

DEFORMATIONS OF Q -CURVATURE II

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ABSTRACT. This is the second article of a sequence of research on deformations of Q -curvature. In the previous one, we studied local stability and rigidity phenomena of Q -curvature. In this article, we mainly investigate the volume comparison with respect to Q -curvature. In particular, we show that volume comparison theorem holds for metrics close to strictly stable positive Einstein metrics. This result shows that Q -curvature can still control the volume of manifolds under certain conditions, which provides a fundamental geometric characterization of Q -curvature. Applying the same technique, we derive the local rigidity of strictly stable Ricci-flat manifolds with respect to Q -curvature, which shows the non-existence of metrics with positive Q -curvature near the reference metric.

1. INTRODUCTION

The Q -curvature is a 4th-order scalar type curvature. It has been studied for decades due to its geometric resemblance to Gaussian and scalar curvature as a higher-order curvature quantity.

For a closed 4-dimensional Riemannian manifold (M^4, g) , Q -curvature is defined to be

$$(1.1) \quad Q_g = -\frac{1}{6}\Delta_g R_g - \frac{1}{2}|Ric_g|_g^2 + \frac{1}{6}R_g^2.$$

It satisfies the *Gauss-Bonnet-Chern Formula*

$$\int_{M^4} \left(Q_g + \frac{1}{4}|W_g|_g^2 \right) dv_g = 8\pi^2 \chi(M),$$

where R_g , Ric_g , and W_g are scalar curvature, Ricci curvature, and Weyl tensor for (M^4, g) respectively. In particular, if (M^4, g) is locally conformally flat, *i.e.* $W_g = 0$, it reduces to

$$\int_{M^4} Q_g dv_g = 8\pi^2 \chi(M),$$

which can be viewed as a generalization of the classic *Gauss-Bonnet Theorem* for closed surfaces.

Branson ([2]) extended (1.1) and defined the Q -curvature for manifolds with dimension at least three to be

$$(1.2) \quad Q_g = A_n \Delta_g R_g + B_n |Ric_g|_g^2 + C_n R_g^2,$$

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where $A_n = -\frac{1}{2(n-1)}$, $B_n = -\frac{2}{(n-2)^2}$ and $C_n = \frac{n^2(n-4)+16(n-1)}{8(n-1)^2(n-2)^2}$. With the aid of the Paneitz operator ([17])

$$P_g = \Delta_g^2 - \operatorname{div}_g [(a_n R_g g + b_n \operatorname{Ric}_g) d] + \frac{n-4}{2} Q_g,$$

where $a_n = \frac{(n-2)^2+4}{2(n-1)(n-2)}$ and $b_n = -\frac{4}{n-2}$, Q -curvature shares a similar conformal transformation law as scalar curvature:

$$\begin{aligned} Q_{\hat{g}} &= e^{-4u} (P_g u + Q_g), & \text{for } n = 4 \text{ and } \hat{g} = e^{2u} g \\ Q_{\hat{g}} &= \frac{2}{n-4} u^{-\frac{n+4}{n-4}} P_g u, & \text{for } n \neq 4 \text{ and } \hat{g} = u^{\frac{4}{n-4}} g \end{aligned}$$

This suggests that Q -curvature is an 4th-order analogue of scalar curvature.

For a long time, mathematicians are seeking for a better understanding about geometric interpretations of Q -curvature especially in dimensions five and above. In the field of conformal geometry, there has been many excellent works regarding Q -curvature (see [9] for a great survey). Without the restrictions in conformal classes, there are not many results on Q -curvature from the viewpoint of Riemannian geometry so far.

The main purpose of the authors' research on Q -curvature is to investigate the Riemannian geometric properties of Q -curvature. For instances, the authors studied local stability and rigidity phenomena and derived some interesting geometric results about Q -curvature (see [13, 14] for more details). These results strongly suggest that Q -curvature shares analogous geometric properties as scalar curvature.

Volume comparison theorem is a fundamental result in differential geometry. It is important both theoretically and practically in the analysis of geometric problems. The classic volume comparison states that a lower bound for Ricci curvature implies the volume comparison of geodesic balls with those in the model spaces. A natural question is that whether we can replace the assumption on Ricci curvature by a weaker one? As for the scalar curvature, this idea has been proved to be feasible in some special situations ([20]).

Inspired by the second author's work [20], we consider the volume comparison for Q -curvature. As the first step, we give the definition of model spaces:

Definition 1.1. A Riemannian manifold (M^n, \bar{g}) is Q -critical, if there is a nontrivial function $f \in C^\infty(M)$ and a constant $\kappa \in \mathbb{R}$ such that

$$\Gamma_{\bar{g}}^* f = \kappa \bar{g},$$

where $\Gamma_{\bar{g}}^* : C^\infty(M) \rightarrow S_2(M)$ is the L^2 -formal adjoint of $\Gamma_{\bar{g}} := DQ_{\bar{g}}$, the linearization of Q -curvature at \bar{g} .

The concept of Q -critical metrics provides a standard model for volume comparison of Q -curvature. That is, we only need to consider the volume comparison with respect to Q -critical metrics. The reason is that for non- Q -critical metrics, one can perturb Q -curvature and the volume simultaneously without any constraint:

Theorem 1.2 (Case-Lin-Yuan [4]). *Let (M^n, g) be a closed Riemannian manifold. If (M^n, g) is not Q -critical, then there are neighborhoods U of Riemannian metrics of g and $V \subset C^\infty(M) \oplus \mathbb{R}$ of $(Q_g, V_M(g))$ such that for any $(\psi, v) \in V$, there is a metric $\hat{g} \in U$ such that $Q_{\hat{g}} = \psi$ and $V_M(\hat{g}) = v$.*

Remark 1.3. Corvino, Eichmair, and Miao first showed such type of theorem holds for scalar curvature ([6]). For a more general version regarding *conformally variational invariants*, please refer to [4].

As a special case of Q -critical metrics, J -Einstein metrics are the most basic examples ([14]):

Definition 1.4. Let (M^n, g) be a Riemannian manifold ($n \geq 3$). We define a symmetric 2-tensor associated to Q -curvature called J -tensor to be

$$J_g := -\frac{1}{2}\Gamma_g^*(1).$$

A metric is called J -Einstein, if $J_g = \Lambda g$ for some constant $\Lambda \in \mathbb{R}$.

In particular, if \bar{g} is Einstein, one can check that

$$J_{\bar{g}} = \frac{1}{n}Q_{\bar{g}}\bar{g} = \frac{(n-2)(n+2)}{8n^2(n-1)^2}R_{\bar{g}}^2\bar{g}.$$

Therefore, Einstein metrics are J -Einstein and hence Q -critical. However, under certain conditions, a J -Einstein metric has to be Einstein. We present a characterization of Einstein metrics in terms of the spectrum of the Einstein operator

$$\Delta_E^{\bar{g}} = \Delta_{\bar{g}} + 2Rm_{\bar{g}}$$

defined on the space of symmetric 2-tensors.

Theorem 1.5. *Suppose (M^n, \bar{g}) is an n -dimensional closed J -Einstein manifold and the Einstein operator $\Delta_E^{\bar{g}}$ on $S_2(M)$ satisfies*

$$\Lambda_E^{\bar{g}} := \inf_{h \in S_2(M) \setminus \{0\}} \frac{\int_M \langle h, -\Delta_E^{\bar{g}} h \rangle_{\bar{g}} dv_{\bar{g}}}{\int_M |h|_{\bar{g}}^2 dv_{\bar{g}}} > -\frac{(n-2)^3(n+2)}{8n(n-1)^2} \min_M R_{\bar{g}}.$$

Furthermore, we assume the scalar curvature $R_{\bar{g}}$ is a constant when $3 \leq n \leq 8$. Then \bar{g} is an Einstein metric.

This result is an important motivation for one to take Einstein metrics to be reference metrics when considering the volume comparison with respect to Q -curvature. Unfortunately, being an Einstein metric is not sufficient for volume comparison to hold. In fact, one needs to impose a stronger assumption on Einstein metrics.

Definition 1.6 (Stability of Einstein manifolds [1, 10]). For $n \geq 3$, suppose (M^n, \bar{g}) is a closed Einstein manifold. The Einstein metric \bar{g} is said to be *strictly stable*, if the *Einstein operator*

$$\Delta_E^{\bar{g}} = \Delta_{\bar{g}} + 2Rm_{\bar{g}}$$

is a negative operator on $S_{2,\bar{g}}^{TT}(M)\setminus\{0\}$, where

$$S_{2,\bar{g}}^{TT}(M) := \{h \in S_2(M) \mid \delta_{\bar{g}}h = 0, \operatorname{tr}_{\bar{g}}h = 0\}$$

is the space of transverse-traceless symmetric 2-tensors on (M^n, \bar{g}) .

Now we state our main result in this article, which concerns a volume comparison with respect to Q -curvature for closed strictly stable Einstein manifolds.

Theorem 1.7. *For $n \geq 3$, suppose (M^n, \bar{g}) is an n -dimensional closed strictly stable Einstein manifold with Ricci curvature*

$$\operatorname{Ric}_{\bar{g}} = (n-1)\lambda\bar{g},$$

where $\lambda > 0$ is a constant. Then there exists a constant $\varepsilon_0 > 0$ such that for any metric g on M satisfying

$$Q_g \geq Q_{\bar{g}}$$

and

$$\|g - \bar{g}\|_{C^4(M,\bar{g})} < \varepsilon_0,$$

the following volume comparison holds

$$V_M(g) \leq V_M(\bar{g}),$$

with the equality holds if and only if g is isometric to \bar{g} .

Remark 1.8. The above volume comparison does not hold for Ricci flat metrics. This is easy to see by taking $g = c^2\bar{g}$ for some constant $c \neq 0$. Clearly, the Q -curvature is $Q_g = Q_{\bar{g}} = 0$, but the volume $V_M(g)$ can be either larger or smaller than $V_M(\bar{g})$ depending on $c > 1$ or $c < 1$.

Remark 1.9. The strictly stability condition in Theorem 1.7 is the same as the scalar curvature case (see [20]). It is also necessary for Q -curvature. See Remark 6.1 for more details regarding a counterexample.

In particular, Theorem 1.7 implies the volume comparison for metrics near the spherical metric, since the reference metric is strictly stable. This is a 4th-order analogue of *Bray's conjecture* for scalar curvature (see [20] for more details).

Corollary 1.10. *For $n \geq 3$, let (\mathbb{S}^n, \bar{g}) be the canonical sphere with Ricci curvature*

$$\operatorname{Ric}_{\bar{g}} = (n-1)\bar{g}.$$

Then there exists a constant $\varepsilon_0 > 0$ such that for any metric g on \mathbb{S}^n satisfying

$$Q_g \geq \frac{1}{8}n(n-2)(n+2)$$

and

$$\|g - \bar{g}\|_{C^4(M,\bar{g})} < \varepsilon_0,$$

the following volume comparison holds

$$V_M(g) \leq V_{\mathbb{S}^n},$$

with the equality holds if and only if g is isometric to \bar{g} .

According to Remark 1.8, the volume comparison for Ricci-flat manifolds can not be expected. However, applying the same idea as proof of Theorem 1.7, we can show that strictly stable Ricci-flat manifolds admits local rigidity with respect to Q -curvature. This extends our previous local rigidity result for tori and answers the question proposed by the referee of our earlier article [13].

Theorem 1.11. *Suppose (M^n, \bar{g}) is a strictly stable Ricci-flat manifold, then there exists a constant $\varepsilon_0 > 0$ such that any metric g satisfying*

$$Q_g \geq 0$$

and

$$\|g - \bar{g}\|_{C^4(M, \bar{g})} < \varepsilon_0$$

implies g has to be Ricci-flat. In particular, there is no metric with positive Q -curvature near \bar{g} .

Remark 1.12. It is not difficult to improve this local rigidity result by a weaker assumption that the Ricci-flat metric \bar{g} is only stable instead. It would be interesting to ask whether we can find an example of unstable Ricci-flat manifold which admits a metric of positive Q -curvature.

This article is organized as follows: In Section 2, we introduce the notation and useful formulas needed throughout the article. In Section 3, we show the rigidity of Einstein metrics in the category of J -Einstein metrics and calculate the first variation of J -tensor at Einstein metrics which will be used in Section 5. In Section 4, we calculate the variational formulas for the main functional. In Section 5, we prove our main results (Theorem 1.7, 1.11) by showing nonpositivity of second variation of the functional and using Morse lemma argument. In Section 6, we provide a counterexample showing that the strictly stability of Einstein metric is necessary for our main result. We also make some observations about a global volume comparison of Q -curvature for a locally conformally flat 4-manifold.

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2. PRELIMINARY AND NOTATION

2.1. Notations. Throughout this article, we will always assume (M^n, g) to be an n -dimensional closed Riemannian manifold ($n \geq 3$) unless otherwise stated. Also, we list notations involved in this article:

\mathcal{M} - the set of all smooth metrics on M ;

$\mathcal{D}(M)$ - the set of all smooth diffeomorphisms $\varphi : M \rightarrow M$;
 $\mathcal{X}(M)$ - the set of all smooth vector fields on M ;
 $S_2(M)$ - the set of all smooth symmetric 2-tensors on M ;
 $V_M(g)$ - the volume of manifold M with respect to the metric g .

We adopt the following convention for Ricci curvature tensor

$$R_{jk} = R_{ijk}^i = g^{il} R_{ijkl}.$$

and denote its traceless part as

$$E_g := Ric_g - \frac{1}{n} R_g g.$$

The Schouten tensor is defined to be

$$S_g = \frac{1}{n-2} \left(Ric_g - \frac{1}{2(n-1)} R_g g \right)$$

and \mathring{S}_g is denoted to be its traceless part.

For Laplacian operator, we use the convention as follows

$$\Delta_g := g^{ij} \nabla_i \nabla_j.$$

For simplicity, we introduce following operations:

$$(h \times k)_{ij} := g^{kl} h_{ik} k_{jl} = h_i^l k_{lj}, \quad h \cdot k := tr_g(h \times k) = g^{ij} g^{kl} h_{ik} k_{jl} = h^{jk} k_{jk}$$

and

$$(Rm \cdot h)_{jk} := R_{ijkl} h^{il}$$

for any $h, k \in S_2(M)$.

Let $X \in \mathcal{X}(M)$ and $h \in S_2(M)$, we use following notations for the operator

$$(\delta_g h)_i := -(div_g h)_i = -\nabla^j h_{ij},$$

which is the L^2 -formal adjoint of Lie derivative (up to a scalar multiple)

$$\frac{1}{2} (L_g X)_{ij} = \frac{1}{2} (\nabla_i X_j + \nabla_j X_i).$$

The Einstein operator acting on $h \in S_2(M)$ is defined to be

$$\Delta_E^g h = \Delta_g h + 2Rm_g \cdot h.$$

The J -tensor ([14]) is defined to be

$$J_g = \frac{1}{n} Q_g g - \frac{1}{n-2} B_g - \frac{n-4}{4(n-1)(n-2)} T_g,$$

where

$$B_g = \Delta_E^g \mathring{S}_g - \nabla_g^2 (tr_g S_g) + \frac{1}{n} g \Delta_g (tr_g S_g) - (n-4) \mathring{S}_g^2 - |\mathring{S}_g|_g^2 g - \frac{2(n-2)}{n} (tr_g S_g) \mathring{S}_g$$

is the *Bach tensor* and

$$T_g := (n-2) \left(\nabla_g^2(\text{tr}_g S_g) - \frac{1}{n} g \Delta_g(\text{tr}_g S_g) \right) + 4(n-1) \left(\mathring{S}_g^2 - \frac{1}{n} |\mathring{S}_g|_g^2 g \right) - \frac{(n-2)(n^2+2n-4)}{n} (\text{tr}_g S_g) \mathring{S}_g.$$

2.2. Basic variational formulae. We list several formulas for linearization of geometric quantities that will be useful for later sections (see [7, 13, 20] for detailed calculations).

The linearization of Ricci tensor is

$$(DRic_g) \cdot h = -\frac{1}{2} \left[\Delta_E^g h - (Ric_g \times h + h \times Ric_g) + \nabla_g^2(\text{tr}_g h) + (\nabla_j(\delta_g h)_k + \nabla_k(\delta_g h)_j) dx^j \otimes dx^k \right],$$

and the linearization of scalar curvature is

$$(DR_g) \cdot h = -\Delta_g(\text{tr}_g h) + \delta_g^2 h - Ric_g \cdot h.$$

The linearization of Q -curvature is

$$\begin{aligned} \Gamma_g h &:= (DQ_g) \cdot h \\ &= A_n \left[-\Delta_g^2(\text{tr}_g h) + \Delta_g \delta_g^2 h - \Delta_g(Ric_g \cdot h) + \frac{1}{2} dR_g \cdot (d(\text{tr}_g h) + 2\delta_g h) - \nabla_g^2 R_g \cdot h \right] \\ &\quad - B_n \left[Ric_g \cdot \Delta_E^g h + Ric_g \cdot \nabla_g^2(\text{tr}_g h) + 2Ric_g \cdot \nabla(\delta_g h) \right] \\ &\quad + 2C_n R_g \left[-\Delta_g(\text{tr}_g h) + \delta_g^2 h - Ric_g \cdot h \right]. \end{aligned}$$

and the L^2 -formal adjoint of Γ_g is

$$\begin{aligned} \Gamma_g^* f &:= A_n \left[\nabla_g^2 \Delta_g f - g \Delta_g^2 f - Ric_g \Delta_g f + \frac{1}{2} g \delta_g(f dR_g) + \nabla_g(f dR_g) - f \nabla_g^2 R_g \right] \\ &\quad - B_n \left[\Delta_g(f Ric_g) + 2f(Rm_g \cdot Ric_g) + g \delta_g^2(f Ric_g) + 2\nabla_g \delta_g(f Ric_g) \right] \\ &\quad - 2C_n \left[g \Delta_g(f R_g) - \nabla_g^2(f R_g) + f R_g Ric_g \right], \end{aligned}$$

where A_n, B_n, C_n are defined in (1.2). The first and second variations of the volume functional are

$$(DV_{M,g}) \cdot h = \frac{1}{2} \int_M (\text{tr}_g h) dv_g$$

and

$$(D^2 V_{M,g}) \cdot (h, h) = \frac{1}{4} \int_M [(\text{tr}_g h)^2 - 2|h|_g^2] dv_g.$$

3. J -TENSOR AND EINSTEIN METRICS

In this section, we will discuss some involved topics about J -tensor and Einstein metrics.

3.1. Rigidity of Einstein metrics in the category of J -Einstein metrics.

As we have stated in the introduction, Einstein metrics can be identified with a characterization in the spectrum of Einstein operator in the category of J -Einstein metrics. Now we present a simple proof here.

Proof of Theorem 1.5. By definition, the J -Einstein metric \bar{g} satisfies the equation

$$\begin{aligned} \mathring{J}_{\bar{g}} &= -\frac{1}{n-2} \left(B_{\bar{g}} + \frac{n-4}{4(n-1)} T_{\bar{g}} \right) \\ &= -\frac{1}{n-2} \left[\Delta_E^{\bar{g}} \mathring{S}_{\bar{g}} + \frac{n^2-10n+12}{4(n-1)} \left(\nabla_{\bar{g}}^2(\text{tr}_{\bar{g}} S_{\bar{g}}) - \frac{1}{n} \bar{g} \Delta_{\bar{g}}(\text{tr}_{\bar{g}} S_{\bar{g}}) \right) \right] + \frac{2}{n} |\mathring{S}_{\bar{g}}|_{\bar{g}}^2 \bar{g} \\ &\quad + \frac{(n-2)^2(n+2)}{4n(n-1)} (\text{tr}_{\bar{g}} S_{\bar{g}}) \mathring{S}_{\bar{g}} \\ &= 0. \end{aligned}$$

It implies

$$\begin{aligned} 0 &= \int_M \langle \mathring{J}_{\bar{g}}, \mathring{S}_{\bar{g}} \rangle_{\bar{g}} dv_{\bar{g}} \\ &= \frac{1}{n-2} \int_M \left[-\langle \mathring{S}_{\bar{g}}, \Delta_E^{\bar{g}} \mathring{S}_{\bar{g}} \rangle_{\bar{g}} + \frac{n^2-10n+12}{4n} |d(\text{tr}_{\bar{g}} S_{\bar{g}})|_{\bar{g}}^2 + \frac{(n-2)^3(n+2)}{8n(n-1)^2} R_{\bar{g}} |\mathring{S}_{\bar{g}}|_{\bar{g}}^2 \right] dv_{\bar{g}} \\ &\geq \frac{1}{n-2} \left(\Lambda_{\bar{g}} + \frac{(n-2)^3(n+2)}{8n(n-1)^2} \min_M R_{\bar{g}} \right) \int_M |\mathring{S}_{\bar{g}}|_{\bar{g}}^2 dv_{\bar{g}} \\ &\geq 0 \end{aligned}$$

due to assumptions on the spectrum of $(-\Delta_E^{\bar{g}})$ and

$$\text{tr}_{\bar{g}} S_{\bar{g}} = \frac{R_{\bar{g}}}{2(n-1)}$$

is a constant for $3 \leq n \leq 8$. Therefore, we conclude that

$$E_{\bar{g}} = (n-2) \mathring{S}_{\bar{g}} = 0,$$

which shows \bar{g} is an Einstein metric. \square

In particular, we have

Corollary 3.1. *Suppose (M^n, \bar{g}) is a closed J -Einstein manifold with non-negative constant scalar curvature and the Einstein operator $\Delta_E^{\bar{g}}$ on $S_2(M)$ is negative, then \bar{g} is a strictly stable Einstein metric.*

Proof. It is straightforward that \bar{g} is an Einstein metric according to Theorem 1.5. Moreover, we obtain

$$\inf_{h \in S_{2,\bar{g}}^{TT}(M) \setminus \{0\}} \frac{\int_M \langle h, -\Delta_E^{\bar{g}} h \rangle_{\bar{g}} dv_{\bar{g}}}{\int_M |h|_{\bar{g}}^2 dv_{\bar{g}}} \geq \inf_{h \in S_2(M) \setminus \{0\}} \frac{\int_M \langle h, -\Delta_E^{\bar{g}} h \rangle_{\bar{g}} dv_{\bar{g}}}{\int_M |h|_{\bar{g}}^2 dv_{\bar{g}}} > 0,$$

since $S_{2,\bar{g}}^{TT} \subsetneq S_2(M)$. Thus the metric \bar{g} is strictly stable Einstein by definition. \square

3.2. Variations of J -tensor at Einstein metrics.

As a geometric symmetric 2-tensor, J -tensor has a natural connection with Q -curvature as we have discussed in the introduction. It is crucial to investigate variational properties of J -tensor, when considering variational problems associated to Q -curvature. In this section, we will obtain the first variation of the traceless part of J -tensor at an Einstein metric along the TT -direction, which is critical in our further discussion.

For simplicity, we may use $'$ to denote the first variation in the space of metrics. The direction of the variation will be clear from the context.

Lemma 3.2. *Suppose \bar{g} is an Einstein metric, then*

$$(DE_{\bar{g}}) \cdot \mathring{h} = -\frac{1}{2}\Delta_{\bar{g}}\mathring{h}$$

and

$$(DR_{\bar{g}}) \cdot \mathring{h} = 0$$

for any $\mathring{h} \in S_{2,\bar{g}}^{TT}(M)$.

Proof. This is straightforward from first variations of Ricci and scalar curvature in Section 2.2 together with facts that $E_{\bar{g}} = 0$ and $\mathring{h} \in S_{2,\bar{g}}^{TT}(M)$. \square

For Einstein metrics, the connection Laplacian is commutative with first variation:

Lemma 3.3. *Suppose \bar{g} is an Einstein metric, then*

$$(\Delta_{\bar{g}}E)' = \Delta_{\bar{g}}E'.$$

Proof. It is straightforward that

$$\begin{aligned} & \Delta_{\bar{g}}E_{jk} \\ &= \bar{g}^{il}\nabla_i\nabla_lE_{jk} \\ &= \bar{g}^{il}\left[\partial_i(\nabla_lE_{jk}) - \Gamma_{il}^p\nabla_pE_{jk} - \Gamma_{ij}^p\nabla_lE_{pk} - \Gamma_{ik}^p\nabla_lE_{jp}\right] \\ &= \bar{g}^{il}\left[\partial_i(\partial_lE_{jk} - \Gamma_{lj}^pE_{pk} - \Gamma_{lk}^pE_{jp}) - \Gamma_{il}^p\nabla_pE_{jk} - \Gamma_{ij}^p\nabla_lE_{pk} - \Gamma_{ik}^p\nabla_lE_{jp}\right] \\ &= \bar{g}^{il}\left[\partial_i\partial_lE_{jk} - (\partial_i\Gamma_{lj}^p)E_{pk} - (\partial_i\Gamma_{lk}^p)E_{jp} - \Gamma_{lj}^p\partial_iE_{pk} - \Gamma_{lk}^p\partial_iE_{jp} - \Gamma_{il}^p\nabla_pE_{jk} - \Gamma_{ij}^p\nabla_lE_{pk} - \Gamma_{ik}^p\nabla_lE_{jp}\right]. \end{aligned}$$

This shows

$$\begin{aligned} & (\Delta_{\bar{g}}E_{jk})' \\ &= \bar{g}^{il}\left[\partial_i\partial_lE'_{jk} - (\partial_i\Gamma_{lj}^p)E'_{pk} - (\partial_i\Gamma_{lk}^p)E'_{jp} - \Gamma_{lj}^p\partial_iE'_{pk} - \Gamma_{lk}^p\partial_iE'_{jp} - \Gamma_{il}^p\nabla_pE'_{jk} - \Gamma_{ij}^p\nabla_lE'_{pk} - \Gamma_{ik}^p\nabla_lE'_{jp}\right] \\ &= \Delta_{\bar{g}}E'_{jk}, \end{aligned}$$

when \bar{g} is Einstein. \square

Now the first variation of Bach tensor at an Einstein metric \bar{g} is given as follows:

Proposition 3.4. *Suppose \bar{g} is an Einstein metric, then for any $\mathring{h} \in S_{2,\bar{g}}^{TT}(M)$,*

$$(3.1) \quad (DB_{\bar{g}}) \cdot \mathring{h} = -\frac{1}{2(n-2)}\left(-\Delta_{\bar{g}}\mathring{h} + \frac{n-2}{n(n-1)}R_{\bar{g}}\right)(-\Delta_{\bar{g}}\mathring{h}).$$

Proof. Rewriting the Bach tensor in terms of scalar curvature and traceless Ricci tensor, we have

$$B_g = \frac{1}{n-2} \Delta_E^g E_g - \frac{1}{2(n-1)} \left(\nabla_g^2 R_g - \frac{1}{n} g \Delta_g R_g \right) - \frac{n-4}{(n-2)^2} E_g^2 - \frac{1}{(n-2)^2} |E_g|_g^2 g - \frac{R_g}{n(n-1)} E_g.$$

From Lemma 3.3 and $E_{\bar{g}} = 0$,

$$B'_{\bar{g}} = \frac{1}{n-2} \Delta_E^{\bar{g}} E'_{\bar{g}} - \frac{1}{2(n-1)} \left(\nabla_{\bar{g}}^2 R'_{\bar{g}} - \frac{1}{n} \bar{g} \Delta_{\bar{g}} R'_{\bar{g}} \right) - \frac{R_{\bar{g}}}{n(n-1)} E'_{\bar{g}},$$

where all variations are taken along the TT -direction $\dot{h} \in S_{2,\bar{g}}^{TT}(M)$. According to Lemma 3.2, we have $R'_{\bar{g}} = 0$ and hence

$$(DB_{\bar{g}}) \cdot \dot{h} = -\frac{1}{2(n-2)} \left(-\Delta_E^{\bar{g}} + \frac{n-2}{n(n-1)} R_{\bar{g}} \right) (-\Delta_E^{\bar{g}} \dot{h}).$$

□

Remark 3.5. Note that when $n = 4$ and

$$Ric_{\bar{g}} = 3\lambda \bar{g},$$

then equation (3.1) becomes

$$(DB_{\bar{g}}) \cdot \dot{h} = -\frac{1}{4} (-\Delta_E^{\bar{g}} + 2\lambda) (-\Delta_E^{\bar{g}} \dot{h}).$$

Recall that Bach tensor appears to be the obstruction tensor in the study of ambient space ([8]). In fact, Matsumoto has shown that for $2m^{\text{th}}$ -order obstruction tensor $\mathcal{O}_{\bar{g}}^{(2m)}$, its first variation at a $2m$ -dimensional Einstein metric with Ricci curvature

$$Ric_{\bar{g}} = (2m-1)\lambda \bar{g}$$

is given by

$$\left(D\mathcal{O}_{\bar{g}}^{(2m)} \right) \cdot \dot{h} = \frac{(-1)^{m-1}}{4(m-1)} \left[\prod_{k=0}^{m-1} (-\Delta_E^{\bar{g}} + 2k(2m-2k-1)\lambda) \right] \dot{h},$$

for any $\dot{h} \in S_{2,\bar{g}}^{TT}(M)$. Please see [15] for further information.

For the rest part of $\dot{J}_{\bar{g}}$, we have

Lemma 3.6. *Suppose \bar{g} is an Einstein metric, then*

$$(DT_{\bar{g}}) \cdot \dot{h} = \frac{n^2 + 2n - 4}{4n(n-1)} R_{\bar{g}} \Delta_E^{\bar{g}} \dot{h}$$

for any $\dot{h} \in S_{2,\bar{g}}^{TT}(M)$.

Proof. Rewriting the tensor T_g in terms of scalar curvature R_g and traceless Ricci curvature E_g , we get

$$T_g = \frac{n-2}{2(n-1)} \left(\nabla_g^2 R_g - \frac{1}{n} g \Delta_g R_g \right) + \frac{4(n-1)}{(n-2)^2} \left(E_g^2 - \frac{1}{n} |E_g|_g^2 g \right) - \frac{n^2 + 2n - 4}{2n(n-1)} R_g E_g.$$

Then the result follows from the fact that $E_{\bar{g}} = 0$ and Lemma 3.2. □

Now combining first variations of B_g and T_g , we derive

Proposition 3.7. Suppose \bar{g} is an Einstein metric, then

$$(D\mathring{J}_{\bar{g}}) \cdot \mathring{h} = \frac{1}{2(n-2)^2} \left(-\Delta_{\bar{g}} + \frac{(n-2)^3(n+2)}{8n(n-1)^2} R_{\bar{g}} \right) (-\Delta_{\bar{g}} \mathring{h}),$$

for any $\mathring{h} \in S_{2,\bar{g}}^{TT}(M)$.

According to Definition 1.6, immediately we obtain the following characterization of the operator $D\mathring{J}_{\bar{g}}$ for strictly stable Einstein metrics \bar{g} :

Corollary 3.8. Suppose \bar{g} is an Einstein metric, then $D\mathring{J}_{\bar{g}}$ is a formally self-adjoint operator on $S_{2,\bar{g}}^{TT}(M)$. Moreover, $D\mathring{J}_{\bar{g}}$ is a positive operator, if \bar{g} is a strictly stable Einstein metric with non-negative scalar curvature.

4. THE KEY FUNCTIONAL AND ITS VARIATIONS

For completeness, we start with generic J -Einstein metrics instead of more restricted Einstein metrics.

Suppose (M^n, \bar{g}) is a closed J -Einstein manifold and consider the functional

$$\mathcal{F}_{M,\bar{g}}(g) = V_M(g)^{\frac{4}{n}} \int_M Q(g) dv_{\bar{g}}.$$

Note that the volume form $dv_{\bar{g}}$ is independent of g and thus the functional $\mathcal{F}_{M,\bar{g}}$ is *scaling invariant*:

$$\mathcal{F}_{M,\bar{g}}(c^2 g) = \mathcal{F}_{M,\bar{g}}(g)$$

for any real number $c \neq 0$.

The functional $\mathcal{F}_{M,\bar{g}}$ is particularly designed for our purpose. This can be glimpsed from its variational properties.

Proposition 4.1. The J -Einstein metric \bar{g} is a critical point of the functional $\mathcal{F}_{M,\bar{g}}$.

Proof. For any $h \in S_2(M)$,

$$\begin{aligned} (D\mathcal{F}_{M,\bar{g}}) \cdot h &= V_M(\bar{g})^{\frac{4}{n}} \int_M ((DQ_{\bar{g}}) \cdot h) dv_{\bar{g}} + \frac{4}{n} V_M(\bar{g})^{\frac{4}{n}-1} V'_{M,\bar{g}} \int_M Q_{\bar{g}} dv_{\bar{g}} \\ &= V_M(\bar{g})^{\frac{4}{n}} \left[\int_M \langle \Gamma_{\bar{g}} h, 1 \rangle_{\bar{g}} dv_{\bar{g}} + \frac{2}{n} \left(\int_M (tr_{\bar{g}} h) dv_{\bar{g}} \right) V_M(\bar{g})^{-1} \int_M Q_{\bar{g}} dv_{\bar{g}} \right] \\ &= V_M(\bar{g})^{\frac{4}{n}} \left[\int_M \langle h, \Gamma_{\bar{g}}^* 1 \rangle_{\bar{g}} dv_{\bar{g}} + \frac{2}{n} \overline{Q_{\bar{g}}} \int_M (tr_{\bar{g}} h) dv_{\bar{g}} \right] \\ &= -2 V_M(\bar{g})^{\frac{4}{n}} \int_M \left\langle h, J_{\bar{g}} - \frac{1}{n} \overline{Q_{\bar{g}}} \bar{g} \right\rangle_{\bar{g}} dv_{\bar{g}} \end{aligned}$$

where

$$\overline{Q_{\bar{g}}} := V_M(\bar{g})^{-1} \int_M Q_{\bar{g}} dv_{\bar{g}}$$

is the average of $Q_{\bar{g}}$ on M .

According to Definition 1.4, a J -Einstein metric has constant Q -curvature and hence

$$J_{\bar{g}} - \frac{1}{n} \overline{Q_{\bar{g}}} \bar{g} = 0,$$

which implies \bar{g} is a critical point of $\mathcal{F}_{M, \bar{g}}$. □

In principle, one can obtain the second variation of $\mathcal{F}_{M, \bar{g}}$ from a formal calculation. However, due to its complicated expression, the calculation would be rather messy. Instead, with an elegant trick, we can minimize the work through the first variation of J -tensor. Analogous to the previous section, we adopt the convention that ' stands for first and '' for second variations with certain $h \in S_2(M)$.

We start with a useful observation:

Lemma 4.2. *For any metric g and $h \in S_2(M)$,*

$$tr_g \mathring{J}'_g = \langle \mathring{J}_g, \mathring{h} \rangle_g,$$

where \mathring{J}'_g is the first variation of traceless J -tensor and \mathring{h} is the traceless part of h .

Proof. Since $tr_g \mathring{J}_g = 0$, differentiating both sides of the equation gives

$$0 = tr_g \mathring{J}'_g + (g^{-1})' \cdot \mathring{J}_g = tr_g \mathring{J}'_g - h \cdot \mathring{J}_g.$$

That is,

$$tr_g \mathring{J}'_g = \langle \mathring{J}_g, h \rangle_g = \left\langle \mathring{J}_g, \mathring{h} + \frac{1}{n} (tr_g h) g \right\rangle_g = \langle \mathring{J}_g, \mathring{h} \rangle_g.$$

□

With the above observation, we can express the integral of second variation of Q -curvature in a compact form.

Proposition 4.3. Suppose \bar{g} is a J -Einstein metric, then we have

$$\int_M Q''_{\bar{g}} dv_{\bar{g}} = -2 \int_M \left[\langle \mathring{J}'_{\bar{g}}, \mathring{h} \rangle_{\bar{g}} - \frac{1}{n} Q_{\bar{g}} |\mathring{h}|_{\bar{g}}^2 + \left(\frac{n+4}{4n} Q'_{\bar{g}} + \frac{n-2}{2n^2} Q_{\bar{g}} (tr_{\bar{g}} h) \right) (tr_{\bar{g}} h) \right] dv_{\bar{g}}$$

holds for any $h \in S_2(M)$.

Proof. We start with an arbitrary Riemannian metric g on M . It is straightforward that for any $h \in S_2(M)$,

$$\begin{aligned}
& \left(\int_M Q_g dv_g \right)'' \\
&= \left(\int_M Q'_g dv_g + \int_M Q_g (dv_g)' \right)' \\
&= \left(\int_M \langle \Gamma_g^* 1, h \rangle_g dv_g \right)' + \int_M Q'_g (dv_g)' + \int_M Q_g (dv_g)'' \\
&= -2 \int_M \left[\langle J'_g, h \rangle_g + 2 \langle (g^{-1})', J_g \times h \rangle_g + \frac{1}{2} \langle J_g, h \rangle_g (tr_g h) \right] dv_g + \int_M Q'_g (dv_g)' + \int_M Q_g (dv_g)'' \\
&= -2 \int_M \left[\langle J'_g, h \rangle_g - 2 \langle J_g, h^2 \rangle_g + \frac{1}{2} \langle J_g, h \rangle_g (tr_g h) \right] dv_g + \int_M Q'_g (dv_g)' + \int_M Q_g (dv_g)'' .
\end{aligned}$$

Furthermore, the decomposition

$$J_g = \mathring{J}_g + \frac{1}{n} Q_g g$$

yields

$$J'_g = \mathring{J}'_g + \frac{1}{n} Q'_g g + \frac{1}{n} Q_g h.$$

Together with variational formulas in Section 2.2, we have

$$\begin{aligned}
& \int_M Q''_g dv_g \\
&= \left(\int_M Q_g dv_g \right)'' - 2 \int_M Q'_g (dv_g)' - \int_M Q_g (dv_g)'' \\
&= -2 \int_M \left[\left\langle \mathring{J}'_g + \frac{1}{n} Q'_g g + \frac{1}{n} Q_g h, h \right\rangle_g - 2 \left\langle \mathring{J}_g + \frac{1}{n} Q_g g, h^2 \right\rangle_g \right] dv_g \\
&\quad - \int_M \left[\left\langle \mathring{J}_g + \frac{1}{n} Q_g g, h \right\rangle_g + \frac{1}{2} Q'_g \right] (tr_g h) dv_g \\
&= -2 \int_M \left[\langle \mathring{J}'_g, h \rangle_g - 2 \left\langle \mathring{J}_g + \frac{1}{2n} Q_g g, h^2 \right\rangle_g + \frac{1}{2} \left(\frac{n+4}{2n} Q'_g + \frac{1}{n} Q_g (tr_g h) + \langle \mathring{J}_g, h \rangle_g \right) (tr_g h) \right] dv_g.
\end{aligned}$$

From the decomposition $h = \mathring{h} + \frac{1}{n}(tr_g h)g$ and Lemma 4.2, we can further simplify the expression to the following form:

$$\begin{aligned}
& \int_M Q'_g dv_g \\
&= -2 \int_M \left[\left\langle \mathring{J}'_g, \mathring{h} + \frac{1}{n}(tr_g h)g \right\rangle_g - 2 \left\langle \mathring{J}_g + \frac{1}{2n}Q_g g, \mathring{h}^2 + \frac{2}{n}(tr_g h)\mathring{h} + \frac{1}{n^2}(tr_g h)^2 g \right\rangle_g \right] dv_g \\
&\quad - \int_M \left[\frac{n+4}{2n}Q'_g + \frac{1}{n}Q_g(tr_g h) + \left\langle \mathring{J}_g, \mathring{h} + \frac{1}{n}(tr_g h)g \right\rangle_g \right] (tr_g h) dv_g \\
&= -2 \int_M \left[\langle \mathring{J}'_g, \mathring{h} \rangle_g - 2 \left\langle \mathring{J}_g + \frac{1}{2n}Q_g g, \mathring{h}^2 \right\rangle_g \right] dv_g \\
&\quad - \int_M \left[\frac{n+4}{2n}Q'_g + \frac{(n-2)}{n^2}Q_g(tr_g h) + \frac{(n-6)}{n} \langle \mathring{J}_g, \mathring{h} \rangle_g \right] (tr_g h) dv_g.
\end{aligned}$$

In particular for J -Einstein \bar{g} , we have $\mathring{J}_{\bar{g}} = 0$ and hence

$$\int_M Q''_{\bar{g}} dv_{\bar{g}} = -2 \int_M \left[\langle \mathring{J}'_{\bar{g}}, \mathring{h} \rangle_{\bar{g}} - \frac{1}{n}Q_{\bar{g}}|\mathring{h}|_{\bar{g}}^2 + \left(\frac{n+4}{4n}Q'_{\bar{g}} + \frac{n-2}{2n^2}Q_{\bar{g}}(tr_{\bar{g}} h) \right) (tr_{\bar{g}} h) \right] dv_{\bar{g}}.$$

□

Now we calculate the second variation of the functional $\mathcal{F}_{M, \bar{g}}$ at a J -Einstein metric \bar{g} .

Proposition 4.4. For $n \geq 3$, suppose (M^n, \bar{g}) is J -Einstein, then for any $h \in S_2(M)$ we have

$$\begin{aligned}
& V_M(\bar{g})^{-\frac{4}{n}} ((D^2 \mathcal{F}_{M, \bar{g}}) \cdot (h, h)) \\
&= -2 \int_M \left[\langle \mathring{h}, (D\mathring{J}_{\bar{g}}) \cdot \mathring{h} \rangle_{\bar{g}} + \frac{1}{n} \left(tr_{\bar{g}}((D\mathring{J}_{\bar{g}})^* \cdot \mathring{h}) + \frac{n+4}{4}(\Gamma_{\bar{g}} \mathring{h}) \right) (tr_{\bar{g}} h) \right] dv_{\bar{g}} \\
&\quad - \frac{n+4}{2n^2} \int_M [(tr_{\bar{g}} h - \overline{tr_{\bar{g}} h}) \mathcal{L}_{\bar{g}}(tr_{\bar{g}} h - \overline{tr_{\bar{g}} h})] dv_{\bar{g}},
\end{aligned}$$

where the operator $\mathcal{L}_{\bar{g}}$ is defined to be

$$\mathcal{L}_{\bar{g}} u := tr_{\bar{g}} \Gamma_{\bar{g}}^* u = \frac{1}{2} \left(P_{\bar{g}} - \frac{n+4}{2} Q_{\bar{g}} \right) u.$$

Proof. Applying variational formulae of volume in Section 2.2, we have

$$\begin{aligned}
& (D^2 \mathcal{F}_{M, \bar{g}}) \cdot (h, h) \\
&= \left(V_M(g)^{\frac{4}{n}} \int_M Q_g dv_{\bar{g}} \right)'' \\
&= \left((V_M(g)^{\frac{4}{n}})' \int_M Q_g dv_{\bar{g}} + V_M(g)^{\frac{4}{n}} \int_M Q'_g dv_{\bar{g}} \right)' \\
&= V_M(\bar{g})^{\frac{4}{n}} \int_M Q''_{\bar{g}} dv_{\bar{g}} + \frac{8}{n} V_M(\bar{g})^{\frac{4}{n}-1} V'_{M, \bar{g}} \int_M Q'_{\bar{g}} dv_{\bar{g}} + \frac{4}{n} V_M(\bar{g})^{\frac{4}{n}-1} V''_{M, \bar{g}} \int_M Q_{\bar{g}} dv_{\bar{g}} \\
&\quad - \frac{4(n-4)}{n^2} V_M(\bar{g})^{\frac{4}{n}-2} (V'_{M, \bar{g}})^2 \int_M Q_{\bar{g}} dv_{\bar{g}} \\
&= V_M(\bar{g})^{\frac{4}{n}} \int_M Q''_{\bar{g}} dv_{\bar{g}} - \frac{8}{n} V_M(\bar{g})^{\frac{4-n}{n}} \int_M (tr_{\bar{g}} h) dv_{\bar{g}} \int_M \langle J_{\bar{g}}, h \rangle_{\bar{g}} dv_{\bar{g}} \\
&\quad + \frac{1}{n} V_M(\bar{g})^{\frac{4}{n}} Q_{\bar{g}} \left[\int_M \left(\frac{n-2}{n} (tr_{\bar{g}} h)^2 - 2|\mathring{h}|_{\bar{g}}^2 \right) dv_{\bar{g}} - \frac{n-4}{n} V_M(\bar{g})^{-1} \left(\int_M (tr_{\bar{g}} h) dv_{\bar{g}} \right)^2 \right],
\end{aligned}$$

where we used the fact $Q_{\bar{g}}$ is a constant, when \bar{g} is J -Einstein.

Now applying Lemma 4.3, it can be simplified as

$$\begin{aligned}
& V_M(\bar{g})^{-\frac{4}{n}} ((D^2 \mathcal{F}_{M, \bar{g}}) \cdot (h, h)) \\
&= -2 \int_M \langle \mathring{J}'_{\bar{g}}, \mathring{h} \rangle_{\bar{g}} dv_{\bar{g}} - \frac{n+4}{2n} \int_M Q'_{\bar{g}} (tr_{\bar{g}} h) dv_{\bar{g}} - \frac{n+4}{n^2} Q_{\bar{g}} V_M(\bar{g})^{-1} \left(\int_M (tr_{\bar{g}} h) dv_{\bar{g}} \right)^2.
\end{aligned}$$

Rewriting its first term as

$$\begin{aligned}
-2 \int_M \langle \mathring{J}'_{\bar{g}}, \mathring{h} \rangle_{\bar{g}} dv_{\bar{g}} &= -2 \int_M \left\langle (D\mathring{J}_{\bar{g}}) \cdot \left(\mathring{h} + \frac{1}{n} (tr_{\bar{g}} h) \bar{g} \right), \mathring{h} \right\rangle_{\bar{g}} dv_{\bar{g}} \\
&= -2 \int_M \langle \mathring{h}, (D\mathring{J}_{\bar{g}}) \cdot \mathring{h} \rangle_{\bar{g}} dv_{\bar{g}} - \frac{2}{n} \int_M [tr_{\bar{g}}((D\mathring{J}_{\bar{g}})^* \cdot \mathring{h})] (tr_{\bar{g}} h) dv_{\bar{g}},
\end{aligned}$$

we obtain

$$\begin{aligned}
& V_M(\bar{g})^{-\frac{4}{n}} ((D^2 \mathcal{F}_{M, \bar{g}}) \cdot (h, h)) \\
&= -2 \int_M \langle \mathring{h}, (D\mathring{J}_{\bar{g}}) \cdot \mathring{h} \rangle_{\bar{g}} dv_{\bar{g}} - \frac{2}{n} \int_M [tr_{\bar{g}}((D\mathring{J}_{\bar{g}})^* \cdot \mathring{h})] (tr_{\bar{g}} h) dv_{\bar{g}} - \frac{n+4}{2n} \int_M [Q'_{\bar{g}}(tr_{\bar{g}} h)] dv_{\bar{g}} \\
&\quad - \frac{n+4}{n^2} Q_{\bar{g}} V_M(\bar{g})^{-1} \left(\int_M (tr_{\bar{g}} h) dv_{\bar{g}} \right)^2 \\
&= -2 \int_M \langle \mathring{h}, (D\mathring{J}_{\bar{g}}) \cdot \mathring{h} \rangle_{\bar{g}} dv_{\bar{g}} - \frac{2}{n} \int_M [tr_{\bar{g}}((D\mathring{J}_{\bar{g}})^* \cdot \mathring{h})] (tr_{\bar{g}} h) dv_{\bar{g}} - \frac{n+4}{2n} \int_M [(tr_{\bar{g}} h)(\Gamma_{\bar{g}} \mathring{h})] dv_{\bar{g}} \\
&\quad - \frac{n+4}{2n^2} \int_M (tr_{\bar{g}} h) [tr_{\bar{g}} \Gamma_{\bar{g}}^* (tr_{\bar{g}} h)] dv_{\bar{g}} - \frac{n+4}{n^2} Q_{\bar{g}} V_M(\bar{g})^{-1} \left(\int_M (tr_{\bar{g}} h) dv_{\bar{g}} \right)^2.
\end{aligned}$$

Denote

$$\mathring{P}_{\bar{g}} := P_{\bar{g}} - \frac{n-4}{2} Q_{\bar{g}}$$

to be the divergence part of Paneitz operator. Clearly the space of constants are contained in $\ker \mathring{P}_{\bar{g}}$. This implies

$$\begin{aligned}
& -\frac{n+4}{2n^2} \int_M (tr_{\bar{g}}h) [tr_{\bar{g}}\Gamma_{\bar{g}}^*(tr_{\bar{g}}h)] dv_{\bar{g}} - \frac{n+4}{n^2} Q_{\bar{g}} V_M(\bar{g})^{-1} \left(\int_M (tr_{\bar{g}}h) dv_{\bar{g}} \right)^2 \\
&= -\frac{n+4}{4n^2} \int_M (tr_{\bar{g}}h) \left[\left(\mathring{P}_{\bar{g}} - 4Q_{\bar{g}} \right) (tr_{\bar{g}}h) \right] dv_{\bar{g}} - \frac{n+4}{n^2} Q_{\bar{g}} V_M(\bar{g})^{-1} \left(\int_M (tr_{\bar{g}}h) dv_{\bar{g}} \right)^2 \\
&= -\frac{n+4}{4n^2} \int_M (tr_{\bar{g}}h) \left(\mathring{P}_{\bar{g}}(tr_{\bar{g}}h) \right) dv_{\bar{g}} + \frac{n+4}{n^2} Q_{\bar{g}} \int_M (tr_{\bar{g}}h - \overline{tr_{\bar{g}}h})^2 dv_{\bar{g}} \\
&= -\frac{n+4}{4n^2} \int_M \left[(tr_{\bar{g}}h - \overline{tr_{\bar{g}}h}) \left(\mathring{P}_{\bar{g}}(tr_{\bar{g}}h - \overline{tr_{\bar{g}}h}) \right) - 4Q_{\bar{g}}(tr_{\bar{g}}h - \overline{tr_{\bar{g}}h})^2 \right] dv_{\bar{g}} \\
&= -\frac{n+4}{2n^2} \int_M \left[(tr_{\bar{g}}h - \overline{tr_{\bar{g}}h}) \mathcal{L}_{\bar{g}}(tr_{\bar{g}}h - \overline{tr_{\bar{g}}h}) \right] dv_{\bar{g}}.
\end{aligned}$$

Therefore,

$$\begin{aligned}
& V_M(\bar{g})^{-\frac{4}{n}} \left((D^2\mathcal{F}_{M,\bar{g}}) \cdot (h, h) \right) \\
&= -2 \int_M \left[\langle \mathring{h}, (D\mathring{J}_{\bar{g}}) \cdot \mathring{h} \rangle_{\bar{g}} + \frac{1}{n} [tr_{\bar{g}}((D\mathring{J}_{\bar{g}})^* \cdot \mathring{h})](tr_{\bar{g}}h) + \frac{n+4}{4n} (tr_{\bar{g}}h)(\Gamma_{\bar{g}}\mathring{h}) \right] dv_{\bar{g}} \\
&\quad - \frac{n+4}{2n^2} \int_M \left[(tr_{\bar{g}}h - \overline{tr_{\bar{g}}h}) \mathcal{L}_{\bar{g}}(tr_{\bar{g}}h - \overline{tr_{\bar{g}}h}) \right] dv_{\bar{g}}. \\
&= -2 \int_M \left[\langle \mathring{h}, (D\mathring{J}_{\bar{g}}) \cdot \mathring{h} \rangle_{\bar{g}} + \frac{1}{n} \left(tr_{\bar{g}}((D\mathring{J}_{\bar{g}})^* \cdot \mathring{h}) + \frac{n+4}{4} (\Gamma_{\bar{g}}\mathring{h}) \right) (tr_{\bar{g}}h) \right] dv_{\bar{g}} \\
&\quad - \frac{n+4}{2n^2} \int_M \left[(tr_{\bar{g}}h - \overline{tr_{\bar{g}}h}) \mathcal{L}_{\bar{g}}(tr_{\bar{g}}h - \overline{tr_{\bar{g}}h}) \right] dv_{\bar{g}}.
\end{aligned}$$

□

Remark 4.5. In general, for a J -Einstein metric \bar{g} , the term

$$\int_M [tr_{\bar{g}}((D\mathring{J}_{\bar{g}})^* \cdot \mathring{h})](tr_{\bar{g}}h) dv_{\bar{g}}$$

in the above proposition may not necessarily vanish unless the dimension $n = 4$. But if \bar{g} is an Einstein metric, this term would be zero due to the operator $D\mathring{J}_{\bar{g}}$ is formally self-adjoint and Lemma 4.2. This case is addressed in Corollary 4.7.

In order to remove the crossing term in the previous proposition and obtain an elegant variational formula for Einstein metrics, we first rewrite the operator Γ_g in a nice way.

Proposition 4.6. Suppose g is an arbitrary Riemannian metric and

$$h = \mathring{h} + \frac{1}{n} (tr_g h) g \in S_{2,g}^{TT}(M) \oplus (C^\infty(M) \cdot g),$$

then

$$\Gamma_g h = div_g[U_g(\mathring{h})] - 2\mathring{J}_g \cdot \mathring{h} + \frac{1}{n} \mathcal{L}_g(tr_g h),$$

where

$$U_g(\mathring{h}) := \frac{1}{2(n-1)(n-2)} \left[n^2 \nabla_g(\mathring{S}_g \cdot \mathring{h}) - 8(n-1) \mathring{h}_{ij} \nabla_g \mathring{S}_g^{ij} + (n^2 + 4n - 8) \mathring{h}(\nabla_g(\text{tr}_g S_g)) \right].$$

Proof. By the relation between Ricci and Schouten tensor

$$\text{Ric}_g = (n-2)S_g + (\text{tr}_g S_g)g,$$

we can express $\Gamma_g \mathring{h}$ in terms of Schouten tensor

$$\begin{aligned} \Gamma_g \mathring{h} &= \frac{n-2}{2(n-1)} \Delta_g(\mathring{S}_g \cdot \mathring{h}) + \nabla^2(\text{tr}_g S_g) \cdot \mathring{h} + \frac{2}{n-2} \mathring{S}_g \cdot \Delta_g \mathring{h} + \frac{4}{n-2} \mathring{S}_g \cdot (Rm_g \cdot \mathring{h}) \\ &\quad - \frac{(n-2)^2(n+2)}{2n(n-1)} (\text{tr}_g S_g)(\mathring{S}_g \cdot \mathring{h}), \end{aligned}$$

which has simpler coefficients instead. Recall the traceless part of J -tensor is given by

$$\begin{aligned} \mathring{J}_g &= -\frac{1}{n-2} \left(B_g + \frac{n-4}{4(n-1)} T_g \right) \\ &= -\frac{1}{n-2} \left(\Delta_g \mathring{S}_g + \frac{n^2 - 10n + 12}{4(n-1)} \left(\nabla_g^2(\text{tr}_g S_g) - \frac{1}{n} g \Delta_g(\text{tr}_g S_g) \right) + 2Rm_g \cdot \mathring{S}_g \right) \\ &\quad + \frac{(n-2)^2(n+2)}{4n(n-1)} (\text{tr}_g S_g) \mathring{S}_g + \frac{2}{n} |\mathring{S}_g|_g^2 g. \end{aligned}$$

Now we have

$$\begin{aligned} &\Gamma_g \mathring{h} + 2\mathring{J}_g \cdot \mathring{h} \\ &= \frac{n-2}{2(n-1)} \Delta_g(\mathring{S}_g \cdot \mathring{h}) + \frac{n^2 + 4n - 8}{2(n-1)(n-2)} \nabla^2(\text{tr}_g S_g) \cdot \mathring{h} + \frac{2}{n-2} \left[\mathring{S}_g \cdot \Delta_g \mathring{h} - (\Delta_g \mathring{S}_g) \cdot \mathring{h} \right] \\ &= \frac{n-2}{2(n-1)} \text{div}_g \left[\nabla_g(\mathring{S}_g \cdot \mathring{h}) + \frac{n^2 + 4n - 8}{(n-2)^2} \mathring{h}(\nabla_g(\text{tr}_g S_g)) + \frac{4(n-1)}{(n-2)^2} (\mathring{S}_g^{ij} \nabla_g \mathring{h}_{ij} - \mathring{h}_{ij} \nabla_g \mathring{S}_g^{ij}) \right] \\ &= \frac{1}{2(n-1)(n-2)} \text{div}_g \left[n^2 \nabla_g(\mathring{S}_g \cdot \mathring{h}) - 8(n-1) \mathring{h}_{ij} \nabla_g \mathring{S}_g^{ij} + (n^2 + 4n - 8) \mathring{h}(\nabla_g(\text{tr}_g S_g)) \right] \\ &= \text{div}_g[U_g(\mathring{h})], \end{aligned}$$

where

$$U_g(\mathring{h}) = \frac{1}{2(n-1)(n-2)} \left[n^2 \nabla_g(\mathring{S}_g \cdot \mathring{h}) - 8(n-1) \mathring{h}_{ij} \nabla_g \mathring{S}_g^{ij} + (n^2 + 4n - 8) \mathring{h}(\nabla_g(\text{tr}_g S_g)) \right]$$

That is,

$$\Gamma_g \mathring{h} = \text{div}_g[U_g(\mathring{h})] - 2\mathring{J}_g \cdot \mathring{h}.$$

As for the trace part, we have

$$\int_M \left[u \Gamma_g \left(\frac{1}{n} (\text{tr}_g h) g \right) \right] dv_g = \int_M \left\langle \Gamma_g^* u, \frac{1}{n} (\text{tr}_g h) g \right\rangle_g dv_g = \frac{1}{n} \int_M [(\text{tr}_g h)(\text{tr}_g \Gamma_g^* u)] dv_g$$

for any $u \in C^\infty(M)$. Recall that

$$\mathcal{L}_g = \text{tr}_g \Gamma_g^*$$

as defined in Proposition 4.4 and \mathcal{L}_g is formally self-adjoint, then

$$\int_M \left[u \Gamma_g \left(\frac{1}{n} (tr_g h) g \right) \right] dv_g = \frac{1}{n} \int_M [(tr_g h)(\mathcal{L}_g u)] dv_g = \frac{1}{n} \int_M [u \mathcal{L}_g (tr_g h)] dv_g.$$

Since u is arbitrary, we conclude that

$$\Gamma_g \left(\frac{1}{n} (tr_g h) g \right) = \frac{1}{n} \mathcal{L}_g (tr_g h).$$

Combining these two parts, we obtain

$$\Gamma_g h = \text{div}_g [U_g(\mathring{h})] - 2\mathring{J}_g \cdot \mathring{h} + \frac{1}{n} \mathcal{L}_g (tr_g h).$$

□

In particular, when \bar{g} is restricted to be an Einstein metric, the second variation of $\mathcal{F}_{M,\bar{g}}$ can be expressed in an elegant way:

Corollary 4.7. *Suppose (M^n, \bar{g}) is an Einstein manifold, then*

$$\begin{aligned} & (D^2 \mathcal{F}_{M,\bar{g}}) \cdot (h, h) \\ &= -2 V_M(\bar{g})^{\frac{4}{n}} \left[\int_M \langle \mathring{h}, (D\mathring{J}_{\bar{g}}) \cdot \mathring{h} \rangle_{\bar{g}} dv_{\bar{g}} + \frac{n+4}{4n^2} \int_M [(tr_{\bar{g}} h - \overline{tr_{\bar{g}} h}) \mathcal{L}_{\bar{g}}(tr_{\bar{g}} h - \overline{tr_{\bar{g}} h})] dv_{\bar{g}} \right], \end{aligned}$$

for any $h = \mathring{h} + \frac{1}{n} (tr_{\bar{g}} h) \bar{g} \in S_{2,\bar{g}}^{TT}(M) \oplus (C^\infty(M) \cdot \bar{g})$.

Proof. This result follows from Corollary 3.8, Lemma 4.2, and the fact that $U_{\bar{g}}(\mathring{h}) = 0$ and $\mathring{J}_{\bar{g}} = 0$ when \bar{g} is an Einstein metric. □

5. VOLUME COMPARISON WITH RESPECT TO Q -CURVATURE

In this section, we give the proof of our main results. As the first step, we recall some fundamental results involved (c.f. [1, 18]):

Lemma 5.1 (Lichnerowicz-Obata's eigenvalue estimate). *Suppose (M^n, \bar{g}) is an n -dimensional closed Riemannian manifold with*

$$\text{Ric}_{\bar{g}} \geq (n-1)\lambda \bar{g},$$

where $\lambda > 0$ is a constant. Then for any function $u \in C^\infty(M) \setminus \{0\}$ with

$$\int_M u dv_{\bar{g}} = 0,$$

we have

$$\int_M |du|^2 dv_{\bar{g}} \geq n\lambda \int_M u^2 dv_{\bar{g}},$$

where equality holds if and only if (M^n, \bar{g}) is isometric to the round sphere $\mathbb{S}^n(r)$ with radius $r = \frac{1}{\sqrt{\lambda}}$.

Lemma 5.2 (Berger-Ebin's splitting lemma for Einstein manifolds). *Suppose (M^n, \bar{g}) is an n -dimensional closed Einstein manifold with Ricci curvature*

$$\text{Ric}_{\bar{g}} = (n - 1)\lambda\bar{g},$$

then we have the direct sum decomposition

$$S_2(M) = \text{Im } \mathcal{L}_{\bar{g}} \oplus (C^\infty(M) \cdot \bar{g}) \oplus S_{2,\bar{g}}^{TT}(M)$$

unless (M^n, \bar{g}) is isometric to the round sphere $\mathbb{S}^n(r)$ up to a scaling, where $\mathcal{L}_{\bar{g}}$ is the Lie derivative. For the round sphere $\mathbb{S}^n(r)$ with radius $r = \frac{1}{\sqrt{\lambda}}$, we have

$$S_2(M) = \text{Im } \mathcal{L}_{\bar{g}} \oplus (E_{n\lambda}^\perp \cdot \bar{g}) \oplus S_{2,\bar{g}}^{TT}(M),$$

where

$$E_{n\lambda} := \{u \in C^\infty(\mathbb{S}^n(r)) \mid \Delta_{\mathbb{S}^n(r)}u + n\lambda u = 0\}$$

is the space of first eigenfunctions for the spherical metric and $E_{n\lambda}^\perp$ is its L^2 -orthogonal complement.

With the aid of *Implicit Function Theorem*, *Berger-Ebin's splitting lemma* suggests that one can define a concept named *local slice* $\mathcal{S}_{\bar{g}}$, which is very helpful in understanding the local structure of Einstein metrics in \mathcal{M} . Simply speaking, a local slice is a set of equivalent classes of metrics near the reference metric \bar{g} modulo diffeomorphisms. The process of pulling back metrics on the local slice is also known as *gauge fixing*.

The above splitting lemma does not provide an orthogonal decomposition despite being a direct sum decomposition. To overcome this issue, we need a refined decomposition which involves the splitting of vector fields as well. As a result, we obtain the following improved version of the traditional *Ebin-Palais slice theorem*. The proof is very similar to the traditional one (see [3, 18]). It seems like such a treatment did not appear in the literature to the best of our knowledge and we hope it would benefit researches of similar topics.

Theorem 5.3 (Ebin-Palais slice theorem). *Suppose (M^n, \bar{g}) is a closed n -dimensional Einstein manifold with Ricci curvature tensor*

$$\text{Ric}_{\bar{g}} = (n - 1)\lambda\bar{g},$$

*where $\lambda \in \mathbb{R}$ is a constant. There exists a local slice $\mathcal{S}_{\bar{g}}$ through \bar{g} in \mathcal{M} . That is, for a fixed real number $p > n$, one can find a constant $\varepsilon_1 > 0$ such that for any metric $g \in \mathcal{M}$ with $\|g - \bar{g}\|_{W^{2,p}(M,\bar{g})} < \varepsilon_1$, there is a diffeomorphism $\varphi \in \mathcal{D}(M)$ with $\varphi^*g \in \mathcal{S}_{\bar{g}}$. Moreover, for a smooth local slice $\mathcal{S}_{\bar{g}}$, we have the decomposition*

$$S_2(M) = T_{\bar{g}}\mathcal{S}_{\bar{g}} \oplus (T_{\bar{g}}\mathcal{S}_{\bar{g}})^\perp,$$

where the tangent space of $\mathcal{S}_{\bar{g}}$ at \bar{g} and its L^2 -orthogonal complement are given by

$$T_{\bar{g}}\mathcal{S}_{\bar{g}} = S_{2,\bar{g}}^{TT}(M) \oplus (C^\infty(M) \cdot \bar{g})$$

and

$$(T_{\bar{g}}\mathcal{S}_{\bar{g}})^\perp = \{\mathcal{L}_{\bar{g}}(X) \mid \langle X, \nabla_{\bar{g}}u \rangle_{L^2(M,\bar{g})} = 0, \forall u \in C^\infty(M)\},$$

when (M^n, \bar{g}) is not isometric to the round sphere $\mathbb{S}^n(r)$ up to a scaling;

$$T_{\bar{g}}\mathcal{S}_{\bar{g}} = S_{2,\bar{g}}^{TT}(M) \oplus (E_{n\lambda}^\perp \cdot \bar{g})$$

and

$$(T_{\bar{g}}\mathcal{S}_{\bar{g}})^\perp = \{\mathcal{L}_{\bar{g}}(X) \mid \langle X, \nabla_{\bar{g}}u \rangle_{L^2(M, \bar{g})} = 0, \forall u \in E_{n\lambda}^\perp\},$$

when (M^n, \bar{g}) is isometric to the round sphere $\mathbb{S}^n(r)$ with $r = \frac{1}{\sqrt{\lambda}}$. Here

$$E_{n\lambda} = \{u \in C^\infty(\mathbb{S}^n(r)) : \Delta_{\mathbb{S}^n(r)}u + n\lambda u = 0\}$$

is the space of first eigenfunctions for the spherical metric.

In order to estimate the second variation of $\mathcal{F}_{M, \bar{g}}$, we also need to investigate the analytic properties of the operator $\mathcal{L}_{\bar{g}}$ (as defined in Proposition 4.4):

Proposition 5.4. Suppose \bar{g} is an Einstein metric with Ricci curvature

$$\text{Ric}_{\bar{g}} = (n-1)\lambda\bar{g},$$

where $\lambda \geq 0$ is a constant, then the operator $\mathcal{L}_{\bar{g}}$ is a non-negative operator. Moreover, $\mathcal{L}_{\bar{g}}$ admits non-trivial kernel when

- $\lambda > 0$ and \bar{g} is spherical:

$$\ker \mathcal{L}_{\bar{g}} = E_{n\lambda},$$

- $\lambda = 0$ and \bar{g} is Ricci-flat:

$$\ker \mathcal{L}_{\bar{g}} = \mathbb{R}.$$

Proof. By definition,

$$\mathcal{L}_{\bar{g}}u = \frac{1}{2} \left(P_{\bar{g}} - \frac{n+4}{2} Q_{\bar{g}} \right) u = \frac{1}{2} \left(-\Delta_{\bar{g}} + \frac{(n-2)(n+2)}{2} \lambda \right) (-\Delta_{\bar{g}} - n\lambda) u.$$

For $\lambda > 0$, the first eigenvalue of $(-\Delta_{\bar{g}})$ is at least $n\lambda$ by Lemma 5.1, which implies the operator $\mathcal{L}_{\bar{g}}$ is non-negative and

$$\ker \mathcal{L}_{\bar{g}} = \ker (-\Delta_{\bar{g}} - n\lambda).$$

This shows $\mathcal{L}_{\bar{g}}$ has non-trivial kernel if and only if \bar{g} is spherical and $\ker \mathcal{L}_{\bar{g}}$ consisted of first eigenfunctions of $\Delta_{\bar{g}}$.

For $\lambda = 0$,

$$\mathcal{L}_{\bar{g}}u = \frac{1}{2} \Delta_{\bar{g}}^2 u.$$

It is clear that $\mathcal{L}_{\bar{g}}$ is non-negative and

$$\ker \mathcal{L}_{\bar{g}} = \ker \Delta_{\bar{g}} = \mathbb{R}.$$

□

With these preparations, we summarize variational properties of $\mathcal{F}_{M, \bar{g}}$ at a strictly stable Einstein metric \bar{g} :

Proposition 5.5. Suppose (M^n, \bar{g}) is a strictly stable Einstein manifold with

$$\text{Ric}_{\bar{g}} = (n-1)\lambda\bar{g},$$

where $\lambda \geq 0$ is a constant, then \bar{g} is a critical point of $\mathcal{F}_{M, \bar{g}}$ and

$$(D^2\mathcal{F}_{M, \bar{g}}) \cdot (h, h) \leq 0$$

for any $h = \dot{h} + \frac{1}{n}(\text{tr}_{\bar{g}}h)\bar{g} \in S_{2, \bar{g}}^{TT}(M) \oplus (C^\infty(M) \cdot \bar{g})$. Moreover, the equality holds if and only if

- $h \in \mathbb{R}\bar{g}$, when (M^n, \bar{g}) is not isometric to the round sphere up to a rescaling of the metric.
- $h \in (\mathbb{R} \oplus E_{n\lambda})\bar{g}$, when (M^n, \bar{g}) is isometric to the round sphere $\mathbb{S}^n(r)$ with radius $r = \frac{1}{\sqrt{\lambda}}$,

where

$$E_{n\lambda} := \{u \in C^\infty(\mathbb{S}^n(r)) \mid \Delta_{\mathbb{S}^n(r)}u + n\lambda u = 0\}$$

is the space of first eigenfunctions for the spherical metric.

Proof. According to Proposition 4.1, we can conclude that \bar{g} is a critical point of $\mathcal{F}_{M, \bar{g}}$, since Einstein metrics are J -Einstein. Recall in Corollary 4.7, we showed that

$$\begin{aligned} & (D^2\mathcal{F}_{M, \bar{g}}) \cdot (h, h) \\ &= -2V_M(\bar{g})^{\frac{4}{n}} \left[\int_M \langle \dot{h}, (D\dot{J}_{\bar{g}}) \cdot \dot{h} \rangle_{\bar{g}} dv_{\bar{g}} + \frac{n+4}{4n^2} \int_M [(tr_{\bar{g}}h - \overline{tr_{\bar{g}}h}) \mathcal{L}_{\bar{g}}(tr_{\bar{g}}h - \overline{tr_{\bar{g}}h})] dv_{\bar{g}} \right] \end{aligned}$$

holds for any $h \in S_{2, \bar{g}}^{TT}(M) \oplus (C^\infty(M) \cdot \bar{g})$. It is obvious that $D^2\mathcal{F}_{M, \bar{g}}$ is non-positive definite according to Corollary 3.8 and Proposition 5.4. Furthermore, $D^2\mathcal{F}_{M, \bar{g}}$ vanishes if and only if $\dot{h} = 0$ and

$$(tr_{\bar{g}}h - \overline{tr_{\bar{g}}h}) \in \ker \mathcal{L}_{\bar{g}}.$$

Now the conclusion follows from Proposition 5.4. \square

Another fundamental result we need is the following version of *Morse lemma* on Banach manifold for degenerate functions:

Lemma 5.6 (Fisher-Marsden [7]). *Let \mathcal{P} be a Banach manifold and $f : \mathcal{P} \rightarrow \mathbb{R}$ a C^2 function. Suppose that $\mathcal{Q} \subset \mathcal{P}$ is a submanifold, $f = 0$ and $df = 0$ on \mathcal{Q} and that there is a smooth normal bundle neighborhood of \mathcal{Q} such that if \mathcal{E}_x is the normal complement to $T_x\mathcal{Q}$ in $T_x\mathcal{P}$ then $d^2f(x)$ is weakly negative definite on \mathcal{E}_x (i.e. $d^2f(x)(v, v) \leq 0$ with equality only if $v = 0$). Let $\langle \langle \cdot, \cdot \rangle \rangle_x$ be a weak Riemannian structure with a smooth connection and assume that f has a smooth $\langle \langle \cdot, \cdot \rangle \rangle_x$ -gradient, $Y(x)$. Assume $DY(x)$ maps \mathcal{E}_x to \mathcal{E}_x and is an isomorphism for $x \in \mathcal{Q}$. Then there is a neighborhood U of \mathcal{Q} such that $y \in U$, $f(y) \geq 0$ implies $y \in \mathcal{Q}$.*

According to Theorem 5.3, we can find a local slice $\mathcal{S}_{\bar{g}}$ through \bar{g} and identify $\mathcal{Q}_{\bar{g}}$ to be the submanifold of $\mathcal{S}_{\bar{g}}$ consisted of homothetic metrics, that is,

$$\mathcal{Q}_{\bar{g}} := \{c^2\bar{g} \in \mathcal{S}_{\bar{g}} \mid c \neq 0\}.$$

Consider the restriction of $\mathcal{F}_{M,\bar{g}}$ on the local slice $\mathcal{S}_{\bar{g}}$, denoted by $\mathcal{F}_{M,\bar{g}}|_{\mathcal{S}_{\bar{g}}}$. Applying the previous Morse lemma, we obtain the following rigidity result:

Proposition 5.7. Suppose (M^n, \bar{g}) is a strictly stable Einstein manifold with Ricci curvature

$$\text{Ric}_{\bar{g}} = (n-1)\lambda\bar{g},$$

where $\lambda \geq 0$ is a constant. There is a neighborhood of \bar{g} in the local slice $\mathcal{S}_{\bar{g}}$, denoted by $U_{\bar{g}}$, such that any metric $g_s \in U_{\bar{g}}$ satisfying

$$\mathcal{F}_{M,\bar{g}}|_{\mathcal{S}_{\bar{g}}}(g_s) \geq \mathcal{F}_{M,\bar{g}}|_{\mathcal{S}_{\bar{g}}}(\bar{g})$$

implies that $g_s = c^2\bar{g}$ for some constant $c > 0$.

Proof. From Proposition 5.5, we conclude that \bar{g} is a critical point of $\mathcal{F}_{M,\bar{g}}|_{\mathcal{S}_{\bar{g}}}$ and $D^2\mathcal{F}_{M,\bar{g}}|_{\mathcal{S}_{\bar{g}}}$ is non-positive definite on $T_{\bar{g}}\mathcal{S}_{\bar{g}}$. Moreover, it is obvious that $D^2\mathcal{F}_{M,\bar{g}}|_{\mathcal{S}_{\bar{g}}}$ is degenerate if and only if when restricted on

$$T_{\bar{g}}\mathcal{Q}_{\bar{g}} = \mathbb{R}\bar{g}.$$

Let $\mathcal{E}_{\bar{g}}$ be the L^2 -orthogonal complement of $T_{\bar{g}}\mathcal{Q}_{\bar{g}}$ in $T_{\bar{g}}\mathcal{S}_{\bar{g}}$. By Theorem 5.3, we can identify

$$\mathcal{E}_{\bar{g}} = \left\{ h \in S_{2,\bar{g}}^{TT}(M) \oplus (C^\infty(M) \cdot \bar{g}) \mid \int_M (\text{tr}_{\bar{g}}h) dv_{\bar{g}} = 0 \right\},$$

if \bar{g} is not spherical;

$$\mathcal{E}_{\bar{g}} = \left\{ h \in S_{2,\bar{g}}^{TT}(M) \oplus (E_{n\lambda}^\perp \cdot \bar{g}) \mid \int_M (\text{tr}_{\bar{g}}h) dv_{\bar{g}} = 0 \right\},$$

if \bar{g} is spherical. Therefore, $D^2\mathcal{F}_{M,\bar{g}}|_{\mathcal{S}_{\bar{g}}}$ is strictly negative definite on $\mathcal{E}_{\bar{g}}$.

We introduce a weak Riemannian structure

$$\langle\langle h, h \rangle\rangle_{g_s} := \int_M [\langle h, h \rangle_{g_s} + \langle \nabla_{g_s} h, \nabla_{g_s} h \rangle_{g_s}] dv_{g_s} = \int_M \langle (1 - \Delta_{g_s})h, h \rangle_{g_s} dv_{g_s}$$

on $\mathcal{S}_{\bar{g}}$. As in [20], it has a smooth connection and the $\langle\langle \cdot, \cdot \rangle\rangle_{g_s}$ -gradient of $\mathcal{F}_{M,\bar{g}}|_{\mathcal{S}_{\bar{g}}}$ is given by

$$Y(g_s) = P_{g_s}(1 - \Delta_{g_s})^{-1} \left[V_M(g_s)^{\frac{4}{n}} \left(\Gamma_{g_s}^*(\rho_{g_s}) + \frac{2}{n} g_s V_M(g_s)^{-\frac{n+4}{n}} \mathcal{F}_{M,\bar{g}}(g_s) \right) \right],$$

where P_{g_s} is the orthogonal projection to $T_{g_s}\mathcal{S}_{\bar{g}}$ and $\rho_{g_s} > 0$ is a smooth function on M satisfying $dv_{\bar{g}} = \rho_{g_s} dv_{g_s}$. Obviously, $Y(g_s)$ is a smooth vector field on $\mathcal{S}_{\bar{g}}$. Now we define an auxiliary vector field on $\mathcal{S}_{\bar{g}}$,

$$Z(g_s) := V_M(g_s)^{\frac{4}{n}} \left(\Gamma_{g_s}^*(\rho_{g_s}) + \frac{2}{n} g_s V_M(g_s)^{-\frac{n+4}{n}} \mathcal{F}_{M,\bar{g}}(g_s) \right).$$

It is straightforward that $Z(\bar{g}) = 0$ due to the fact that \bar{g} is Einstein and furthermore,

$$(DZ_{\bar{g}}) \cdot h = (D^2\mathcal{F}_{M,\bar{g}}|_{\mathcal{S}_{\bar{g}}}) \cdot (h, \cdot)$$

for any $h \in \mathcal{E}_{\bar{g}}$. Thus we have

$$DY_{\bar{g}} = P_{\bar{g}}(1 - \Delta_{\bar{g}})^{-1}(DZ_{\bar{g}}),$$

which implies $DY_{\bar{g}}$ is an isomorphism on $\mathcal{E}_{\bar{g}}$ due to the reason that $D^2\mathcal{F}_{M,\bar{g}}|_{\mathcal{S}_{\bar{g}}}$ is strictly negative definite on $\mathcal{E}_{\bar{g}}$ from previous discussions.

According to Lemma 5.6, we can find a neighborhood $U_{\bar{g}} \subset \mathcal{S}_{\bar{g}}$ such that any metric $g_s \in U_{\bar{g}}$ satisfying

$$\mathcal{F}_{M,\bar{g}}|_{\mathcal{S}_{\bar{g}}}(g_s) \geq \mathcal{F}_{M,\bar{g}}|_{\mathcal{S}_{\bar{g}}}(\bar{g}) = \mathcal{F}_{M,\bar{g}}(\bar{g})$$

implies that $g_s \in \mathcal{Q}_{\bar{g}}$, which means we can find a constant $c > 0$ such that $g_s = c^2\bar{g}$. \square

Now we can prove our volume comparison theorem:

Proof of Theorem 1.7. Applying Theorem 5.3, we can find a positive constant $\varepsilon_0 < \varepsilon_1$ such that for any metric \hat{g} satisfies that

$$\|\hat{g} - \bar{g}\|_{C^4(M,\bar{g})} < \varepsilon_0,$$

there exists a diffeomorphism φ such that $\varphi^*\hat{g} \in U_{\bar{g}} \subseteq \mathcal{S}_{\bar{g}}$, where $U_{\bar{g}}$ is defined in Proposition 5.7.

For $\lambda > 0$, suppose g is a Riemannian metric on M satisfying

$$Q_g \geq Q_{\bar{g}}$$

and

$$\|g - \bar{g}\|_{C^4(M,\bar{g})} < \varepsilon_0,$$

but with *reversed volume comparison*:

$$(5.1) \quad V_M(g) \geq V_M(\bar{g}).$$

We are going to show that g has to be isometric to \bar{g} and hence the claimed volume comparison holds.

According to the argument in the previous paragraph, there exists a diffeomorphism φ such that $\varphi^*g \in U_{\bar{g}} \subseteq \mathcal{S}_{\bar{g}}$ and

$$\mathcal{F}_{M,\bar{g}}|_{\mathcal{S}_{\bar{g}}}(\varphi^*g) = V_M(\varphi^*g)^{\frac{4}{n}} \int_M (Q_g \circ \varphi) dv_{\bar{g}} \geq V_M(\bar{g})^{\frac{4}{n}} \int_M Q(\bar{g}) dv_{\bar{g}} = \mathcal{F}_{M,\bar{g}}|_{\mathcal{S}_{\bar{g}}}(\bar{g})$$

due to our assumptions and the fact that $Q_{\bar{g}}$ is a constant. Thus, we conclude that $\varphi^*g = c^2\bar{g}$ for some positive $c \in \mathbb{R}$ by Proposition 5.7. Now the reversed volume comparison (5.1) becomes

$$V_M(g) = V_M(\varphi^*g) = c^n V_M(\bar{g}) \geq V_M(\bar{g}),$$

which implies $c \geq 1$. However, the curvature comparison assumption implies $c \leq 1$, since

$$Q_{\bar{g}} = Q_{\bar{g}} \circ \varphi \leq Q_g \circ \varphi = Q_{\varphi^*g} = c^{-4}Q_{\bar{g}}.$$

Therefore, $\varphi^*g = \bar{g}$ and it concludes the theorem. \square

With the same idea, we can prove the rigidity of strictly stable Ricci-flat manifolds:

Proof of Theorem 1.11. Similar to the proof of Theorem 1.7, we can find an $\varepsilon_0 > 0$ such that for any metric \hat{g} satisfies

$$\|\hat{g} - \bar{g}\|_{C^4(M,\bar{g})} < \varepsilon_0,$$

there is a diffeomorphism φ such that $\varphi^*\hat{g} \in U_{\bar{g}} \subset \mathcal{S}_{\bar{g}}$, where $U_{\bar{g}}$ is given by Proposition 5.7.

Suppose g is a metric satisfying

$$Q_g \geq 0$$

and

$$\|g - \bar{g}\|_{C^4(M, \bar{g})} < \varepsilon_0,$$

then we can find a diffeomorphism φ such that $\varphi^*g \in U_{\bar{g}}$ and

$$\mathcal{F}_{M, \bar{g}}|_{\mathcal{S}_{\bar{g}}}(\varphi^*g) = V_M(\varphi^*g)^{\frac{4}{n}} \int_M (Q_g \circ \varphi) dv_{\bar{g}} \geq 0.$$

However, the metric \bar{g} is Ricci-flat and hence

$$\mathcal{F}_{M, \bar{g}}|_{\mathcal{S}_{\bar{g}}}(\bar{g}) = V_M(\bar{g})^{\frac{4}{n}} \int_M Q(\bar{g}) dv_{\bar{g}} = 0.$$

This shows $\varphi^*g = c^2\bar{g}$ for some positive constant $c \in \mathbb{R}$ by Proposition 5.7, which means g is homothetic to \bar{g} up to a diffeomorphism. \square

6. REMARKS AND FURTHER DISCUSSIONS

In this section, we address some important observations and remarks regarding our main theorem. First, we make a comment on the stability assumption:

Remark 6.1. The stability condition in Theorem 1.7 is necessary. This is the same phenomenon observed in [20] for the volume comparison of scalar curvature.

Let \bar{g} be the canonical product metric on $\mathbb{S}^2 \times \mathbb{S}^2 \times \mathbb{S}^2$. It is well-known that this manifold is unstable (c.f [10]). Consider the metric

$$g_t = (1 + t^2)^{-1}g_{\mathbb{S}^2}^1 + (1 - t)^{-1}g_{\mathbb{S}^2}^2 + (1 + t)^{-1}g_{\mathbb{S}^2}^3$$

with $t \in (0, 1)$ sufficiently small. Then its Q -curvature is given by

$$Q_{g_t} = \frac{1}{50}(-3t^4 + 7t^2 + 48) > \frac{24}{25} = Q_{\bar{g}}$$

However, its volume satisfies

$$V_M(g_t) = (1 - t^4)^{-1}V_M(\bar{g}) > V_M(\bar{g}).$$

It shows that the volume comparison fails in this case. In fact, the volume comparison is not expected to hold for unstable Einstein manifolds due to Corollary 4.7.

Now we turn to the locality assumption. For general dimensions $n \geq 3$, we have proved a volume comparison result for metrics sufficiently closed to a strictly stable positive Einstein metric. It turns out that for dimension $n = 4$, we do have a global volume comparison result as follows:

Proposition 6.2. Let (M^4, \bar{g}) be a closed 4-dimensional locally conformally flat Riemannian manifold with positive constant Q -curvature. Then for any metric g on M satisfies

$$Q_g \geq Q_{\bar{g}}$$

pointwisely on M , we have

$$V_M(g) \leq V_M(\bar{g}) - \frac{1}{4Q_{\bar{g}}} \|W_g\|_{L^2(M,g)}^2 \leq V_M(\bar{g}),$$

where equality holds if and only if g is also locally conformally flat.

Proof. By Gauss-Bonnet-Chern formula,

$$8\pi^2 \chi(M) = \int_M Q_g dv_g + \frac{1}{4} \int_M |W_g|^2 dv_g = \int_M Q_{\bar{g}} dv_{\bar{g}} = Q_{\bar{g}} V_M(\bar{g}).$$

Then

$$\begin{aligned} V_M(\bar{g}) &= Q_{\bar{g}}^{-1} \left(\int_M Q_g dv_g + \frac{1}{4} \int_M |W_g|^2 dv_g \right) \\ &\geq V_M(g) + \frac{1}{4Q_{\bar{g}}} \int_M |W_g|^2 dv_g. \end{aligned}$$

□

As a straightforward application, we have

Corollary 6.3. *Let (M^4, \bar{g}) be either the standard round 4-sphere or a closed hyperbolic 4-manifold. Then for any metrics g satisfies*

$$Q_g \geq Q_{\bar{g}},$$

we have

$$V_M(g) \leq V_M(\bar{g}).$$

Moreover, in case of the round 4-sphere, the equality holds if and only if g is isometric to the spherical metric \bar{g} .

Proof. We only need to prove the rigidity part of 4-sphere. According to Proposition 6.2, the metric g has to be locally conformally flat and hence $g \in [\bar{g}]$ by *Kuiper's theorem*[11], since \mathbb{S}^4 is simply connected. On the other hand, due to our assumption $Q_g \geq Q_{\bar{g}}$ and Gauss-Bonnet-Chern formula, we have $Q_g = Q_{\bar{g}}$. Now the conclusion follows from a uniqueness result in [5, 12, 19]. □

Based on this global volume comparison observed above for 4-dimensional hyperbolic manifolds, we would like to propose the following conjecture:

Conjecture 6.4. For any $n \geq 3$, let (M^n, \bar{g}) be a closed hyperbolic manifold. Suppose g is a metric on M with

$$Q_g \geq Q_{\bar{g}},$$

then we have

$$V_M(g) \leq V_M(\bar{g}).$$

Remark 6.5. This is a corresponding version of *Schoen's conjecture* on scalar curvature (see [20] for more details). As a first step, we would be interested in the question that whether this conjecture holds for metrics C^4 -closed to the hyperbolic metric \bar{g} . In this case, it depends on a further research on the spectrum of the operator $\mathcal{L}_{\bar{g}}$.

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