A PROBABILISTIC DYNAMIC PROGRAMING MODEL FOR DETERMINING OPTIMUM INVENTORIES

John Wang[1]
Steve Bin Zhou[2]
Zhi Pei[3]
[1]Montclair State University
[2]University of Houston-Downtown
[3]The University of Texas at Tyler
wangj@montclair.edu; drbinzhou@yahoo.com; zpei@uttyler.edu

Abstract

The occurrence and size of snowfall, and therefore, the level of demand for salt, are uncertain for each week. Therefore, a probabilistic inventory model is required. This will be a periodic review model - the inventory of salt at the beginning of each week will determine the optimal quantity to be purchased to satisfy expected demand. Dynamic programming with a backward recursion is utilized to determine weekly optimal costs/inventory level.

1. Introduction

The United States is significantly dependent on its road system to support the rapid, reliable, reasonable movement of people, goods, and services. In many states, this requires substantial planning, training, manpower, equipment, and material resources to clear roads and streets through the winter (EPA, 1999). The most common deicing method is the use of sodium chloride (salt) in the form of crushed rock salt, which is inexpensive, abundant, and easy to mine, store, distribute, and apply. Salt brines are increasingly used in some areas, but the vast majority is still rock salt.

In areas inclined to winter precipitation, transportation network must be able to rapidly respond to snow and ice on roadways. Ice removal is a vital service in these communities. Deicing chemicals melt ice by lowering the temperature at which it melts. They can also prevent new ice from forming and improve traction. Salt is a popular deicing chemical because it is cheap and abundant. As (Lilek, 2017) indicated that the United States produces 22.9 and needs 24.5 million metric tons of rock salt for roadway deicing. Top consuming states are New York, Ohio, and Illinois, the total cost of rock salt used for roadway deicing was $1.18 billion.

Winter road maintenance is a critical function of public works and transportation agencies in states with harsh winter climate or in cold regions. The use of salts represents a cost-effective snowfighting technique, having beneficial impact on public safety, essential mobility and travel costs (Autelitano, Rinaldi, and Giuliani, 2019).

In line with Tiwari and Rachlin (2018), in areas of North America that receive snow, road-salt runoff leads to many problem, such as groundwater salinization, loss or reduction in lake turnover, and changes in soil structure. It also poses threats to birds, mammals, and roadside vegetation. Furthermore, road-salt runoff can affect biotic communities by causing changes in the composition of fish or aquatic invertebrate assemblages.

Having faced an increase in extreme winter weather, many places in the U.S. have experienced increased demand for roadway snow and ice control (RSIC) operations. According to Sullivan, Dowds, Novak, Scott, and Ragsdale (2019), as the number and severity of extreme weather events increases, the costs associated with winter roadway maintenance materials, plow operator time, equipment maintenance and replacement, and fuel use will also increase.

The process involved in salting the roads of a county in Northern New Jersey included the involvement of thirteen inspectors. During the winter periods these inspectors are all assigned to certain parts of the county, and inspect the conditions of the roads during a snowstorm. Once they see the conditions of the roads and determine if they need to be salted, they send trucks out. These trucks are contracted out for their services. The county uses around 25 to 30 contractors paying them ninety dollars an hour for their services. These trucks are filled with 10 tons of mineral rock salt and 24 gallons of calcium chloride which is sprayed onto the salt and acts as a catalyst for melting the snow. The county also uses a mixture of pea gravel with salt when the conditions of the roads are not that bad or when they are low on salt. The percentage of pea gravel to salt is 3:2. In salting the roads, the hills are the first priority followed by the main streets and level areas.

When the snow conditions are less than 3 inches the county salts the roads. If there is more than three inches, they usually do one run before the snow starts to accumulate and then plow the roads, and if necessary the roads are resalted again when the snow stops depending on the intensity of the storm. This process helps set up a base on the roadway preventing a dangerous situation called rib ice and provides better traction for the motorists. After they salt the roads, the salt will last for about an hour and fifteen minutes before they have to resalt. Again, this situation is based on the intensity of the storm and it may or may not have to be done. Some of the costs involved in this process includes the purchasing of the salt, the construction of the dome and all costs associated with it, and other related costs involved with the salting process.

2. The Model

When it comes to winter weather specifics, the only sure thing is that nobody is sure. Any scientist will tell you it is risky to predict temperature more than thirty days in advance and impossible to predict precipitation. Snow, especially the heavy one, has caused too much trouble and frustration.

Many scholars focus on the specific methods and/or materials for deicing. For instance, Hossain, Fu, & Lake (2015) conducted approximately 300 tests in a real-world environment, which cover four alternative materials, and 21 snow events. They presented the results of an extensive field study on the comparative performance of alternative materials for snow and ice control of transportation facilities. Each of the alternatives tested were compared to regular rock salt in terms of snow melting performance — bare-pavement regain time.

In order to design a road de-icing device by heating, Bernardin and Munch (2019) consider in the one dimensional setting the optimal control of a parabolic equation with a nonlinear boundary condition of the Stefan– Boltzmann type. This control problem models the heating of a road during a winter period to keep the road surface temperature above a given threshold. Their model allows to quantify the minimal energy to be provided to keep the road surface without frost or snow.

However, there is a shortage of research articles dealing with the specific dilemma that is faced by the county level officials. The problem is to determine the optimal inventory of salt (required to melt snow) that the Essex County should carry on a weekly basis during the winter months (November to April). The optimal inventory will be determined based on the inventory on hand at the beginning of the week, and the optimal amount to be purchased during the week can be determined based on the expected demand for salt for the week. The expected demand will be determined using historical probability distribution of snowfalls in the county for each week.

The occurrence and size of snowfall, and therefore, the level of demand for salt, are uncertain for each week. Therefore, a probabilistic inventory model is required. This will be a periodic review model - the inventory of salt at the beginning of each week will determine the optimal quantity to be purchased to satisfy expected demand. Dynamic programming with a backward recursion is utilized to determine weekly optimal costs/inventory level.

Dynamic programming is both a mathematical optimization method and a computer programming method (Beuchat, Georghiou, & Lygeros, 2020; Sauré, Begen, & Patrick, 2020). Operational management of reservoirs at hourly/daily timescales is challenging due to the uncertainty associated with the inflow forecasts and the volumes in the reservoir. Ramaswamy and Saleh (2020) apply a multi-objective dynamic programming model and obtain optimized release strategies accounting for the inflow uncertainties. Their study provides perspectives on the benefits of the proposed forecasting and optimization framework in reducing the decision-making burden on the operator by providing the uncertainties associated with the inflows, releases and the water levels in the reservoir.

2.1. Basis and assumptions

1. Salt can be ordered at the beginning of each week and delivered immediately.
2. Inventory management is over a 24-week period, from the second week in November to the third week in April.
3. Bulk purchase of salt is allowed at the beginning of week one or any other week and there is limited storage capacity.
4. There is a holding cost of \$h/ton/week and a shortage cost of \$s/ton/week. Holding costs include cost of funds tied up in financing inventory, cost of storage, insurance and maintenance costs. Shortage costs include additional costs incurred when salt (or substitutes such as sand, gravel, etc.) has to be obtained in an emergency, the political costs and damage to the reputation of the county management, accident costs, etc., when there is not enough salt to melt snow in the event of a snowstorm. (Some of these costs are difficult to quantify.)
5. Salt may be purchased at a unit cost of \$u per ton; there are no fixed costs.
6. Inventory may be left over at the end of the 24-week period and can be stored and used to begin the next season.
7. There are no financial constraints regarding the quantity of salt that can be purchased in any week.

2.2. The mathematical model

There are twelve (discrete) possible levels of snowfall. The probability that the level of snowfall will be j during any particular week is P_j (j = 1, ... n). The corresponding expected salt demand for that level of snow is D_j so that for any week t, the expected demand for salt will be

$$E(Dt) = \sum_{j=1}^{n} PjDj$$

i.e., for any week:

Level of Snowfall, j:	1	2	3	...	n
Probability, P_j:	P_1	P_2	P_3	...	P_n
Salt Demand, D_j:	D_1	D_2	D_3	...	D_n

Let

\$u = unit cost per ton of salt

\$h = holding cost per ton of salt per week

$s = shortage cost per ton of salt per week
c(x) = u * x = purchase cost of x tons of salt
i = inventory (in tons) of salt on hand at the beginning of week t

Then, the minimum cost of meeting demands for weeks t, t + 1, …, 24, given by $f_t(i)$, is

(i) For week twenty-four

$$f_{24}(i) = \min[c(x) + \sum_{i+x \geq Dj} Pj(i + x - Dj) * \square + \sum_{i+x<Dj} Pj(Dj - i - x) * s\,]$$

(ii) For week t ≤ 23

$$f_t(i) = \min[c(x) + \sum_{i+x \geq Dj} Pj(i + x - Dj) * \square + \sum_{i+x<Dj} Pj(Dj - i - x) * s + \sum_{i+x \geq Dj} Pj * f(t+1)(i + x - Dj) * \square + \sum_{i+x<Dj} Pj * f(t+1)(0) * s\,]$$

$$(i, x, D_j \geq 0)$$

The future cost component for $i + x < D_j$ is f_{t+1} (0) because, in the event of a shortage, beginning inventory in the following period will be zero.

2.3. Data

Snowfall distribution

Weekly snowfall figures were compiled from daily records of snowfall obtained from the National Weather Service. Data was obtained from Winter 1993/94 through Winter 2014/2015 (21 years), each period beginning first week in October, i.e.. Week 1, is October 1 through 7 of each year, Week 2 is October 8 through 14, etc. The end of the 24th week is within the third week in April.The snowfall levels (in inches) were simply summed up for the seven days in each week and a probability distribution was derived from the frequency distribution of snowfall levels in each week over the 21-year period. Three snow size/level ranges (points of the discrete probability density function) were obtained which suited the data reasonably. These ranged from zero (or trace of snow) to 30 inches or more.

Salt demand

Exact figures of salt required to melt each inch of snow could not be obtained from the county. According to a county official, this was difficult to estimate because salt use may continue one or two days after a heavy snowstorm (for maintenance) and salt is also used for fire-fighting. However, from charts of annual snow costs provided by the county, we were able to estimate that an average of 1920 tons of salt were used per inch of snow in Winter 2014/15. The county official confirmed that this figure (approximately 2000 tons) may be a reasonable estimate. Salt demand of 2000 tons/inch was therefore considered reasonable for use in the calculations.

Salt costs

A purchase cost of $31.25 per ton of salt was given by the county official.

Holding costs

Storage costs are relatively low because salt is easily stored in a simple warehouse. The county has two major warehouses, each with a storage capacity of 10, 000 tons. The approximate cost of each building was $150, 000. If we assume that the buildings can be used for 25 years, and ignoring time value of money considerations, the cost of each building (amortized over 25 years) is

$150, 000/25= $6, 000/year

Assuming insurance and maintenance costs of $5000 a year, we obtain

Cost of storage per ton per week = ($6, 000 + $5, 000)/(10,000 x 52) = $0.021

If the opportunity cost of funds used to finance inventory is assumed to be 13% p.a., then

Financing cost per ton per week = (31.25 x .13)/52 = $0.078

Addition of the above two costs results in a holding cost of approximately 10 cents per ton per week, which is the cost used in the computations.

Shortage costs

The county has no figures on these costs that could be made available, because of the difficulty in quantifying certain components of these costs. For this reason, shortage cost is initially assumed to be about twice the unit cost of salt ($60/ton). This translates into a total shortage cost of $120, 000 per inch of snow not cleared by the county. It can be argued that political costs alone will be higher than this amount, especially with heavy snowfall. (The shortage cost - snow size relationship is, in reality, nonlinear, but a linear relationship is assumed here.) Therefore, a range of higher (and also lower) shortage costs are used to sensitize the analysis.

3. Computations

A computer program was written to undertake the recursive computations. The following costs have been used in the computations:

Unit cost of salt - $31.25/ton
Holding cost - 10 cents/ton/week
Storage cost - $60/ton/week

The first set of data is the printout of the data file containing the probability distributions for each snow level for the 24-week period.

The computations are shown for total inventory each week (beginning inventory (BEG. INV.) plus purchases (QUANT) for the week not exceeding 25, 000 tons.

An initial run of the program had shown that even for a shortage cost as high as $l, 000/ton, the maximum expected weekly salt requirement would be 22, 000 tons during periods with the highest probabilities of heavy snowfall. The restriction also saves computer time and eliminates redundant data for inventory levels over 25, 000 tons.

It is assumed that minimum purchases of salt will be 1, 000 tons at a time and larger purchases will be in multiples of 1, 000 tons, so increments in purchases/Inventory levels are by 1, 000 tons.

4. Results and discussion

Results show that for Weeks 18, 17, and 16, it is optimal not to make any purchases for any level of beginning inventory, including zero. The Expected total Cost (TOT. COST) which is equal to Purchase Cost (PCOST) plus Expected Holding/Shortage Cost (H/S COST) plus Expected Future Cost (FUT. COST) is minimum for purchase quantity of zero, and is shown in the last column as Optimal Cost (OPT. COST). The optimum cost for each inventory level is highlighted. For subsequent weeks, the optimal level of inventory increases from 2000 tons/week in week 15 to 20,000 tons/week during week 8 and then down to 2000 tons/week in week 1.

The results here show that it is optimal to begin the season by purchasing 2000 tons of salt in the first week if no inventory was left over from last season (BEG. INV. = 0) or to purchase 1000 tons if the inventory left over was 1000 tons. If 2000 or more tons were left over, then no purchases need to be made. For the following weeks, salt can be purchased as required to maintain the optimal inventory level required for each week.

It is observed that, for this shortage cost of $60/ton, it is optimal not to keep any inventory of salt at all (in the case where beginning inventory = 0 and purchases = 0) even though there is a small probability (0.06) that there might be an average snowfall of up to 1 inch during weeks 16, 17, and 18. Because of the inability to quantify shortage costs especially political costs, it is reasonable to assume that the county cannot afford any salt shortage whatsoever and must therefore keep a minimum inventory at all times sufficient to melt at least one inch of snow, i.e., 2000 tons. Two alternative approaches were used to deal with this:

1. A modified program was run to give optimal total weekly inventory values for different shortage costs. The lowest shortage cost which gives optimal total weekly inventory of at least 2000 tons for weeks 16, 17, and 18 was found to be $300/ton. This translates into total shortage cost of $600, 000 per inch of snow not melted. The original program was then run with shortage cost = $300 and holding cost remaining at 10c.
2. A restriction was introduced into the program to ensure that the minimum total inventory for each week is 2000 tons. (For ease of computation, increments in inventory of 2000 tons at a time were used in the recursions here.)

The results show only the optimal values for each amount of beginning inventory, for shortage costs of $100, $200, and $300 per ton. Table 1 shows a comparison of the minimum optimal level of total inventory required each week for various shortage costs ranging from $30 to $1, 000. Shortage cost of $1, 000 is included to illustrate the effect of imposition of a much larger shortage cost, to imply very little allowance for shortage. The inventory levels in Table 1 are the results for the case without restriction of minimum inventory to 2000 tons.

The results show that for all shortage costs, an optimum inventory level of 2000 tons should be maintained in week 1 as in the case with S = $60. From then on, as would be expected, higher optimal levels are required as the probabilities of occurrence and size of snowfall increase in the following weeks. The optimal inventory levels are not much different during peak snowfall periods for the different shortage costs ranging from $100 to $1, 000. They are somewhat lower for a shortage cost of $60. For periods of sparse snowfall (the beginning and ending weeks) the differences across shortage costs are more obvious especially between S = $60 and S =$1, 000.

The B column for S = $60 shows the inventory levels with the restriction of minimum inventory to $2, 000 tons. The difference is that for those weeks in the unrestricted case where inventory equals zero, the inventory levels are now 2000 tons.

The maximum level of inventory recommended by the model is 22,000 tons during the peak periods of snowfall. Optimal inventory levels determined by the model generally correspond to the probability density functions. This is easily observable in week nine, even though it is a period of relatively high snowfall, the optimal inventory is 8000 tons, compared with 22,000 is because the probability distribution cuts off at snow level 5 (3-4 inches of snow) where the salt demand would be 8000 tons. On the other hand, optimal inventory for weeks 8, 10, 11, and 12 amount to 22,000 tons because their probability distributions are spread out up to snow level 12 where the salt demand would be 22,000 tons. The model determines this maximum because of the relatively high shortage costs. It is seen from Table 1 that for lower shortage costs ($60 or less), the optimal inventories are less than 22,000 tons (down to between 6000 and 20,000 tons at S = $40). Howler, the optimal inventories for weeks 6 and 7 remain at 20,000 tons across all shortage costs from $40 to $1000 because they have the most disperse probability distributions as well as being two of the weeks with highest probabilities of snowfall.

Table 1. Optimal inventory levels (in thousand tons) for different shortage cost ($S/ton)

Week	$30	$40	$50	$60 (A)	$60(B)	$100	$200	$300	**$1000**
18	0	0	0	0	2	0	0	2	4
17	0	0	0	0	2	2	2	2	2
16	0	0	0	0	2	2	2	4	10
15	0	0	0	2	2	8	10	10	14
14	0	2	2	4	4	14	16	16	18
13	0	4	4	6	6	14	14	14	14
12	0	6	8	8	10	18	20	22	22
11	0	12	12	12	12	22	22	22	22
10	0	14	14	16	16	22	22	22	22
9	0	8	8	8	8	8	8	8	8
8	0	20	22	22	22	22	22	22	22
7	0	20	20	20	20	20	20	20	20
6	0	20	20	20	20	20	20	20	20
5	0	8	8	8	8	S	8	8	8
4	0	6	6	6	6	6	6	6	6
3	0	10	10	10	10	10	10	10	10
2	0	14	14	14	14	14	14	14	14
1	0	2	2	2	2	2	2	2	2

The results obtained for the other weeks can be similarly explained, e.g., inventory is 14, 000 tons for week 2 and 10, 000 tons for week 3 because there are (small) outlying probabilities of snowfall at snow levels 8 and 6, respectively. The "outliers" seem to bias the model's results, but this is due to the dominant effect of shortage costs which represents the avoidance of shortage in practice, however small the probability and/or size of snowfall.

For a shortage cost of a $30/ton ($60, 000 per inch of snow), the model recommends that no salt be kept. This is because this amount is lower than the purchase plus holding costs of salt per ton, and therefore justifies the assumed shortage costs of $60 or more as realistic.

In view of the likelihood that holding costs may be higher or lower than the 10c per ton assumed, the program was modified to undertake computations for holding costs varying from 5c to 20c, for shortage costs of $60 as well as $300. Results are summarized in Table 2.

Not surprisingly, for a given holding cost, the inventory levels determined by the model are the same for both S = $60 and S = $300, since the shortage costs component, being relatively much higher, dominates holding cost. There would be a significant difference if holding and shortage costs were much closer, which is not the case with the realistic costs in this study.

For the same reason, for a given shortage cost, the variation in holding costs from 5c to 20c does not give significantly different results. For s=$60, there is a difference at week 14 where optimum inventory is 4000 tons for h = 5c, 10c, and 15c, and 2000 tons for h = 20c. For both s = $60 and s = $300, the optimum inventory at week 6 is 22, 000 tons when h = 5c/ton, 20, 000 tons when h = 10c and 18, 000 tons at h = 15c and h = 20c. This shows the effect of the relative dominance of shortage cost (in higher inventory levels when holding costs are lower).

From the above, it can be deduced that any error in estimating holding costs will not significantly affect the results; the major source of problems is the estimation of shortage costs.

Table 2. Optimal inventory levels (in 10^3 tons) for different shortage & holding costs

Shortage	$60	$60	$60	$60	$300	$300	$300	$300
Holding	5c	10c	15c	20c	5c	10c	15c	20c
WK 18	0	0	0	0	2	2	2	2
17	0	0	0	0	2	2	2	2
16	0	0	0	0	4	4	4	4
15	2	2	2	2	10	10	10	10
14	4	4	4	2	16	16	16	16
13	6	6	6	6	14	14	14	14
12	8	8	8	8	22	22	22	22
11	12	12	12	12	22	22	22	22
10	16	16	16	16	22	22	22	22
9	8	8	8	8	8	8	8	8
8	22	22	22	22	22	22	22	22
7	20	20	20	20	20	20	20	20
6	22	20	18	18	22	20	18	18
5	8	8	8	8	8	8	8	8
4	6	6	6	6	6	6	6	6
3	10	10	10	10	10	10	10	10
2	14	14	14	14	14	14	14	14
1	2	2	2	2	2	2	2	2

5. Conclusions

The model's results are reasonable, to the extent that holding and shortage costs can be reasonably quantified. A major limitation is that because of the dominance of the relatively high shortage costs, the model determines a high inventory level even when there is only a very small probability of heavy snowfall, e.g., in weeks 10, 11, and 12, the probability is only .06 that snowfall of over 10 inches will occur (one winter out of 21 years in the sample). Since it is unrealistic to just drop out such isolated probabilities, the model can only be improved by using a larger sample, say with data for 50 to 100 years, which may provide lower probabilities for such isolated cases, e.g., if this occurs only once in 50 or 100 years. The model's results may then be more realistic, but would still depend on the relative dominance of shortage costs.

In conclusion, the model is applicable in practice; even if certain costs are difficult to quantify, it is possible to undertake sensitivity analysis under various cost estimates, as has been done in this study. Overall, an optimal solution was reached, but further study needs to be done to juxtapose this optimal solution with real values obtained over some time period in the future.

6. References

Autelitano, F., Rinaldi, M., Giuliani, F. (2019). Winter highway maintenance strategies: Are all the sodium

chloride salts the same? *Construction & Building Materials.* Nov. 2019, 226, 945-952. DOI: 10.1016/j.conbuildmat.2019.07.292

Bernardin, F. and Munch, A. (2019). Modeling and optimizing a road de-icing device by a nonlinear heating.

ESAIM: Mathematical Modelling & Numerical Analysis, 53(3), 775-803. DOI: 10.1051/m2an/2018056

Beuchat, P. N., Georghiou, A., Lygeros, J. (2020). Performance Guarantees for Model-Based Approximate Dynamic Programming in Continuous Spaces. Transactions on Automatic Control, 65(1), 143-158.

Chi, A. and Lohse, D. (1996, Jan. 10). *Hoist It, Heap It and Haul It: Managing Snow Removal.* Wall Street Journal, p B1.

Hossain, S.M.K., Fu, L.P., Lake, R. (2015). Field evaluation of the performance of alternative deicers for winter maintenance of transportation facilities. Canadian Journal of Civil Engineering. Jul2015, 42(7), 437-448.

Lilek, J. (2017). Roadway deicing in the United States: How a few industrial minerals supply a vital transportation service. AGI Critical Issues Program: www.americangeosciences.org/critical-issues

Ou, J. and Wein, L.M. (1995). Dynamic scheduling of a production/inventory system with products and random yield.

Management Science, 41,1000-17.

Peet, J. (1995, No. 10). Weather or not: predicting winter's impact is cloudy science. *The Star Ledger.*

Puterman, M.L. (1995). Markov decision processes: Discrete stochastic dynamic programming. *Interfaces,* 25, 93-4.

Ramaswamy, V. & Saleh, F. (2020). Ensemble Based Forecasting and Optimization Framework to Optimize

Releases from Water Supply Reservoirs for Flood Control. Water Resources Management, 34(3), 989-1004. DOI:10.1007/s11269-019-02481-8

Underwood, E. (1991). Every time it snows, it's pennies from Heaven. AdWeek's Marketing Week, 32, 9.

White, D.J. (1995). Finite Horizon Markov Decision Processed with Uncertain Terminal Payoffs. *Operations Research,* 43, 862-9.

Sauré, A.; Begen, M., Patrick, J. (2020). Dynamic multi-priority, multi-class patient scheduling with stochastic service times. European Journal of Operational Research, 280(1), p254-265.

Sullivan, J. L., Dowds, J., Novak, D. C., Scott, D. M., Ragsdale, C. (2019). Development and application of an iterative heuristic for roadway snow and ice control. *Transportation Research Part A: Policy & Practice,* 127, 18-31. DOI: 10.1016/j.tra.2019.06.021

Tiwari, A. and Rachlin, J.W. (2018). A review of road salt ecological impacts. Northeastern Naturalist, 25(1), 123-142. DOI: 10.1656/045.025.0110