KALMAN FILTER DESIGN FOR LARGE-SCALE SYSTEMS BY USING UNIFIED APPROACH

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The following faculty members have examined the final copy of this thesis for form and content, and recommend that it be accepted in partial fulfillment of the requirement for the degree of Master of Science, with a major in Electrical Engineering

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John Watkins, Committee Chair

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M. Edwin Sawan, Committee Member

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Ziqi Sun, Committee Member

DEDICATON

To my parents, my wife, and my dear friends

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ABSTRACT

In this thesis, a method of designing a Kalman filter for a linear, discrete-time, singularly perturbed stochastic system using the delta operator was introduced. This unified approach, which is based on the delta operator, was the main method used to unify the continuous-time system and the discrete-time system. This method has many advantages over the q-operator: it makes the system simpler, it has better finite word-length characteristics, and the truncation and round-off error is less. One of the singular perturbation techniques, *quasi-steady state* approximation, was used to separate the system into a slow subsystem and a fast subsystem. Then, the exact solution of the Kalman filter and the minimized mean square error for the full-order system, and the composite solution of the Kalman filter and minimized mean square error for the two subsystems were solved. This method was applied using a numerical example in which the steady-state Kalman filter solution was found.

TABLE OF CONTENTS

TABLE OF CONTENTS (continued)

CHAPTER 1

INTRODUCTION

In the early 1950s, classical control theory was established. Many procedures were presented by engineers to analyze and design a system (Mohammad, 1983). As a result of that theory, controls have become very commonplace in people's lives. However, they do not make things; rather, they make things work better by changing existing parameters or adding new parameters. This thesis will introduce a method to analyze and design a complex system that is common in our lives in order to make it safe and convenient.

Since the world is developing rapidly, science and technology are very popular topics in many countries. People like to talk about the advanced technique, which is based on a large-scale system. What is a large-scale system? A simple answer is that it is a very complex system. Relative to this question, there are two viewpoints: One is that a large-scale system can be separated into a number of different subsystems that have some connection with each other by some method. The other is that a large-scale system has at least two controllers. For now, these two viewpoints refer to hierarchical structures and decentralized structures. In our normal lives, these two kinds of structures represent all systems: energy, network, power, space structure, transportation, and so on (Mohammad, 1983).

Large-scale systems are very common in people's lives. As a result, many related problems are also emerging. For example, how do you control central air conditioning? How do you control or observe a power station that is far from a downtown area? How do you find a place that repairs aircraft? All of these problems are concerned with large-scale systems research.

In this thesis, a large-scale system, also called a stochastic system, was modeled, analyzed, and designed using a unified approach. Three key concepts here are unified approach, large-scale system, and stochastic system. This thesis provides a method to analyze these three concepts in detail and then combine them to obtain a state feedback gain to improve the system.

CHAPTER 2

UNIFIED APPROACH

2.1 Introduction

The unified approach is a method to unify the continuous-time system and the discrete-time system. It more resembles a transform, which is based on the delta operator. The differential operator, d/dt, is used in the continuous-time system, and the q-operator is used in the discrete-time system (Shim & Sawan, 2005). However, it is easy to see that these two kinds of operators have no connection, if some of the properties are linked together. For example, consider the stability of a linear system. A continuous-time system is stable if the real part of the poles is less than zero, but a discrete-time system is stable when the magnitude of the poles is less than one.

Another weakness of the q-operator involves finite word-length (Shim & Sawan, 2005). Poles of the discrete-time system are more complicated than that of the continuous-time system because they are always less than one. If some poles that are closer to one are used on a computer, the code will be very complex (Shim & Sawan, 2005). Middleton and Goodwin (1986) found that finite word-length characteristics are better when using the delta operator than when using the q-operator. Also, Janecki (1988) claimed that the low resolution of the stability circle may cause instability and the pole/zero cancellation problem. However, if the discrete-time system is reduced by the delta operator, then the new model will lose nothing and act more like a continuous-time system. Also, round-off and truncation errors will not be a problem affecting the finite word-length characteristics.

In other words, for the Kalman filter, which is the focus of this thesis, the delta operator, rather than the q-operator, is the better way to design the Kalman filter design (Salagodo, Middleton, & Goodwin, 1988). For the q-operator, the discrete Riccati equation has no connection with the continuous Riccati equation. However, if the delta operator is used for the discrete Riccati equation, then it is easy to determine the connection between the continuous and discrete Riccati equations. It also means that both the continuous and discrete Riccati equations can be shown on the delta form of the Riccati equation (Salagodo et al. 1988). Li and Gevers (1993b) also provide some advantages of the delta operator in the transfer function part.

The main idea in this thesis is that the continuous-time system and the discrete-time system are unified by the delta operator.

2.2 Delta Operator (Middleton & Goodwin, 1990)

Consider a linear and time-invariant continuous system as

$$\frac{dx}{dt} = Ax(t) + Bu(t), y(t) = Cx(t) \tag{2.1}$$

where x is an $n \times 1$ state vector, u is an $r \times 1$ control vector, A is an $n \times n$ matrix, and B is an $n \times r$ martix. According to Middleton and Goodwin (1986), given a sampling interval Δ and zero-order hold, the system shown by equation (2.1) can be changed to

$$x(k+1) = qx(k) = A_q x(k) + B_q u(k) \tag{2.2}$$

$$y(k) = C_q x(k) \tag{2.3}$$

where

$$A_q = e^{A\Delta} \tag{2.4}$$

$$B_q = \int_0^{\Delta} e^{A(\Delta-\tau)} B d\tau = A^{-1}(e^{A\Delta} - I)B \tag{2.5}$$

$$C_q = C \tag{2.6}$$

From this transformation, it is easy to see the relationship between the continuous-time system and the discrete-time system.

Then, by using the same sampling interval Δ, the delta operator, as defined by Middleton and Goodwin (1990), is

$$\delta = \frac{q-1}{\Delta} \tag{2.7}$$

where q is the q-operator notion, and the relationship is given as

$$qx(k) = x(k+1) \tag{2.8}$$

From equations (2.7) and (2.8), the relationship of the q-operator to the delta operator is

$$\delta x(k) = \frac{qx(k)-x(k)}{\Delta} \tag{2.9}$$

Using equation (2.9), the system shown by equations (2.2) and (2.3) can be rewritten as

$$\delta x(k) = A_\delta x(k) + B_\delta u(k) \tag{2.10}$$

$$y(k) = C_\delta x(k) \tag{2.11}$$

where the matrixes of equations (2.2), (2.3), (2.10), and (2.11) have the same relation as

$$A_\delta = \frac{A_q - I}{\Delta},\ B_\delta = \frac{B_q}{\Delta},\ C_\delta = C_q \tag{2.12}$$

Now consider the continuous-time system, and the relationship between the matrix of equation (2.1) and the matrix of equation (2.10) defined as

$$A_\delta = \Omega A, B_\delta = \Omega B \tag{2.13}$$

where

$$\Omega = \frac{1}{\Delta}\int_0^\Delta e^{A\tau} d\tau = I + \frac{A\Delta}{2!} + \frac{A^2\Delta^2}{3!} + \cdots \tag{2.14}$$

It is easy to determine that when $\Delta \to 0$, Ω goes to I and $A_\delta = A$.

As shown, the relationships between the differential operator, q-operator, and delta operator are represented. Also, it is easy to see that both the continuous-time and discrete-time systems could be transformed into a unified model based on the delta operator.

2.3 Unified Approach

By using the delta operator to unify the continuous-time and discrete-time systems, the new model, called the unified model, is shown as

$$\rho x(\tau) = A_\rho x(\tau) + B_\rho u(\tau) \tag{2.15}$$

$$y(\tau) = C_\rho x(\tau) \tag{2.16}$$

where

$$A_\rho = \begin{Bmatrix} A \\ A_\delta \end{Bmatrix},\ B_\rho = \begin{Bmatrix} B \\ B_\delta \end{Bmatrix},\ \rho = \begin{Bmatrix} \frac{d}{dt} \\ \delta \end{Bmatrix},\ \tau = \begin{Bmatrix} t \\ k \end{Bmatrix} \tag{2.17}$$

The first row denotes the continuous-time system, and the second row denotes the discrete-time system. For the unified model, the operator is the delta operator, and the transform variable is γ. The stability regions are defined as

$$\frac{\Delta}{2}|\gamma|^2 + Re\{\gamma\} < 0 \tag{2.18}$$

2.4 Summary

This chapter introduced the delta operator and how to use it to transform the continuous-time and discrete-time systems into the unified model, which contains both systems. This is called the unified approach.

CHAPTER 3

SINGULAR PERTURBATION

3.1 Introduction

The system focused on in this research is a large-scale system. As mentioned in the introduction, a large-scale system has two types of structures: hierarchical and decentralized. This thesis is only concerned with the hierarchical structure, which is a two-time-scale system.

A two-time-scale system is a system that has two different types of eigenvalues, slow and fast. Because of this, the system can be separated into a slow subsystem and a fast subsystem by using the matrix block diagonalization method and *quasi-steady state* approximation (Chang, 1974; Kokotovic, 1975; Chow & Kokotovic 1976). These two methods together are called the singular perturbation technique. In 1987 and 1988, Naidu and Price gave an explanation for singularly perturbed discrete systems, and some applications were made (Mahmoud, 1982; Mahmoud & Chen, 1983; Tran & Sawan, 1983, 1984; Mahmoud & Singh, 1985; Mahmoud, Chen, & Singh, 1986). In this thesis, the *quasi-steady state* approximation was used along with the unified approach to unify the continuous and discrete singularly perturbed systems (Shim & Sawan, 2001, 2002; Lee , Shim & Sawan., 2005).

3.2 Singularly Perturbed Systems

Consider the following two-time-scale system:

$$\begin{bmatrix} \dot{x(t)} \\ \varepsilon \dot{z(t)} \end{bmatrix} = \begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix} \begin{bmatrix} x(t) \\ z(t) \end{bmatrix} + \begin{bmatrix} B_1 \\ B_2 \end{bmatrix} u(t), y(t) = \begin{bmatrix} C_1 & C_2 \end{bmatrix} \begin{bmatrix} x(t) \\ z(t) \end{bmatrix} \tag{3.1}$$

where $x(t)$ and $z(t)$ are $n \times m$ state vectors, $u(t)$ is a $r \times 1$ control vector, A_{ij} are appropriate dimension matrices, and A_{22} must be nonsingular. The eigenvalues λ are defined as

$$0 < |\lambda_{s1}| \leq |\lambda_{s2}| \cdots \leq |\lambda_{sn}| < |\lambda_{f1}| \leq |\lambda_{f2}| \cdots \leq |\lambda_{fm}| < \left|\frac{2}{\Delta}\right| \tag{3.2}$$

The singular perturbation parameter ε is defined as

$$\varepsilon = \frac{|\lambda_{sn}|}{|\lambda_{f1}|} \ll 1 \tag{3.3}$$

Therefore,

$$|\lambda_{max}(A_s)| \ll |\lambda_{min}(A_f)| \tag{3.4}$$

From the norm properties of the invertible matrices, this can be equivalent to

$$|A_f|^{-1} \ll |A_s|^{-1} \tag{3.5}$$

Now, the system shown by equation (3.1) could be separated into a slow subsystem and a fast subsystem (Shim & Sawan, 2005).

3.3 *Quasi-Steady State* Approximation (Chang, 1974; Kokotovic, 1975; Chow & Kokotovic 1976)

Letting $\varepsilon \dot{z(t)}$ go to zero, the system shown by equation (3.1) becomes

$$\begin{bmatrix} \dot{\bar{x}(t)} \\ 0 \end{bmatrix} = \begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix} \begin{bmatrix} \bar{x}(t) \\ \bar{z}(t) \end{bmatrix} + \begin{bmatrix} B_1 \\ B_2 \end{bmatrix} \bar{u}(t), \bar{y}(t) = \begin{bmatrix} C_1 & C_2 \end{bmatrix} \begin{bmatrix} \bar{x}(t) \\ \bar{z}(t) \end{bmatrix} \tag{3.6}$$

where the state and control vectors with bars are called the *quasi-steady state.* The remaining part of the system is the slow part of the system.

From equation (3.6), the system can be rewritten as

$$\dot{\bar{x}(t)} = A_{11}\bar{x}(t) + A_{12}\bar{z}(t) + B_1\bar{u}(t) \tag{3.7}$$

$$0 = A_{21}\bar{x}(t) + A_{22}\bar{z}(t) + B_2\bar{u}(t) \tag{3.8}$$

$$\bar{y}(t) = C_1\bar{x}(t) + C_2\bar{z}(t) \tag{3.9}$$

Equation (3.8) can be rewritten as

$$\bar{z}(t) = -A_{22}^{-1}(A_{21}\bar{x}(t) + B_2\bar{u}(t)) \tag{3.10}$$

Then equation (3.10) can be put into equations (3.7) and (3.9), resulting in

$$\dot{\bar{x}(t)} = (A_{11} - A_{22}^{-1}A_{12}A_{21})\bar{x}(t) + (B_2 - A_{22}^{-1}A_{12}B_2)\bar{u}(t) \tag{3.11}$$

$$\bar{y}(t) = (C_1 - C_2 A_{22}^{-1} A_{21})\bar{x}(t) \tag{3.12}$$

From equations (3.11) and (3.12), the slow subsystem can be defined as

$$\dot{x_s}(t) = A_0 x_s(t) + B_0 u_s(t), x_s(0) = x_{s0} \tag{3.13}$$

$$y_s(t) = C_0 x_s(t) \tag{3.14}$$

where

$$A_0 = A_{11} - A_{22}^{-1} A_{12} A_{21}, B_0 = B_2 - A_{22}^{-1} A_{12} B_2, C_0 = C_1 - C_2 A_{22}^{-1} A_{21} \tag{3.15}$$

For the fast subsystem, it is assumed that the slow variables are constant during fast modes. Therefore, using $z(t)$ and $y(t)$ in equation (3.1) to subtract equations (3.8) and (3.9), the result is

$$\varepsilon(\dot{z(t)} - \dot{\bar{z}(t)}) = A_{21}\big(x(t) - \bar{x}(t)\big) + A_{22}\big(z(t) - \bar{z}(t)\big) + B_2\big(u(t) - \bar{u}(t)\big) \tag{3.16}$$

$$y(t) - \bar{y}(t) = C_1\big(x(t) - \bar{x}(t)\big) + C_2(z(t) - \bar{z}(t)) \tag{3.17}$$

Assuming that $x(t) = \bar{x}(t)$, and let $z_f(t) = z(t) - \bar{z}(t)$, $u_f(t) = u(t) - \bar{u}(t)$ and $y_f(t) = y(t) - \bar{y}(t)$, then the fast subsystem is defined as

$$\varepsilon \dot{z_f}(t) = A_{22} z_f(t) + B_2 u_f(t), z_f(0) = z_0 - \bar{z_0} \tag{3.18}$$

$$y_f(t) = C_2 z_f \tag{3.19}$$

3.4 Summary

This chapter introduced the singular perturbation technique to reduce the order of a two-time-scale system, which is one type of a large-scale system. For the singular perturbation technique, the *quasi-steady state* approximation method (Chang, 1974; Kokotovic, 1975; Chow & Kokotovic 1976) was chosen to separate the two-time-scale system into a slow subsystem and a fast subsystem. After this transformation, the analysis and design will focus on subsystems.

CHAPTER 4

KALMAN FILTER

4.1 Introduction

Noise exists in every system. If a system is concerned with a noise, it can be called a stochastic system. This chapter will introduce a method to analyze an unknown stochastic system, which is called the Kalman filter design.

4.2 Observer

For some unavailable systems, the only way to know what is inside the system is by way of state feedback from the system. A new system was established to attach to the original system. This new system, called observer, and the state variables of the observer are estimates of them from the original system.

Consider the following system:

$$\dot{x(t)} = Ax(t) + Bu(t), y(t) = Cx(t) \tag{4.1}$$

The observer of the system shown by equation (4.1) is defined as

$$\dot{\overline{x}(t)} = Ax(t) + Bu(t) + L(y(t) - \overline{y}(t)) \tag{4.2}$$

where $x(t)$ and $\overline{x}(t)$ have the same dimensions, and L is the observer gain, which needs to be computed.

The estimation error is defined as

$$e(t) = x(t) - \bar{x}(t) \tag{4.3}$$

Then, using equations (4.1) and (4.2), it is easy to find that the error equation is defined as

$$\dot{e(t)} = (A - LC)e(t) \tag{4.4}$$

In order to guarantee that

$$\lim_{t\to\infty} e(t) = 0 \tag{4.5}$$

the observer gain L needs to ensure that the real part of all eigenvalues of $(A - LC)$ are negative.

4.3 Kalman Filter

This thesis was concerned with a stochastic system. An observer, which is attached to a stochastic system, and the observer gain allow the mean square error to be minimized, and this is called the Kalman filter.

Consider a stochastic system, as shown by equation (4.1), with some noises. The new system is defined as

$$\dot{x}(t) = Ax(t) + Bu(t) + w_1(t), y(t) = Cx(t) + w_2(t) \tag{4.6}$$

where w_1 and w_2 are vectors and White Gaussian noises with zero mean and V_1 and V_2 intensities. Also, w_1 and w_2 are independent, $V_1$is positive semi definite, V_2 is positive definite, and $x(0)$ is a random vector with mean $m_x(0)$ and covariance matrix Σ.

Using the observer equation (4.2) for equation (4.6), the error equation is defined as

$$\dot{e}(t) = (A - LC)e(t) + w_1(t) - Lw_2(t) \tag{4.7}$$

For now, L needs to be found to guarantee that the mean square error is minimized. This means that the covariance matrix K of the error must be found, as well as the trace of K. The solution for this problem is

$$\dot{\bar{x}}(t) = (A - L^*(t)C)\bar{x}(t) + Bu(t) + L^*(t)y(t) \tag{4.8}$$

$$L^*(t) = K(t)C^T V_2^{-1} \tag{4.9}$$

$$\dot{K}(t) = K(t)A + A^T K(t) + V_1 - K(t)C^T V_2^{-1} CK(t), K(0) = K_0 \tag{4.10}$$

where $L^*(t)$ is the optimal observer gain, and $Tr[K(t)]$ is the minimized mean square error. Equation (4.8) is called the Kalman filter.

Consider the steady-state situation when time goes to infinity and $\dot{K(t)}$ goes to zero. Then equation (4.10) can be rewritten as

$$0 = KA + A^T K + V_1 - KC^T V_2^{-1} CK \tag{4.11}$$

This equation is the solution of K in the steady-state situation. Of importance is that the system is asymptotically stable. This thesis is only concerned with the steady-state solution.

4.4 Kalman Filter for Delta Form Systems (Middleton & Goodwin, 1990)

Consider a stochastic system by using q-operator as

$$qx(k) = Ax(k) + w(k) \tag{4.12}$$

$$y(k) = Cx(k) + v(k) \tag{4.13}$$

where $w(k)$ and $v(k)$ are independent and white Gaussian noises with zero mean and W and V intensities. Because these two variables are independent, the covariance matrix of $w(k)$ and $v(k)$ is defined as

$$S = cov\{v, w\} = 0 \tag{4.14}$$

Then the observer and the optimal discrete-time observer gain of the system, shown by equations (4.12) and (4.13), are given as

$$q\tilde{x}(k) = A\tilde{x}(k) + L(k)(y(k) - C\tilde{x}(k)) \tag{4.15}$$

$$L(k) = (AP(k)C^T + S)(CP(k)C^T + V)^{-1} \tag{4.16}$$

where $\tilde{x}(0) = x_0$. The term $P(k)$ is defined as

$$P(k+1) = W + AP(k)A^T - AP(k)C^T(V + CP(k)C^T)^{-1}CP(k)A^T \tag{4.17}$$

The steady-state solution is

$$P = W + APA^T - APC^T(V + CPC^T)^{-1}CPA^T \tag{4.18}$$

Now using the unified approach to unify the system shown by equations (4.12) and (4.13) based on the delta operator, the unified model is given as

$$\rho x(\tau) = A_\rho x(\tau) + w_\rho(\tau) \tag{4.19}$$

$$y(\tau) = C_\rho x(\tau) + v_\rho(\tau) \tag{4.20}$$

where $w_\rho(\tau)$ and $v_\rho(\tau)$ are independent and white Gaussian noises with zero mean and W_ρ and V_ρ intensities, and

$$A_\rho = \frac{A-I}{\Delta}, C_\rho = C, W_\rho = \frac{W}{\Delta}, V_\rho = \Delta V \tag{4.21}$$

Therefore,

$$A = (\Delta A_\rho + I), C = C_\rho \tag{4.22}$$

Putting equations (4.22) and (4.21) into equations (4.15), (4.16), and (4.17), the solution of the Kalman filter of the delta form unified model is given as (Middleton and Goodwin, 1990)

$$\rho\bar{x}(\tau) = A_\rho\bar{x}(\tau) + L_\rho(y(\tau) - C_\rho\bar{x}(\tau)) \tag{4.23}$$

where

$$L_\rho(\tau) = \left[(\Delta A_\rho + I)K_\rho(\tau)C_\rho^T + S\right](\Delta C_\rho K_\rho(\tau)C_\rho^T + V_\rho)^{-1} \tag{4.24}$$

where $S = 0$ and

$$\rho K_\rho(\tau) = W_\rho + K_\rho(\tau){A_\rho}^T + A_\rho K_\rho(\tau) + \Delta A_\rho K_\rho(\tau){A_\rho}^T - L_\rho(\tau)(V_\rho + \Delta C_\rho K_\rho(\tau)C_\rho^T)L_\rho(\tau)^T \tag{4.25}$$

4.5 Summary

This chapter introduced the Kalman filter design. This filter is an estimator that has the optimal value of the mean square error. For the determined system, the value of the error was determined by eigenvalues of $(A - LC)$. For the stochastic system, the error equation was different because of noise. Therefore, the mean square error guaranteed that the error was minimized. The Riccati equation was used to find the solution of the mean square error. This chapter also showed the method of the Kalman filter design for the unified model, which is the main focus of this thesis.

CHAPTER 5

KALMAN FILTER DESIGN FOR LARGE-SCALE SYSTEM BY USING UNIFIED APPROACH

5.1 Definition

This thesis focused on designing a Kalman filter for a stochastic large-scale system by using the unified approach, where the system is a linear, discrete-time, shift-invariant singularly perturbed stochastic system with white Gaussian noises (Naidu 1988). In the first step, the discrete-time model was unified to a unified model by using the delta operator (Middleton and Goodwin 1990). Then the *quasi-steady state* approximation method (Chang, 1974; Kokotovic, 1975; Chow & Kokotovic 1976) was used to separate the new model into a slow subsystem and a fast subsystem. Last, a Kalman filter was designed for the full-order system on a delta form and two subsystems, and then the mean square error was compared to the exact result and the composite result.

5.2 Conditions and Limitations

Consider a discrete-time, shift-invariant, singularly perturbed stochastic system:

$$\begin{bmatrix} x(k+1) \\ \varepsilon z(k+1) \end{bmatrix} = \begin{bmatrix} A_{d11} & A_{d12} \\ A_{d21} & A_{d22} \end{bmatrix} \begin{bmatrix} x(k) \\ z(k) \end{bmatrix} + \begin{bmatrix} B_{d1} \\ B_{d2} \end{bmatrix} u(k) + \begin{bmatrix} G_{d1} \\ G_{d2} \end{bmatrix} w_d(k) \tag{5.1}$$

$$y(k) = \begin{bmatrix} C_{d1} & C_{d2} \end{bmatrix} \begin{bmatrix} x(k) \\ z(k) \end{bmatrix} + v_d(k) \tag{5.2}$$

where

- $x(k)$ is an $n \times 1$ slow-state vector, $z(k)$ is an $m \times 1$ fast state vector, $u(t)$ is an $r \times 1$ input vector, and $y(t)$ is an $q \times 1$ output vector.
- A_{dij}, B_{di}, G_{di}, and C_{dj} are appropriate dimension matrices, and A_{d22} must be nonsingular.

- $w_d(k)$ and $v_d(k)$ are independent and White Gaussian noises with zero mean and W_d and V_d intensities.
- ε is the singular perturbation parameter defined as

$$\varepsilon = \frac{|\lambda_{f1}|}{|\lambda_{sn}|} \ll 1 \tag{5.3}$$

- eigenvalues λ of the system are

$$1 > |\lambda_{s1}| \geq |\lambda_{s2}| \cdots \geq |\lambda_{sn}| > |\lambda_{f1}| \geq |\lambda_{f2}| \cdots \geq |\lambda_{fm}| \tag{5.4}$$

5.3 Objective and Design Specification

For the system shown by equations (5.1) and (5.2), the main objective was designing the Kalman filter. The Kalman filter estimator is given as

$$\rho\bar{X} = (A - LC)\bar{X} + Bu + Ly \tag{5.5}$$

In this design, L is the optimal Kalman gain. It is an optimal value because the solution of the Riccati equation of the covariance matrix of error e, which contains L, is minimized. Equation (5.5) is a general form of the Kalman filter. In this problem, the exact solution of the full-order system was solved, meaning that matrixes A and B were of the full order based on the delta form. Then, because that system is a singularly perturbed system, the composite solution was sought. Therefore, the next step was the Kalman filter design for the different subsystems, which were separated using *quasi-steady state* approximation (Chang, 1974; Kokotovic, 1975; Chow & Kokotovic 1976). In these subsystems, the orders were reduced so that the matrix in equation (5.5), which was used for subsystems, would be different. This different matrix was the matrix of the subsystems. Also, all parameters were different from the Riccati equations.

The final objective was the mean square error because the optimal value of L came from the minimized mean square error. This means that only L can ensure that the mean square error is minimized, that L can be called the optimal value, and that the estimator, which is designed like equation (5.5), can be called the Kalman filter. Finally, by comparing the solution of the mean square error, the best value was found.

The method and exact matrixes of the models are shown in detail in section 5.4.

5.4 Analysis

With system equations (5.1) and (5.2), it is evident that this system is a very common discrete-time system. The most common method for designing a Kalman filter for this type of system is by solving the discrete Riccati differential equation to obtain the mean square error. Then the optimal Kalman gain is found by using the solution of the Riccati equation. However, in this thesis, a new method was introduced. The delta operator was very important for developing this method because the system was unified by the delta operator and then transformed into the unified model. The delta form of the differential Riccati equation was used to solve the mean square error of the unified model. Because this system is a singularly perturbed system, it concerned the composite solution, and this thesis used the *quasi-steady state* approximation method (Chang, 1974; Kokotovic, 1975; Chow & Kokotovic 1976) to separate the system into a fast part and a slow part; then the same method was used to solve both subsystems and to obtain the composite solution. This is the core part of the new method. The details of it will follow.

5.5 System Unification

Consider the system shown in equations (5.1) and (5.2). System unification was accomplished by the delta operator in the following steps.

Step 1: Rewrite the system by using the q-operator as

$$\begin{bmatrix} qx(k) \\ \varepsilon qz(k) \end{bmatrix} = \begin{bmatrix} A_{q11} & A_{q12} \\ A_{q21} & A_{q22} \end{bmatrix} \begin{bmatrix} x(k) \\ z(k) \end{bmatrix} + \begin{bmatrix} B_{q1} \\ B_{q2} \end{bmatrix} u(k) + \begin{bmatrix} G_{q1} \\ G_{q2} \end{bmatrix} w_q(k) \tag{5.6}$$

$$y(k) = \begin{bmatrix} C_{q1} & C_{q2} \end{bmatrix} \begin{bmatrix} x(k) \\ z(k) \end{bmatrix} + v_q(k) \quad (5.7)$$

where A_{qij}=A_{dij}, $B_{qi} = B_{di}$, $G_{qi} = G_{di}$, $C_{qj} = C_{dj}$, $w_q(k) = w_d(k)$, and $v_q(k) = v_d(k)$. In this step, nothing has changed.

Step 2: Given a sampling intervalΔ, use the delta operator to unify the system as

$$\begin{bmatrix} \rho x(\tau) \\ \varepsilon \rho z(\tau) \end{bmatrix} = \begin{bmatrix} A_{\delta 11} & A_{\delta 12} \\ A_{\delta 21} & A_{\delta 22} \end{bmatrix} \begin{bmatrix} x(\tau) \\ z(\tau) \end{bmatrix} + \begin{bmatrix} B_{\delta 1} \\ B_{\delta 2} \end{bmatrix} u(\tau) + \begin{bmatrix} G_{\delta 1} \\ G_{\delta 2} \end{bmatrix} w_\delta(\tau) \quad (5.8)$$

$$y(\tau) = \begin{bmatrix} C_{\delta 1} & C_{\delta 2} \end{bmatrix} \begin{bmatrix} x(\tau) \\ z(\tau) \end{bmatrix} + v_\delta(\tau) \quad (5.9)$$

where $A_{\delta ij} = \frac{A_{qij-I}}{\Delta}$, $B_{\delta i}$=$\frac{B_{qi}}{\Delta}$, $C_{\delta j} = C_{qj}$, $G_{\delta i} = \frac{G_{qi}}{\Delta}$, $W_\delta = \frac{W_d}{\Delta}$, $V_\delta = \Delta V_d$, and $S_\delta = S_q = 0$. The main method used here came from Middleton and Goodwin (1990). Using this formula, a new delta-form is obtained. Because of this transform, the eigenvalues changed, which is a benefit of this system, as mentioned previously.

Step 3: Now let

$$\begin{bmatrix} A_{\delta 11} & A_{\delta 12} \\ A_{\delta 21} & A_{\delta 22} \end{bmatrix} = \begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix} \quad (5.10)$$

$$\begin{bmatrix} B_{\delta 1} \\ B_{\delta 2} \end{bmatrix} = \begin{bmatrix} B_1 \\ B_2 \end{bmatrix} \quad (5.11)$$

$$\begin{bmatrix} G_{\delta 1} \\ G_{\delta 2} \end{bmatrix} = \begin{bmatrix} G_1 \\ G_2 \end{bmatrix} \quad (5.12)$$

$$\begin{bmatrix} C_{\delta 1} & C_{\delta 2} \end{bmatrix} = \begin{bmatrix} C_1 & C_2 \end{bmatrix} \quad (5.13)$$

$$W = W_\delta, V = V_\delta, S = S_\delta \quad (5.14)$$

The system shown by equations (5.1) and (5.2) can be rewritten as

$$\begin{bmatrix} \rho x(\tau) \\ \varepsilon \rho z(\tau) \end{bmatrix} = \begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix} \begin{bmatrix} x(\tau) \\ z(\tau) \end{bmatrix} + \begin{bmatrix} B_1 \\ B_2 \end{bmatrix} u(\tau) + \begin{bmatrix} G_1 \\ G_2 \end{bmatrix} w(\tau) \quad (5.15)$$

$$y(\tau) = \begin{bmatrix} C_1 & C_2 \end{bmatrix} \begin{bmatrix} x(\tau) \\ z(\tau) \end{bmatrix} + v(\tau) \quad (5.16)$$

where $w(\tau)$is a white Gaussian noise with zero mean and $W = \frac{W_d}{\Delta}$ is intensity, and $v(\tau)$ is a white Gaussian noise with zero mean and $V = \Delta V_d$ intensity. This is the unified model.

Because the eigenvalues changed, the two-time-scale property for equations (5.8) and (5.9) also changed (Shim & Sawan, 2005) as

$$0 < |P_{s1}| \leq |P_{s2}| \cdots \leq |P_{sn}| < |P_{f1}| \leq |P_{f2}| \cdots \leq |P_{fm}| < \left|\frac{2}{\Delta}\right| \tag{5.17}$$

The parameter ε is defined as

$$\varepsilon = \frac{|P_{sn}|}{|P_{f1}|} \ll 1 \tag{5.18}$$

This system, shown by equations (5.8) and (5.9), is the most official form of the unified approach, so the Kalman filter was designed for this system.

5.6 Exact Solution

Consider the system model shown by equations (5.8) and (5.9), and let

$$X = \begin{bmatrix} x(\tau) \\ z(\tau) \end{bmatrix}, \begin{bmatrix} A_{11} & A_{12} \\ \frac{A_{21}}{\varepsilon} & \frac{A_{22}}{\varepsilon} \end{bmatrix} = A, \begin{bmatrix} B_1 \\ \frac{B_2}{\varepsilon} \end{bmatrix} = B, \begin{bmatrix} G_1 \\ \frac{G_2}{\varepsilon} \end{bmatrix} = G, [C_1 \quad C_2] = C \tag{5.19}$$

The exact solution of this system is given as (Middleton & Goodwin, 1990)

$$\rho\bar{X}(\tau) = (A - LC)\bar{X}(\tau) + Bu(\tau) + Ly(\tau) \tag{5.20}$$

$$L = [(\Delta A + I)K(\tau)C^T + S](\Delta CK(\tau)C^T + V)^{-1} \tag{5.21}$$

where K is

$$\rho K(\tau) = GWG^T + K(\tau)A^T + AK(\tau) + \Delta AK(\tau)A^T - L(\Delta CK(\tau)C^T + V)^{-1}L^T \tag{5.22}$$

Equation (5.20) is the Kalman filter, L is the Kalman gain, and $Tr[K(\tau)]$ is the minimized mean square error.

5.7 Composite Solution

Rewrite the system shown by equations (5.8) and (5.9) as

$$\rho x(\tau) = A_{11}\, x(\tau) + A_{12}z(\tau) + B_1u(\tau) + G_1w(\tau) \tag{5.23}$$

$$\varepsilon\rho z(\tau) = A_{21}x(\tau) + A_{22}z(\tau) + B_1u(\tau) + G_2w(\tau) \tag{5.24}$$

$$y(\tau) = C_1x(\tau) + C_2z(\tau) + v(\tau) \tag{5.25}$$

Now using the singular perturbation technique, *quasi-steady state* approximation (Chang, 1974; Kokotovic, 1975; Chow & Kokotovic 1976), let $\rho z(\tau) = 0$. Then equations (5.23), (5.24), and (5.25) become

$$\rho\tilde{x}(\tau) = A_{11}\tilde{x}(\tau) + A_{12}\tilde{z}(\tau) + B_1\tilde{u}(\tau) + G_1w(\tau) \tag{5.26}$$

$$0 = A_{21}\tilde{x}(\tau) + A_{22}\tilde{z}(\tau) + B_1\tilde{u}(\tau) + G_2w(\tau) \tag{5.27}$$

$$\tilde{y}(\tau) = C_1\tilde{x}(\tau) + C_2\tilde{z}(\tau) + v(\tau) \tag{5.28}$$

Therefore,

$$\tilde{z}(\tau) = -A_{22}^{-1}(A_{21}\tilde{x}(\tau) + B_1\tilde{u}(\tau) + G_2w(\tau)) \tag{5.29}$$

Then, putting equation (5.29) into equations (5.26) and (5.28), the slow subsystem is defined as

$$\rho x_s(\tau) = A_0x_s(\tau) + B_0u(\tau) + G_0w(\tau) \tag{5.30}$$

$$y_s(\tau) = C_0x_s(\tau) + D_0u(\tau) + v_0(\tau) \tag{5.31}$$

where

$$x_s(\tau) = \tilde{x}(\tau)$$

$$A_0 = A_{11} - A_{12}A_{22}^{-1}A_{21}, B_0 = B_1 - A_{12}A_{22}^{-1}B_2$$

$$C_0 = C_1 - C_2A_{22}^{-1}A_{21}, D_0 = C_2A_{22}^{-1}B_2$$

$$G_0 = G_1 - A_{12}A_{22}^{-1}G_2$$

$$v_0(\tau) = C_2A_{22}^{-1}G_2w(\tau) + v(\tau) \tag{5.32}$$

Then

$$cov\{v_0(\tau)\} = C_2A_{22}^{-1}G_2WG_2^TA_{22}^{-1^T}C_2^T + V = R_s \tag{5.33}$$

The observer of the slow subsystem is defined as

$$\rho\bar{x}_s(\tau) = A_0\overline{x_s}(\tau) + B_0u(\tau) + L_s(y_s(\tau) - \overline{y_s}(\tau)) \tag{5.34}$$

$$\bar{y}_s(\tau) = C_0\bar{x}_s(\tau) + D_0u(\tau) \tag{5.35}$$

Therefore, the error equation of the slow subsystem is given as

$$\rho e_s(\tau) = \rho x_s(\tau) - \rho\bar{x}_s(\tau) = (A_0 - LC_0)e_s + G_0w(\tau) - Lv_0(\tau) \tag{5.36}$$

and the solution of slow subsystem is given as

$$\rho K_s = G_0WG_0^T + K_sA_0^T + A_0K_s + \Delta A_0K_sA_0^T - L_s(R_s + \Delta C_0K_sC_0)L_s^T \tag{5.37}$$

where

$$L_s = [(\Delta A_0 + I)K_sC_0^T + S_s](\Delta C_0K_sC_0 + R_s)^{-1} \tag{5.38}$$

The minimized mean square error is $Tr[K_s]$, and L_s is the optimal Kalman gain for the slow subsystem.

Now it is easy to obtain the fast subsystem, which is given as

$$\varepsilon\rho z_f(\tau) = A_{22}z_f(\tau) + B_2u_f(\tau) + G_2w(\tau) \tag{5.39}$$

$$y_f(\tau) = C_2z_f(\tau) + v(\tau) \tag{5.40}$$

Using the same method as for obtaining the slow subsystem, the solution of the Kalman filter of the fast subsystem is

$$\rho K_f = G_2WG_2^T + K_f\frac{A_{22}}{\varepsilon}^T + \frac{A_{22}}{\varepsilon}K_f + \Delta\frac{A_{22}}{\varepsilon}K_f\frac{A_{22}}{\varepsilon}^T - L_f(V + \Delta C_2K_fC_2)L_f^T \tag{5.41}$$

$$L_f = \left[\left(\Delta\frac{A_{22}}{\varepsilon} + I\right)K_fC_2^T + S_f\right](\Delta C_2K_fC_2 + V)^{-1} \tag{5.42}$$

$$\varepsilon\rho\bar{z}_f(\tau) = A_{22}\bar{z}_f(\tau) + B_2u_f(\tau) - L_f(y_f(\tau) - C_2\bar{z}_f(\tau)) \tag{5.43}$$

The minimized mean square error is $Tr[K_f]$, and the optimal Kalman gain of the fast subsystem is L_f. Equation (5.38) is the fast subsystem Kalman filter.

Combining the slow and fast subsystems, the composite solution of minimized mean square error is $Tr[K_s + K_f]$.

5.8 Summary

This chapter introduced the method of designing a Kalman filter for a linear, discrete-time, shift-invariant two-time-scale stochastic large-scale system by using the unified approach. First, the original system was unified by the delta operator (Middleton & Goodwin, 1990). Second, the delta form equation of the Kalman filter (Middleton & Goodwin, 1990) was used to obtain the exact solution of the full-order system. Third, the full-order system in delta form was separated into a slow subsystem and a fast subsystem by using the *quasi-steady state* approximation (Chang, 1974; Kokotovic, 1975; Chow & Kokotovic, 1976), and then the Kalman filters of both the slow and fast subsystems were solved by the delta form of the Kalman filter formula. Finally, the minimized mean square error of the exact solution and composite solution were compared.

CHAPTER 6

APPLICATION FOR KALMAN FILTER DESIGN FOR DISCRETE-TIME STOCHASTIC SYSTEM

6.1 Numerical Example

Consider a singularly perturbed discrete-time system (Shim & Sawan, 2001):

$$\begin{bmatrix} x(k+1) \\ z(k+1) \end{bmatrix} = \begin{bmatrix} 1 & 0.0399 & 0.0006 & 0 \\ 0 & 0.9922 & 0.0275 & 0.0028 \\ 0 & -0.4178 & 0.6215 & 0.1270 \\ 0 & 0 & 0 & 0.3679 \end{bmatrix} \begin{bmatrix} x(k) \\ z(k) \end{bmatrix} + \begin{bmatrix} 0 \\ 1 \\ 0 \\ 1 \end{bmatrix} w(k) \tag{6.1}$$

$$y(k) = \begin{bmatrix} 1 & 1 & 1 & 1 \end{bmatrix} \begin{bmatrix} x(k) \\ z(k) \end{bmatrix} + v(k) \tag{6.2}$$

and let $\Delta = 0.1$. Also assume $\varepsilon = 0.1$, which are included in the matrix of equation (6.1). The terms $w(k)$ and $v(k)$ are independent white Gaussian noises with zero mean and 1 and 1 intensities, respectively. Then design a steady-state Kalman filter for this system and compare the mean square errors of the exact and composite solutions.

The original system of this example was taken from the work of Shim & Sawan (2001). In this paper, some changes were made to ensure that system, shown by equations (6.1) and (6.2) had the same form as the system shown by equations (5.1) and (5.2).

6.2 Analysis

As mentioned previously, this system is a discrete Riccati algebraic equation. Keeping this in mind, the solution can be solved directly, but it is based on the q-operator, of which the weaknesses during designing were discussed earlier. Therefore, the new method shown in Chapter 5 will be used for solving this problem. This new method is based on the delta operator, which is more advantageous than the q-operator, as discussed previously.

However, if the method of the delta form is used for the full-order system, then the solution will be the exact solution for the system. For a singularly perturbed system, the composite solution is of main concern; therefore, the main point of this example is that the delta form method will be used for the subsystems after using the singular perturbation technique.

6.3 Solution

Let a 4×4 matrix be matrix A, a 4×1 matrix be matrix B, and a 1×4 matrix be matrix C. System unification using the delta operator is the first step. Actually, this method is based on the matrix, so the new matrix for the unified model is

$$A_\delta = \begin{bmatrix} 0 & 0.3990 & 0.006 & 0 \\ 0 & -0.0780 & 0.2750 & 0.0280 \\ 0 & -4.1780 & -3.7850 & 1.2700 \\ 0 & 0 & 0 & -6.3210 \end{bmatrix}, G_\delta = \begin{bmatrix} 0 \\ 10 \\ 0 \\ 10 \end{bmatrix}, C_\delta = [1 \quad 1 \quad 1 \quad 1]$$

For this new model, the equation shown in Chapter 5 can be used. The exact solution is

$$L^* = \begin{bmatrix} 0.0925 \\ 0.0129 \\ -0.0125 \\ 0.0019 \end{bmatrix}$$

and the mean square error of the exact solution is

$$e = Tr[K^*] = 1.2504$$

The result is the exact mean square error for the full-order unified model. Then, because the system is a singularly perturbed system, the new model also contains fast and slow eigenvalues; therefore, the model is also a two-time-scale model. The composite solution will be sought.

The next step, using the singular perturbation technique, separates the delta form model into slow and fast parts:

For the slow subsystem,

$$A_{\rho 0} = \begin{bmatrix} 0 & 0.3314 \\ 0 & -0.2598 \end{bmatrix}, \ G_{\rho 0} = \begin{bmatrix} 3.1850 \times 10^{-5} \\ 1.0019 \end{bmatrix}, \ C_{\rho 0} = [1 \quad -0.1038]$$

In this slow part, the measurement noise will be changed because of the transfer. Therefore, the new measurement noise is given as

$$v_{\rho 0} = -0.0211w(\tau) + v(\tau)$$

Therefore, the slow subsystem is

$$\rho x_{\rho s}(\tau) = A_{\rho 0}x_{\rho s}(\tau) + G_{\rho 0}w(\tau)$$

$$y_s = C_{\rho 0}x_{\rho s}(\tau) + v_0(\tau)$$

For this subsystem, the same method as used for the whole system could be used. Then, the Kalman filter for the slow subsystem is given as

$$L_s = \begin{bmatrix} 0.1079 \\ 0.0156 \end{bmatrix}$$

For the fast subsystem

$$A_{\rho f} = \begin{bmatrix} -3.7850 & 1.2700 \\ 0 & -6.3210 \end{bmatrix} \text{ and } C_{\rho f} = \begin{bmatrix} 1 & 1 \end{bmatrix}$$

The same steps will be used for the fast subsystem:

$$\rho z_{\rho f}(\tau) = A_{\rho f}z_{\rho f}(\tau) + G_{\rho f}w(\tau)$$

$$y_f = C_{\rho f}x_{\rho s}(\tau) + v(\tau)$$

Then the solution of the Kalman fitler for the fast subsystem is

$$L_f = \begin{bmatrix} 0.1328 \times 10^{-5} \\ 0.8904 \times 10^{-5} \end{bmatrix}$$

In designing a Kalman filter, the mean square error must be considered. For a perfect Kalman filter estimator, the mean square error must be the minimized. Therefore, for the composite solution the mean square error is the trace of both the solution of the delta form Riccati algebraic equation of the slow subsystem and the solution of the delta form Riccati algebraic equation of the fast subsystem. Therefore,for this system, the composite solution of the mean square error is

$$e_c = Tr\left[K_s + K_f\right] = 1.2866$$

6.4 Summary

From the results, it is easy to see that the exact solution was always the optimal solution for the system, because the mean square error e was minimized. On the other hand, this method, which is based on the delta operator, was a good one for designing the Kalman filter. The next concern was to determine which method is the best. By comparing the mean square errors, the method that obtains the smallest mean square error was the best method for the Kalman filter design of a discrete-time system.

CHAPTER 7

CONCLUSION AND FUTURE WORK

By comparing the mean square error from an exact solution and a composite solution, it is easy to find that the exact solution of the full-order system, which is the optimal solution, because that the mean square error of the exact solution is less than that of the composite solution, and the exact solution is the minimized mean square error. Also, the exact Kalman gain L^* is the optimal Kalman gain of the original system.

From the numerical example provided in Chapter 6, it can be seen that the new unified method is a good design for the Kalman filter for singularly perturbed, discrete-time, stochastic systems. For the continuous-time system, the only difference is that the sampling interval Δ is zero. In this condition, it is easy to determine that all equations on the delta form are the same as the original equations of the continuous-time system. As mention previously, this method is based on the delta operator. Any linear systems can be transformed into a unified model by using the delta operator. The delta form method reduces many notions and equations and simplifies the process of the calculation. However, this method will keep the accuracy during the calculation. Other benefits of using the delta operator include a better word-length characteristic and a decrease in the round-off error.

Future work could concern the best way to design a Kalman filter. For now, a discrete-time system involves three methods of designing a Kalman filter. The first method, based on the q-operator, is the most common. The second method is the one introduced by this thesis. A third method could be when the system is unified by a delta operator, where the unified model would be more like a continuous-time system, and then the method for continuous-time systems could also be used on the unified model. The best method would be found by comparing the solutions and other conditions.

Another topic for future work might be a linear-quadratic-Gaussian (LQG) design. This work would also be based on the delta operator. At this time, the unified approach will be used in a linear-quadratic-Gaussian system, but future work could involve the design of an optimal control feedback gain of a singularly perturbed LQG system by using the unified approach based on the delta operator.

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