Implementing a Toffoli Gate in an N-Qubit Linear Nearest Neighbor Array

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Abstract. Most proposals for quantum computers are based on linear nearest neighbor (LNN) arrangements where qubits only interact with neighbors. A Toffoli gate is a three-qubit operation, which is used in many quantum applications such as error correction, and algorithms like Shor’s factorization. Typically, to implement a Toffoli gate in an LNN architecture, additional operations called swap gates are required to bring the qubits adjacent to each other. We propose a new method to implement a Toffoli gate in LNN arrays without using swap gates. As such, compared to other circuits, the quantum cost of our circuit is much lower.

1. Introduction

In quantum computing, a qubit (or quantum bit) is represented by the state $\alpha|0\rangle + \beta|1\rangle$ where $|\alpha|^2 + |\beta|^2 = 1$. A Toffoli gate is a three-qubit gate, which has two controls and one target qubit. When the two controls are each in the $|1\rangle$ state, the state of the target is flipped. Currently, to implement a Toffoli gate in a three-qubit system 11 elementary gates are required (single qubit gates and controlled-\textsc{not} gates) [1]. As the number of qubits increases, the total number of elementary gates increases. This is because most physical arrangements of qubits are along an LNN array where a qubit only interacts with its neighbors. As such, to interact two non-NN qubits, additional gate operations are required to bring them adjacent to each other. Here, we show methods for implementing a three-qubit Toffoli gate efficiently in an N-qubit system.

2. Toffoli gate in a three qubit system

Fig. 1 shows a Toffoli gate for a three-qubit system with qubit 3 as the target. Here, we implement the gate by sandwiching a $C^Z (C^H)$ gate between two controlled-Hadamard ($C^H (C^H)$) gates. A $C^Z$ gate has two controls and 1 target [2]. A $\pi$ phase is picked up by the target only when the two controls are each in the $|1\rangle$ state. Overall, a $C^Z$ gate is equivalent to a controlled-Z ($C^1(Z)$) gate, which is a symmetric gate (either qubit can be the target or the control). The $C^1(Z)$ and $C^1(H)$ gates implement a Z and an Hadamard (H) gate, respectively, on the target qubit when the control qubit is in the $|1\rangle$ state. The Z and H gates are defined as:

$$Z = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \quad H = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}$$

Fig.1. Implementation of a Toffoli gate with qubit 3 as the target, using 3 elementary gates [2].

3. Toffoli gate in a four qubit system

Fig. 2 shows our method for implementing a Toffoli gate in a four qubit system where two of participants are not adjacent. Hereafter, we will refer to the control and target qubits as participants. The Toffoli gate is realized by sandwiching a $C^Z$ gate between two H gates applied on the target qubit. For realizing the $C^Z$ gate, two Toffoli gates between qubits 1, 2 and 3 with qubit 2 as the target and two $C^1(Z)$ gates between qubits 2 and 4, are implemented. The two $C^1(Z)$ gates between qubits 2 and 4 are replaced by two $C^1(-I)$ gates (Fig. 2).

Fig. 2. Implementation of a Toffoli gate in a four qubit system using qubit 2 as the dummy.
4. Toffoli gate in an $N$ qubit system

Fig. 3 shows a Toffoli gate with three participants in an $N$ qubit system, where qubits $p$, $q$ and $r$ are the participants. Here and throughout, $p$ is the 1st qubit, $q$ is in the middle, and $r$ is the $N^{th}$ qubit. As before, to perform a Toffoli gate, a $C^2(Z)$ gate is realized on these three participants which is sandwiched between two $H$ gates applied on the target qubit. In our method, we divide qubits into two major groups of “$m$” and “$(N-m)+1$” qubits. If $N$ is odd, we choose “$m=(N+1)/2$”, and if $N$ is even, we choose “$m=(N+2)/2$”. In either case, if $m=q$, we change $m=q+1$.

The dummy qubit is selected on $m$’s position (a dummy qubit is one that is used as a target that is finally restored to its initial state). Depending on whether participant $q$ is in group “$m$” (case 1) or in group “$(N-m)+1$” (case 2), each $C^2(Z)$ gate is decomposed into four gates. In case 1, the $C^2(Z)$ gate is replaced by two Toffoli gates between $p$, $q$ and $m$, with $m$ as the target, and two $C^1(Z)$ gates between $m$ and $r$. In case 2, the $C^2(Z)$ gate is substituted by two Toffoli gates between $r$, $q$ and $m$ ($m$ as the target), and two $C^1(Z)$ gates between $m$ and $p$.

Following the procedure of decomposing the $N$ qubit system into groups and subgroups, the total gate count for the Toffoli gate was calculated for different $N$. Fig. 4 shows the plot (blue line). From the plot, a second order polynomial trend line was calculated (black line), which gave an equation for the gate count, $G_N$ (total number of elementary gates) as a function of $N$:

$$G_N = 0.2519N^2 + 6.6723N - 36.973$$  \hspace{1cm} (3)

As a measure of the closeness of this equation to the actual gate count, for $N = 40$, Eq. (3) gives $G_N = 633$, while the exact value of $G_N$ using our method is 626.

5. Conclusions

In this paper, we introduced a new method to implement a Toffoli gate in an LNN array efficiently, by reducing the number of elementary gates. A quadratic as well a linear variation of the total gate count with the number of qubits has been presented.