

Increased Stability of solutions to the Helmholtz Equation

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1. Introduction

Study of the Cauchy problem for Helmholtz equation is stimulated by the inverse scattering theory and more generally by remote sensing. Remote sensing is a general idea that involves obtaining information of a far away object using the reflected waves from the object. The waves used could be acoustical or electromagnetic waves. This particular physical phenomenon is modeled by the Helmholtz equation

$$(\Delta + k^2)u = f \quad \text{in } \Omega \quad (1)$$

The main goal of the inverse scattering theory is to obtain the properties of the object using the information provided by the reflected waves. The information in the reflected waves constitutes the Cauchy data for Helmholtz equation, which is given by

$$u = u_0 \quad \text{and} \quad \partial_\nu u = u_1 \quad \text{on } \Gamma \quad (2)$$

where Ω is a domain in \mathbf{R}^n , $n = 2, 3$ and Γ is a Lipschitz open part of the boundary. Finding the solution to the partial differential equation (1) with the given the given Cauchy data is known as the Cauchy problem. This paper addresses the stability issues associated with the solution to the Cauchy problem and explains the increased stability of the Cauchy problem for Helmholtz equation when the frequency increases, including a numerical example. The numerical algorithm used is based on spherical harmonics.

2. Experiment, Results, Discussion, and Significance

Theoretical Results:

In [1] a stability estimate for the solution u of (1) and (2) is obtained inside the domain $\Omega(d)$, which is as follows:

$$\|u\|(\Omega(d)) \leq C \left(F + \frac{M^{1-\lambda} F(k, d)^\lambda}{d^{2-2\lambda} k} \right) \quad (3)$$

where $\Omega(d)$ is the part of the domain Ω which is at a distance d from the obstacle. One important observation is that the stability and hence the resolution of the Cauchy problem increases as the frequency k increases. In [2] a stability estimate for the solution u of (1) and (2) is obtained inside the whole domain Ω which is as follows:

$$\|u\|(\Omega) \leq CM^2 \left(\delta^2 + \delta_k^2 e^{-\frac{\sqrt{2}\delta^4}{\sqrt{h}}} + \delta_k^{-\frac{1}{2}} \right) \quad (4)$$

where $\delta_k = (-\ln \delta)(\ln k)$ and $\delta = \frac{F}{M}$. Again one can observe that the stability is increasing as the frequency k increases.

Numerical Experiment and Results

The physical setting for the numerical experiment is as follows:

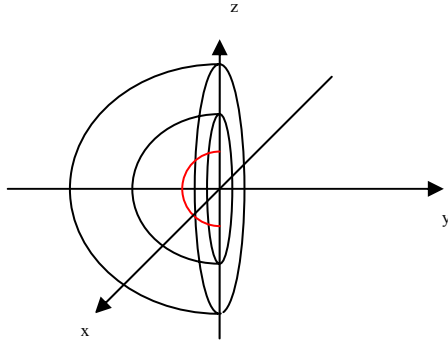


Figure 1. Description of the experimental setting.

The outer semi sphere (Γ) is of radius 2 and the inner semi sphere (Γ_0) is of radius 1. The sensors are placed on Γ , about 100 equally spaced on Γ . The sources (Acoustical) are placed on the semi circle (red colored in Fig 1) of radius 0.5. Now using the noisy measurements (noise is 1%) by the sensors we will calculate the acoustical field on the surface Γ_0 using the spherical harmonics as explained in [3]. The noisy calculations are made using the fundamental solution of the Helmholtz equation:

$$u(x, y) = \frac{e^{ik|x-y|}}{4\pi |x-y|} \quad (5)$$

The solution is reconstructed on Γ_0 using the following equation.

$$u(x) = \sum_{n=0}^N \sum_{m=-n}^n u_{n,m} h_n^{(1)} Y_n^m(\theta, \varphi) \quad (6)$$

Table: 1

Error percentages for the reconstructed solutions at various frequencies:

	k=2	k=4	k=8	k=16
N=9	245%	77.8%	3.4%	1.6%
N=10	3917%	288%	6.4%	0.98%

3. Conclusions

One important conclusion is that the stability and hence the resolution of the Cauchy problem increases as the frequency k increases. It is also stable up to the boundary of the object.

4. Acknowledgements

[1] Hrycak T. and Isakov V., "Increased stability in the continuation of solutions to the Helmholtz equation," Inverse Problems, volume (20), (2004): 697-712.

[2] Aralumallige D., "Increased Stability of solutions to the Helmholtz Equation", Thesis (2005), Wichita State University.

[3] Colton D. and Kress R., "Inverse Acoustic and Electromagnetic Scattering Theory", Springer 1998.