INVERSION FORMULAS FOR TIME-DISTANCE HELIOSEISMOLOGY

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The following faculty members have examined the final copy of this thesis for form and content, and recommended that it be accepted in partial fulfillment of the requirement for the degree of Master of Science with a major in Applied Mathematics.

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DEDICATION

I would like to dedicate the thesis to my grandparents Clyde and Loyce Maddox, who nurtured my youthful curiosity and started me on the path of higher learning.

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ABSTRACT

In this thesis we examine the inverse kinematic problem presented in time-distance helioseismology. Acoustic wave oscillations in the Sun travel along ray paths below the Sun's surface. These ray paths are defined by Fermat's principle of least time. Differences from the expected travel times of these oscillations to different points on the Sun's surface is indicative of inhomogeneities in the solar structure. Measuring these perturbations in the travel times allow the recovery of information about acoustic sources and flows within the Sun. Our goal is to accurately recover functions describing these perturbations, and hence information about solar features. An inverse problem is solved using the transport equation in conjunction with a first order approximation of the ray path geometry. We obtain results that show this inversion process is unique and stable, as well as explicit formulas for the solutions to the scalar and vector tomography problems considered.

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Chapter 1

1 Introduction

1.1 Time-Distance Helioseismology

Time-Distance Helioseismology by Duvall et al. (1993) introduced the use of terrestrial seismic techniques to interpret the travel times of acoustic waves between points on the Sun's surface. By accurately measuring variations from expected travel times and applying an inversion technique, it is possible to recover information about the inhomogeneities beneath the surface which perturb the time-distance curves, or ray paths, the waves have taken. In the Earth, the travel times of acoustic waves depend on depth and horizontal variables. However, in the solar plasma, travel times depend on two factors: the acoustic speed profile and the velocity of subsurface flows. Acoustic speed increases with depth. Eventually the speed will increase to a point where the wave is completely refracted and returns to the surface some distance from its origin. This defines the general shape of the ray path. Flow velocities along ray paths either shorten or lengthen travel times, depending on the direction of flow. Measuring the travel times of waves propagating in opposite directions along the same ray path reveals information about the acoustic speed structure and of subsurface flows [13].

Kosovichev (1996) developed an inversion technique based upon these principles, borrowing from work done in seismic tomography. From that paper, the travel time τ_i along a single ray path Γ_i can be expressed in a standard geometric optics approximation as a function of the acoustic speed $c(\boldsymbol{x}, t)$ and the flow velocity $\boldsymbol{v}(\boldsymbol{x}, t)$:

$$\tau_i(t) = \int_{\Gamma_i} \frac{ds}{c(\boldsymbol{x}, t) + \boldsymbol{v}(\boldsymbol{x}, t) \cdot \boldsymbol{n}}, \tag{1.1}$$

with $\boldsymbol{x}$ as the spatial coordinate, t as time, s the distance along the ray and $\boldsymbol{n}$ the unit vector tangent to the ray. The direction of propagation determines the sign of $\boldsymbol{v} \cdot \boldsymbol{n}$, so flow effects cause travel time differences when waves propagate in opposite directions. Variations in the travel times are small (less than 5% in the areas examined by Kosovichev), so inhomogeneity in the acoustic speed can be examined in a first approximation as perturbations to a static reference acoustic speed profile $c_0(\boldsymbol{x})$. For this static case, the reference travel time is $\tau_{0,i} = \int_{\Gamma_{0,i}} ds/c_0(\boldsymbol{x})$.

Fermat's principle, which states that the path information travels along is the path of least time, is then applied. The variation of the travel times may be expressed as

$$\delta\tau_i^{\pm} \equiv \tau_i^{\pm}(t) - \tau_{0,i} = -\int_{\Gamma_{0,i}} \frac{\delta c(\boldsymbol{x},t) \pm \boldsymbol{v}(\boldsymbol{x},t)\cdot\boldsymbol{n}}{c_0^2(\boldsymbol{x})} ds. \tag{1.2}$$

Here the plus sign signifies wave travel away from the center of the examined region and the minus sign is for travel towards the center. Since travel time varies due to the direction of propagation, it is possible to recover the two functions from (1.2) by splitting the travel time residuals into two equations. The average and difference between travel times are

$$\delta\tau_i^{\text{aver}}(t) \equiv \frac{1}{2}\left[\delta\tau_i^{+}(t) + \delta\tau_i^{-}(t)\right] = -\int_{\Gamma_{0,i}} \frac{\delta c(\boldsymbol{x},t)}{c_0^2(\boldsymbol{x})} ds, \tag{1.3}$$

$$\delta\tau_i^{\text{diff}}(t) \equiv \delta\tau_i^{+}(t) - \delta\tau_i^{-}(t) = -2\int_{\Gamma_{0,i}} \frac{\boldsymbol{v}(\boldsymbol{x},t)\cdot\boldsymbol{n}}{c_0^2(\boldsymbol{x})} ds. \tag{1.4}$$

Here it is assumed that δc and $\boldsymbol{v}$ did not change during the observation. The separation of this problem into the recovery of two functions, the acoustic speed perturbation $\delta c(\boldsymbol{x},t)$ and the tangent component of the ray path flow $\boldsymbol{v}(\boldsymbol{x},t)\cdot\boldsymbol{n}$, permits the usage of scalar and vector tomography, respectively. Our approach involves recovering as much information about these functions as possible by treating them as separate cases.

The final preliminary consideration for the problem is to determine the nature of $c_0(\boldsymbol{x})$ for a solar problem. In seismic cases, acoustic speed increases linearly with depth which results in an explicit inversion formula for the acoustic speed profile in the scalar tomography case [2], [4]. Unfortunately, the solar case does not allow the use of this solution because the geodesic ray paths that the acoustic information travels along are not circular in nature. Duvall et al. (1997) states that the typical distance between surface reflections of waves is π times the depth of the ray and the length along the ray path is four times the depth. Geometrically, this defines a cycloid, the classical answer to the brachistochrone problem famously solved by Bernoulli in 1696 [19]. Cycloidal ray paths mean that the square of the acoustic speed increases linearly with depth. With an acoustic profile for the unperturbed case, we may now design an inversion method to recover the acoustic perturbation function $\delta c(\boldsymbol{x}, t)$.

1.2 Past Research

Active research in time-distance helioseismology has been ongoing since 1993, when *Time-Distance Helioseismology* was published. The premise of this paper was to determine the structure of local solar phenomena by using information obtained by direct imaging of the Sun. From these images, it is possible to obtain information about the intensity and direction of acoustic waves that have reached the solar surface. The acoustic intensity manifests itself much like a wave on the open ocean. Direct images of these waves may then be processed by means of spherical harmonics to extract the intensities of acoustic waves at the surface for a number of frequencies. This frequency decomposition provides the necessary information for tomographic analysis.

Duvall's 1993 work focused on discussing the basic methodology of and challenges involved with applying the techniques of seismology to helioseismology. The first challenge, the necessity of vector tomography in addition to scalar tomography, was discussed in the previous section. In addition, inversion methods must take into account accuracy in measuring wave travel times and must choose an optimal geometry to extract information from solar imaging. These core issues form the basis for evaluating the validity of contemporary inversion models. Kosovichev (1996), D'Silva (1996) and others used these considerations to expand on Duvall's implementation of geometric optics. Their efforts were designed to increase the diagnostic power and stability of the ray path analysis and time-distance inversion method. They accomplished this by solving various forward problems to test the validity of results obtained by inversion [8], [13], [5], [11]. In particular, many of these results have shown that ray path analyses may not be able to account fully for the transmission of acoustic information. A number of recent works focus on developing so-called sensitivity kernels to handle scattering of particles from ray paths. This is beyond the scope of the geometric optics interpretation of the problem and will not be considered in this paper.

Data for helioseismology comes primarily from the Solar and Heliospheric Observatory (SOHO) and Solar Dynamics Observatory (SDO) satellites currently in orbit around the Sun. SOHO has been in operation since 1996, whereas SDO was launched in 2010. Both devices use Doppler imagers to measure variations in the solar magnetic field, the Magnetic Doppler Imager (MDI) and Helioseismic and Magnetic Imager (HMI), respectively. The aforementioned spherical harmonic analysis is then applied to images of the magnetic field to provide helioseismic data. Typically, the data used in helioseismic inversions is taken over several hours and then averaged to reduce noise [17]. The addition of the HMI on the SDO has provided significantly increased resolution for observations and made time-distance helioseismology an even more attractive branch of stellar physics. A coalition of scientists have developed a data-analysis pipeline for time-distance measurements [21]. Future investigations into the validity of inversion methods and solar structure will be much more fruitful with this increased volume of data.

In order to properly utilize this data, it is necessary to develop the most accurate inversion algorithms possible. Researchers have made use of the data provided in this pipeline as well as numerical simulations to evaluate the validity of their algorithms. These investigations have also been done to other techniques developed for local helioseismology, not just time-distance inversions, which have shown that ray path approximations are still valid when compared to other techniques [18]. The goal of this paper is to provide the theoretical framework for a ray path approximation inversion and to show that this algorithm behaves in a stable manner.

To do this, the inverse kinematic tomography technique developed by [3], [2] will be adapted for use in the solar tomography case.

Chapter 2

2 The Integral Geometry Problem

2.1 The Transport Equation in 2D

We consider a stationary two-dimensional transport equation [2];

$$Pu(x,\omega) = \langle \omega, \nabla_x u(x,\omega) \rangle + \mu(x)u(x,\omega) = a(x), \qquad x \in \Omega \subset \mathbb{R}^2. \tag{2.1}$$

Here $\langle \cdot, \cdot \rangle$ is the usual scalar product in $\mathbb{R}^2$, $\nabla_x = (\partial_1, \partial_2)$, $u(x,\omega)$ is the density of particles at a point x moving in direction ω for $\omega = (\cos\alpha, \sin\alpha) \in S = \{\omega \in \mathbb{R}^2 : |\omega| = 1\}$, $\mu(x)$ is the attenuation in the body at point x and $a(x)$ is the acoustic source (perturbation) in the body at point x. The domain Ω is a strictly convex subset of $\mathbb{R}^2$ with a $C^1(\mathbb{R}^2)$ boundary $\partial\Omega$.

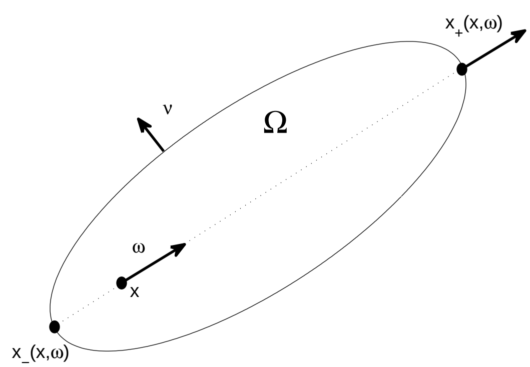

Figure 2.1

For $x \in \partial\Omega$, let

$$u(x,\omega)|_{\partial\Omega\times S} = f(x,\omega) = \begin{cases} f_-(x,\omega) & \langle x,\nu\rangle < 0 \\ f_+(x,\omega) & \langle x,\nu\rangle > 0 \end{cases}. \tag{2.2}$$

Here ν is the outward unit normal, as indicated in Figure 2.1. Physically, f_- represents the incoming particle flow into the domain Ω and f_+ is the outgoing particle flow from Ω. The direct problem for this system is to find the particle density $u(x,\omega)$ and hence f_+ given a, μ and f_-. The inverse problem is determining a given the trace of u on $\partial\Omega \times S$ and the attenuation μ. Taking the variables to be in the complex plane, $z = x_1 + \imath x_2$ and writing $u(x,\omega) = u(z,\alpha)$, $f(x,\omega) = f(z,\alpha)$, $\mu(x) = \mu(z)$ and $a(x) = a(z)$ for $z \in \mathbb{C}$ and $\omega = \omega(\alpha) = (\cos\alpha, \sin\alpha)$, we may apply Euler's formulae

$$\cos\alpha = \frac{e^{\imath\alpha} + e^{-\imath\alpha}}{2}, \qquad \sin\alpha = \frac{e^{\imath\alpha} - e^{-\imath\alpha}}{2\imath}$$

to the equation (2.1) to obtain

$$e^{\imath\alpha} P u = \left(\bar{\partial} + e^{2\imath\alpha}\partial + e^{\imath\alpha}\mu\right) u = e^{\imath\alpha} a. \tag{2.3}$$

for the complex differential operators $\partial = \frac{1}{2}(\partial_1 - \imath\partial_2)$ and $\bar{\partial} = \frac{1}{2}(\partial_1 + \imath\partial_2)$. The Fourier series expansion of $u(z,\alpha)$ with respect to α is of the form

$$u(z,\alpha) = \sum_{n\in\mathbb{Z}} \hat{u}_n(z) e^{\imath n\alpha}, \qquad \hat{u}_n = \int_0^{2\pi} u(z,\alpha) e^{-\imath n\alpha} d\alpha.$$

For notational convenience, we define $u_n(z) = \hat{u}_{-n}(z)$ and obtain

$$u(z,\alpha) = \sum_{n\in\mathbb{Z}} u_n e^{-\imath n\alpha}. \tag{2.4}$$

Substituting this into (2.3) yields

$$e^{\imath\alpha} P u = \sum_{n\in\mathbb{Z}} \left(\bar{\partial} u_n e^{-\imath n\alpha} + \partial u_n e^{-\imath(n-2)\alpha} + \mu u_n e^{\imath(n-1)\alpha}\right) = e^{\imath\alpha} a.$$

Finally, we change indices and the expression simplifies to

$$e^{\imath\alpha} Pu = \sum_{n\in\mathbb{Z}} \left(\bar{\partial} u_n + \partial u_{n+2} + \mu u_{n+1}\right) e^{-\imath n\alpha} = e^{\imath\alpha} a. \tag{2.5}$$

The function u is real-valued, so $u_n = \bar{u}_{-n}$. Define the vector $\boldsymbol{u} = (u_0, u_1, u_2, \ldots) \in l_2(0,\infty)$. Using the shift operator

$$U : (u_0, u_1, u_2, \ldots) \mapsto (0, u_0, u_1, u_2, \ldots),$$

its adjoint U^* in $l_2(0,\infty)$, $A = -(U^*)^2$ and $\bar{\partial}_A = \bar{\partial} - A\partial$, we may cast this inverse problem as follows:

Problem 2.1. *Let $\boldsymbol{u}$ be the solution of the Cauchy problem*

$$\begin{aligned} \bar{\partial}_A \boldsymbol{u} + \mu U^* \boldsymbol{u} = 0 \quad \textit{in } \Omega, \\ \boldsymbol{u}|_{\partial\Omega} = \boldsymbol{f}, \end{aligned} \tag{2.6}$$

where $\bar{\partial}_A$ is a Beltrami-type operator with operator coefficient A. Reconstruct the function $\boldsymbol{u}$ from its trace $\boldsymbol{f}$.

This problem is solved by Bukhgeĭm and Bukhgeĭm (2006) using an analog for the standard Cauchy integral formula [1].

$$C_A u(z) = \frac{1}{2\pi} \int_{\partial\Omega} \nu_A \left(\zeta - z\right)^{-1}_{-A} u(\zeta) ds$$

Here $\nu_A = \nu - \bar{\nu} A$ where $\nu = \nu_1 + \imath\nu_2$ is the outward unit normal to $\partial\Omega$, and $z_A^{-1} = (z + \bar{z}A)^{-1}$. In our case it is not possible to fully recover $\boldsymbol{u}$, and the data in each case will dictate how many of its components may be determined.

2.2 Weighted Transport Equation

Problem 2.1 supposes that the domain the information is passing through is flat. Since we are examining acoustic waves that pass beneath the surface, it is necessary to introduce a way to make the physical problem compatible with the mathematical one. The ray path Γ_{n_0} for the acoustic "slowness" $n_0(z) = 1/c_0(z)$ at depth z is given by the intersection of the planes

$$p_1 := \langle x, \nu \rangle - h = 0$$
$$p_2 := z - \varphi(|x|, r) = 0,$$

or $p(x, z) = (p_1, p_2) = 0$. Here h is the distance of the vertical cutting plane from the origin and the function $\varphi(|x|, r)$ is the two-dimensional ray path. In the solar case, this means φ is a cycloid and must be defined parametrically [19], [9]:

$$\varphi(|x|, r) = \begin{cases} |x| = R\,(t - \sin(t)) \\ z = R\,(1 - \cos(t)) - \frac{a}{b} \end{cases} \tag{2.7}$$

where R and t_0 are the solutions of the equations $|x(t_0)| = R$ and $z(t_0) = 0$ and $t \in [\pi, t_0]$ is a parameter.

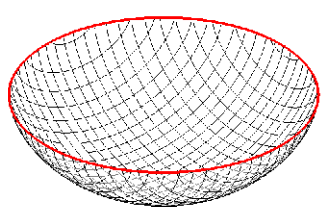

Figure 2.2

In Chapter 3, the Riemannian geometry prescribed by this type of geodesic surface is discussed in more detail.

We now introduce a weight function ρ that projects the radially symmetric surface to the x-plane,

$$\rho(x,\omega) = \sqrt{1 + (\varphi')^2 \frac{\langle x,\omega\rangle^2}{|x|^2}}. \tag{2.8}$$

In practice, this expression is very difficult to work with. Hence, we make use of the geometry of our problem to approximate ρ. Calling again on the fact that the acoustic speed $c_0(z) = \sqrt{a+bz}$ for $a > b$, we can guarantee that $|\varphi'| < 1$. Figure 2.3 shows this pictorially.

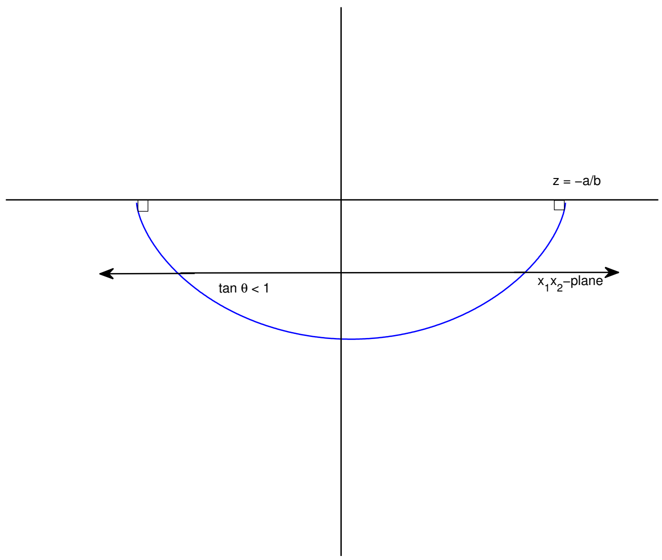

Figure 2.3

Under this condition, the binomial theorem is applied to (2.8) to yield

$$\rho(x,\omega) \approx 1 + \frac{1}{2}(\varphi')^2 \frac{\langle x,\omega\rangle^2}{|x|^2}, \tag{2.9}$$

and thus we have a first order approximation for ρ. In general, it is possible to take arbitrarily many terms of the binomial expansion. However, this is only recommended when it is known that $|\varphi'|$ is small.

2.3 The Scalar Tomography Problem

Problem 2.2. *Scalar Tomography: Recover the scalar acoustic perturbation function* $a(x)$ *by solving Problem 2.1 for equation (2.5) with the right hand side weighed by* $\rho(x,\omega)$*:*

$$e^{\imath\alpha}Pu = \sum_{n\in\mathbb{Z}} \left(\bar{\partial}u_n + \partial u_{n+2} + \mu u_{n+1}\right) e^{-\imath n\alpha} = e^{\imath\alpha}\rho(x,\omega)a(x). \tag{2.10}$$

Using the approximated weight function (2.9) in equation (2.10), we obtain

$$e^{\imath\alpha}Pu = \sum_{n\in\mathbb{Z}} \left(\bar{\partial}u_n + \partial u_{n+2} + \mu u_{n+1}\right) e^{-\imath n\alpha} = e^{\imath\alpha}\left(1 + \frac{1}{2}\left(\varphi'\right)^2 \frac{\langle x,\omega\rangle^2}{|x|^2}\right) a(x). \tag{2.11}$$

Here, our goal becomes to rewrite the right hand side so that we can determine a in terms of the Fourier coefficients u_n. To do this, let us examine $\langle x,\omega\rangle$:

$$\begin{aligned}
\langle x,\omega\rangle &= x_1\left(\frac{e^{\imath\alpha}+e^{-\imath\alpha}}{2}\right) + x_2\left(\frac{e^{\imath\alpha}-e^{-\imath\alpha}}{2\imath}\right) \\
&= \frac{1}{2}\left[\left(x_1 - \imath x_2\right)e^{\imath\alpha} + \left(x_1 + \imath x_2\right)e^{-\imath\alpha}\right], \\
\langle x,\omega\rangle^2 &= \frac{1}{4}\left[\left(x_1 - \imath x_2\right)^2 e^{2\imath\alpha} + 2\left(x_1^2 + x_2^2\right) + \left(x_1 + \imath x_2\right)^2 e^{-2\imath\alpha}\right] \\
&= \frac{1}{4}\left[\bar{x}^2 e^{2\imath\alpha} + 2|x|^2 + x^2 e^{-2\imath\alpha}\right].
\end{aligned}$$

Above, $x = x_1 + \imath x_2$, $\bar{x}$ is the typical complex conjugate and $|x|^2 = x\bar{x}$. From here, we may express (2.11) as

$$\sum_{n\in\mathbb{Z}} \left(\bar{\partial}u_n + \partial u_{n+2} + \mu u_{n+1}\right) e^{-\imath n\alpha} = ae^{\imath\alpha} + \frac{a}{8|x|^2}(\varphi')^2\left[\bar{x}^2 e^{3\imath\alpha} + 2|x|^2 e^{\imath\alpha} + x^2 e^{-\imath\alpha}\right]. \tag{2.12}$$

Equation (2.12) expresses our problem as an infinite system of equations. However, this system only has finitely many equations with a nonzero right hand sides. Because of this, we may match the indices of $e^{-\imath n\alpha}$ to the right hand side and form the following system using the identity $u_n = \bar{u}_{-n}$:

$$\bar{\partial}\bar{u}_3 + \partial\bar{u}_1 + \mu\bar{u}_2 = \frac{a\bar{x}}{8x}(\varphi')^2 \qquad n = -3 \tag{2.13}$$

$$\bar{\partial}\bar{u}_1 + \partial u_1 + \mu u_0 = a\left(1 + \frac{1}{4}(\varphi')^2\right) \qquad n = -1 \tag{2.14}$$

$$\bar{\partial}u_1 + \partial u_3 + \mu u_2 = \frac{ax}{8\bar{x}}(\varphi')^2 \qquad n = 1 \tag{2.15}$$

$$\bar{\partial}u_n + \partial u_{n+2} + \mu u_{n+1} = 0 \qquad n \in \mathbb{Z} \setminus \{-3, -1, 1\}. \tag{2.16}$$

In this system, (2.13) and (2.15) are conjugates of one another, which leaves two independent equations with nonzero right hand sides. For this analysis, we will take $\mu \equiv 0$ assuming that the effects of attenuation in the solar plasma beneath the surface are negligible [11]. In this case, the nontrivial elements of the system become

$$\bar{\partial}u_1 + \partial u_3 = \frac{ax}{8\bar{x}}(\varphi')^2 \qquad n = 1 \tag{2.17}$$

$$\bar{\partial}\bar{u}_1 + \partial u_1 = a\left(1 + \frac{1}{4}(\varphi')^2\right) \qquad n = -1. \tag{2.18}$$

To obtain an expression for a, we must appeal to the solution of Problem 2.1 and recall that $\boldsymbol{f} = \boldsymbol{u}|_{\partial\Omega\times S} = (f_2, f_3, \ldots)$ only allows us to reconstruct $\boldsymbol{u} = (u_2, u_3, \ldots)$. This means that the inversion formula for a must be expressible in terms of $u_n, n \geq 2$. Deriving a simple expression algebraically is not possible, but we may transform this system into a Beltrami-like partial differential equation. Solving (2.18) for a,

$$a = \left(\frac{4 + (\varphi')^2}{4}\right)^{-1} 2\Re\partial u_1, \tag{2.19}$$

where $\Re\partial u_1 = \frac{1}{2}\left(\partial u_1 + \bar{\partial}\bar{u}_1\right)$. Substitute this into (2.17), then

$$\bar{\partial}u_1 - \left(\frac{(\varphi')^2}{4 + (\varphi')^2}\right)\frac{x}{\bar{x}}\Re\partial u_1 = -\partial u_3. \tag{2.20}$$

For simplicity, we set $u = u_1$, $q(x) = -\left(\frac{(\varphi')^2}{4+(\varphi')^2}\right)\frac{x}{\bar{x}}$ and $g = -\partial u_3$.

Problem 2.3. *Solve the Cauchy problem*

$$\begin{aligned} \bar{\partial} u + q(x)\Re\partial u &= g \\ u|_{\partial\Omega} &= f. \end{aligned} \tag{2.21}$$

for u given $q(x)$, g and f.

To analyze the solution u of this problem for stability and uniqueness, it is necessary to introduce some notation from [2]. Suppose that Ω is a simply connected bounded domain with smooth boundary $\partial\Omega = \{z = z(s) : s \in [0, l]\}$, where s is a natural parameter and l is the length of $\partial\Omega$. We introduce the Hilbert transform $\mathcal{H}$ on the space of l-periodic functions on $\partial\Omega$. This transform multiplies the n-th Fourier coefficient by $-\imath \cdot \mathrm{sign}(n)$. More importantly for our purposes, we introduce the non-negative self-adjoint operator

$$\Lambda = \frac{l}{2\pi}\mathcal{H}\frac{d}{ds}.$$

In addition, we also define

$$u_+ = \frac{(I + \imath\mathcal{H})u}{2}, \quad u_- = \frac{(I - \imath\mathcal{H})u}{2}.$$

Heuristically, we can think of the operator $\Lambda^{1/2}$ as "half of a derivative."

Lemma 2.1. *Uniqueness and stability of the function u from Problem 2.3 is guaranteed by the estimate*

$$\left\|\sqrt{2}\partial u\right\|^2_{L_2(\Omega)} \leq (1 - \max|q|)^{-1}\left(\sqrt{2}\,\|g\|_{L_2(\Omega)} + \left\|\Lambda^{1/2} f_+\right\|^2_{L_2(\partial\Omega)}\right). \tag{2.22}$$

Proof. The identity

$$2\left\|\partial u\right\|_{L_2(\Omega)}^2+\left\|\Lambda^{1/2}u_-\right\|_{L_2(\partial\Omega)}^2=2\left\|\bar{\partial}u\right\|_{L_2(\Omega)}^2+\left\|\Lambda^{1/2}u_+\right\|_{L_2(\partial\Omega)}^2 \tag{2.23}$$

is proven by Bukhgeĭm and Bukhgeĭm (2006). Noting from Problem 2.3 that $\bar{\partial}u=g-q(x)\Re\partial u$ and $u_+|_{\partial\Omega}=f_+$,

$$\begin{aligned}
\sqrt{2}\left\|\partial u\right\|_{L_2(\Omega)} &\leq \sqrt{2\left\|\bar{\partial}u\right\|_{L_2(\Omega)}^2+\|\Lambda^{1/2}u_+\|_{L_2(\partial\Omega)}^2} && (2.24)\\
&\leq \sqrt{2}\left\|\bar{\partial}u\right\|_{L_2(\Omega)}+\left\|\Lambda^{1/2}u_+\right\|_{L_2(\partial\Omega)} && (2.25)\\
&\leq \sqrt{2}\left\|g-q(x)\Re\partial u\right\|_{L_2(\Omega)}+\left\|\Lambda^{1/2}u_+\right\|_{L_2(\partial\Omega)} && (2.26)\\
&\leq \sqrt{2}\left\|g\right\|_{L_2(\Omega)}+\sqrt{2}\max|q(x)|\left\|\partial u\right\|_{L_2(\Omega)}+\left\|\Lambda^{1/2}u_+\right\|_{L_2(\partial\Omega)} && (2.27)\\
&\leq (1-\max|q|)^{-1}\left(\sqrt{2}\left\|g\right\|_{L_2(\Omega)}+\left\|\Lambda^{1/2}f_+\right\|_{L_2(\partial\Omega)}\right). && (2.28)
\end{aligned}$$

□

Theorem 2.1. *Let $\Omega=\mathbb{D}=\{z\in\mathbb{C}:|z|<1\}$, and let $g\in C^{\alpha}(\Omega)$ and $f\in C^{1+\alpha}(\partial\bar{\Omega})$. Then the solution for Problem 2.3, $u\in C^{\alpha}(\Omega)\cap C^{1+\alpha}(\partial\Omega)$, is unique and given by the formula*

$$u=C_0\Re f+T_0g+T_0v+\frac{\imath}{2\pi}\int_{\partial\Omega}\Im(f-T_0v)ds. \tag{2.29}$$

Proof. Under these hypotheses, our problem becomes

$$\bar{\partial}u+q(x)\Re\partial u=g \quad \text{in } \mathbb{D} \tag{2.30}$$

$$u|_{\partial\mathbb{D}}=f. \tag{2.31}$$

We will show that proving uniqueness for this problem only requires the real parts of f and

$\int_{\partial\mathbb{D}} \Im f ds$. Let

$$Cu(z) = \frac{1}{2\pi}\int_{\partial\mathbb{D}} \zeta\frac{u(\zeta)}{\zeta - z}ds, \tag{2.32}$$

$$Tu(z) = -\frac{1}{\pi}\int_{\mathbb{D}} \frac{u(\zeta)}{\zeta - z}d\zeta, \tag{2.33}$$

$$C_0u(z) = \frac{1}{2\pi}\int_{\partial\mathbb{D}} u(\zeta)\frac{\zeta + z}{\zeta - z}ds, \tag{2.34}$$

$$T_0u(z) = Tu(z) - \overline{Tu\left(\frac{1}{\bar{z}}\right)}, \tag{2.35}$$

for $\zeta = \zeta_1 + \imath\zeta_2$ and $d\zeta = d\zeta_1 d\zeta_2$. Here C and T are the typical Cauchy integral operator and Teodorescu transform. Using these, we may write the Cauchy-Pompeiu formula

$$u = Cu + T\bar{\partial}u. \tag{2.36}$$

Furthermore, we have

$$u = C_0\Re u + T_0\bar{\partial}u + \frac{\imath}{2\pi}\int_{\partial\mathbb{D}} \Im u(\zeta)ds \tag{2.37}$$

for any $u \in C^1(\bar{\mathbb{D}})$. Next, we use a well-known theorem [20] that provides the solution of the Riemann-Hilbert boundary value problem for operator $\bar{\partial}$ based on (2.37).

Theorem 2.2. *Let $g \in C^{\alpha}(\bar{\mathbb{D}})$ and $f \in C^{1+\alpha}(\partial\mathbb{D})$. Then there is only one solution of $u \in C^1(\partial\mathbb{D}) \cap C(\mathbb{D})$ of the Riemann-Hilbert boundary value problem for $\bar{\partial}$:*

$$\begin{aligned} \bar{\partial}u &= g \quad \text{in } \mathbb{D} \\ \Re u|_{\partial\mathbb{D}} &= f, \quad \int_{\partial\mathbb{D}} \Im u ds = 0. \end{aligned} \tag{2.38}$$

Moreover, we have $u \in C^{1+\alpha}(\bar{\mathbb{D}})$ and the formulae

$$u = C_0 f + T_0 g \tag{2.39}$$

$$\|u\|_{1+\alpha} \leq K \left(\|g\|_\alpha + \|f\|_{1+\alpha} \right) \tag{2.40}$$

for Hölder spaces $C^\alpha, C^{1+\alpha}$ and some constant K.

For $\alpha \in (0, 1]$ and $\Omega \subset \mathbb{C}$, $f : \bar{\Omega} \to \mathbb{C}$ belongs to $C^\alpha(\bar{\Omega})$ if and only if

$$\|f\|_\alpha = \sup_{x \in \Omega} |f| + \sup_{\substack{x', x'' \in \Omega \\ x' \neq x''}} \frac{|f(x') - f(x'')|}{|x' - x''|^\alpha} < \infty. \tag{2.41}$$

Similarly, $C^{1+\alpha}(\bar{\Omega}) = \left\{ u \in \mathbb{C}^1(\bar{\Omega}) : \partial u, \bar{\partial} u \in C^\alpha(\bar{\Omega}) \right\}$. Now we may return to (2.30) assuming that we only have the boundary data

$$\Re u|_{\partial\mathbb{D}} = \Re f, \quad \int_{\partial\mathbb{D}} \Im u ds = \int_{\partial\mathbb{D}} \Im f ds = c \tag{2.42}$$

for some constant c. Replacing u above with $u := u - \imath \frac{c}{2\pi}$, the problem of (2.30) with data (2.42) reduces to the case where $c = 0$. Hence,

$$\bar{\partial} u + q(x) \Re \partial u = g \quad \text{in } \mathbb{D} \tag{2.43}$$

$$\Re u|_{\partial\mathbb{D}} = \Re f, \quad \int_{\partial\mathbb{D}} \Im u ds = 0. \tag{2.44}$$

With this simplified problem, we seek to construct a formula for u. First, we set

$$u = C_0 \Re f + T_0 g + T_0 v + \imath c_1.$$

Defining a new function $u_0 = C_0 \Re f + T_0 g$, we now have

$$\bar{\partial} u = \bar{\partial} u_0 + \bar{\partial} T_0 v = g + v,$$
$$\partial u = \partial u_0 + \partial T_0 v.$$

This is true because of the identities $\bar{\partial} C_0 u = 0$, $\bar{\partial} T_0 u = u$ and $\Re T_0 u|_{\partial \mathbb{D}} = 0$. Our equation (2.43) is now

$$g + v + q(x) \Re \left(\partial u_0 + \partial T_0 v \right) = g,$$
$$v + q(x) \Re \partial T_0 v = -q(x) \Re \partial u_0.$$

For $|\varphi'| \leq 1$,

$$|q(x)| \leq \frac{1}{4 + (\varphi')^2} \leq \frac{1}{4},$$

and since it is easy to check $\|\partial T_0 v\|_{L_2(\mathbb{D})} \leq \|v\|_{L_2(\mathbb{D})}$ for operator $Sv := -q \Re \partial T_0 v$ we have $\|S\| \leq \frac{1}{4}$ [20]. We may now solve for v:

$$v = (I + S)^{-1} \left(-q \Re \partial u_0 \right).$$

Here I is the identity operator. After determining v, we must choose c_1 so that

$$\int_{\partial \mathbb{D}} \Im u ds = \int_{\partial \mathbb{D}} \Im T_0 v ds + 2\pi c_1 = 0.$$

Hence

$$c_1 = -\frac{1}{2\pi} \int_{\partial \mathbb{D}} \Im T_0 v ds.$$

Therefore, the solution of problem (2.30), (2.42) is given by

$$u = C_0 \Re f + T_0 g + T_0 v + \frac{\imath}{2\pi} \int_{\partial\Omega} \Im(f - T_0 v) ds,$$

where

$$\begin{aligned} v &= (I+S)^{-1} \left(-q \Re \partial u_0\right), \qquad (2.45) \\ u_0 &= C_0 \Re f + T_0 g. \end{aligned}$$

□

As a result of Theorem 2.1 and Lemma 2.1, we may recover the scalar acoustic perturbation function a uniquely and stably using equation (2.19).

2.4 The Vector Tomography Problem

Problem 2.4. *Vector Tomography: Recover the subsurface flow function $\boldsymbol{v}(x) = (v_1(x), v_2(x))$ by solving Problem 2.1 for equation (2.5) with the right hand side $e^{\imath\alpha}\rho(x,\omega)\langle\omega, \boldsymbol{v}\rangle$.*

Recalling (2.9), the right hand side for our problem may be written as

$$\begin{aligned} e^{\imath\alpha}\rho(x,\omega)\langle\omega,\boldsymbol{v}\rangle &= e^{\imath\alpha}\left[1 + \frac{(\varphi')^2}{8|x|^2}\left(\bar{x}^2 e^{2\imath\alpha} + 2|x|^2 + x^2 e^{-2\imath\alpha}\right)\right]\left[v_1\cos\alpha + v_2\sin\alpha\right] \\ &= e^{\imath\alpha}\left[1 + \frac{(\varphi')^2}{8|x|^2}\left(\bar{x}^2 e^{2\imath\alpha} + 2|x|^2 + x^2 e^{-2\imath\alpha}\right)\right]\left[v_1\left(\frac{e^{\imath\alpha}+e^{-\imath\alpha}}{2}\right) + v_2\left(\frac{e^{\imath\alpha}-e^{\imath\alpha}}{2\imath}\right)\right] \\ &= \frac{1}{2}\left[1 + \frac{(\varphi')^2}{8|x|^2}\left(\bar{x}^2 e^{2\imath\alpha} + 2|x|^2 + x^2 e^{-2\imath\alpha}\right)\right]\left[v_1\left(e^{2\imath\alpha}+1\right) - \imath v_2\left(e^{2\imath\alpha}-1\right)\right] \\ &= \frac{1}{2}\left[1 + \frac{(\varphi')^2}{8|x|^2}\left(\bar{x}^2 e^{2\imath\alpha} + 2|x|^2 + x^2 e^{-2\imath\alpha}\right)\right]\left[\bar{v}e^{2\imath\alpha} + v\right] \\ &= \frac{1}{2}\left[\bar{v}e^{2\imath\alpha} + v + \frac{(\varphi')^2}{8|x|^2}\left(\bar{v}e^{2\imath\alpha} + v\right)\left(\bar{x}^2 e^{2\imath\alpha} + 2|x|^2 + x^2 e^{-2\imath\alpha}\right)\right] \\ &= \frac{1}{2}\left\{ \begin{array}{c} e^{4\imath\alpha}\left[\frac{(\varphi')^2}{8|x|^2}\bar{v}\bar{x}^2\right] + e^{2\imath\alpha}\left[\bar{v} + \frac{(\varphi')^2}{8|x|^2}\left(2\bar{v}|x|^2 + v\bar{x}^2\right)\right] \\ + \left[v + \frac{(\varphi')^2}{8|x|^2}\left(2v|x|^2 + \bar{v}x^2\right)\right] + e^{-2\imath\alpha}\left[\frac{(\varphi')^2}{8|x|^2}vx^2\right] \end{array} \right\}. \end{aligned}$$

For simplicity, we have taken $v = v_1 + \imath v_2$ and $\bar{v} = v_1 - \imath v_2$. As with Problem 2.2, this now reduces to a system of equations where finitely many have nonzero right hand sides. Again taking $\mu \equiv 0$,

$$\bar{\partial}\bar{u}_4 + \partial\bar{u}_2 = \frac{(\varphi')^2}{16|x|^2}\bar{v}\bar{x}^2 \qquad n = -4 \tag{2.46}$$

$$\bar{\partial}\bar{u}_2 + \partial\bar{u}_0 = \frac{1}{2}\left[\bar{v} + \frac{(\varphi')^2}{8|x|^2}\left(2\bar{v}|x|^2 + v\bar{x}^2\right)\right] \qquad n = -2 \tag{2.47}$$

$$\bar{\partial}u_0 + \partial u_2 = \frac{1}{2}\left[v + \frac{(\varphi')^2}{8|x|^2}\left(2v|x|^2 + \bar{v}x^2\right)\right] \qquad n = 0 \tag{2.48}$$

$$\bar{\partial}u_2 + \partial u_4 = \frac{(\varphi')^2}{16|x|^2}vx^2 \qquad n = 2 \tag{2.49}$$

$$\bar{\partial}u_n + \partial u_{n+2} = 0 \qquad n \in \mathbb{Z} \setminus \{-4, -2, 0, 2\} \tag{2.50}$$

In the above system, we may only recover $(u_3, u_4, u_5, \ldots)$ from our boundary data. Furthermore, this system is much more complicated than the one that arose in the scalar tomography case. However, in the first approximation the contribution of $\boldsymbol{v}(x)$ should be small in relation to $a(x)$. Furthermore, $|\varphi'| < 1$, and the product $v(\varphi')^2$ should be negligible, taken here as zero. Equations (2.47), (2.48) and (2.49) then reduce to

$$\bar{\partial}\bar{u}_2 + \partial u_0 = \frac{1}{2}\bar{v} \qquad n = -2 \tag{2.51}$$

$$\bar{\partial}u_0 + \partial u_2 = \frac{1}{2}v \qquad n = 0 \tag{2.52}$$

$$\bar{\partial}u_2 + \partial u_4 = 0. \qquad n = 2 \tag{2.53}$$

Under this approximation, it is now possible to recover the function u_2 from our boundary data, since it possible to determine $u_n,\ n \geq 3$ by solving Problem 2.1. We then apply $\bar{\partial}$ to (2.51) and ∂ to (2.52) and take the difference:

$$\bar{\partial}^2\bar{u}_2 + \bar{\partial}\partial u_0 - \partial\bar{\partial}u_0 - \partial^2 u_2 = \frac{1}{2}\left(\bar{\partial}\bar{v} - \partial v\right).$$

The operators $\bar{\partial}$ and ∂ commute and u_0 is real, so $\bar{\partial}\partial u_0 - \partial\bar{\partial}u_0 = 0$. Using the definitions of $\bar{\partial}$ and ∂, the equation simplifies to

$$\bar{\partial}^2\bar{u}_2 - \partial^2 u_2 = 2\imath\Im\partial^2 u_2 = -\frac{\imath}{2}\left(\partial_2 v_1 - \partial_1 v_2\right). \tag{2.54}$$

The right hand side is the solenoidal part of $\boldsymbol{v}(x)$, the two-dimensional analog of the curl. To fully recover the function $\boldsymbol{v}$, we must also know its divergence $\partial_1 v_1 + \partial_2 v_2$. Existing helioseismic techniques are able to measure horizontal divergence quite accurately [10], so we take this divergence to be known. With this data, we may take the sum of $\bar{\partial}$(2.51) and ∂(2.52) to obtain

$$2\Re\partial^2 u_2 + 2\bar{\partial}\partial u_0 = 2\Re\partial v.$$

Using the identity $\Delta = 4\bar{\partial}\partial$ gives rise to a new problem.

Problem 2.5. *Solve the following Poisson problem with the given Dirichlet data:*

$$\Delta u_0 = 4\Re\left(\partial v - \partial^2 u_2\right). \tag{2.55}$$

The solution to this problem is well known and uses the method of reflection [15]. We have the two dimensional case, so we take the domain Ω to be a disk in $\mathbb{R}^2$ with radius r and $\xi \in \Omega$. The reflection of ξ, $\xi^* = \dfrac{r^2\xi^2}{|\xi|^2}$ is not in Ω. This fact is exploited to determine the Green's function $G(x,\xi)$ and hence the solution of the Poisson problem u_0:

$$u_0(\xi) = \int_\Omega G(x,\xi)\Delta u_0 dx + \int_{\partial\Omega} u(x)\frac{\partial G(x,\xi)}{\partial \nu_x} dS_x, \quad \text{where} \tag{2.56}$$

$$G(x,\xi) = \frac{1}{2\pi}\left(\ln\frac{|\xi|}{r}|x-\xi| - \ln\frac{|\xi|}{r}|x-\xi^*|\right). \tag{2.57}$$

Theorem 2.3. *The subsurface flow function $\boldsymbol{v}(x) = (v_1(x), v_2(x)) \in C^1(\bar{\Omega})$ may be recovered* *uniquely and stably by the formula*

$$v = v_1 + \imath v_2 = 2\bar{\partial} u_0 + 2\partial u_2. \tag{2.58}$$

Proof. The solutions to Problems 2.1 and 2.5 are known and stable, so u_2 and u_0 may be determined without issue. Theorem 3.1 states that a vector field defined in a subset of $\mathbb{R}^3$ may be decomposed uniquely into curl-free and divergence-free parts, and so formula (2.58) follows from (2.52), (2.56) and (2.57). □

Chapter 3

3 Technical Details and Theorems

3.1 Riemannian Geometry

Remark 3.1. *The discussion in this section provides background information on the geometric considerations necessary to properly define the geodesic surfaces used in the context of geometric optics [12], [16]. Also, we take for simplicity that a differentiable manifold M is an open set in $\mathbb{R}^d$ with (local) coordinates $x = (x^1, x^2, \ldots, x^d)$.*

Definition 1 A Riemannian metric on a differentiable manifold M is given by a scalar product on each tangent space T_pM which depends smoothly on the base point. This means that for each of the two tangent vectors $v, w \in T_pM \simeq \mathbb{R}^d$ with coordinates $(v^1, v^2, \ldots, v^d)$ and $(w^1, w^2, \ldots, w^d)$ the scalar product is given by

$$\langle v, w \rangle := g_{ij}(x) v^i w^j. \tag{3.1}$$

Here we use the Einstein summation rule: an index occurring twice in a product is to be summed from 1 up to the space dimension. Also $G = [g_{ij}(x)]$ is a symmetric real positive definite matrix with $g_{ij} = g_{ji}$ for all i, j and $g_{ij}\xi^i\xi^j > 0$ for all $\xi = (\xi^1, \xi^2, \ldots, \xi^d) \in \mathbb{R}^d$ where $\xi \neq 0$. This allows us to identify the Riemannian metric with matrix G.

According to (3.1), the length of v is given by

$$\|v\| := \langle v, v \rangle^{1/2}.$$

In Riemannian geometry, it is useful to write tangent vectors $v, w \in T_pM$ in the form

$$v = v^i \frac{\partial}{\partial x^i}, \quad w = w^j \frac{\partial}{\partial x^j}.$$

Here $\frac{\partial}{\partial x^i}$ is considered as a vector field along $e_i = (0, \ldots, 1, 0, \ldots, 0)$ where the 1 is in the i-th place. In this language $\frac{\partial}{\partial x^i} = e_i$, so

$$\left\langle \frac{\partial}{\partial x^i}, \frac{\partial}{\partial x^j} \right\rangle = \langle Ge_i, e_j \rangle = g_{ij}.$$

Now, we let $[a, b]$ be a closed interval in $\mathbb{R}$ and $\gamma : [a, b] \to M$ be a smooth curve. The length of γ is defined as

$$L(\gamma) := \int_a^b \left\| \frac{d}{dt}\gamma(t) \right\| dt, \tag{3.2}$$

and the energy of γ as

$$E(\gamma) := \frac{1}{2} \int_a^b \left\| \frac{d}{dt}\gamma(t) \right\|^2 dt. \tag{3.3}$$

In physics, $E(\gamma)$ is typically called the "action of γ," where γ is considered to be the orbit of a point mass. Working now with coordinates $(x^1\gamma(t), x^2\gamma(t), \ldots, x^1\gamma(t))$, we employ the abbreviation

$$\dot{x}^i(t) := \frac{d}{dt}\left(x^i(\gamma(t))\right).$$

Then

$$L(\gamma) = \int_a^b \left[g_{ij}\left(x(\gamma)\right)\dot{x}^i(t)\dot{x}^j(t)\right]^{1/2} dt$$
$$E(\gamma) = \int_a^b g_{ij}\left(x(\gamma)\right)\dot{x}^i(t)\dot{x}^j(t)dt$$

Lemma 3.1. *For each smooth curve $\gamma : [a, b] \to M$,*

$$L(\gamma)^2 \leq 2(b - a)E(\gamma),$$

and equality holds if and only if $\left\| \frac{d}{dt}\gamma(t) \right\|$ *is constant.*

Proof. By the Cauchy-Schwartz inequality,

$$\int_a^b \left\| \frac{d}{dt}\gamma(t) \right\| dt \le (b-a)^{1/2} \left(\int_a^b \left\| \frac{d}{dt}\gamma(t) \right\|^2 dt \right)^{1/2}$$

with equality precisely when $\left\| \frac{d}{dt}\gamma(t) \right\|$ is constant. □

Lemma 3.2. *If* $\gamma : [a,b] \to M$ *is a smooth curve and* $\psi : [\alpha, \beta] \to [a,b]$ *is a change of parameter, then*

$$L(\gamma \circ \psi) = L(\gamma)$$

.

Proof. Let $t = \psi(\tau)$. Then

$$L(\gamma \circ \psi) = \int_a^b \left[g_{ij}\left(x\left(\gamma\left(\psi(\tau)\right)\right)\right) \dot{x}^i\left(\psi(\tau)\right) \dot{x}^j\left(\psi(\tau)\right) \left(\frac{d\psi}{dt}\right)^2 \right]^{1/2} d\tau = L(\gamma)$$

by the chain rule and a change of variables from τ to t. □

Lemma 3.3. *The Euler-Lagrange equations for the energy* E *are*

$$\ddot{x}(t) + \Gamma^i_{jk}(x(t))\dot{x}^j(t)\dot{x}^k(t) = 0, \quad i = 1, 2, \ldots, d \tag{3.4}$$

with

$$\Gamma^i_{jk} = \frac{1}{2} g^{il}\left(g_{jl,k} + g_{kl,j} - g_{jk,l}\right),$$
$$G^{-1} = \left[g^{ij}\right].$$

Here $g^{il}g_{e_j} = \delta_{ij} = \begin{cases} 1, i = j \\ 0, i \neq j \end{cases}$ *, and the subscript* $,k$ *means the partial derivative* $\frac{\partial}{\partial x^k}$*:*

$$g_{jl,k} = \frac{\partial}{\partial x^k} g_{jl}(x).$$

The expressions Γ^i_{jk} *are called the Christoffel symbols.*

Proof. The Euler-Lagrange equations of a functional

$$I(x) = \int f(t, x(t), \dot{x}(t))dt$$

are given by

$$\frac{d}{dt}\frac{\partial}{\partial \dot{x}^i} f - \frac{\partial}{\partial x^i} f = 0, \quad i = 1, 2, \ldots, d.$$

In the case of $E(\gamma)$ we have

$$f = f(x(t), \dot{x}(t)) = \frac{1}{2} g_{jk}(x(t))\dot{x}^j \dot{x}^k$$
$$\tfrac{d}{dt}\left[g_{jk}(x(t))\dot{x}^k(t) + g_{jk}(x(t))\dot{x}^j(t)\right] - g_{jk,i}(x(t))\dot{x}^j(t)\dot{x}^k(t) = 0,$$

for $i = 1, 2, \ldots, d$, hence

$$g_{ik}\ddot{x}^k + g_{jk}\ddot{x}^j + g_{ik,l}\dot{x}^l\dot{x}^k + g_{ji,l}\dot{x}^l\dot{x}^i - g_{jk,i}\dot{x}^j\dot{x}^k = 0$$

By renaming some indices and using the symmetry $g_{ik} = g_{ki}$, we obtain

$$2g_{lm}\ddot{x}^m + \left(g_{lk,j} + g_{jl,k} - g_{jk,l}\right)\dot{x}^j\dot{x}^k = 0, \quad l = 1, 2, \ldots, d. \tag{3.5}$$

Recalling $g^{ij}g_{lm} = \delta_{im}$, we see that $g^{ij}g_{lm}\ddot{x}^m = \ddot{x}^i$, and therefore the above equation becomes

$$\ddot{x}^m + \frac{1}{2} g^{il}\left(g_{lk,j} + g_{jl,k} - g_{jk,l}\right)\dot{x}^j\dot{x}^k = 0, \quad i = 1, 2, \ldots, d.$$

Equation (3.4) follows directly. □

Definition 2 A smooth curve $\gamma : [a, b] \to M$ which satisfies

$$\ddot{x}^i(t) + \Gamma(x(t))\dot{x}^j(t)\dot{x}^k(t) = 0,$$

for $i = 1, 2, \ldots, d$ is called a geodesic.

Thus geodesics are the critical points of the energy functional. By Lemma 3.2, the length functional is invariant under parameter changes. As in the Euclidean case, it is easy to see that regular ($\dot{x}(t) \neq 0$) curves can by parametrized by arc length. Our goal is to minimize the length within the class of regular smooth curves, with this discussion culminating in Lemma 3.4 below. Since length is invariant under representation, it is sufficient to consider curves parametrized by arc length when seeking the curve of shortest length. Furthermore, Lemma 3.1 states that energy may be minimized instead of length. Conversely, every critical point of the energy functional, i.e. every solution of (3.4) or geodesic, is parametrized proportionally to arc length.

Namely, for a solution of (3.4)

$$\begin{aligned}
\frac{d}{dt}\langle \dot{x}, \ddot{x} \rangle &= \frac{d}{dt}\left(g_{ij}(x(t))\dot{x}^i(t)\dot{x}^j(t)\right) \\
&= g_{ij}\ddot{x}^i\dot{x}^j + g_{ij}\dot{x}^i\ddot{x}^k + g_{ij,k}\dot{x}^i\dot{x}^j\dot{x}^k \\
&= -\left(g_{ik,l} + g_{lj,k} - g_{lk,j}\dot{x}^l\dot{x}^j\dot{x}^k\right) + g_{lj,k}\dot{x}^l\dot{x}^j\dot{x}^k \\
&= 0
\end{aligned}$$

by formula (3.5). We also used the fact that $g_{jk,l}\dot{x}^i\dot{x}^k\dot{x}^j = g_{lk,j}\dot{x}^l\dot{x}^k\dot{x}^j$ by interchanging indices l and j. Hence $\|\dot{x}\|$ is constant, and so the curve is parametrized proportionally to arc length.

Lemma 3.4. *Each geodesic is parametrized proportionally to arc length.*

The Riemannian curvature is given by

$$R^i_{jkl} = -\Gamma^i_{jk,l} + \Gamma^i_{jl,k} - \Gamma^h_{jk}\Gamma^i_{hl} + \Gamma^h_{jl}\Gamma^i_{hk}. \tag{3.6}$$

The Ricci curvature is given by the formula $R_{jl} = R^i_{jil}$, and the scalar curvature by $R = g^{jl}R_{jl}$. If M is two-dimensional, its Gaussian curvature $K_g = R/2$. For a conformal metric $ds^2 = g(x,y)\,((dx)^2 + (dy)^2)$,

$$K_g = -\frac{1}{2g(x,y)}\Delta \ln g(x,y). \tag{3.7}$$

If $g = g(y)$, then from (3.7) we have

$$K_g = -\frac{1}{g(y)^3}\left[g''(y)\cdot g(y) - (g'(y))^2\right]. \tag{3.8}$$

Example 3.1. *Hyperbolic Geometry: Let* $M = H$*, where* $H = \{(x,y) \in \mathbb{R}^2 : y > 0\}$ *with metric* $g_{ij} = y^{-2}\delta_{ij}$*, that is* $ds^2 = \frac{1}{y^2}\,((dx)^2 + (dy)^2)$*.*

Clearly, $g^{ij} = y^2\delta^{ij}$. By the formulae for Christoffel symbols from Lemma 3.3 we have

$$\Gamma^1_{12} = \Gamma^1_{21} = \frac{1}{2}y^2\,(g_{12,1} + g11,2 - g_{12,1}) = \frac{1}{2}y^2\frac{\partial}{\partial y}y^{-2} = -y^{-1}. \tag{3.9}$$

Similarly, $-\Gamma^2_{22} = \Gamma^2_{22} = -\frac{1}{y}$ and all other Christoffel symbols are 0. In addition, we have the identities $R^1 212 = R^2_{212} = -y^{-2}$, $R^1_{111} = R^2_{222} = 0$, $R_{11} = R^1_{111} + R^2_{121} = R_{22} + -y^{-2}$, $R = -2$ and $K_g = -1$. Therefore the hyperbolic space H has constant Gaussian curvature -1. Geodesics parametrized by arc length are determined using Lemma 3.3:

$$\ddot{x} - 2y^{-1}\dot{x}\dot{y} = 0,$$

$$\ddot{y} + y^{-1}\dot{x}^2 - y^{-1}\dot{y}^2 = 0.$$

Let $p = \dfrac{dx}{dy}$; then

$$\dot{x} = p\dot{y}, \quad \ddot{x} = \frac{dp}{dy}\dot{y}^2 + p\ddot{y}.$$

Substituting for $\ddot{y}$ from the second equation into the first yields

$$\frac{dp}{dy} = y^{-1}\left(p^3 + p\right).$$

Integrating by partial fractions then gives

$$\frac{dx}{dy} = p = \pm\frac{cy}{\sqrt{1 - c^2y^2}}.$$

For $c = 0$, we obtain vertical lines as geodesics. Otherwise, letting $c = \dfrac{1}{a}$ and integrating gives $(x - b)^2 + y^2 = a^2$. Hence geodesics here are semicircles centered on the x-axis.

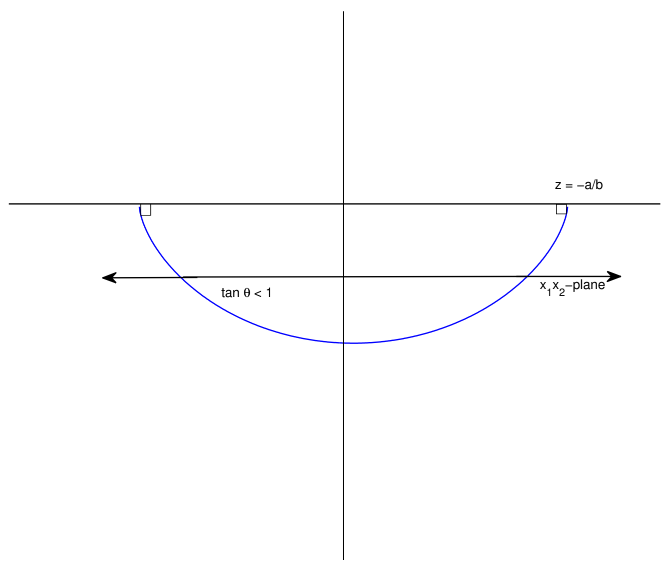

Figure 3.1

Interestingly, the hyperbolic distance from any point (a, b), $b > 0$ to the x-axis, measured along a vertical geodesic, is $\displaystyle\int_a^b \frac{dy}{y} = \infty$. As a result, hyperbolic space has no boundary; it extends infinitely far in all directions.

Scaling and translating the linear velocity case c_0 yields the metric $g_{ij}(y) = (a+by)^{-2}\delta_{ij}$ and $M = \{(x,y) \in \mathbb{R}^2 : a+by > 0\}$ which will also be a hyperbolic space with $K_g = -b^2$ and geodesics will be semicircles centered at the line $y = -\frac{a}{b}$.

When $g_{ij}(y) = (a+by)^{-1}\delta_{ij}$, we have a conformal metric with $g(y) = (a+by)^{-1}$. Taking derivatives,

$$g'(y) = (-1)\,(a+by)^{-2}\,b, \quad g''(y) = 2\,(a+by)^{-3}\,b^2,$$

and we find from (3.7)

$$\begin{aligned} K_g &= -\frac{1}{2(a+by)^{-3}}\left[2b^2\,(a+by)^{-3}\,(a+by)^{-1} - (a+by)^{-4}\,b^2\right] \\ &= -\frac{b^2}{2(a+by)} \\ &= -\frac{b^2}{2}g(y) \end{aligned}$$

From this, we see that the Gaussian curvature K_g is a function of the depth y. Variable curvature makes the corresponding tomography problem more difficult than in a hyperbolic case. For $a = 0$ and $b = 1$, $g(y) = y^{-1}$ and geodesics are only expressible in parametric form

$$\begin{cases} x = r(t - \sin t) + x_0 \\ y = r(1 - \cos t) \end{cases} \qquad t \in [0, 2\pi].$$

These are the equations defining a cycloid generated by the motion of a fixed point on the circumference of a circle of radius r rolling on the x-axis in the region $y > 0$. If $g(y) = (a+by)^{-1}$, the circle rolls on the positive side of the line $y = -\frac{a}{b}$:

$$\begin{cases} x = r(t - \sin t) + x_0 \\ y = r(1 - \cos t) - \frac{a}{b} \end{cases}$$

3.2 Recovering the Vector Field

Remark 3.2. *The following theorem [15] shows that there is a unique decomposition of a vector field into divergence-free and curl-free components. This decomposition allows the vector field $\boldsymbol{v}(x)$ from the Vector Tomography Problem 2.4 to be fully reconstructed. Note that this theorem holds in subsets of* $\mathbb{R}^3$ and $\mathbb{R}^2$.

Theorem 3.1. *If $\Omega \subset \mathbb{R}^3$ (or $\mathbb{R}^2$) is a smooth bounded function and $\boldsymbol{u}$ is a smooth vector field on $\bar{\Omega}$, then there is a unique orthogonal decomposition $\boldsymbol{u} = \boldsymbol{u}_1 + \boldsymbol{u}_2$, where $\boldsymbol{u}_1$ is a smooth gradient field and $\boldsymbol{u}_2$ is both divergence-free and parallel to the boundary $\partial\Omega$ (i.e., $\boldsymbol{u}_2 \cdot \boldsymbol{\nu} = 0$ on $\partial\Omega$, where $\boldsymbol{\nu}$ is the outward unit normal).*

Proof. We must find a potential function $\phi(x)$ such that

$$\boldsymbol{u} = \nabla\phi + \boldsymbol{u}_2, \tag{3.10}$$

where $\boldsymbol{u}_2$ is divergence free and parallel to $\partial\Omega$. If we take the divergence of this equation, we have $\Delta\phi = f$ in Ω, where $f = \nabla \cdot \boldsymbol{u}$. If we take the scalar product of this equation with $\boldsymbol{\nu}$, we find that $\dfrac{\partial\phi}{\partial\boldsymbol{\nu}} = h$ on $\partial\Omega$, where $h = \boldsymbol{u} \cdot \boldsymbol{\nu}$. Therefore, ϕ must satisfy a Neumann problem. The compatibility condition

$$\begin{aligned}\int_{\partial\Omega} h\, dS = \int_{\partial\Omega} \boldsymbol{u} \cdot \boldsymbol{\nu}\, dS \quad &= \int_{\partial\Omega} \boldsymbol{u}_1 \cdot \boldsymbol{\nu}\, dS = \int_{\partial\Omega} \nabla\phi \cdot \boldsymbol{\nu}\, dS \\ = \int_{\Omega} \nabla \cdot (\nabla\phi)\ dx = \int_{\Omega} \Delta\phi\, dx \quad &= \int_{\Omega} f\, dx\end{aligned}$$

gives that the Neumann problem is solvable for $\phi \in C^\infty(\bar{\Omega})$. We let $\boldsymbol{u}_1 = \nabla\phi$. By construction we have that $\boldsymbol{u}_2 = \boldsymbol{u} - \boldsymbol{u}_1$ is divergence-free.

Orthogonality is easily verified:

$$\int_{\Omega} \boldsymbol{u}_1 \cdot \boldsymbol{u}_2\, dx = \int_{\Omega} \nabla \cdot (\phi\boldsymbol{u}_2)\, dx = \int_{\partial\Omega} \phi\boldsymbol{u}_2 \cdot \boldsymbol{\nu}\, dS = 0.$$

Finally, we verify uniqueness. Suppose that we also have $\boldsymbol{u} = \boldsymbol{v}_1 + \boldsymbol{v}_2$ with $\boldsymbol{v}_1 = \nabla \psi$ and $\boldsymbol{v}_2$ divergence-free and parallel to $\partial\Omega$. Then

$$0 = \boldsymbol{u}_1 - \boldsymbol{v}_1 + \boldsymbol{u}_2 - \boldsymbol{v}_2 = \nabla\left(\phi - \psi\right) + \boldsymbol{u}_2 - \boldsymbol{v}_{\mathbf{2}}.$$

We may now take the scalar product of this with $\boldsymbol{u}_2 - \boldsymbol{v}_2$ and integrate over Ω to obtain

$$0 = \int_\Omega \left((\boldsymbol{u}_2 - \boldsymbol{v}_2) \cdot \nabla\left(\phi - \psi\right) + |\boldsymbol{u}_2 - \boldsymbol{v}_{\mathbf{2}}|^2 \right) dx.$$

By the divergence theorem and the properties of $|\boldsymbol{u}_2 - \boldsymbol{v}_{\mathbf{2}}|$, $(\boldsymbol{u}_2 - \boldsymbol{v}_{\mathbf{2}}) \cdot \nabla\left(\phi - \psi\right) dx = 0$. This means that $\int |\boldsymbol{u}_2 - \boldsymbol{v}_{\mathbf{2}}|^2 dx = 0$, and hence $\boldsymbol{u}_2 \equiv \boldsymbol{v}_2$. This also implies that $\boldsymbol{u}_1 \equiv \boldsymbol{v}_1$, so uniquess is proven. $\square$

Chapter 4

4 Conclusion

4.1 Main Results

We have obtained explicit inversion formula for the scalar acoustic source function $a(x)$ [(2.19), (2.29)] and the subsurface flow vector function $\boldsymbol{v}(x)$ [(2.56), (2.58)]. Furthermore, we have proven that the formulae are unique and stable with respect to the initial data. These formulae are the tools necessary to determine the functions which describe the perturbations to the expected travel times of acoustic waves beneath the surface of the Sun. This is the "acoustic speed profile" of the Sun and is indicative of internal structure. Density, temperature, pressure and convection patterns of the solar plasma may be inferred from this profile. These structural properties allow for the physical investigation of local solar phenomena such as sunspots, granulation, giant cell convection, meridional circulation and solar rotation.

4.2 Future Work

For these formulas to be used to study the Sun's interior, it is necessary to develop a numerical algorithm which uses the solar acoustic data now available from the Solar Dynamics Observatory and its Helioseismic and Magnetic Imager. The algorithm will determine the functions $a(x)$ and $\boldsymbol{v}(x)$ on a series of surfaces in order to reconstruct the solar structure in solid of revolution determined by the geodesic curves. Currently, existing local helioseismology algorithms are being used to process the SDO data, forming the "helioseismic data pipeline." This will provide a good basis of comparison for our algorithm. The introduction of more physical information, such as the equations of state and stellar structure could aid in obtaining more accuracy in our inversion algorithm. Several physical factors may damage the validity of simplifying assumptions made in this paper, such as scattering, high flow regions, local turbulence due to global convection, etc. These are but a small list of local phenomena of the Sun, and all deserve consideration.

Mathematically, improvements are possible as well. Approximations were used to simplify the problem of determining $\boldsymbol{v}(x)$, but it is likely that the system of differential equations which arises in the Vector Tomography Problem 2.4 is solvable without them. In addition, it should be possible to keep arbitrarily many of the terms in the expansion of the weight function $\rho(x, \omega)$. Studying this problem is general could potentially result in a much more accurate inversion algorithm. Also, the scalar and vector tomography problems for nonzero attenuation $\mu(x)$ need to be considered. This may be accomplished by either introducing more physical information to the problem as mentioned above or by studying the sensitivity kernels currently being used by helioseismologists.

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