Locating three-term arithmetic progressions in a closed subset of the unit interval

LOCATING THREE-TERM ARITHMETIC PROGRESSIONS IN A CLOSED SUBSET OF THE UNIT INTERVAL

A Thesis by

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Bachelor of Science, Wichita State University, May 2022

Submitted to the Department of Mathematics, Statistics, and Physics
and the faculty of the Graduate School of
Wichita State University
in partial fulfillment of
the requirements for the degree of
Master of Science

July 2024

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The following faculty members have examined the final copy of this thesis for form and content, and recommend that it be accepted in partial fulfillment of the requirement for the degree of Master of Science with a major in Mathematics.

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ABSTRACT

This thesis investigates a closed subset E of the unit interval $[0, 1]$ supporting a probability measure with size and Fourier decay constraints. We show that if these constraints are met, then E must contain at least one three-term arithmetic progression.

ACKNOWLEDGEMENTS

First and foremost, I would like to express gratitude to my advisor, Dr. Robert Fraser, for his guidance, support, and encouragement throughout my research and writing of this thesis. His insight, expertise, and passion for the field of harmonic analysis has been essential for the entirety of my work.

I am also profoundly grateful to Dr. Buma Fridman who has helped me develop a deep love for the field of analysis and mathematics more broadly. His passion for mathematics and enthusiastic dedication to teaching have been a source of motivation and inspiration for me.

Finally, I owe my deepest appreciation to my wife, Stephanie, for her unwavering support, patience, and love from the beginning of my educational journey. Her constant encouragement has been instrumental in overcoming every challenge along the way.

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CHAPTER 1

Introduction

1.1 Literature Review

In 1953 Roth [8] showed that any subset of the natural numbers with positive upper density will contain infinitely many three-term arithmetic progressions. This result by Roth played a pivotal role in developing the study of arithmetic progressions by providing a result that could be both extended and improved upon. In 1975, Szemerédi [9] expanded upon Roth's theorem to show that any subset of the integers with positive upper density will not only contain infinitely many three-term arithmetic progressions, but infinitely many k-term arithmetic progressions for $k \in \mathbb{N}$.

Much of the work since Roth's Theorem has centered around improving quantitative results. Roth's theorem showed that for $\epsilon > 0$, any subset A of $\{1, 2, \cdots, N\}$ with at least $N \cdot \epsilon$ elements must contain a three-term arithmetic progression if $|A| > C_\epsilon \cdot N$

Extensions have also been made to other spaces as well. In 2008, Green and Tao [5] extended Roth's result to the prime numbers. Here, they showed that the prime numbers contain k-term arithmetic progressions for any $k \in \mathbb{N}$. Some results have also been published extending Roth's Theorem to finite fields. In the space of finite fields, significantly better statements have been shown. In 2017, Ellenberg and Gijswijt [3] showed that the number of elements in the largest subset of $\mathbb{F}_3^n$ that contains no three-term arithmetic progressions is $\mathcal{O}(2.756^n)$. Therefore, any set larger than this will contain a three-term arithmetic progression.

Roth's theorem has also been extended to subsets of $\mathbb{R}$ of fractional dimension. Łaba and Pramanik [6] showed that for a closed set $E \subset \mathbb{R}$ of Hausdorff dimension α, if α is close enough to 1, and if E supports a probability measure μ with certain size and Fourier decay constraints, then E must contain non-trivial three-term arithmetic progressions. The result of this thesis is in line with this result.

1.2 Outline

We begin by stating the main theorem we will prove in this thesis.

Theorem 1.1. *Let $\beta > 0$ and let μ be a probability measure supported on a closed set $E \subset [0,1]$ such that for each $x \in \mathbb{R}$ and $\epsilon > 0$*

1. $\mu(B(x,\epsilon)) \lesssim \epsilon \log^{1/3}(1/\epsilon)$
2. $|\widehat{\mu}(\xi)| \lesssim |\xi|^{-\beta/2}$.

Then the set E contains a three-term arithmetic progression.

Many of the methods used to prove this theorem are adapted from a result by Fraser on three-term arithmetic progressions in a subset of $\mathbb{F}_q^\infty$ with sufficiently large Fourier dimension. [4] We will utilize the following strategy. First, divide the unit interval $[0,1]$ into 2^{n+1} sub-intervals each with Lebesgue measure $2^{-n}/2$. We will need to determine a choice of n so that there will be enough sub-intervals supporting μ which contain points as "near" three-term arithmetic progressions. To find this value of n we will utilize Varnavides's Theorem. We will call this choice n_1. We can then determine a subset of these intervals on which μ is sufficiently large. Using a pigeon-hole argument, we will show this subset is large enough to satisfy Varnavides's Theorem.

If we think of each of the sub-intervals on which μ has large enough support as discrete elements, we will have our result. To do this, we consider the left endpoints of each of our intervals to represent the interval. If we find three-term arithmetic progressions out of intervals with large support, then these endpoints of the intervals represent "near" three-term arithmetic progressions. To show E actually contains three-term arithmetic progressions, we choose n_2 to be significantly larger than n_1. We then convolve μ with a compactly supported smooth function whose support is determined by n_i, $i \in 1,2$. This convolution will effectively smooth out the support of μ. We will then consider a trilinear form Λ_{n_1} which picks out three-term arithmetic progressions. Since we have convolved μ with a function, we can use

this convolution as an input for Λ. We then consider the difference $\Lambda_{n_1}(\mu_{n_1}, \mu_{n_1}, \mu_{n_1}) - \Lambda_{n_1}(\mu_{n_2}, \mu_{n_2}, \mu_{n_2})$ as n_2 gets arbitrarily large. Once we have shown that this difference is small we can conclude that E must contain a three-term arithmetic progression. It should be noted that this strategy can be extended to any closed set $F \subset \mathbb{R}$ by simply rescaling and translating the conditions to more properly fit the desired set.

1.3 Background

We will now state a few theorems, definitions, and properties that will be useful throughout this thesis. First we define a common notation used in this thesis. The expression

$$\alpha \lesssim \beta$$

implies $\alpha < C \cdot \beta$ where C is a constant. The expression

$$\alpha \lesssim_\epsilon \beta$$

implies $\alpha < C_\epsilon \cdot \beta$ where C is a constant depending on ϵ.

Since our goal is to locate three-term arithmetic progressions, and we will be encountering arithmetic progressions of arbitrary length, we will define the general arithmetic progression.

Definition 1.2 (Arithmetic Progression)**.** *A sequence of real numbers is an* ***arithmetic progression*** *if the difference between each successive term remains constant throughout the sequence. An arithmetic progression can be written as*

$$a_n = a_1 + (n-1)d$$

Theorem 1.1 also requires the use of a specific type of measure: a probability measure.

Definition 1.3 (Probability Measure)**.** *Let $(X, \mathcal{M})$ be a couple consisting of a set X and a σ-algebra $\mathcal{M}$ of subsets of X. A measure μ defined on $(X, \mathcal{M})$ is a* ***probability measure*** *if it satisfies the following conditions.*

1. *μ takes on values in $[0,1]$ where $\mu(X) = 1$ and $\mu(\emptyset) = 0$.*

2. *μ is countably additive. Specifically, for all countable collections of pairwise disjoint measurable subsets of $E_1, E_2, E_3, \cdots$ of X*

$$\mu\left(\bigcup_{i\in\mathbb{N}} E_i\right) = \sum_{i\in\mathbb{N}} \mu(E_i).$$

When we take the Fourier transform of a probability measure μ on $\mathbb{R}$, we get several very nice properties. One particularly helpful property is the normalization property. Specifically, this is the fact that $\widehat{\mu}(0) = 1$. The Fourier transform of μ is given by

$$\widehat{\mu}(\xi) = \int_{\mathbb{R}} e^{-2\pi i x\xi} d\mu(x).$$

Therefore,

$$\begin{aligned}\widehat{\mu}(0) &= \int_{\mathbb{R}} e^{-2\pi i x\cdot 0}\, d\mu(x) \\ &= \int_{\mathbb{R}} 1\, d\mu(x) \\ &= 1\end{aligned}$$

This is because the total mass of a probability measure over $\mathbb{R}$ is always equal to 1 by definition.

While proving Theorem 1.1, we need to construct some functions that have a certain decay and smoothness property. Namely, we need these functions to decay faster than any polynomial function. Functions that possess this property are called Schwartz functions.

Definition 1.4 (Schwartz Space)**.** *The **Schwartz space** $\mathcal{S}(\mathbb{R})$ is the space of functions ϕ such that for each $\alpha \in \mathbb{N}$ and $\beta \in \mathbb{N}$ the expression*

$$\sup_{x\in\mathbb{R}} |x|^{\alpha} \left|\frac{d^{\beta}}{dx^{\beta}}\phi(x)\right|$$

is finite. We define this expression to be the seminorm $||\cdot||_{\alpha,\beta}$ on $\mathcal{S}(\mathbb{R})$. In other words, $\phi \in \mathcal{S}(\mathbb{R})$ if and only if ϕ is a smooth function such that ϕ and all of its derivatives decay faster than any polynomial.

Schwartz functions are often useful for the purposes of controlling the decay of other functions and maps. Note that the Fourier transform is a continuous bijection from the Schwartz space to itself, meaning that the Fourier transform of a Schwartz function is also a Schwartz function. Additionally, any compactly supported smooth function is a Schwartz function.

One mathematical operation that we will find helpful is the convolution.

Definition 1.5 (Convolution)**.** *Let f and g be measurable complex-valued functions on $\mathbb{R}$. Then the* ***convolution*** *of f and g is given by*

$$f * g(x) = \int f(y)g(x-y)dy.$$

The convolution operator effectively blends two functions and produces a third function showing how one function modifies the other. Convolution possesses the properties of commutativity, associativity, and distributivity over addition. We are also able to convolve a function f with a measure μ to create a new function given as

$$f * \mu(x) = \int f(x-y)\, d\mu(y).$$

Additionally, if we convolve a Schwartz function with a compactly supported measure, the end result will also be a Schwartz function.

Another particularly helpful property of the convolution is that the Fourier transform of a convolution of two functions is equal to the product of the Fourier transforms of each function. Specifically, assuming f and g are both L^1 functions, we have

$$\begin{aligned}
\widehat{f * g}(\xi) &= \int (f * g)(x) e^{-2\pi i x\xi}\, dx \\
&= \int \Big(\int f(x-y)g(y)dy \Big) e^{-2\pi i x\xi}\, dx \\
&= \int g(y) \Big(\int f(x-y) e^{-2\pi i x\xi}\, dx \Big) dy
\end{aligned}$$

Here, make a change of variables letting $u = x - y$. Therfore, $du = dx$ and $x = u + y$. We

then have

$$\begin{aligned}
\widehat{f * g}(\xi) &= \int g(y)\Big(\int f(u) e^{-2\pi i(u+y)\xi}\, du\Big)\, dy \\
&= \int g(y)\Big(\int f(u) e^{-2\pi i u\xi} e^{-2\pi i y\xi}\, du\Big)\, dy \\
&= \int g(y) e^{-2\pi i y\xi} \widehat{f}(\xi)\, dy \\
&= \widehat{f}(\xi) \int g(y) e^{-2\pi i y\xi}\, dy \\
&= \widehat{f}(\xi)\widehat{g}(\xi).
\end{aligned}$$

We shall also find the following theorem quite useful. We state it here without proof.

Theorem 1.6. *(Plancherel's Theorem) The linear map $\mathcal{F}$ defined on $L^1(\mathbb{R}) \cap L^2(\mathbb{R})$ given by*

$$\mathcal{F}f = \widehat{f}$$

has a unique extension to all of L^2 as a bounded linear operator. The operator $\mathcal{F}$ is an isometry; that is, for all $f \in L^2(\mathbb{R})$ we have

$$||\mathcal{F}f||_{L^2(\mathbb{R})} = ||f||_{L^2(\mathbb{R})}$$

Additionally, the operator T defined on $L^1(\mathbb{R}) \cap L^2(\mathbb{R})$ by

$$Tf(\xi) = \int f(x) e^{2\pi i x\xi}\, dx$$

has a unique extension to L^2. Lastly, we have $T = \mathcal{F}^{-1}$.

Lastly we shall prove a few basic properties of the Fourier transform. The first property we will prove is the scaling property.

Theorem 1.7. *(Scaling Property) If $f(x)$ has the Fourier transform $\widehat{f}(x)$ then the Fourier transform of $f(\gamma x)$ is given by*

$$\mathcal{F}(f(\gamma\, \cdot))(\xi) = \frac{1}{\gamma}\widehat{f}\Big(\frac{\xi}{\gamma}\Big)$$

where $\mathcal{F}$ denotes the Fourier transform.

Proof. Let $g(x) = f(\gamma x)$. Then

$$\begin{aligned}\widehat{g}(\xi) &= \int g(x)e^{-2\pi i x\xi}\,dx \\ &= \int f(\gamma x)e^{-2\pi i x\xi}\,dx.\end{aligned}$$

Now let $u = \gamma x$. Therefore $x = \frac{u}{\gamma}$ and $du = \gamma dx$ and $dx = \frac{du}{\gamma}$. We then have

$$\begin{aligned}\widehat{g}(\xi) &= \int \frac{f(u)e^{-2\pi i\frac{u}{\gamma}\xi}}{\gamma}\,du \\ &= \frac{1}{\gamma}\int f(u)e^{-2\pi i\frac{u}{\gamma}\xi}\,du \\ &= \frac{1}{\gamma}\widehat{f}\Big(\frac{\xi}{\gamma}\Big)\end{aligned}$$

□

Next we show the translation property.

Theorem 1.8. *(Translation Property) If $f(x)$ has the Fourier transform $\widehat{f}(\xi)$ then the Fourier transform of $f(x-\nu)$ is given by*

$$\mathcal{F}(f(\cdot - \nu))(\xi) = e^{-2\pi i\nu\xi}\widehat{f}(\xi)$$

where $\mathcal{F}$ denotes the Fourier transform.

Proof. let $g(x) = f(x-\nu)$. Therefore

$$\begin{aligned}\widehat{g}(\xi) &= \int g(x)e^{-2\pi i x\xi}\,dx \\ &= \int f(x-\nu)e^{-2\pi i x\xi}\,dx.\end{aligned}$$

Now let $u = x - \nu$. Therefore $x = u + \nu$ and $du = dx$. We then have

$$\begin{aligned}\widehat{g}(\xi) &= \int f(u)e^{-2\pi i(u+\nu)\xi}\,du \\ &= \int f(u)e^{-2\pi i u\xi}e^{-2\pi i\nu\xi}\,du \\ &= e^{-2\pi i\nu\xi}\widehat{f}(\xi)\end{aligned}$$

□

CHAPTER 2

Discretization

2.1 Dividing the Interval

Our first task is to divide the unit interval $[0,1]$ into 2^{n+1} sub-intervals each with Lebesgue measure $\frac{1}{2^{n+1}}$. We wish to treat each interval discrete element. To do this, we consider the left endpoint of each interval where we have a bijection from the set of left endpoints to the set of dyadic sub-intervals. If we are able to show that there are three-term arithmetic progressions consisting of these intervals as discrete elements, then we will have collections of triplets of left endpoints where each point is within $2^{-n}/2$ of a point in an actual three-term arithmetic progression.

Next, let $\mathcal{I}$ be the full collection of dyadic sub-intervals I_i of $[0,1]$ each with Lebesgue measure $2^{-n}/2$. Let $\mathcal{I}'$ be the set of sub-intervals on which μ has large support. We now need to make a choice of n so that we can guarantee the presence of three term arithmetic progressions in $\mathcal{I}'$. To do this, we will utilize a modification of a theorem by Varnavides.

Theorem 2.1. *Let Ω be a subset of $\{1,\cdots,N\}$. for $\epsilon > 0$, if*

$$|\Omega| \geq \frac{N}{\log^{2/5}(N)}, \tag{2.1}$$

then the number of three-term arithmetic progressions in Ω is $\gtrsim_\epsilon N^{2-\epsilon}$.

The proof of this variant of Varnavides's Theorem relies heavily on an extension of Roth's theorem by Bloom and Sisask. [2]

Theorem 2.2. *Let $N \geq 2$ and $A \subset \{1,2,\ldots,N\}$ be such that $|A| \geq \frac{N}{\log(N)}$. Then A contains a three-term arithmetic progression.*

Let $\delta > 0$ and let $P \subset \{1,2,\ldots,N\}$ be an arithmetic progression of length k which will be chosen at a later step. Let $\mathcal{P}_k$ be the set of all progressions P of length k. We will

consider the sets $\Omega \cap P$. We will now need to prove several lemmas in order to provide a proof of Theorem 2.1.

Lemma 2.3. *The total number of elements of $\mathcal{P}_k$ is at most $\lesssim \frac{N^2}{k}$.*

Proof. We define a progression P to be

$$x_0, x_0 + d, \ldots, x_0 + (k-1)d$$

where x_0 is the starting point of P and $d > 0$. Since

$$x_0 + (k-1)d \leq N$$

we have the number of starting places is given by $N - (k-1)d$. Observe that if we assume $d < \frac{N}{2k}$ then we see that $N - (k-1)d \geq N/3$. We then have

$$\sum_{1 \leq d \leq N/k} N - (k-1)d \geq \sum_{1 \leq d \leq N/(2k)} \frac{N}{3} = \frac{N^2}{6k}$$

and

$$\sum_{1 \leq d \leq N/k} N - (k-1)d \leq \sum_{1 \leq d \leq N/k} N = \frac{N^2}{k}.$$

Therefore

$$\frac{N^2}{6k} \leq |\mathcal{P}_k| \leq \frac{N^2}{k}.$$

□

We define an incidence to be a pair $(\omega, P) \in \Omega \times \mathcal{P}_k$ such that $\omega \in P$.

Lemma 2.4. *The number of incidences is at least $\gtrsim |\Omega|^2$.*

Proof. Let $\omega \in \Omega$. We want to obtain a lower bound on the number of incidences that ω will contribute. Specifically, we want to find the number of progressions $P \in \mathcal{P}_k$ that will contain ω. There are N/k possible step sizes d for the progression P.

We now want to count the number of possible positions in P for which we can find ω. Without loss of generality, by symmetry we can choose $\omega \leq N/2$. We will only consider step sizes $d < \frac{N}{100 \cdot k}$. Let

$$\omega = x_0 + n \cdot d.$$

We then have

$$\begin{aligned} n &= \frac{\omega - x_0}{d} \\ &\leq \frac{\omega - 1}{d}. \end{aligned}$$

We also have $n \leq k$. Therefore the number of positions of P where ω can appear is

$$\min(k, \lceil (\omega - 1)/d \rceil).$$

We now narrow our choice of ω so that ω is far from the endpoints 1 and N. Ω contains at least $|\Omega|/2$ points that are at least $|\Omega|/4$ from both 1 and N. To see this let $|\Omega| = \alpha$. We can divide the set $\{1, 2, \cdots, N\}$ into the regions of points within $\alpha/4$ of 1, points within $\alpha/4$ of N and those that are at least $\alpha/4$ away from both 1 and N. We can write the first region as

$$\{1, 2, \cdots \lfloor \alpha/4 \rfloor\}.$$

The second region can be expressed as

$$\{N - \lfloor \alpha/4 \rfloor + 1, N - \lfloor \alpha/4 \rfloor + 2, \cdots, N\}.$$

Each of these regions cover $2 \cdot \lfloor \alpha/4 \rfloor$ points. Therefore at most $\alpha/2$ points can be considered in the excluded regions and there must be at least $\alpha/2$ points at least $\alpha/4$ away from both 1 and N simultaneously. Call this set of points Ω'. Suppose $\omega \in \Omega'$. For each $1 \leq d \leq \frac{N}{100 \cdot k}$ the number of progressions of step size d that contain ω is $\gtrsim \min(k, \omega/d)$. In the case where $d \leq \omega/k$ the minimum will be k. However, $\omega \geq \frac{1}{4}|\Omega|$. Therefore the minimum will be k so long as $d \leq \frac{|\Omega|}{100 \cdot k}$. In this case ω will be contained in k progressions of step size equal to d. Therefore the total number of elements in $\mathcal{P}_k$ that contain ω is at least $k \cdot \frac{|\Omega|}{100 \cdot k} = \frac{|\Omega|}{100}$. Since there are $\frac{|\Omega|}{2}$ choices for ω, the total number of incidences is at least $\gtrsim |\Omega|^2$. $\square$

Now let $\mathcal{P}_k^1$ be the set of progressions P such that $|\Omega \cap P| > \frac{k}{\log(k)}$. Let $\mathcal{P}_k^2$ be the rest of all progressions P. Our choice in defining $\mathcal{P}_k^1$ allows us to use Roth's theorem on these progressions to ensure the presence of three-term arithmetic progressions in the intersection with Ω. We now prove one last lemma before we prove Theorem 2.1.

Lemma 2.5. *For any $\delta > 0$ each three-term arithmetic progression is contained in at most $\lesssim_\delta N^\delta k$ elements of $\mathcal{P}_k$.*

Proof. let $\delta > 0$. Let Q be a three-term arithmetic progression contained in Ω. Let d be the common difference of Q. We now count the number of k-term arithmetic progressions P containing Q. Let s be the common difference of P. We now find an upper bound on the number of possible choices for s. The step size s must be a divisor of d. Otherwise P could not contain Q. Let σ be the divisor function. We then have $\sigma(d) \lesssim_\delta d^\delta \lesssim_\delta N^\delta$. [1] Additionally, for a given common difference s, the first element of Q can occur in at most k positions of P; hence, at most k progressions Q of step size s contain P. Therefore the total number of progressions containing Q is at most $\lesssim N^\delta \cdot k$. □

We can now prove our version of Varnavides's Theorem.

Proof of Theorem 2.1. We have split the progressions $\mathcal{P}_k$ into two subsets $\mathcal{P}_k^1$ and $\mathcal{P}_k^2$. Recall that $\mathcal{P}_k^1$ is defined as the set of progressions that contain at least $\frac{k}{\log(k)}$ elements of Ω. We can say that the total number of incidences is $\gtrsim |\Omega|^2$ and is less than or equal to $k \cdot |\mathcal{P}_k^1| + \frac{k}{\log(k)}|\mathcal{P}_k^2|$. We can use these facts to obtain a lower bound on $|\mathcal{P}_k^1|$. We know that $|\mathcal{P}_k^2| \leq |\mathcal{P}_k|$ and that $|\mathcal{P}_k| \lesssim \frac{N^2}{k}$. Therefore

$$
\begin{aligned}
|\Omega|^2 &\lesssim k|\mathcal{P}_k^1| + \frac{k}{\log k}\frac{N^2}{k} \\
|\Omega|^2 &\lesssim k|\mathcal{P}_k^1| + \frac{N^2}{\log k} \\
\frac{|\Omega|^2 - \frac{N^2}{\log k}}{k} &\lesssim |\mathcal{P}_k^1| \\
\frac{\frac{N^2}{\log^{4/5} N} - \frac{N^2}{\log k}}{k} &\lesssim |\mathcal{P}_k^1|
\end{aligned}
$$

We can then choose $k = N^\delta$. Therefore

$$\frac{\frac{N^2}{\log^{4/5} N} - \frac{N^2}{\delta \log N}}{N^\delta} \lesssim |\mathcal{P}_k^1|$$

Therefore, if we choose N appropriately we can conclude that $|\mathcal{P}_k^1| \gtrsim N^{2-\delta}$. Our choice of N is determined as follows:

$$\begin{aligned}
&\frac{1}{\log^{4/5}(N)} \gg \frac{1}{\delta \log(N)} \\
&\implies \delta \log(N) \gg \log^{4/5}(N) \\
&\implies \delta^5 \log(N) \gg 1 \\
&\implies N \gg e^{1/\delta^5}.
\end{aligned}$$

Each element of $\mathcal{P}_k^1$ contains a three-term arithmetic progression of elements of Ω by Roth's theorem. Therefore we can count the number of three-term arithmetic progressions in Ω by dividing the number of elements of $\mathcal{P}_k^1$ by the number of times a given three-term arithmetic progression appears inside an element of $\mathcal{P}_k^1$. This implies the number of three-term arithmetic progressions in Ω is at least $\gtrsim_\delta \frac{|\mathcal{P}_k^1|}{N^\delta k} \gtrsim_\delta \frac{N^{2-\delta}}{N^{2\delta}} = N^{2-3\delta}$. Here we can choose δ small enough so that $N^{2-3\delta} \leq N^{2-\epsilon}$ □

Now we choose n such that 2^n will satisfy condition 2.1 for a given $\epsilon > 0$. Call this choice n_1. We define $\mathcal{I}$ to be the set sub-intervals I_i of $[0,1]$ and $\mathcal{I}'$ to be the sub-intervals I_k of $\mathcal{I}$ such that $\mu(I_k) \geq \frac{2^{-n_1}}{100}$. Our next goal is to show that $\mathcal{I}'$ is large enough so we can use Theorem 2.1.

Lemma 2.6. *Let $\mathcal{I}'$ be the subset of sub-intervals of the unit interval such that for each $I_k \in \mathcal{I}'$, $\mu(I_k) \geq \frac{2^{-n_1}}{100}$. Then $|\mathcal{I}'| \geq \frac{2^{n_1}}{\log^{2/5}(2^{n_1})}$.*

Proof. The proof of this lemma will utilize the pigeonhole principle. Since μ is a probability measure, $\mu([0,1]) = 1$. Assume there are k intervals with measure at least $\frac{2^{-n_1}}{100}$. All other

intervals will have measure less than this. We then have

$$\begin{aligned}
\mu\left(\bigcup_{I_i\in\mathcal{I}} I_i\right) = 1 &\leq k\cdot(2^{-n_1}\log^{1/3}(2^{n_1})) + (2^{n_1}-k)\left(\frac{2^{-n_1}}{100}\right)\\
&\implies 1 \leq k\cdot(2^{-n_1}\log^{1/3}(2^{n_1})) + \frac{1}{100}\\
&\implies \frac{99}{100} \leq k\cdot(2^{-n_1}\log^{1/3}(2^{n_1}))\\
&\implies k \geq \left(\frac{99}{100\cdot\log^{1/3}(2^{n_1})}\right)2^{n_1}\\
&\geq \frac{2^{n_1}}{\log^{2/5}(2^{n_1})}
\end{aligned}$$

Here, the first line utilizes the ball condition 1 of Theorem 1.1:

$$\mu(B(x,\frac{1}{2^{n_1}})) \lesssim \frac{1}{2^{n_1}}\log^{1/3}(2^{n_1}).$$

□

Therefore, considering each interval as a discrete element, we are able to use Theorem 2.1 to guarantee the set of intervals with large measure contains a three-term arithmetic progression.

CHAPTER 3

Introduction and Lower Bounds on a Trilinear Form

3.1 Introduction of the Trilinear Form

We start by introducing ψ, a compactly supported, smooth function supported on the interval $[-1,1]$. We then renormalize this function to obtain the function

$$\psi_{n_1}(x) = \frac{2}{5-2^{-n_1(1+\epsilon)}}\psi\left(\frac{2}{5-2^{-n_1(1+\epsilon)}}\Big(x-(\frac{2^{-n_1(1+\epsilon)}+5}{2})\Big)\right).$$

Note that the support for ψ_{n_1} is on the interval $[2^{-n_1(1+\epsilon)},5]$. We now show an important bound for this function.

Lemma 3.1. *Let* $\psi_{n_1}(x) = \frac{2}{5-2^{-n_1(1+\epsilon)}}\psi\left(\frac{2}{5-2^{-n_1(1+\epsilon)}}\Big(x-(\frac{2^{-n_1(1+\epsilon)}+5}{2})\Big)\right)$. *be a compactly supported, smooth, Schwartz function supported on* $[2^{-n_1(1+\epsilon)},5]$. *Then*

$$|\widehat{\psi_{n_1}}(\xi)| \lesssim (1+|\xi|)^{-100}$$

Where the implied constant is independent of n_1.

Proof. let $\gamma = \frac{2}{5-2^{-n_1}}$ and $\nu = \frac{2^{-n_1}+5}{2}$. We then have

$$\psi_{n_1} = \gamma\psi(\gamma(x-\nu)).$$

By the scaling property and the translation property of the Fourier transform we have

$$\begin{aligned}\widehat{\psi_{n_1}}(\xi) &= \gamma e^{-2\pi i\nu\xi}\frac{1}{\gamma}\psi(\frac{\xi}{\gamma})\\ &= e^{-2\pi i\frac{2^{-n_1}+5}{2}\xi}\widehat{\psi}\Big(\frac{5-2^{-n_1(1+\epsilon)}}{2}\xi\Big).\end{aligned}$$

Taking the absolute value of both sides, we obtain

$$\begin{aligned}|\widehat{\psi_{n_1}}(\xi)| &= \left|e^{-2\pi i\frac{2^{-n_1}+5}{2}\xi}\widehat{\psi}\Big(\frac{5-2^{-n_1(1+\epsilon)}}{2}\xi\Big)\right|\\ &= \left|\widehat{\psi}\Big(\frac{5-2^{-n_1(1+\epsilon)}}{2}\xi\Big)\right|.\end{aligned}$$

Here, $\frac{5-2^{-n_1}}{2}$ is bounded below by 2 when n_1 is small and bounded above by $\frac{5}{2}$ when n_1 is large. Because ψ is a Schwartz function, $\widehat{\psi}$ is also a Schwartz function. Therefore we can say

$$\widehat{\psi}(\xi) \lesssim (1+|\xi|)^{-100}.$$

Therefore

$$\begin{aligned}\widehat{\psi_{n_1}}(\xi) &\lesssim \left(1+\left|(\frac{5-2^{-n_1(1+\epsilon)}}{2})\xi\right|\right)^{-100} \\ &\lesssim (1+|\xi|)^{-100}.\end{aligned}$$

□

This gives us a bound for $\widehat{\psi_{n_1}}$ which is independent of our choice of n_1. We now introduce the following trilinear form.

$$\Lambda_{n_1}(f,g,h) = \int_{a>2^{-n_1(1+\epsilon)}} \psi_{n_1}(a) \int f(x)g(x+a)h(x+2a)\,dx\,da$$

Here, $a \in \mathbb{R}$ is positive. We now define another compactly supported smooth function ϕ supported on $[-1,1]$. Here, $\phi(0)=1$, $\int \phi = 1$, and $\phi(x) \geq \frac{1}{2}$ whenever $|x| \leq \frac{1}{2}$. Since ϕ and ψ_{n_1} are compactly supported smooth functions, they possess the particularly useful property that their Fourier transforms will be Schwartz functions. This means that we will be able to utilize these functions to control the decay of Λ. We next consider the L^1- normalized dilation of ϕ given by

$$\phi_k = 2^k \phi(2^k x).$$

We can now take the Fourier transform of ϕ_k.

$$\begin{aligned}\widehat{\phi}_k(\xi) &= \int 2^k \phi(2^k x) e^{-2\pi i x\xi}\,dx \\ &= \int \phi(u) e^{-2\pi i 2^{-k} u\xi}\,du \\ &= \widehat{\phi}(2^{-k}\xi).\end{aligned}$$

We now define the following convolution:

$$\mu_{n_k} = \mu * \phi_k.$$

In a certain sense, this convolution smooths out the support of μ. Additionally, the convolution of μ with a function ϕ_k is a genuine function. Therefore we can consider it as an input for Λ. Consider

$$\Lambda_{n_1}(\mu_{n_k}, \mu_{n_k}, \mu_{n_k}) = \int_{a>2^{-n_1(1+\epsilon)}} \psi_{n_1}(a) \int \mu_{n_k}(x)\mu_{n_k}(x+a)\mu_{n_k}(x+2a)\, dx\, da.$$

For small n_k, the support of μ_{n_k} is quite spread out. Therefore if $\Lambda(\mu_{n_k}, \mu_{n_k}, \mu_{n_k})$ is positive, we have near three-term arithmetic progressions but our error is quite large. As n_k gets large, the support of μ_{n_k} tightens. Therefore if $\Lambda(\mu_{n_k}, \mu_{n_k}, \mu_{n_k}) > 0$ we know we have near three-term arithmetic progressions with an error that has an inverse relationship to n_k.

3.2 Writing Λ in Terms of the Fourier Transform

We now express $\Lambda_{n_1}(\mu * \phi_k, \mu * \phi_k, \mu * \phi_k)$ in terms of the Fourier transform. For simplicity, we use the notation $\mu_i = \mu * \phi_{n_i}$ where $i \in \{1, 2\}$.

$$\begin{aligned}
\Lambda_{n_1}(\mu_i, \mu_i, \mu_i) &= \int_{a>2^{-n_1(1+\epsilon)}} \psi_{n_1}(a) \int \mu * \phi_i(x)\mu * \phi_i(x+a)\mu * \phi_i(x+2a)\, dx\, da \\
&= \int_{a>2^{-n_1(1+\epsilon)}} \psi_{n_1}(a) \int \widehat{\mu_i(\cdot)}(\xi)\overline{\widehat{\mu_i(\cdot+a)\mu_i(\cdot+2a)}(\xi)}\, d\xi\, da \\
&= \int_{a>2^{-n_1(1+\epsilon)}} \psi_{n_1}(a) \int \widehat{\mu_i(\cdot)}(\xi)\overline{\widehat{\mu_i(\cdot+a) * \mu_i(\cdot+2a)}(\xi)}\, d\xi\, da \\
&= \int_{a>2^{-n_1(1+\epsilon)}} \psi_{n_1}(a) \int \widehat{\mu_i(\cdot)}(\xi)\overline{\int \widehat{\mu_i(\cdot+a)}(\eta)\widehat{\mu_i(\cdot+2a)}(\xi-\eta)\, d\xi\, d\eta\, da}
\end{aligned}$$

From here, note that we can rewrite $\widehat{\mu_i(\cdot+a)}(\eta)$ and $\widehat{\mu_i(\cdot+2a)}(\xi-2\eta)$ using Theorem 1.8 to obtain

$$\begin{aligned}
\Lambda_{n_1}(\mu_i, \mu_i, \mu_i) &= \int_{a>2^{-n_1(1+\epsilon)}} \psi_{n_1}(a) \int \widehat{\mu_i}(\xi)\overline{\int e^{2\pi i a\eta}\widehat{\mu_i(\eta)}e^{4\pi i a(\xi-\eta)}\widehat{\mu_i(\xi-\eta)}\, d\xi\, d\eta\, da} \\
&= \int_{a>2^{-n_1(1+\epsilon)}} \psi_{n_1}(a) \int \widehat{\mu_i}(\xi) \int e^{-2\pi i a\eta}\widehat{\mu_i(-\eta)}e^{-4\pi i a(\xi-\eta)}\widehat{\mu_i(\eta-\xi)}\, d\xi\, d\eta\, da
\end{aligned}$$

Here $-2\pi i a\eta - 4\pi i a(\xi-\eta) = -2\pi i a(2\xi-\eta)$. Therefore

$$\begin{aligned}
\Lambda_{n_1}(\mu_i, \mu_i, \mu_i) &= \iint \widehat{\mu_i}(\xi)\widehat{\mu_i}(-\eta)\widehat{\mu_i}(\eta-\xi) \int_{a>2^{-n_1(1+\epsilon)}} e^{-2\pi i a(2\xi-\eta)}\psi_{n_1}(a)\, d\xi\, d\eta\, da \\
&= \iint \widehat{\mu_i}(\xi)\widehat{\mu_i}(-\eta)\widehat{\mu_i}(\eta-\xi)\widehat{\psi_{n_1}}(2\xi-\eta)\, d\xi\, d\eta.
\end{aligned}$$

Let $\theta = 2\xi - \eta$. Therefore we have $-\eta = \theta - 2\xi$ and $\eta - \xi = \xi - \theta$. This is to write everything in terms of θ and ξ. We can then write our trilinear form as

$$\Lambda_{n_1}(\mu_i, \mu_i, \mu_i) = \iint \widehat{\mu_i}(\xi)\widehat{\mu_i}(\theta - 2\xi)\widehat{\mu_i}(\xi - \theta)\widehat{\psi_{n_1}}(\theta)\, d\xi\, d\theta.$$

3.3 Lower Bounds on Λ

Our next goal is to demonstrate a lower bound on Λ. Recall that we have defined $\mathcal{I}'$ to be the set of intervals I_k such that $\mu(I_k) \geq \frac{2^{-n_1}}{100}$.

Lemma 3.2. *If the number of three-term arithmetic progressions in $\mathcal{I}'$ is $\gtrsim 2^{n_1(2-\epsilon)}$ then $\Lambda_{n_1}(\mu_{n_1}, \mu_{n_1}, \mu_{n_1}) \gtrsim 2^{-n_1\epsilon}$.*

Proof. Assume $\mathcal{I}'$ has $N^{2-\epsilon}$ three-term arithmetic progressions.

$$\begin{aligned}\Lambda_{n_1}(\mu_{n_1}, \mu_{n_1}, \mu_{n_1}) &= \int_{a > 2^{-n_1(1+\epsilon)}} \psi_{n_1}(a) \int \mu_{n_1}(x)\mu_{n_1}(x+a)\mu_{n_1}(x+2a)\, dx\, da \\ &\gtrsim \sum_{I_1, I_2, I_3} 2^{3\cdot n_1} \iint 1_{I_1}(x) 1_{I_2}(x+a) 1_{I_3}(x+2a) \mu(I_1)\mu(I_2)\mu(I_3)\, dx\, da\end{aligned}$$

To see why this holds, consider

$$\begin{aligned}\mu_k(x) &= \mu * \phi_k \\ &= \int \phi_n(x - \lambda)\, d\mu(\lambda) \\ &= 2^n \int \phi(2^n(x - \lambda))\, d\mu(\lambda)\end{aligned}$$

Recall that we have defined ϕ so that $\phi(0) = 1$, and $\phi(x) \geq 1/2$ whenever $|x| \leq 1/2$. Therefore, whenever $|x - \lambda| \leq \frac{2^{-n}}{2}$ we have $\phi(2^n(x - \lambda)) \geq 1/2$. Therefore, we have

$$\begin{aligned}2^n \int \phi(2^n(x - \lambda))\, d\mu(\lambda) &\geq 2^{n_1} \int_{B(x, \frac{2^{-n}}{2})} \frac{1}{2}\, d\mu(\lambda) \\ &= 2^n \cdot \frac{1}{2}\mu(B(x, \frac{2^{-n}}{2}))\end{aligned}$$

Therefore, since the intervals I_k are chosen so that the length of each interval is $2^{-n}/2$, the ball $B(x, \frac{2^{-n}}{2})$ will contain that interval I containing x. Therefore, we can say

$$\mu_n(x) \geq \frac{1}{2} \cdot 2^n \sum_I 1_I(x)\mu(I).$$

Applying this argument to both $x+a$ and $x+2a$ will give us the bound we need. Next we have

$$\begin{aligned}
\Lambda_{n_1}(\mu_{n_1},\mu_{n_1},\mu_{n_1}) &\gtrsim \sum_{I_1,I_2,I_3\,\in\,\mathcal{I}'} 2^{3\cdot n_1} \iint 1_{I_1}(x)1_{I_2}(x+a)1_{I_3}(x+2a)(100\cdot 2^{n_1})^{-3}\,dx\,da \\
&\gtrsim \sum_{I_1,I_2,I_3\,\in\,\mathcal{I}'} 2^{3\cdot n_1}2^{-3\cdot n_1} \iint 1_{I_1}(x)1_{I_2}(x+a)1_{I_3}(x+2a)\,dx\,da \\
&\gtrsim \sum_{\substack{I_1,I_2,I_3\,\in\,\mathcal{I}' \\ I_1,I_2,I_3\,\in\,\text{3-term AP}}} 2^{-2\cdot n_1} \\
&\gtrsim 2^{n_1(2-\epsilon)}2^{-2\cdot n_1} \\
&= 2^{-n_1\epsilon}
\end{aligned}$$

Here we have utilized our version of Varnavides's theorem showing there are at least $N^{2-\epsilon}$ three-term arithmetic progressions made out of each of the intervals in $\mathcal{I}'$ where $N=2^{n_1}$. $\square$

In the next section we will choose n_2 to be significantly larger than n_1. We will now prove a lemma that will be especially helpful in proving the Theorem 1.1. Specifically, we will show if $\Lambda_{n_1}(\mu_{n_2},\mu_{n_2},\mu_{n_2})$ is bounded below we can guarantee a certain number of near three-term arithmetic progressions.

Lemma 3.3. *If $\Lambda_{n_1}(\mu_{n_2},\mu_{n_2},\mu_{n_2}) > 2^{-n_1\epsilon}$, then there exist $\gtrsim 2^{-n_1\epsilon+n_2(2-\epsilon)}$ pairs (x,a), where $a>2^{-n_1(1+\epsilon)}$, that are $100\cdot 2^{-n_2}$ separated such that x, $x+a$, $x+2a$ are within 2^{-n_2} of the support of μ.*

Proof. First, recall that

$$\Lambda_{n_1}(\mu_{n_2},\mu_{n_2},\mu_{n_2}) = \int_{a>2^{-n_1(1+\epsilon)}} \psi_{n_1}(a) \int \mu_{n_2}(x)\mu_{n_2}(x+a)\mu_{n_2}(x+2a)\,dx\,da.$$

Observe that the pairs (x,a) such that x, $x+a$, or $x+2a$ are not within 2^{-n_2} of the support of μ will cause $\mu_{n_2}(x)$, $\mu_{n_2}(x+a)$, or $\mu_{n_2}(x+2a)$ to vanish causing the integrand to vanish. Therefore, fix x_0 and a_0 so that x_0, x_0+a_0, and x_0+2a_0 are all within 2^{-n_1} of the support

of μ and $a_0 > 2^{-n_1(1+\epsilon)}$. Therefore we have the following estimate.

$$\begin{aligned}
&\int_{|a-a_0|<100\cdot 2^{-n_2}} \psi_{n_1}(a) \int_{|x-x_0|<100\cdot 2^{-n_2}} \mu_{n_2}(x)\mu_{n_2}(x+a)\mu_{n_2}(x+2a)\, dx\, da \\
&\qquad = \int_{|a-a_0|<100\cdot 2^{-n_2}} \psi_{n_1}(a) \int_{|x-x_0|<100\cdot 2^{-n_2}} \int \phi_{n_2}(x-t_1)\, d\mu(t_1) \\
&\qquad\qquad \cdot \int \phi_{n_2}(x+a-t_2)\, d\mu(t_2) \int \phi_{n_2}(x+2a-t_3)\, d\mu(t_3)\, dx\, da \\
&\qquad \leq \int_{|a-a_0|<100\cdot 2^{-n_2}} \psi_{n_1}(a) \int_{|x-x_0|<100\cdot 2^{-n_2}} \int 2^{n_2} 1_{[x-2^{-n_2},x+2^{-n_2}]}(t_1)\, d\mu(t_1) \\
&\qquad\qquad \cdot \int 2^{n_2} 1_{[x+a-2^{-n_2},x+a+2^{-n_2}]}(t_2)\, d\mu(t_2) \int 2^{n_2} 1_{[x+2a-2^{-n_2},x+2a+2^{-n_2}]}(t_3)\, d\mu(t_3)\, dx\, da
\end{aligned}$$

This inequality holds because we have bounded ϕ_{n_2} above by 2^{n_2}. Therefore we can consider the indicator function over the given intervals. We also bound ψ_{n_1} above to obtain

$$\begin{aligned}
&\int_{|a-a_0|<100\cdot 2^{-n_2}} \int_{|x-x_0|<100\cdot 2^{-n_2}} 2^{3\cdot n_2} \mu([x-2^{-n_2}, x+2^{-n_2}]) \\
&\qquad \cdot \mu([x+a-2^{-n_2}, x+a+2^{-n_2}])\mu([x+2a-2^{-n_2}, x+2a+2^{-n_2}])\, dx\, da \\
&\leq 2^{3\cdot n_2} \int_{|a-a_0|<100\cdot 2^{-n_2}} \int_{|x-x_0|<100\cdot 2^{-n_2}} \left(2^{-n_2} \log^{1/2}(2^{n_2})\right)^3 dx\, da \\
&\lesssim_\epsilon 2^{-n_2(2-\epsilon)}
\end{aligned}$$

The last line holds since $n_2 \log(2) \lesssim 2^{\epsilon n_2}$. This shows that each $100 \cdot 2^{-n_2}$ neighborhood of some (x_0, a_0) will contribute $\lesssim 2^{-n_2(2-\epsilon)}$ to the integral. Therefore, since we have assumed $\Lambda_{n_1}(\mu_{n_2}, \mu_{n_2}, \mu_{n_2}) > 2^{-n_1\epsilon}$, we can divide $2^{-n_1\epsilon}$ by $2^{-n_2(2-\epsilon)}$ to show the total number of pairs is $\gtrsim 2^{-n_1\epsilon + n_2(2-\epsilon)}$. □

Next, we will choose a large n_2 and show $|\Lambda_{n_1}(\mu_{n_1}, \mu_{n_1}, \mu_{n_1}) - \Lambda_{n_1}(\mu_{n_2}, \mu_{n_2}, \mu_{n_2})|$ is small.

CHAPTER 4

Refining Arithmetic Progressions

4.1 Bounds on θ

We have chosen n_1 so that Theorem 2.1 will hold. The result remains valid for any $n \geq n_1$. Therefore we can choose an n_2 significantly larger than n_1. We then allow n_2 to get arbitrarily large and show the difference $|\Lambda_{n_1}(\mu_2, \mu_2, \mu_2) - \Lambda_{n_1}(\mu_1, \mu_1, \mu_1)|$ is small. However, before we do this, we want to show that we are able to restrict our consideration of Λ_{n_1} over θ to the region where θ is small. We start by proving a lemma that will help with bounding several integrals seen in the remainder of this paper.

Theorem 4.1. *Let $\tau > 0$. Then the following inequality holds.*

$$\int |\widehat{\mu_i}(\theta - 2\xi)|^2 (1 + |\xi|)^{-\tau/2} \, d\xi \lesssim_\tau 1 \tag{4.1}$$

where the implicit constant is independent of θ.

We start with a lemma that will be key for this result.

Lemma 4.2. *Define the function*

$$T_R(\xi) = \begin{cases} 1 - \frac{|\xi|}{R} & |\xi| \leq R \\ 0 & |\xi| > R \end{cases}$$

Then the function $\widehat{T_R}(\xi) \geq 0$. Moreover, for any θ, we have

$$\int_\xi |\widehat{\mu}(\theta - 2\xi)|^2 T_R(\xi) d\xi \leq \int_\xi |\widehat{\mu}(\xi)|^2 T_R(\xi) d\xi.$$

Proof. We start by showing from the definition of the Fejer Kernel, that $\widehat{T}_R(\zeta) \geq 0$. We do

this by direct calculation:

$$\begin{aligned}
\widehat{T}_R(\zeta) &= \int T_R(\xi) e^{-2\pi i\xi\zeta} d\xi \\
&= \int_{-R}^{R} (1 - \frac{|\xi|}{R}) e^{-2\pi i\xi\zeta} d\xi \\
&= \int_{-R}^{R} e^{-2\pi i\xi\zeta} d\xi + \int_{-R}^{0} \frac{\xi}{R} e^{-2\pi i\xi\zeta} d\xi - \int_{0}^{R} \frac{\xi}{R} e^{-2\pi i\xi\zeta} d\xi \\
&= \frac{\sin(2R\pi\zeta)}{\pi\zeta} + \frac{1}{R}\Big(\frac{\xi e^{-2\pi i\xi\zeta}}{-2\pi i\zeta}\Big|_{-R}^{0} - \int_{-R}^{0} \frac{e^{-2\pi i\xi\zeta}}{-2\pi i\zeta} d\xi\Big) - \frac{1}{R}\Big(\frac{\xi e^{-2\pi i\xi\zeta}}{-2\pi i\zeta}\Big|_{0}^{R} - \int_{0}^{R} \frac{e^{-2\pi i\xi\zeta}}{-2\pi i\zeta} d\xi\Big) \\
&= \frac{\sin(2R\pi\zeta)}{\pi\zeta} + \frac{1}{R}\Big(\frac{Re^{2\pi iR\zeta}}{-2\pi i\zeta} + \frac{1 - e^{2\pi iR\zeta}}{4\pi^2\zeta^2}\Big) - \frac{1}{R}\Big(\frac{Re^{-2\pi iR\zeta}}{-2\pi i\zeta} + \frac{e^{-2\pi iR\zeta} - 1}{4\pi^2\zeta^2}\Big) \\
&= \frac{\sin(2R\pi\zeta)}{\pi\zeta} + \frac{e^{2\pi iR\zeta} - e^{-2\pi iR\zeta}}{-2\pi i\zeta} + \frac{2 - e^{2\pi iR\zeta} - e^{-2\pi iR\zeta}}{4\pi^2 R\zeta^2} \\
&= \frac{\sin(2R\pi\zeta)}{\pi\zeta} + \frac{e^{-2\pi iR\zeta} - e^{2\pi iR\zeta}}{2\pi i\zeta} + \frac{2 - e^{2\pi iR\zeta} - e^{-2\pi iR\zeta}}{4\pi^2 R\zeta^2} \\
&= \frac{\sin(2R\pi\zeta)}{\pi\zeta} - \frac{\sin(2R\pi\zeta)}{\pi\zeta} + \frac{2 - 2\cos(2\pi R\zeta)}{4R\pi^2\zeta^2} \\
&= \frac{1 - \cos(2\pi R\zeta)}{2R\pi^2\zeta^2} \\
&\geq 0.
\end{aligned}$$

Since μ is real valued, we have that $\mu(x) = \overline{\mu(x)}$. Therefore,

$$\begin{aligned}
&\int |\widehat{\mu}(\theta - 2\xi)|^2 T_R(\xi) d\xi \\
&= \iint e^{-2\pi i(\theta-2\xi)x}\, d\mu(x) \overline{\int e^{-2\pi i(\theta-2\xi)y} d\mu(y)}\, T_R(\xi)\, d\xi \\
&= \iiint e^{-2\pi i(x-y)(\theta-2\xi)}\, d\mu(x)\, d\mu(y)\, T_R(\xi)\, d\xi \\
&= \iint e^{-2\pi i\theta(x-y)} \int T_R(\xi) e^{-2\pi i(x-y)2\xi}\, d\xi\, d\mu(x)\, d\mu(y) \\
&\leq \iint e^{-2\pi i\theta(x-y)} \int T_R(\xi) e^{-2\pi i(x-y)\xi}\, d\xi\, d\mu(x)\, d\mu(y) \\
&= \iint e^{-2\pi i\theta(x-y)} \widehat{T_R}(x-y)\, d\mu(x)\, d\mu(y) \\
&\leq \iint |e^{-2\pi i\theta(x-y)} \widehat{T_R}(x-y)|\, d\mu(x)\, d\mu(y) \\
&= \iint \widehat{T_R}(x-y)\, d\mu(x)\, d\mu(y).
\end{aligned}$$

Here, the last line comes from the fact that $\widehat{T_R}$ is nonnegative. If we reverse our calculations,

we can write this result as

$$\int |\widehat{\mu}(\xi)|^2 T_R(\xi)\, d\xi.$$

□

Proof of Theorem 4.1. We rewrite the integral in 4.1 as

$$\int_\xi (1+|\xi|)^{-\tau/2} |\widehat{\mu}^2(\theta - 2\xi)| d\xi \;=\; I_\theta^* + \sum_{k=0}^{\infty} I_{k,\theta}$$

where

$$I_\theta^* = \int_{|\xi|\le 1} (1+|\xi|)^{-\tau/2} |\widehat{\mu}^2(\theta - 2\xi)| d\xi$$

and

$$I_{k,\theta} = \int_{2^k \le |\xi| \le 2^{k+1}} (1+|\xi|)^{-\tau/2} |\widehat{\mu}^2(\theta - 2\xi)| d\xi$$

However, since the integral is symmetric, we can simply consider the integral

$$J_{k,\theta} = \int_{2^k \le \xi \le 2^{k+1}} (1+|\xi|)^{-\tau/2} |\widehat{\mu}^2(\theta - 2\xi)| d\xi.$$

Next, we show that

$$J_{k,\theta} \lesssim 2^{-\tau k/4}$$

where the implied constant is not dependent on k or on θ.

We see immediately due to the range of integration

$$J_{k,\theta} \le 2^{-k\tau/2} \int_{2^k \le \xi \le 2^{k+1}} |\widehat{\mu}^2(\theta - 2\xi)|\, d\xi$$

Since the integrand is nonnnegative, we have an upper bound of

$$\begin{aligned}
&100 \cdot 2^{-k\tau/2} \int |\widehat{\mu}^2(\theta - 2\xi)| T_{2^{k+2}}(\xi)\, d\xi \\
&\lesssim 2^{-k\tau/2} \int |\widehat{\mu}^2(\xi)| T_{2^{k+2}}(\xi)\, d\xi \\
&\lesssim 2^{-k\tau/4} \int (1+|\xi|)^{-k\tau/4} |\widehat{\mu}^2(\xi)| T_{2^{k+2}}(\xi)\, d\xi \\
&\lesssim 2^{-k\tau/4} \int (1+|\xi|)^{-k\tau/4} |\widehat{\mu}^2(\xi)|\, d\xi \\
&\lesssim 2^{-k\tau/4}
\end{aligned}$$

Here, the fact that $\int (1+|\xi|)^{-k\tau/4}|\widehat{\mu}^2(\xi)|\, d\xi$ is finite is can be seen in Theorem 2.8 and 3.10 in Mattila's Fourier Analysis and Hausdorff Dimension. [7]

□

Next, recall that we have expressed Λ_{n_1} in terms of the Fourier transform as

$$\Lambda_{n_1}(\mu_i,\mu_i,\mu_i) = \iint \widehat{\mu_i}(\xi)\widehat{\mu_i}(\theta-2\xi)\widehat{\mu_i}(\xi-\theta)\widehat{\psi_{n_1}}(\theta)\, d\xi\, d\theta.$$

Now, we split up the integral into two regions as follows.

$$\Lambda(\mu_i,\mu_i,\mu_i) = I + II$$

Where

$$I = \int_{|\theta|\leq 2^{n_1/2}} \int \widehat{\mu_i}(\xi)\widehat{\mu_i}(\theta-2\xi)\widehat{\mu_i}(\xi-\theta)\widehat{\psi_{n_1}}(\theta)\, d\xi\, d\theta$$

and

$$II = \int_{|\theta|> 2^{n_1/2}} \int \widehat{\mu_i}(\xi)\widehat{\mu_i}(\theta-2\xi)\widehat{\mu_i}(\xi-\theta)\widehat{\psi_{n_1}}(\theta)\, d\xi\, d\theta.$$

Recall that we have shown the following bound on $\widehat{\psi_{n_1}}$ in lemma 3.1.

$$|\widehat{\psi_{n_1}}(\theta)| \lesssim (1+|\theta|)^{-100}$$

where the constant here is independent of our choice of n_1. We then have that

$$\begin{aligned}
II &= \int_{|\theta|>2^{n_1/2}} \int \widehat{\mu_i}(\xi)\widehat{\mu_i}(\theta-2\xi)\widehat{\mu_i}(\xi-\theta)\widehat{\psi_{n_1}}(\theta)\, d\xi\, d\theta. \\
&\lesssim \int_{|\theta|>2^{n_1/2}} (1+|\theta|)^{-100} \int \widehat{\mu_i}(\xi)\widehat{\mu_i}(\theta-2\xi)\widehat{\mu_i}(\xi-\theta)\, d\xi\, d\theta. \\
&\leq \int_{|\theta|>2^{n_1/2}} (1+|\theta|)^{-100} \int (1+|\xi|)^{-\beta/2}\widehat{\mu_i}(\theta-2\xi)\widehat{\mu_i}(\xi-\theta)\, d\xi\, d\theta. \\
&\leq \int_{|\theta|>2^{n_1/2}} (1+|\theta|)^{-100} \Big(\int (1+|\xi|)^{-\beta/2}\widehat{\mu_i}^2(\theta-2\xi)\, d\xi\Big)^{1/2} \Big(\int (1+|\xi|)^{-\beta/2}\widehat{\mu_i}^2(\xi-\theta)\, d\xi\Big)^{1/2} d\theta \\
&\lesssim \int_{|\theta|>2^{n_1/2}} (1+|\theta|)^{-100}\, d\theta \\
&\lesssim \int_{|\theta|>2^{n_1/2}} |\theta|^{-100}\, d\theta \\
&= 2^{-99n_1/2}
\end{aligned}$$

Here, each of the integrals in ξ is bounded utilizing Theorem 4.1. Therefore, moving forward, we are able to write our trilinear form as

$$\Lambda_{n_1}(\mu_i,\mu_i,\mu_i)=\int_{|\theta|<2^{n_1/2}}\int \widehat{\mu_i}(\xi)\widehat{\mu_i}(\theta-2\xi)\widehat{\mu_i}(\xi-\theta)\widehat{\psi_{n_1}}(\theta)\,d\xi\,d\theta+\mathcal{O}(2^{-99\cdot n_1/2}).$$

4.2 Determining the Difference

Now that we have chosen n_1, we can choose n_2 to be significantly larger than n_1. We can make n_2 as large as we wish. With the choice of n_2 we now show that $\Lambda_{n_1}(\mu_1,\mu_1,\mu_1)-\Lambda_{n_1}(\mu_2,\mu_2,\mu_2)$ has a small difference. For the sake of simplifying notation, we define the following function.

$$F_\theta(\xi)=\widehat{\phi_1}(\xi)\widehat{\phi_1}(\theta-2\xi)\widehat{\phi_1}(\xi-\theta)-\widehat{\phi_2}(\xi)\widehat{\phi_2}(\theta-2\xi)\widehat{\phi_2}(\xi-\theta).$$

The difference then becomes

$$\Lambda_{n_1}(\mu_1,\mu_1,\mu_1)-\Lambda_{n_1}(\mu_2,\mu_2,\mu_2)=\int_{|\theta|<2^{n_1/2}}\int \widehat{\psi_{n_1}}(\theta)\widehat{\mu}(\xi)\widehat{\mu}(\theta-2\xi)\widehat{\mu}(\xi-\theta)F_\theta(\xi)\,d\xi d\theta.$$

In order to show this difference is small, we will split this integral into the sum of two integrals in order to separate our integral over ξ into two regions: one where ξ is large and the other where ξ is small. The difference will then become

$$\Lambda_{n_1}(\mu_1,\mu_1,\mu_1)-\Lambda_{n_1}(\mu_2,\mu_2,\mu_2)=\mathcal{A}+\mathcal{B},$$

where

$$\mathcal{A}=\int_{|\theta|<2^{n_1/2}}\int_{|\xi|<100\cdot 2^{n_1/2}}\widehat{\psi_{n_1}}(\theta)\widehat{\mu}(\xi)\widehat{\mu}(\theta-2\xi)\widehat{\mu}(\xi-\theta)F_\theta(\xi)\,d\xi\,d\theta$$

and

$$\mathcal{B}=\int_{|\theta|<2^{n_1/2}}\int_{|\xi|>100\cdot 2^{n_1/2}}\widehat{\psi_{n_1}}(\theta)\widehat{\mu}(\xi)\widehat{\mu}(\theta-2\xi)\widehat{\mu}(\xi-\theta)F_\theta(\xi)\,d\xi\,d\theta.$$

We first consider the case where ξ is small. Recall that $\phi_{n_1}=2^{n_1}\phi(2^{n_1}x)$. Therefore, $\widehat{\phi_{n_1}}=\widehat{\phi}(2^{-n_1}\xi)$. Therefore we can express $F_\theta(\xi)$ as

$$F_\theta(\xi)=\widehat{\phi}(2^{-n_1}\xi)\widehat{\phi}(2^{-n_1}(\theta-2\xi))\widehat{\phi}(2^{-n_1}(\xi-\theta))-\widehat{\phi}(2^{-n_2}\xi)\widehat{\phi}(2^{-n_2}(\theta-2\xi))\widehat{\phi}(2^{-n_2}(\xi-\theta))$$

We can then Taylor expand $\widehat{\phi}(\xi)$ centered at 0. This gives us

$$\widehat{\phi}(\xi) = \widehat{\phi}(0) + \widehat{\phi}'(0)\xi + \mathcal{O}(\xi^2).$$

Therefore we can then write $F_\theta(\xi)$ using this Taylor expansion.

$$\begin{aligned} F_\theta(\xi) = &\left(\widehat{\phi}(0) + 2^{-n_1}\widehat{\phi}'(0)\xi + \mathcal{O}(2^{-2n_1}\xi^2)\right)\left(\widehat{\phi}(0) + 2^{-n_1}\widehat{\phi}'(0)(\theta - 2\xi) + \mathcal{O}(2^{-2n_1}(\theta-2\xi)^2)\right) \\ &\cdot \left(\widehat{\phi}(0) + 2^{-n_1}\widehat{\phi}'(0)(\xi - \theta) + \mathcal{O}(2^{-2n_1}(\xi-\theta)^2)\right) \\ &- \left(\widehat{\phi}(0) + 2^{-n_2}\widehat{\phi}'(0)\xi + \mathcal{O}(2^{-2n_2}\xi^2)\right)\left(\widehat{\phi}(0) + 2^{-n_2}\widehat{\phi}'(0)(\theta - 2\xi) + \mathcal{O}(2^{-2n_2}(\theta-2\xi)^2)\right) \\ &\cdot \left(\widehat{\phi}(0) + 2^{-n_2}\widehat{\phi}'(0)(\xi - \theta) + \mathcal{O}(2^{-2n_2}(\xi-\theta)^2)\right) \end{aligned}$$

Now, since we are only considering the region where $|\xi| \lesssim 2^{n_1/2}$ and $\theta \lesssim 2^{n_1/2}$ we can say that each of the error terms become $\mathcal{O}(2^{-n_1})$ and $\mathcal{O}(2^{-n_2})$ and therefore

$$\begin{aligned} F_\theta(\xi) = &\left(\widehat{\phi}(0) + 2^{-n_1}\widehat{\phi}'(0)\xi + \mathcal{O}(2^{-n_1})\right)\left(\widehat{\phi}(0) + 2^{-n_1}\widehat{\phi}'(0)(\theta - 2\xi) + \mathcal{O}(2^{-n_1})\right) \\ &\cdot \left(\widehat{\phi}(0) + 2^{-n_1}\widehat{\phi}'(0)(\xi - \theta) + \mathcal{O}(2^{-n_1})\right) \\ &- \left(\widehat{\phi}(0) + 2^{-n_2}\widehat{\phi}'(0)\xi + \mathcal{O}(2^{-n_2})\right)\left(\widehat{\phi}(0) + 2^{-n_2}\widehat{\phi}'(0)(\theta - 2\xi) + \mathcal{O}(2^{-n_2})\right) \\ &\cdot \left(\widehat{\phi}(0) + 2^{-n_2}\widehat{\phi}'(0)(\xi - \theta) + \mathcal{O}(2^{-n_2})\right) \end{aligned}$$

We then expand these products. For simplicity, we only show the expansion for n_1. The product for n_2 will look similar

$$\begin{aligned} &\left(\widehat{\phi}(0) + 2^{-n_1}\widehat{\phi}'(0)\xi + \mathcal{O}(2^{-n_1})\right)\left(\widehat{\phi}(0) + 2^{-n_1}\widehat{\phi}'(0)(\theta - 2\xi) + \mathcal{O}(2^{-n_1})\right) \\ &\qquad \cdot \left(\widehat{\phi}(0) + 2^{-n_1}\widehat{\phi}'(0)(\xi - \theta) + \mathcal{O}(2^{-n_1})\right) \\ &= \widehat{\phi}^3(0) + \widehat{\phi}^2(0)\widehat{\phi}'(0)2^{-n_1}\xi + \widehat{\phi}^2(0)\widehat{\phi}'(0)2^{-n_1}(\theta - 2\xi) + \widehat{\phi}^2(0)\widehat{\phi}'(0)2^{-n_1}(\xi - \theta) + \mathcal{O}(2^{-n_1}) \\ &= \widehat{\phi}^3(0) + \mathcal{O}(2^{-n_1}) \end{aligned}$$

Therefore we can write

$$\begin{aligned} F_\theta(\xi) &= \widehat{\phi}^3(0) + \mathcal{O}(2^{-n_1}) - \left(\widehat{\phi}^3(0) + \mathcal{O}(2^{-n_2})\right) \\ &= \mathcal{O}(2^{-n_1}). \end{aligned} \tag{4.2}$$

Now we can write $\mathcal{A}$ as

$$\begin{aligned}
\mathcal{A} &= \int_{|\theta|<2^{n_1/2}} \int_{|\xi|<100\cdot 2^{n_1/2}} \widehat{\psi_{n_1}}(\theta)\widehat{\mu}(\xi)\widehat{\mu}(\theta-2\xi)\widehat{\mu}(\xi-\theta)F_\theta(\xi)\,d\xi\,d\theta \\
&\lesssim 2^{-n_1} \int_{|\theta|<2^{n_1/2}} \int_{|\xi|<100\cdot 2^{n_1/2}} \widehat{\psi_{n_1}}(\theta)\widehat{\mu}(\xi)\widehat{\mu}(\theta-2\xi)\widehat{\mu}(\xi-\theta)\,d\xi\,d\theta \\
&\leq 2^{-n_1} \int_{|\theta|<2^{n_1/2}} \widehat{\psi_{n_1}}(\theta) \int_{|\xi|<100\cdot 2^{n_1/2}} (1+|\xi|)^{-\beta/2}\widehat{\mu}(\theta-2\xi)\widehat{\mu}(\xi-\theta)\,d\xi\,d\theta \\
&\leq 2^{-n_1} \int_{|\theta|<2^{n_1/2}} \widehat{\psi_{n_1}}(\theta)\Big(\int_{|\xi|<100\cdot 2^{n_1/2}} (1+|\xi|)^{-\beta/2}|\widehat{\mu}^2(\theta-2\xi)|\,d\xi\Big)^{1/2} \\
&\qquad\qquad \cdot \Big(\int_{|\xi|<100\cdot 2^{n_1/2}} (1+|\xi|)^{-\beta/2}|\widehat{\mu}^2(\xi-\theta)|\,d\xi\Big)^{1/2} d\theta.
\end{aligned}$$

Using Theorem 4.1 we are able to bound each of these integrals in ξ. Therefore we have

$$\begin{aligned}
\mathcal{A} &\lesssim 2^{-n_1} \int_{|\theta|<2^{n_1/2}} \widehat{\psi_{n_1}}(\theta)\,d\theta \\
&\lesssim 2^{-n_1}.
\end{aligned}$$

This holds since ψ_{n_1} is a Schwartz function and we are integrating over a finite region.

Next consider the case where ξ is large. We consider the integral

$$\mathcal{B} = \int_{|\theta|<2^{n_1/2}} \int_{|\xi|>100\cdot 2^{n_1/2}} \widehat{\psi_{n_1}}(\theta)\widehat{\mu}(\xi)\widehat{\mu}(\theta-2\xi)\widehat{\mu}(\xi-\theta)F_\theta(\xi)\,d\xi\,d\theta.$$

We have defined ϕ to be a Schwartz function. Since the space of Schwartz functions is closed under the Fourier transform, $\widehat{\phi}$ is also a Schwartz function. Therefore, $F_\phi(\xi)$ is bounded. We

then have the inequality

$$\begin{aligned}
\mathcal{B} \lesssim & \int_{|\theta|<2^{n_1/2}} \widehat{\psi_{n_1}}(\theta) \left(\int_{|\xi|>100\cdot 2^{n_1/2}} (1+|\xi|)^{-\beta/2} |\widehat{\mu}^2(\theta-2\xi)| \, d\xi \right)^{1/2} \\
& \cdot \left(\int_{|\xi|>100\cdot 2^{n_1/2}} (1+|\xi|)^{-\beta/2} |\widehat{\mu}^2(\xi-\theta)| \, d\xi \right)^{1/2} d\theta \\
& \lesssim \int_{|\theta|<2^{n_1/2}} \widehat{\psi_{n_1}}(\theta) \left(\int_{|\xi|>100\cdot 2^{n_1/2}} (1+|\xi|)^{-\beta/4} (2^{n_1/2})^{-\beta/4} |\widehat{\mu}^2(\theta-2\xi)| \, d\xi \right)^{1/2} \\
& \cdot \left(\int_{|\xi|>100\cdot 2^{n_1/2}} (1+|\xi|)^{-\beta/4} (2^{n_1/2})^{-\beta/4} |\widehat{\mu}^2(\xi-\theta)| \, d\xi \right)^{1/2} d\theta \\
& \lesssim 2^{-n_1\beta/8} \int_{|\theta|<2^{n_1/2}} |\widehat{\psi_{n_1}(\theta)}| d\theta \\
& \lesssim 2^{-n_1\beta/8} \int_{|\theta|<2^{n_1/2}} |1+\theta|^{-100} d\theta \\
& < 2^{-n_1\beta/8} \int |1+\theta|^{-100} d\theta \\
& \lesssim 2^{-n_1\beta/8}
\end{aligned}$$

Using the Lemma 4.1 we can bound each integral in ξ and we are able to bound $\widehat{\psi_{n_1}}$ by Lemma 3.1. Therefore we have the following bound on the difference

$$\begin{aligned}
|\Lambda_{n_1}(\mu_1,\mu_1,\mu_1) - \Lambda_{n_1}(\mu_2,\mu_2,\mu_2)| &\lesssim 2^{-n_1} + 2^{-n_1\beta/8} + 2^{-99n_1/2} \\
&\lesssim 2^{-n_1\beta/8}.
\end{aligned}$$

This is because 2^{-n_1} and $2^{-99n_1/2}$ are both smaller than $2^{-n_1\beta/8}$. We also have by Lemma 3.2 that $\Lambda_{n_1}(\mu_1,\mu_1,\mu_1) \gtrsim 2^{-n_1\epsilon}$. Therefore if we choose $\epsilon = \beta/100$, we have by the triangle inequality that $\Lambda_{n_1}(\mu_2,\mu_2,\mu_2) \gtrsim 2^{-n_1\epsilon} - 2^{-n_1\beta/8} \gtrsim 2^{-n_1\epsilon}$. From Theorem 3.3 we know that there are triples of points in the support of μ that are within 2^{-n_2} of a three term arithmetic progression. As we take n_2 to infinity, we obtain a sequence of near three-term arithmetic progressions. Since the support of μ is compact, we can find a sub sequence that will converge to an actual three-term arithmetic progression. This guarantees the presence of an actual three-term arithmetic progression in the support of μ, thus completing the proof of Theorem 1.1.

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