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# Uniqueness and Stability of Determining the Residual Stress by One Measurement

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*In this paper we prove a Hölder and Lipschitz stability estimates of determining the residual stress by a single pair of observations from a part of the lateral boundary or from the whole boundary. These estimates imply first uniqueness results for determination of residual stress from few boundary measurements.*

**Keywords** Continuation of solutions; Elasticity theory; Inverse problems.

**Mathematics Subject Classification** 35R30; 74B10; 35B60.

## 1. Introduction

We consider an elasticity system with residual stress. Let $\Omega$ be an open bounded domain in $\mathbb{R}^3$ with smooth boundary $\partial\Omega$. The residual stress is modelled by a symmetric second-rank tensor $R(x) = (r_{jk}(x))_{j,k=1}^3 \in C^7(\overline{\Omega})$ which is divergence free

$$\nabla \cdot R = 0 \quad \text{in } \Omega \tag{1.1}$$

and satisfies the boundary condition

$$R\nu = 0 \quad \text{on } \partial\Omega, \tag{1.2}$$

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where $\nabla \cdot R$ is a vector-valued function with components given by

$$(\nabla \cdot R)_j = \sum_{k=1}^{3} \partial_k r_{jk}, \quad 1 \leq j \leq 3.$$

In this paper $x \in \mathbf{R}^3$ and $\nu = (\nu_1, \nu_2, \nu_3)^\top$ is the unit outer normal vector to $\partial\Omega$. Here and below, differential operators $\nabla$ and $\Delta$ without subscript are with respect to $x$ variables. Let $\mathbf{u}(x, t) = (u_1, u_2, u_3)^\top : Q \to \mathbb{R}^3$ be the displacement vector in $Q := \Omega \times (-T, T)$. We assume that $\mathbf{u}(x, t)$ solves the initial boundary value problem:

$$\mathbf{A}_R\mathbf{u} := \rho\partial_t^2\mathbf{u} - (\lambda + \mu)\nabla(\nabla \cdot \mathbf{u}) - \mu\Delta\mathbf{u} - \nabla \cdot ((\nabla\mathbf{u})R) = 0 \quad \text{in } Q, \tag{1.3}$$

$$\mathbf{u} = \mathbf{u}_0, \quad \partial_t\mathbf{u} = \mathbf{u}_1 \quad \text{on } \Omega \times \{0\}, \tag{1.4}$$

$$\mathbf{u} = \mathbf{g}_0 \quad \text{on } \partial\Omega \times (-T, T), \tag{1.5}$$

where $\rho$ is density and $\lambda$ and $\mu$ are Lamé constants satisfying

$$0 < \mu, \quad 0 < \rho, \quad 0 < \lambda + \mu. \tag{1.6}$$

The system (1.3) can be written as

$$\rho\partial_t^2\mathbf{u} - \nabla \cdot \sigma(\mathbf{u}) = 0,$$

where $\sigma(\mathbf{u}) = \lambda(\operatorname{tr}\epsilon)I + 2\mu\epsilon + R + (\nabla\mathbf{u})R$ is the stress tensor and $\epsilon = (\nabla\mathbf{u} + \nabla\mathbf{u}^\top)/2$ is the strain tensor. Note that the term $\nabla \cdot R$ does not appear in (1.3) due to (1.1). Also, by the same condition, we can see that

$$(\nabla \cdot ((\nabla\mathbf{u})R))_i = \sum_{j,k=1}^{3} r_{jk}\partial_j\partial_k u_i, \quad 1 \leq i \leq 3.$$

Since we are only concerned with the residual stress and we are motivated by applications to the material science we suppose that density $\rho$ and Lamé coefficients $\lambda$ and $\mu$ are constants. To make sure that the problem (1.3) with (1.4), (1.5) is well-posed, we assume that

$$\|R\|_{C^1(\overline{\Omega})} < \varepsilon_0 \tag{1.7}$$

for some small constant $\varepsilon_0 > 0$. The assumption (1.7) is also physically motivated Man (1998). It is not hard to see that if $\varepsilon_0$ is sufficiently small, then the boundary value problem (1.3), (1.4), (1.5) is hyperbolic, and hence for any initial data $(\mathbf{u}_0, \mathbf{u}_1) \in H^1(\Omega) \times L^2(\Omega)$ and lateral Dirichlet data $\mathbf{g}_0 \in C^1([-T, T]; H^1(\Omega))$, $\mathbf{u}_0 = \mathbf{g}_0$ on $\partial\Omega \times \{0\}$, there exists a unique solution $\mathbf{u}(\cdot; R; (\mathbf{u}_0, \mathbf{u}_1, \mathbf{g}_0)) \in C([-T, T]; H^1(\Omega))$ to (1.3)–(1.5).

In this paper we are interested in the following inverse problem:

Determine the residual stress $R$ by a single pair of Cauchy data $(\mathbf{u}, \sigma(\mathbf{u})\nu)$ on $\Gamma \times (-T, T)$, where $\mathbf{u} = \mathbf{u}(\cdot; R; (\mathbf{u}_0, \mathbf{u}_1, \mathbf{g}_0))$ and $\Gamma \subset \partial\Omega$.

We will address uniqueness and stability issues. The focus is on the stability since the uniqueness follows immediately from it. Our method is based on Carleman estimates techniques initiated by Bukhgeim and Klibanov (1981). For works on

Carleman estimates and related inverse problems for scalar equations, we refer to books Bukhgeim (2000) and Klibanov and Timonov (2004) for further details and references. Here we only want to mention some related results for the dynamical Lamé system and the residual stress system (1.3). For the Lamé system, the first step has been made by Isakov (1986) where he proved the Carleman estimate and established the uniqueness for the inverse source problem. It should be noted that Isakov (1986) transformed the principal part of the system into a composition of two scalar wave operators. It is well-known that the Lamé system is principally diagonalized as a system of equations for $\mathbf{u}$ and $div\,\mathbf{u}$. Based on this fact, $L^2$-Carleman estimates were derived in Eller et al. (2002) and Ikehata et al. (1998) for the Lamé system and applications of to the Cauchy problem and the inverse problem were given. Recently, Imanuvilov et al. (2003) obtained a Carleman estimate for the Lamé system by considering a new principally diagonalized system for $(\mathbf{u}, div\,\mathbf{u}, curl\,\mathbf{u})$. In Imanuvilov et al. (2003), they used this estimate to study the problem of identifying the density and Lamé coefficients by two sets of data measured in a boundary layer and a Hölder-type stability estimate. The continuation of this work is in Imanuvilov and Yamamoto (2005).

For the dynamical residual stress model (1.3), an $L^2$-Carleman estimate has been proved when the residual stress is small (Isakov et al., 2003). The system with the residual stress is no longer isotropic. In other words, this system is strongly coupled, and it is not possible to decouple the leading part without increasing the order of the system. There are almost no results on Carleman estimates and inverse problems for anisotropic systems which are very important in applications. In Isakov et al. (2003), we used the standard substitution $(\mathbf{u}, div\,\mathbf{u}, curl\,\mathbf{u})$ and reduced (1.3) to a new system where the leading part is a special lower triangular matrix differential operator with the wave operators in the diagonal. The key point is that the coupled terms in the leading part contain only the second derivatives of $\mathbf{u}$ with respect to $x$ variables and they can be absorbed by $div\,\mathbf{u}, curl\,\mathbf{u}$ when the residual stress is small. Using similar Carleman estimates, Lin and Wang (2003) studied the problem of uniquely determining the density function by a single set of boundary data. The unique continuation property for the stationary case of (1.3) was proved in Nakamura and Wang (2003).

In this work we study the problem of recovering the (small) residual stress in(1.3)–(1.5) by single set of Cauchy data. We will derive a Hölder stability estimate in convex hull of the observation surface $\Gamma$ and a Lipschitz stability estimate for $R$ in $\Omega$ when $\Gamma = \partial\Omega$ and observation time $T$ is large. There are other results concerning the determination of the residual stress by infinitely many boundary measurements, i.e. by the Dirichlet-to-Neumann map, we refer to Hansen and Uhlmann (2003), Rachele (2003), and Robertson (1977, 1988).

We are now ready to state the main results of the paper. Denote $d = \inf|x|$ and $D = \sup|x|$ over $x \in \Omega$. We assume that

$$0 < d. \tag{1.8}$$

Let $\mathcal{R}(\varepsilon_0, E)$ be the class of residual stresses defined by

$$\mathcal{R}(\varepsilon_0, E) = \{\|R\|_{C^6(\overline{\Omega})} < E : R \text{ is symmetric and satisfies (1.1), (1.2), and (1.7)}\}.$$

To study the inverse problem, we need not only the well-posedness of (1.3)–(1.5) but also some extra regularity of the solution $\mathbf{u}$. To achieve the latter property, the initial data $(\mathbf{u}_0, \mathbf{u}_1)$ and the Dirichlet data $\mathbf{g}_0$ are required to satisfy some smoothness and compatibility conditions). More precisely we will assume that $\mathbf{u}_0 \in H^9(\Omega), \mathbf{u}_1 \in H^8(\Omega)$ and $\mathbf{g}_0 \in C^8([-T, T]; H^1(\partial\Omega)) \cap C^5([-T, T]; H^4(\partial\Omega))$ and they satisfy standard compatibility assumptions of order 8 at $\partial\Omega \times \{0\}$. By using energy estimates (Duvaut and Lions, 1976) and Sobolev embedding theorems as in Imanuvilov et al. (2003) one can show that

$$\|\partial_x^\alpha \partial_t^\beta \mathbf{u}\|_{C^0(\overline{\Omega})} \leq C \tag{1.9}$$

for $|\alpha| \leq 2$ and $0 \leq \beta \leq 5$.

By examining the equation (1.3), we can see that the residual stress tensor appears in the equation without first derivatives because of (1.1). It turns out that a single set of Cauchy data is sufficient to recover the residual stress. To guarantee the uniqueness, we impose some non-degeneracy condition on the initial data $(\mathbf{u}_0, \mathbf{u}_1)$. More precisely, we assume that

$$\det \mathbf{M} = \det \begin{pmatrix} \partial_1^2 \mathbf{u}_0 & 2\partial_1\partial_2\mathbf{u}_0 & 2\partial_1\partial_3\mathbf{u}_0 & \partial_2^2\mathbf{u}_0 & 2\partial_2\partial_3\mathbf{u}_0 & \partial_3^2\mathbf{u}_0 \\ \partial_1^2 \mathbf{u}_1 & 2\partial_1\partial_2\mathbf{u}_1 & 2\partial_1\partial_3\mathbf{u}_1 & \partial_2^2\mathbf{u}_1 & 2\partial_2\partial_3\mathbf{u}_1 & \partial_3^2\mathbf{u}_1 \end{pmatrix} > E^{-1} \quad \text{on } \overline{\Omega}. \tag{1.10}$$

Note that $\mathbf{M}(x)$ is a $6 \times 6$ matrix-valued function. For example, one can check that $\mathbf{u}_0(x) = (x_1^2, x_2^2, x_3^2)^\top$ and $\mathbf{u}_1(x) = (x_2x_3, x_1x_3, x_1x_2)^\top$ satisfy (1.10).

We will use the following notation: $C, \gamma$ are generic constants depending only on $\Omega, T, \rho, \lambda, \mu, \varepsilon_0, E, \mathbf{u}_0, \mathbf{u}_1, \mathbf{g}_0$, any other dependence is indicated, $\|\cdot\|_{(k)}(Q)$ is the norm in the Sobolev space $H^k(Q)$. $Q(\varepsilon) = Q \cap \{\varepsilon < |x|^2 - \theta^2 t^2 - d_1^2\}$ and $\Omega(\varepsilon) = \Omega \cap \{\varepsilon < |x|^2 - d_1^2\}$. Here $d_1$ is some positive constant. $\mathbf{u}(\,;1)$ and $\mathbf{u}(\,;2)$ denote solutions of the initial boundary value problems (1.3), (1.4) associated with $R(\,;1)$ and $R(\,;2)$. Finally, we introduce the norm of the differences of the data

$$F = \sum_{\beta=2}^{4} (\|\partial_t^\beta(\mathbf{u}(\,;2) - \mathbf{u}(\,;1))\|_{(\frac{5}{2})}(\Gamma \times (-T, T)) + \|\partial_t^\beta \sigma_\nu(\mathbf{u}(\,;2) - \mathbf{u}(\,;1))\|_{(\frac{3}{2})}(\Gamma \times (-T, T)))$$

Due to (1.6) we can choose positive $\theta$ so that

$$\theta^2 < \frac{\mu}{\rho}, \quad \theta^4 < \frac{\mu}{\rho}\frac{d^2}{T^2}. \tag{1.11}$$

**Theorem 1.1.** *Assume that the domain $\Omega$ satisfies* (1.8), *$\theta$ satisfies* (1.11), *and for some $d_1$*

$$|x|^2 - d_1^2 < 0 \quad \textit{when } x \in (\partial\Omega \backslash \Gamma), \quad \textit{and} \quad D^2 - \theta^2 T^2 - d_1^2 < 0. \tag{1.12}$$

*Let the initial data $(\mathbf{u}_0, \mathbf{u}_1)$ satisfy* (1.10).

*Then there exist $\varepsilon_0$ and constants $C, \gamma < 1$, depending on $\varepsilon$, such that for $R(\,;1), R(\,;2) \in \mathcal{R}(\varepsilon_0, E)$ one has*

$$\|R(\,;2) - R(\,;1)\|_{(0)}(\Omega(\varepsilon)) \leq CF^\gamma. \tag{1.13}$$

The domain $\Omega(\varepsilon)$ is discussed in Isakov (2006, Sec. 3.4).

If $\Gamma$ is the whole lateral boundary and $T$ is sufficiently large, then a much stronger (and in a certain sense best possible) Lipschitz stability estimate holds.

**Theorem 1.2.** *Let $d_1 = d$. Assume that the domain $\Omega$ satisfies* (1.8),

$$D^2 < 2d^2, \tag{1.14}$$

*and*

$$\frac{D^2 - d^2}{\theta^2} < T^2. \tag{1.15}$$

*Let the initial data $(\mathbf{u}_0, \mathbf{u}_1)$ satisfy* (1.10). *Let $\Gamma = \partial\Omega$.*

*Then there exist an $\varepsilon_0$ and $C$ such that for $R(\,;1), R(\,;2) \in \mathcal{R}(\varepsilon_0, E)$ satisfying the condition*

$$R(\,;1) = R(\,;2) \quad on\ \Gamma \times (-T, T), \tag{1.16}$$

*one has*

$$\|R(\,;2) - R(\,;1)\|_{(0)}(\Omega) \leq CF \tag{1.17}$$

Let us show compatibility of conditions (1.15) and (1.11). From conditions (1.11) and (1.14) we have

$$\frac{D^2 - d^2}{\theta^2} < \frac{\mu}{\rho}\frac{d^2}{\theta^4}$$

and hence we can find $T^2$ between these two numbers.

As mentioned previously, the proofs of these theorems rely on Carleman estimates. Using the results of Isakov et al. (2003) we will derive needed Carleman estimates in Section 2. Using this estimate we will prove in Section 3 the Hölder stability estimate (1.13). In Section 4, we demonstrate the Lipschitz stability of the Cauchy problem for the residual stress model. This estimate is one of key ingredients to derive the Lipschitz stability estimate for our inverse problem in Section 5.

## 2. Carleman Estimate

In this section we will describe Carleman estimates needed to solve our inverse problem. Their proofs can be found in Isakov et al. (2003). Let $\psi(x,t) = |x|^2 - \theta^2 t^2 - d_1^2$ and $\varphi(x,t) = \exp(\frac{\eta}{2}\psi(x,t))$, where $\theta$ is choosen in (1.11) and $\eta < C$ is a large constant to be fixed later.

**Theorem 2.1.** *There are constants $\varepsilon_0$ and $C$ such that for $R$ satisfying* (1.7)

$$\int_Q (\tau|\nabla_{x,t}\mathbf{u}|^2 + \tau|\nabla_{x,t}v|^2 + \tau|\nabla_{x,t}\mathbf{w}|^2 + \tau^3|\mathbf{u}|^2 + \tau^3|v|^2 + \tau^3|\mathbf{w}|^2)e^{2\tau\varphi}$$
$$\leq C\int_Q (|\mathbf{A}_R\mathbf{u}|^2 + |\nabla(\mathbf{A}_R\mathbf{u})|^2)e^{2\tau\varphi} \tag{2.1}$$

*for all* $\mathbf{u} \in H_0^3(Q)$ *and*

$$\int_Q (\tau^2|\mathbf{u}|^2 + |div\,\mathbf{u}|^2 + |curl\,\mathbf{u}|^2 + \tau^{-1}|\nabla\mathbf{u})e^{2\tau\varphi} \leq C\int_Q |\mathbf{A}_R\mathbf{u}|^2 e^{2\tau\varphi} \tag{2.2}$$

*for all* $\mathbf{u} \in H_0^2(Q)$.

Carleman estimates of Theorem 2.1 is our basic tool for treating the inverse problem.

**Lemma 2.2.** *For* $u \in H_0^2(\Omega)$ *satisfying* $\Delta u = f_0 + \sum_{j=1}^3 \partial_j f_j$ *with* $f_0, f_j$ $(1 \leq j \leq 3)$ *belonging to* $L^2$, *we have*

$$\frac{1}{\tau}\int_\Omega |\nabla u|^2 e^{2\tau\varphi}dx \leq C\int_\Omega \left(\frac{f_0^2}{\tau^2} + \sum f_j^2\right)e^{2\tau\varphi}dx.$$

*Proof.* For the weight function $\varphi$ with large $\eta$, we can use Theorem 3.1 of Fabre and Lebeaux (1996) to get

$$\int_\Omega \tau|u|^2 e^{2\tau\varphi}dx \leq C\int_\Omega \left(\frac{f_0^2}{\tau^2} + \sum f_j^2\right)e^{2\tau\varphi}dx. \tag{2.3}$$

Now we will bound $\nabla u$. Observe that $\nabla u e^{\tau\varphi} = \nabla(ue^{\tau\varphi} - \tau u\nabla\varphi e^{\tau\varphi})$ and hence

$$\frac{1}{\tau}\int_\Omega |\nabla u|^2 e^{2\tau\varphi}dx \leq \frac{2}{\tau}\int_\Omega |\nabla(ue^{2\tau\varphi})|^2 dx + C\tau\int_\Omega |u|^2 e^{2\tau\varphi}dx \tag{2.4}$$

We have

$$\begin{aligned}\Delta(ue^{\tau\varphi}) &= (\Delta u)e^{\tau\varphi} + 2\tau\nabla u\cdot(\nabla\varphi)e^{\tau\varphi} + (\tau^2|\nabla\varphi|^2 + \tau\Delta\varphi)ue^{\tau\varphi}\\ &= \left(f_0 + \sum\partial_j f_j\right)e^{\tau\varphi} + 2\tau\nabla u\cdot(\nabla\varphi)e^{\tau\varphi} + (\tau^2|\nabla\varphi|^2 + \tau\Delta\varphi)ue^{\tau\varphi}.\end{aligned}$$

Multiplying this equality by $-\frac{1}{\tau}ue^{\tau\varphi}$, integrating by parts, and using the Cauchy-Schwarz inequality, we obtain

$$\begin{aligned}&\frac{1}{\tau}\int_\Omega |\nabla(ue^{2\tau\varphi})|^2 dx\\ &= -\int_\Omega \frac{1}{\tau}\left(f_0 + \sum\partial_j f_j\right)ue^{2\tau\varphi}dx - 2\int_\Omega (\nabla u\cdot\nabla\varphi)ue^{2\tau\varphi}dx - \int_\Omega (\tau|\nabla\varphi|^2 + \Delta\varphi)u^2 e^{2\tau\varphi}dx\\ &\leq \frac{1}{2}\int_\Omega \frac{f_0^2}{\tau^2}e^{2\tau\varphi}dx + \frac{1}{2}\int_\Omega |u|^2 e^{2\tau\varphi}dx + \frac{1}{\tau}\left|\int_\Omega \sum f_j(\partial_j u + 2\tau u\partial_j\varphi)e^{2\tau\varphi}dx\right|\\ &\quad + 2\int_\Omega \nabla u\cdot\nabla\varphi ue^{2\tau\varphi}dx + C\tau\int_\Omega u^2 e^{2\tau\varphi}dx\\ &\leq \frac{1}{2}\int_\Omega \frac{f_0^2}{\tau^2}e^{2\tau\varphi}dx + \frac{1}{2}\int_\Omega |u|^2 e^{2\tau\varphi}dx + \frac{1}{\tau\delta}\int_\Omega \sum f_j^2 e^{2\tau\varphi}dx\\ &\quad + \frac{\delta}{\tau}\int_\Omega |\nabla u|^2 e^{2\tau\varphi}dx\frac{C\tau}{\delta}\int_\Omega |u|^2 e^{2\tau\varphi}dx,\end{aligned}$$

where $\delta > 0$ is arbitrary and we used that $|ab| \le \delta|a|^2 + \frac{1}{4\delta}|b|^2$. Choosing sufficiently small $\delta > \frac{1}{C}$ to absorb the term with $\nabla u$ in the right side by the left side and using (2.3), (2.4) we yield the bound of Lemma 2.2. □

Now we are ready to prove Theorem 2.1.

*Proof of Theorem* 2.1. We consider (1.3) with a source term,

$$\mathbf{A}_R \mathbf{u} = \mathbf{f} \quad \text{in } Q. \tag{2.5}$$

Using that $\rho, \lambda, \mu$ are constants we have from Isakov et al. (2003), Section 2, the new system of equations

$$\begin{cases} P_1 \mathbf{u} = \dfrac{\lambda + \mu}{\rho} \nabla v + \dfrac{\mathbf{f}}{\rho}, \\ P_2 v = \displaystyle\sum_{j,k=1}^{3} \nabla \frac{r_{jk}}{\rho} \cdot \partial_j \partial_k \mathbf{u} + div\, \frac{\mathbf{f}}{\rho}, \\ P_1 \mathbf{w} = \displaystyle\sum_{j,k=1}^{3} \nabla \frac{r_{jk}}{\rho} \times \partial_j \partial_k \mathbf{u} + curl\, \frac{\mathbf{f}}{\rho}, \end{cases} \tag{2.6}$$

where $P_1 = \partial_t^2 - \sum_{j,k=1}^{3} \rho^{-1}(\mu\delta_{jk} + r_{jk})\partial_j\partial_k$, $P_2 = \partial_t^2 - \sum_{j,k=1}^{3} \rho^{-1}((\lambda + 2\mu)\delta_{jk} + r_{jk})\partial_j\partial_k$, where $\delta_{jk}$ is the Kronecker delta. Due to (1.6) and (1.7) with small $\varepsilon_0$, $P_1$ and $P_2$ are hyperbolic operators. Using (1.6), (1.11) by standard calculations one can show that $\varphi$ is strongly pseudo-convex in $\overline{Q}$ provided $\eta < C$ is sufficiently large and $\varepsilon_0$ is sufficiently small (see Isakov et al., 2003, or Isakov, 2006). Observe that according to the first condition (1.11) the function $\psi(x, t) = |x|^2 - \theta^2 t^2 - d^2$ is pseudo-convex on $\overline{Q}$ and according to the second condition (1.11) the gradient of $\psi$ is non-characteristic on $\overline{Q}$ with respect to operators $\rho\partial_t^2 - \mu\Delta$, $\rho\partial_t^2 - (\lambda + 2\mu)\Delta$, and hence with respect to $P_1, P_2$ provided $\varepsilon_0$ is sufficiently small. We fix such $\eta$ observing that it depends only on $Q, \rho, \lambda, \mu$ and $\theta$. It follows from Theorem 3.1 in Isakov et al. (2003) that there exists a constant $C > 0$ such that for all $\tau > C$ we have

$$\begin{aligned} &\int_Q (\tau|\nabla_{x,t}\mathbf{u}|^2 + \tau|\nabla_{x,t} v|^2 + \tau|\nabla_{x,t}\mathbf{w}|^2 + \tau^3|\mathbf{u}|^2 + \tau^3|v|^2 + \tau^3|\mathbf{w}|^2)e^{2\tau\varphi} \\ &\quad \le C\int_Q (|\mathbf{f}|^2 + |\nabla \mathbf{f}|^2)e^{2\tau\varphi} + C\varepsilon_0 \int_Q \sum_{j,k=1}^{3} |\partial_j\partial_k \mathbf{u}|^2 e^{2\tau\varphi} \end{aligned} \tag{2.7}$$

for all $\mathbf{u} \in H_0^3(Q)$. As well known, $\Delta\mathbf{u} = \nabla v - curl\,\mathbf{w}$. Therefore, by Theorem 3.2 in Isakov et al. (2003) and by (2.7),

$$\begin{aligned} \int_Q \sum_{j,k=1}^{3} |\partial_j\partial_k \mathbf{u}|^2 e^{2\tau\varphi} &\le C\int_Q \tau|\Delta\mathbf{u}|^2 e^{2\tau\varphi} \\ &= C\int_Q \tau|\nabla v - curl\,\mathbf{w}|^2 e^{2\tau\varphi} \\ &\le C\int_Q (|\mathbf{f}|^2 + |\nabla\mathbf{f}|^2)e^{2\tau\varphi} + C\varepsilon_0 \int_Q \sum_{j,k=1}^{3} |\partial_j\partial_k \mathbf{u}|^2 e^{2\tau\varphi}. \end{aligned}$$

Thus for small $\varepsilon_0$, we yield

$$\int_Q \sum_{j,k=1}^{3} |\partial_j \partial_k \mathbf{u}|^2 e^{2\tau\varphi} \leq C \int_Q (|\mathbf{f}|^2 + |\nabla \mathbf{f}|^2) e^{2\tau\varphi}.$$

and the estimate (2.7) leads to the first Carleman estimate (2.1).

To prove the second estimate we will use Carleman estimates for elliptic and hyperbolic operators in Sobolev norms of negative order. Applying Theorem 3.2 in Imanuvilov et al. (2003) to each of scalar hyperbolic operators in (2.6) we obtain

$$\tau \int_Q |\mathbf{u}|^2 e^{2\tau\varphi} \leq C \int_Q (v^2 + \tau^{-2} |\mathbf{f}|^2) e^{2\tau\varphi},$$
$$\tau \int_Q v^2 e^{2\tau\varphi} \leq C \int_Q (\varepsilon_0 |\nabla \mathbf{u}|^2 + |\mathbf{f}|^2) e^{2\tau\varphi},$$
$$\tau \int_Q |\mathbf{w}|^2 e^{2\tau\varphi} \leq C \int_Q (\varepsilon_0 |\nabla \mathbf{u}|^2 + |\mathbf{f}|^2) e^{2\tau\varphi}.$$

Adding these inequalities we arrive at

$$\tau \int_Q (|\mathbf{u}|^2 + |v|^2 + |\mathbf{w}|^2) e^{2\tau\varphi} \leq C \int_Q (\varepsilon_0 |\nabla \mathbf{u}|^2 + |\mathbf{f}|^2) e^{2\tau\varphi}. \tag{2.8}$$

To eliminate the first term in the right side we use again the known identity $\Delta \mathbf{u} = \nabla v - \mathit{curl}\, \mathbf{w}$, apply Lemma 2.2, and integrate with respect to $t$ over $(-T, T)$ to get

$$\int_Q |\nabla \mathbf{u}|^2 e^{2\tau\varphi} \leq C\tau \int_Q (v^2 + |\mathbf{w}|^2) e^{2\tau\varphi}.$$

Using this estimate and choosing $\varepsilon_0$ small and $\tau > C$ we complete the proof of (2.2). □

In order to use (2.1), it is required that the Cauchy data of the solution and the source term vanish on the lateral boundary. To handle non-vanishing Cauchy data, the following lemma is useful.

**Lemma 2.3.** *For any pair of* $(\mathbf{g}_0, \mathbf{g}_1) \in H^{\frac{5}{2}}(\Gamma \times (-T, T)) \times H^{\frac{3}{2}}(\Gamma \times (-T, T))$, *we can find a vector-valued function* $\mathbf{u}^* \in H^3(Q)$ *such that*

$$\mathbf{u}^* = \mathbf{g}_0, \quad \sigma_\nu(\mathbf{u}^*) = \mathbf{g}_1, \quad \mathbf{A}_R \mathbf{u}^* = 0 \quad on\ \Gamma \times (-T, T),$$

*and*

$$\|\mathbf{u}^*\|_3(Q) \leq C(\|\mathbf{g}_0\|_{(\frac{5}{2})}(\Gamma \times (-T, T)) + \|\mathbf{g}_1\|_{(\frac{3}{2})}(\Gamma \times (-T, T))) \tag{2.9}$$

*for some* $C > 0$ *provided* $\varepsilon_0$ *in* (1.7) *is sufficiently small.*

*Proof.* By standard extensions theorems for any $\mathbf{g}_2 \in H_{(\frac{1}{2})}(\Gamma \times (-T, T)$ we can find $\mathbf{u}^{**} \in H^3(Q)$ so that

$$\mathbf{u}^{**} = \mathbf{g}_0, \quad \sigma_\nu(\mathbf{u}^{**}) = \mathbf{g}_1, \quad \partial_\nu^2 \mathbf{u}^{**} = \mathbf{g}_2 \quad \text{on } \Gamma \times (-T, T)$$

and

$$\|\mathbf{u}^{**}\|_{(3)}(Q) \leq C(\|\mathbf{g}_2\|_{(\frac{1}{2})}(\Gamma \times (-T,T)) + \|\mathbf{g}_1\|_{(\frac{3}{2})}(\Gamma \times (-T,T)) + \|\mathbf{g}_0\|_{(\frac{5}{2})}(\Gamma \times (-T,T))).$$

Since $\Gamma \times (-T,T)$ is non-characteristic with respect to $\mathbf{A}_R$ provided $\varepsilon_0$ is small, the condition $\mathbf{A}_R\mathbf{u}^{**} = 0$ on $\Gamma \times (-T,T)$ is equivalent to the fact that $\mathbf{g}_2$ can be written as a linear combination (with $C^1$ coefficients) of $\partial_t^2\mathbf{g}_0$ and tangential derivatives of $\mathbf{g}_0$ (of second order) and of $\mathbf{g}_1$ (of first order) along $\Gamma$. In particular,

$$\|\mathbf{g}_2\|_{(\frac{1}{2})}(\Gamma \times (-T,T)) \leq C(\|\mathbf{g}_1\|_{(\frac{3}{2})}(\Gamma \times (-T,T)) + \|\mathbf{g}_0\|_{(\frac{5}{2})}(\Gamma \times (-T,T)).$$

Choosing $\mathbf{g}_2$ as this linear combination we obtain (2.9). □

## 3. Hölder Stability for the Residual Stress

In this section we prove the first main result of the paper, Theorem 1.1. Let $\mathbf{u}(\,;1)$ and $\mathbf{u}(\,;2)$ satisfy (1.3), (1.4), (1.5) corresponding to $R(\,;1)$ and $R(\,;2)$, respectively. Denote $\mathbf{u} = \mathbf{u}(\,;2) - \mathbf{u}(\,;1)$ and $\mathbf{F} = R(\,;2) - R(\,;1) = (f_{jk})$, $j,k = 1,\dots,3$. By subtracting equations (1.3) for $\mathbf{u}(\,;1)$ from the equations for $\mathbf{u}(\,;2)$ we yield

$$\mathbf{A}_{R(\,;2)}\mathbf{u} = \mathscr{A}(\,;\mathbf{u}(\,;1))\mathbf{F} \quad \text{on } Q \quad \text{where } \mathscr{A}(\,;\mathbf{u}(\,;1))\mathbf{F} = \sum_{j,k=1}^{3} f_{jk}\partial_j\partial_k\mathbf{u}(\,;1) \tag{3.1}$$

and

$$\mathbf{u} = \partial_t\mathbf{u} = 0 \quad \text{on } \Omega \times \{0\}. \tag{3.2}$$

Differentiating (3.1) in $t$ and using time-independence of the coefficients of the system, we get

$$\mathbf{A}_{R(\,;2)}\mathbf{U} = \mathscr{A}(\,;\mathbf{U}(\,;1))\mathbf{F} \quad \text{on } Q, \tag{3.3}$$

where

$$\mathbf{U} = \begin{pmatrix} \partial_t^2\mathbf{u} \\ \partial_t^3\mathbf{u} \\ \partial_t^4\mathbf{u} \end{pmatrix} \quad \text{and} \quad \mathbf{U}(\,;1) = \begin{pmatrix} \partial_t^2\mathbf{u}(\,;1) \\ \partial_t^3\mathbf{u}(\,;1) \\ \partial_t^4\mathbf{u}(\,;1) \end{pmatrix}.$$

By extension theorems for Sobolev spaces there exists $\mathbf{U}^* \in H^2(Q)$ such that

$$\mathbf{U}^* = \mathbf{U}, \quad \partial_\nu\mathbf{U}^* = \partial_\nu\mathbf{U} \quad \text{on } \Gamma \times (-T,T), \tag{3.4}$$

and

$$\|\mathbf{U}^*\|_{(2)}(Q) \leq C(\|\mathbf{U}\|_{(\frac{3}{2})}(\Gamma \times (-T,T)) + \|\partial_\nu\mathbf{U}\|_{(\frac{1}{2})}(\Gamma \times (-T,T))) \leq CF. \tag{3.5}$$

due to the definitions of $\mathbf{u}$, $\mathbf{U}$, and $F$. We now introduce $\mathbf{V} = \mathbf{U} - \mathbf{U}^*$. Then

$$\mathbf{A}_{R(\,;2)}\mathbf{V} = \mathscr{A}\mathbf{F} - \mathbf{A}_{R(\,;2)}\mathbf{U}^* \quad \text{on } Q \tag{3.6}$$

and

$$\mathbf{V} = \partial_\nu(\mathbf{V}) = 0 \quad \text{on } \Gamma \times (-T, T). \tag{3.7}$$

To use the Carleman estimate (2.1), we introduce a cut-off function $\chi \in C^2(\mathbf{R}^4)$ such that $0 \le \chi \le 1$, $\chi = 1$ on $Q(\frac{\varepsilon}{2})$ and $\chi = 0$ on $Q \backslash Q(0)$. By Leibniz' formula

$$\mathbf{A}_{R(;2)}(\chi\mathbf{V}) = \chi\mathbf{A}_{R(;2)}(\mathbf{V}) + \mathbf{A}_1\mathbf{V} = \chi\mathscr{A}\mathbf{F} - \chi\mathbf{A}_{R(;2)}\mathbf{U}^* + \mathbf{A}_1\mathbf{V}$$

due to (3.6). Here (and below) $\mathbf{A}_1$ denotes a first order matrix differential operator with coefficients uniformly bounded by $C(\varepsilon)$. By the choice of $\chi$, $\mathbf{A}_1\mathbf{V} = 0$ on $Q(\frac{\varepsilon}{2})$. Because of (3.7) the function $\chi\mathbf{V} \in H_0^2(Q)$, so we can apply to it the Carleman estimate (2.2) to get

$$\begin{aligned}\int_Q \tau|\chi\mathbf{V}|^2 e^{2\tau\varphi} &\le C\int_Q (|\mathbf{F}|^2 + |\mathbf{A}_{R(;2)}(\mathbf{U}^*)|^2)e^{2\tau\varphi} + C\int_{Q\backslash Q(\frac{\varepsilon}{2})} |\mathbf{A}_1\mathbf{V}|^2 e^{2\tau\varphi}\\ &\le C\left(\int_Q |\mathbf{F}|^2 e^{2\tau\varphi} + F^2 e^{2\tau\Phi} + C(\varepsilon)e^{2\tau\varepsilon_1}\right)\end{aligned} \tag{3.8}$$

where $\Phi = \sup\varphi$ over $Q$ and $\varepsilon_1 = e^{\frac{\eta\varepsilon}{4}}$. To get the last inequality we used the bounds (3.5) and (1.9).

On the other hand, from (1.3), (3.1), (3.2) we have

$$\begin{aligned}\rho\partial_t^2\mathbf{u} &= \sum f_{jk}\partial_j\partial_k\mathbf{u},\\ \rho\partial_t^3\mathbf{u} &= \sum f_{jk}\partial_t\partial_j\partial_k\mathbf{u}\end{aligned}$$

on $\Omega \times \{0\}$. So using the definitons of $\mathbf{M}$, $\mathbf{F}$ we obtain $\rho(\partial_t^2\mathbf{u}, \partial_t^3\mathbf{u}) = \mathbf{MF}$ on $\Omega \times \{0\}$, and from the condition (1.10) we have

$$|\mathbf{F}|^2 \le C\sum_{\beta=2,3} \left|\partial_t^\beta\mathbf{u}(,0)\right|^2. \tag{3.9}$$

Since $\chi(,T) = 0$,

$$\begin{aligned}\int_\Omega \left|\chi\partial_t^\beta\mathbf{u}(x,0)\right|^2 e^{2\tau\varphi(x,0)}dx &= -\int_0^T \partial_t\left(\int_\Omega |\chi\partial_t^\beta\mathbf{u}(x,t)|^2 e^{2\tau\varphi(x,t)}\right)dx\,dt\\ &\le \int_Q 2\chi^2\left(|\partial_t^{\beta+1}\mathbf{u}||\partial_t^\beta\mathbf{u}| + \tau|\partial_t\varphi||\partial_t^\beta\mathbf{u}|^2\right)e^{2\tau\varphi}\\ &\quad + 2\int_{Q\backslash Q(\frac{\varepsilon}{2})} |\partial_t^\beta\mathbf{u}|^2\chi|\partial_t\chi|e^{2\tau\varphi}\end{aligned}$$

where $\beta = 2, 3$. The right side does not exceed

$$\begin{aligned}&C\left(\int_Q \tau|\chi\mathbf{U}|^2 e^{2\tau\varphi} + C(\varepsilon)\int_{Q\backslash Q(\frac{\varepsilon}{2})} |\mathbf{U}|^2 e^{2\tau\varphi}\right)\\ &\quad \le C\left(\int_Q \tau|\chi\mathbf{V}|^2 e^{2\tau\varphi} + C(\varepsilon)\int_{Q\backslash Q(\frac{\varepsilon}{2})} |\mathbf{U}|^2 e^{2\tau\varphi} + \tau\int_Q |\mathbf{U}^*|^2 e^{2\tau\varphi}\right)\end{aligned}$$

because $\mathbf{U} = \mathbf{V} + \mathbf{U}^*$. Using that $\chi = 1$ on $\Omega(\frac{\varepsilon}{2})$, $\varphi < \varepsilon_1$ on $Q\backslash Q(\frac{\varepsilon}{2})$ and $\varphi < \Phi$ on $Q$ from these inequalities, from (3.8), from (3.5), and from (1.9) we yield

$$\int_{\Omega(\frac{\varepsilon}{2})} |\partial_t^\beta \mathbf{u}|^2(,0)e^{2\tau\varphi(,0)} \leq C\bigg(\int_Q |\mathbf{F}|^2 e^{2\tau\varphi} + C(\varepsilon)e^{2\tau\varepsilon_1} + \tau e^{2\tau\Phi}F^2\bigg). \tag{3.10}$$

Using that $\chi = 1$ on $\Omega(\frac{\varepsilon}{2})$, from (3.9) and (3.10) we obtain

$$\int_{\Omega(\frac{\varepsilon}{2})} |\mathbf{F}|^2 e^{2\tau\varphi(,0)} \leq C\bigg(\int_{Q(\frac{\varepsilon}{2})} |\mathbf{F}|^2 e^{2\tau\varphi} + \tau e^{2\tau\Phi}F^2 + C(\varepsilon)e^{2\tau\varepsilon_1}\bigg) \tag{3.11}$$

where we also split $Q$ in the right side of (2.7) into $Q(\frac{\varepsilon}{2})$ and its complement, and used that $|\mathbf{F}| \leq C$ and $\varphi < \varepsilon_1$ on the complement. To eliminate the integral in the right side of (3.11) we observe that

$$\int_{Q(\frac{\varepsilon}{2})} |\mathbf{F}|^2(x)e^{2\tau\varphi(x,t)}dx\,dt \leq \int_{\Omega(\frac{\varepsilon}{2})} |\mathbf{F}|^2(x)e^{2\tau\varphi(x,0)}\bigg(\int_{-T}^{T} e^{2\tau(\varphi(x,t)-\varphi(x,0))}dt\bigg)dx.$$

Due to our choice of function $\varphi$ we have $\varphi(x,t) - \varphi(x,0) < 0$ when $t \neq 0$. Hence by the Lebesgue Theorem the inner integral (with respect to $t$) converges to 0 as $\tau$ goes to infinity. By reasons of continuity of $\varphi$, this convergence is uniform with respect to $x \in \Omega$. Choosing $\tau > C$ we therefore can absorb the integral over $Q(\frac{\varepsilon}{2})$ in the right side of (3.11) by the left side arriving at the inequality

$$\int_{\Omega(\varepsilon)} |\mathbf{F}|^2 e^{2\tau\varphi(,0)} \leq C(\tau e^{2\tau\Phi}F^2 + C(\varepsilon)e^{2\tau\varepsilon_1}).$$

Letting $\varepsilon_2 = e^{\frac{\eta\varepsilon}{2}} \leq \varphi$ on $\Omega(\varepsilon)$ and dividing the both parts by $e^{2\tau\varepsilon_2}$ we yield

$$\int_{\Omega(\varepsilon)} |\mathbf{F}|^2 \leq C(\tau e^{2\tau(\Phi-\varepsilon_2)}F^2 + e^{-2\tau(\varepsilon_2-\varepsilon_1)}) \leq C(\varepsilon)(e^{2\tau\Phi}F^2 + e^{-2\tau(\varepsilon_2-\varepsilon_1)}) \tag{3.12}$$

since $\tau e^{-2\tau\varepsilon_2} < C(\varepsilon)$. To prove (1.13) it suffices to assume that $F < \frac{1}{C}$. Then $\tau = \frac{-\log F}{\Phi+\varepsilon_2-\varepsilon_1} > C$ and we can use this $\tau$ in (3.12). Due to the choice of $\tau$,

$$e^{-2\tau(\varepsilon_2-\varepsilon_1)} = e^{2\tau\Phi}F^2 = F^{2\frac{\varepsilon_2-\varepsilon_1}{\Phi+\varepsilon_2-\varepsilon_1}}$$

and from (3.12) we obtain (1.13) with $\gamma = \frac{\varepsilon_2-\varepsilon_1}{\Phi+\varepsilon_2-\varepsilon_1}$. The proof of Theorem 1.1 is now complete. □

## 4. Lipschitz Stability in the Cauchy Problem

Now we will prove a Lipschitz stability estimate for the Cauchy problem for the system (2.5). This estimate is a key to prove the estimate (1.17) in the inverse problem. Before going to the main result of this section, we state a lemma concerning the boundary condition for auxiliary functions $v$ and $\mathbf{w}$. We refer to Lin and Wang (2003) for the proof.

**Lemma 4.1.** *Let a solution $\mathbf{u} \in H^3(Q)$ to system $\mathbf{A}_R\mathbf{u} = \mathbf{f}$ in $Q$ satisfy*

$$\mathbf{f} = \mathbf{u} = \sigma_\nu(\mathbf{u}) = 0 \quad on\ \Gamma \times (-T, T)$$

*and let $R$ satisfy* (1.7) *with $\varepsilon_0$ sufficiently small.*

*Then*

$$\partial_k \mathbf{u} = \partial_j \partial_k \mathbf{u} = 0 \quad on\ \Gamma \times (-T, T),\ for\ 1 \le i, j, k \le 3.$$

Now we can prove the following result.

**Theorem 4.2.** *Suppose that* $\Omega$ *and* $T$ *satisfy the assumptions of Theorem* 1.2. *Let* $\mathbf{u} \in (H^3(Q))^3$ *solve the Cauchy problem*

$$\begin{cases} \mathbf{A}_R \mathbf{u} = \mathbf{f} & in\ Q \\ \mathbf{u} = \sigma_\nu(\mathbf{u}) = 0 & on\ \partial\Omega \times (-T, T) \end{cases} \tag{4.1}$$

*with* $\mathbf{f} \in L^2((-T, T); H^1(\Omega))$ *and* $\mathbf{f} = 0$ *on* $\partial\Omega \times (-T, T)$. *Furthermore, assume that* (1.7) *holds for sufficiently small* $\varepsilon_0$.

*Then there exists a constant* $C > 0$ *such that*

$$\|\mathbf{u}\|^2_{H^1(Q)} + \|v\|^2_{H^1(Q)} + \|\mathbf{w}\|^2_{H^1(Q)} \le C\|\mathbf{f}\|^2_{L^2((-T,T);H^1(\Omega))}. \tag{4.2}$$

By virtue of (4.2) and equivalence of the norms $\|\mathbf{u}\|_{(1)}(\Omega)$ and of

$$\|div\, \mathbf{u}\|_{(0)}(\Omega) + \|curl\, \mathbf{u}\|_{(0)}(\Omega)$$

in $H^1_0(\Omega)$ (e.g., Duvaut and Lions, 1976, pp. 358–369), it is not hard to derive the following

**Corollary 4.1.** *Under the conditions of Theorem* 4.2

$$\|\mathbf{u}\|_{(0)}(Q) + \|\nabla_{x,t}\mathbf{u}\|_{(0)}(Q) + \|\partial_t \nabla \mathbf{u}\|_{(0)}(Q) \le C\|\mathbf{f}\|^2_{L^2((-T,T);H^1(\Omega))}. \tag{4.3}$$

*Proof of Theorem* 4.2. By standard energy estimates for the system (2.6) we get

$$C^{-1}\left(E(0) - C\int_{\Omega\times(0,t)} (|\mathbf{f}|^2 + |\nabla \mathbf{f}|^2)\right) \le E(t) \le C\left(E(0) + \int_{\Omega\times(0,t)} (|\mathbf{f}|^2 + |\nabla \mathbf{f}|^2)\right) \tag{4.4}$$

for some $C$ where

$$E(t) = \int_\Omega (|\partial_t \mathbf{u}|^2 + |\partial_t v|^2 + |\partial_t \mathbf{w}|^2 + |\nabla \mathbf{u}|^2 + |\nabla v|^2 + |\nabla \mathbf{w}|^2 + |\mathbf{u}|^2 + |v|^2 + |\mathbf{w}|^2)(, t).$$

To use the Carleman estimate (2.1) we need to cut off $\mathbf{u}$ near $t = T$ and $t = -T$. We first observe that from the definition

$$1 \le \varphi(x, 0), \quad x \in \Omega,$$

and from the condition (1.15)

$$\varphi(x, T) = \varphi(x, -T) < 1 \quad \text{when } x \in \overline{\Omega}.$$

So there exists a $\delta > \frac{1}{C}$ such that

$$1-\delta < \varphi \quad \text{on } \Omega \times (0,\delta), \quad \varphi < 1-2\delta \quad \text{on } \Omega \times (T-2\delta, T). \tag{4.5}$$

We now choose a smooth cut-off function $0 \le \chi_0(t) \le 1$ such that $\chi_0(t) = 1$ for $-T+2\delta < t < T-2\delta$ and $\chi(t) = 0$ for $|t| > T-\delta$. It is clear that

$$\mathbf{A}_R(\chi_0 \mathbf{u}) = \chi_0 \mathbf{f} + 2\rho \partial_t \chi_0 \partial_t \mathbf{u} + \rho \partial_t^2 \chi_0 \mathbf{u}.$$

By Lemma 4.1 and basic facts about Sobolev spaces $\chi_0 \mathbf{u} \in H_0^3(Q)$, we can use the Carleman estimate (2.1) to get

$$\int_Q (\tau^3(|\chi_0\mathbf{u}|^2 + (\chi_0 v)^2 + |\chi_0 \mathbf{w}|^2) + \tau(|\nabla_{x,t}(\chi_0\mathbf{u})|^2 + |\nabla_{x,t}(\chi_0 v)|^2 + |\nabla_{x,t}(\chi_0\mathbf{w})|^2))e^{2\tau\varphi}$$
$$\le C\left(\int_Q (|\mathbf{f}|^2 + |\nabla \mathbf{f}|^2)e^{2\tau\varphi} + \int_{\Omega\times\{T-2\delta<|t|<T\}} (|\partial_t \mathbf{u}|^2 + |\partial_t \nabla \mathbf{u}|^2 + |\mathbf{u}|^2 + |\nabla \mathbf{u}|^2)e^{2\tau\varphi}\right).$$

Shrinking the integration domain on the left side to $\Omega \times (0,\delta)$ where $\chi = 1$ and $1-\delta < \varphi$ and using that $\varphi < 1-2\delta$ on $\Omega \times (T-\delta, T)$ we derive that

$$e^{2\tau(1-\delta)} \int_0^\delta E(t)dt$$
$$\le C\left(\int_Q (|\mathbf{f}|^2 + |\nabla \mathbf{f}|^2)e^{2\tau\varphi} + Ce^{2\tau(1-2\delta)} \int_{T-2\delta}^T \int_\Omega (|\partial_t \mathbf{u}|^2 + |\partial_t \nabla \mathbf{u}|^2 + |\mathbf{u}|^2 + |\nabla \mathbf{u}|^2)\right). \tag{4.6}$$

To eliminate the last integral in (4.6) we remind that

$$curl\,\partial_t \mathbf{u} = \partial_t \mathbf{w}, \quad div\,\partial_t \mathbf{u} = \partial_t v, \quad \Delta \partial_t \mathbf{u} = \nabla(\partial_t v) - curl(\partial_t \mathbf{w})$$

and use the standard elliptic $L^2$-estimate

$$\int_\Omega |\nabla \partial_t \mathbf{u}|^2(,t) \le C\left(\int_\Omega (|\partial_t v|^2 + |\partial_t \mathbf{w}|^2)\right)(,t).$$

Now using the energy bound (4.4) we derive from (4.6)

$$e^{2\tau(1-\delta)}\frac{\delta}{C}E(0) - Ce^{2\tau\Phi}\int_Q (|\mathbf{f}|^2 + |\nabla \mathbf{f}|^2) \le Ce^{2\tau\Phi}\int_Q (|\mathbf{f}|^2 + |\nabla \mathbf{f}|^2) + Ce^{2\tau(1-2\delta)}E(0).$$

Choosing $\tau$ so large that $e^{-2\tau\delta} < \frac{\delta}{C^2}$ and fixing this $\tau$ we eliminate the term with $E(0)$ on the right side. Using again (4.4) we complete the proof. □

## 5. Lipschitz Stability for the Residual Stress

In this section we prove the second main result of the paper, Theorem 1.2. We will use the notation of Section 4.

In view of Lemma 2.3, there exists $\mathbf{U}^* \in H^3(Q)$ such that

$$\mathbf{U}^* = \mathbf{U}, \quad \partial_\nu \mathbf{U}^* = \partial_\nu \mathbf{U}, \quad \mathbf{A}_{R(;2)}\mathbf{U}^* = 0 \quad \text{on } \partial\Omega \times (-T,T), \tag{5.1}$$

and

$$\|\mathbf{U}^*\|_{(3)}(Q) \leq C(\|\mathbf{U}\|_{(\frac{5}{2})}(\Gamma \times (-T, T)) + \|\partial_\nu \mathbf{U}\|_{(\frac{3}{2})}(\partial\Omega \times (-T, T))) \leq CF. \tag{5.2}$$

due to the definition of $F$. We introduce $\mathbf{V} = \mathbf{U} - \mathbf{U}^*$. Due to (5.1),

$$\mathbf{V} = \partial_\nu \mathbf{V} = 0, \quad \mathbf{A}_{R(\,;2)}\mathbf{V} = 0 \quad \text{on } \partial\Omega \times (-T, T). \tag{5.3}$$

Applying Corollary 4.1 to (3.6), (3.7) and using (5.2) gives

$$\|\mathbf{V}\|_{(0)}^2(Q) + \|\nabla_{x,t}\mathbf{V}\|_{(0)}^2(Q) + \|\partial_t \nabla \mathbf{V}\|_{(0)}^2(Q) \leq C\big(\|\mathbf{F}\|_{(1)}^2(\Omega) + F^2\big). \tag{5.4}$$

On the other hand, as in the proof of Theorem 1.1 we will bound the right side by $\mathbf{V}$.

We will use the cut off function $\chi_0$ of Section 4. According to Lemma 4.1 and (5.3), $\chi_0\mathbf{V} \in H_0^3(Q)$. By Leibniz' formula

$$\mathbf{A}_{R(\,;2)}(\chi_0\mathbf{V}) = \chi_0\mathscr{A}(\,;\mathbf{U}(\,;1))\mathbf{F} - \chi_0\mathbf{A}_{R(\,;2)}\mathbf{U}^* + 2\rho(\partial_t\chi_0)\partial_t\mathbf{V} + \rho(\partial_t^2\chi_0)\mathbf{V}$$

and by the Carleman estimate (2.1)

$$\begin{aligned}
&\int_Q \chi_0^2(\tau^3|\mathbf{V}|^2 + \tau|\nabla\mathbf{V}|^2)e^{2\tau\varphi} \\
&\leq C\bigg(\int_Q (|\mathbf{F}|^2 + |\nabla\mathbf{F}|^2 + |\mathbf{A}_{R(\,;2)}\mathbf{U}^*|^2 + |\nabla(\mathbf{A}_{R(\,;2)}\mathbf{U}^*)|^2)e^{2\tau\varphi} \\
&\qquad + \int_{\Omega\times\{T-2\delta<|t|<T\}} (|\mathbf{V}|^2 + |\nabla_{x,t}\mathbf{V}|^2 + |\partial_t\nabla\mathbf{V}|^2)e^{2\tau\varphi}\bigg) \\
&\leq \bigg(\int_Q (|\mathbf{F}|^2 + |\nabla\mathbf{F}|^2)e^{2\tau\varphi} + e^{2\tau\Phi}F^2 + e^{2\tau(1-2\delta)}\int_\Omega (|\mathbf{F}|^2 + |\nabla\mathbf{F}|^2)\bigg),
\end{aligned}$$

where we let $\Phi = \sup_Q \varphi$ and used (4.5) and (5.4). Since $\mathbf{U} = \mathbf{V} + \mathbf{U}^*$ from (5.2) we obtain

$$\begin{aligned}
&\int_Q \chi_0^2(|\mathbf{U}|^2 + |\nabla\mathbf{U}|^2)e^{2\tau\varphi} \\
&\leq C(e^{2\tau\Phi}F^2 + \int_\Omega \bigg(\int_{-T}^T e^{2\tau\varphi(x,t)}dt + e^{2\tau(1-2\delta)}\bigg)(|\mathbf{F}|^2 + |\nabla\mathbf{F}|^2)(x))dx
\end{aligned} \tag{5.5}$$

Utilizing (3.2) and (1.10), similarly to deriving (3.9), we get from (3.1) that $\rho(\partial_t^2\mathbf{u}, \partial_t^3\mathbf{u}) = \mathbf{MF}$ on $\Omega \times \{0\}$. Therefore, using (1.10) we will have

$$\begin{aligned}
\int_\Omega (|\mathbf{F}|^2 + |\nabla\mathbf{F}|^2)e^{2s\varphi(,0)} &\leq C\int_\Omega \sum_{\beta=2,3;k=0,1} |\partial_t^\beta \nabla^k \mathbf{u}(,0)|^2 e^{2\tau\varphi(,0)} \\
&= -C\int_0^T \partial_t\bigg(\int_\Omega \sum \chi_0^2 |\partial_t^\beta \nabla^k \mathbf{u}|^2 e^{2\tau\varphi} dx\bigg)dt
\end{aligned}$$

$$\leq C\int_Q \chi_0^2 \sum(|\partial_t^\beta \nabla^k \mathbf{u}||\partial_t^{\beta+1}\nabla^k\mathbf{u}| + \tau|\partial_t\varphi||\partial_t^\beta\nabla^k\mathbf{u}|^2)e^{2\tau\varphi}$$
$$+ C\int_{\Omega\times(T-2\delta,T)} \chi_0|\partial_t\chi_0|\sum|\partial_t\beta\nabla^k\mathbf{u}|^2)e^{2\tau\varphi}.$$

Now as in the proofs of Section 3 the right side is less than

$$C\bigg(\int_Q \tau\chi_0^2(|\mathbf{U}|^2 + |\nabla\mathbf{U}|^2)e^{2\tau\varphi} + \int_{\Omega\times(T-2\delta,T)}(|\mathbf{U}|^2+|\nabla\mathbf{U}|^2)e^{2\tau\varphi}\bigg)$$
$$\leq C\bigg(\int_Q \tau\chi_0^2(|\mathbf{U}|^2 + |\nabla\mathbf{U}|^2)e^{2\tau\varphi} + e^{2\tau(1-2\delta)}(\|F\|^2_{(1)}(\Omega) + F^2)\bigg).$$

where we used equality $\mathbf{U} = \mathbf{U}^* + \mathbf{V}$ and (5.2), (5.4). From two previous bounds we conclude that

$$\int_\Omega(|\mathbf{F}|^2 + |\nabla\mathbf{F}|^2)e^{2\tau\varphi(,0)} \leq C\bigg(\tau e^{2\tau\Phi}F^2 + \int_\Omega\bigg(\int_{-T}^T e^{2\tau\varphi(,t)}dt + e^{2\tau(1-2\delta)}\bigg)(|\mathbf{F}|^2+|\nabla\mathbf{F}|^2)\bigg). \tag{5.6}$$

Due to our choice of $\varphi$, $1 \leq \varphi(,0), \varphi(,t) - \varphi(,0) < 0$ when $t \neq 0$. Thus by the Lebesgue theorem as in the proofs of Section 3, we have

$$2C\bigg(\int_{-T}^T e^{2\tau\varphi(,t)}dt + e^{2\tau(1-\delta)}\bigg) \leq e^{2\tau\varphi(,0)}$$

uniformly on $\Omega$ when $\tau > C$. Hence choosing and fixing such large $\tau$ we eliminate the second term on the right side of (5.6). The proof of Theorem 1.2 is now complete. □

By using Carleman estimates on functions satisfying the homogeneous zero boundary conditions ($\mathbf{g}_0 = 0$ or zero stress boundary condition) one can replace $\partial\Omega$ in Theorem 1.2 by its "large" part $\Gamma$ (Isakov, 2006, Section 4.5, for scalar equations).

## 6. Conclusion

Using similar methods one can expect to demonstrate uniqueness and stability for both variable $\rho, \lambda, \mu$ and residual stress most likely from two sets of suitable boundary data. Motivation is coming from geophysical problems. Our assumptions exclude zero initial data. So far it looks like a very difficult question to show uniqueness from few sets of boundary data when the initial data are zero. The stability guaranteed by Theorems 1.1 and especially by Theorem 1.2 indicates a possibility of a very efficient algorithms with high resolution for numerical idenitification of residual stress from single lateral measurements. It would a very good idea to run some numerical experiments to understand possibilities of practical applications of these stability properties.

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