Uniqueness of inverse source problems for some semilinear elliptic equations

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Abstract
Uniqueness of positive f is established in the inverse source problem $-\Delta u + c(x,u) = f(x)$ under conditions on the known coefficient *c*. This inverse problem is significant in the areas of semiconductor manufacturing, and in the study of ion channels in biology and health care.

1 Introduction

Solving inverse source problems allows one to detect the presence of bodies or discontinuities which are not directly observable, like submarines or earthquake fault lines. The unknown object is called the "source," and is represented by a density function in a system of partial differential equations. Information can be gathered about the source by measuring signals from some distance away—for instance, from the surface of the earth or through the water. In the terminology of inverse problems these signals are known as "boundary data."

Physical applications such as detecting semiconductor defects in manufacturing and modeling ion channels in biological systems provide motivation for studying the partial differential equations

$$-\Delta u + c(x,u) = f(x) \text{ in } \Omega \subseteq R^n, \quad n \geq 2, \tag{1}$$

$$u = g \text{ on } \partial\Omega \tag{2}$$

Quality control of semiconductors in the computer industry, as well as biological research and health care technology, require the solving of this particular inverse source problem: determining the values of the function *f*. At each point *x*, the value of *f*(*x*) indicates a physical property, e.g., concentration of germanium in a semiconductor, the density of white blood cells, etc.

The nature of each of these inverse source problems imposes certain conditions of physics which must be interpreted as a mathematical description of coefficient c. Under these physically meaningful conditions it is possible to establish uniqueness of the density *f*.

Establishing uniqueness of the solution to the inverse source problem gives evidence that there is a reasonable way to produce equipment that measures the signals, or "boundary data," and determines the properties of the source which cannot be found directly.

If the solution is not unique, it means that there is no hope to solve the problem even using the aid of a computer. The problem is said to be "ill-posed" in the sense that it is unreasonable to expect that any solution obtained will be the desired one. This gives scientists a clear indication that another method should be used for detecting properties of the unknown source.

2 Discussion and results

In equation (1) we consider Ω to be a bounded domain, *f* to be a nonnegative function, the boundary to be smooth, and we require that Neumann data (the rate of change of the signal at the boundary) may be measured for each set of prescribed Dirichlet data (the value of the signal at the boundary), i.e., that the Dirichlet-to-Neumann map be known. A less sophisticated term for the Dirichlet-to-Neumann map is that "many boundary measurements" are known.

The inverse problem is highly ill-conditioned, and yet there does exist a unique solution.

Theorem. *Let Ω have a connected complement $R^n - \overline{\Omega}$. Assume that c is a known real-valued function which is convex and non-decreasing in u. Then f(x) is uniquely determined in* Ω

To prove the theorem we use boundary data $g_1 = t + \tau g$ and $g_2 = t$ with $t \in R$ and $\tau \in [0,1]$. We form finite differences to the solution obtained with boundary data $g_2 - g_1 = \tau g$.

Upon dividing the equation by τ and using elliptic estimates, one concludes that "differentiation from the boundary" or "differentiation with respect to a parameter" is justified using the parameter τ, and so one obtained a linearized equation $-\Delta u_\tau + c_u u_\tau = 0 \text{ in } \Omega$ for the function $v = u_\tau$.

From known theory of elliptic equations, there is a unique solution v. Since $c(x, u)$ is convex, $c_u(\cdot, u)$ is a one-to-one function in u. Thus u is determined uniquely, and we insert this u into the original equation which gives us a unique function $f(x) = -\Delta u(x;t) + c(x, u(x;t))$.

The major portion of the proof deals with justification of differentiation from the boundary in equations (1), (2). The maximum principle plays an important role in the proof, as well as functional spaces and the tracing of delicate dependencies of function compositions on the boundary data.

3 Conclusion

When c is a known function, then there is a unique function *f* which solves the inverse source problem $-\Delta u + c(x, u) = f(x)$. There may be hope for showing uniqueness when c is an unknown function subject to further restrictions.

Perhaps the assumption which is most critical to the proof is the knowledge of the Dirichlet-to- Neumann map, or "many boundary measurements." In the laboratory one can find this situation. However, in applications like radar for defense maneuvers this assumption is less realistic, because who is going to surround the enemy airplanes with sensors?

Most importantly, uniqueness permits us to reach the conclusion that there will be a convergent computational algorithm for solving the inverse source problem successfully.

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