Product invariant properties of topological spaces

PRODUCT INVARIANT PROPERTIES OF TOPOLOGICAL SPACES

BY

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N. L.

INTRODUCTION

This is a collection of some product invariant properties of topological spaces. It is presented in three main parts, and the conclusion. The first part uses the product of two topological spaces, and it consists of both the properties that are given by a number of sources to be invariant as well as the verification of the invariance of others which were suggested in part by William J. Pervin [5, P. 134], and partly by Dr. R. E. Douglas Jones, the thesis advisor. Following this is a collection of properties of products in the general case and in the case involving finitely many spaces. The final part considers those properties which are preserved in the product of two spaces having different properties. In the conclusion the whole collection is given in two tables.

With the exception of definition 1.39 which derives from Hall and Spencer [2, P. 95] the definitions in part one are largely consistent with those of Pervin [5]; also, throughout the paper, the use of symbols is consistent with Pervin [5]. According to Mendelson [4] open sets of product spaces contain sets which consist of products of sets that are open in the coordinate spaces. Hence, in the space $X_1 \times X_2$ an open set G contains an open set of the form $G_1 \times G_2$ with G_i open in X_i, i = 1, 2. The contraction "iff" is used for the phrase "if and only if"; and the end of a proof is marked by the symbol |. Wherever "I" appears as an index set it is intended to represent the positive integers.

The classical compactness properties are considered in this opening part of the paper as follows:

Definition 1.1. A subset E of a topological space $(X, \mathcal{T})$ is compact iff every open covering of E is reducible to a finite subcovering of E.

Theorem 1.2. $X_1 \times X_2$ is compact iff X_1 and X are compact.

Proof. See Pervin [5, P. 133].

Definition 1.3. A topological space is countably compact iff every countable open cover has a finite subcover.

Theorem 1.4. $X_1 \times X_2$ is countably compact iff X_1 and X_2 are countably compact.

Proof. Let X_1 and X_2 be countably compact and let $\{G_i\}$ $i \epsilon I$ be a countable open cover for $X_1 \times X_2$. For every point (x_1, x_2) of $X_1 \times X_2$ and open set $G_i \epsilon \{G_i\}$ containing it there are open sets $G_1 \subseteq X_1$, $G_2 \subseteq X_2$ such that $(x_1, x_2) \epsilon G_1 \times G_2 \subset G_i$ for some i. For any fixed $x'_2 \epsilon X_2$ the countable family of sets $\{G_1\}$ such that $(x_1, x'_2) \epsilon G_1 \times G_2$ forms a countable open cover of the countably compact space X_1 so that there is a finite subcover, call it $\{G_{n,1}\}_{n=1}^{m}$. Considering $\left\{G_{n,2} \mid (x_1, x'_2) \epsilon G_{n,1} \times G_{n,2}\right\}$, $0 = \bigcap_n G_{n,2} \neq \phi$ and $x'_2 \epsilon 0 \subseteq X_2$ where 0 is open. Therefore, the finite family $\left\{G_{n,1} \times 0\right\}_{n=1}^{m}$ covers $\left\{(\xi, x'_2) \mid \xi \epsilon X_1\right\}$. Now, $\left\{0 \mid x'_2 = x_2 \epsilon X_2\right\}$ is a countable cover for the countably compact X_2 and must have a finite cover, call it, $\left\{O_k\right\}_{k=1}^{n}$. Then, for any particular k, $\left\{G_{n,1} \times O_k\right\}_{n=1}^{m}$ is a finite cover for $X_1 \times O_1$ and hence the family $\left\{G_{n,1} \times O_k\right\}_{1 \leq n \leq m,\ 1 \leq k \leq q}$, is a finite cover for $X_1 \times X_2$. Since $O \subset G_{n,2}$ it follows that the countable family $\left\{G_1 \times G_2\right\}$, where each of the sets $G_1 \times G_2$ is a subset of one of the open sets G_i. Hence there is a finite subcover of $\left\{G_i\right\}_{i \epsilon I}$. Conversely, let $X_1 \times X_2$ be countably compact and let $\left\{G_i\right\}_{i \epsilon I}$ be a countable open cover for X_1, and $\left\{O_j\right\}$ $j \epsilon I$ for X_2. Define $\left\{G^*\right\} = \left\{G_i \times O_j \mid i, j \epsilon I\right.$. $\left\{G^*\right\}$ is a countable open cover for $X_1 \times X_2$, so that for some k=1, ..., n, $\left\{G^*_k\right\}$ covers $X_1 \times X_2$. Fixing x'_2 in X_2, the family $\left\{G_{i_k} \times O_{x'_2}\right\}$ is a finite open cover for $\left\{(\xi, x'_2) \mid \xi \epsilon X_1\right\}$ where $O_{x'_2}$ is a member of $\left\{O_j\right\}_{j \epsilon I}$ which contains x'_2 and hence $\left\{G_{i_k}\right\}_{k=1}^{n}$ is a finite open cover of X_1. Similarly, there is a finite subset of $\left\{O_j\right\}_{j \epsilon I}$ which covers X_2. |

Definition 1.5. A topological space, (X, Υ) will be called locally compact iff each point of X is contained in a compact neighborhood.

Theorem 1.6. $X_1 \times X_2$ is locally compact iff X_1 and X_2 are locally compact.

Proof. Suppose that X_1 and X_2 are locally compact and (x_1, x_2) is any point $X_1 \times X_2$. Since X_i is locally compact there exists a compact neighborhood N_i of x_i in X_i, i=1, 2 such that by theorem 1.2 $N_1 \times N_2$ is a compact neighborhood of (x_1, x_2) in $X_1 \times X_2$. Hence, $X_1 \times X_2$ is locally compact. Conversely, let $X_1 \times X_2$ be locally compact and let x_i be any point in X_i, i=1,2. Then there exists a compact neighborhood, N, of $(x_1, x_2) \epsilon X_1 \times X_2$ whose continuous projection image, $N_i = \Pi_i(N)$ is a compact neighborhood of x_i in X_i, i=1,2. Therefore, X_1 and X_2 are locally compact. |

Definition 1.7. A subset E of a topological space X is sequentially compact iff every sequence of points of E contains a subsequence which converges to a point of E.

Theorem 1.8. $X_1 \times X_2$ is sequentially compact iff X_1 and X_2 are sequentially compact.

Proof. Let X_1 and X_2 be sequentially compact and let $\left\{(x_n, y_n)\right\}$ be a sequence of points of $X_1 \times X_2$. Suppose that $\left\{x_{n_i}\right\}$ is the subsequence of $\left\{x_n\right\}$ which converges to x' in X_1; let $\left\{y_{n_i}\right\} = \left\{y_{n_i} \in \{y_n\} \mid (x_{n_i}, y_{n_i}) \in \{ x_n, y_n \}\right\}$. Since X_2 in sequentially compact there exists a subsequence $\left\{y_{n_j}\right\}$ of $\left\{y_{n_i}\right\}$ which converges to some point $y' \in X_2$. A corresponding subsequence $\left\{x_{n_j}\right\}$ of $\left\{x_{n_i}\right\}$ converges to $x' \in X_1$. Therefore $(x_{n_j}, y_{n_j}) \to (x', y') \in X_1 \times X_2$ so that $\left\{(x_{n_j}, y_{n_j})\right\}$ is a subsequence of $\left\{(x_n, y_n)\right\}$ which converges to a point of $X_1 \times X_2$. Therefore, $X_1 \times X_2$ is sequentially compact. Conversely, let $X_1 \times X_2$ be sequentially compact and let $\{x_n\}$, $\{y_n\}$ be a sequence in X_1 and X_2 respectively. Let $\left\{(x_{n_i}, y_{n_i})\right\}$ be the subsequence of $\left\{(x_n, y_n)\right\}$ which converges to $(x', y') \in X_1 \times X_2$. This implies that $x_{n_i} \to x' \in X_1$ and $y_{n_i} \to y' \in X_2$, so that $\left\{x_{n_i}\right\}$ is a subsequence of $\{x_n\}$ which converges to $y' \in X_2$. Therefore, both X_1 and X_2 are sequentially compact. |

Definition 1.9. A family E of subsets of a topological space X will be called locally finite iff every point of the space has a neighborhood which has a nonempty intersection with at most a finite number of the members of E.

Definition 1.10. A topological space X is paracompact iff for every open covering of X there is a locally finite open cover which refines it.

Using the right half-open interval topology on the set of nonnegative real numbers Gaal [1, P. 157] has outlined a way of showing that the product of a paracompact space with itself need not be paracompact. However, if one space is compact and the other paracompact then the product is paracompact, See Theorem 3.3.

Definition 1.11. A topological space is σ-compact iff it is the union of a countable collection of compact sets.

Theorem 1.12. The product of finitely many σ-compact spaces is σ-compact.

Proof. Gaal [1, P. 152] outlines the proof of this theorem using the product of two spaces. |

Definition 1.13. A topological space is called a Lindelof Space iff every open covering of the space is reducible to a countable subcovering.

Gaal [1, P. 145] has shown that if X_1 and X_2 are both Lindelof, $X_1 \times X_2$ is not necessarily Lindelof.

The finite intersection property as well as the first and the second axioms of countability are product invariant for two spaces.

Definition 1.14. A family of sets will be said to have the finite intersection property iff every finite subfamily of the family has a nonempty intersection.

Theorem 1.15. $\left\{Y_\alpha \times Z_\beta \mid \epsilon A, \beta \epsilon B\right\}$ has the finite intersection property iff $\left\{Y_\alpha\right\}_{\alpha \epsilon A}$ and $(Z\,\beta)_{\beta \epsilon B}$ have the finite intersection property.

Proof. Let $(Y_\alpha)_{\alpha\epsilon A}$, $Z\beta)_{\beta\epsilon B}$ be families of sets that have the finite intersection property, and let $\left\{Y_{\alpha_i} \times Z_{\beta_i}\right\}_{i=1}^{m}$ be an arbitrary finite subfamily of the collection $\left\{Y_\alpha \times Z\beta \,|\, \alpha\epsilon A, \beta\epsilon B\right\}$. Since $(Y_\alpha)\epsilon A$ and $(Z\beta)\beta\epsilon B$ have the finite intersection property $\bigcap_{i=1}^{n} Y_\alpha i \neq \Phi$, $\bigcap_{i=1}^{n} Z\beta_i \neq \Phi$ so that if $y\epsilon \bigcap_{i=1}^{n} Y_{\alpha_i}$, $z\epsilon \bigcap_{i=1}^{n} Z_{\beta_i}$ it follows that $(Y,Z)\epsilon \bigcap_{i=1}^{n} (Y_{\alpha_i} \times Z_{\beta_i})$ and that the subfamily $\left\{Y_{\alpha_i} \times Z_{\beta_i}\right\} \bigcap_{i=1}^{n}$ of $\left\{Y_\alpha \times Z_\beta \,|\, \epsilon A, \beta\epsilon B\right\}$ has a nonempty intersection. Conversely, let $\left\{Y_\alpha \times Z_\beta \,|\, A_1 \beta\epsilon B\right\}$ be a family of sets having the finite intersection property, and suppose that $\left\{Y_{\alpha_i}\right\}_{i=1}^{n}$ is an arbitrary finite subfamily of $\left\{Y_\alpha\right\}_{\alpha\epsilon A}$. Then $\bigcap_{i=1}^{n} (Y_{\alpha_i} \times Z_{\beta_0}) \neq \Phi$, with β_0 fixed in B. Let $(Y,Z)\epsilon \bigcap_{i=1}^{n} (Y_{\alpha_i} \times Z_{\beta_0})$. Then $(y,z)\epsilon (Y_{\alpha_i} \times Z_{\beta_0})$ $i=1,2,\ldots, n$ and hence $y\epsilon Y_{\alpha_i}, \forall i=1,\ldots, n$. Therefore, $\bigcap_{i=1}^{n} Y_{\alpha_i} \neq \Phi$, and so the family $(Y_\alpha)_{\alpha\epsilon A}$ has the finite intersection property. Similarly, $(Z\beta)\beta\epsilon B$ has the finite intersection property.|

Definition 1.16. A topological space X is a first axiom space iff for every point $x\epsilon X$, there exists a countable family $\left\{B_n(x)\right\}$ of open sets containing x such that whenever x belongs to an open set G, $B_n(x)$ G for some n.

Theorem 1.17. $X_1 \times X_2$ is a first axiom space iff X_1 and X_2 are first axiom spaces.

Proof. Let X_1 and X_2 be first axiom spaces, $(x_1, x_2)\epsilon X_1 \times X_2$ and let G be an open set in $X_1 \times X_2$ containing (x_1, x_2). Since G contains an open set of the form $G_1 \times G_2$ with G_i open in X_i and containing x_i, i=1,2, by the first axiom property of X_i, $x_i\epsilon B_{n_i,i}(x_i) \subseteq G_{i_a}$ for some n_i, with $B_{n_i,i}(x_i)\epsilon\left\{B_{n_i,i}(x\ )\right\}$ a family of countable open sets of X_i containing x_i.

Then G contains $B_{n_{1,1}}(x_1) \times B_{n_{2,2}}(x_2)$ in $X_1 \times X_2$ and hence $X_1 \times X_2$ is a first axiom space. Conversely, let $X_1 \times X_2$ be a first axiom space, $x_i \in X_i$, i=1,2, and G_1 an open set in X_1 containing x_1. Then, $G_1 \times X_2$ is an open set containing (x_1, x_2) in $X_1 \times X_2$. Since $X_1 \times X_2$ is first axiom there exists a countable family $\left\{B_n(x_1, x_2)\right\}$ of open sets of $X_1 \times X_2$ containing (x_1, x_2) such that for some n $B_n(x_1, x_2) \subseteq G_1 \times X_2$. Let $B_{n,1}(x_1) = \pi\left[B_n(x_1, x_2)\right]$, thus, $x_1 \in B_{n,1}(x_1)$ G_1 in X_1. It follows that X_1 is a first axiom space, and similarly X_2 is a first axiom space.|

Definition 1.18. A topological space (X, Υ) is a second axiom space iff there exists a countable base for the topology.

Theorem 1.19. $X_1 \times X_2$ is a second axiom space iff X_1 and X_2 are second axiom.

Proof. Let X_1 and X_2 be second axiom spaces, $(x_1, x_2) \in X_1 \times X_2$, and G an open set of $X_1 \times X_2$ containing (x_1, x_2). G contains an open set of the form $G_1 \times G_2$ with $x_i \in G_i$, and G_i open in X_i, i=1,2. The families $\{B_m\}$ and $\{B_n\}$ of countable open sets forming the bases for the topologies of X_1 and X_2 respectively exist such that for some m and n $(x_1, x_2) \in B_m \times B_n \subseteq G_1 \times G_2 \subseteq G$, $\left\{B_m \times B_n\right\}$ a family of countable open sets of $X_1 \times X_2$. Conversely, let $X_1 \times X_2$ be second axiom, $x_1 \in X_1$, and G_1 an open set of X_1. There exists at least one n such that $x_1 \in \pi_1(B_n)$, $B_n \subseteq G_1 \times X_2$ with B_n a member of a countable family $\{B_n\}$ which is a base for the open sets of $X_1 \times X_2$. For such a number, n, B_n contains an open set of the form $B_{n,1} \times B_{n,2}$ with $B_{n,i}$ open in X_i, i=1,2. Then, $\{ B_{n,1} \mid B_{n,1} \times B_{n,2} \subseteq B_n \}$ is a countable base for the open sets of X_1. Therefore, X_1 is a second axiom. Similarly, X_2 is second axiom.|

Among the separation axioms the first four, T_0, T_1, T_2, and T_3 are considered here as follows.

<u>Definition 1.20</u>. A topological space X is a <u>To - space</u> iff for two distinct points x and y of X, there exists an open set which contains one of them but not the other.

<u>Theorem 1.21</u>. $X_1 \times X_2$ is To iff X_1 and X_2 are To.

<u>Proof</u>. Let X_1 and X_2 be To - spaces and let (x_1, x_2), (y_1, y_2) be two distinct points of $X_1 \times X_2$. In particular, let $x_1 \neq y_1$. Since X_1 is To $\exists$ an open set $O_1 \subset X_1 \ni O_1$ contains one of the points and not the other. Suppose that $x_1 \in O_1$ and $y_1 \notin O_1$. Then $O_1 \times X_2$ is an open set of $X_1 \times X_2$, containing (x_1, x_2), but not (y_1, y_2). Then, $X_1 \times X_2$ is To. Conversely, let $X_1 \times X_2$ be a To - space and let $x_i, y_i \in X_i$, i=1,2, $x_1 = y_1$ and $x_2 = y_2$ and (x_1, x_2), (y_1, y_2) are distinct points of $X_1 \times X_2$. Then $\exists$ an open set of $X_1 \times X_2$ containing one of the points and not the other. Suppose that O contains (x_1, x_2) $\exists$ $O_1 \times O_2 \subseteq O \ni (x_1, x_2) \in O_1 \times O_2$ and (y_1, y_2) and $O_1 \times O_2$, with O_i open in X_i, i=1,2. Then O_1 contains x_1 and not y_1 and so X_1 is a To-space. Similarly, X_2 is a To - space.|

<u>Definition 1.22</u>. A topological space X is a <u>T_1 - space</u> iff for x and y, two distinct points in X_1 there exist two open sets, one containing x but not y, and the other containing y but not x.

Theorem 1.23. $X_1 \times X_2$ is a T_1 - space iff X_1 and X_2 are T_1 spaces.

Proof. Let X_1 and X_2 be T_1 - spaces and let $(x_1, x_2), (y_1, y_2)$ be two distinct points of $X_1 \times X_2$. If $x_1 \neq y_1$, then there exist two open sets G_1 and O_1 in X_1 such that $x_1 \in G \subseteq G_1$ and $y_1 \notin O \subseteq O_1$. Then for all $z \in X_2$ $(x_1, z) \in G_1 \times X_2$, $(y_1, z) \in G_1 \times X_2$, $(y_1, z) \in O_1 \times X_2$ and $(x_1, z) \notin O_1 \times X_2$. Therefore, $O_1 \times X_2$ is an open set of $X_1 \times X_2$ containing (y_1, y_2) and not (x_1, x_2) and $G_1 \times X_2$ is an open set of $X_1 \times X_2$ containing (x_1, x_2) and not (y_1, y_2). Therefore, $X_1 \times X_2$ is a T_1 - space. Conversely, let $X_1 \times X_2$ be a T_1 - space and let $x_i, y_i \in X_i$, i=1,2. If $(x_1, x_2) \neq (y_1, y_2)$ then $\exists G$, O open in $X_1 \times X_2 \ni (x_1, x_2) \in G$, $(y_1, y_2) \notin G$, $(y_1, y_2) \in O$ and $(x_1, x_2) \notin O$. But there exist open sets $G_1 \times G_2$, $O_1 \times O_2$ in $X_1 \times X_2$ such that $G_1 \times G_2 \subset G$, $O_1 \times O_2 \subset O$ and G_i, O_i open in x_i. If $x_2 \neq y_2$ and $x_1 = y_1$ then $x_2 \in G_2$, $y_2 \notin G_2$, $x_2 \notin O_2$ and $y_2 \in O_2$; hence, X_2 is a T_1 - space. Simarly, X_1 is T_1. |

Definition 1.24. A topological space X is a T_2 - space or Hausdorff Space iff whenever x and y are two distinct points of X, then there exist two disjoint open sets, one containing x and the other containing y.

Theorem 1.25. $X_1 \times X_2$ is a T_2 - space iff X_1 and X_2 are T_2 - spaces.

Proof. Let X_1 and X_2 be T_2 - spaces and let (x_1, x_2), (y_1, y_2) be two distinct points of $X_1 \times X_2$. If $x_1 \neq y_1$ $\exists$ two disjoint open sets G_1, O_1 in X_1 containing x_1 and y_1 respectively and setting $G = G_1 \times X_2$, $O = O_1 \times X_2$ it follows that G and O are two disjoint open sets of $X_1 \times X_2$ such that (x_1, x_2) is contained in G and (y_1, y_2) in O. Conversely, let $X_1 \times X_2$ be a T_2 - space and let $x_i, y_i \in X_i$, i=1, 2, and let (x_1, x_2), (y_1, y_2) be distinct points in $X_1 \times X_2 \ni x_1 \neq y_1$ and $y_2 = x_2$. Then there exist two disjoint open sets G and O in $X_1 \times X_2$ containing (x_1, x_2) and (y_1, y_2) respectively. Let $G_1 \times G_2 \subset G$, $O_1 \times O_2 \subset O$ with G_i, O_i open in X_i, $(x_1, x_2) \in G_1 \times G_2$ and $(y_1, y_2) \in O_1 \times O_2$. Since $x_2 \in G_2 \cap O_2$, $G_1 \cap O_1 = \Phi$ and hence X_1 is a T_2 - space. Similarly, X_2 is T_2.

<u>Definition 1.26</u>. A topological space X is <u>regular</u> iff for a closed subset F of X and a point x of X not contained in F there exist two disjoint open sets, one containing F and the other containing x.

In one of his characterizations of a regular space, Pervin [5, P. 87] proves that a topological space X is regular iff for every point $x \in X$ and open set G containing x there exists an open set G* such that $x \in G^*$ and $\bar{G}^* \subseteq G$.

<u>Theorem 1.27</u>. $X_1 \times X_2$ is regular iff X_1 and X_2 are regular.

<u>Proof</u>. Let X_1 and X_2 be regular spaces and let G be an open containing x in $X_1 \times X_2$. G contains an open set of the form $G_1 \times G_2$ with G_i open in X_i and defining $x_i = \pi_i(x)$, $x_i \in G_i$, i=1,2. Since X_i is regular, by Pervin's characterization of regular spaces mentioned above, $\exists G^*_i \ni x_i \in G^*_i$ and $\bar{G}^*_i \subseteq G_i$. Hence, $x \in G^*_1 \times G^*_2$ and $\bar{G}^*_1 \times \bar{G}^*_2 \subseteq G_1 \times G_2 \subseteq G$. Therefore, $X_1 \times X_2$ is regular. Conversely, let $X_1 \times X_2$ be regular and let $x_1 \notin F_1$, F_1 closed and x_1, f_1 are both in X_1. For any $y \in X_2$, $(x_1, y) \notin F_1 \times X_2$ and by the regularity of $X_1 \times X_2$ there are two disjoint open sets G, $O \subset X_1 \times X_2$ such that $(x_1, y) \in G$ and $F_1 \times X_2 \subset O$. Since $y \in \pi_2(G) \cap \pi_2(O) \subseteq X_2$, then $\pi_1(G) \cap \pi_1(O) = \Phi$, and setting $G_1 = \pi_1(G)$, $O_1 = \pi_1(O)$ is follows that G_1 and O_1 are two disjoint open sets of X_1 containing x_1 and F_1 respectively. Therefore X_1 is regular and similarly X_2 is regular.|

Definition 1.28. A topological space X is completely regular iff for a closed subset F of X and a point x of X not in F there exists a continuous mapping $f : X \to [0, 1]$ such that $f(x) = 0$ and $f(F) = \{1\}$.

Theorem 1.29. $X_1 \times X_2$ is completely regular iff X_1 and X_2 are completely regular.

Proof. Let X_1 and X_2 be completely regular and let P and F be in $X_1 \times X_2$ with F a closed subset of the product space and P is not contained in F. Let $G = X_1 \times X_2 \setminus F$, $P_i = \pi_i(P)$, and let $G_1 \times G_2$ be an open set of $X_1 \times X_2$ contained in $G \ni G_i$ is open in X_i and contains P_i, $i = 1,2$. Since $X_i \setminus G_i$ is closed in X_i and does not contain P_i, $\exists$ a continuous mapping $g_i : X_i \to [0, 1] \ni g_i(P_i) = 0$, $g_i(x_i) = \{1\} \forall x_i \epsilon (X \setminus G_i)$. Define $f(x) = \left|\frac{g_1(x_1) + g_2(x_2)}{2}\right| + \left|\frac{g_1(x_1) - g_2(x_2)}{2}\right|$ where $x_i = \pi_i(x)$, i=1,2. Since f merely sums up the absolute values of the sum and difference of two continuous functions then f is continuous. Therefore, f is a continuous mapping of $X_1 \times X_2$ into $[0, 1]$ such that $f(P) = 0$, and $f(x) = \{1\} \forall x \epsilon F \subseteq \left[(X_1 \times X_2) \setminus (G_1 \times G_2)\right]$. Therefore $X_1 \times X_2$ is completely regular. Conversely, let $X_1 \times X_2$ be completely regular and let F_i be a closed subset of X_i. Suppose that $x_i \epsilon G_i = X_i \setminus F_i$, $i = 1,2$. Then there exists a continuous mapping of $X_1 \times X_2$ into $[0,1]$ such that $f(P_1, P_2) = 0$, and $f(x, y) = \{1\} \forall (x, y) \epsilon \left[(X_1 \times X_2) \setminus (G_1 \times G_2)\right]$. Define $g_1(x) = f(x, P_2)$, $g_2(y) = f(P_1, y)$. Then $g_1 : X_1 \to [0, 1] \ni g_1(P_1) = 0$, $g_1(x) = \{1\}$, $\forall x \epsilon (X_1 \setminus G_1)$; $g_2 : X_2 \to [0, 1] \ni g_2(P_2) = 0$, $g_2(y) = \{1\} \forall y \epsilon (X_2 \setminus G_2)$. Since f is continuous, both the g_i's are continuous, i=1,2. Therefore, X_1 and X_2 are completely regular.|

Definition 1.30. A topological space X is a T_3 - space if it is a regular space which is also a T_1 - space.

Corollary 1.31. $X_1 \times X_2$ is a T_3 - space iff both X_1 and X_2 are T_3 - spaces.

Proof. Both regularity and T_1 are product invariant in the cartesian product of two topological spaces by theorems 1.27, and 1.23 respectively. |

Definition 1.32. A topological space X is normal iff for every two distinct closed sets F_1 and F_2 of X there exist two disjoint open sets, one containing F_1 and the other containing F_2.

Kelley [3, P. 134] gives an example of a normal space X with the property that X x X is not normal. Hence, normality is not a product invariant.

Definition 1.33. Two subsets A and B form a separation or partition of a set E in a topological space (X, Υ) iff $E = A \cup B$, $A \neq \Phi$, $B \neq \Phi$, and $A \cap \overline{B} = \overline{A} \cap B = \Phi$.

Definition 1.34. A set is called connected iff it has no separation.

Theorem 1.35. $X_1 \times X_2$ is connected iff X_1 and X_2 are connected.

Proof. See Pervin [5, P. 132]. |

Definition 1.36. If E is a subset of a topological space X and we let $I = [0, 1]$, then a path in E joining two points x and y is a continuous mapping f of I into E such that $f(0) = x$ and $f(1) = y$.

<u>Definition 1.37</u>. The subset E of a topological space X is <u>arcwise connected</u> iff every two points in E may be joined by some path in E.

<u>Theorem 1.38</u>. $X_1 \times X_2$ is arcwise connected iff both X_1 and X_2 are arcwise connected.

<u>Proof.</u> Let X_1, X_2 be arcwise connected and let $(x_1, x_2), \ni (y_1, y_2) \in X_1 \times X_2$. Then $\exists$ a continuous function $g_i : [0, 1] \to X_i \ni g_i(0) = x_i$, $g_i(1) = y_i$, $i = 1,2$. Define $f(t) = \Big(g_1(t), g_2(t)\Big)$, $t \in [0, 1]$. Then $f : [0, 1] \to X_1 \times X_2 \ni f(0) = (x_1, x_2)$, $f(1) = (y_1, y_2)$. To show that f is continuous, let G be an open set containing $f(t)$; $\exists$ open sets $G_i \subseteq X_i$ $\ni G_1 \times G_2 \subseteq G$ and $g_i\ (G^*_i) \subseteq G_i$ where G^*_i is an open subset of $[0, 1]$. Let $G^* = G^*_1 \cap G^*_2$, then $f(G^*) \subseteq G_1 \times G_2 \subseteq G$ so that f is continuous. Conversely, let $X_1 \times X_2$ be arcwise connected and let $x, y \in X_1 \times X_2$, $x \neq y$. Suppose that $x_i = \pi_i(x)$, $y_i = \pi_i(y)$, $x_i, y_i \in X_i$, $i = 1,2$. Since $\exists$ $f : [0, 1] \to X_1 \times X_2 \ni f(0) = x$, $f(1) = y$, and f is continuous, and since the projection mapping is also continuous, then defining $g_i = \pi_i \circ f$, g_i is a continuous mapping of the closed unit interval into X_i such that $g_i(0) = x_i$ and $g_i(1) = y_i$. Therefore, X_1 and X_2 are arcwise connected. |

<u>Definition 1.39</u>. A space S is said to be <u>locally connected</u> at a point P if and only if, given any neighborhood U of P, there exists a connected neighborhood V of P such that $V \subseteq U$. The space S is said to be locally connected if and only if it is locally connected at each of its points.

<u>Theorem 1.40</u>. $X_1 \times X_2$ is locally connected iff X_1 and X_2 are locally connected.

Proof. Let X_1 and X_2 be locally connected and let $(x_1, x_2) \leftarrow X_1 \times X_2$. Since for any neighborhood N of (x_1, x_2) N contains an open subset of the form $G_1 \times G_2$, with G_i containing x_i and open in X_i, i = 1,2, then $\exists$ a connected set O_i in $X_i \ni x_i \in O_i \subseteq G_i$ by the local connectivity of X_i. Then $O_1 \times O_2$ is a connected subset of $G_1 \times G_2$ by Theroem 1.35 containing (x_1, x_2) and which is contained in N. Since (x_1, x_2) is an arbitrary point of $X_1 \times X_2$, $X_1 \times X_2$ is locally connected. Conversely, let $X_1 \times X_2$ be locally connected and let x_i by any point of X_i, i = 1,2. If N_i is a neighborhood of x_i in X_i then $N_1 \times N_2$ is a neighborhood of (x_1, x_2) in $X_1 \times X_2$ and by the local connectivity of $X_1 \times X_2$ $\exists$ a connected neighborhood E of $(x_1, x_2) \ni (x_1, x_2) \in E \subseteq N_1 \times N_2$. Since $E_i = \pi_i(E)$ is a continuous image of a connected set containing x_i and which is contained in N_i, it follows that X_i is locally connected.|

The properties examined here are isolatedness and separability.

Definition 1.41. A subset E of a topological space is isolated iff no point of E is a limit point of E; that is, $E \cap d(E) = \Phi$.

Theorem 1.42. $A_1 \times A_2$ is isolated iff A_1 and A_2 are isolated, where $A_i \subseteq X_i$, i = 1,2, and $A_1 \times A_2 \subseteq X_1 \times X_2$.

Proof. Let A_1, A_2 be isolated nonempty subsets of topological spaces and let $(A_1 \times A_2)'$ be the derived set of $A_1 \times A_2$. Since $(A_1 \times A_2)' = \left[(A'_1 \times A_2) \cup (A_1 \times A'_2)\right]$, $(A_1 \times A_2) \cap (A_1 \times A_2)' = (A_1 \times A_2) \cap \left[(A'_1 \times A_2) \cup (A_1 \times A'_2)\right] = \left[(A_1 \times A_2) \cap (A'_1 \times A_2)\right] \cup \left[(A_1 \times A_2) \cap (A_1 \times A'_2)\right] = \left[(A_1 \cap A'_1) \times A_2\right] \cup \left[A_1 \times (A_2 \cap A'_2)\right]$. By the isolatedness of both A_1 and

A_2, it follows that $(A_1 \times A_2) \cap (A_1 \times A_2)' = \Phi$ and hence, $A_1 \times A_2$ is isolated. Conversely, let $A_1 \times A_2$ be isolated and let A'_2 be the derived set of A_i, $i = 1,2$. Since $A'_1 \times A_2 \subseteq (A_1 \times A_2)'$ and since $(A_1 \times A_2) \cap (A_1 \times A_2)' = \Phi$, $(A_1 \times A_2) \cap (A'_1 \times A_2) = \Phi$ and hence, $A_1 \cap A'_1 = \Phi$. Therefore A_1 is isolated, and likewise, A_2 is isolated.|

<u>Definition 1.43</u>. A topological space X, will be called <u>separable</u> iff there exists a countable dense subset of X.

<u>Theorem 1.44</u>. $X_1 \times X_2$ is separable iff both X_1 and X_2 are separable.

<u>Proof</u>. Let X_1 and X_2 be separable, A_i a dense countable subset of X_i, $i = 1,2$, and let (x_1, x_2) be an arbitrary point in $X_1 \times X_2$. $A_1 \times A_2$ is countable in $X_1 \times X_2$. Let G be an open set of $X_1 \times X_2$ containing (x_1, x_2). Then for some set G_i open in X_i, $(x_1, x_2) \in G_1 \times G_2 \subset G$. Since A_i is dense in X_i, $\exists y_i \in A_i \ni y_i \in G_i$. Therefore, $G \cap (A_1 \times A_2) \neq \Phi$, and hence $A_1 \times A_2$ is a countable dense subset of $X_1 \times X_2$. Conversely, let $X_1 \times X_2$ be separable and A a countable dense subset of $X_1 \times X_2$. Suppose that G_i is an open set of X_i containing $x_i \in X_i$, $i = 1,2$, $\exists (y_1, y_2) \in A \ni (y_1, y_2) \in (G_1 \times G_2) \cap A$. If $A_i = \pi_i(A)$ then $y_i \in A_i$, so that $y_i \in G_i \cap A_i$, and A_i is a countable dense subset of X_i, $i = 1,2$. Therefore, X_i is separable. |

II

This part of the collection consists of the definitions which have not occurred earlier, theorems, and the sources for their proofs concerning the properties that are invariance in the product of the general and the finite case.

<u>Theorem 2.1</u>. $\pi_\lambda X_\lambda$ is T_0 iff each space X_λ is T_0.

<u>Proof</u>. See Pervin [5, P. 141].|

<u>Theorem 2.2</u>. $\pi_\lambda X_\lambda$ is T_1 iff each space X_λ is T_1.

<u>Proof</u>. See Kelley [3, P. 133].|

<u>Theorem 2.3</u>. $\pi_\lambda X_\lambda$ is Hausdorff iff all spaces X_λ are Hausdorff.

<u>Proof</u>. See Pervin [5, P. 138].|

<u>Theorem 2.4</u>. $\pi_\lambda X_\lambda$ is regular iff each space X_λ is regular.

<u>Proof</u>. See Kelley [3, P. 133].|

<u>Theorem 2.5</u>. $\pi_\lambda X_\lambda$ is completely regular iff each space X_λ is completely regular.

<u>Proof</u>. See Kelley [3, P. 133].|

<u>Theorem 2.6</u>. $\pi_\lambda X_\lambda$ is compact iff each space X_λ is compact.

<u>Proof</u>. See Pervin [5, P. 143].|

Theorem 2.7. $\pi_\lambda \ X_\lambda$ is connected iff each space X_λ is connected.

Proof. See Pervin [5, P. 139].|

Theorem 2.8. $\pi_\lambda \ X_\lambda$ is first axioms iff each space X_λ is first axiom and all but a countable number are indiscrete.

Proof. See Pervin [5, P. 140].|

Theorem 2.9. $\pi_\lambda \ X_\lambda$ is second axiom iff each space X_λ is second axiom and all but a countable number are indiscrete.

Proof. See Pervin [5, P. 141].|

Theorem 2.10. $\pi_\lambda \ X_\lambda$ is locally compact iff each space X_λ is locally compact and all but a finite number are compact.

Proof. See Pervin [5, P. 146].|

Definition 2.11. A topological space is σ- compact iff it is the union of a countable collection of compact sets.

Theorem 2.12. The product of finitely many σ - compact spaces is σ- compact.

Proof. See Gaal [1, P. 152].|

Theorem 2.13. $\pi_\lambda \ X_\lambda$ is locally connected iff each space X_λ is locally connected and all but a finite number are connected.

Proof. See Pervin [5, P. 140].|

Gaal [1, P. 152] has proposed an example to demonstrate that the product of denumerably many σ- compact spaces is not necessarily σ-compact. Furthermore, separability and the two axioms of countability are not product invariant in general, Kelley [3, P. 133].

III

Sometimes two or more topological spaces have different properties among which one is preserved by the product of the spaces, and this part of the collection concerns a few of such combinations.

Definition 3.1. A subset E of a topological space is dense-in-itself iff every point of E is a limit point of E; that is, $E \subseteq d(E)$.

Theorem 3.2. $X_1 \times X_2$ is dense-in-itself iff at least one of the spaces X_1 and X_2 is dense-in-itself.

Proof. See Pervin [5, P. 134].|

Theorem 3.3. If X_1 is compact and X_2 is paracompact, then $X_1 \times X_2$ is paracompact.

Proof. See Pervin [5, P. 168].|

Definition 3.4. A topological space X will be called countably paracompact iff for every countable open covering of X there is a locally finite open cover which refines it.

Theorem 3.5. If X_1 is compact and X_2 is countably paracompact, then $X_1 \times X_2$ is countably paracompact.

Proof. See Gaal [1, P. 157].|

Theorem 3.6. The product of a compact space and of a Lindelöf space is always a Lindelöf space.

Proof. See Gaal [1, P. 145].|

Theorem 3.7. If X_1 is a regular paracompact space and X_2 is a σ- compact space, then $X_1 \times X_2$ is paracompact.

Proof. See Pervin [5, P. 167].|

CONCLUSION

Two tables are used here to categorize the properties discussed above. In the first table the properties are considered under the product topology of two spaces, a finite number of spaces, and the general case; the "plus" sign means that the property is preserved and the "minus" sign means that it is not. In table two, the product of two spaces is the only one given. The properties of each one of the spaces is given and the property which is preserved in the product topology is indicated.

TABLE I

PROPERTY	$X_1 \times X_2$	$\pi_{i=1}^{n} X_i$	GENERAL CASE
Compactness	+	+	+
Countable Compactness	+		
Local Compactness	+	+	
Sequential Compactness	+	+	
σ - Compactness	+	+	-
Paracompactness	-	-	-
Connectivity	+	+	+
Arcwise Connectivity	+		
Local Connectivity	+	+	
First Axiom	+	+	-
Second Axiom	+	+	-
T_0, T_1, T_2	+	+	+
Regularity	+	+	+
Complete Regularity	+	+	+
T_3	+	+	+
Normal	-	-	-
Lindelöf	-	-	-
Finite Intersection Property	+		
Isolatedness	+		
Separability	+		-

TABLE II

X_1	X_2	PRESERVED
Dense-in-itself Paracompact	Compact	Dense-in-itself Paracompact
Countably Paracompact	Paracompact	Countably Paracompact
Regular Paracompact	σ - Paracompact	Regular Paracompact
Lindelöf	Compact	Lindelöf

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