knowledge of quantum mechanics and some familiarity with field theory. The functional analysis necessary to define an interacting QFT and a formal scattering theory is included to a depth to make this document self contained. An appendix contains a statement of important theorems whose results provide the language in common use in this subject. Also, a rigorous definition of quantum fields (for Fermions) is given in an appendix.

A special chapter (Chapter 5) is included to emphasize important results of the theory's mathematical foundation: Haag's theorem, generalized scattering theory and some reasons for doubting the possibility of achieving field quantization by canonical quantization of classical fields.

Also, far more material is included than necessary to solve the problem advertised. There is a long vista downstream and it is well to point the way with some suggestions. The final goal of any endeavor such as this is to prove the existence, within the Wightman formalism, of ANY four dimensional fully interacting quantum field theory. Constructive Quantum Field Theory is over half a century old, and this goal has yet to be achieved. Within the last couple of decades the discipline has become somewhat less active, and it has been fairly said that nowadays a graduate student can start and complete his degree in the subject without seeing the glacier move. (This is a quote of A.S. Wightman found in the Preface of [Strocchi 93]).

A word on organization: occasionally a section will be inserted in single space, slightly smaller type with larger margins. These are explicatory, usually very detailed. The points addressed are intended to avoid seemingly puzzling contradictions that may show up on a first reading: viz: the Fock space defined on the space of smearing functions or on $\mathbb{R}^3$ (or $\mathbb{R}^4$). The first are used to define quantum fields as an algebra, the second as an operator valued distribution defined on space-time. Confusing these points makes understanding theorems impossible.

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CHAPTER 1

INTRODUCTION

1.1 Units

The units used throughout this work are such that the speed of light c and Planck's constant $\hbar$ are unity. This will reduce the units time, length, and mass to a single unit which is usually called the mass: $T = L = M^{-1}$. Thus, when one achieves an answer in units of mass, it will have the form M^a, and the physical units for this result should be $M^b L^c T^d$. Consistency requires $a = b - c - d$ and conversion is achieved by the equation

(Answer in units $M^b L^c T^d$) = (Answer in units M^a) $\times \hbar^{b-a} \times c^{a-b-d}$.

The metric tensor is defined as $g^{\mu\nu} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix}$. Inner products of 4-vectors are written simply as kx with implied meaning $kx = k_0 x_0 - \mathbf{k}\cdot\mathbf{x}$. Fourier transforms have the geometrical factors in front of the k-space formula, viz: $f(x) = \frac{1}{(2\pi)^4}\int f(k)e^{ikx}dk$ where the exponent's signature is as shown and dk is $dk = dk^0 dk^1 dk^2 dk^3 = dk^0 d^3\mathrm{k}$. Occasionally, in definitions, the π factors are split between x and k, but this will always be pointed out.

1.2 Singular Functions

There exist a variety of singular functions which are solutions to $[\partial_t^2 - \Delta]\phi = 0$ or $\delta(x)$. All are derived from the integral $\frac{1}{(2\pi)^4}\int \frac{e^{ikx}}{m^2 - k^2}dk$ and the various ways it is defined. The first we introduce is the Feynman function defined

$$\Delta_F(x-y) = \frac{1}{(2\pi)^4}\int \frac{e^{ik(x-y)}}{m^2 - k^2 - i\eta}\,dk$$

where η is an infinitesimal, the integral is evaluated taking allowance of $\eta > 0$ and then the limit $\eta \to 0$ is taken. Note $\Delta_F(-x) = \Delta_F(x)$. This function solves $(\partial_0^2 - \nabla^2)\Delta_F(x) = \delta(x)$. Similar to this is the Dyson function, defined

$$\Delta_D(x-y) = \frac{1}{(2\pi)^4}\int \frac{e^{ik(x-y)}}{m^2 - k^2 + i\eta}\,dk\,.$$

The difference between them is denoted $\Delta^1(x) \equiv i(\Delta_F - \Delta_D)$. This is also often written

$\Delta^1(x) = \frac{1}{(2\pi)^3}\int \delta(k^2 - m^2)e^{ikx}\,dk$ or in three dimensional form

as $\Delta^1(x) = \frac{1}{(2\pi)^3}\int \frac{\cos\omega_k x_0}{\omega_k}\,e^{-i\mathbf{k}\cdot\mathbf{x}}\,d^3\mathbf{k}$. Another important function, which solves the homogeneous Klein-Gordon equation, is Schwinger's function written as

$$\Delta(x) = \frac{-i}{(2\pi)^3}\int \varepsilon(k_0)\delta(k^2 - m^2)e^{ikx}\,dk\,.$$

Here $\varepsilon(x) = \theta(x) - \theta(-x)$ with $\theta(x)$ the Heaviside function. These have a closed form which we will include:

$$\Delta(x) = \frac{1}{2\pi}\varepsilon(x)\delta(x_\mu x^\mu) + \frac{m}{4\pi}\frac{\Theta(x_\mu x^\mu)\varepsilon(x)}{\sqrt{x_\mu x^\mu}}\,J_1(m\sqrt{x_\mu x^\mu})$$

and $$\Delta^1(x) = \frac{m^2}{4\pi}\,\mathrm{Im}\left(\frac{H^{(1)}(m\sqrt{x_\mu x^\mu})}{m\sqrt{x^\mu x_\mu}}\right).$$

Two other very important singular functions are

$$\Delta_R(x) = \frac{1}{(2\pi)^4}\int \frac{e^{ikx}}{m^2 - k^2 + i\eta k_0} \quad \text{and} \quad \Delta_A(x) = \frac{1}{(2\pi)^4}\int \frac{e^{ikx}}{m^2 - k^2 - i\eta k_0}.$$

These are the "retarded" and "advanced" Green functions for the Klein Gordon equation, so called because $\Delta_R(x) = 0$ if $x_0^2 < 0$ (i.e. outside the forward light cone about x.) Similarly, $\Delta_A(x) = 0$ if $x_0^2 > 0$, (i.e. outside the backward light cone about x). These properties hold for spacelike arguments only (i.e. arguments x where $x^2 < 0$). From these definitions, one sees that $\Delta(x) = \Delta_A(x) - \Delta_R(x)$.

1.3 Notation

Euclidean space is denoted as $\mathbf{R}^4$, with vectors $x = (x_0, x_1, x_2, x_4) = (x_0, \mathbf{x})$ and the vector space defined on $\mathbf{R}^4$ with inner product $xy = x_0 y_0 - \mathbf{x}\cdot\mathbf{y}$ will be denoted as $\mathbf{M}$. We will denote by $\mathscr{S}(\mathbf{R}^n)$ the set of infinitely differentiable complex-valued functions $f(x)$ on $\mathbf{R}^n$ for which $\|f\|_{\alpha,\beta} \equiv \sup_{x\in R^n}\left| x^\alpha \frac{\partial^{|\beta|}}{\partial_{x_1}^{\alpha_1}\ldots\partial_{x_n}^{\alpha_n}} f(x)\right| < \infty$ for all $\alpha, \beta \in I_+^n$ where the last symbol represents the set of all n-tuples of nonnegative integers $\alpha = \langle \alpha_1, \ldots, \alpha_n \rangle$ and $|\alpha| = \sum_{i=1}^{n} \alpha_i$. The topological dual space to $\mathscr{S}(\mathbf{R}^n)$ will be denoted $\mathscr{S}'(\mathbf{R}^n)$ and is called the space of tempered distributions. One sees that functions in $\mathscr{S}(\mathbf{R}^n)$ and their derivatives fall off more quickly than any polynomial.

Conventions for Fourier Analysis of fields are as follows: Consider a scalar field φ,

Set $$\varphi(x) = \frac{1}{(2\pi)^{3/2}}\int \alpha(k) e^{ikx} dk. \tag{1.1}$$

Plugging this into the Klein-Gordon equation we get $\int \alpha(k)(-k^2 + m^2)e^{ikx} dk = 0$, which implies that $\alpha(k) = \delta(k^2 - m^2) d(k)$. Only those coefficients in equation (1.1) appear that satisfy

$k_0{}^2 - \mathbf{k}^2 - m^2 = 0$. Now define $\omega_k = \sqrt{\mathbf{k}^2 + m^2} > 0$ with $d^*(k) = d(-k)$.

Since there are only contributions with $k^2 = m^2 > 0$, we will define

$d(k) = \theta(k)d^+(k) + \theta(-k)d^-(k)$ and utilize the reality condition $[d^+(k)]^* = d^-(-k)$.

Thus (1.1) becomes $$\varphi(x) = \frac{1}{(2\pi)^{3/2}} \int \frac{d^3\mathbf{k}}{2\omega_k} (d^-(-\omega_k, -k) e^{-iw_k x_0} e^{-i\mathbf{k}\cdot\mathbf{x}} + d^+(\omega_k, \mathbf{k}) e^{iw_k x_0} e^{-i\mathbf{k}\cdot\mathbf{x}}) \qquad (1.2)$$

Introduce $c^*(\mathbf{k}) = \dfrac{d^+(\omega_k, \mathbf{k})}{\sqrt{2\omega_k}}$ and we will have the final form

$$\varphi(x) = \frac{1}{(2\pi)^{3/2}} \int \frac{d^3\mathbf{k}}{\sqrt{2\omega_k}} (c(\mathbf{k}) e^{-ikx} + c^*(\mathbf{k}) e^{ikx}) . \qquad (1.3)$$

We will denote the negative and positive frequency components as

$$\varphi^-(x) = \frac{1}{(2\pi)^{3/2}} \int \frac{d^3\mathbf{k}}{\sqrt{2\omega_k}} c(\mathbf{k}) e^{-ikx} \text{ and } \varphi^+(x) = \frac{1}{(2\pi)^{3/2}} \int \frac{d^3\mathbf{k}}{\sqrt{2\omega_k}} c^*(\mathbf{k}) e^{ikx} . \qquad (1.4)$$

The negative and positive frequency parts of the field have the covariant decomposition

$\varphi = \varphi^- + \varphi^+$. Note that $(\varphi^-)^* = \varphi^+$.

It is often convenient to use "box normalization" where we imagine the system to be enclosed in a cube of three volume V. Then the momenta become discrete

$k_i = \kappa\kappa_i$, $\kappa = \dfrac{2\pi}{V^{1/3}}$, $\kappa_i = 0, +/- 1, +/- 2,...$

From a Fourier integral we pass to a Fourier series with the correspondence

$\dfrac{1}{V}\sum_k \to \dfrac{1}{(2\pi)^3}\int d^3\mathbf{k}$. Thus we finally have the representation

$$\varphi(x) = \frac{1}{V^{1/2}} \sum_k \frac{1}{\sqrt{2\omega_k}} (c_\mathbf{k} e^{-ikx} + c_\mathbf{k}{}^* e^{ikx}) \quad \text{with} \quad c_\mathbf{k} = \frac{(2\pi)^3}{V^{1/2}} c(\mathbf{k}) .$$

This formalism applies to all the fields we will discuss.

Vacuum expectation values of fields and products of fields will be denoted as either $\langle 0|\phi(x)\phi(y)....|0\rangle$ or as $\langle \phi(x)\phi(y)....\rangle$, where $|0\rangle$ is the vacuum, or lowest energy state of the theory. Both versions are standard, although we will try to be consistent and employ the first. According to the fundamental Reconstruction Theorem [Streater 00], a quantum field can be completely recovered from its vacuum expectation values.

1.4 Preview of Results

Let us begin with the Dirac-Klein-Gordon coupled system in four dimensions

$$(-\Delta + \partial_t^2 + m_0^2)\varphi = g\overline{\psi}\psi \tag{1.5}$$

$$(-i\gamma^\mu \partial_\mu + \kappa_0)\psi = g\varphi\psi \ . \tag{1.6}$$

Utilizing the Dirac matrix γ^0, we show here that the following is a solution

$$\psi = \exp[ig\gamma^0 \int \Delta(x-y)\varphi(y)d^3\mathbf{y}]\Big|_{y_0=x_0} \psi^0(x) \tag{1.7}$$

of the equation for ψ. We also show that if the exponential is a unitary operator, then

$$\overline{\psi}(x)\psi(x) = \overline{\psi}^0(x)\psi^0(x), \tag{1.8}$$

where the superscript indicates a free field. Under similar arguments, then, the integral equation for φ, namely $\varphi(x) = \varphi_0(x) + g\int \Delta_F(x-y)\overline{\psi}(y)\psi(y)dy$ becomes a formal solution in terms of free fields (for ψ) as well, viz:

$$\varphi(x) = \varphi_0(x) + g\int \Delta_F(x-y)\overline{\psi}^0(y)\psi^0(y)dy \ . \tag{1.9}$$

In brief and subject to renormalization, these statements are, in fact, true. Also, the operators ψ and φ become self adjoint under these conditions, and the consequent spectral theorems for (un)bounded operators may be applied. The procedure of Haag and Ruelle (denoted H-R) can be applied to yield the time development of the system in a transparent way.

Showing when these statements are true, and what mathematical consequences may be derived therefrom, is the subject matter of this dissertation.

CHAPTER 2

FOCK SPACE

Quantum field operators act on a particular Hilbert space called Fock space, which we will here define. The basic idea underlying the Fock representation is that the states of the system (at a given time) are analyzed by making reference to a "ground" state and by specifying the number and type of elementary excitations which characterize a given state, with respect to the ground state. By elementary excitations one means the generic excitation of the system (e.g. a collective excitation in a plasma, the dynamical state of a single particle, etc.). The following discussion is from [Reed 80].

We will let $\mathcal{H}$ be a Hilbert space and let $\mathcal{H}^n$ denote the n-fold tensor product $\mathcal{H}^n = \mathcal{H} \otimes \mathcal{H} \otimes \ldots \otimes \mathcal{H}$. Set $\mathcal{H}^0 = \mathbf{C}$ and define $\mathcal{F}(\mathcal{H}) = \bigoplus_{n=0}^{\infty} \mathcal{H}^n$ as the "Fock" space over $\mathcal{H}$. We demand $\mathcal{H}$ be separable, then $\mathcal{F}(\mathcal{H})$ is also. We now define two important subspaces of Fock space. Let $\mathcal{P}_n$ be the permutation group on n elements of $\mathcal{H}^n$ and $\{\varphi_k\}$ be a basis for $\mathcal{H}$. For each $\sigma \in \mathcal{P}_n$ let us define an operator (also called σ) on basis elements of $\mathcal{H}^n$ by

$$\sigma(\varphi_{k_1} \otimes \varphi_{k_2} \otimes \ldots \otimes \varphi_{k_n}) = \varphi_{k_{\sigma(1)}} \otimes \varphi_{k_{\sigma(2)}} \otimes \ldots \varphi_{k_{\sigma(n)}} .$$

σ can be extended to a bounded operator (of norm 1) on $\mathcal{H}^n$. Define $S_n = \frac{1}{n!} \sum_{\sigma \in \mathcal{P}_n} \sigma$.

<u>Theorem A</u>: $S_n^2 = S_n$ and $S_n^* = S_n$

<u>Corollary</u>: S_n is an orthogonal projection.

<u>Definition</u>: The range of S_n is the n-fold symmetric tensor product of $\mathcal{H}$.

If $\mathcal{H} = L^2(\mathbf{R})$ and $\mathcal{H}^n = L^2(\mathbf{R}) \otimes \ldots \otimes L^2(\mathbf{R}) = L^2(\mathbf{R}^n)$ then $S_n\mathcal{H}^n$ is the subspace of $L^2(\mathbf{R}^n)$ of all functions invariant under any permutation of variables.

Definition: $\mathscr{F}_s(\mathscr{H}) = \bigoplus_{n=0}^{\infty} S_n \mathscr{H}^n$ is the symmetric Fock space or Boson Fock space over $\mathscr{H}$. Let

$\varepsilon: \mathscr{P}_n \to \{1,-1\}$ by $\varepsilon(\sigma) = 1$ for even permutations

$= -1$ for odd permutations

Define $A_n = \frac{1}{n!}\sum_{\sigma\in\mathscr{P}_n} \varepsilon(\sigma)\sigma$; and note A is an orthogonal projection on $\mathscr{H}^n$.

Now define $A_n\mathscr{H}^n$ = n-fold anti-symmetric tensor product of $\mathscr{H}$.

If $\mathscr{H} = L^2(\mathbf{R})$ $A_n\mathscr{H}^n$ is the subspace of $L^2(\mathbf{R}^n)$ consisting of functions odd under interchange of any two coordinates.

Definition: $\mathscr{F}_a(\mathscr{H}) = \bigoplus_{n=0}^{\infty} A_n \mathscr{H}^n$ is the anti-symmetric Fock space or the Fermion Fock space over $\mathscr{H}$.

Obviously $\mathscr{F}(\mathscr{H}) = \mathscr{F}_a(\mathscr{H}) + \mathscr{F}_s(\mathscr{H})$

CHAPTER 3

SMEARED FIELDS

What are commonly referred to as “Quantum Fields” Φ(x) are highly singular objects. We here introduce a procedure for rendering these singular mathematical expressions finite and/or well behaved by a “smearing” procedure. We are interested in producing mathematical objects that are members of either $L^2(\mathbf{R}^4)$ or $L^2(\mathscr{S}(\mathbf{R}^n)$ for n = 3 or 4. The following formalism is due to [Roman 69], and allows us to make statements about quantum fields in $L^2(\mathbf{R}^4)$.

Consider a Lorentz invariant measure on k-space, Ω_k, defined by

$d\Omega_k(\mu) = \delta(k^2 - \mu^2)\theta(k)dk$. Introduce the family of “wave packet” functions defined by

$$f_a(x) = \frac{1}{(2\pi)^{3/2}} \int \tilde{f}_a(k) e^{-ikx} d\Omega_k(\mu) \tag{3.1}$$

where kx is a short form for $kx = k_0 x_0 - \mathbf{k} \cdot \mathbf{x}$.

The $\tilde{f}_a(k)$ may be arbitrary, but the presence of $\delta(k^2 - m^2)$ insures that equation (3.1) obeys the Klein-Gordon equation and the Heaviside function $\theta(k)$ insures that

$$(\partial^2{}_t - \nabla^2 + m^2) f_a(x) = 0 \quad \text{and } f_a \text{ are positive energy solutions.} \tag{3.2}$$

If the coefficients $\tilde{f}_a(k)$ in equation (3.1) satisfy

$$\int \frac{1}{2k_0} \tilde{f}_a^*(k) \tilde{f}_\beta(k) d^3\mathbf{k} = \delta_{\alpha\beta} \,, \tag{3.3}$$

then it is certainly true that

$$(f_a, f_\beta) = i \int f_a^*(x) \overleftrightarrow{\partial}_0 f_\beta(x) d^3\mathbf{x} = \delta_{\alpha\beta} \tag{3.4}$$

(the symbol $\overleftrightarrow{\partial}$ means $A\overleftrightarrow{\partial}B = A(\partial B) - (\partial A)B$). Equation (3.2), and because $k_0 > 0$, has the consequence that

$$(f_a^*, f_\beta) = i\int f_a(x)\overleftrightarrow{\partial}_0 f_\beta(x)d^3\mathbf{x} = 0\,. \tag{3.5}$$

Finally, with equation (3.1) one sees that the completeness relations below are fulfilled

$$\sum_a f_a^*(x)f_a(y) = i\Delta^+(x-y) \quad , \tag{3.6a}$$

$$\sum_a f_a(x)f_a^*(y) = i\Delta^-(x-y) \quad . \tag{3.6b}$$

Here formally $\displaystyle\int_{k_0=+\omega_k} \frac{d^3\mathbf{k}}{(2\pi)^3 2\omega_k} e^{ik(x-y)} \equiv \Delta^+(x-y)$ and

$$\int_{k_0=-\omega_k} \frac{d^3\mathbf{k}}{(2\pi)^3 2\omega_k} e^{ik(x-y)} \equiv \Delta^-(x-y) \ .$$

The preceding observations tell us that any normalizable solution F of the Klein-Gordon equation can be expanded in terms of the wave packets f_a(x) and f_a*(x), i.e.

$$F(x) = \sum_z (d_a f_a(x) + h_a f_a^*(x))\,\textbf{.} \tag{3.7}$$

Now let us show how we may construct these types of functions, and use them to generate a "field algebra". Pick any f(x) a member of $\mathscr{S}(\mathbf{R}^4)$. Take the Fourier Transform in $\mathbf{R}^4$ and get $f(p)$. Let $H_m = \{p \in R^4 | \, p\cdot\tilde{p} = m^2, x_0 > 0\}$ be **mass hyperboloids** and define a measure Ω_m on H_m for each $m \geq 0$ as follows: let j_m be the homeomorphism of H_m onto $\mathbf{R}^3$ given by j_m:$\langle p_0, p_1, p_2, p_3\rangle \mapsto \langle p_1, p_2, p_3\rangle = \mathbf{p}$. Define a measure Ω_m on H_m by setting

$\displaystyle\Omega_m(E) = \int_{j_m(E)} \frac{d^3p}{\sqrt{m^2+|\mathbf{p}|^2}}$ for any measurable set $E \subset H_m$. Ω_m is invariant under the restricted

Lorentz group. [This is essentially the same measure introduced prior to equation (3.1), but a little more carefully defined.] Let $\mathcal{H}=L^2(H_m, d\Omega_m)$. For each $f \in \mathscr{S}(\mathbf{R}^4)$ define Ef in $\mathcal{H}$ by

$Ef = \sqrt{2\pi}\,\hat{f} \uparrow H_m$ where the Fourier transform $\hat{f}(p) = \frac{1}{(2\pi)^2}\int e^{ip\cdot\tilde{x}} f(x)dx$ is defined in terms of the Lorentz invariant inner product $p\cdot\tilde{x} = p_0x^0 - p_1x^1 - p_2x^2 - p_3x^3$. Note that if f is the distribution $f(x) = g(\mathbf{x})\delta(t)$, then $\sqrt{2\pi}\,\hat{f}$ is the ordinary three-dimensional Fourier transform of g. The reason for these manipulations is that the $Ef = \sqrt{2\pi}\,\hat{f} \uparrow H_m$ will satisfy the Klein Gordon equation.

This will allow us to smear a field $\Psi(x)$ with a function that automatically satisfies the Klein Gordon equation; viz: $\Psi(f) = \int \Psi(x)f(x)dx$, and leave a time dependence in $\Psi(f)$. The resulting objects form an algebra which comprise the mathematical objects whose Hilbert space is $L^2(\mathscr{S}(\mathbf{R}^4)$.

CHAPTER 4

DEFINITION OF LOCAL QUANTUM FIELD THEORY

4.1 Basic Assumptions

The structure of the theory is a generalization of observations about "free" quantum fields, which have been successfully defined and represent a complete and absolutely satisfactory description of isolated, non-interacting particles. These observations provide a natural and obvious framework for formulating both space-time and internal symmetries [Roman 69]. The mathematical physics community is now in the process of working out a complete and consistent definition of "interacting" fields, checking to see if the assumptions make sense and/or have solutions that are complete, etc.

The most concise statement of the assumptions of General Quantum Field Theory is the original one [Streater 64]. The axioms for a quantum field theory [e.g. ($\mathcal{H}, c, \Gamma, \Omega_0^f$), see appendix A1 to this dissertation for a full definition in the case of Fermion fields] fall into four groups:

1. Assumptions of Relativistic Quantum Theory

The states of the theory are described by unit rays in a separable Hilbert space $\mathcal{H}$. The relativistic transformation law of states is given by a continuous unitary representation of the inhomogeneous $SL(2,C)$:

$$\{a, A\} \rightarrow U(a, A) .$$

[NB A here is a representation of the group (operator), and a is a constant vector].

Since $U(a,1)$ is unitary it can be written as $U(a,1) = \exp(iP^{\mu} a_{\mu})$, where P^{μ} is an unbounded Hermitian (i.e. symmetric) operator, interpreted as the energy momentum operator of the theory.

The operator $P^{\mu}P_{\mu} = m^2$ is interpreted as the square of the mass. The eigenvalues of P^{μ} lie in or on the forward light cone (spectral condition). There is an invariant state, ψ_0, $U(a, A)\psi_0 = \psi_0$ unique up to a constant phase factor (uniqueness of the vacuum).

Next are the properties defining a field, with transformation law under $SL(2,C)$ given by the *n x n* matrix representation $S: A \to S(A)$.

2. Assumptions about the Domain and Continuity of the Field

For each test function $f \in \mathscr{S}$ defined on space-time, there exists a set $\varphi_1(f),\ldots\varphi_n(f)$ of operators. These operators, together with their adjoints $\varphi_1(f)^*,\ldots\varphi_n(f)^*$ are defined on a domain *D* of vectors, dense in $\mathcal{H}$. Furthermore, *D* is a linear set containing ψ_0,

$$\psi_0 \in D,$$

and the U(a,A) and the $\varphi_f(f)$ and $\varphi_f(f)^*$ carry vectors in D into vectors in D

$U(a, A)D \subset D$, $\varphi_f(f)D \subset D$, $\varphi_f(f)^* D \subset D$, *where* $j = 1, 2,\ldots n$.

If $\Phi, \Psi \in D$, then $(\Phi, \varphi_f(f)\psi)$ is a tempered distribution, regarded as a functional of *f*.

[NB *D* always contains the domain D_0, consisting of those vectors which are obtained from the vacuum state by applying polynomials in the smeared fields. Also, since an unbounded operator, defined as Hermitean on a domain, may have several self-adjoint extensions to vectors not in the domain (even if the domain is dense) one requires self-adjoint operators *ab initio* so one may assert the usual spectral condition, completeness etc.]

3. Transformation Law of the Field

The equation $U(a, A)\varphi_f(f)U(a, A)^{-1} = \sum S_{jk}(A^{-1})\varphi_k(\{a, A\} f)$ is valid when each side is applied to any vector in *D*. Here $\{a, A\} f(x) = f(A^{-1}(x - a))$.

4. Local Commutativity or Microscopic Causality

If the support of *f* and the support of *g* are space-like separated [i.e. if $f(x)g(y) = 0$ for all pairs of points such that $(x-y)^2 \geq 0$], then one or the other of

$$[\varphi_j(f), \varphi_k(g)]_\pm \equiv \varphi_j(f)\varphi_k(g) \pm \varphi_k(g)\varphi_j(f) = 0$$

holds for all *j,k* when the left-hand side is applied to any vector in *D*. Similarly

$$[\varphi_f(f), \varphi_k(g)^*]_\pm = 0.$$

Put in terms of unsmeared fields, this assumption is simply

$$[\varphi_j(x), \varphi_k(y)]_\pm = 0 \quad \text{if } (x-y)^2 < 0 \text{ and } [\varphi j(x), \varphi_k{}^*(y)]_\pm = 0$$

where $\varphi^*(y)$ is the field that to the test function $g(y)$ gives the operator $[\varphi(\bar{g})]^*$,

i.e. $\varphi^*(g) = \varphi(\bar{g})^*$.

<u>Definition</u>: A relativistic quantum theory satisfying axiom *1* with a field φ_j, *j=1,...n* satisfying *2*, *3*, and *4* is a **field theory** if the vacuum state is cyclic for the smeared fields, that is, if polynomials in the smeared field components $P(\varphi_1(f), \varphi_2(g), \ldots)$, when applied to the vacuum state, yield a set D_0 of vectors dense in the Hilbert space of states.

We here mention one additional assumption often made enabling a contact with collision theory; it is the notion of asymptotic completeness: $\mathcal{H} = \mathcal{H}^{in} = \mathcal{H}^{out}$ or the Hilbert spaces describing asymptotic and interpolating fields are identical.

4.2 Three Important Mathematical Problems in Quantum Mechanics

(1) *Show Self Adjointness* of H the Hamiltonian operator (and all other operators representing observables):

physical reasoning gives a formal expression for H and other "observables" as operators on a realization of $\mathcal{H}$ as $L^2(M,d\mu)$. We use the word "formal" because domains are not specified. It is usually easy to find a domain on which a given formal expression is a well-defined symmetric operator. The first mathematical problem is to prove essential self-adjointness or, if the operator is not essentially self-adjoint, to investigate the various self-adjoint extensions and choose the "right one" to be the observable. Of course the reason to show this property is to enable the application of the spectral theorems, which apply to these self adjoint operators.

(2) Spectral analysis: The second problem is to investigate the spectra of the observables (in particular the Hamiltonian) and to estimate the position and multiplicity of the point spectra.

(3) Scattering theory: The third problem is to describe in some way the behavior of the system for large *t* (time).

CHAPTER 5

BASIC MATHEMATICAL ISSUES

The purpose of this section is to summarize the issues dealing with the lack of boundedness and divergence of some operators occurring in quantum field theory. A careful definition of the basic quantities involved is included in appendix A1. Here we present a very brief synopsis of the difficult issues and methods of dealing with them.

Now we will attempt to introduce the reader to some of the most important obstacles one must surmount in the attempt to insert interacting fields into the Wightman formalism. The following discussion is heuristic.

In the transition from Quantum Mechanics to Quantum Field Theory, the formalism evolves from a finite number of degrees of freedom to an infinite number. Nearly all mathematical difficulties associated with QFT are related to this fact. It is of note that a countable infinity suffices to undermine the foundation.

In standard quantum mechanics, starting from the canonical formalism of classical mechanics, there is a configuration space whose points are described by coordinates q_k, ($k = 1,\ldots n$) and for each position coordinate there is a conjugate momentum coordinate p_k. The "quantization" consists in replacing these variables by self-adjoint operators satisfying [Haag 96]

$$[q_k, q_l] = [p_k, p_l] = 0; [p_k, q_l] = \frac{\hbar}{i}\delta_{kl}.$$

Now, by substituting the following

$$a_k = 2^{-1/2}(p_k - iq_k);\ \ a_k^* = 2^{-1/2}(p_k + iq_k)$$

one defines an occupation number n_k as an eigenvalue of the positive operator $N_k = a_k^* a_k$ and the commutation relations imply that n_k must be a non-negative integer. An occupation number distribution (n) is an infinite sequence of such integers. Let us divide the set of such sequences into classes, saying that ($n^{(1)}$ and $n^{(2)}$) are in the same class if the sequences differ only in a finite number of places. The application of a creation or annihilation operator changes (n) in only one place. Therefore any set of basis vectors $\Psi_{(n)}$ with (n) restricted to one class already spans a representation space of $N_k = a_k^* a_k$ and it is evident that representations belonging to different classes cannot be unitarily equivalent. A study of all such representations [Gard 54] has shown that the number of such is at least as large as the continuum.

The preceding has devastating consequences for formal perturbation theory, as it is normally applied. Let us take the simplest case:

> *Haag's Theorem*: Suppose $\phi^f(\mathbf{x},t)$ defines a free (neutral) scalar field theory with mass $m > 0$. Suppose $\phi(\mathbf{x},t)$ defines an interacting field obtained from the former by a small perturbation, and that both ϕ^f and ϕ satisfy the Wightman axioms. Now suppose that (at any instant t) the operators $\phi^f(\mathbf{x})$and $\phi(\mathbf{x})$ both provide an irreducible operator ring. Finally suppose that $\phi^f(\mathbf{x}) = V\phi(\mathbf{x})V^{-1}$, where V is unitary and time independent. Then it follows that $\phi(x)$ is a *free field*, describing a free-field theory with mass *m*. [Roman 69]

The so-called interaction picture in formal Quantum Field Theory is based on just such a structure as described above. This is a very inconvenient circumstance.

Much of the formalism of Quantum Field Theory is transcribed from quantum mechanics by substituting continuous variables for discrete variables. The canonical commutation relations

$$[p_i, q_j] = -i\delta_{ij} \tag{5.1}$$

become, for a scalar field φ,

$$[\partial_0 \varphi(\mathbf{x},t), \varphi(\mathbf{y},t)] = -i\delta(\mathbf{x}-\mathbf{y}) \tag{5.2}$$

the equal time commutation relations. One sees immediately that the field variables are very singular entities. Equation (5.2) is the form for a free, or non-interacting field. It is assumed, and verified by results from soluble models, that the singularity *at the same point* for a product of interacting fields is at least as severe as their non-interacting counterparts. Interactions are nearly always defined by products of fields at the same point. For this reason the fields are not regarded as space-time functions but rather as distributions: operator valued distributions.

We deal with two species of fields: Bosons satisfying commutation relations of the form

here ({ , } = [,]$_+$)

$$[\varphi(\mathbf{x},t),\varphi(\mathbf{y},t)] = [\pi(\mathbf{x},t),\pi(\mathbf{y},t)] = 0 \tag{5.3}$$

$$[\pi(\mathbf{x},t),\varphi(\mathbf{y},t)] = -i\delta^3(\mathbf{x}-\mathbf{y}) \quad \text{where} \quad \pi = \dot{\varphi} \tag{5.4}$$

and Fermions satisfying (anti)commutation relations of the form

$$\{\psi^*_\alpha(\mathbf{x}),\psi^*_\beta(\mathbf{y})\} = \{\psi_\alpha(\mathbf{x}),\psi_\beta(\mathbf{y})\} = 0\,, \tag{5.5}$$

$$\{\psi^*_\alpha(\mathbf{x}),\psi_\beta(\mathbf{y})\} = \delta_{\alpha\beta}\delta(\mathbf{x}-\mathbf{y})\,. \tag{5.6}$$

When the fields are Fourier analyzed, we will get formulae resembling

$$\psi^0(x) = \frac{1}{V^{1/2}}\sum_{\mathrm{k}}\sum_{s=\pm}\left(\frac{m}{\Omega_k}\right)^{1/2}[a_{\mathbf{ks}}u(\mathbf{k};s)e^{-ikx} + b^*_{\mathbf{ks}}v(\mathbf{k};s)e^{ikx}]\,, \tag{5.7}$$

$$\overline{\psi}^0(x) = \frac{1}{V^{1/2}}\sum_{\mathrm{k}}\sum_{s=\pm}\left(\frac{m}{\Omega_k}\right)^{1/2}[a^*_{\kappa s}\overline{u}(\mathbf{k};s)e^{-ikx} + b_{\mathrm{k}s}\overline{v}(\mathbf{k};s)e^{ikx}]\,, \tag{5.8}$$

or

$$\varphi^0(x) = \frac{1}{V^{1/2}}\sum_{\mathrm{k}}\frac{1}{\sqrt{2\omega_k}}[c_{\mathbf{k}}e^{-ikx} + c^*_{\mathbf{k}}e^{ikx}]\,. \tag{5.9}$$

[We always use the summation/integration conventions for Fourier analyzed fields that were introduced in Chapter 1].

Here $\Omega_k = +\sqrt{\kappa_0^2 + \mathbf{k}^2}$, $\omega_k = +\sqrt{m_0^2 + \mathbf{k}^2}$, and u and v satisfy the algebra $(-\gamma^\mu k_\mu + \kappa)u(\mathbf{k};s) = 0$ and $(\gamma^\mu k_\mu + \kappa)v(\mathbf{k};s) = 0$, and the variable "s" is such that $u^*(\mathbf{k};s)u(\mathbf{k};s') = v^*(\mathbf{k};s)v(\mathbf{k};s') = \frac{\Omega_k}{\kappa}\delta_{ss'}$.

c_k or a_k are "annihilation operators" and c_k^* and b_k^* are "creation" operators, raising or lowering the particle content of Fock space by one. Let us review the typical machinery by the quote from [Feynman 72] below:

"Here is a very general example to demonstrate what is going on in the Fermion AND Boson cases. We have written $\mathcal{H}^n$ for a general *"n-particle"* state which is made up of products of one particle states, e.g. $\psi^n = \psi(x_1)\psi(x_2)\ldots\psi(x_n)$; where we have indicated variables in configuration space. We have written $\mathcal{H}^{(n)}$ for the space made up of symmetrized or anti-symmetrized space products of one particle states. Thus for a properly symmetrized vector $\psi^{(n)}$ in $\mathcal{F}_a(\mathcal{H})$ we could write $\psi^{(n)}(x_1,\ldots x_n) = \begin{vmatrix} \psi_1(x_1) & \cdots & \psi_1(x_n) \\ \cdots & \cdots & \cdots \\ \psi_n(x_1) & \cdots & \psi_n(x_n) \end{vmatrix}$. Let $|\varphi\rangle$ be any one-particle state. Define $a^+(\varphi)$ to be that linear operator which satisfies

$$a^+(\varphi)|\psi_1,\ldots\psi_n\rangle = |\varphi,\psi_1,\ldots\psi_n\rangle \qquad (5.10)$$

for any n-particle (properly symmetrized) state $|\psi_1,\ldots\psi_n\rangle$. For $n = 0$ this is understood to mean $a(\varphi)|vacuum\rangle = 0$. We call $a^+(\varphi)$ the creation operator for the state $|\varphi\rangle$ and its adjoint $a(\varphi)$ the annihilation operator. To find the effect of $a^+(\varphi)$ on an n-particle state $|\psi_1,\ldots\psi_n\rangle$ multiply on the left by an arbitrary (n-1) particle state $\langle\chi_1,\ldots\chi_{n-1}|$.

$$\langle\chi_1,\ldots\chi_{n-1}|a(\varphi)|\psi_1,\ldots\psi_n\rangle = \langle\psi_1,\ldots\psi_n|a^+(\varphi)|\chi_1,\ldots\chi_{n-1}\rangle^*$$
$$= \langle\psi_1,\ldots\psi_n|\varphi,\chi_1,\ldots\chi_{n-1}\rangle^*$$

$$= \begin{vmatrix} \langle\psi_1|\varphi\rangle\langle\psi_1|\chi_1\rangle & \cdots & \langle\psi_1|\chi_{n-1}\rangle \\ \cdots & \cdots & \cdots \\ \langle\psi_{n1}|\varphi\rangle\langle\psi_n|\chi_1\rangle & \cdots & \langle\psi_n|\chi_{n-1}\rangle \end{vmatrix}^*$$

$$= \left\{ \sum_{k=1}^{n} \zeta^{k-1} \langle \psi_k | \varphi \rangle \begin{vmatrix} \langle \psi_1 | \varphi \rangle \langle \psi_1 | \chi_1 \rangle & \cdots & \langle \psi_1 | \chi_{n-1} \rangle \\ \cdots & (no\, \psi_k) & \cdots \\ \langle \psi_{n1} | \varphi \rangle \langle \psi_n | \chi_1 \rangle & \cdots & \langle \psi_n | \chi_{n-1} \rangle \end{vmatrix} \right\}^*$$

(an expansion by minors) ζ = *-1* for Fermi, +1 for Bose case.

$$= \sum_{k=1}^{n} \zeta^{k-1} \langle \psi_k | \varphi \rangle^* \langle \psi_1 \cdots (no\, \psi_k) \cdots \psi_n | \chi_1 \cdots \chi_{n-1} \rangle^*$$

$$= \sum_{k=1}^{n} \zeta^{k-1} \langle \varphi | \psi_k \rangle \langle \chi_1 \cdots \chi_{n-1} | \psi_1 \cdots (no\, \psi_k) \cdots \psi_n \rangle$$

This is for arbitrary $\langle \chi_1, \ldots \chi_{n-1} |$ so we can write

$$a(\varphi) | \psi_1 \cdots \psi_n \rangle = \sum_{k=1}^{n} \zeta^{k-1} \langle \varphi | \psi_k \rangle | \psi_1 \cdots (no\, \psi_k \cdots \psi_n \rangle \qquad (5.11)$$

Equations (5.10) and (5.11) describe the annihilation and creation operators on many-particle states. It follows that $a^+(\varphi_1) a^+(\varphi_2) = \zeta a^+(\varphi_2) a^+(\varphi_1)$. Taking the adjoint one gets a similar relation for annihilation operators."

Sometimes a notation is used that specifies how many particles are present in a given state. For example $| n_1, n_2, \ldots . n_k, \ldots \rangle$ would indicate that there are n_1 particles in state 1, n_2 particles in state 2 and so on. For Boson particles, any state n_k may have an infinite number of particles occupying it. For this reason, Boson operators are unbounded with respect to the Fock Hilbert space on which they act. For Fermion particles, n_k may be only 1 or 0, as a result of the anti-commutation relations. Therefore, the Fermi operators are bounded with respect to their Fock space. There are an infinite number of states available; but we consider the set of all the finite particle states, and we only define operators on this set. The set is called F_0 in the case of Bosons and G_0 in the case of Fermions. The consequence of this is that the operators are only densely defined. We can only prove self-adjointness of the operators on F_0 or G_0, which will be the core for each. Thus, in this terminology the Boson operators may only be proved *essentially* self-adjoint and unbounded. Fermion operators may only be proved *essentially* self-adjoint and bounded.

A final series of remarks regards the operator products evaluated at a single point. Equations (5.7), (5.8) and (5.9) along with equations (5.3) thru (5.6) imply (in the Fourier transformed space)

$$[c_{\mathbf{k}}, c_{\mathbf{k}'}{}^*] = \delta_{\mathbf{k}\mathbf{k}'} \quad \text{and} \quad [c_{\mathbf{k}}, c_{\mathbf{k}'}] = [c_{\mathbf{k}}{}^*, c_{\mathbf{k}'}{}^*] = 0. \tag{5.12}$$

Let us consider the correlation function in configuration space $\langle 0|\varphi(x)\varphi(y)|0\rangle$. This, along with equations (5.9) and (5.12) will give us

$$\int \frac{d^3\mathbf{k}\, d^3\mathbf{k}'}{(2\pi)^3\sqrt{2\omega_k 2\omega_{k'}}} e^{ikx} e^{-ik'y} \langle 0|c(k)c^*(k)|0\rangle$$

$$= \int_{k_0=+\omega_k} \frac{d^3\mathbf{k}}{(2\pi)^3 2\omega_k} e^{ik(x-y)} \equiv \Delta^+(x-y)\ . \tag{5.13}$$

Now as $x \to y$, this approaches a quadratically diverging expression

$$\langle 0|\varphi^2(x)|0\rangle = \Delta^+(0) = \int \frac{d^3\mathbf{k}}{(2\pi)^3 2\omega_k} \quad . \tag{5.14}$$

It is for this reason that objects occurring as "measurable" such as currents, that involve products of field operators are normal ordered, which simply means that they are defined so that their vacuum expectation values vanish; viz:

$$:\phi(x)\phi(y): = \phi(x)\phi(y) - \langle 0|\phi(x)\phi(y)|0\rangle\ . \tag{5.15}$$

The normal ordering, or Wick ordering, is indicated by the colons. This ordering has another, extremely useful interpretation. The free fields φ and ψ may have their Fourier components split into positive and negative frequency parts in a covariant way:

$$\psi(x) = \psi^+(x) + \psi^-(x) \quad \text{and} \quad \varphi(x) = \varphi^+(x) + \varphi^-(x) \tag{5.16}$$

Let $a_\alpha(k^\alpha), a_\beta(k^\beta), \ldots$ refer to a set of arbitrary momentum-space operators, all referring to either positive or negative frequency parts. Define the normal product of these operators as

$$N(a_\alpha(k^\alpha)a_\beta(k^\beta)\ldots) = (-1)^p a_\rho(k^\rho)a_\sigma(k^\sigma) \tag{5.17}$$

where the RHS has the same set of operators, but they are arranged such that reading from right to left we have first all negative frequency operators and then the positive frequency operators. The number p denotes the number of transpositions among *Fermion* operators that are necessary to achieve the ordering. For example, let α and β refer to Boson operators, γ to a Fermion operator, then $N(a_\alpha^+ a_\beta^- a_\gamma^+) = a_\alpha^+ a_\gamma^+ a_\beta^-$. If all three indices refer to Fermion operators we have $N(a_\alpha^+ a_\beta^- a_\gamma^+) = -a_\alpha^+ a_\gamma^+ a_\beta^-$. If we consider an operator $d(k)$ that is not pure positive or pure negative frequency, but is $d(k) = d(k)^+ + d(k)^-$ then we define $N(a_\alpha^+ a_\beta^- \ldots d \ldots) = -a_\alpha^+ a_\gamma^+ a_\beta^- d^+ - a_\alpha^+ a_\gamma^+ a_\beta^- d^-$. Note that

$$N(\psi(x)\psi(y)) = \psi(x)\psi(y) - i\Delta^+(x-y) =: \psi(x)\psi(y): \tag{5.18}$$

Now the interesting thing is that equation (5.15) and equation (5.17) are the same operation. This result, not obvious, is a standard theorem in the context of quantum field theory.

Do the commutation rules hold for interacting fields? How is normal ordering used in practical cases? Let us discuss an example where these problems are addressed, called the Derivative Coupling model. This model illustrates where the difficulties we have discussed arise, and is also very similar to the solution we will obtain for our problem, including the difficulties requiring normal ordering. Suppose the field equations are

$$(-i\gamma_\mu \partial^\mu + m)\psi = g\gamma_\mu \partial^\mu \phi(x)\psi(x) \tag{5.19}$$

$$(\partial_0^2 - \Delta + m^2)\phi = \partial^\mu j_\mu = 0\,. \tag{5.20}$$

Here j_μ is the standard Dirac current denoted $j^\mu(x) = \overline{\psi}(x)\gamma^\mu\psi(x)$ and ϕ is a free field.

In the classical case this has the immediate solution

$$\psi(x) = e^{ig\phi(x)}\psi_0(x) \text{ with } \psi_0 \text{ a free Dirac field of mass } m. \tag{5.21}$$

This appears to be well defined, since in the classical realm one may get regular enough free fields so that the above is well defined. Now let us try to quantize the theory by assuming "canonical quantization" rules. The canonical momentum for ϕ is $\pi = \dot{\phi} + gj_0$. The commutation relations are

$$\{\psi,\psi\} = 0 = \{\psi^*,\psi^*\} \qquad \{\psi(\mathbf{x},t),\psi^*(\mathbf{y},t)\} = \delta(\mathbf{x}-\mathbf{y}) \tag{5.22}$$

$$[\phi,\phi] = 0 = [\pi,\pi] \qquad [\phi(\mathbf{x},t),\pi(\mathbf{y},t)] = i\delta(\mathbf{x}-\mathbf{y}) \tag{5.23}$$

$$[\phi,\psi] = 0 = [\pi,\psi]. \tag{5.24}$$

Equations (5.18) and (5.19) lead to (recall $j_0(x) = \overline{\psi}(x)\gamma_0\psi(x)$)

$$[j_0(\mathbf{x},t),\psi(\mathbf{y},t)] = -\delta(\mathbf{x}-\mathbf{y})\psi(\mathbf{y},t) \qquad [\dot{\phi}(\mathbf{x},t),\psi(\mathbf{y},t)] = g\delta(\mathbf{x}-\mathbf{y})\psi(\mathbf{y},t). \tag{5.25}$$

Consider the unequal time commutator $[\phi(x),\psi(y)]$; it is determined by the initial data and equations (5.15) and (5.16).

$$[\phi(x),\psi(y)] = -g\Delta(x-y)\psi(y). \tag{5.26}$$

From this, the result for the three point function is

$$\langle\phi(x)\psi(y)\overline{\psi}(z)\rangle = \langle[\phi_-(x),\psi(y)\overline{\psi}(z)]\rangle = -g(\Delta^+(x-y)-\Delta^+(x-z))\langle\psi(y)\overline{\psi}(z)\rangle. \tag{5.27}$$

Thus, the right hand side of equation (5.15) is not well defined since the limit $y \to x$ in the correlation function diverges like $\partial_\mu\Delta^+(\varepsilon)$ as $\varepsilon \to 0$. A subtraction (or renormalization) is required to properly define the RHS of equation (5.15). A candidate that works is

$$(\gamma_\mu\partial^\mu\psi)(x) = \lim_{y\to x}\gamma_\mu\partial_x^\mu(\phi(x)\psi(y) + g\Delta^+(x-y)\psi(y)). \tag{5.28}$$

The renormalized field which solves equation (5.19) with RHS defined by equation (5.27) is

$$\psi_R(x) = \lim_{\Lambda\to\infty} e^{\frac{1}{2}g^2\Delta^+_\Lambda(0)} e^{ig\phi_\Lambda(x)}\psi_0(x) =: e^{ig\phi}:(x)\psi_0(x) \tag{5.29}$$

where the notation $\Lambda \to \infty$ is taken to mean the ultraviolet limit, and $\phi_\Lambda(x)$ is a scalar field with ultraviolet cutoff Λ and $\Delta^+{}_\Lambda(0)$ the corresponding two point function evaluated at the origin. Formally this is written as

$$\psi_R(x) = \lim_{\Lambda\to\infty} Z_\Lambda^{-1/2}\psi_{\Lambda,un}(x) = Z^{-1/2}\psi_{un}(x). \tag{5.30}$$

We have, formally for ψ_R

$$\{\psi_R(\mathbf{x},t), \psi_R{}^*(\mathbf{y},t)\} = Z^{-1}\delta(\mathbf{x}-\mathbf{y}) \tag{5.31}$$

and the renormalization "constant" is divergent. The unrenormalized field ψ_{un} *does* obey equal time anti-commutation rules, but it is ill-defined.

Let us discuss this solution in some detail. The details are interesting for this model problem, but they are imperative for understanding the solution of our Y4 problem. First note that in the derivative coupling model, ϕ is a free field.

Consider any operator $ig\phi$, for which we will define the Wick exponential by

$$:e^{ig\phi}: = \sum_{n=0}^{\infty} \frac{(ig)^n}{n!} :\phi^n: \tag{5.32}$$

and the "Wick monomial" $:\phi^n:$ is defined recursively by

$$:\phi:(x) = \phi(x) \tag{5.33}$$

$$:\phi^2:(x) = \lim_{y\to x}[\phi(x)\phi(y) - \langle\phi(x)\phi(y)\rangle] \tag{5.34}$$

$$:\phi^n:(x) = \lim_{y\to x}[:\phi^{n-1}:(x)\phi(y) - (n-1)\langle\phi(x)\phi(y)\rangle :\phi^{n-2}:(x). \tag{5.35}$$

The use of these formulae are as follows: if we were to approximate the exponential in equation (5.29) term by term, we would employ the formulae above. The odd powered terms will always have a vanishing vacuum expectation value after wick ordering because $<\phi(x)> = 0$.

Let us review the origin of the terms in equation (5.28). Recall that for two operators $\hat{f}$ and $\hat{g}$, with $[\hat{f},[\hat{f},\hat{g}]]=0$ and $[\hat{g},[\hat{f},\hat{g}]]=0$ the Baker, Campbell Hausdorff formula reads

$$e^{\hat{f}}e^{\hat{g}} = e^{\hat{f}+\hat{g}+\frac{1}{2}[\hat{f},\hat{g}]} \tag{5.36}$$

Now define $\phi = \phi^{+} + \phi^{-}$ where

$$\phi^{+}(x) = \frac{1}{(2\pi)^{3/2}}\int \frac{1}{\sqrt{2k_0}}c^{*}(\mathbf{k})e^{ikx}d\mathbf{k} \text{ and } \phi^{-}(x) = \frac{1}{(2\pi)^{3/2}}\int \frac{1}{\sqrt{2k_0}}c(\mathbf{k})e^{-ikx}d\mathbf{k}\,.$$

Finally, let $\hat{f} = \phi^{+}$ and $\hat{g} = \phi^{-}$; note that

$$[\phi^{+}(x),\phi^{-}(y)] = -i\Delta^{+}(x-y) \text{ and } [\phi^{-}(x),\phi^{+}(y)] = -i\Delta^{-}(x-y)\,. \tag{5.37}$$

With these definitions one sees that the LHS of equation (5.35) is, in fact, the normal ordered version of the total field ϕ, and by the commutation rules for $c(\mathbf{k})$ the result follows. Since the equation of x and y in relation (5.36) is undefined, one may fourier transform to k space, and define the limit $x \to y$ or the limit $k_0 \to \infty$.

CHAPTER 6

CLASSICAL Y4 THEORY

Let us begin with a few observations regarding the coupled system known as the Dirac-Klein-Gordon (DKG) equations.

$$(-i\gamma^{\mu}\partial_{\mu} + \kappa)\psi = g\varphi\psi \qquad (\kappa, g > 0) \tag{6.1}$$

$$(-\Delta + \partial_t^2 + m^2)\varphi = g\overline{\psi}\psi \qquad (m > 0) \tag{6.2}$$

Equation (6.1) is the Dirac equation for a bi-spinor ψ (of mass κ) and (6.2) is the Klein-Gordon equation for a scalar field of mass *m* with source term on RHS. We will discuss some relevant (to our thesis research) classical results for 1+1 and 3+1 dimensions.

In the 3+1 case, the γ's are operators in "spin space" satisfying $\gamma^{\mu}\gamma^{\nu} + \gamma^{\nu}\gamma^{\mu} = 2g^{\mu\nu}$

($g^{00} = 1$; $g^{kk} = -1$ *for* $k = 1,2,3$; $g^{\mu\nu} = 0$ *for* $k \neq 0$).

In the 1+1 dimensional case, DKG takes the following form

$$\partial_t\psi = \hat{\alpha}\partial_x\psi + (\kappa - g\varphi)\hat{\beta}\psi \tag{6.3}$$

$$\partial_t^2\varphi - \partial_x^2\varphi + m^2\varphi = g\overline{\psi}\psi \tag{6.4}$$

where the α and β matrices are defined as

$$\alpha = \begin{pmatrix} 0 & -1 \\ -1 & 0 \end{pmatrix} \quad \beta = -i\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}. \tag{6.5}$$

This system is hyperbolic, with the usual properties (finite propagation speed and real eigenvalues for the first order form of the system). It is always natural to seek solutions for the non-linear equations in terms of (perturbations of) solutions of the linear equations.

Denote by $L^p(\mathbf{R}^n)$ the Lebesgue space of functions whose p^{th} powers are integrable, with norm $\|u\|_p = (\int_{\mathbf{R}^n} |u(x)|^p dx)^{\frac{1}{p}}$, $\|u\|_\infty$ = ess sup $|u(x)|$.

The space $H^m(\mathbf{R}^n)$ is the Sobolev space of functions whose distributional derivatives of order $\leq m$ lie in $L^2(\mathbf{R}^n)$, with norm $\|u\|_{H^m} = \{\sum_{|j|\leq m} \int_{R^n} |D^j u(x)|^2 dx\}^{\frac{1}{2}}$.

When written as a system, the appropriate solutions of equations (6.1) and (6.2) [or (6.3) and (6.4)] are pairs $\left[\psi(t), \begin{pmatrix}\varphi(t) \\ \varphi_t(t)\end{pmatrix}\right]$ lying in the spaces $H^m \oplus (H^m \oplus H^{m-1})$ for each t. There is a conserved "current" for the DKG system, denoted $\overline{\psi}\psi$, which is equal to $|\psi_1 - \overline{\psi}_2|^2$ in 1 + 1 dimensions and $|\psi_1 - \overline{\psi}_4|^2 + |\psi_2 + \overline{\psi}_3|^2$ in 3 + 1 dimensions.

Now let us review some known facts regarding DKG systems: regarding equations (6.3) and (6.4), [Chad 73] proves an existence theorem as follows:

Theorem B: For the given data (at time $t = t_0$) $\left(\psi^0, \begin{pmatrix}\varphi^0 \\ \dot{\varphi}^0\end{pmatrix}\right) \in H^1 \oplus (H^1 \oplus L^2)$ there exist unique functions ψ and φ, the map $t \to \left(\psi(t), \begin{pmatrix}\varphi(t) \\ \dot{\varphi}(t)\end{pmatrix}\right) : (t_0, \infty) \to H^1 \oplus (H^1 \oplus L^2)$ being continuous, which for all t, $t_0 < t < \infty$, satisfy the integrated form of equations (6.3) and (6.4).

Regarding regularity, [Chad 73] goes on to show that if the Cauchy data $\left(\psi^0, \begin{pmatrix}\varphi^0 \\ \dot{\varphi}^0\end{pmatrix}\right)$ have arbitrarily many L^2-derivatives then the solutions ψ and φ are in the class C^∞ and satisfy the differential equations in the classical sense.

Concerning DKG in 3 + 1 dimensions (e.g. equations (6.1) and (6.2)) issues of existence and regularity are somewhat more complicated. [d'Anconia 05] has addressed this issue and finds: for “small” data, given smooth, compactly supported initial data DKG does have a smooth solution for all times $t > 0$, with “small” being appropriately defined. For arbitrary (and “large”) initial data, global existence questions are not resolved except for the rather limited case of the theorem (due to Chadam) quoted in Chapter 9.

Using “null form” arguments (field variables vanish on the “light-cone”) d’Anconia shows that DKG is Locally Well Posed for data $\left(\psi^0, \begin{pmatrix} \varphi^0 \\ \dot{\varphi}^0 \end{pmatrix}\right) \in H^{\varepsilon} \oplus (H^{1/2+\varepsilon} \oplus H^{-1/2+\varepsilon})$, for arbitrarily small, but not zero, ε. [N.B. We have been rather free with D’Anconia’s notation.] Locally Well Posed refers to data within a compact, causal region (the downward lightcone from the point in consideration). It is known [Lin 96] that “optimal” regularity cannot be reached unless Klainerman’s [Klan 02] “null form” condition is satisfied. Optimal here refers to that achieved by the massless (κ, $m = 0$) scale invariant case… $\left(\psi^0, \begin{pmatrix} \varphi^0 \\ \dot{\varphi}^0 \end{pmatrix}\right) \in L^2 \oplus (H^{1/2} \oplus H^{-1/2})$.

This issue might take us far afield. We will be defining a scattering theory, wherein the solutions for the *Quantum Theory* are represented as Fourier Transforms of Schwarz space functions in $\mathbf{R}^4$ (or the Minkowski version thereof). These are guaranteed smooth, but the issue of using them to solve the full, coupled theory is not addressed. Thus the global existence, and regularity, will be issues in the full *quantum field theory*, just as they are in the classical field theory.

We are now in a position to introduce a “solution” to the 3 + 1 DKG system. Let ψ^k be the k^{th} component of a Dirac spinor. Let $\Delta(x)$ be the Schwinger Green Function solution to the free Klein-Gordon equation with the following properties:

Definition: $\Delta(x) = \frac{1}{(2\pi)^4}\oint \frac{e^{ikx}}{m^2 - k^2}\,dk$ for a simple, closed circular contour (clockwise) enclosing the points $+\omega_k$ and $-\omega_k$. $\omega_k = \sqrt{\mathbf{k}^2 + m^2}$

$$\Delta(0) = \Delta(0, \mathrm{x}) = 0\,; \qquad \left.\frac{\partial\Delta(x)}{\partial x^\ell}\right|_{x_0=0} = 0 \text{ for } \ell = 1, 2, 3; \quad \left.\frac{\partial\Delta(x)}{\partial x^0}\right|_{x_0=0} = \delta(\mathbf{x})\,. \tag{6.6}$$

This has the explicit configuration space representation

$$\Delta(x) = \frac{1}{2\pi}\varepsilon(x)\delta(x_\mu x^\mu) + \frac{m}{4\pi}\frac{\Theta(x_\mu x^\mu)\varepsilon(x)}{\sqrt{x_\mu x^\mu}}J_1(m\sqrt{x_\mu x^\mu}) \quad \varepsilon(x) = \theta(x) - \theta(-x)\,. \tag{6.7}$$

Let $\psi^0(x)$ be a solution to the free (sourceless) Dirac equation. We assert that DKG has the "formal" solution

$$\psi = \exp[ig\gamma^0 \int \Delta(x-y)\varphi(y)d^3\mathbf{y}]\Big|_{y_0=x_0}\psi^0(x) \tag{6.8}$$

using the Dirac matrix γ^0. One merely has to plug equation (6.8) into DKG to prove the assertion.

We propose to use the relation (6.8) to rigorously define the time development of the Y4 system as a scattering theory. The objects ψ and φ will be fully quantized, and we will explain a number of puzzling issues associated with the "standard" definition (Haag-Ruelle) of the quantum scattering. With the exact solution available, this will turn out to be a rather simple exercise.

At this point it is worthwhile to state a theorem due to Wightman on the difficulties of defining *any* kind of interacting quantum field.

Theorem C: If properties mentioned in Axioms 1 through 4 hold for a Hermitean scalar field $\phi(x)$, then $\phi(x)$ cannot be defined as an operator in a Hilbert space $\mathcal{H}$ for any x, otherwise the Hilbert space consists only of the vacuum state, i.e. the theory is trivial.
For a proof, see [Strocchi 93], page 7.

The meaning of this theorem is clear: quantum fields cannot be defined as "functions" on a Hilbert space, but rather must be defined as operator-valued distributions. Therefore, when interactions (or any other object) require a multiplication of fields evaluated at a point, some kind of regularization (e.g. renormalization) is required.

CHAPTER 7

QUANTUM THEORY FOR THE DIRAC-KLEIN-GORDON SYSTEM

We will now attempt to make mathematical sense of the quantum field theory whose defining equations are

$$(-i\gamma^{\mu}\partial_{\mu}+\kappa)\psi = g\varphi\psi \qquad (\kappa, g>0) \qquad 7.1)$$

$$(-\Delta+\partial_t^2+m^2)\varphi = g\overline{\psi}\psi \qquad (m>0) \qquad (7.2)$$

and whose nominal solution for the ψ field has been given above as

$$\psi = \exp[ig\gamma^0\int\Delta(x-y)\varphi(y)d^3\mathbf{y}]\Big|_{y_0=x_0}\psi^0(x) \qquad (7.3)$$

using the Dirac matrix γ^0.

φ and ψ are interacting quantum fields, their rigorous definition as Wightman fields (in 4 dimensions) has been sought for half a century. We will not be able to provide this definition (yet) but will be able to proceed far enough to define the time development of the object in equation (7.3) (now viewed as a fully quantized field). This is one of our three stated goals of Quantum Field Theory. For this task we assume the other two goals are already satisfied. In addition, we will rely upon a major assumption that is always made in scattering theory: that is *asymptotic completeness* (defined in Section 4.1). Some final remarks at the end of this dissertation will speculate on approaches to the first two problems.

First, the ψ^0(x) above is a free field, and thus already defined previously. The exponential then becomes the problem. Prior to commenting on the Δ, which is a "c-number" (complex valued function) we note the same difficulty with the exponential applies as mentioned in the Introduction section, from [Strocchi 93]. Our line of argument will proceed as follows:

1. We will introduce some formalism to define the "time zero" fields. The purpose for this maneuver is to facilitate the discussion of Haag-Ruelle scattering theory, which utilizes field structures smeared only in $\mathbf{R}^3$, and leaves the time explicitly present. (There is more than one way to do this, but we will follow the method explained in [Reed 79]). We want our operators to be "functions" of time.
2. We will see from #1 that for every well-defined quantum field (i.e. operator valued distribution) we can define a quadratic form on a suitable domain that corresponds one for one to the field and is, in fact, the "field" that is normally introduced in physics texts. This latter "field" is the object of Haag-Ruelle scattering theory.

Definition: Consider a Hilbert space $\mathcal{H}$. An antilinear map $C\colon \mathcal{H} \to \mathcal{H}$ $(C(\alpha\varphi + \beta\psi) = \bar{\alpha}C\varphi + \bar{\beta}C\psi)$ is called a **conjugation** if it is norm-preserving and $C^2=I$.

Definition: Let $\mathcal{H}$ be a complex Hilbert space, $\Phi_S(.)$ the associated Segal (See Appendix A2) quantization for Boson fields. Let C be a conjugation on $\mathcal{H}$ and define $\mathcal{H}_C = \{f \in \mathcal{H} | Cf = f\}$. For each $f \in \mathcal{H}_C$ define $\varphi(f) = \Phi_S(f)$ and $\pi(f) = \Phi_S(if)$. The map $f \mapsto \varphi(f)$ is called the **canonical free field** and the map $f \mapsto \pi(f)$ is called the **canonical conjugate momentum** for *Boson* fields**.** [N.B. there is no connection between this variable and the physical momentum operator *P.* Also, the set of elements in $\mathcal{H}$ for which these maps are defined depends on the conjugation *C*.] For *Dirac* fields, for each $f \in \mathcal{H}_C$ define $\psi(f) = \Psi(f)$ and $\pi_\psi(f) = i\Psi^*(f)$ $= i\bar{\Psi}(f)\gamma^0$ on $\mathcal{H}_C = \{f \in \mathcal{H} | Cf = f\}$.

We will now introduce terminology to define "time zero" fields or what is called the Schrodinger representation. Let us begin with the [Reed 75]

Theorem D: Let $\mathcal{H}$ be a complex Hilbert space with conjugation *C*. Let $\varphi(\cdot)$ and $\pi(\cdot)$ be the corresponding canonical (Boson) fields. Then:

(a) (i) For each $f \in \mathcal{H}_C$, $\varphi(f)$ is essentially self adjoint on F_0.
(ii) $\{\varphi(f) | f \in \mathcal{H}_C\}$ is a commuting family of self-adjoint operators.
(iii) The vacuum Ω_0 is a cyclic vector for the family $\{\varphi(f) | f \in \mathcal{H}_C\}$.
(iv) In $\mathcal{H}_C$, if $f_n \to f$, then $\varphi(f_n)\psi \to \varphi(f)\psi$ for all $\psi \in F_0$ and $e^{i\varphi(f_n)}\psi \to e^{i\varphi(f)}\psi$ for all $\psi \in \mathcal{F}_s(\mathcal{H})$.

(b) Properties (a)(i) through (a)(iv) hold with $\varphi(.)$ replaced by $\pi(.)$.

(c) If $f, g \in \mathcal{H}_C$, then $\varphi(f)\pi(g)\psi - \pi(g)\varphi(f)\psi = i(f,g)\psi$ for all $\psi \in F_0$
$$e^{i\varphi(f)}e^{i\pi(g)} = e^{-i(f,g)}e^{i\pi(g)}e^{i\varphi(f)}.$$

We are here referring to the objects that were written in equations (5.7) through (5.9). The main point is that the algebraic operators in those equations (i.e. the *a*'s and *b*'s and *c*'s) had a time dependence explicit only through the exponential e^{ikx}, e^{-ikx}. These remarks are the reasoning behind that notation. The conjugation used for the free (non-interacting) fields is now: write $f \in L^2(H_m, d\Omega_m)$ as $f(p_0, \mathbf{p})$ and define $(Cf)(p_0, \mathbf{p}) = \overline{f(p_0, -\mathbf{p})}$. The fields corresponding to C will be denoted by $\varphi(.)$ and $\pi(.)$ and defined as $\varphi_m(f) = \varphi(Ef)$ or $\pi_m(f) = \pi(Ef)$ for all real-valued $f \in \mathscr{S}(\mathbf{R}^4)$ in the case of Bosons and $\psi_m(f) = \psi(Ef)$ in the case of Fermions. This will extend to all of $\mathscr{S}(\mathbf{R}^4)$ by linearity. Because of the projection E the class of functions on which φ and π are defined can be extended to include distributions of the form $\delta(t\text{-}t_0)g(x_1,x_2,x_3)$ where the $g \in \mathscr{S}(\mathbf{R}^3)$.

Definition: The maps $g \mapsto \varphi_m(\delta g)$ and $g \mapsto \pi_m(\delta g)$ are called the **time-zero** (or Schrodinger) fields.

We must now transfer the quantum fields from the Fock space built by the Segal operators to the Fock space built by the time-zero operators, which will be that space built up from $L^2(H_m, d\Omega_m)$ to that built up from $L^2(\mathbf{R}^3)$. For $f \in L^2(H_m, d\Omega_m)$ we define $a^*(f) = (a^-(f))^*$ and $a(f) = (a^-(Cf))$ Each $f(p) \in L^2(H_m, d\Omega_m)$ may be viewed as a function $f(\mathbf{p}) = f(\mu(\mathbf{p}),\mathbf{p})$ on $\mathbf{R}^3$. Thus for each $f(p) \in L^2(H_m, d\Omega_m)$ define $(Jf)(\mathbf{p}) = f(\mu(\mathbf{p}))/\sqrt{\mu(\mathbf{p})}$. Here J is a unitary map of $\mathcal{F}_S(L^2(H_m, d\Omega_m))$ onto $\mathcal{F}_S(L^2(\mathbf{R}^3))$. Denote as $\tilde{a}^*$ and $\tilde{a}$ the creation and annihilation operators in $\mathcal{F}_S(L^2(\mathbf{R}^3))$ associated to a^* and a in $\mathcal{F}_S(L^2(H_m, d\Omega_m))$. They are related by

$$\tilde{a}\left(\frac{f(\mathbf{p})}{\sqrt{\mu(\mathbf{p})}}\right) = \Gamma(J)a(f)\Gamma(J)^{-1} \quad \text{and} \quad \tilde{a}^*\left(\frac{f(\mathbf{p})}{\sqrt{\mu(\mathbf{p})}}\right) = \Gamma(J)a^*(f)\Gamma(J)^{-1}.$$

The (unitary) map $\Gamma(J)$ relates the Segal fields we define in appendix A2 to the time-zero fields as (for real-valued $f \in \mathscr{S}(\mathbf{R}^4)$)

$$\tilde{\Phi}_m(f) = \Gamma(J)\Phi_m(f)\Gamma(J)^{-1} \quad = \quad \frac{1}{\sqrt{2}}\left\{\tilde{a}\left(\tilde{C}\frac{Ef}{\sqrt{\mu}}\right) + \tilde{a}^*\left(\frac{Ef}{\sqrt{\mu}}\right)\right\}.$$

Here $\tilde{C} = JCJ^{-1}$ operates as follows: $(\tilde{C}g)(\mathbf{p}) = \overline{g(-\mathbf{p})}$. In the sequel, we will drop the tilde since all fields will be time-zero fields under the conjugation C.

Consider $D_{\mathscr{S}} = \{\psi | \ \psi \in F_0,\ \psi^{(n)} \in \mathscr{S}(\mathbf{R}^{3n})$ for all $n\}$. For each $p \in \mathbf{R}^3$ define an operator $a(p)$ on $\mathscr{F}_S(L^2(\mathbf{R}^3))$ with domain $D_{\mathscr{S}}$ by

$$(a(\mathbf{p})\psi)^{(n)}(\mathbf{k}_1,\ldots\mathbf{k}_n) = \sqrt{n+1}\,\psi^{(n+1)}(\mathbf{p},\mathbf{k}_1,\ldots\mathbf{k}_n)\,. \qquad (7.4a)$$

Note the adjoint of the operator $a(\mathbf{p})$ is not densely defined since it is given by

$$(a^*(\mathbf{p})\psi)^{(n)}(\mathbf{k}_1,\ldots\mathbf{k}_n) = \frac{1}{\sqrt{n}}\sum_{l=1}^{n}\delta(\mathbf{p}-\mathbf{k}_l)\psi^{(n-1)}(\mathbf{k}_1,\ldots\mathbf{k}_{l-1},\mathbf{k}_{l+1},\ldots\mathbf{k}_n) \qquad (7.4b)$$

and the RHS is non-zero only on a finite group of measure-zero sets. But a^* is a well defined quadratic form on $D_{\mathscr{S}} \times D_{\mathscr{S}}$. So that as quadratic forms on $D_{\mathscr{S}}$ we may express the free scalar field and time zero fields as

$$\Phi_m(\mathbf{x},t) = \frac{1}{(2\pi)^{3/2}}\int_{R^3} e^{i(\mu(\mathbf{p})t-\mathbf{p}\cdot\mathbf{x})}a^*(\mathbf{p}) + e^{-i(\mu(\mathbf{p})t-\mathbf{p}\cdot\mathbf{x})}a(\mathbf{p})\frac{d^3\mathbf{p}}{\sqrt{2\mu(\mathbf{p})}}\,, \qquad (7.5)$$

$$\varphi_m(\mathbf{x}) = \frac{1}{(2\pi)^{3/2}}\int_{R^3} e^{-i\mathbf{p}\cdot\mathbf{x}}a^*(\mathbf{p}) + e^{i\mathbf{p}\cdot\mathbf{x}}a(\mathbf{p})\frac{d^3\mathbf{p}}{\sqrt{2\mu(\mathbf{p})}}\,, \qquad (7.6)$$

$$\pi_m(\mathbf{x}) = \frac{i}{(2\pi)^{3/2}}\int_{R^3} e^{-i\mathbf{p}\cdot\mathbf{x}}a^*(\mathbf{p}) - e^{i\mathbf{p}\cdot\mathbf{x}}a(\mathbf{p})\sqrt{\frac{\mu(\mathbf{p})}{2}}\,d^3\mathbf{p}\,. \qquad (7.7)$$

Recall $\mu(\mathbf{p})$ is the frequency on the mass hyperboloid, $\mu(\mathbf{p}) = +\sqrt{m^2 + \mathbf{p}^2}$. The remarks regarding our defined conjugation and time zero fields apply in an obvious way to the Fermion fields.

Let us attempt to clarify some issues regarding the rigorous, Segal quantization of field theory and the time zero fields we have introduced to reconcile with the formalism of most physics texts. We are referring to Fock spaces $\mathcal{F}_S(\mathcal{H})$, where $\mathcal{H} = \mathcal{S}(\mathbf{R}^4)$ in the Segal quantization, and $\mathcal{H} = L^2(\mathbf{R}^3)$ in equation (7.5). In the latter case, the creation and annihilation operators are defined in equations (7.4ab). Now $a(\mathbf{p})$ is defined as an operator on $\mathcal{F}_s(L^2(\mathbf{R}^3))$ with domain $D_{\mathcal{S}}$. $a^*(\mathbf{p})$ is a well-defined quadratic form on $D_{\mathcal{S}} \times D_{\mathcal{S}}$; e.g. if $\psi_1 = \{0,\psi^{(1)},...\}$ and $\psi_2 = \{0,0,\psi^{(2)},0,...\}$ then

$$(\psi_2, a^*(\mathbf{p})\psi_1) = \frac{1}{\sqrt{2}}\int (\overline{\psi^{(2)}(\mathbf{k}_1,\mathbf{p})}\psi^{(1)}(\mathbf{k}_1) + \overline{\psi^{(2)}(\mathbf{p},\mathbf{k}_1)}\psi^{(1)}(\mathbf{k}_1))d\mathbf{k}_1 . \qquad (7.8)$$

The reason we are going to all this trouble has to do with defining an operator product such as $\Phi(x)\Psi(x)$ that might occur in an interaction term, viz: $\int \Phi(x)\Psi(x)dx$, so that the result is a quantum field (operator valued distribution). If we use the expansions in equations (7.5) or (7.6) we can define an inner product as in equation (7.8). If this gives a reasonable (finite) result then our operator product is a quadratic form. The relevant question is: does this form arise from an operator? Our ultimate goal is proving that these operator products are self adjoint operators on a suitable Fock space. Now formally expressions like $\int \Phi(x)\Psi(x)dx$ will simply yield product terms like $a^*(\mathbf{p})c^*(\mathbf{k})$ or $a(\mathbf{p})c^*(\mathbf{k})$. Integrations over d$\mathbf{p}$ and d$\mathbf{k}$ will produce a delta function in $\mathbf{x}$. Even if we smear the term, $\int g(x)\Phi(x)\Psi(x)dx$ there is no guarantee the result will be convergent. An example will clarify. Suppose we have an interaction term like $\varphi(x)^4$. In Chapter 5 we saw that we should write this as $:\varphi(x)^4:$. This operation will assure that the product is, at least, a well defined quadratic form on $D_{\mathcal{S}} \times D_{\mathcal{S}}$. With φ defined as in equation (7.6), there will be sixteen terms in the creation/annihilation operators. Suppose $\psi_i = \{0,0,0,\psi_i^{(3)},0...\}$for i = 1,2, then the term in $(\psi_1, \varphi(x)^4\psi_2)$ with two a^*'s and two a's

is $$\int\left(\int\int\int\int\left(\frac{\exp(-i\mathbf{x}\cdot(\mathbf{k}_1+\mathbf{k}_2-\mathbf{k}_3-\mathbf{k}_4))}{\prod_{j=1}^4(2\mu(k_j))^{1/2}(2\pi)^{3/2}}\right)\times \left(\int\int\overline{\psi_1^{(3)}(\mathbf{p}_1,\mathbf{k}_1,\mathbf{k}_2)}\psi_2^{(3)}(\mathbf{p}_2,\mathbf{k}_3,\mathbf{k}_4)d\mathbf{p}_1 d\mathbf{p}_2\right)d\mathbf{k}_1\ldots d\mathbf{k}_4\right)d\mathbf{x} .$$

This expression is a well defined integral since $\psi_1^{(3)} \in \mathcal{S}(\mathbf{R}^3)$. But it does not arise from an operator. Consider $:\varphi(x)^4:\Omega_0$. This is a vector $(0,0,0,0,\psi^{(4)},0,...)$ with $\psi^{(4)}(\mathbf{k}_1,\mathbf{k}_2,\mathbf{k}_3,\mathbf{k}_4) = \int_{R^3}\frac{\exp(-i\mathbf{x}\cdot(\sum_{i=1}^4\mathbf{k}_i)}{\prod_{i=1}^4(2\pi)^{3/2}(2\mu(\mathbf{k}_i))^{1/2}}d\mathbf{x} = \frac{\delta(\sum_{i=1}^4\mathbf{k}_i)}{(2\pi)^{9/2}\prod_{i=1}^4(2\mu(\mathbf{k}_i))^{1/2}}$. Now, this expression is not in $L^2(\mathbf{R}^{(3)(4)})$; first because of the delta function, but even if we restrict the x integration by letting $g \in C_0^\infty(\mathbf{R}^3)$ and consider

$$\int_{R^3} \frac{g(\mathbf{x})\exp(-i\mathbf{x}\cdot(\sum_{i=1}^4 \mathbf{k}_i)}{\prod_{i=1}^4 (2\pi)^{3/2}(2\mu(\mathbf{k}_i))^{1/2}} dx = \frac{\hat{g}(\sum_{i=1}^4 \mathbf{k}_i)}{(2\pi)^{9/2}\prod_{i=1}^4 (2\mu(\mathbf{k}_i))^{1/2}}$$ we still do not get an L^2 function because $\mu(k_i)$ grows too slowly at infinity. There are two kinds of infinities, the infinite volume (x space) divergence and the ultraviolet (large k) divergence. We need a theorem for operator products that tells us when the quadratic forms we get arise from an operator in $D_{\mathscr{S}}$. We thus introduce from [Reed 75] the following

Theorem E: Let n_1 and n_2 be nonnegative integers and suppose that $W \in L^2(\mathbf{R}^{3(n_1+n_2)})$. Then there is a unique operator T_W on $\mathcal{F}_s(L^2(\mathbf{R}^3)$ so that $D_{\mathscr{S}} \subset D(T_W)$ is a core for T_W and

$$T_W = \int_{R^{3(n1+n2)}} W(\mathbf{k}_1,\ldots\mathbf{k}_{n_1},\mathbf{p}_1,\ldots\mathbf{p}_{n_2})\left(\prod_{i=1}^{n_1} a^*(\mathbf{k}_i)\right)\left(\prod_{i=1}^{n_2} a(\mathbf{p}_i)\right) d\mathbf{k}d\mathbf{p}$$ as quadratic forms on

$D_{\mathscr{S}} \times D_{\mathscr{S}}$. The theorem also holds for Fermi fields defined on $\mathcal{F}_a(L^2(\mathbf{R}^3)$ with $n_1 = n_2 = 1$.

In our case, Y4, the product terms we will be considering will be $\int \varphi(x)\overline{\psi}(x)\psi(x)dx$ as well as $\varphi(x) = \varphi^0(x) + g\int \Delta_F(x-y)\overline{\psi}(y)\psi(y)dy$.

Let us try to define equation (7.3) in a form suitable for a quantum field theory. To simplify computations let us define

$$\eta(x, y_0) = \exp[ig\gamma^0 \int \Delta(x-y)\varphi(y)d^3\mathbf{y}] \tag{7.9}$$

$$\partial_t \eta(x, y_0 = x_0) = ig\varphi(x)\gamma^0\eta(x) = \partial_t\eta(x).$$

The $\Delta(x\text{-}y)$ appearing in equation (7.9) is probably a classical artifact. The $\partial_t\eta(x)$ term appears to introduce a "mass" term (a term linear in φ) in the field equation. With this in mind let us introduce the

Proposition: In equation (7.3) let us replace $\Delta(x\text{-}y)$ with the quantity $i\langle 0|[\varphi(x),\varphi(y)]|0\rangle$, where the fields $\varphi(x)$ are the exact, interacting fields.

Let us note that for "free" Boson fields, we have the following commutation relation:

$[\varphi_f(x),\varphi_f(y)] = -i\Delta(x-y)$, i.e. the commutator is a c-number (complex valued function). As such, since $\langle 0|0\rangle = 1$, we may take the vacuum expectation value of the commutator without changing its significance: $\langle 0|[\varphi_f(x),\varphi_f(y)]|0\rangle = -i\Delta(x-y)$. Now an easy generalization of this equation is to remove the "free field" requirement and get $\langle 0|[\varphi(x),\varphi(y)]|0\rangle = -i\Delta'(x-y)$, with Δ' a c-number factor on the RHS. The reason for introducing this is the following

Theorem F: The vacuum expectation value of the quantity $\langle 0|[\varphi(x),\varphi(y)]|0\rangle$, the commutator of the exact interacting field $\varphi(x)$, is (from [Bjorken Drell 65]

$$\int_0^\infty d\sigma^2 \rho(\sigma^2)\Delta(x-y,\sigma) = \Delta'(x,y) = \langle 0|[\varphi(x),\varphi(y)]|0\rangle \tag{7.10}$$

where the Δ inside the integral is the same Schwinger function with mass σ. The ρ term is the "spectral weight" defined as $\rho(q) = (2\pi)^3 \sum_n \delta^4(p_n - q)|<0|\varphi(0)|n>|^2$ in terms of the momenta p_n.

Proof: Use displacement invariance of the theory to write
$\langle n|\varphi(x)|m\rangle = \langle n|e^{iP\cdot x}\varphi(0)e^{-iP\cdot x}|m\rangle = e^{i(p_n-p_m)\cdot x}\langle n|\varphi(0)|m\rangle$. Then

$$\Delta'(x,y) = -i\sum_n \langle 0|\varphi(0)|n\rangle\langle n|\varphi(0)|0\rangle(e^{-ip_n(x-x')} - e^{ip_n(x-x')}) = \Delta'(x-x')\,. \tag{7.11}$$

Let us group together all states corresponding to the same eigenvalue p_n. Using $1 = \int d^4q\delta^4(p_n - q)$, equation (7.11) becomes

$$\Delta'(x-y) = \frac{-i}{(2\pi)^3}\int d^4q\left[(2\pi)^3\sum_n \delta^4(p_n-q)|\langle 0|\varphi(0)|n\rangle|^2\right](e^{-ip_n(x-x')} - e^{ip_n(x-x')}) \tag{7.12}$$

where the quantity in brackets is the spectral weight introduced above. An immediate consequence of Lorentz invariance is that $\rho(q)$ is a scalar function of q^2 only.

Let a be the matrix of coefficients for a proper Lorentz transformation. Then $U(a)|0\rangle = |0\rangle$ and for the scalar field $\varphi(0)$ we have $U(a)\varphi(0)U^{-1}(a) = \varphi(0)$ and $\rho(q) = (2\pi)^3\sum_n \delta^4(p_n - q)|<0|\varphi(0)|U(a)n>|^2$. Lorentz invariance of the "δ function" allows us to write $\delta^4(p_n - q) = \delta^4[(p_n - q)a^{-1}]$. Select a complete set of states labeled $|m\rangle = |U(a)n\rangle$, with eigenvalues $p_m^\mu = \langle m|P^\mu|m\rangle = \langle n|U^{-1}(a)P^\mu U(a)|n\rangle = (p_n a^{-1})^\mu$ and carry out the sum for $\rho(q)$. The result is $\rho(q) = (2\pi)^3\sum_m \delta^4(p_m - qa^{-1})|\langle 0|\varphi(0)|m\rangle|^2 = \rho(qa^{-1})$.

$\rho(q)$ vanishes outside the forward light cone (from Wightman's assumptions) and so one can write $\rho(q) = \rho(q^2)\theta(q_0)$ where $\rho(q^2)$ vanishes for $q^2 < 0$, and is real and positive semi-definite for $q^2 \geq 0$. Finally, equation (7.12) can be written as a weighted integral over the mass parameter of the *free-field* commutator function

$$\Delta'(x-x') = \frac{-i}{(2\pi)^3}\int d^4q\left[(2\pi)^3\sum_n \delta^4(p_n - q)\left|\langle 0|\varphi(0)|n\rangle\right|^2\right](e^{-ip_n(x-x')} - e^{ip_n(x-x')})$$
$$= \frac{-i}{(2\pi)^3}\int_0^\infty ds^2\rho(s^2)\int d^4q\delta(q^2 - s^2)\varepsilon(q_{0)}e^{-iq\cdot(x-x')}$$
$$= \int_0^\infty ds^2\rho(s^2)\Delta(x-x',s) \quad \blacksquare$$

NB The justification for interchanging orders of integration in the second step, and related issues, will be addressed in Chapter 9; cf. equation (9.15).

The reason for this detail is to note that these same arguments apply to *all* the various Green functions of QFT. From all this equation (7.3) becomes

$$\psi = \exp[ig\gamma^0\int \Delta'(x-y)\varphi(y)d^3\mathbf{y}]\Big|_{y_0=x_0}\psi^0(x)\,. \tag{7.13}$$

Now let us insert this primed term into the definition in equation (7.9) to get

$$\eta(x, y_0) = \exp[ig\gamma^0\int \Delta'(x-y)\varphi(y)d^3\mathbf{y}] \tag{7.14}$$

and similarly for

$$\partial_t\eta(x, y_0 = x_0) = ig\varphi(x)\gamma^0 = \partial_t\eta(x)\,. \tag{7.15}$$

The "solution" for equation (7.2) is usually written as the integral equation

$$\varphi(x) = \varphi^0(x) + g\int \Delta_F(x-y)\overline{\psi}(y)\psi(y)dy \tag{7.16}$$

and here the Δ function is an appropriate Feynman propagator, and the first term on RHS satisfies the free field equation. In the sequel we will see that under proper conditions, $\overline{\psi}(x)\psi(x) \to \overline{\psi}^0(x)\psi^0(x)$ and then equation (7.16) becomes exact.

Let us interject a few remarks. To make sense of equations (7.13) through (7.15) requires some knowledge (whether assumed or derived) of the spectral function $\rho(q)$. In nearly all calculations, we will want to assume that the integral defining Δ' converges, so that the steps in the proof of Theorem E regarding the exchange of integration variables $d^3\boldsymbol{y}$ and ds^2 can be justified. This almost certainly is not generally true. But, supposing the necessary conditions are met, and we can get $\varphi(x)$ as a Wightman field; then the work to do on ψ becomes quite simple. We already have ψ^0, and with $\varphi(x)$ defined one only needs to make sense of the exponential. One hopes that a (presumably infinite) renormalization constant emerges as a product, leaving the resulting exponential factor a well defined operator on some specifiable domain. The manifold difficulties of the Fermion field are thereby overcome.

To a further extent, we will show that the masses of the propagating particles (here labeled ψ and φ) are NOT generated by radiative corrections within the model we are considering. [They will, of course, be modified by such corrections].

Chapter 8

SCATTERING THEORY FOR Y4

The problem we will address in this section is scattering, or the time development of a localized system. Thus we will be examining structures in the asymptotic limit $t \to \pm\infty$. We have said that "fields" are operator valued distributions defined by smearing space-time structures with four-dimensional localized functions, and using a measure defined such that they satisfy the Klein Gordon equation. Immediately, then, we must say what we mean by "scattering" of a field. The common approach, which will be followed here, is to use time-zero fields or the "Schrodinger Representation"; these will be operator valued distributions in $\mathbf{R}^3$, for fixed t, and then we will vary the time. It is worthwhile to note that the fact that this can be done is a special feature of free quantum fields, and it is not true in general Wightman theories.

Scattering theory is described by the LSZ method or Haag-Ruelle method, the former being simpler in that it assumes somewhat less stringent axioms than Wightman and also assumes a convergence criterion between *Interpolating* (interacting) fields and *Asymptotic* (in, out) fields, (the regions of definition are shown in the cartoon below). Haag-Ruelle (HR) theory assumes the fields satisfy the Wightman axioms (plus two others) and derives the convergence criterion therefrom. HR is therefore more satisfactory from the point of view of *Proving* or *Demonstrating* the time development of a particular theory. It is the one we follow here.

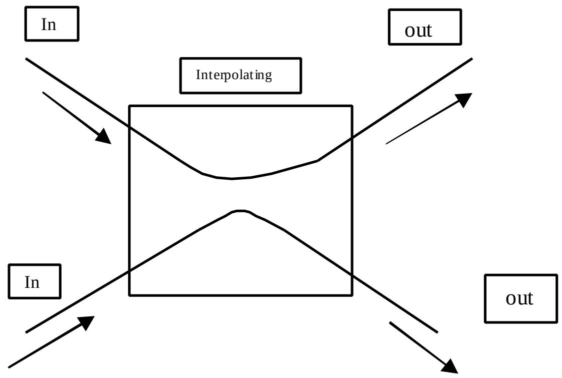

Figure 1 Geometry for scattering

We will now attempt to derive the scattering properties of Y4. Let us begin with the notion of the scattering, or “S” matrix.

Assuming the “in” and “out” algebras are equivalent, (asymptotic completeness) there exists a (proper) unitary operator S in the Hilbert space so that (for the fields concerned)

$$\varphi^{out}(x) = S^{-1}\varphi^{in}(x)S \quad \text{with } S^{-1} = S^{*}. \tag{8.1}$$

Using our normalized wave packet expansions [equation (3.7) and its Fermion counterpart] we can write

$$a^{out}(p) = S^{-1}a^{in}(p)S \quad \text{and} \quad a^{out}(p)^{*} = S^{-1}a^{\mathrm{in}}(p)^{*}S\ . \tag{8.2}$$

Under the Wightman postulate regarding the uniqueness of the vacuum (and its normalizability) we have the following [Roman 69]. It is important to show that the S operator, although well defined, actually is not trivial: it produces non-zero vectors on the Fock Hilbert space.

Theorem G: Relations (8.1) and (8.2) determine *S*. (*S* is the unitary operator demonstrating equivalence between the in and out states) and *S* contains at least one non-vanishing (and finite) matrix element (i.e. it is not an improper operator).

Proof: Apply equation (8.2) to the vacuum (here labeled Ω_0) and get

$$0 = S^{-1}a^{in}(p)S|\Omega_0\rangle$$

However, by definition $a^{in}(p)|\Omega_0\rangle = 0$, so uniqueness of Ω_0 leads to

$$S|\Omega_0\rangle = |\Omega_0\rangle \text{ (up to a phase factor). Hence} \tag{8.3}$$

$$\langle\Omega_0|S|\Omega_0\rangle = \langle\Omega_0|\Omega_0\rangle = 1 \tag{8.4}$$

Thus *S* has at least one non-vanishing (and finite) matrix element. ▌

For a state α with specification (α), relations (8.2) lead to the

Theorem H:

$$|(\alpha)\rangle_{in} = S|(\alpha)\rangle_{out} \tag{8.5}$$

Proof: Because of relation (8.2)

$$|(a)\rangle_{in} \equiv a(p_{\alpha_1})^{in} * a(p_{\alpha_2})^{in} * \ldots a(p_{\alpha_s})^{in} * |\Omega_0\rangle = Sa(p_{\alpha_1})^{out} * S^{-1}Sa(p_{\alpha_2})^{out} * \ldots a(p_{\alpha_s})^{out} * S^{-1}|\Omega_0\rangle$$
$$= Sa(p_{\alpha_1})^{out} * a(p_{\alpha_2})^{out} * \ldots a(p_{\alpha_s})^{out} * |\Omega_0\rangle = S|(\alpha)\rangle_{out}. \quad ▌$$

From equation (8.5) we derive the two relations

$$_{out}\langle(\alpha)|(\beta)\rangle_{in} = {}_{out}\langle(\alpha)|S||(\beta)\rangle_{out} \quad \text{and} \quad {}_{out}\langle(\alpha)|(\beta)\rangle_{in} = {}_{in}\langle(\alpha)|S||(\beta)\rangle_{in}. \tag{8.6}$$

Thus the matrix elements of the *S*-matrix, for the transition $(\beta) \to (\alpha)$ between a specified initial state and final state, are computed as the overlap of the in-state [with the initial specification (β)] and the out-state [with the final specification (α)].

Haag-Ruelle scattering is an “S Matrix” theory; it does not rely on the exact form of the field equations solved by the interpolating fields. We are given this in our solutions to Y4, and consequently we hope to have something to gain from this. To derive a consistent scattering theory, we will assume that our fields satisfy Wightman's axioms. But first let us consider the mathematical properties of equations (7.13) and (7.16), our nominal solutions.

Theorem 1: Consider the object $\eta = \exp[ig\gamma^0 \int \Delta'(x-y)\varphi(y)d^3\mathbf{y} \equiv \eta(x, y_0)$. Because of the Δ', η is unity at any point outside the forward light-cone centered at *x*, including *x* itself. In this region, ψ is equal to ψ^0.

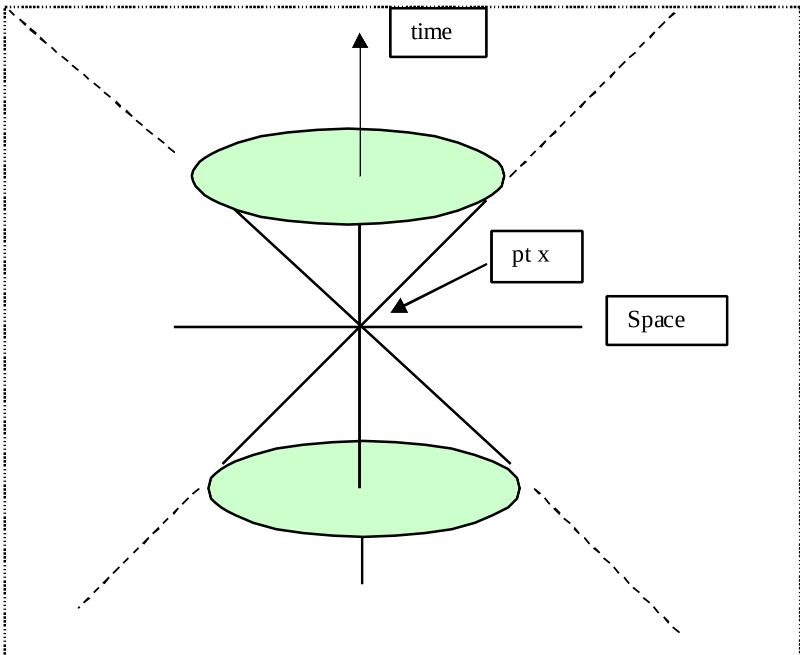

Figure 2 Region of definition of Δ.

Define $\chi[f,t] = g\gamma^0 \int f(\mathrm{x})\delta(x_0 - t)\int \Delta'(x-y)\varphi(y)\delta(y_0 - t')dydx$ and so (8.7)

$$\exp[ig\gamma^0 \int f(\mathrm{x})\delta(x_0 - t)\int \Delta'(x-y)\varphi(y)\delta(y_0 - t')dydx \equiv \eta[f,t] = e^{i\chi}. \quad (8.8)$$

For these definitions we take $f \in \mathcal{H}_C \cap \mathscr{S}(\mathbf{R}^3)$ and *f(0) = 1*. (See the discussion succeeding theorem D for definition of $\mathcal{H}_C$ and other matters.) Before proving some interesting results regarding these definitions, let us quote a few theorems from [Reed 80]

"**<u>Theorem I</u>:** (Basic criteria for self-adjointness) Let T be a symmetric operator on a Hilbert space $\mathcal{H}$. Then the following three statements are equivalent

(a) T is self-adjoint

(b) T is closed and Ker($T^* \pm i$) = {0}

(c) Ran($T \pm i$) = $\mathcal{H}$

For a proof see [Reed 80].

<u>Corollary to self-adjointness criteria</u>: Let T be a symmetric operator on a Hilbert Space. Then the following are equivalent:

(a) T is essentially self adjoint

(b) Ker($T^* \pm i$) = {0}

(c) Ran($T \pm i$) = is dense

<u>Spectral Theorem</u> – (Functional calculus form):

Let A be a self-adjoint operator on $\mathcal{H}$. Then there is a unique map $\hat{\phi}$ from the bounded Borel functions on $\mathbf{R}$ into $\mathcal{L}(\mathcal{H})$ so that

(a) $\hat{\phi}$ is an algebraic *-homomorphism

(b) $\hat{\phi}$ is norm continuous, that is, $\left\|\hat{\phi}(h)\right\|_{\mathcal{L}(\mathcal{H})} \le \|h\|_\infty$

(c) Let $h_n(x)$ be a sequence of bounded Borel functions with $h_n(x) \to x$ as $n \to \infty$ for all x and n. Then, for any $\psi \in D(A)$, $\lim_{n\to\infty} \hat{\phi}(h_n)\psi = A\psi$.

(d) If $h_n(x) \to h(x)$ pointwise and if the sequence $\|h_n\|_\infty$ is bounded, then $\hat{\phi}(h_n) \to \hat{\phi}(h)$ strongly. In addition:

(e) If $A\psi = \lambda\psi$, $\hat{\phi}(h)\psi = h(\lambda)\psi$.

(f) If $h \ge 0$, then $\hat{\phi}(h) \ge 0$.

Let us discuss projection valued measures (p.v.m.) for unbounded operators. Let P_Ω be the operator on $\chi_\Omega(A)$ where χ_Ω is the characteristic function of the measureable set

$\Omega \subset \mathbf{R}$. The family of operators $\{P_\Omega\}$ has the properties

(a) Each P_Ω is an orthogonal projection

(b) $P_\varnothing = 0, P_{(-\infty,\infty)} = I$

(c) If $\Omega = \bigcup_{n=1} \Omega_n$ with $\Omega_n \bigcap \Omega_m = \varnothing$ if $n \ne m$, then $P_\Omega = \mathrm{s}-\lim_{N\to\infty} \sum_{n=1}^{N} P_{\Omega_n}$

(d) $P_{\Omega_1} P_{\Omega_2} = P_{\Omega_1 \bigcap \Omega_2}$

This is a generalization of bounded p.v.m. with (b) replacing $P_{(-a,a)} = I$ for some finite a. Now, given a bounded Borel function g one can define $g(A)$ by (using previous spectral theorem)

$(\alpha, g(A)\alpha) = \int_{-\infty}^{\infty} g(\lambda) d(\alpha, P_\lambda \alpha)$ for any $\alpha \in \mathcal{H}$, and $(\alpha, P_\Omega \alpha)$ a well defined Borel measure on $\mathbf{R}$, which is denoted $d(\alpha, P_\lambda \alpha)$. Now suppose g is an *unbounded* complex valued Borel function. Let $D_g = \{\alpha | \int_{-\infty}^{\infty} |g(\lambda)|^2 d(\alpha, P_\lambda \alpha) < \infty\}$. Then D_g is dense in $\mathcal{H}$ and an operator $g(A)$ is defined on D_g by $(\alpha, g(A)\alpha) = \int_{-\infty}^{\infty} g(\lambda) d(\alpha, P_\lambda \alpha)$ which is often written symbolically as $g(A) = \int g(\lambda) dP_\lambda$. This terminology is the continuum version of the familiar finite matrix operator. With this we have the

Theorem J: There is a one to one correspondence between self-adjoint operators A and projection valued measures $\{P_\Omega\}$ on $\mathcal{H}$, the correspondence being given by $A = \int \lambda dP_\lambda$

If $g(.)$ is a real-valued Borel function on $\mathbb{R}$, then $g(A) = \int g(\lambda) dP_\lambda$ defined on D_g is self-adjoint. If g is bounded, $g(A)$ coincides with $\hat{\phi}$ in the spectral theorem. "

Now we would like to claim

Theorem 2: χ, as defined above, is self-adjoint if φ is self adjoint.

Proof: Let us consider Ker($\chi^* \pm i$).

Let $S=\{\alpha$ in $\mathcal{H}$ | $\alpha \in$Dom(φ)}; thus the domain for χ is the domain for φ. If we note that for any $\alpha, \beta \in S$, the inner product is calculated as (using the symmetry of χ)

$$(\alpha, \chi[f,t]\beta) = g\gamma^0 \int f(\mathrm{x})\delta(x_0 - t) \int \Delta'(x-y)(\alpha, \varphi(y)\beta)\delta(y_0 - t')d^3\mathrm{y}dy_0 dx$$

then in S, and for each $f \in \mathcal{H}_C$ the proof is straightforward. Pick $\alpha \in S$. Consider

$$-i(\alpha,\alpha) = (i\alpha,\alpha) = (\chi\alpha,\alpha) = (\alpha,\chi^*\alpha) = (\alpha,\chi\alpha) = i(\alpha,\alpha)$$

so α must be 0; i.e. there are no solutions. By the criteria above (e.g. Theorem I, part (b)), the theorem is proved. ▮

Note that if we replace "self adjoint" for φ with "essentially self adjoint", the same basic argument holds using the corollary above. This would be applicable for the case (which is usual in quantum field theory) when φ is self adjoint but unbounded and we can only specify a core on which φ is defined.

Query: Under what condition is $\eta = e^{i\chi}$ a unitary operator in S?

Suppose that our φ is self adjoint and *unbounded*.

Theorem 3: Let $V(t) = e^{i\chi}$. For χ as defined in equation (8.7) t' is arbitrary. Define a set $S^0_\Lambda = \{x \in \mathbf{M} \mid \|\chi[f,t]\| < \infty\}$.

(a) For each $t \in \mathbf{R}$, $V(t)$ is a unitary operator.
(b) If $\alpha \in \mathcal{H}$ and $t \to t'$, then $V(t)\alpha \to V(t')\alpha = \alpha$
(c) For $\alpha \in D(\chi)$, $\dfrac{V(t)\alpha - \alpha}{t-t'} \to i\varphi\alpha$ as $t - t' \to 0$ NB $D(\chi) = D(\varphi)$
(d) If $\lim\limits_{t-t'\to 0} \dfrac{V(t)\alpha - \alpha}{t-t'}$ exists, then $\alpha \in D(\chi)$.

Proof:

(a) Using the spectral theorem for unbounded operators with $\varphi|\alpha\rangle = \lambda(t')|\alpha\rangle$ we get $\chi[f,t] = g\gamma^0 \int f(\mathrm{x})\delta(x_0 - t)\int \Delta'(x-y)\varphi(y)\delta(y_0 - t')dydx$ with $e^{i\chi}|\alpha\rangle = \exp[ig\iint f(\mathrm{x})\delta(x_0 - t)\Delta'(x,y)\lambda(t')\delta(y_0 - t')dydx]|\alpha\rangle$.

Define the *finite* quantity $R_p \equiv \int_0^p ds^2 \rho(s^2)$, $p < \infty$, (i.e. a cut-off spectral integral)

together with the formula for the Δ function $\Delta(x-y) = \dfrac{t-t'}{(2\pi)^3}\displaystyle\int \dfrac{\sin\omega_k(t-t')}{\omega_k(t-t')} e^{i\mathrm{k}\cdot(\mathrm{x}-\mathrm{y})} d^3\mathrm{k}$ and

we will get

$$\chi_\lambda[f,t] = g\gamma^0 \frac{t-t'}{(2\pi)^3}\int f(\mathbf{x})\int R_p \lambda(t')\frac{\sin\omega_k(t-t')}{\omega_k(t-t')} e^{i\mathbf{k}\cdot(\mathbf{x}-\mathbf{y})} d^3\mathbf{y}d^3\mathbf{x}d^3\mathbf{k}\,. \tag{8.9}$$

Completion of the calculation shows $e^{i\chi_\lambda}e^{-i\chi_\lambda} = 1$. Note we have denoted the eigenvalue for the φ operator as $\lambda(t')$, to remind us of the arbitrariness of the parameter t'.

(b) Consider $\|e^{i\chi[f,t]}\alpha - e^{i\chi[f,t']}\alpha\| = \|e^{i\chi[f,t]}\alpha - \alpha\|$. We have $\|e^{i\chi[f,t]}\alpha - e^{i\chi[f,t']}\alpha\|^2 = \displaystyle\int_{\mathbf{R}} |e^{i\chi_\lambda[f,t]} - 1|^2 d(\alpha, P_\lambda\alpha)$ by the spectral theorem.

Since $|e^{i\chi_\lambda[f,t]} - 1|^2$ is dominated by $g(\lambda)$ where $g(\lambda) = 2$ and g is integrable (on D_g) and since $|e^{i\chi_\lambda[f,t]} - 1|^2 \to 0$ as $t \to t'$ for each $|\lambda(t')| \in \mathbf{R}$ the result follows by the Lebesgue dominated convergence theorem.

(c) Let us write equation ((8.7) as

$$\chi[f,t] = g\gamma^0 \frac{1}{(2\pi)^3}\int f(\mathbf{x})\int R_p \varphi(\mathbf{y},t')\frac{\sin\omega_k(t-t')}{\omega_k} e^{i\mathbf{k}\cdot(\mathbf{x}-\mathbf{y})} d^3\mathbf{y}d^3\mathbf{x}d^3\mathbf{k} \tag{8.10}$$

Now let us write (define)

$$\chi_\lambda[f,t] = g\gamma^0 \frac{1}{(2\pi)^3}\int f(\mathbf{x})\int R_p\lambda(t')\frac{\sin\omega_k(t-t')}{\omega_k}e^{i\mathbf{k}\cdot(\mathbf{x}-\mathbf{y})}d^3\mathbf{y}d^3\mathbf{x}d^3\mathbf{k}\,. \tag{8.11}$$

Let $\alpha \in D(\chi) = D(\varphi)$. Consider $\frac{V(t)\alpha-\alpha}{t-t'}$. Now we have

$\lim_{t\to t'}\frac{e^{i\chi}-1}{t-t'} = \lim_{t-t'\to 0}\frac{e^{i\chi}-1}{i\chi(t)}\frac{i\chi(t)}{t-t'} = \lim_{t-t'\to 0}\frac{i\chi(t)}{t-t'}$. Applying this (as an operator) to α,

and using (8.11) we get, when we take the limit, that $\frac{V(t)\alpha-\alpha}{t-t'}$ goes to $i\chi\alpha$.

(d) Define $D(C) = \{\alpha|\lim_{t-t'\to 0}\frac{V(t)\alpha-\alpha}{t-t'}$ exists$\}$, $D(C) \subset D(\chi)$, and let $iC\alpha = \lim_{t-t'\to 0}\frac{V(t)\alpha-\alpha}{t-t'}$ for $\alpha \in D(C)$. A computation shows that C is symmetric.
By (c) , $D(C) \supset D(\chi)$, so $C = \chi$. ▮

This theorem proves how the total field ψ solves the coupled field equation: by defining the time derivative and showing that its existence is proven. Recall $\psi = e^{i\chi_\lambda}\psi^0$, then $\partial_t\psi = i\varphi\psi$

Let us turn to φ.

<u>Theorem 4:</u> On $S_\Lambda^0 = \{x \in \mathbf{M} \| \chi[f,t]| < \infty\}$ above, and $f \in \mathcal{H}_C \cap \mathscr{S}(\mathbf{R}^3)$, φ is essentially self adjoint.

Proof:

From the proof of 3(a) we have $e^{i\chi_\lambda}e^{-i\chi_\lambda} = 1$ and thus $\overline{\psi}(x)\psi(x) = \overline{\psi}^0(x)\psi^0(x)$. We are therefore considering $\varphi(x) = \varphi^0(x) + g\int\Delta_F(x-y){:}\overline{\psi}^0(y)\psi^0(y){:}\,dy$. This is entirely in terms of free fields. By definition, ${:}\psi(x)\psi(y){:} = \psi(x)\psi(y) - \langle 0|\psi(x)\psi(y)|0\rangle$, and we will write

$${:}\overline{\psi}^0(x)\psi^0(x){:} = \lim_{\varepsilon\to 0}\frac{1}{2}[\overline{\psi}^0(x+\varepsilon)\psi^0(x) + \overline{\psi}(x)\psi(x+\varepsilon) - \langle 0|\overline{\psi}^0(x+\varepsilon)\psi^0(x)|0\rangle - \langle 0|\overline{\psi}(x)\psi(x+\varepsilon)|0\rangle]$$

and the set S_Λ^0 does not contain the (infinite) limit for ψ. Let us state as a lemma a theorem from [Reed 75]

<u>Lemma:</u> The two-point function $\Delta^+(x,m^2)$ of the free field has the following properties:
(a) Δ^+ is Lorentz invariant
(b) There exist C^∞ functions f_s, $f_t^{t,+}$, and $f_t^{t,-}$ so that

$$\Delta^+(x;m^2) = \begin{cases} f_s(x^2;m) & x^2 < 0 \\ f_t^+(x^2;m) & x^2 > 0, x_0 > 0 \\ f_t^-(x^2;m) & x^2 > 0, x_0 < 0 \end{cases}$$

where $x^2 = x \cdot \tilde{x} = x_0^2 - x_1^2 - x_2^2 - x_3^2$.

(c) For $y > 1$, $f_s(y^2) \le C_\varepsilon e^{-(m-\varepsilon)y}$

(d) $\lim_{y\to\infty} |y|^{2n} f_t^{\pm}(y) = 0$

(e) $f_t^+(y) = \overline{f_t^-(y)}$

Now note that $\Delta_F(x) = \begin{cases} \Delta^+(-x) & x_0 > 0 \\ \Delta^+(x) & x_0 < 0 \end{cases}$

and also we have the

Corollary to Lemma: The product $\theta(x_0)\Delta^+(x, m^2)$ exists where $\theta(x_0)$ is defined by

$$\theta(f) = \int_{x_0 \ge 0} f(x) d^3x$$

Now note that $:\psi(x)\psi(x):$ as defined in the above limit exists and is no more singular than the Heaviside distribution θ in the following regions

Region I $\quad -\infty < t < x_0$

Region II $\quad x_0 < t < x_0 + \varepsilon_0$

Region III $\quad x_0 + \varepsilon_0 < t < \infty$

From the Corollary and the definition of $:\psi(x)\psi(x):$ we see that the product $\Delta_F(x-y):\overline{\psi}^0(x)\psi^0(x):$ exists as a distribution in the set $S_\Lambda^0 = \{x \in \mathbf{M} \| \chi[f,t] \| < \infty\}$ with $f \in \mathcal{H}_C \cap \mathscr{S}(\mathbf{R}^3)$. We may therefore use $\Delta_F(x-y):\overline{\psi}^0(x)\psi^0(x):$ as an integrand and take the integral. φ is therefore a well-defined quadratic form. φ is obviously symmetric. Now allow $g \in L^2(\mathbf{R}^3)$, define $\varphi(g) = \int g(x)\varphi(x)dx$, and we may apply Theorem E and get φ as an operator. By the same reasoning that proved Theorem 2 we see that $\mathrm{Ker}(\theta \pm i) = 0$ and therefore the result follows. ∎

We thus have shown that φ and ψ are essentially self adjoint, with suitable restrictions. With these, we shall begin to define the Haag-Ruelle theory for our Y4 solution. In addition to the Wightman axioms, we assume the following:

Assumption 1: (Mass operator has discrete eigenvalues) Let P_μ be the generators of the translation subgroup $U(a,I)$ of the Poincare representation $U(a,\Lambda)$. For some $m,\kappa > 0$ and some $\varepsilon > 0$, the spectrum of P_μ is contained in $\{0\} \cup H_m \cup \overline{V}_{m+\varepsilon,+} \cup H_\kappa \cup \overline{V}_{\kappa+\varepsilon,+}$

$\equiv \{0\} \cup \{p | p^2 = m^2; p_0 > 0\} \cup \{p | \kappa^2 > p^2 \geq (m+\varepsilon)^2; p_0 > 0\}$

$\cup \{p | p^2 = \kappa^2; p_0 > 0\} \cup \{p | p^2 \geq (\kappa + \varepsilon)^2; p_0 > 0\}$

where $p^2 = p_0^2 - p_1^2 - p_2^2 - p_3^2$ and $\kappa^2 > (2m)^2$. Moreover, the set of vectors $\mathbb{S}$ which are eigenvectors for p^2 with eigenvalue m^2 is nonempty, and there is a cyclic vector for the action of $U(a,I)$ on $\mathbb{S}$.

<u>Assumption 2</u>: (coupling of the vacuum to the one particle states) The spectral weight $d\rho$ for the Kallen-Lehmann (equation (7.5)) representation has the form $d\rho(s) = \delta(s-m) + d\tilde{\rho}(s)$ where $d\tilde{\rho}(s)$ has support in $[m+\varepsilon, \infty)$, i.e. there are isolated eigenvalues of momentum. The spectrum of the ψ operator will be written as $\sigma(s)$ and will have the form $d\sigma(s) = \delta(s-\kappa) + d\tilde{\sigma}(s)$ where $d\tilde{\sigma}(s)$ has support in $\{p | p^2 \geq (\kappa+\varepsilon)^2; p_0 > 0\}$. Arguments similar to those leading to theorem **E** provide the spectral form for a spinor field, viz: $\rho_{\alpha\beta}(q) = \rho_1(q^2) q_{\alpha\beta} + \rho_2(q^2)\delta_{\alpha\beta}$ for the 4 x 4 matrix representing ρ (see [Bjorken Drell 65]; page 155).

The spectra has the appearance resembling the graph below

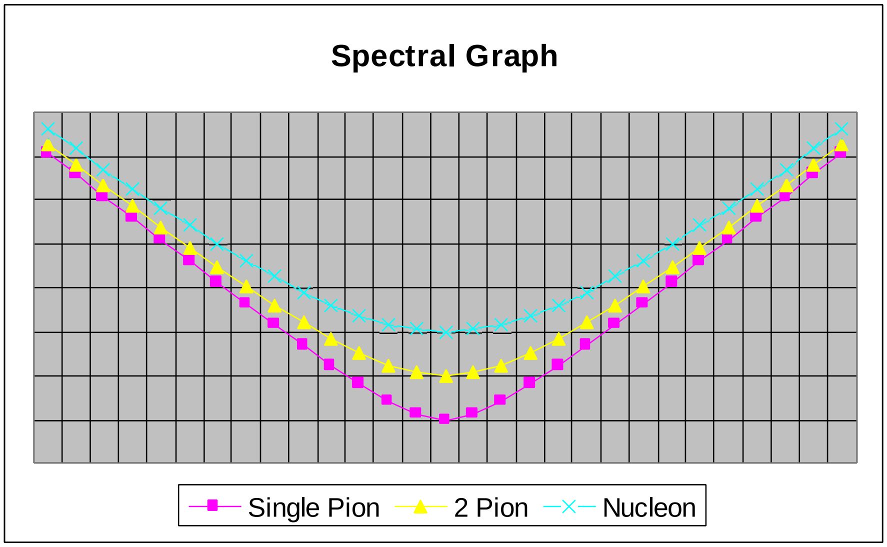

Figure 3. Spectra of DKG System

Before defining rigorous scattering, let us sketch here the complete quantum field theory for Y4. This will take the form of a "problem description" for proving that the theory we introduce satisfies the Wightman axioms. We start with the Boson field.

$$\varphi(x) = \varphi^0(x) + g\int \Delta_F(x-y) : \overline{\psi}(y)\psi(y) : dy$$

where we understand $:\overline{\psi}(x)\psi(x) := \lim_{\varepsilon\to 0} \overline{\psi}(x+\varepsilon)\psi(x) - \langle\overline{\psi}(x+\varepsilon)\psi(x)\rangle$ and we note that writing $\psi(y,t=y_0) = \psi^0(y)$. φ is now a well defined quantity (in terms of the free fields and the Feynman Green function). We are in the Schrodinger representation, and in absence of derivative coupling the canonical coordinate to φ is still $\dot{\varphi}^0$, just as in the case of a free Boson field. We can now define creation and annihilation operators as follows:

We still have $\pi(\mathbf{x}) = \dot{\varphi}^0(\mathbf{x})$ (in the interacting case) and so set

$$[\varphi(\mathbf{x}), \pi(\mathbf{x}')] = i\delta(\mathbf{x}-\mathbf{x}') \quad \text{along with} \tag{8.12}$$

$$[\pi(\mathbf{x}),\pi(\mathbf{x}')] = [\varphi(\mathbf{x}), \varphi(\mathbf{x}')] = 0.$$

Consider the the F.T. $\int [\varphi^0(x) + g\int \Delta_F(x-y) : \overline{\psi}^0(y)\psi^0(y) : dy] e^{i\nu_q t} e^{-i\mathbf{q}\cdot\mathbf{x}} d^3\mathbf{x}$. Re-write this as

$\int [\varphi^0(x) + g\int\int \{\Delta_F(k) e^{-ik(x-y)} dk\} : \overline{\psi}^0(y)\psi^0(y) : dy] e^{i\nu_q t} e^{-i\mathbf{q}\cdot\mathbf{x}} d^3\mathbf{x}$ and separating the $k(x\text{-}y)$ into components one sees that $v_q = k_0 = \omega_0 = +\sqrt{\mathbf{k}\cdot\mathbf{k} + m_0{}^2}$. Thus we may take Fourier transforms and define the k-space commutation rules for the creation and annihilation operators, viz:

$$[c_{\mathbf{q}}, c^*_{\mathbf{q}'}] := \frac{1}{V}\iint d^3x d^3y \frac{e^{i(\mathbf{q}\cdot\mathbf{x}-\mathbf{q}'\cdot\mathbf{y})}}{\sqrt{4\nu_{\mathbf{q}}\nu_{\mathbf{q}'}}} [\pi(\mathbf{x}) - i\nu_{\mathbf{q}}\varphi(\mathbf{x}), \pi(\mathbf{y}) + i\nu_{\mathbf{q}'}\varphi(\mathbf{y})]$$

$$\equiv \frac{1}{V}\iint d^3x d^3y \frac{e^{i(\mathbf{q}\cdot\mathbf{x}-\mathbf{q}'\cdot\mathbf{y})}}{\sqrt{4\nu_{\mathbf{q}}\nu_{\mathbf{q}'}}} [c(\mathbf{x}), c^*(\mathbf{y})]$$

with π as defined above and φ from equation (7.16). Using equation (8.12) we will get

$$[c_{\mathbf{q}}, c_{\mathbf{q'}}^*] = \frac{1}{V} \iint d^3x d^3y \frac{e^{i(\mathbf{q}\cdot\mathbf{x} - \mathbf{q'}\cdot\mathbf{y})}}{\sqrt{4\nu_{\mathbf{q}}\nu_{\mathbf{q'}}}} (\nu_{\mathbf{q}} + \nu_{\mathbf{q'}}) = \delta_{\mathbf{qq'}} \frac{(\nu_{\mathbf{q}} + \nu_{\mathbf{q'}})}{\sqrt{4\nu_{\mathbf{q}}\nu_{\mathbf{q'}}}} = \delta_{\mathbf{qq'}} \tag{8.13}$$

To arrive at equation (8.13) we have used the relation

$$[\overline{\psi}(\mathbf{x})\psi(\mathbf{x}), \overline{\psi}(\mathbf{y})\psi(\mathbf{y})] = 0 \tag{8.14}$$

for equal times and the fact that $\psi(\mathbf{x},t=x_0) = \psi^0(\mathbf{x},t)$. Also $[\varphi^0,\psi^0] = 0$. This is all suggested by the relation

$$[\pi(\mathbf{x}) - i\nu_{\mathrm{q}}\varphi(\mathbf{x}), \pi(\mathbf{y}) + i\nu_{\mathrm{q'}}(\mathbf{y})] = i\delta(\mathbf{x} - \mathbf{y}) \tag{8.15}$$

which can be easily verified (for any $\nu_q = q_0$) by direct calculation.

The assumption here is that the symmetric Fock space for the interacting system is generated from the vacuum by the operators in equation (8.14). We are, in fact, showing how to define the *in-out* operators to describe the scattering process. These are normally assumed, but here they may be determined from exact solutions to the field problem. In the course of this work [Chapter 9] we will attempt to formally define, up to renormalization, the full, interacting Y4 theory. This is the reason that we described our work as a "Problem Description" for the Wightman problem. Here we are describing the scattering mechanism portion of our problem, and saving the actual proof of "Wightmanism" for another day. A partial solution to the latter in a quite general setting, using a "dressing transformation", is described in [Shirikov 07].

Let us now return to rigorous Haag-Ruelle theory. We will do the following: introduce the *in-out* fields from a physical perspective, motivating the computations. Then, using H-R theory, we will rigorously define the *in-out* fields in functional analytic terms. This will be done for the Boson fields first, the Fermion fields are quite a bit simpler in our formalism.

Our motivation is that the incoming, or outgoing, fields are subject only to self-interactions via the Fermion-Boson coupling and are therefore free fields with the physical mass. Assumptions for the analysis include: (1) Intrinsic mass is not generated in the context of the Yukawa interaction. (2) Renormalization will emerge naturally, and the renormalization "constants" can be given as the limit of a well defined expression. Let us start with [Bjorken Drell 65]

$$(\partial_0^2 - \Delta + m_b^2)\varphi = j(x) \tag{8.16}$$

where $j(x) = g\overline{\psi}(x)\psi(x)$. The m_b is traditionally called the "bare" mass. Let us define

$$\delta m^2\varphi(x) = (m^2 - m_b^2)\varphi(x) \tag{8.17}$$

and add this to equation (8.16) to get

$$(\partial_0^{\,2} - \Delta + m^2)\varphi = j(x) + \delta m^2 \varphi(x) \equiv \bar{j}(x) \tag{8.18}$$

where the last term is regarded as a "source" to the scattered field. Subtracting this scattered field from $\varphi(x)$ leaves the free or incoming field with physical mass m; thus

$$\sqrt{Z}\varphi_{in}(x) = \varphi(x) - \int d^4y \Delta_{ret}(x-y)\,\bar{j}(y) \tag{8.19}$$

introducing the retarded Green function, which is identically zero when $x_0 < y_0$. Z will be defined so that the normalization of φ_{in} is unity. Let us now limit $\rho(s)$ to $\rho(s) = \delta(s-m)$, for m the physical mass. Then equation (8.19) becomes

$$\sqrt{Z}\varphi_{in}(x) = \varphi(x) - \int d^4y \Delta_{ret}(x-y;m)\,\bar{j}(y)\,. \tag{8.20}$$

The computation of $\rho(s^2)$ can now be performed with only a one-particle matrix element, viz:

$\rho(s^2) = \langle 0|\varphi(x)|\, p\rangle$. From this and equation (8.20) we get the important relation

$$\langle 0|\varphi(x)|\, p\rangle = \sqrt{Z}\langle 0|\varphi_{in}(x)|\, p\rangle + \int d^4y \Delta_{ret}(x-y;m)\langle 0|\,\bar{j}(y)|\, p\rangle, \tag{8.21}$$

and from this we have the

<u>**Theorem K:**</u> The relation between interpolating (φ) and asymptotic (φ_{in}) fields is

$$\langle 0|\varphi(x)|\, p\rangle = \sqrt{Z}\langle 0|\varphi_{in}(x)|\, p\rangle \tag{8.22}$$

Proof: Notice that the last term in equation (8.21) vanishes [Bjorken Drell 65]

$$\langle 0|\,\bar{j}(y)|\, p\rangle = \langle 0|(\partial_0^2 - \Delta + m^2)\varphi(y)|\, p\rangle = (\partial_0^2 - \Delta + m^2)e^{-ip\cdot y}\langle 0|\varphi(0)|\, p\rangle$$
$$= (m^2 - p^2)\langle 0|\varphi(y)|\, p\rangle = 0 \quad ∎$$

Using similar arguments with Δ_{adv}, it should come as no surprise that

$$\langle 0|\varphi(x)|\, p\rangle = \sqrt{Z}\langle 0|\varphi_{out}(x)|\, p\rangle \tag{8.23}$$

Equations (8.16) through (8.23) comprise what is known as standard LSZ theory [LSZ 55].

Let us pursue this issue with more rigor (Haag-Ruelle theory [Ruelle 62]), and define an object similar to the above which we will label ϕ_{in}. Later, a theorem will relate ϕ_{in} to φ_{in}. The objects we are referring to are quadratic forms, defined on $D_{\mathscr{S}}$ as we defined earlier. They are operator valued distributions defined in the Schrodinger picture. Pick a function h in $C_0^\infty(\mathbf{R})$ so that $h(y)$ is 1 near $y = m^2$ and supp $h \subset (0, m^2 + \varepsilon)$. We will define ϕ_{in} as an operator valued distribution whose Fourier transform is

$$\hat{\phi}_{in}(p) = h(p^2)\hat{\varphi}(p)\,. \tag{8.24}$$

That is, if we let Tg be an object whose Fourier Transform is $[Tg](p) = h(p^2)\hat{g}(p)$ then

$$\phi_{in}(g) = \varphi(Tg)\,. \tag{8.25}$$

Let $f \in \mathscr{S}(\mathbf{R}^3)$. Then $\hat{f}(\mathbf{p})e^{-ip_0t_0}h(\mathbf{p}^2)$ is in $\mathscr{S}(\mathbf{R}^4)$, so g in equation (8.25) can be chosen to have the form $f(\mathbf{x})\delta(t - t_0)$; then $\phi_{in}(\mathbf{x}, t)$ is a distribution of $\mathbf{x}$ that is smooth in t. Actually, for $f \in \mathscr{S}(\mathbf{R}^3)$, $\phi_{in}(f, t)$ is C^∞ in t.

From equation (8.25), $\dot{\phi}_{in}(\mathbf{x}, t)$ is a distribution in x. For any $f \in C^\infty(\mathbf{R}^4)$ with $f(\cdot, t)$ and $\partial_0 f(\cdot, t)$ in $\mathscr{S}(\mathbf{R}^3)$ for each t, one may form $(f\overleftrightarrow{\partial_0}\phi_{in})(t)$, as described in the following:

Let $\Phi_m(x,t)$ be a free field of mass $m > 0$, i.e. an operator valued distribution described in the Schrodinger picture. Let $f(x,t)$ be a regular wave packet for the Klein-Gordon equation such that the Fourier transforms of the initial data $f = F.T.[(\phi(\cdot,0)]$ and $g = F.T.[\phi_t(\cdot,0)]$ (where $F.T.$ = Fourier Transform) are C^∞ and have compact support. Let $\overleftrightarrow{\partial_0}$ be the symbol defined by

$$(g\overleftrightarrow{\partial_0}k)(t) = \int [g(\mathbf{x},t)\frac{\partial}{\partial t}k(\mathbf{x},t) - k(\mathbf{x},t)\frac{\partial}{\partial t}g(\mathbf{x},t)]d^3\mathbf{x} \tag{8.26}$$

that maps functions of x and t to a function of t. $f\overleftrightarrow{\partial_0}\Phi_m$ is independent of time since both f and Φ_m obey the Klein Gordon equation. If the Fourier Transform of f has the form

$$\hat{f}(\mathbf{p}, t) = (2\mu(\mathbf{p}))^{-1/2}h(\mathbf{p})e^{-i\mu(\mathbf{p})t}\,, \tag{8.27a}$$

then

$$f\overleftrightarrow{\partial}_0\Phi_m = i\int h(\mathbf{p})a^*(\mathbf{p})d^3\mathbf{p} \quad , \tag{8.27b}$$

and if

$$\hat{f}(\mathbf{p},t) = (2\mu(\mathbf{p}))^{-1/2}h(\mathbf{p})e^{+i\mu(\mathbf{p})t}, \tag{8.27c}$$

then

$$f\overleftrightarrow{\partial}_0\Phi_m = -i\int h(\mathbf{p})a(\mathbf{p})d^3\mathbf{p}, \tag{8.27d}$$

where we recognize a* and a as creation/annihilation operators for the symmetric Fock space.

Also, from equation (8.27b) we see that as N runs through 0, 1, …, and f_i runs through all choices obeying (8.27a), the vectors $(f_1\overleftrightarrow{\partial}_0\Phi_m)(f_2\overleftrightarrow{\partial}_0\Phi_m)\ldots(f_N\overleftrightarrow{\partial}_0\Phi_m)\Omega_0$ run through a total set of $\mathcal{F}_s(\mathcal{H})$, the Hilbert space of the free Boson field. (Ω is the vacuum).

We have set up the machinery for the Bose field. Now let us do the same for the Fermion field ψ.

Starting again with heuristics (LSZ), write the Dirac equation as

$$(i\partial\!\!\!/ - \kappa)\psi(x) = \bar{j}(x) \quad , \qquad \bar{j}(x) \equiv j(x) - (\kappa - \kappa_0)\psi(x) \tag{8.28}$$

and assume the equal time anti-commutation relations

$$\{\psi_\alpha(\mathbf{x},t), \psi_\beta(\mathbf{y},t)\} = 0, \qquad \{\psi_\alpha(\mathbf{x},t), \psi_\beta{}^*(\mathbf{y},t)\} = \delta^3(\mathbf{x}-\mathbf{y})\delta_{\alpha\beta} \ . \tag{8.29}$$

In analogy to equation (8.19) we will write

$$\sqrt{Z'}\psi_{in}(x) = \psi(x) - \int d^4y S_{ret}(x-y)\bar{j}(y) \tag{8.30}$$

with Z' a factor to normalize the in-coming field to unity. S_{ret} is the Fermion equivalent of the corresponding quantity in equation (8.19) and has properties

$$(i\partial\!\!\!/ - \kappa)S_{ret}(x-y;\kappa) = \delta^4(x-y), \qquad S_{ret}(x-y) = 0 \ \text{ for } x_0 < y_0 \quad . \tag{8.31}$$

We will also get the result

$$\langle 0|\psi(x)|\, ps\rangle = \sqrt{Z'}\langle 0|\psi_{in}|\, ps\rangle . \tag{8.32}$$

For the out fields we will then get

$$\sqrt{Z'}\psi_{out}(x) = \psi(x) - \int d^4y S_{adv}(x-y)\bar{j}(y) \quad \text{with } S_{adv}(x-y) = 0 \text{ for } x_0 > y_0 . \tag{8.33}$$

We will now state a theorem due to Haag which is the keystone of Haag-Ruelle scattering theory. In this form it relates to Boson fields. We will repeat it for Fermion fields presently. Note that if one writes

$$\psi^0(x) = \frac{1}{V^{1/2}}\sum_{\mathbf{k}}\sum_{s=\pm}\left(\frac{m}{\Omega_k}\right)^{1/2}[a_{\mathbf{ks}}u(\mathbf{k};s)e^{-ikx} + b^*_{\mathbf{ks}}v(\mathbf{k};s)e^{ikx}],$$

$$\bar{\psi}^0(x) = \frac{1}{V^{1/2}}\sum_{\mathbf{k}}\sum_{s=\pm}\left(\frac{m}{\Omega_k}\right)^{1/2}[a^*_{\kappa s}\bar{u}(\mathbf{k};s)e^{ikx} + b_{\mathrm{k}s}\bar{v}(\mathbf{k};s)e^{-ikx}],$$

then the inversion formulas one finds in physics texts are

$$a_{\mathbf{k}s}{*} = \frac{1}{\sqrt{V}}\left(\frac{\kappa}{\omega_k}\right)^{1/2}\int e^{-ikx}\bar{\psi}(x)\gamma_0 u(\mathbf{k};s)d^3\mathbf{x} \tag{8.34}$$

and similarly for the other coefficients. This is inconvenient for making general statements about Boson fields and Fermion fields simultaneously. From this we see that, to satisfy the Dirac equation, we must have $(\gamma^\mu k_\mu - \kappa)u(\mathbf{k};s) = 0$, and from the identity $(\gamma^\mu k_\mu - \kappa)(\gamma^\mu k_\mu + \kappa)u = 0$ we see that each component of $u(\mathbf{k};s)$ must satisfy the Klein Gordon equation. This each component of u may be computed from an equation like (8.27b) or (8.27d), using the same rules for smearing. Let us generalize these equations as follows:

Pick a function h in $C_0^\infty(\mathbb{R})$ so that $h(y)$ is 1 near $y = \kappa^2$ and supp $h \subset (0, \kappa^2 + \varepsilon)$. We will define ξ_{in} as an operator valued distribution whose Fourier transform is

$$\hat{\xi}_{in}(p) = h(p^2)\hat{\psi}(p). \tag{8.35}$$

That is, if we let Tg = object whose $F.T.[Tg](p) = h(p^2)\hat{g}(p)$ then

$$\xi_{in}(g) = \varphi(Tg)\,. \tag{8.36}$$

Let $f_{i,a} \in \mathscr{S}(\mathbf{R}^3)$. Then $\hat{f}_{i,\alpha}(\mathbf{p})e^{-ip_0t_0}h(\mathbf{p}^2)$ is in $\mathscr{S}(\mathbf{R}^4)$, so g in equation (8.36) can be chosen to have the form $f(\mathbf{x})\delta(t - t_0)$; then $\xi_{in}(\mathbf{x},t)$ is a distribution of $\mathbf{x}$ that is smooth in t. Actually, for $f_{i,\alpha} \in \mathscr{S}(\mathbf{R}^3)$, $\xi_{in}(f,t)$ is C^∞ in t. If the Fourier Transform of f has the form

$$\hat{f}_{i,\alpha}(\mathbf{p},t) = (2\mu(\mathbf{p}))^{-1/2}h(\mathbf{p})e^{-i\mu(\mathbf{p})t} \tag{8.37a}$$

then

$$f_{i,\alpha}\overleftrightarrow{\partial}_0\Psi_m = i\int h(\mathbf{p})a_{i,\alpha}{}^*(\mathbf{p})d^3\mathbf{p} \tag{8.37b}$$

and if

$$\hat{f}_{i,\alpha}(\mathbf{p},t) = (2\mu(\mathbf{p}))^{-1/2}h(\mathbf{p})e^{+i\mu(\mathbf{p})t} \tag{8.37c}$$

then

$$f_{i,\alpha}\overleftrightarrow{\partial}_0\Psi_m = -i\int h(\mathbf{p})a_{i,\alpha}(\mathbf{p})d^3\mathbf{p}\,. \tag{8.37d}$$

Here $f_{i,\alpha}$ satisfies the same conditions as in the Boson case, the index i will refer to the component and α to the spin index. We have defined the Fermion creation/annihilation operators. The vectors $(f_{1i,\alpha}\overleftrightarrow{\partial}_0\Psi_m)(f_{2i,\alpha}\overleftrightarrow{\partial}_0\Psi_m)\ldots(f_{Ni,\alpha}\overleftrightarrow{\partial}_0\Psi_m)\Omega_0$ run through a total set of $\mathcal{F}_a(\mathcal{H})$, the Hilbert space of the free Fermi field. (Ω_0 is the vacuum). We have set up the machinery for the Fermi field. With this we now have the

Lemma: Let $f_{i,a}$ be a regular wave packet for the mass m (or k) Klein Gordon equation. Then $(f\overleftrightarrow{\partial}_0\phi_{in})(t)\Omega_0$ is independent of t where Ω_0 is the vacuum for Boson theory and $(f_{i,\alpha}\overleftrightarrow{\partial}_0\xi_{in})(t)\Omega_0$ is independent of t in the Fermion case.

With this, the main theorem of Haag-Ruelle theory, when referring to our Yukawa theory here, states the following:

Theorem L: Let φ and ψ be Hermitian scalar and spinor fields obeying the Wightman axioms and satisfying assumptions 1 and 2 above. Define ϕ_{in} and ξ_{in} as above. Then

1. For any regular wave packets $f^{(1)}{}_{i,a},\ldots,f^{(n)}{}_{i,a}$ the limits

$$\lim_{t\to\mp\infty}(f^{(1)}_{i,\alpha}\overleftrightarrow{\partial}_0\xi_{in})(t)\cdots(f^{(n)}_{i,\alpha}\overleftrightarrow{\partial}_0\xi_{in})(t)\Omega_0 \equiv \eta^{Fermion}_{in/out}(f^{(1)}_{i,\alpha},\ldots f^{(n)}_{i,\alpha}) \text{ and}$$

$$\lim_{t\to\mp\infty}(f\overleftrightarrow{\partial}_0\phi_{in})(t)\cdots(f^{(n)}\overleftrightarrow{\partial}_0\phi_{in})(t)\Omega_0 \equiv \eta^{Boson}_{in/out}(f^{(1)},\ldots f^{(n)})$$ exist in the norm topology on $\mathcal{H}$

and are independent of the choice of h. Let $\mathcal{H}_{in}$ and $\mathcal{H}_{out}$ denote the closed span of n_{in} and n_{out}.

2. $\mathcal{H}_{in}$ and $\mathcal{H}_{out}$ are invariant by the representation U of the Poincare group for both particle types.

3. There are operator-valued distributions φ_{in} on $\mathcal{H}_{in}$ and φ_{out} on $\mathcal{H}_{out}$ so that $\langle \mathcal{H}_{in}, U, \varphi_{in}\rangle$ and $\langle \mathcal{H}_{out}, U, \varphi_{out}\rangle$; ψ_{in} on $\mathcal{H}_{in}$ and ψ_{out} on $\mathcal{H}_{out}$ so that $\langle \mathcal{H}_{in}, U, \psi_{in}\rangle$ and $\langle \mathcal{H}_{out}, U, \psi_{out}\rangle$ are all unitarily equivalent to the free fields of mass m (or k) and so that

$$\eta^{Fermion}_{in/out}(f^{(1)}_{i,\alpha},\ldots f^{(n)}_{i,\alpha}) = (f^{(1)}_{i,\alpha}\overleftrightarrow{\partial}_0\psi_{in})\cdots(f^{(n)}_{i,\alpha}\overleftrightarrow{\partial}_0\psi_{in})\Omega_0 \text{ and}$$

$$\eta^{Boson}_{in/out}(f^{(1)},\ldots f^{(n)}) = (f^{(1)}\overleftrightarrow{\partial}_0\varphi_{in})\cdots(f^{(n)}\overleftrightarrow{\partial}_0\varphi_{in})\Omega_0 .$$

With this theorem, the theoretical relations between φ_{in} and ϕ_{in}, ξ_{in} and ψ_{in} are established.

The proof of this theorem is one of the major accomplishments of constructive quantum field theory. A sketch can be found in [Haag 93], a full proof is stated in [Reed 79]. A beautiful and quite general (i.e. not dependent on Schrodinger picture), water-tight proof is found in [Ruelle 62].

This completes the description of the time development of the coupled system, one of the stated three goals in Chapter 4. The final reduction formulae that are used to compute cross sections are the stuff of traditional Quantum Field Theory and well known.

Let us review what we have done. We have presented a closed form solution for the Dirac Klein Gordon coupled system. It is in closed form in a certain "picture", the time zero or Schrodinger picture, where the Fermion current becomes equal to its uncoupled form. We have given a rigorous (modulo our solution) definition of the asymptotic fields (which still satisfy a coupled non-linear equation) and shown how they relate to the asymptotic fields utilized in LSZ, which are, in fact, free fields. The formalism shows why these objects both give the same results when computed. (They are always computed in the $t \to \infty$ limit.) We have shown that if the φ (Boson) field can be reduced to a self-adjoint form, then the coupled ψ field will also be self adjoint. From these, it would seem that the proof that the Hamiltonian itself is self adjoint would follow without a great deal of trouble. This would be a major accomplishment.

An interesting point is to recollect comments made with regard to the classical theory and regularity. It was remarked that this might be an open issue in quantum theory, and yet we have glibly discussed solutions that are not only regular, but serve as a basis for tempered distributions. Clearly the extra requirements of quantum theory have something to say about this issue: it greatly "tempers" the behavior of possible solutions.

The entire issue of renormalization needs to be examined, and after this is done, the agreement of our theory with standard theory established.

CHAPTER 9

RENORMALIZATION

9.1 Complete Y4

We have seen that the smooth objects about which we may prove theorems are the quantum fields, or operator valued distributions, and the objects specified as functions of *x* and *t* are quadratic forms associated with these fields; the so-called fields referred to in physics texts. These forms are the objects used in calculations for reducing theory to numbers to compare with experiment. These numbers are usually obtained by perturbation theory on the associated forms. Now, to *define the theory* we practice smearing and define operator valued distributions. To *produce numbers* we must utilize the highly singular associated forms to somehow make sense of the calculational procedures. This requires renormalization, because of the various types of divergences that inevitably occur. So, in order to completely specify Y4, we must address the issue of renormalization.

The first type of unwanted divergence that occurs is the so-called "zero point energy". The usual creation and annihilation operators plugged into the classical field equations will lead to an energy $E = \sum (n_k + \frac{1}{2})\omega_k$. For an infinite system, the ½ term will provide an infinite energy. This difficulty has already been surmounted by normal ordering, which was covered in Chapter 5. Since $n_k = c_k{}^*c_k$ the previous formula will become $E = \sum \frac{1}{2}\omega_k(c_k^* c_k + c_k^* c_k)$ as a result of the normal ordering operation. In the DKG case, whenever a product such as $\overline{\psi}(x)\psi(x)$ occurs, we will substitute $:\overline{\psi}(x)\psi(x):$. In addition, a product of the form $\overline{\psi}(x)\psi(x)$ will be understood to stand for $\lim_{\varepsilon\to 0}\overline{\psi}(x+\varepsilon)\psi(x)$ or $\lim_{\varepsilon\to 0}\frac{1}{2}[\overline{\psi}(x+\varepsilon)\psi(x)+\overline{\psi}(x)\psi(x+\varepsilon)]$.

Our knowledge of any four dimensional quantum field theory is largely due to calculating results for interacting fields by perturbing the system and utilizing free fields to compute the results. The perturbative theory as practiced by physicists produces results of unparalleled accuracy. But it also produces infinite results for simple objects that are re-defined in the theory as a result of interactions, viz: coupling constants, masses, field function renormalizations etc. Rigor must bow to pragmatism and explain why this is so. We shall see that in the Y4 theory presented here, it is possible to posit a renormalization procedure that (presumably) will agree with that generated by the standard perturbation procedure, with the happy result that it need be done only once at the outset, and therefore our theory may serve as a candidate for a four dimensional system to obey the Wightman axioms. Let us start with

$$(-i\gamma^{\mu}\partial_{\mu}+\kappa_0)\psi = g\varphi\psi \qquad (\kappa_0, g>0) \tag{9.1}$$

$$(-\Delta+\partial_t^2+m_0^2)\varphi = g\overline{\psi}\psi \qquad (m_0>0) \tag{9.2}$$

where we have explicitly indicated by the 0 subscript that the mass terms, m and κ, are yet to be defined. Then we will have the

Proposition: There exists a renormalization scheme such that

$$(-i\gamma^{\mu}\partial_{\mu}+\kappa_0)\psi_R = g_R\varphi_R\psi_R \quad \text{and} \quad (-\Delta+\partial_t^2+m_0^2)\varphi_R = g_R\overline{\psi}_R\psi_R \tag{9.3}$$

where the terms ψ_R, φ_R and g_R are finite, well defined quantities.

Proof: Let $\varphi=\sqrt{Z_3}\,\varphi_R$, $\psi=\sqrt{Z_2}\,\psi_R$, $g_R=\dfrac{Z_2}{\sqrt{Z_3}}g$ and $Z_2 = Z_3$ (9.4)

Absent the criterion "*well defined*", insertion of these quantities into equations (9.1) and (9.2) yields the desired result. ∎

We will now address the issue of being well-defined. The quantities Z_2 and Z_3 are actually suitably defined limits. The line of reasoning is the following: equations (9.3) and (9.4) are defined in terms of renormalized (i.e. finite) quantities; if we can isolate the "infinities" in φ and ψ in terms of multiplicative quantities, via equations (9.1) and (9.2) which we actually solve, and then define these constants as the limits of well-defined expressions, we are done. Measurable terms such as g_R can be defined as a finite limit of the respective entities Z_2, Z_3 and g when arranged as $g_R = \frac{Z_2}{\sqrt{Z_3}} g$. Let us turn to this.

We have found a formal solution for equations (9.1) and (9.2), at least at equal times. (This means that the fields φ and ψ, and their products or correlation functions, all have the same time argument). The solutions are, with $\chi[x,t] = g\gamma^0 \int \Delta'(x-y)\varphi(y)\delta(y_0 - t)dydx$,

$$\psi(x) = e^{i\chi}\psi^0(x), \text{ and } \varphi(x) = \varphi^0(x) + g\int \Delta_F(x-y){:}\overline{\psi}^0(y)\psi^0(y){:}\,dy\,, \tag{9.5}$$

the 0 superscripts denoting solutions to the sourceless field equations. Note that $\chi(x,t)$, although usually evaluated at $t = x_0$, must be defined rigorously for arbitrary t. We concentrate on $e^{i\chi}$.

The exponential will be written as $:e^{i\chi}:$, the colon symbol is read as Wick exponential or Wick ordering and is defined for any operator so that the vacuum expectation for equal arguments vanishes; viz: $\langle 0|\phi(x)\phi(x)|0\rangle = 0$. This is accomplished by the definition (see Chapter 5 for a parallel discussion)

$$:\phi(x)\phi(y): = \phi(x)\phi(y) - \langle 0|\phi(x)\phi(y)|0\rangle\,. \tag{9.6}$$

Now for any operator $i\chi$, we will define the Wick exponential by

$$:e^{ig\chi}:=\sum_{n=0}^{\infty}\frac{(ig)^n}{n!}:\chi^n: \tag{9.7a}$$

and the "Wick monomial" $:\chi^n:$ is defined recursively by

$$:\chi:(x)=\chi(x) \tag{9.7b}$$

$$:\chi^2:(x)=\lim_{y\to x}[\chi(x)\chi(y)-\langle\chi(x)\chi(y)\rangle] \tag{9.7c}$$

$$:\chi^n:(x)=\lim_{y\to x}[:\chi^{n-1}:(x)\chi(y)-(n-1)\langle\chi(x)\chi(y)\rangle:\chi^{n-2}:(x)\,. \tag{9.7d}$$

In conventional renormalization theory, and by the same reasoning that led to equation (5.25) this yields the UV limit ($\Lambda\to\infty$)

$$\lim_{\Lambda\to\infty}\exp[\frac{1}{2}g^2\Delta_\Lambda^+{}'(0)]e^{ig\chi_\Lambda(x)} \tag{9.8}$$

where $\chi_\Lambda(x)$ is the scalar field with ultraviolet cut off Λ and $\Delta_\Lambda^+(0)$ the corresponding two point function $\langle\chi(x)\chi(y)\rangle$ evaluated at the origin. In our parlance $\chi_\Lambda(x)$ is as in equation (9.5) with its Fourier frequencies cut off at Λ, and $\Delta_\Lambda^+{}'(0)=\lim_{x\to 0}\frac{1}{(2\pi)^4}\oint\{\int\rho(s^2)e^{ikx}\Delta(k;s)ds^2\}d^4k$ with $|k|<\Lambda$ also. We can now define the solution to equation (9.4) as

$$\lim_{\Lambda\to\infty}\exp[\frac{1}{2}g^2\Delta_\Lambda^+{}'(0)]e^{ig\chi_\Lambda(x)}\psi^0(x)=\lim_{\Lambda\to\infty}\frac{1}{\sqrt{Z_2}}e^{ig\chi_\Lambda(x)}\psi^0(x) \tag{9.9}$$

and thus we finally define $\psi(x)=\sqrt{Z_2}\,\psi_R(x)$ with $\lim_{\Lambda\to\infty}\exp[-\frac{1}{2}g^2\Delta_\Lambda^+{}'(0)]=\sqrt{Z_2}$ and $\psi_{un}(x)=\lim_{\Lambda\to\infty}e^{ig\chi_\Lambda(x)}\psi^0(x)$ as the un-renormalized solution.

In the development of our theory, while utilizing the spectral representation, we made an assumption on the convergence of certain integrals. It is now time to be more specific regarding this matter. It will turn out that this issue and renormalization are intimately related.

Note that $$\partial_{x_0}\langle 0|[\varphi(x),\varphi(y)]|0\rangle=\partial_{x_0}\left[\int_0^\infty\rho(s^2)\Delta(x-y;s^2)ds^2\right]$$

which leads to $$i\langle 0|[\dot{\varphi}(x),\varphi(y)]|0\rangle = \int_0^\infty ds^2 \rho(s^2)\frac{\partial}{\partial(x^0-y^0)}\Delta(x-y;s^2)\,.$$

Taking $x_0 = y_0$ the result becomes (since $\left[\frac{\partial\Delta(x-y;s^2)}{\partial(x_0-y_0)}\right]_{x_0=y_0} = \delta(\mathbf{x}-\mathbf{y})$ for any s^2)

$$i\langle 0|[\dot{\varphi}(x),\varphi(y)]|0\rangle = -i\delta(\mathbf{x}-\mathbf{y})\int_0^\infty ds^2\rho(s^2) \tag{9.10}$$

using the spectral representation. If we assume (merely in a heuristic sense) that the interacting field satisfies a standard equal time commutation relation, we can write

$$[\dot{\varphi}(x),\varphi(y)] = -iZ_3^{-1}\delta(\mathbf{x}-\mathbf{y}) \;\;. \tag{9.11}$$

Taking the vacuum expectation value of this relation and comparing with equation (9.10) we see that we must have

$$Z_3^{-1} = \int_0^\infty ds^2\rho(s^2)\,. \tag{9.12}$$

Experience tells us that this quantity may not be finite. However, the quantity that must exist to justify our manipulation is not Z_3 but Δ_F'. Consider the Fourier representation of this quantity

$$\Delta_F'(x-y) = \frac{1}{(2\pi)^4}\int dk e^{ik(x-y)}\Delta_F'(k^2)\,. \tag{9.13}$$

Thus the form of the transform is

$$\Delta_F'(k^2) = \int_0^\infty ds^2\,\frac{\rho(s^2)}{s^2-k^2-i\eta}\,. \tag{9.14}$$

Now if this quantity remains finite, then our previous manipulations are justified. Thus

$$\int_0^\infty ds^2\,\frac{\rho(s^2)}{s^2} < \infty \tag{9.15}$$

will insure that our manipulations are sound. Multiply equation (9.14) by k^2 and take the limit $k^2 \to \infty$. Using equation (9.12) we will get

$$\int ds^2 \frac{\rho(s^2)k^2}{s^2 - k^2 - i\eta} \to -\int ds^2 \rho(s^2) = -Z_3^{-1} \tag{9.16}$$

and so finally

$$Z_3^{-1} = -\lim_{k^2 \to \infty} k^2 \Delta_F'(k^2) . \tag{9.17}$$

This relation, from [Roman 69], is quite useful. It gives us a dynamical definition of Z_3 in terms of a well-defined quantity. It will suggest approximation methods in proofs of Wightmanism. We may utilize this relation in the traditional manner and write

$$\varphi(x) = \sqrt{Z_3}\varphi_R(x) \tag{9.18}$$

to isolate the infinity in the φ field in a well-defined manner.

The equations (9.3) and (9.4) present a relation between the spectral representations and other dynamical quantities that can presumably be calculated. The reader will appreciate that the actual computation of Z_2 and Z_3 presents a formidable task. Nevertheless, in principle we may isolate the infinities as well defined limits, define φ (say) as a (possibly unbounded) self-adjoint operator and thus define ψ as well from Theorem 3.

There is one additional point. The manipulations leading from equation (9.1) and (9.2) to (9.4) do not seem to involve m_0 and κ_0. In traditional renormalization theory the masses also must be renormalized. In our formalism, this is handled by the replacement of $\Delta(x,y)$ by $\Delta'(x,y)$ and the definition equation (9.12). Let us examine this issue in detail.

In equation (9.12) let us split off the one particle contribution as follows: write $\rho(s^2) = \delta(s^2 - m^2) + \sigma(s^2)$ and define [m is here the physical value]

$$Z_3^{-1} = 1 + \int_{4m^2}^{\infty} ds^2 \rho(s^2) \ . \tag{9.19}$$

Observe that in terms of the bare [unrenormalized] field φ^0 one may write the commutator

$$i\langle 0|[\varphi^0(x),\varphi^0(y)]|0\rangle = Z_3\int_0^\infty ds^2\rho(s^2)\Delta(x-y;s^2). \tag{9.20}$$

Write the φ^0equation concerning an unrenormalized source $j = g\overline{\psi}(x)\psi(x)$ as follows

$$j^0(x) \equiv (\partial_0^{\ 2} + \Delta^2 + m_0^{\ 2})\varphi^0(x). \tag{9.21}$$

Applying the operator $(\partial_0^{\ 2} + \Delta^2 + m_0^{\ 2})_x$ to equation (9.21) and noting that

$$(\partial_0^2 + \Delta^2)_x\Delta(x-y;s^2) = -s^2\Delta(x-y;s^2) \qquad \text{we will get}$$

$i\langle 0|[j^0(x),\varphi^0(x)]|0\rangle = Z_3\int ds^2\rho(s^2)(m_0^{\ 2}-s^2)\Delta(x-y;s^2)$. Taking $\frac{\partial}{\partial x^0}$ and letting $x^0 = y^0$ one

then will find the result

$$i\langle 0|[\frac{\partial j^0}{\partial x^0}(\mathbf{x}),\varphi^0(\mathbf{y})]|0\rangle = \delta(\mathbf{x}-\mathbf{y})Z_3\int ds^2\rho(s^2)(m_0^2-s^2). \tag{9.22}$$

If j^0 does not depend on φ^0 or its derivatives, then j^0 commutes with φ^0 and the LHS of equation (9.22) vanishes. As a consequence $Z_3 m_0^{\ 2}\int ds^2\rho(s^2) - Z_3\int ds^2 s^2\rho(s^2) = 0$.

Using equation (9.12) we see that $m_0^2 = Z_3\int ds^2 s^2\rho(s^2)$. Using equation (9.19) we can also say

$$m_0^2 = Z_3 m^2 + Z_3\int_{4m^2}^{\infty} ds^2 s^2\rho(s^2)\ . \tag{9.23}$$

Now to renormalize m_0 we simply replace it with (9.23) when it occurs. An interesting consequence of equation (9.23) is that if the bare mass m_0 is zero, then the physical mass must also be zero (ρ is always positive). Thus "self interactions" cannot build up a non-vanishing rest mass from nothing. This fact is important for the development presented here. Suppose that self interactions *could* build a non-vanishing rest mass from the vacuum. Equivalently, this implies

[Roman 69] that $\langle 0|\varphi(0)|0\rangle \neq 0$. Let us say it is a constant equal to $\frac{c}{(2\pi)^{3/2}}$. This gives a contribution c^2 to $\rho(s^2)$ so that $\int_0^\infty ds^2 \rho(s^2)/s^2 = c^2\int_0^\infty ds^2/s^2 + \cdots = \infty$, which violates (9.15).

With the foregoing in mind, we will discuss the traditional renormalization scheme for the entire DKG system, including mass corrections. In elastic (energy conserving) processes, the S matrix is often written $S_{fi} = \delta_{fi} + iT_{fi}$. The Hamiltonian operator for the DKG system is written

$$H_B^0[\varphi,\psi] + H'_B[\varphi,\psi] \equiv \frac{1}{2}\int(\partial_\mu\varphi)^2 + m_0^2\varphi^2)d^3\mathbf{x} + i\int \overline{\psi}\gamma_0\dot{\overline{\psi}}d^3\mathbf{x} + H'_B[\varphi,\psi]$$

with $H'_B[\varphi,\psi] = -g\int \varphi(x)\overline{\psi}(x)\psi(x)d^3\mathbf{x}$; the subscript B denoting a "bare" or un-renormalized quantity. In terms of the bare fields the T matrix is $T_{fi} = {}_B\langle f|H'[\varphi,\psi]|i\rangle_B$. Now, it is demanded that this transition matrix produce the same results when renormalized quantities are used. [NB it is also demanded that $T_{fi} = {}_B\langle f|H'[\varphi,\psi]|i\rangle_B = \frac{g_R}{g}Z_2^{-1}Z_3^{-1/2}{}_R\langle f|H'[\varphi,\psi]|i\rangle_R$ and introduce $g_R = Z_1^{-1}Z_2Z_3^{1/2}g$ so that $T_{fi} = {}_B\langle f|H'[\varphi,\psi]|i\rangle_B = Z_1^{-1}{}_R\langle f|H'[\varphi,\psi]|i\rangle_R$]. The point here is that, while renormalization is forced upon the formalism in order to define the product of operators at the same point, it is generally dealt with only in the context of perturbation theory. The requirement above is enforced on matrix elements of the T matrix operator, for each term in a power series expansion in g. Since this is evaluated term by term, other corrections, such as mass renormalizations, are only introduced on an order by order basis. Thus, there is no corresponding formalism in perturbation theory to our equation (9.23). But the relation is demanded by consistency. This relation is not new, but its use in the definition of an exact, renormalized theory is new.

9.2 Speculations for Further Work

The most obvious program for further work is to try to rigorously define Z_2 and Z_3, prove the essential self-adjointness and boundedness of the Hamiltonian defined by the φ_R and ψ_R operators and then go on to complete the existence proof of Y4 as a Wightman system. One at least has a plan of attack in terms of formal solutions, which has not existed before.

The next step would be to examine a more realistic system, such as the pseudo-scalar Yukawa theory, defined as

$$(-i\gamma^{\mu}\partial_{\mu}+\kappa)\psi = i\gamma^{0} g\varphi\psi \qquad (\kappa, g>0) \tag{9.24}$$

$$(-\Delta+\partial_{t}^{2}+m^{2})\varphi = ig\overline{\psi}\gamma^{0}\psi \qquad (m>0)\ . \tag{9.25}$$

A solution similar to equation (9.5) can be written down, but the exponential involves an object like $\varphi(y)[\chi_i(y)/\phi_j(y)]$ where we write the Dirac bi-spinor as $\psi(x)=\begin{pmatrix}\phi_1(x)\\ \phi_2(x)\\ \chi_1(x)\\ \chi_2(x)\end{pmatrix}$. The result is

$$\phi_1 = \phi_1^0 e^{-g\int \Delta(x-y)\varphi(y)\delta(y_0-x_0)[\chi_1/\phi_1](y)dy} \tag{9.26a}$$

$$\phi_2 = \phi_2^0 e^{-g\int \Delta(x-y)\varphi(y)\delta(y_0-x_0)[\chi_2/\phi_2](y)dy} \tag{9.26b}$$

$$\chi_1 = \chi_1^0 e^{+g\int \Delta(x-y)\varphi(y)\delta(y_0-x_0)[\phi_1/\chi_1](y)dy} \tag{9.26c}$$

$$\chi_2 = \chi_2^0 e^{+g\int \Delta(x-y)\varphi(y)\delta(y_0-x_0)[\phi_2/\chi_2](y)dy}\ . \tag{9.26d}$$

We thus are forced to attempt to define the inverse of a spinor-valued operator as an *exponential.* This does not seem to be too formidable but is also a dissertation-length endeavor.

In the section on the classical DKG problem we referred to a special theorem of Chadam. This is:

<u>Chadam's Theorem</u>: (3 + 1 Dim version) Whenever a solution to DKG exists we have $\int [|\psi_1 - \overline{\psi}_4|^2 + |\psi_2 + \overline{\psi}_3|^2]dx$ = const. Hence, if the initial data satisfies $\psi_1(x,0) = \overline{\psi}_4(x,0)$, $\psi_2(x,0) = -\overline{\psi}_3(x,0)$ for all $x \in \mathbf{R}^3$, then $\overline{\psi}\psi(x,t) \equiv 0$ for all $t \geq 0$ and φ remains a free solution.

(1 + 1 D version) Let $\psi = \begin{pmatrix} \psi_1 \\ \psi_2 \end{pmatrix}$ be the two component spinor in equations (6.3) and (6.4). In the representation (6.5) we have

$$\int |\psi_1 - \overline{\psi}_2|^2 dx = \text{const.} \tag{9.27}$$

Hence if $\psi_1(x,0) = \overline{\psi}_2(x,0)$ then $\psi_1(x,t) = \overline{\psi}_2(x,t)$ for all $t \geq 0$ and φ is thus a free solution.

<u>Proof</u>: We will prove the 1 + 1 Dimensional version, since the other is a similar version of the same argument. In component form DKG is

$$\frac{\partial \psi_1}{\partial t} = -\frac{\partial \psi_2}{\partial x} - iM\psi_1 + ig\varphi\psi_1 \tag{9.28}$$

$$\frac{\partial \psi_2}{\partial t} = -\frac{\partial \psi_1}{\partial x} + iM\psi_2 - ig\varphi\psi_2 \tag{9.29}$$

(Here a bar over a coordinate will denote complex conjugate).

Hence the function $w = \psi_1 - \overline{\psi}_2$ will satisfy $\frac{\partial w}{\partial t} = \frac{\overline{\partial w}}{\partial x} - i(M - g\varphi)w$. Multiplication by $\overline{w}$ yields $\overline{w}\frac{\partial w}{\partial t} = \overline{w}\frac{\overline{\partial w}}{\partial x} - i(M - g\varphi)|w|^2$. Conjugation of this produces $w\frac{\overline{\partial w}}{\partial t} = w\frac{\partial w}{\partial x} + i(M - g\varphi)|w|^2$. Summing the last two displayed equations yields $\frac{\partial}{\partial t}|w|^2 = \frac{1}{2}\frac{\partial}{\partial x}(w^2 + \overline{w}^2)$, and an integration of this over all space gives the desired result. The final assertion that φ is free follows from the fact that under the representation (6.5) we have $\overline{\psi}\psi = |\psi_1|^2 - |\psi_2|^2$. ▮

There are issues in 1 + 1 dimensions to discuss with respect to scattering. It is a common practice to investigate model theories of lower dimension in order to enjoy simpler mathematics while still covering essential physical phenomena. In Y2 (Yukawa two dimensional theory) this may be a dangerous enterprise. Chadam proves another remarkable theorem [Chad 74] which we will simply paraphrase. In one space dimension, conditions exist under which the spinor field does not decay uniformly to zero, thus precluding a scattering theory in the $H^1(\mathbf{R}^1)$ norm.

The great utility of Chadam's Theorem is that there is coupling introduced through equation (9.24), but equation (9.25) is rigorously a free field. I believe this situation leads to an unstable vacuum and thus there exists a "super fluid" phase in Y4. This would be another interesting research problem.

CHAPTER 10

SUMMARY AND STATEMENT OF NEW RESULTS

Starting with the Dirac-Klein Gordon coupled system in four dimensions

$$(-\Delta + \partial_t^2 + m_0^2)\varphi = g\overline{\psi}\psi \tag{10.1}$$

$$(-i\gamma^{\mu}\partial_{\mu} + \kappa_0)\psi = g\varphi\psi \tag{10.2}$$

we have found a representation of a solution in the classical case as

$$\psi_k = \psi_k^0(x)\exp[ig\varepsilon_k \int \Delta(x-y)\varphi(y)d^3\mathbf{y}]\Big|_{y_0=x_0} \tag{10.3}$$

which is more conveniently represented as

$$\psi = \exp[ig\gamma^0 \int \Delta(x-y)\varphi(y)d^3\mathbf{y}]\Big|_{y_0=x_0} \psi^0(x) \tag{10.4}$$

utilizing the Dirac matrix γ^0.

The immediate integral equation resulting from equation (10.4) in the φ variable is

$$\varphi(x) = \varphi^0(x) + g\int \Delta_F(x-y)\overline{\psi}(y)\psi(y)dy\,. \tag{10.5}$$

In the quantum case equation (10.4) becomes exact by replacing $i\Delta(x-y)$ with

$[\varphi_f(x), \varphi_f(y)] = -i\Delta(x-y)$ and postulating that the commutator relates to the interacting fields.

The result is

$$\psi = \exp[ig\gamma^0 \int \Delta'(x-y)\varphi(y)d^3\mathbf{y}]\Big|_{y_0=x_0} \psi^0(x)\,, \tag{10.6}$$

more usefully written as $\psi = \exp[ig\gamma^0 \int \Delta'(x-y)\varphi(y)\delta(y_0 - t)dy]\Big|_{t=x_0} \psi^0(x)$

where we regard the variables φ and ψ as operators and Δ' utilizes the quantum spectral

representation. We have proved theorems showing under what conditions the operator corresponding to the quadratic form $\eta = \exp[ig\gamma^0 \int \Delta'(x-y)\varphi(y)d^3\mathbf{y} \equiv \eta(x, y_0)$ is unitary, reducing the expression in equation (10.5) to an exact solution in terms of free fields

$$\varphi(x) = \varphi^0(x) + g\int \Delta_F(x-y) : \overline{\psi}^0(y)\psi^0(y) : dy \,. \tag{10.7}$$

We have indicated that the operator product must be normal ordered. With equation (10.6) written as $\exp[ig\gamma^0 \int f(\mathrm{x})\delta(x_0 - t)\int \Delta'(x-y)\varphi(y)\delta(y_0 - t')dydx \equiv \eta[f,t]$ we have shown that the quantum solutions (10.6) and (10.7) are essentially self adjoint (Theorems 3 and 4) on the cores labeled F_0 and G_0, the set of all finite particle states in the total Fock space. We have stated the conditions under which this self adjointness holds. Utilizing traditional theorems from Haag-Ruelle scattering theory, we have derived the time development of the DKG system up to a point where known theorems can be used to compute the results. We have also defined the full, interacting creation/annihilation operators for the φ field [equations (8.13) through (8.15)], although we reserve the use of this information for completion of a proof that Y4 is a Wightman system. Finally, we have shown how our theory may be renormalized and compared to the results of traditional quantum field theory on the Yukawa system obtained by perturbation theory.

REFERENCES

Baez, J.C., Segal, I.E., Zhou, Z.
[Baez 92] "Introduction to Algebraic and Constructive Quantum Field Theory", Princeton University Press, Princeton, New Jersey; 1992.

Bjorken, James D. and Drell, Sidney D.
[Bjorken Drell 65] "Relativistic Quantum Fields", McGraw Hill Book Co., New York, 1965. Physics Vol 51, World Scientific, New Jersey, 1993.
Chadam, J.M.

[Chad 73] *Global Solutions of the Cauchy Problem for the Classical Coupled Maxwell-Dirac Equation in One Space Dimension,* J. Funct. Anal, 13, 173-184, 1973.

Chadam, J and Glassey, Robert T.
[Chad 74] *On Certain Global Solutions of the Cauchy Problem for the (Classical) Coupled Klein-Gordon-Dirac Equations in One and Three Space Dimensions,* Arch. Rat. Mech. Anal. 54 (1974), 223-237.

Chalmers, John M. and Glassey, Robert T.
[Chalm 97] *Quantization of Non-Polynomial Field Theories*, arXiv:hep- th/9712065v2 11 Dec 1997.

D'Ancona, Piero, Damiano, Foschi and Selberg, Sigmund
[D'Ancona 05] *Null Structure and Almost Optimal Local Regularity for the Dirac-Klein-Gordon System*; arXiv:math.AP/0509545.

Feynman, R.
[Feyn 72] "Statistical Mechanics", W.A. Benjamin, Reading Mass. 1972.

Garding, L. and Wightman, A.S.
[Gard 54] *Representations of the anticommutation relations*, Proc Nat. Acad. Sci. 40, 617 (1954).

Haag, R.
[Haag 96] "Local Quantum Physics", Springer-Verlag 2nd Edition, Berlin, 1996.

Itzykson, Claude and Zuber, Jean-Bernard
[Itzyk 80] "Quantum Field Theory", McGraw-Hill, Inc, New York, 1980.

Kallen G.
[Kallen 64] "Elementary Particle Physics", Addison-Wesley, Reading, Mass. 1964.

Klannerman, S. and Selberg, S.
[Klan 02] *Bilinear Estimates and Applications to Nonlinear Wave Equations*, Commun. Contemp. Math, 4, No. 2, 223-295, 2002.
Lehmann, H., Symanzik, K. and Zimmerman
[LSZ 55] , W., *Nuovo Cimento*, **1**, (1055).

Lindblad, H.

[Lin 96] "*Counterexamples to Local Existence for Semi-linear Wave Equations*, Amer. J. Math. 118, 1 – 16, 1996.

Reed, M and Simon, B

[Reed 75] "Fourier Analysis", Academic Press Ltd, London, 1975.
[Reed 79] "Scattering Theory", Academic Press Ltd, London, 1979.
[Reed 80) "Functional Analysis", Academic Press Ltd, London, 1980.

Roman, P.

[Roman 69] *"*Introduction to Quantum Field Theory*"*, John Wiley & Sons, New York, N.Y. 1969.

Ruelle, D

[Ruelle 62] . "*On the asymptotic condition in Quantum Field Theory,*" Helv. Phys. Acta. **35**, 147 (1962).

Shirikov, M.I.

[Shirikov 07] *"Dressing and Haag's Theorem,"* arXiv.math-ph/0703021v1, 6 March 2007.

Streater, Raymond F. and Wightman, Arthur S.

[Streater 00], [Streater 64] *"*PCT, Spin and Statistics, and All That*"*, Princeton University Press, Princeton, New Jersey, 2000. Originally W.A. Benjamin, Reading, Mass, 1964.

Strocchi, F.

[Strocchi 93] *"*General Properties of Quantum Field Theory*"*, Lecture Notes in Physics, Vol 51, World Scientific, Singapore, 1993.

Wightman, A.S

[Wightman 70] , in Partial Differential Equations, ed. by Avez, Van Nostrand, 1970.

APPENDICES

APPENDIX 1

THEOREMS AND DEFINITIONS

Dominated Convergence Theorem: If $f_n \in L^1(M,d\mu)$, $\lim_{n\to\infty} f_n(x) = f(x)$ a.e. in $[\mu]$, and if there is a $G \in L^1$ with $|f_n(x)| \le G(x)$ a.e. in $[\mu]$, for all n, then $f \in L^1$ and $\lim_{n\to\infty} \|f - f_n\|_1 = 0$.

Denote by $\mathscr{L}(\mathcal{H},\mathcal{H}')$ the set of bounded linear transformation from one Hilbert space $\mathcal{H}$ to another $\mathcal{H}'$. $\mathscr{L}$ is a vector space and becomes a Banach space under the norm $\|T\| = \sup_{\|x\|_{\mathcal{H}}=1} \|Tx\|_{\mathcal{H}'}$.

Definition: The space $\mathscr{L}(\mathcal{H},\mathbf{C})$ is called the dual space of $\mathcal{H}$ and is denoted by $\mathcal{H}^*$. The elements of $\mathcal{H}^*$ are called continuous linear functionals.

Definition: A Hilbert space which has a countable dense subset is said to be separable.

Theorem: A Hilbert space $\mathcal{H}$ is separable if and only if it has a countable orthonormal basis S.

Riesz Lemma: For each $T \in \mathcal{H}^*$, there is a unique $y_T \in \mathcal{H}$ such that $T(x) = (y_T,x)$ for all $x \in \mathcal{H}$. In addition $\|y_T\|_{\mathcal{H}} = \|T\|_{\mathcal{H}^*}$

Definition: A quadratic form is a map $q: Q(q) \times Q(q) \to \mathbf{C}$ where $Q(q)$ is a dense linear subset of H called the form domain, such that $q(\ .\ ,\psi)$ is conjugate linear and $q(\varphi,\ .\ )$ is linear for $\varphi,\psi \in Q(q)$. If $q(\varphi,\psi) = \overline{q(\psi,\varphi)}$ we say that q is symmetric.

A consequence of the Riesz lemma is that there is a one-to-one correspondence between bounded quadratic forms and bounded operators. What is the relation if q is unbounded? This introduces the great utility of quadratic forms to formal Quantum Field Theory.

Let P be a self-adjoint operator on H. In a spectral representation, P becomes multiplication by x on $\oplus_{n=1}^N L^2(\mathbf{R},\mu_n)$. Let $Q(q) = \left\{ \{\psi_n(x)\}_{n=1}^N \middle| \sum_{n=1}^N \int_{-\infty}^{\infty} |x| |\psi_n(x)|^2 d\mu_n < \infty \right\}$ and for $\psi, \varphi \in Q(q)$ define $q(\varphi,\psi) = \sum_{n=1}^N \int_{-\infty}^{\infty} \overline{x\varphi_n(x)} \psi_n(x) d\mu_n$. Here, q is the **quadratic form** associated with the operator P and one writes $Q(q) = Q(\mathrm{P})$. $Q(\mathrm{P})$ is called the **form domain** of the operator P [Reed 80].

These formal definitions may detract from the utility of these objects in Quantum Field Theory. The ψ and φ above will be vectors in Fock space representing a state or a given number of particles. The operator P could be the Hamiltonian, or any other self adjoint operator of interest (confusingly labeled by φ and ψ in the text of this dissertation). The great utility of these objects is as follows:

Definition: Let q be a semibounded quadratic form, (that is $q(\psi,\psi) \geq -m\|\psi\|^2$). q is called closed if $Q(q)$ is complete under the norm $\|\psi\|_{+1} = \sqrt{q(\psi,\psi)+(M+1)\|\psi\|^2}$. If q is closed and $D \subset Q(q)$ is dense in $Q(q)$ in the $\|\ .\ \|_{+1}$ norm, then D is called a **form core** for q.

Theorem: The form q associated with a semibounded self-adjoint operator is closed. Furthermore, any operator core for P is a form core for q.

Theorem: If a semibounded quadratic form is self-adjoint, it must be closed and symmetric.

This is a very handy fact, but in the mathematical analysis of quantum field theory the most useful observation is the following:

For symmetric operators there is no problem finding closed extensions. A smallest closed extension (e.g. the double adjoint) always exists. But, it may be that NONE of these extensions is self adjoint. On the other hand, semi-bounded quadratic forms need not have any closed extension. But when they exist and are semi-bounded, then they are quadratic forms associated with self-adjoint operators.

APPENDIX 2

FREE QUANTUM FIELDS

We have introduced the formal apparatus necessary to rigorously define a quantum field. We will describe here the Fermion field where the operators are bounded and therefore somewhat different to deal with than Boson fields. Boson fields are described in nearly every standard reference, and the one we have used to profit is [Reed 75]. For this reason, and because the Fermion field presents many notational complications we will include its description here rather than the Boson field.

In Chapter 2 we introduced Fock space on a Hilbert space $\mathcal{H}$, and defined two proper subsets according to the symmetric or anti-symmetric interchange of two variables. These two subsets are the Hilbert spaces upon which the "free field" quantum field operators will be defined.

The various smearing functions f_a were defined next. These will always be defined so that they are in L^2 and satisfy the free Klein Gordon equation of mass m. They have nice properties: they are in $\mathscr{S}(\mathbf{R}^4)$, the functions of rapid decrease in configuration space. The necessity of "smearing" is for a physical reason: we restrict the variance of the momentum of a wave packet to be very nearly a single value, having either finite or infinite support, but to be such that it satisfies the Klein Gordon equation. When calculations in configuration space are performed, these are the smearing functions employed. The apparatus we have introduced so far is standard, and typical of the machinery utilized in current research. The definitions and theorems are well known, and since the references are given most of the formal proofs are omitted. The goal here is to arrive at the place where we may state our original work in a common language, and all statements referring to new results will then be rigorously proved. Boson fields correspond in nearly all particulars to the formalism we are introducing here, and operate on $\mathcal{F}_s(\mathcal{H})$.

For Fermion fields let $\mathcal{H}$ be a complex Hilbert space and let $\mathcal{F}_a(\mathcal{H}) = \bigoplus_{n=0}^{\infty} A_n \mathcal{H}^n$ be the anti-symmetric Fock space defined above. Let $f \in \mathcal{H}$ be fixed. For vectors in $\mathcal{H}^n$ of the form $\eta = \psi_1 \otimes \psi_2 \otimes \ldots \otimes \psi_n$ define a map $d^-(f)\colon \mathcal{H}^n \to \mathcal{H}^{n-1}$ by

$$d^-(f)\eta = (f, \psi_1)(\psi_2 \otimes \ldots \otimes \psi_n).$$

$d^-(f)$ is linear and so extends to finite linear combinations of all such η, the extension is well defined and $\left\| d^-(f)\eta \right\| \le \|f\| \cdot \|\eta\|$. $d^-(f)$ thus extends to a bounded map (of norm $\|f\|$) of $\mathcal{H}^n$ into $\mathcal{H}^{n-1}$. This is true for each n, (except n=0 in which case we define $d^-(f)\colon \mathcal{H}^n \to 0$) and so $d^-(f)$ is a bounded operator of norm $\|f\|$ from $\mathcal{F}(\mathcal{H})$ to $\mathcal{F}(\mathcal{H})$. Consider the adjoint d^+ defined as $d^+(f) \equiv (d^-(f))^*$ taking $\mathcal{H}^n$ into $\mathcal{H}^{n+1}$ by the action $d^+(f)\psi_1 \otimes \ldots \otimes \psi_n) = f \otimes \psi_1 \otimes \ldots \otimes \psi_n$ on product vectors. Note, $d^+(f)$ is linear but $f \mapsto d^-(f)$ is anti-linear. To shorten notation, we will write $A_n\mathcal{H}^n$ from above as $\mathcal{H}_a^{(n)}$ and call this term the *n-particle* subspace of $\mathcal{F}_a(\mathcal{H})$. A vector $\psi = \{\psi^{(n)}\}_{n=0}^{\infty} \in \mathcal{F}_a(\mathcal{H})$ for which $\psi^{(n)}$=0 for all but finitely many n is called a *finite particle vector* in the Fermion subspace. Denote the set of finite particle vectors by G_0. The vector $\Omega_0^f = \langle 1,0,0,\ldots \rangle$ is called the *vacuum*.

Let B be any self-adjoint operator on $\mathcal{H}$ whose domain of self-adjointness is D. Let $D_A = \{\psi \in G_0 | \psi^{(n)} \in \otimes_{k=1}^{n} D$ for each $n\}$ and define $d\Gamma(B)$ on $D_B \cap \mathcal{H}_a^{(n)}$ as

$$d\Gamma(B) = B \otimes I \otimes \cdots \otimes I + I \otimes B \otimes \cdots \otimes I + \cdots + I \otimes \cdots \otimes I \otimes B \quad .$$

Theorem A2.1: The Fermion $d\Gamma(B)$ is self adjoint on D_B.
$d\Gamma(B)$ is called the *second quantization* of *B*.

If $B = I$, then its second quantization $N = d\Gamma(I)$ is essentially self adjoint on G_0 and for $\psi \in \mathcal{H}_a^{(n)}$, $N\psi = n\psi$. *N* is called the *number operator*.

Let *U* be a unitary operator on $\mathcal{H}$. Define $\Gamma(U)$ to be that unitary operator on $\mathcal{F}_a(\mathcal{H})$ which equals $\otimes_{k=1}^{n} U$ when restricted to $\mathcal{H}_a^{(n)}$ for $n > 0$, and which equals the identity on $\mathcal{H}_s^{(0)}$. If e^{itA} is a continuous unitary group on $\mathcal{H}$, then $\Gamma(e^{itA})$ is the group generated by $d\Gamma(I)$, that is $\Gamma(e^{itA}) = e^{itd\Gamma(A)}$. Similar remarks, for this notation, apply to the Boson case.

Definition: The *annihilation operator* $c^-(f)$ on $\mathcal{F}_a(\mathcal{H})$ with domain G_0 is given by

$$c^-(f) = \sqrt{N+1}d^-(f). \tag{A2.1}$$

This operator takes each $(n+1)$-particle subspace into the *n*-particle subspace. Note that for each ψ and η in G_0, $(c^-(f)\psi, \eta) = (\psi, Ac^+(f), \eta)$. [*A* is here the anti-symmetrizing operator] This has the result

$$(c^-(f))^* \uparrow G_0 = Ac^+(f) \tag{A2.2}$$

where the up arrow means "restricted to".

The operator in equation (A2.2) is called a *creation* operator. Both $c^-(f)$ and $(c^-(f))^* \uparrow G_0$ are closable, and their closures will be denoted $c^-(f)$ and $c^{\vartheta}(f)^*$.

Definition: The operator $\Psi_S(f)$ defined on G_0 and denoted

$$\Psi_S(f) = \frac{1}{\sqrt{2}}(c^-(f) + c^-(f)^*) \tag{A2.3}$$

will be called the *Segal Fermion field operator*.

The Segal field operator is symmetric (e.g. Hermitean), and is a real (but not complex) linear map since $f \mapsto c^-(f)$ is antilinear while $f \mapsto (c^-(f))^*$ is linear. Some properties of the creation and annihilation operators are

$$c(f)c(g)^* + c(g)^*c(f) = \langle f, g \rangle, \quad c(if) = ic(f) \tag{A2.4}$$

$$c(f)c(g) + c(g)c(f) = 0.$$

It is convenient to introduce the following

<u>Definition</u>: Let $\mathcal{H}$ be a given complex Hilbert space. A *free Fermion field* over $\mathcal{H}$ is a system $(\mathcal{K}, c, \Gamma, \Omega_0^f)$ consisting of:

1) a complex Hilbert space $\mathcal{K} = \left\langle \mathcal{F}_a(L^2(H_m, d\Omega_m)) \right\rangle$,
2) a linear mapping c from $\mathcal{H}$ to the bounded operators on $\mathcal{K}$ satisfying equations (A2.4),
3) a unit vector Ω_0^f in $\mathcal{K}$ that is cyclic for the $c(f)$, $f \in \mathcal{H}$ and
4) a continuous representation Γ of the unitary group on $\mathcal{H}$ into the unitary group on $\mathcal{K}$ satisfying the conditions of Theorem E.

<u>Theorem A2.2</u>: Let $\mathcal{H}$ be a complex Hilbert space; $\Psi_S(\cdot)$ the corresponding Segal quantization. Then:

(a) (self adjointness) For each $f \in \mathcal{H}$, $\Psi_S(f)$ is essentially self-adjoint on G_0, the finite particle vectors.

(b) (cyclicity of the vacuum) Ω_0^f is in the domain of all finite products $\Psi_S(f_1)\cdots\Psi_S(f_n)$ and the set $\{\Psi_S(f_1)\cdots\Psi_S(f_n)\,\Omega_0 \mid f_i$ and n arbitrary$\}$ and the set of all these products is dense in $\mathcal{F}_a(\mathcal{H})$, (or is total in $\mathcal{F}_a(\mathcal{H})$).

(c) (anti-commutation relations) For each $\Psi \in G_0$ and $f, g \in \mathcal{H}$,

$$\Psi_S(f)\Psi_S(g)\psi + \Psi_S(g)\Psi_S(f)\psi = i\,\mathrm{Im}(f,g)_{\mathcal{H}}\,\psi \tag{A2.5}$$

(d) (continuity) If $f_n \to f$ in $\mathcal{H}$, then

$\Psi_S(f_n)\psi \to \Psi_S(f)\psi$ for all $\psi \in G_0$.

(e) For every unitary operator U on $\mathcal{H}$, $\Gamma(U)$: $D(\overline{\Psi_S(f)}) \to D(\overline{\Psi_S(Uf)})$ and for $\psi \in D(\overline{\Psi_S(Uf)})$,

$\Gamma(U)\overline{\Psi_S(f)}\Gamma(U)^{-1}\psi = \overline{\Psi_S(Uf)}\psi$ for all $\psi \in \mathcal{H}$.

The proof of this theorem is similar to one presented in [Reed 75] page 210. The Fermion operators are bounded. In the above, the overbar refers to closure of the operator. We will use $\Psi_S(f)$ for the closure of the operator as well as the operator itself.

Definition: For each real-valued $f \in \mathscr{S}(\mathbf{R}^4)$ $\quad \Psi_m(f) = \Psi_S(Ef)$
and for arbitrary $f \in \mathscr{S}(\mathbf{R}^4)$ $\quad$ write $\quad \Psi_m(f) = \Psi_m(\mathrm{Re}\, f) + i\Psi_m(\mathrm{Im}\, f)$
with Ef defined as the projection onto mass hyperboloids as in Chapter 4.

The mapping $f \mapsto \Psi_m(f)$ is called the free Fermion field of mass m. With this definition

Theorem A2.3: The quadruple $\left\langle \mathscr{F}_a(L^2(H_m, d\Omega_m)), \Gamma(U_m(\cdot,\cdot)), \Psi_m(\cdot), G_0 \right\rangle$

satisfies the Wightman axioms. For each $f \in \mathscr{S}(\mathbf{R}^4)$

$$\Psi_m((\partial_t^2 - \Delta + m^2)f) = 0 .$$

The Klein Gordon operator might come as a surprise to a casual reader, knowing full well that when we speak of Fermion fields we are speaking of spinors and solutions of the Dirac equation. We have yet to define spinors, but each component thereof is a Fermion field as defined above, and each spinor component is required (for free fields) to solve the Klein Gordon equation; this follows by simple algebra of the Dirac operator.

Fermionic fields are the essential quantum ingredient in the classical spinor field, just as Bosonic fields are inserted in classical vector (or scalar, or tensor) fields. The following is from [Baez 92].

> **Definition:** A *spinor* field $\psi(x)$ is a function on Minkowski space-time $\mathbf{M}$ with values in a finite-dimensional "spin" space. Under a transformation g in the Poincare group $\mathscr{P}$, such a function $\psi(x)$ transforms not only directly, displacing x into $g^{-1}(x)$, but the vector in spin space, $\psi(x)$, is transformed according to a given representation $\mathbf{R}$ of the universal two-fold cover $\overline{\mathscr{L}}$ of the homogeneous Lorentz group $\mathscr{L}$, which we identify with the group $SL(2,\mathbf{C})$ when $dim\, \mathbf{M}$ = 4.

We will always consider normal Dirac spinors, for which the "spin" space is four dimensional. In infinitesimal form, the action of the homogeneous Lorentz group $\boldsymbol{\mathcal{L}}$ can be described as follows: let $Q(x,y)$ denote the symmetric form on **M**, $x \cdot \tilde{y} = x_0 y_0 - x_1 y_1 - x_2 y_2 - x_3 y_3$ and γ denote the unique linear map from **M** (as a vector space) to 4 x 4 matrices, such that $\gamma(x)^2 = Q(x,x) I$. Let the e_j $(j = 0,1,2,3)$ be the vectors in the original four-space whose components are all 0 except 1 in the j^{th} position. $\boldsymbol{\mathcal{L}}$ can be defined as that homogeneous linear transformation leaving Q invariant, and its action on **M** is generated by the vector fields $L_{jk} = \varepsilon_j x_j \partial_k - \varepsilon_k x_k \partial_j$ where $\varepsilon_j = Q(e_j, e_j)$. The spin representation **R** of $\boldsymbol{\mathcal{L}}$ is then defined as that for which the corresponding infinitesimal representation carries L_{jk} into $\frac{1}{2}\gamma_j \gamma_k$.

We may make an intuitive clarification of this idea as follows. Consider [Itzyk 80] an infinitesimal homogeneous Lorentz transformation under which we must have (in the Dirac representation of the γ matrices)

$$\psi(x) \to \psi'(x) = S(\Lambda)\psi(\Lambda^{-1}x) = \psi(x) + \delta\psi(x) \quad \text{where}$$

$$\Lambda = I - \frac{i}{2}\delta\omega_{\alpha\beta} M^{\alpha\beta} \qquad M^{\alpha\beta}{}_{\mu\nu} = i(g^{\alpha}{}_{\nu} g^{\beta}{}_{\nu} - g^{\alpha}{}_{\nu} g^{\beta}{}_{\mu}) \quad \text{for Minkowski metric } g.$$

Here $S(\Lambda) = I - \frac{i}{2}\delta\omega_{\alpha\beta}(\frac{1}{2}[\gamma^{\alpha}\gamma^{\beta}]) = I - \frac{i}{4}\delta\omega_{\alpha\beta}\sigma^{\alpha\beta}$

Then

$$\delta\psi(x) = -\frac{i}{2}\delta\omega_{\alpha\beta}\left[\frac{\sigma^{\alpha\beta}}{2} + i(x^{\alpha}\partial^{\beta} - x^{\beta}\partial^{\alpha})\right]\psi(x)$$

$$= -\frac{i}{2}\delta\omega_{\alpha\beta} L^{\alpha\beta}\psi(x).$$

<u>Definition</u>: Let **D** be the operator in the space of functions over **M** with values in the four dimensional spin space given by $\mathbf{D} = \sum \varepsilon_j \gamma_j \partial_j$. The equation $\mathbf{D}\psi + \mathrm{i}m\psi = 0$ $(m > 0)$ is called the *Dirac* equation.