Arbuzov E. V., Ph.D., Sobolev Institute of Mathematics SB RAS
(4, Acad. Koptyug avenue 630090, Novosibirsk, Russia)
tel. (383)3332188, e-mail: arbuzov@math.nsc.ru
Bukhgeim A. A., Ph.D., Schlumberger
(Aslakveien 14E, 0753 Oslo, Norway)
e-mail: ABoukhgueim@slb.com
Bukhgeim A. L., Ph.D., Wichita State University,
Department of Mathematics and Statistics
(1845 N. Fairmount, Wichita, KS, 67260-0033, USA) and
Sobolev Institute of Mathematics SB RAS
(4, Acad. Koptyug avenue 630090, Novosibirsk, Russia)
e-mail: boukhgueim@math.wichita.edu

The Carleman formula and recovering the refractive index of the two-dimensional Helmholtz equation

Abstract. We consider the Cauchy problem for the Helmholtz equation with given bounded refractive index in an arbitrary planar domain with Cauchy data only on part of the boundary of the domain. We derive a Carleman-type formula for a solution to this problem.

From another side we obtain the inversion formula for smooth unknown refractive index using the Cauchy date set on the boundary. Based of this inversion formula we numerically reconstruct the refractive index on the unit disk.

Let Ω be a simply connected domain with C^2 boundary. We identify $\mathbb{R}^2$ with the complex plane $\mathbb{C}$ and denote

$$\mathcal{D} = \begin{pmatrix} 2\bar{\partial} & 0 \\ 0 & 2\partial \end{pmatrix}, \quad A = \begin{pmatrix} 0 & b(x) \\ a(x) & 0 \end{pmatrix}, \quad \mathbf{u} = \begin{pmatrix} u_1(x) \\ u_2(x) \end{pmatrix}.$$

Here a, b and u_j are complex-valued functions. We consider the equation

$$P\mathbf{u} = \mathcal{D}\mathbf{u} + A\mathbf{u} = 0.$$

The Helmholtz equation $\Delta u(x) + k^2 n(x) u(x) = 0$ is reduced to $P\mathbf{u} = 0$, if we set

$$u_1 = u, \quad u_2 = 2\bar{\partial} u, \quad b = -1, \quad a(x) = k^2 n(x).$$

Let $M \subset \partial\Omega$ be a union of finitely many closed arcs, matrix

$$\Phi(x) = \begin{pmatrix} \psi(x) & 0 \\ 0 & \bar{\psi}(x) \end{pmatrix}, \quad \bar{\partial}\psi = 0, \quad \varphi = 2\mathrm{Im}\psi,$$

For real $\tau > 0$ let $a_{\pm\tau} = e^{\pm \mathrm{i}\tau\varphi} a$, and matrix $A_\tau^t = \begin{pmatrix} 0 & a_{-\tau} \\ b_\tau & 0 \end{pmatrix}$.

For the solution of the Cauchy problem $P\mathbf{u} = 0, \quad \mathbf{u}|_M = \mathbf{f}$, we have the formula

$$u_1(x_0) = \lim_{\tau\to\infty} \int_M e^{-\tau\psi(x_0)} (e^{\tau\Phi(x)} \mathrm{N}\mathbf{f}, \widetilde{\mathbf{v}} + \mathbf{w}) \mathrm{d}s,$$

where

$$\operatorname{Re}\psi(x) = \begin{cases} 1, & x \in M; \\ 0, & x \in \partial\Omega \setminus M. \end{cases}$$

Here $a \in L_\infty$, $b \in W^1_\infty$, $x_0 \in \Omega$, $\widetilde{\mathbf{v}}(x) = \begin{pmatrix} 2\bar{\partial}\mathcal{E}(x - x_0) \\ 0 \end{pmatrix}$, $\mathcal{E}(x) = -\frac{1}{2\pi}\ln(x_1^2 + x_2^2)$, and $\mathbf{w} \in W^1_p$, $1 < p < 2$, is a solution to

$$\mathcal{D}\mathbf{w} - A^t_\tau \mathbf{w} = A^t_\tau \widetilde{\mathbf{v}},$$

$\mathrm{N} = \begin{pmatrix} \nu & 0 \\ 0 & \bar{\nu} \end{pmatrix}$, $\nu = \nu_1 + \mathrm{i}\nu_2$ — the unit outward normal. The proof is given in [1, 2].

It is also possible to derive an inversion formula for smooth potentials $A \in C^1(\overline{\Omega})$. Let $P\mathbf{u} = 0$ in $\Omega = \{x \in \mathbb{R}^2 : |x| < 1\}$ and $\varphi = \frac{1}{2}[(x_1 - x_{01})^2 - (x_2 - x_{02})^2]$, $x_0 = (x_{01}, x_{02}) \in \Omega$. We create the boundary conditions

$$\operatorname{Re}(e^{-\tau\Phi}\mathbf{u})|_{\partial\Omega} = e_1 = \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \quad \int_{\partial\Omega} \operatorname{Im}(e^{-\tau\Phi}\mathbf{u})\,ds = 0,$$

and measure the response

$$\operatorname{Im}(e^{-\tau\Phi}\mathbf{u})|_{\partial\Omega} = \mathbf{f}(x, t) = \begin{pmatrix} f_1 \\ f_2 \end{pmatrix}.$$

Existence and uniqueness of such special solutions follows from [3].

Then we have

$$a(x_0) = \frac{1}{2\pi i} \lim_{\tau\to\infty} \tau \int_{\partial\Omega} \bar{\nu} f_2(x, \tau)\,ds.$$

In a similar way, switching $e_1 \leftrightarrow e_2$, we obtain b.

Some numerical reconstructions of potential a from [4] are given on the next pages.

Acknowledgements. The first author was partially funded by RFBR grant 08-01-00207 and the State Maintenance Programs for the Leading Scientific Schools of the Russian Federation (GrantNSh-7157.2006.1). The third author was partially funded by NSF grant DMS-0505470.

References

[1] *Arbuzov E. V. and Bukhgeim A. L.* The Carleman formula for the Helmholtz equation on the plane. // Siberian Mathematical Journal. 2006. v. 47, 3. P. 1–8.

[2] *Arbuzov E. V. and Bukhgeim A. L.* The Cauchy problem for elliptic equation of second order on the plane // J. Inv. Ill-Posed Problems. (submitted).

[3] *Bukhgeim A. L.* Recovering a potential from the Cauchy data in the two-dimensional case // J. Inv. Ill-Posed Problems. 2008. v. 16, 1. P. 19–33.

[4] *Bukhgeim A. A. and Bukhgeim A. L.* Numerical recovery of a potential from the Cauchy data in the two-dimensional case // J. Inv. Ill-Posed Problems. (submitted).

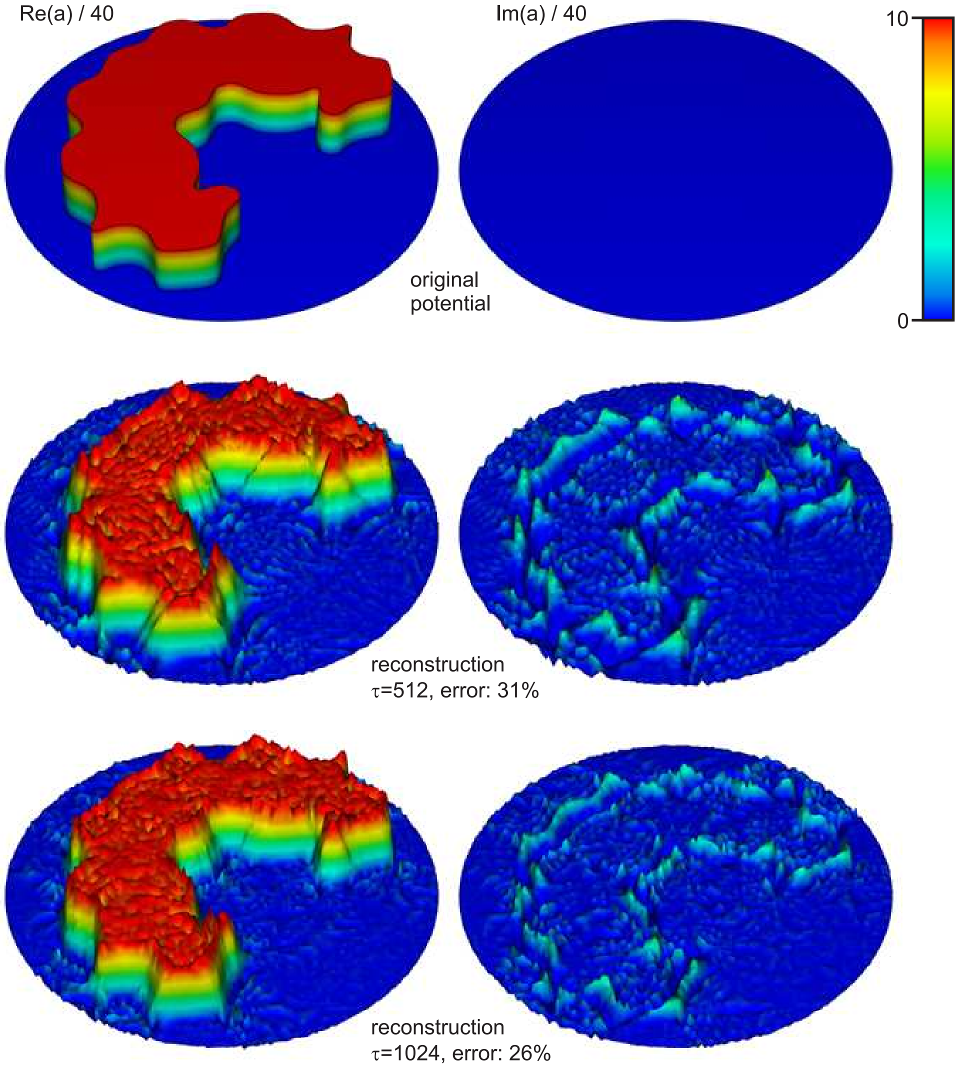

Figure 1: Reconstructions of a real-valued potential a, which forms a characteristic function of a non-convex subset of the unit disc with smooth boundaries. Imaginary part (shown on the right) equals to zero. Note, that on the image we show scaled functions, the true ones are 40 times bigger. $\mathrm{Re}(a)/40$ is shown on the left, and $\mathrm{Im}(a)/40$ is shown on the right. We show reconstructions with $\tau = 512, 1024$. We can observe some Gibbs effects on the boundary of our stair-step function, in both real and imaginary components.

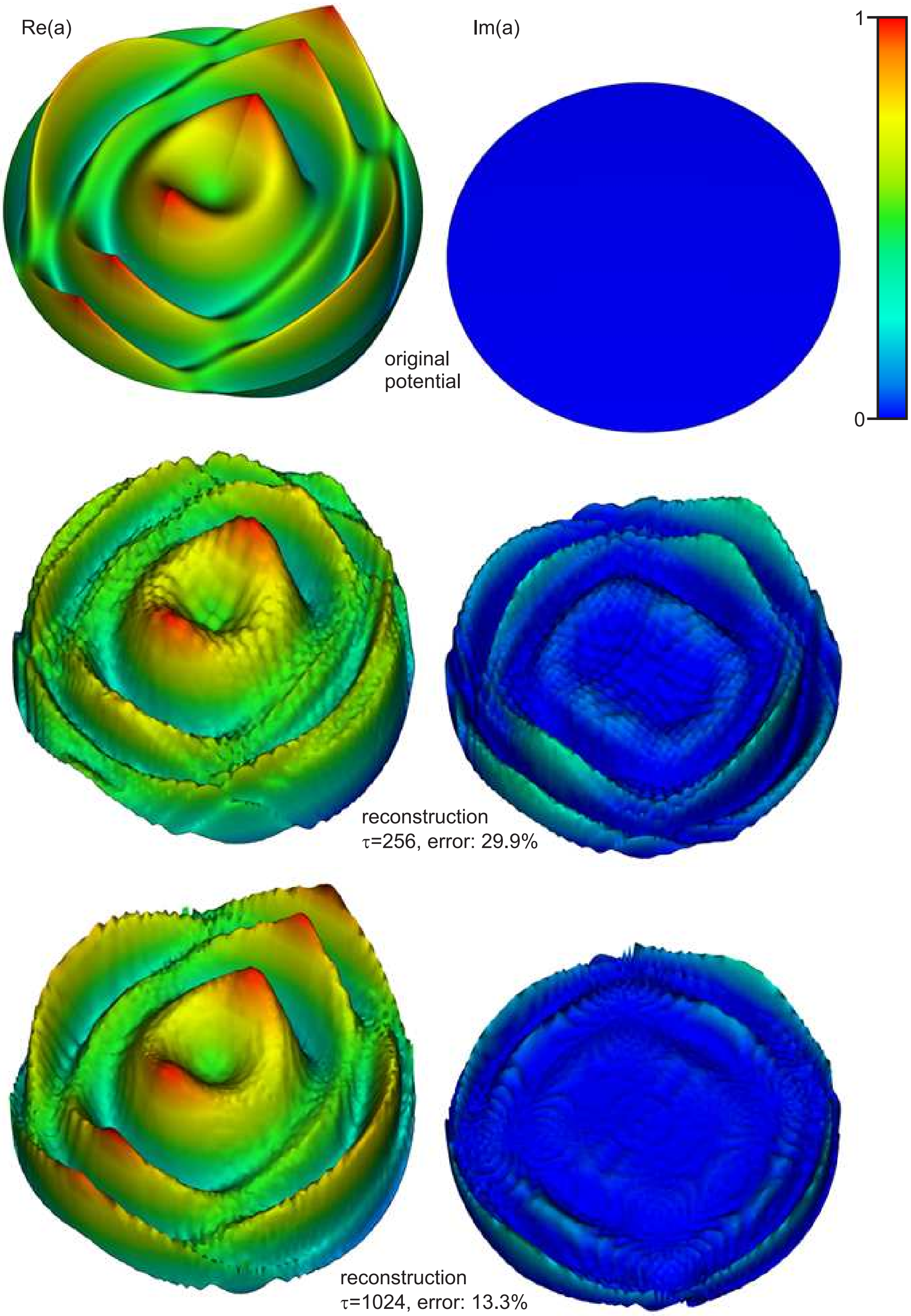

Figure 2: Reconstructions of a real-valued potential a, where real part (shown on the left) has a non-smooth "crease" along the $y-$axis, and imaginary part (shown on the right) equals to zero. We show reconstructions with $\tau = 256, 1024$. The real component again has an impact on the reconstruction of the imaginary component, especially for small values of τ. But, as τ gets larger, reconstruction error becomes smaller.

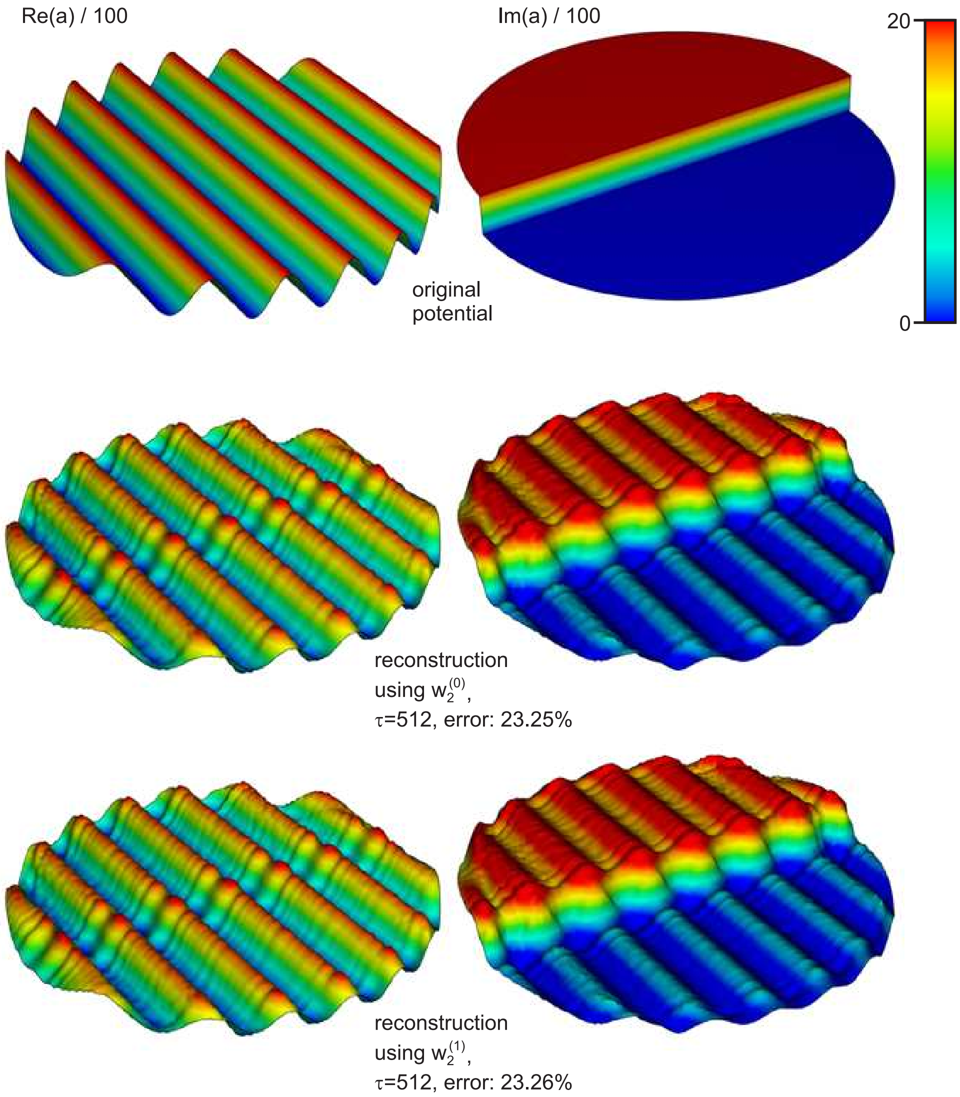

Figure 3: Reconstructions of a complex-valued potential, where real part is smooth and imaginary part follows the discontinuous stair-step shape. Note, that on the image we show scaled functions, the true ones are 100 times bigger. $\mathrm{Re}(a)/100$ is shown on the left, and $\mathrm{Im}(a)/100$ is shown on the right. We reconstructed the potential with $\tau = 512$, using first and second iterations of w_2 during the solution of the forward problem. The difference between reconstruction errors is only one hundredth of a per cent, which shows that usually it's sufficient to use the value of w_2 from the first iteration. Again, we observe that traces of the real part are visible in the reconstruction of the imaginary part, and vice versa: the imaginary stair-step is reflected in the reconstruction of the real part.

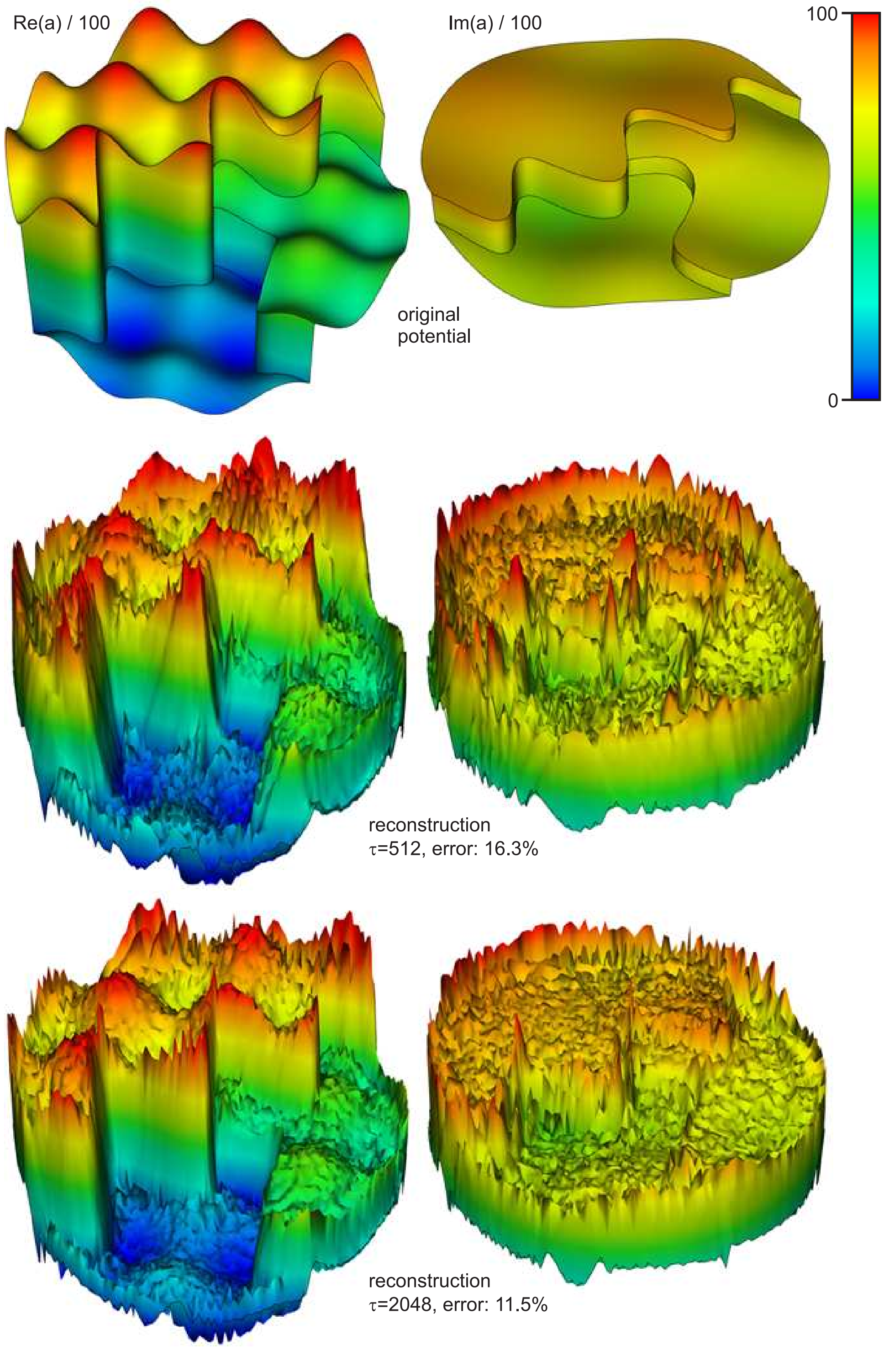

Figure 4: Reconstructions of a discontinuous complex-valued potential, where discontinuities (illustrated with vertical "walls") occur along the same curves both for real and imaginary components. Note, that on the image we show scaled functions, the true ones are 100 times bigger. $\mathrm{Re}(a)/100$ is shown on the left, and $\mathrm{Im}(a)/100$ is shown on the right. We show reconstructions with $\tau = 512, 2048$. Abrupt variations of reconstructed functions mimic original discontinuities, inspite of the present Gibbs effect.