On uniqueness of obstacles and boundary conditions from restricted dynamical and scattering data

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1 Introduction.

In this paper we are interested in uniqueness of time independent domain and of coefficients of the boundary condition from transient or scattering data. The transient case is described either by the wave equation or by the heat equation in the exterior of a bounded domain D with the given initial data and the boundary value condition $\partial_\nu u - a_0 \partial_t u - bu = 0$ or $u = 0$ on the boundary ∂D of D. Here ν is the outer unit normal to ∂D. These boundary conditions model different characters of reflection from ∂D and they are of interest for modeling various types of reflecting boundaries (coated, colored, with imperfections etc). The direct transient problems include also initial value data and some boundary condition at an outer boundary of a given domain Ω containing inclusion D. Mathematically, they are represented by initial boundary value problems for hyperbolic or parabolic equations which are well understood ([15]). The direct scattering problem is also sufficiently well studied [16], [17], [4], [19], [20]. Typically, for well defined scattering one needs dissipativity condition (like $0 \leq a_0, 0 \leq b$), and then scattering manifests itself in a certain behavior at infinity. The first term of asymptotics is given by the so-called scattering amplitude (or pattern) $\mathcal{A}(\sigma, \xi, k)$ where σ is the direction of a receiver, ξ is the direction of incident wave, and k is frequency of wave.

Inverse problems in transient case consist of finding D, a_0, b from the additional boundary data on $\partial\Omega$. The inverse scattering problem by obstacle is to find D, a_0, b from the scattering data $\mathcal{A}(\sigma, \xi, k)$. Both problems are of importance since they model recovery of some objects from reflected acoustic, electromagnetic, and similar signals. Contrary to the direct problems, theory of inverse problems has many challenges, and practical algorithms of reconstruction are far from satisfactory, although there is recent progress in both theory and numerics.

One of first uniqueness results in inverse obstacle scattering is due to Schiffer [16] who showed uniqueness of D from $\mathcal{A}$ given at all σ, ξ, k in case of Dirichlet boundary condition on ∂D (soft obstacle). Schiffer's method needs the scattering data only at fixed ξ and sufficiently large (depending on size of D) number of frequencies k [4]. This method is not applicable to Neumann data (even when $a_0 = 0, b = 0$). When complete dynamical data (the hyperbolic Neumann-to- Dirichlet map) is available uniqueness of a general (time independent) hyperbolic equation, of obstacle D and of the boundary condition is obtained (or can be obtained) by the boundary control method [2], [12]. Since the scattering amplitude at all σ, ξ, k uniquely determines the hyperbolic Neumann-to Dirihclet map it is also true for quite general scattering problems. However, the complete data are redundant and there are numerous difficulties in uniqueness from restricted data (k or ξ are fixed) In 1988 the author [7] proposed use of singular solutions to handle more difficult case of transparent obstacles D. Kirsch and Kress [13] realized that singular solutions can be a tool to prove uniqueness for Neumann boundary condition from scattering data at fixed k and all σ, ξ. Later on this method was applied to a variety of boundary conditions. Moreover, Colton and Kirsch [3] used singular solutions in designing a new efficient numerical algorithm for finding D from scattering data at fixed frequency. In practice, scattering data are easily available at many k while it is harder to collect them at different ξ. However, so far it was difficult to use these additional frequencies either in theoretical or in numerical work. Observe that use of several (high) frequencies might in some cases substantially increase stability in inverse scattering problem. This stability is notoriously (logarithmically) weak for small k. This weak stability forbids high resolution of numerical reconstruction and severly restricts applications of inverse scattering problems. Stability of continuation of solutions of the wave equation was studied in the fundamental paper of John [11]. The increased stability of continuation of solutions to the Helmholtz equation was recently considered in [6], [10] .

In this paper we propose some modification of the Schiffer's proof. As a result we are able to demonstrate uniqueness of D with mixed boundary condition from a special single set of boundary data for the wave and heat equations, and from the scattering data $\mathcal{A}$ at fixed ξ, all σ and k on some subinterval of $(0, \infty)$. For finite observation times T this modification involves simple Carleman type estimate and trace theorems which allow to bound boundary traces of solutions of the wave and heat equations. We need trace theorems in differences of domains which in general can have arbitrary cusps. So far we only can handle D whose boundary is union of finitely many plane pieces. Uniqueness from transient data on $(0, \infty)$ and in inverse scattering problem is obtained for smooth domain and (unknown) general boundary condition. It is derived by using the Fourier-Laplace transform into frequency domain, analyticity of solutions with respect to k, and choosing k to be real or complex.

We remind some standard notation. ν is the outer unit normal to the boundary ∂D of an open set D. D_e is $\mathbf{R}^n \setminus \bar{D}$. $\sigma = \frac{x}{|x|}$, $\partial_j = \frac{\partial}{\partial x_j}$, $\partial^\alpha = \partial_1^{\alpha_1} ... \partial_n^{\alpha_n}$ where α is multiindex $(\alpha_1, ..., \alpha_n)$. $H_{(l)}(\Omega)$ denotes the Sobolev space with the norm

$$\|u\|_{(l)}(\Omega) = (\sum_{|\alpha| \leq l} \int_\Omega |\partial^\alpha u|^2)^{\frac{1}{2}}.$$

Generic constants C depend on D_1, D_2, b, but not on u_1, u_2, v, τ. Any additional dependence is indicated.

2 Main results

We consider the wave equation

$$\partial_t^2 u - \Delta u = 0 \text{ on } (\Omega \setminus \bar{D}) \times (0, T) \tag{2.1}$$

with zero initial condition

$$u = \partial_t u = 0 \text{ on } (\Omega \setminus \bar{D}) \times \{0\} \tag{2.2}$$

and with the boundary value data

$$\partial_\nu u = g_1 \text{ on } \partial\Omega \times (0, T), \tag{2.3}$$

$$\partial_\nu u - a_0\partial_t u - bu = 0 \text{ on } \partial D \times (0,T), \tag{2.4}$$

$$g_1, \partial_t g_1 \in C([0,T]; H_{(\frac{1}{2})}(\partial\Omega)),\ g_1 = 0 \text{ on } \partial\Omega \times \{0\}.$$

We will assume that Ω is a convex bounded domain in $\mathbf{R}^n$ with the boundary $\partial\Omega \in C^2$ and D is an open set with $\bar{D} \subset \Omega$, with connected $\Omega \setminus \bar{D}$, and with Lipschitz boundary. Moreover, let $a_0, b \in L_\infty(\partial D)$ and be time independent, $0 \leq a_0$. Then the energy integrals guarantee existence and uniqueness of a solution $u \in H_{(1)}((\Omega \setminus \bar{D}) \times (0,T))$ to the initial boundary value problem (2.1), (2.2), (2.3) (see eg [15], I, p.275), [15], II, p.103). More precisely, $u, \partial_t u \in C([0,T]; H_{(1)}(\Omega \setminus \bar{D})), \partial_t^2 u \in C([0,T]; L_2(\Omega \setminus \bar{D}))$due to additional regularity of the g_1 with respect to t. When $\partial D \in C^2$ the elliptic regularity theory applied to the equation (2.1) at fixed t implies that $u \in C([0,T]; H_{(2)}(\Omega \setminus \bar{D}))$.

We denote by Γ an (relatively) open nonvoid subset of $\partial\Omega$. Let $l(x,y)$ be infimum of lengths of curves in $\bar{\Omega} \setminus D$ joining points x and y. We introduce $L(D)$ as $sup(inf\ l(x,y))$ where infimum is over $x \in \Gamma$ and supremum is over $y \in \partial D$. Observe that generally $L(D) \leq diam\ \Omega + p(D)$ where $p(D)$ is perimeter of D, i.e. $sup\ l(x,y)$ over $x, y \in \partial D$. If D is convex (or only star-shaped), then $L(D) \leq p(\Omega) + diam\ \Omega$. Finally, if in addition $\Gamma = \partial\Omega$, then $L(D) \leq \frac{1}{2} diam\ \Omega$.

Theorem 2.1 *Let $\partial D \in C^2$. Assume that*

$$2L(D) < T \tag{2.5}$$

and that

$$g_1 \neq 0 \textit{ near any point of } \Gamma \times (0, t_0) \textit{ for any positive } t_0. \tag{2.6}$$

Let $a_0 = b = 0$.

Then the additional data

$$u = g_0 \textit{ on } \Gamma \times (0,T). \tag{2.7}$$

for a solution u to the problem (2.1), (2.2), (2.3), (2.4) uniquely determine D.

We can show uniqueness of b under additional assumptions that D is a polyhedron. A polyhedron is an open set whose boundary is the union of finitely many parts of hyperplanes. Then D, D_e are Lipschitz domains.

Theorem 2.2 *Assume that the conditions (2.5) and (2.6) are satisfied . Let D be a polyhedron. Let $a_0 = 0$ and $b \in C^1(\bar{\Omega})$.*

Then the additional data (2.7) for a solution u to the initial boundary value problem (2.1), (2.2), (2.3), (2.4) uniquely determine D and b on ∂D.

If $T = \infty$ then one can uniquely identify general smooth D and b on ∂D. We introduce the following dissipativity type condition on the boundary coefficients:

$$0 \leq a_0, \; 0 \leq b. \tag{2.8}$$

Theorem 2.3 *Let $\partial D \in C^2$. Let $a_0 = 0$ and let condition (2.8) hold. Let u be a solution to the problem (2.1), (2.2), (2.3), (2.4) with $T = \infty$. Let $g_1 = 0$ on $\partial\Omega \times (T_2, \infty)$ for some T_2.*

Then a domain D and the boundary coefficient $b \in C^1(\bar{\Omega})$ on ∂D are uniquely determined by the function $g_0 = u$ on $\Gamma \times (0, \infty)$.

These results are related to uniqueness of an obstacle and of a boundary coefficients a_0, b in the inverse scattering problem at fixed incident direction and on an interval of frequencies. Let $n = 3$. Let U solve the reduced wave equation

$$\Delta U + k^2 U = 0 \text{ in } D_e, k > 0 \tag{2.9}$$

with the boundary data

$$\partial_\nu U + (ika_0 - b)U = 0 \text{ on } \partial D \tag{2.10}$$

and

$$U(x) = e^{ikx\cdot\xi} + U_s(x), \tag{2.11}$$

where U_s satisfies the Sommerfeld radiation condition. Under the condition (2.8) the scattered solution $U_s \in C^1(\bar{\Omega}) \cap H_{(2)}(D_e \cap \Omega)$ exists, is unique, and has the form

$$U_s(x) = \mathcal{A}(\sigma, \xi, k)\frac{e^{ik|x|}}{|x|} + U_{s,1}(x), \; \sigma = \frac{x}{|x|}, \tag{2.12}$$

where $U_{s,1}(x) \leq C(|x| + 1)^{-2}$ [3], [17] . The inverse scattering problem is to determine an obstacle (D and a_0, b) from the scattering amplitude $\mathcal{A}$.

Theorem 2.4 *Let $n = 3$. Let $\partial D \in C^2$. Let the dissipativity condition (2.8) be satisfied.*

Then scattering data $\mathcal{A}(\sigma, \xi, k)$ given for all $\sigma \in S^2$, all $k \in I$ for some interval I and for fixed incident direction ξ uniquely determine D and $a_0, b \in C^1(\bar{\Omega})$ on ∂D.

This theorem has a short proof given in section 4. Despite obvious interest in identification of both obstacle and boundary conditions, we are aware of only some partial uniqueness results when the incident direction ξ is fixed [14]. Theorem 2.4 implies the following stronger uniqueness statements for the wave equation when the observation time $T = \infty$ and when one uses special boundary data g_1.

Let φ be a function in $C_0^2(\mathbf{R})$ supported in $(-\infty, d)$ where $d = inf\ x \cdot \xi$ over $x \in D$. We will use the free space solution of the wave equation

$$u_i(x, t; \xi) = \varphi_0(x \cdot \xi - t). \tag{2.13}$$

Let $u(x, t; \xi)$ be the solution to the initial boundary value problem

$$\partial_t^2 u - \Delta u = 0 \text{ on } (\Omega \setminus \bar{D}) \times \mathbf{R} \tag{2.14}$$

with zero initial conditions

$$u = 0 \text{ on } (\Omega \setminus \bar{D}) \times (-\infty, 0) \tag{2.15}$$

and with boundary value data

$$\partial_\nu u = \partial_\nu u_i(\ ; \xi) \text{ on } \partial\Omega \times \mathbf{R},$$

$$\partial_\nu u - a_0 \partial_t u - bu = 0 \text{ on } \partial D \times (0, T), \tag{2.16}$$

generated by the plane wave $u_i(\ ; \xi)$.

Corollary 2.5 *Let $n = 3$. Let the dissipativity condition (2.8) be satisfied. Let ξ be fixed.*

Then a domain D and the boundary coefficients a_0, b on ∂D are uniquely determined by the function $u(\ ; \xi)$ on $\Gamma \times (0, \infty)$.

A similar result under less restrictive conditions (arbitrary positive T instead of (2.5)) holds for parabolic equations provided the boundary data on $\partial\Omega \times (0,T)$ are time constant on (T_1, T) for some $T_1 < T$. Let u solve the heat equation

$$\partial_t u - \Delta u = 0 \text{ on } (\Omega \setminus \bar{D}) \times (0,T) \tag{2.17}$$

with zero initial condition

$$u = 0 \text{ on } (\Omega \setminus \bar{D}) \times \{0\} \tag{2.18}$$

and with the boundary value data

$$\partial_\nu u = g_1 \text{ on } \partial\Omega \times (0,T), \ \ \partial_\nu u - bu = 0 \text{ on } \partial D \times (0,T), \tag{2.19}$$

where

$$g_1, \partial_t g_1 \in L_2((0,T); H_{(\frac{1}{2})}(\partial\Omega)), \ g_1 = 0 \text{ on } \partial\Omega \times \{0\}.$$

As known ([15],II,p.27) there is a unique solution u to (2.17),(2.18),(2.19) with $\partial_t^{\alpha_0}\partial^\alpha u \in L_2((\Omega \setminus \bar{D}) \times (0,T))$ when $2\alpha_0 + |\alpha| \leq 2$.

Theorem 2.6 *Let $\partial D \in C^2$. Assume that condition (2.6) is satisfied. Let g_1 be time independent on (T_1, T) for some positive $T_1 < T$. Let $b \in C^1(\Omega)$ and $0 \leq b$.*

Then additional data (2.7) for the solution u to problem (2.17), (2.18), (2.19) uniquely determine D and b on ∂D.

3 Proofs of Theorems 2.1 and 2.2

We start with a simpler proof of Theorem 2.1 which demonstrates some ideas of more complicated proof of Theorem 2.2.

Proof of Theorem 2.1

First we will show that $D_1 = D_2$.

Let us assume the opposite. Let D^∞ be the connected component of $\Omega \setminus (\bar{D}_1 \cup \bar{D}_2)$ whose boundary contains Γ. Let $D_\infty = \Omega \setminus \bar{D}^\infty$. We can assume that there is a point in $D_2 \setminus \bar{D}_1$. Using connectedness of $\Omega \setminus \bar{D}_1$ and joining this point with a point of D^∞ we can conclude that there is a connected component D_0 of $D_\infty \setminus \bar{D}_1$. Obviously, $\partial D_0 \subset \partial D_1 \cup \bar{D}^\infty$.

The function $u = u_2 - u_1$ solves the wave equation in the domain $D^\infty \times (-\infty, T)$ and it has zero Cauchy data on $\Gamma \times (-\infty, T)$. By using sharp uniqueness of the continuation results (like Lemma 3.4.7 and Exercise 3.4.12 in [9]) we obtain that $u = 0$ on $D^\infty \times (0, T - L)$. Hence

$$u_1 = u_2 \text{ on } D^\infty \times (0, T - L). \tag{3.20}$$

We denote by $\nu(1), \nu(2)$ exterior unit normals to $\partial D_1, \partial D_2$ and by ν the exterior unit normal to ∂D_0. Observe that $\nu = -\nu(1)$ on $\partial D_0 \cap \partial D_1$ and $\nu = \nu(2)$ on $\partial D_0 \setminus \bar{D}_1$. From (3.20) we have $\partial_\nu u_1 = \partial_\nu u_2 = 0$ on $(\partial D_0 \cap \partial D_2) \times (0, T - L)$ and due to (2.4) $\partial_\nu u_1 = 0$ on the remainder of $\partial D_0 \times (0, T - L)$ (since it is contained in $\partial D_1 \times (0, T - L)$).

Forming standard energy integral we have

$$0 = \int_{D_0 \times (0, T-L)} (\partial_t^2 u_1 - \Delta u_1) \partial_t u_1 e^{-t} =$$

$$\frac{1}{2} \int_{D_0 \times (0, T-L)} \partial_t ((\partial_t u_1)^2) e^{-t} - \int_{\partial D_0 \times (0, T-L)} \partial_\nu u_1 \partial_t u_1 e^{-t} +$$

$$\frac{1}{2} \int_{D_0 \times (0, T-L)} \partial_t |\nabla u_1|^2 e^{-t}$$

where we integrated by parts over D_0. Integrating by parts with respect to t and using that $\partial_\nu u_1 = 0$ on $\partial D_0 \times (0, T - L)$ we yield

$$0 = \frac{1}{2} \int_{D_0 \times \{T-L\}} (\partial_t u_1)^2 e^{-(T-L)} + \frac{1}{2} \int_{D_0 \times (0, T-L)} (\partial_t u_1)^2 e^{-t} +$$

$$\frac{1}{2} \int_{D_0 \times \{T-L\}} |\nabla u_1|^2 e^{-(T-L)} + \frac{1}{2} \int_{D_0 \times (0, T-L)} |\nabla u_1|^2 e^{-t}$$

which implies that $u_1 = 0$ on $D_0 \times (-\infty, T - L)$. By applying again sharp uniqueness of the continuation results for u_1 and using condition (2.5) we conclude that $u_1 = 0$ near $\Gamma \times \{0\}$ which contradicts assumption (2.6).

□

Proof of Theorem 2.2

Let u_1, u_2 be solutions to the initial boundary value problems (2.1), (2.2), (2.3), (2.4) for domains D_1, D_2 and boundary coefficients b_1, b_2.

First we will show that $D_1 = D_2$.

As in the proof of Theorem 2.1, let us assume the opposite and make use of notation of this proof.

Let $b_0 = -b_1$ on $\partial D_1 \cap \bar{D}_2$, $b_0 = b_2$ on $\partial D_2 \setminus \bar{D}_1$. It is clear that b_0 is measurable and bounded on ∂D_0, provided the surface measure is induced by surface measures on ∂D_1 and ∂D_2.

Lemma 3.1 *Under conditions of Theorem 2.2*

$$e^{-\tau(T-L)}(\int_{D_0\times\{T-L\}}((\partial_t u_1)^2 + |\nabla u_1|^2) - \int_{\partial D_0\times\{T-L\}} b_0 u_1^2)+$$

$$\tau(\int_{D_0\times(0,T-L)}((\partial_t u_1)^2 + |\nabla u_1|^2)e^{-\tau t} - \int_{\partial D_0\times(0,T-L)} b_0 u_1^2 e^{-\tau t}) = 0 \qquad (3.21)$$

for any $\tau > 0$.

Proof: As in the proof of Theorem 2.1 conditions of Theorem 2.2 imply the equality (3.20). Hence

$$\partial_\nu u_1 = b_0 u_1 \text{ on } \partial D_0 \times (0, T-L). \qquad (3.22)$$

As we stated before Theorem 2.1, $\partial_t^2 u \in C([0,T]; L_2(\Omega \setminus \bar{D}))$, so from the equation (2.1) $\Delta u \in C([0,T]; L_2(\Omega \setminus D))$ and by interior elliptic regularity $u \in C([0,T]; H_{(2)}(D_{0m}))$ for any subdomain D_{0m} of D_0 with closure in D_0. For such a domain (with C^1-boundary) we have

$$0 = \int_{D_{0m}\times(0,T-L)} (\partial_t^2 u_1 - \Delta u_1)\partial_t u_1 e^{-\tau t} =$$

$$\frac{1}{2}\int_{D_{0m}\times(0,T-L)} \partial_t((\partial_t u_1)^2)e^{-\tau t} - \int_{\partial D_{0m}\times(0,T-L)} \partial_\nu u_1 \partial_t u_1 e^{-\tau t}+$$

$$\frac{1}{2}\int_{D_{0m}\times(0,T-L)} \partial_t |\nabla u_1|^2 e^{-\tau t},$$

where we integrated by parts over D_{0m} which was possible since $u \in C([0,T]; H_{(2)}(D_{0m}))$. Approximating D_0 by D_{0m} so that the distances between their boundaries go to zero and functions representing boundaries in appropriate local coordinate systems have bounded (with respect to m) Lipschitz constants (like in [9], p.102) we can replace D_{0m} by D_0. Integrating by parts with respect to t and using (3.22) we obtain

$$0 = \frac{1}{2}\int_{D_0\times\{T-L\}} (\partial_t u_1)^2 e^{-\tau t} -$$

$$\frac{\tau}{2}\int_{D_0\times(0,T-L)}(\partial_t u_1)^2 e^{-\tau t}+\int_{\partial D_0\times(0,T-L)} b_0 u_1\partial_t u_1 e^{-\tau t}+$$

$$\frac{1}{2}\int_{D_0\times\{T-L\}}|\nabla u_1|^2 e^{-\tau t}+\frac{\tau}{2}\int_{D_0\times(0,T-L)}|\nabla u_1|^2 e^{-\tau t}=$$

$$\frac{1}{2}\int_{D_0\times\{T-L\}}((\partial_t u_1)^2+|\nabla u_1|^2)e^{-\tau t}-\frac{1}{2}\int_{\partial D_0\times\{T-L\}} b_0 u_1^2 e^{-\tau t}+$$

$$\frac{\tau}{2}\int_{D_0\times(0,T-L)}((\partial_t u_1)^2+|\nabla u|^2)e^{-\tau t}-\frac{\tau}{2}\int_{\partial D_0\times(0,T-L)} b_0 u_1^2 e^{-\tau t}$$

which gives (3.21).

□

Lemma 3.2 *We have*

$$\frac{\tau}{2}\int_{D_0\times\{T_2\}} u^2 e^{-\tau T_2}+\frac{\tau^2}{4}\int_{D_0\times(0,T_2)} u^2 e^{-\tau t}\leq\int_{D_0\times(0,T_2)}(\partial_t u)^2 e^{-\tau t} \qquad (3.23)$$

for all functions $u\in L_2(D_0\times(0,T_2))$ *with* $\partial_t u\in L_2(D_0\times(0,T_2))$, $u=0$ *on* $D_0\times\{0\}$.

Proof: Let $v=e^{-\frac{\tau}{2}t}u$. Obviously,

$$\int_{(0,T_2)}(\partial_t u)^2 e^{-\tau t}dt=\int_0^{T_2}(\partial_t v+\frac{\tau}{2}v)^2=\int_0^{T_2}((\partial_t v)^2+\tau v\partial_t v+\frac{\tau^2}{4}v^2)\geq$$

$$\int_0^{T_2}(\frac{\tau}{2}\partial_t(v^2)+\frac{\tau^2}{4}v^2)=\frac{\tau}{2}v^2(T)+\frac{\tau^2}{4}\int_0^{T_2}v^2.$$

Integrating with respect to x over D_0 and substituting back $v=e^{-\frac{\tau}{2}}u$ we complete the proof.

□

Lemma 3.3 *There is a constant* C *such that*

$$|\int_{\partial D_0} b_0 v^2|\leq C\int_{D_0}(\tau^{\frac{2}{3}}v^2+\tau^{-\frac{1}{3}}|\nabla v|^2)$$

for all functions $v\in H_{(1)}(D_0)$.

Proof.

Since D_1, D_2 are polyhedrons, D_0 is a Lipschitz domain. In Lipschitz domains we have interpolation and trace theorems for Sobolev spaces $H_{(s)}, 0 \leq s \leq 1$ [8], [15], I, Chapter 1. By interpolation theorems

$$\|v\|^2_{(\frac{2}{3})}(D_0) \leq C\|\tau^{-\frac{1}{6}}v\|^{\frac{4}{3}}_{(1)}(D_0)\|\tau^{\frac{2}{6}}v\|^{\frac{2}{3}}_{(0)}(D_0) \leq$$

$$C(\tau^{-\frac{1}{3}}\|v\|^2_{(1)}(D_0) + \tau^{\frac{2}{3}}\|v\|^2_{(0)}(D_0)),$$

due to the Holder inequality $AB \leq \frac{A^p}{p} + \frac{B^q}{q}$ with $p = \frac{3}{2}, q = 3$. Using trace theorems

$$\int_{\partial D_0} v^2 \leq C\|v\|^2_{(\frac{2}{3})}(D_0) \leq C(\tau^{-\frac{1}{3}}\|v\|^2_{(1)}(D_0) + \tau^{\frac{2}{3}}\|v\|^2_{(0)}(D_0)). \quad (3.24)$$

Since that $|b_0|$ is bounded by some C, from (3.24) we get statement of Lemma 3.3.

□

Proof of Theorem 2.2.

From Lemma 3.3 we conclude that

$$-C\int_{D_0\times\{t\}}(\tau^{\frac{2}{3}}u_1^2 + \tau^{-\frac{1}{3}}|\nabla u_1|^2) \leq -\int_{\partial D_0\times\{t\}} b_0u_1^2$$

and multiplying by $e^{-\tau t}$ and integrating with respect to t,

$$-C\int_{D_0\times(0,T-L)}(\tau^{\frac{2}{3}}u_1^2 + \tau^{-\frac{1}{3}}|\nabla u_1|^2)e^{-\tau t} \leq -\int_{\partial D_0\times(0,T-L)} b_0u_1^2e^{-\tau t}.$$

From these inequalities, Lemma 3.1, and Lemma 3.2 we yield

$$e^{-\tau(T-L)}(\int_{D_0\times\{T-L\}}(|\nabla u_1|^2 - C\tau^{-\frac{1}{3}}|\nabla u_1|^2 - C\tau^{\frac{2}{3}}u_1^2) + \frac{\tau^2}{2}\int_{D_0\times\{T-L\}} u_1^2)+$$

$$\frac{\tau^3}{4}\int_{D_0\times(0,T-L)} u_1^2e^{-\tau t} + \tau\int_{D_0\times(0,T-L)}(|\nabla u_1|^2 - C\tau^{-\frac{1}{3}}|\nabla u_1|^2 - C\tau^{\frac{2}{3}}u_1^2)e^{-\tau t} \leq 0.$$

Regrouping the terms in these integrals, we obtain

$$e^{-\tau(T-L)}\int_{D_0\times\{T-L\}}((1 - C\tau^{-\frac{1}{3}})|\nabla u_1|^2 + (\frac{\tau^2}{2} - C\tau^{\frac{2}{3}})u_1^2)+$$

$$\int_{D_0\times(0,T-L)}((\tau - C\tau^{\frac{2}{3}})|\nabla u_1|^2 + (\frac{\tau^3}{4} - C\tau^{\frac{5}{3}})u_1^2)e^{-\tau t} \leq 0.$$

Choosing τ to be large we conclude that $u_1 = 0$ on $D_0 \times (0, T-L)$. We can extend u_1 as zero onto $(\Omega \setminus D_1) \times (-\infty, 0)$ while preserving the homogeneous wave equation. By using condition (2.5) and the sharp uniqueness of the continuation results for the wave equation [8], section 3.4, we conclude that $u_1 = 0$ near some point of $\Gamma \times \{0\}$ which contradicts assumption (2.6) on boundary data g_1.

This completes the proof.

□

4 Uniqueness when $T = \infty$

By $U(\tau)$ we will denote the Laplace transformation

$$\int_{\mathbf{R}} u(t)e^{-\tau t}dt, \ \Re\tau > 0,$$

of a locally integrable function (of polynomial growth) which is zero on some interval $(-\infty, b)$.

Due to assumption (2.8) standard energy integrals guarantee (polynomial) boundedness of solution u to (2.1), (2.2), (2.3), (2.4) with respect to t. Hence we can apply the Laplace transformation and yield

$$\Delta U - \tau^2 U = 0 \text{ in } D_e, \Re\tau > 0, \tag{4.25}$$

with the boundary condition

$$\partial_\nu U - bU = 0 \text{ on } \partial D. \tag{4.26}$$

As above using uniqueness of the continuation results for the wave equation we conclude that for some neighborhood V of Γ the function U is uniquely determined on V.

Proof of Theorem 2.3.

As above let us assume that there are two different domains D_1, D_2 with the boundary coefficients b_1 and b_2 generating the same boundary data. Let U_1, U_2 be solutions of the problem (4.25), (4.26) with $D = D_1, D_2$ and with boundary coefficients b_1, b_2. We will use the notation of domains from the proof of Theorem 2.1. As observed above $U_1 = U_2$ on V and hence by the uniqueness of the continuation on D^∞. So

$$\partial_{\nu(2)} U_1 = \partial_{\nu(2)} U_2 = b_2 U_2 = b_2 U_1 \text{ on } \partial D_0 \cap \partial D_2.$$

Obviously,

$$\partial_{\nu(1)} U_1 = b_1 U_1 \text{ on } \partial D_0 \cap \partial D_1.$$

Hence letting $\nu = \nu(2)$ on $\partial D_0 \cap \partial D_2$, $\nu = -\nu(1)$ on $\partial D_0 \cap \partial D_1$ and $b_0 = b_2$ on $\partial D_0 \cap \partial D_2$, $b_0 = -b_1$ on $\partial D_0 \cap \partial D_1$ we yield

$$\partial_\nu U_1 = b_0 U_1 \text{ on } \partial D_0. \tag{4.27}$$

Regularity assumptions on the boundary data and known results on hyperbolic problems as above imply that

$$\partial_t u_1 \in L_\infty((0,\infty); H_{(1)}(\Omega \setminus \mathbf{D}_1)),\ \partial_t^2 u_1 \in L_\infty((0,\infty); L_2(\Omega \setminus \mathbf{D}_1)),$$

so writing $\partial_t^2 u_1$ in the wave equation (2.1) into the right side and using standard elliptic theory we conclude that $u_1 \in L_\infty((0,\infty); H_{(2)}(D_e \cap B)$ for a ball B containing D_1, D_2. Hence $U_1 \in H_{(2)}(D_0)$. Multiplying equality $(\Delta - \tau^2)U_1 = 0$ by $\bar{U}_1$ and integrating by parts we yield

$$0 = \int_{D_0} (\Delta - \tau^2) U_1 \bar{U}_1 = \int_{\partial D_0} \partial_\nu U_1 \bar{U}_1 - \int_{D_0} |\nabla U|_1^2 - \tau^2 \int_{D_0} |U_1|^2.$$

Using boundary condition (4.27) for U_1 we arrive at equality

$$0 = \int_{\partial D_0} b_0 |U_1|^2 - \int_{D_0} |\nabla U_1|^2 - \tau^2 \int_{D_0} |U_1|^2. \tag{4.28}$$

We will use (4.28) with complex τ^2, for example $\tau = 1 + i$. Taking imaginary part we conclude from (4.28) that

$$\int_{D_0} |U_1|^2 = 0$$

and hence $U_1 = 0$ on D_0. By uniqueness of the continuation $U_1 = 0$ on $\Omega \setminus \bar{D}_1$ [7], section 3.3. By uniqueness in the inverse Laplace transformation $u_1 = 0$ on $D_{1e} \times \mathbf{R}$. This contradicts condition (2.6).

This contradiction shows that $D_1 = D_2$. Now we will demonstrate uniqueness of boundary condition. Since $U_1 = U_2$ on D_e by subtracting two sets of the boundary conditions we yield $(b_2 - b_1)U_1 = 0$ on ∂D_1. As above, due to uniqueness in the Cauchy problem for elliptic equations and to the boundary condition for U_1 this function can not be zero on any open subset of ∂D_0. Hence $b_1 = b_2$ on ∂D_0.

$\Box$

5 Uniqueness in the inverse scattering problem

We will use some results of scattering theory for dissipative hyperbolic initial boundary value problems given by Lax and Phillips [17], [18] with further additions described in [20]. Indeed, as the Hilbert space H in [17] we use the completion of $C^\infty(\bar{D}_e)$-vector functions $\mathbf{u} = (u_1, u_2)$ with bounded supports with respect to the energy norm

$$\|\mathbf{u}\|_E^2 = \int_{D_e} (|\nabla u_1|^2 + |u_2|^2) + \int_{\partial D} b|u_1|^2$$

and the group $U(t)$ with the inifinitesimal generator

$$A = \begin{pmatrix} 0 & I \\ \Delta & 0 \end{pmatrix}$$

with the domain consisting of $(u_1, u_2) \in H$ such that $u_1 \in H_{(2)}(D_e), u_2 \in H_{(1)}(D_e)$ and $\partial_\nu u_1 - a_0 u_1 - b u_1 = 0$ on ∂D. Integrating by parts and using the boundary condition we yield

$$(A\mathbf{u}, \mathbf{u})_E = \int_{D_e} (\nabla u_2 \cdot \nabla u_1 + (\Delta u_1) u_2) + \int_{\partial D} b u_1 u_2 =$$

$$\int_{\partial D} (-\partial_\nu u_1 u_2 + b u_1 u_2) = \int_{\partial D} (-a_0 u_2^2) \le 0$$

and hence A is dissipative.

In [17] the abstract scattering theory is applied in case of scattering by obstacles with the boundary condition (2.4) where $b = 0$, however using the above remarks the argument in [17] is valid when $0 \le b$. As follows from section 10 in [17], the scattering amplitude $\mathcal{A}(\xi, \sigma, k)$ is well-defined and has a meromorphic continuation with respect to k in the complex plane. This continuation has no poles in the upper half-plane $\Im k > 0$ and no real poles $k \neq 0$. Moreover, for any ξ and k with $\Im k > 0$ there is a unique solution $U(,\xi,k)$ to the Helmholtz equation

$$\Delta U + k^2 U = 0 \text{ in } D_e, \Im k > 0, \tag{5.29}$$

with boundary condition

$$\partial_\nu U + (ika_0 - b)U = 0 \text{ on } \partial D \tag{5.30}$$

and

$$U(x) = e^{ikx\cdot\xi} + U_s(x) \tag{5.31}$$

where U_s satisfies the Sommerfeld radiation condition. As known the scattered solution has the form

$$U_s(x,\xi,k) = \frac{e^{ik|x|}}{|x|}(\mathcal{A}(\sigma,\xi,k) + O(|x|^{-1})). \tag{5.32}$$

Analytic dependence of the scattering solutions and hence of the scattering amplitude on $k, \Im k \geq 0, k \neq 0$, can be also shown by using integral equations of potential theory on ∂D [4], [19] combined with basic analytic Fredholm theory and with uniqueness of scattering solutions under the dissipativity condition (2.8).

Proof of Theorem 2.4.

As above let us assume that there are two different domains D_1, D_2 with the boundary coefficients a_{01}, b_1 and a_{02}, b_2 generating the same scattering data $\mathcal{A}_1, \mathcal{A}_2$. Let U_1, U_2 be solutions of the scattering problem (5.29), (5.30), (5.32) for $D = D_1, D_2$. We will use notation of domains from the proof of Theorem 2.1. We have $\mathcal{A}_1(,\xi,k) = \mathcal{A}_2(,\xi,k), \Im k \geq 0, k \neq 0$, by uniqueness of the analytic continuation from $k \in I$. Due to a generalization of the Rellich's Theorem given by Vekua [22], p.312, Theorem 4, $U_1 = U_2$ outside Ω and hence by uniqueness of the continuation on D^∞. So

$$\partial_{\nu(2)} U_1 = \partial_{\nu(1)} U_2 = (-ika_{02} + b_2)U_2 = (-ika_{02} + b_2)U_1 \text{ on } \partial D_0 \cap \partial D_2.$$

Obviously,

$$\partial_{\nu(1)} U_1 = (-ika_{01} + b_1)U_1 \text{ on } \partial D_0 \cap \partial D_1.$$

Hence letting $\nu = \nu(2)$ on $\partial D_0 \cap \partial D_2$, $\nu = -\nu(1)$ on $\partial D_0 \cap \partial D_1$ and $a = -a_{02}, b = b_2$ on $\partial D_0 \cap \partial D_2$, $a = a_{01}, b_0 = -b_1$ on $\partial D_0 \cap \partial D_1$ we yield

$$\partial_\nu U_1 = (ika + b_0)U_1 \text{ on } \partial D_0. \tag{5.33}$$

From the definition of a (generalized) solution $U_1 \in H_{(1)}(D_0)$ to the Helmholtz equation $(\Delta + k^2)U_1 = 0$ in D_0 we have

$$0 = \int_{\partial D_0} \partial_\nu U_1 \bar{U}_1 - \int_{D_0} |\nabla U|_1^2 + k^2 \int_{D_0} |U_1|^2.$$

Using the boundary condition (5.33) for U_1 we arrive at the equality

$$0 = \int_{\partial D_0} (ika + b_0)|U_1|^2 - \int_{D_0} |\nabla U_1|^2 + k^2 \int_{D_0} |U_1|^2. \tag{5.34}$$

First we will use scattering data with real nonzero k. Taking imaginary part in (5.34) and dividing by k we conclude that

$$0 = \int_{\partial D_0} a|U_1|^2.$$

Hence from (5.34) we yield

$$0 = \int_{\partial D_0} b_0|U_1|^2 - \int_{D_0} |\nabla U_1|^2 + k^2 \int_{D_0} |U_1|^2. \tag{5.35}$$

for any k with $\Im k \geq 0, k \neq 0$. Now we will use (6.42) with complex k^2, for example $k = 1 + i$. Taking imaginary part we conclude from (6.42) that

$$\int_{D_0} |U_1|^2 = 0$$

and hence $U_1 = 0$ on D_0 and by uniqueness of the continuatuon in $\mathbf{R}^3 \setminus \bar{D}_1$ [7], section 3.3. This contradicts behavior (5.31) of U_1 at infinity. We arrived at a contradiction.

This contradiction shows that $D_1 = D_2$. Now we will demonstrate uniqueness of boundary condition. Since $U_1 = U_2$ on D_e by subtracting two sets of the boundary conditions we yield $(ik(-a_{02} + a_{01}) + (b_2 - b_1))U_1 = 0$ on ∂D_1. As above, due to uniqueness in the Cauchy problem for elliptic equations and to the boundary condition for U_1 this function can not be zero on any open subset of ∂D_1. Hence $ik(-a_{02}+a_{01})+(b_2-b_1)=0$ and $a_{02}=a_{01}, b_1=b_2$ on ∂D_1.

□

Proof of Corollary 2.5

The scattered wave $u_s(\ ;\xi)=u(\ ;\xi)-u_i(\ ;\xi)$ solves the initial boundary value problem

$$\partial_t^2 u_s - \Delta u_s = 0 \text{ on } (\Omega \setminus \bar{D}) \times \mathbf{R},$$

$$u_s = 0 \text{ on } (\Omega \setminus \bar{D}) \times (-\infty, 0),$$

$$\partial_\nu u_s - a_0 \partial_t u_s - b u_s = -(\partial_\nu u_i - a_0 \partial_t u_i - b u_i) \text{ on } \partial D \times \mathbf{R},\ 0 \le a_0,$$

generated by the plane wave $u_i(\ ;\xi)$.

Since the incident wave is smooth and compactly supported, the standard energy integrals imply that the scattered wave is bounded and since $U(,\tau,\xi)$ is bounded on D_e from exponential decay at inifinity of the fundamental solution for the operator $\Delta-\tau^2$ as in [17], proof of Lemma 10.3, we conclude that

$$U_s(x,\tau;\xi) = \frac{e^{-\tau|x|}}{|x|}(\mathcal{A}^*(\sigma,\xi;\tau)+O(|x|^{-1})). \tag{5.36}$$

Obviously,

$$U_i(x,\tau,\xi) = \Phi_0(\tau)e^{-\tau x\cdot\xi},\ \text{with } \Phi_0(\tau) = \int_{\mathbf{R}} \varphi_0(\theta)e^{-\tau\theta}d\theta$$

Summing up we have

$$\Delta U(,\tau,\xi) - \tau^2 U(,\tau,\xi) = 0 \text{ on } D_e,$$

$$U(x,\tau,\xi) = \Phi_0(\tau)e^{-\tau x\cdot\xi} + U_s(x,\tau,\xi). \tag{5.37}$$

Combining (5.36), (5.37) with the standard definition of scattering amplitude we yield

$$\mathcal{A}^*(\sigma,\xi,-i\tau) = \mathcal{A}(\sigma,\xi,\tau). \tag{5.38}$$

Since $u(,\xi,\tau)$ is uniquely determined by the data of the inverse problem , so are $U(,\xi,\tau)$ and the first terms of its asymptotic expansion $\mathcal{A}^*(\sigma,\xi,\tau)$. Due to (5.38) the scattering amplitude $\mathcal{A}(\sigma,\xi;k)$, ξ fixed, is uniquely determined as well. Now uniqueness of D_e and of the boundary coefficients follows from Theorem 2.4.

6 Proof of uniqueness for the heat equation

As in the proof of Theorem 2.3 we assume the opposite and we will use the notation and the argument at the beginning of the proof of Theorem 2.3. Since the boundary data are constant on (T_1, T) we can extend them as contant in time onto (T_1, ∞). Since coefficients of the boundary conditions do not depend on t the solution of extended problem is analytic with respect to t. Due to uniqueness of analytic continuation, the Cauchy data for extended problem are uniquely determined on $\Gamma \times (0, \infty)$ by the data of initial inverse problem. So we can assume that $T = \infty$.

Due to assumption (2.8) maximum principles guarantee that solution u (of an extended problem) is bounded with respect to t. Hence we can apply the Laplace transform and to yield

$$\Delta U - \tau U = 0 \text{ in } D_e, \Re\tau > 0, \tag{6.39}$$

with the boundary condition

$$\partial_\nu U - bU = 0 \text{ on } \partial D. \tag{6.40}$$

As above using the uniqueness of the continuation results for the heat equation we conclude that for some neighborhood V of Γ the function U is uniquely determined on V.

Proof of Theorem 2.6.

As above let us assume that there are two different domains D_1, D_2 with boundary coefficients b_1 and b_2 generating the same Cauchy data on Γ. Let U_1, U_2 be solutions of the problem (6.39), (6.40) with $D = D_1, D_2$ and with boundary coefficients b_1, b_2. We will use the notation of domains from the proof of Theorem 2.1. As observed above $U_1 = U_2$ on V and hence by the uniqueness of the continuation on D^∞. So

$$\partial_{\nu(2)} U_1 = \partial_{\nu(2)} U_2 = b_2 U_2 = b_2 U_1 \text{ on } \partial D_0 \cap \partial D_2.$$

Obviously,

$$\partial_{\nu(1)} U_1 = b_1 U_1 \text{ on } \partial D_0 \cap \partial D_1.$$

Hence letting $\nu = \nu(2)$ on $\partial D_0 \cap \partial D_2$, $\nu = -\nu(1)$ on $\partial D_0 \cap \partial D_1$ and $b = b_2$ on $\partial D_0 \cap \partial D_2$, $b = -b_1$ on $\partial D_0 \cap \partial D_1$ we yield

$$\partial_\nu U_1 = bU_1 \text{ on } \partial D_0 \tag{6.41}$$

From the definition of a (generalized) solution $U_1 \in H_{(1)}(D_0)$ to equation $(\Delta - \tau)U_1 = 0$ in D_0 we have

$$0 = \int_{\partial D_0} \partial_\nu U_1 \bar{U}_1 - \int_{D_0} |\nabla U|_1^2 - \tau \int_{D_0} |U_1|^2.$$

Using the boundary condition (6.41) for U_1 we arrive at the equality

$$0 = \int_{\partial D_0} b|U_1|^2 - \int_{D_0} |\nabla U_1|^2 - \tau \int_{D_0} |U_1|^2, \Re\tau > 0. \quad (6.42)$$

Letting $\tau = 1 + i$ and taking imaginary part of (6.42) we conlude that $U_1 = 0$ on D_0. As in the proof of Theorem 2.3 we arrive at a contradiction. Hence $D_1 = D_2$. Uniqueness of b now follows as in the proof of Theorem 2.3.

This completes the proof.

□

7 Conclusion

Results of this paper are still far from complete. In particular, it is quite possible that in the situation of Theorem 2.1 one can assume more general boundary condition (2.4) and uniquely determine not only domain D but also boundary coefficients a_0, b. At present there is no ideas how to approach this problem. Similar questions arise for parabolic equations. The main difficulty lies in regularity of connected component D_0 which in general is not Lipschitz, so trace and embedding theorems for Sobolev spaces are not valid. That is why we assumed that $\partial D \in C^2$ which guarantees that solution u is in $H_{(2)}$, so we can use (2.1) in D_0. Otherwise we have only $H_{(1)}$-solutions which are defined globally in $(\Omega \setminus \bar{D}) \times (0, T)$ and there difficulties with boundary conditions on ∂D_0. If unknown domains are polyhedrons these difficulties disappear. Most likely Theorem 2.1 holds for mixed boundary conditions ((2.4) on a part of D and the Dirichlet condition on the complementary part, and one can uniquely determine these parts). Generalization of Theorems 2.1-2.4 onto hyperbolic and Helmholtz equations which coincide with the wave equation outside some large sphere are immediate. After applying the scheme of scattering theory [17], [18] to the mixed boundary data when on a part of D we have boundary condition (2.10) and on the remaining part $U = 0$ the proof of Theorem 2.3 (with minor modifications) gives uniqueness of D and of boundary conditions (including uniqueness of parts of the boundary with various types of boundary data). By using recent results on elliptic problems in Lipschitz domains combined with known scattering theory [19] one expects to have Theorem 2.3 for Lipschitz obstacles D. In our opinion, one can probably prove uniqueness of transparent obstacles from scattering data at a fixed incident direction. At present most general uniqueness results for transparent obstacles are obtained from the scattering data at fixed frequency and all directions of receivers and incident waves [7], [21] or from the Dirichlet-to-Neumann map [1] when $k = 0$. Some ideas from the proof of Theorem 2.3 are useful, but substantial modifications are needed. It is also feasible to get an analogue of Theorem 2.1 for any $T > 0$, but then one expects uniqueness of D only in the domain which can be reached from Γ in time $\frac{T}{2}$.

We expect that appropriate versions of Theorems 2.1,2.2,2.3 are valid for interesting positive symmetric systems of mathematical physics including Maxwell's and elasticity systems provided boundary conditions on ∂D are dissipative. Many technical tools for proofs are now available [5], [15], [17].

It is quite important to design range type numerical algorithms for identification of D and of boundary coefficients similar to linear sampling method [3] and probably using some ideas from proofs in sections 3,4,5. It is a realistic task, because all considered inverse problems are overdetermined.

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References

[1] Bal, G. *Reconstructions in impedance and optical tomography with singular interfaces*, Inverse Problems, **21**, 2005, 113-133.

[2] Belishev, M. I. *Boundary Control in Reconstruction of Manifolds and Metrics*, Inverse Problems, **13** (1997), R1-R45.

[3] Colton, D., Kirsch, A. *A simple method of solving inverse scattering problems in the resonance region*, Inverse Problems, **12**, 1996, 383-395.

[4] Colton, D., Kress, R. *Inverse Acoustic and Electromagnetic Scattering Theory*, Springer-Verlag, 1998.

[5] Eller, M., Isakov, V., Nakamura, G., Tataru, D. *Uniqueness and stability in the Cauchy problem for Maxwell's and elasticity systems*, College de France Seminar, 14. "Studies in Appl. Math.", **31**, North-Holland, 2002, 329-349.

[6] Hrycak, T., Isakov, V. *Increased stability in the continuation of solutions to the Helmholtz equation*, Inverse Problems, **20**, 2004, 697-712.

[7] Isakov, V. *On uniqueness in the inverse scattering problem*, Comm. Part. Diff. Equat., **15**, 1990, 1565-1581.

[8] Isakov, V. *Some inverse problems for the diffusion equation*, Inverse Problems, **15**, 1999, 3-11.

[9] Isakov, V. Inverse Problems for Partial Differential Equations Springer-Verlag, New York, 2005.

[10] Isakov, V. *Increased stability in the continuation for the Helmholtz equation with variable coefficient*, Contemp. Math., AMS (to appear).

[11] John, F. *Continuous dependence on data for solutions of partial differential equations with prescribed bound*, Comm. Pure Appl. Math., **13**, 1960, 551-585.

[12] Katchalov, A., Kurylev, Y., Lassas, M. Inverse boundary spectral problems, Chapman and Hall-CRC, 2000.

[13] Kirsch, A., Kress, R. *Uniqueness in Inverse Obstacle Scattering*, Inverse Problems, **9**, 1993, 285-299.

[14] Kress R., Rundell, W. *Inverse scattering for shape and impedance*, Inverse Problems, **17**, 2001, 1075–1085.

[15] Lions, J.-L. and Magenes, E. Non-homogeneous Boundary Value Problems and Applications, Springer-Verlag, Berlin, 1972.

[16] Lax, P., Phillips, R. Scattering theory, Academic Press, 1989.

[17] Lax, P., Phillips, R. *Scattering Theory for Dissipative Hyperbolic Systems*, J. Func. Anal., **14**, 1973, 172-235

[18] Lax, P., Phillips, R. *Scattering theory for the acoustic equation in an even number of space dimensions*, Indiana Univ. Math. J, **22**, 1972, 101-134.

[19] McLean, W. *Strongly Elliptic Systems and Boundary Integral Equations*, Cambridge Univ. Press, 2000.

[20] Petkov, V. *Scattering theory for hyperbolic operators*, North-Holland, 1989.

[21] Valdivia, N. *Uniqueness in Inverse Obstacle Scattering with Conductive Boundary Condition* Appl. Anal., **83**, 2004 , 825-853.

[22] Vekua, I.N. *New Methods for soliving Elliptic Equations*, North-Holland, 1967.

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