COVARIANCE STRUCTURES OF GAUSSIAN AND LOG-GAUSSIAN VECTOR STOCHASTIC PROCESSES

A Dissertation by

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DEDICATION

To my amazing parents, Tim and Trina, my brother, Bryce,
and Timothy Johnson for your continued support and encouraging words

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ABSTRACT

Although the covariance structures of univariate Gaussian and log-Gaussian stochastic processes have been extensively studied in the past few decades, the development of covariance structures for Gaussian and log-Gaussian vector stochastic processes is still in the early stages. Specifically, there has been little discussion about how to construct the covariance matrix functions of multivariate Gaussian time series with long memory, especially ones with power-law and log-law decaying covariance structures. Furthermore, there have been relatively few results about how to determine whether a given matrix function is the covariance matrix function of a log-Gaussian vector random field. This dissertation provides new methods for identifying and constructing covariance matrix functions of Gaussian vector time series and log-Gaussian vector random fields. In particular, research is presented on how to find covariance matrix structures with power-law decaying and log-law decaying direct and cross covariances. Also, the intricate relationship between the mean function and the covariance matrix function of the log-Gaussian vector random field is explored. In additon, operation preserving properties are investigated for the log-Gaussian vector random field.

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CHAPTER 1

INTRODUCTION

One of the most commonly studied areas in probability theory is stochastic processes. A stochastic process is a collection of random variables that represent the evolution of some random system, generally over time or space. Unlike in a deterministic process, even when the initial conditions are known, there are multiple directions or routes in which the process may develop. For example, consider a simulation in which one flips a fair coin fifty times. If a head comes up, the person takes one step to the right, and if a tail comes up, the person takes one step to the left. This process is referred to as the simple random walk and serves as just one model for the randomness we often observe in real data. The randomness we find in life creates a certain difficulty in finding appropriate models for data that can accurately predict real-world outcomes. This is why statisticians are continually searching for ways to expand the current literature on stochastic processes. One may use stochastic processes to model and predict various types of data, such as finanical market fluctuations, climate change measurements like temperature and wind velocity, speech, audio and video signals, medical measurements like blood pressure, and other forms of random movement like the paths of particles and gases.

Of all the types of stochastic processes, Gaussian processes are the most commonly studied because of the extensive amount of knowledge we have about the Gaussian distribution as a whole. We know a considerable amount about its mean and covariance function, as we will discuss in Chapter 2, as well as its role as a limiting distribution for many other distributions. In particular, the central limit theorem not only applies to random variables, but also to stochastic processes, such as martingales. The most well-known Gaussian processes are the Wiener process, the Ornstein-Uhlenbeck process, the Brownian bridge, and fractional Brownian motion, which all consider notions of independence and stationarity among the observations or the increments of the observations. These stochastic processes have been used to model real data, such as those in physics, chemistry, and quantum mechanics, to name a few.

In regard to non-Gaussian stochastic processes, there seems to be little known. Another stochastic process, the log-Gaussian process, is derived by making a logarithm transform to a Gaussian process. Since the log-Gaussian and Gaussian processes are so closely related, each is defined by its mean and covariance function, one may wonder why so little research has been completed on the log-Gaussian stochastic process, except for Matheron (1989). At first glance, we would expect results that hold for Gaussian processes to hold for log-Gaussian processes. For example, we might expect that the sum of two covariance functions of two log-Gaussian processes would be the covariance function of another log-Gaussian process. However, this does not seem to be the case as we find that the relationship between the mean and the covariance function of a log-Gaussian process is much more complicated than that for a Gaussian process.

Over the past one hundred years, there have been many developments regarding the covariance structures of univariate Gaussian and log-Gaussian stochastic processes. However, there are not too many about the covariance structures of Gaussian and log-Gaussian vector stochastic processes. The primary purpose of the current research is to establish several methods for constructing covariance matrix functions of multivariate Gaussian time series with long memory as well as constructing covariance matrix functions of log-Gaussian vector random fields. Obviously, the difficulty in developing such models arises from trying to specify the cross covariance functions among the components, not just the direct covariance of each individual component. Specifically, methods are presented for constructing covariance matrix functions of stationary Gaussian vector time series from univariate stationary Gaussian time series, with the conditionally negative definite matrix playing an important role. As a special case, a stationary Gaussian vector time series with long memory that has a power-law decaying covariance structure is derived, and in another case, one with a log-law decaying covariance structure is derived. Also, a class of power-law decay and long-range dependent log-Gaussian processes is discussed. This dissertation also provides an approach for constructing the covariance matrix function of a nonstationary Gaussian vector time series.

This dissertation is organized as follows. In Chapter 2, past research and practical applications of Gaussian and log-Gaussian stochastic processes are briefly reviewed, with an emphasis placed on processes with long memory. In addition, general background material is provided as well as some necessary definitions. In Chapter 3, a theorem is given that provides a method for generating covariance matrix functions of multivariate stationary Gaussian time series with long memory from univariate stationary Gaussian time series. Then, we follow up with an analysis of some of the conditions of Theorem 3.1, such as the necessity of the gamma function. Next we provide some examples of how to use Theorem 3.1 to create power-law decaying and log-law decaying direct and cross covariance structures for stationary Gaussian vector time series. The last corollary of Theorem 3.1 utilizes a matrix version of Young's theorem (1913) to construct covariance matrix functions of multiple Gaussian time series. Our second theorem also uses univariate stationary Gaussian time series to develop covariance matrix functions of multiple stationary Gaussian time series with long memory, but it allows for the direct and cross covariances to have entries from a conditionally negative definite matrix. The final theorem in Chapter 3 presents a method for the covariance matrix structure of a nonstationary Gaussian vector time series.

In Chapter 4, we develop the necessary and sufficient conditions for a matrix function to be the covariance matrix function of a log-Gaussian vector random field. In addition, we provide some interesting examples of covariance matrix functions of log-Gaussian vector random fields and we further explain the complex relationship between the mean vector and the covariance matrix function of such fields. Then we discuss which basic operations on log-Gaussian vector random fields and their covariance matrix functions, like sums and products, are preserved. We also provide a method for identifying whether a covariance matrix function of a Gaussian vector random field is the covariance matrix function of a log-Gaussian vector random field. Then we illustrate how to construct a covariance matrix function with entries from a conditionally negative definite matrix. Also, in Chapter 4 we present some results that are unique to the univariate log-Gaussian random field, with an emphasis on long-memory and power-law decaying processes. Finally, in Chapter 5, we provide our overall conclusions as well as some remarks about future research.

CHAPTER 2

BACKGROUND MATERIAL

Let $\mathcal{D}$ be a set we take to be an index domain. Given a probability space, $(\Omega, \mathcal{F}, P)$, a univariate stochastic process is a collection of random variables, $\{Z(\mathbf{x}, \boldsymbol{\omega}), \mathbf{x} \in \mathcal{D}, \boldsymbol{\omega} \in \Omega\}$, where, for each $\mathbf{x} \in \mathcal{D}$, $Z(\mathbf{x}, \boldsymbol{\omega})$ is a random variable on the probability space. Typical examples of the index domain $\mathcal{D}$ are $\mathbb{Z}^d$, $\mathbb{R}^d$, $\mathbb{R}^d \times \mathbb{R}$, and $\mathbb{R}^d \times \mathbb{Z}$, where d is a positive integer. For a particular index domain, $\mathcal{D}$, we often give a specific name to the stochastic process. Throughout this dissertation, we call $\{Z(\mathbf{x}, \boldsymbol{\omega}) : \mathbf{x} \in \mathcal{D}, \boldsymbol{\omega} \in \Omega\}$ a time series when $\mathcal{D} = \mathbb{Z}^d$. We call $\{Z(\mathbf{x}, \boldsymbol{\omega}) : \mathbf{x} \in \mathcal{D}, \boldsymbol{\omega} \in \Omega\}$ a stochastic process or a random field, respectively, when $\mathcal{D} = \mathbb{R}$ or $\mathcal{D} = \mathbb{R}^d$, $d \geq 2$. If $\mathcal{D} = \mathcal{S} \times \mathcal{T}$, where $\mathcal{S} = \mathbb{R}^d$ or $\mathbb{Z}^d$, and $\mathcal{T} = \mathbb{R}$ or $\mathbb{Z}$, we say $\{Z(\mathbf{x}, \boldsymbol{\omega}) : \mathbf{x} \in \mathcal{D}, \boldsymbol{\omega} \in \Omega\}$ is a spatio-temporal random field.

Since $Z(\boldsymbol{\omega}), \boldsymbol{\omega} \in \Omega$, is a random variable that maps from $(\Omega, \mathcal{F})$ into a measurable space $(\mathbb{R}, \mathcal{B})$, a stochastic process $\{Z(\mathbf{x}, \boldsymbol{\omega}), \mathbf{x} \in \mathcal{D}, \boldsymbol{\omega} \in \Omega\}$ can be seen as a function of two arguments, $\mathbf{x}$ and $\boldsymbol{\omega}$. For a fixed $\mathbf{x} \in \mathcal{D}$, $Z(\mathbf{x}, \boldsymbol{\omega})$ is a random variable. On the other hand, for a fixed $\boldsymbol{\omega} \in \Omega$, $Z(\mathbf{x}, \omega)$ is a function of $\mathbf{x}$ that represents a possible observation on the stocastic process, and hence, is called a sample function of the stochastic process. From here on, we suppress the argument $\boldsymbol{\omega}$ and we write $\{Z(\mathbf{x}) : \mathbf{x} \in \mathcal{D}\}$ for $\{Z(\mathbf{x}, \boldsymbol{\omega}) : \mathbf{x} \in \mathcal{D}, \boldsymbol{\omega} \in \Omega\}$.

In this dissertation, let m be a positive integer and let $\{\mathbf{Z}(\mathbf{x}) = (Z_1(\mathbf{x}), \ldots, Z_m(\mathbf{x}))' : \mathbf{x} \in \mathcal{D}\}$ denote an m-variate stochastic process, which is a family of m-variate real random vectors on the same probability space. For a real-valued stochastic vector process $\{\mathbf{Z}(\mathbf{x}) : \mathbf{x} \in \mathcal{D}\}$, with finite second-order moments, we denote its mean function and covariance matrix function, respectively, by

$$\boldsymbol{\mu}(\mathbf{x}) = (\mathrm{E}Z_1(\mathbf{x}), \ldots, \mathrm{E}Z_m(\mathbf{x}))', \qquad \mathbf{x} \in \mathcal{D},$$

and

$$\begin{aligned}\mathbf{C}(\mathbf{x}_1,\mathbf{x}_2) &= \mathrm{E}[\{\mathbf{Z}(\mathbf{x}_1)-\mathrm{E}\mathbf{Z}(\mathbf{x}_1)\}\{\mathbf{Z}(\mathbf{x}_2)-\mathrm{E}\mathbf{Z}(\mathbf{x}_2)\}'] \\ &= \begin{pmatrix} \mathrm{cov}(Z_1(\mathbf{x}_1),Z_1(\mathbf{x}_2)) & \dots & \mathrm{cov}(Z_1(\mathbf{x}_1),Z_m(\mathbf{x}_2)) \\ \vdots & \ddots & \vdots \\ \mathrm{cov}(Z_m(\mathbf{x}_1),Z_1(\mathbf{x}_2)) & \dots & \mathrm{cov}(Z_m(\mathbf{x}_1),Z_m(\mathbf{x}_2)) \end{pmatrix}, \ \mathbf{x}_1,\mathbf{x}_2 \in \mathcal{D}.\end{aligned}$$

The diagonal entries of this matrix function are called direct covariance functions and the off-diagonal entries are called cross covariance functions.

Throughout this dissertation, we refer to a vector or matrix with all positive entries as a positive vector or positive matrix, respectively. For two vectors $\mathbf{x},\mathbf{y} \in \mathbb{R}^d$, when we say $\mathbf{x} \leq \mathbf{y}$, we mean $x_i \leq y_i, i = 1,\dots,d$. Also, for $\mathbf{x} \in \mathbb{R}^d$ we use the following conventions: $\ln \mathbf{x} = (\ln x_1,\dots,\ln x_m)'$ and $e^{\mathbf{x}} = (e^{x_1},\dots,e^{x_m})'$.

One objective of the current research is to explore the covariance structures of multivariate Gaussian time series. An m-variate Gaussian time series $\{\mathbf{Z}(x) = (Z_1(x),\dots,Z_m(x))' : x \in \mathbb{Z}\}$ is a time series whose finite-dimensional distributions are Gaussian. One well-known fact about Gaussian processes is that they can be completely characterized by their mean functions and covariance matrix functions. So, when the mean vector $\boldsymbol{\mu}(\mathbf{x})$ is the zero vector, the process' behavior is characterized by its covariance matrix function. Also, since the mean function and the covariance matrix function are not tied together, this allows for many operation preserving properties on both the process and the covariance matrix function, as we will see in Chapter 4. One usage of a Gaussian stochastic process is as a nonlinear interpolation tool, in which one uses a technique called kriging to estimate the values of missing data from nearby data values. See, for example, Stein (1999). We also see Gaussian processes utilized in other areas, including the Black-Scholes formula for modeling option prices and in Brownian motion, or the diffusion of microscopic particles suspended in fluid.

There have been many new results concerning univariate random fields, most notably in the Gaussian case. However, there have been few discoveries regarding non-Gaussian vector random fields. While there has been some research specifically regarding the univariate log-Gaussian random field, there is minimal literature on the log-Gaussian vector random field, and more specifically, the covariance structure of such a random field. For example, Limpert et al. (2001) provided asymptotically efficient estimators for the mean and the covariance of a log-Gaussian process. Perpéte and Schmitt (2011) showed that log-Gaussian time series possess multifractal properties. De Oliveira (2006) found optimal point and block predictors for log-Gaussian vector random fields.

The application of log-Gaussian random fields can be found in a multitude of fields including finance, geology, survival analysis, microbiology, ecology, linguistics, economics, atmospheric sciences, and enviornmental sciences. For example, Yue (2000) used the bivariate lognormal random field to model flood episodes. Broadie et al. (2000) demonstrated how to apply the stochastic mesh method for pricing multidimensional American options, when the underlying model of the options was a lognormal vector process. Haslett et al. (2006) provided an example of using the log-Gaussian vector process to model pollen and climate data for fossils. In particular, they noted that while a log-Gaussian vector process was best for modelling their data, using the log-Gaussian process was computationally expensive and created convergence issues for their MCMC algorithm.

Another primary objective of the current research is to explore the covariance structures of log-Gaussian vector random fields. A log-Gaussian vector random field is a random field whose finite-dimensional distributions are log-Gaussian and it is obtained after making a logarithm transform to a Gaussian vector random field. More precisely, a vector random field, $\{\mathbf{Z}(\mathbf{x}) = (Z_1(\mathbf{x}), \ldots, Z_m(\mathbf{x}))' : \mathbf{x} \in \mathcal{D}\}$, is said to be a log-Gaussian vector random field if $\{\ln \mathbf{Z}(\mathbf{x}) = (\ln Z_1(\mathbf{x}), \ldots, \ln Z_m(\mathbf{x}))' : \mathbf{x} \in \mathcal{D}\}$ is a Gaussian vector random field. The finite-dimensional distribution functions of $\{\mathbf{Z}(\mathbf{x}) : \mathbf{x} \in \mathcal{D}\}$,

$$
\begin{aligned}
&P(\mathbf{Z}(\mathbf{x}_1) \le \mathbf{u}_1, \ldots, \mathbf{Z}(\mathbf{x}_n) \le \mathbf{u}_n) \\
&= \begin{cases} P(\ln \mathbf{Z}(\mathbf{x}_1) \le \ln \mathbf{u}_1, \ldots, \ln \mathbf{Z}(\mathbf{x}_n) \le \ln \mathbf{u}_n), & \text{if } \mathbf{u}_1, \ldots, \mathbf{u}_n > \mathbf{0}, \\ 0, & \text{otherwise,} \end{cases}
\end{aligned}
$$

relate closely to the Gaussian vector random field, $\{\ln \mathbf{Z}(\mathbf{x}) : \mathbf{x} \in \mathcal{D}\}$, and the log-Gaussian vector random field is thus characterized by its mean function and covariance matrix function, just like the Gaussian vector random field.

It is known that an $m \times m$ covariance matrix function $\mathbf{C}(\mathbf{x}_i, \mathbf{x}_j)$ on $\mathcal{D}$ satisfies the following two fundamental properties:

(i) $\{\mathbf{C}(\mathbf{x}_i, \mathbf{x}_j)\}' = \mathbf{C}(\mathbf{x}_j, \mathbf{x}_i)$, and

(ii) the inequality

$$\sum_{i=1}^{n}\sum_{j=1}^{n} \mathbf{a}_i' \mathbf{C}(\mathbf{x}_i, \mathbf{x}_j)\mathbf{a}_j \geq 0 \tag{2.1}$$

holds for every $n \in \mathbb{N}$, for any $\mathbf{a}_k \in \mathbb{R}^m$, and for any $\mathbf{x}_i \in \mathcal{D}, i = 1, \ldots, n$. On the other hand, for any given matrix function $\mathbf{C}(\mathbf{x}_1, \mathbf{x}_2)$ on $\mathcal{D}$ that satisfies these two properties, there always exists a Gaussian or elliptically contoured vector stochastic process with zero mean vector and covariance matrix function $\mathbf{C}(\mathbf{x}_1, \mathbf{x}_2)$ on $\mathcal{D}$ (see Gikhman and Skorokhod, 1969; Ma, 2011a). However, inequality (2.1) is only a necessary condition and not a sufficient condition for a matrix function to be the covariance matrix function of a log-Gaussian vector random field. This is because, unlike a Gaussian vector random field, the mean function of a log-Gaussian vector random field cannot be arbitrarily assumed since it has an interdependent relationship with the covariance matrix function, as we will see in Chapter 4.

It seems that we have few ideas about the covariance structure of other non-Gaussian vector random fields with finite second-order moments. In particular, given a matrix function $\mathbf{C}(\mathbf{x}_1, \mathbf{x}_2)$ on $\mathcal{D}$ that satisfies inequality (2.1), it is often difficult to find a non-Gaussian vector random field whose covariance matrix function is $\mathbf{C}(\mathbf{x}_1, \mathbf{x}_2)$. For instance, the condition under which a function is the covariance function of a random set and of a univariate log-Gaussian process was considered by Matheron (1989), and only partial answers and counter-examples were obtained. See also Armstrong (1992).

Recall that a vector stochastic process $\{\mathbf{Z}(\mathbf{x}) : \mathbf{x} \in \mathcal{D}\}$ is weakly (second-order) stationary if its mean function is a constant vector, and its covariance matrix function

$\mathbf{C}(\mathbf{x}_1, \mathbf{x}_2)$ depends only on $\mathbf{x}_1 - \mathbf{x}_2$, in which case we simply write $\mathbf{C}(\mathbf{x}_1 - \mathbf{x}_2)$ for $\mathbf{C}(\mathbf{x}_1, \mathbf{x}_2)$. In the particular case where $\mathcal{D} = \mathbb{Z}^d$, d is a positive integer, we write $\mathbf{C}(\mathbf{n})$ for $\mathbf{C}(\mathbf{n}_1 - \mathbf{n}_2)$. We say $\{\mathbf{Z}(\mathbf{x}) : \mathbf{x} \in \mathcal{D}\}$ is strictly stationary if for any positive integer n and any $\mathbf{x}_1, \ldots, \mathbf{x}_n, \mathbf{x} \in \mathcal{D}$, the random vectors

$$(\mathbf{Z}(\mathbf{x}_1), \ldots, \mathbf{Z}(\mathbf{x}_n)) \quad \text{and} \quad (\mathbf{Z}(\mathbf{x}_1 + \mathbf{x}), \ldots, \mathbf{Z}(\mathbf{x}_n + \mathbf{x}))$$

are identically distributed. Clearly, weak and strict stationarity are equivalent for a log-Gaussian vector random field.

In general, verifying inequality (2.1) for a given matrix function is not a straightforward task. In the particular case where $\mathcal{D} = \mathbb{Z}$, from Herglotz's Theorem (see Brockwell and Davis (1991)) we obtain that the covariance matrix function $\mathbf{C}(n), n \in \mathbb{Z}$, of an m-variate stationary time series is the function that can be written in the form

$$\mathbf{C}(n) = \int_{-\pi}^{\pi} e^{in\omega} d\mathbf{F}(\omega), \qquad n \in \mathbb{Z},$$

where $\mathbf{F}(\omega), \omega \in [-\pi, \pi]$, is an $m \times m$ right-continuous, bounded matrix function with $\mathbf{F}(-\pi) = \mathbf{0}$, and for any ω_1, ω_2 with $-\pi \le \omega_1 \le \omega_2 \le \pi$, $\mathbf{F}(\omega_2) - \mathbf{F}(\omega_1)$ is Hermitian and positive definite. The matrix function $\mathbf{F}(\omega), \omega \in [-\pi, \pi]$, is called the spectral distribution matrix function of $\mathbf{C}(n)$, or its associated m-variate stationary time series, and it is uniquely determined by $\mathbf{C}(n)$. If $\mathbf{F}(\omega) = \int_{-\pi}^{\omega} \mathbf{f}(u) du$, $u \in [-\pi, \pi]$, then $\mathbf{f}(\omega)$ is called the spectral density function of $\mathbf{C}(n)$. Cramér and Kolmogorov found, independently, that inequality (2.1) holds for a matrix function, $\mathbf{C}(n), n \in \mathbb{Z}$, if and only if its spectral density function

$$\mathbf{f}(\omega) = \sum_{n=-\infty}^{\infty} \mathbf{C}(n) \cos(n\omega)$$

is positive definite for every $\omega \in [0, \pi]$. We refer the reader to Cramér (1940), Cramér and Leadbetter (1967), and Brockwell and Davis (1991) for the Cramér-Kolmogorov characterization. We will demonstrate the usefulness of this charaterization in both Chapters 3 and 4.

In the past several decades there have been many developments on the subject of long memory processes and their role in time series analysis. The applications of stationary and nonstationary processes with long memory can be found in a multitude of fields including agronomy, astronomy, climatology, economics, engineering, geophysics, and hydrology, where the common goal is to create accurate models for the phenomena we find in nature. For example, Erramilli et al. (1996) present evidence of the impact that long memory has on data network trafficking problems and more specifically, queueing performance. Montanari (2003) provides a nice summary of the role long memory plays in hydrology, as well as the continuing debate of using long memory processes versus short memory or even Markovian processes to model rainfall and river discharges. Basak et al. (2001) discuss the accuracy of using short memory ARMA processes to approximate data with long memory behavior, such as the Nile River data. Other studies about long memory and its relation to hydrology can be found in Eltahir (1996) and Mesa and Poveda (1993). Henry and Zaffaroni (2003) provide an overview of the connections found between long memory and U.S. financial history, like income and inflation series, and asset prices. They also discuss the successfulness of applying different long memory time series models to U.S. financial data. Brockwell (2007) introduces a likelihood-based approach that accurately forecasts non-Gaussian long-memory models. For further properties, models, and applications of long-range dependent and power-law Gaussian processes, see Anh et al. (2003), Baillie (1996), Beran (1992, 1994), Gao (2004), Gay and Heyde (1990), Ma (2003b), Robinson (2003), and Zaffaroni and d'Italia (2003), among others.

While it is not easy to check inequality (2.1) directly, we provide a particularly useful method for obtaining covariance matrix functions of multiple stationary Gaussian time series with long memory. In general, short memory or short-range dependent univariate processes have covariance functions that exponentially decay, while long memory or long-range dependent univariate processes have covariance functions that decay at a slower rate, perhaps at the rate of some power-law (Beran, 1994). However, there are several other formal definitions, not all of which are equivalent. We refer the reader to Palma (2007) or Taqqu (2003) for alternative definitions of long memory, some of which include the spectral density of $C(n)$, and for the necessary conditions under which these definitions are equivalent. Formally, we say a univariate stationary time series on $\mathbb{Z}$ has long memory if its covariance function is not absolutely summable, i.e.

$$\sum_{n=1}^{\infty} |C(n)| = \infty,$$

and a univariate stationary stochastic process on $\mathbb{R}$ has long memory if its covariance function is not absolutely integrable. Otherwise, we say the time series or stochastic process has short memory or short-range dependence. Alternatively, a univariate stationary time series has long-range dependence if we can demonstrate a hyperbolic decay of its covariance

$$C(n) \sim n^{-\alpha} L(n), \qquad \text{as } n \to \infty, \qquad 0 < \alpha < 1.$$

Here $L(\cdot)$ is a slowly varying function at infinity, such as $L(x) = \ln(x)$ or $L(x) = c$, where c is a positive constant. Recall that a positive measurable function on $(0, \infty)$ is said to be *slowly varying* at infinity if for all $c > 0$,

$$\lim_{x \to \infty} \frac{L(cx)}{L(x)} = 1.$$

Also, α is called the *long-memory parameter* because as α decreases, the intensity of the long memory increases (Taqqu, 2003).

Although univariate time series with long memory have been extensively studied, not much work has been undertaken about long memory multiple time series, and studies have only begun to emerge in the last ten years or so. These types of models would be useful in studying some dynamical systems, such as analyzing multiple stock market prices simulateneously or data network trafficking. For example, Coeurjolly et al. (2011) employed a wavelet analysis of multivariate fractional Brownian motion with long memory; see also

Amblard et al. (2011). Archard et al. (2008) used the multivariate long memory model to study the convergence conditions of wavelet correlations between two long memory processes. Arianos and Carbone (2009) presented a method for extracting better coupling estimates of correlated series with long memory. Sela and Hurvich (2009) created algorithms that compute the covariance matrices of multivariate time series that are derived from univariate ARFIMA models. We refer the reader to Ma (2011c), Nielsen (2004), Renshaw (1994), and Rodriguez-Iturbe et al. (1998) for further models and applications of vector processes with long memory. We say a multivariate time series has long memory if at least one of its component time series $\{Z_k(x)\}$, $k = 1, \ldots, m$, has long memory and it has short memory if all of its component time series have short memory.

Another area of recent interest is obtaining stationary Gaussian vector processes with power-law decaying or log-law decaying covariance structures. Although there have been many discoveries about covariance functions with exponential decay, research about power-law decaying and log-law decaying covariance structures has become more common in the last twenty five years. We say a covariance function has a power-law decaying structure when

$$C(n) = a|n|^{-k}, \qquad n \in \mathbb{Z},$$

and it has a log-law decaying structure when

$$C(n) = a(\log |n|)^{-k}, \qquad n \in \mathbb{Z},$$

where a and k are scalars and $k > 0$.

Stationary Gaussian processes with power-law decaying or log-law decaying covariance structures can be found in a multitude of fields including agriculture, climatology, and economics, to name a few. For example, Zhang et al. (1988) found that stationary nonequilibrium lattice gases have power-law decaying correlations. Whittle (1956, 1962) observed that data from agricultural uniformity trials had a power-law decaying covariance structure. Stolze et al. (1995) proved that the correlation functions of quantum spin chains exhibit power-law decay at varying degrees of temperature. Cang et al. (2013) show that the orientational correlation function of supercooled liquids has a logarithmic decay feature. Baddeley (1997) provided a new method for visually determining the distance between any two objects by showing that the image intensity correlation of the measurements has a log-law decaying structure. Zumbach (2012) found that a time series with a log-law decaying covariance structure is a better model for the Dow Jones Industrial Average than a series with exponential decay, two-exponential decay, or even power-law decay. So the ability to model data that has power-law decaying or log-law decaying covariance features is becoming more and more in demand.

CHAPTER 3

GAUSSIAN VECTOR TIME SERIES

This chapter explores the covariance structures of Gaussian vector time series. In Section 3.1, we focus on the covariance matrix function of a stationary Gaussian vector time series, whereas in Section 3.2, our objective is to investigate the covariance matrix function of a non-stationary Gaussian vector time series.

3.1 Stationary Time Series

First, we provide some methods for obtaining covariance matrix functions of multiple stationary Gaussian time series from covariance functions of univariate stationary Gaussian time series. In particular, Theorem 3.1 and 3.2 below are useful in constructing power-law decaying and log-law decaying direct and cross covariance structures. Then we utilize these theorems to construct examples of stationary Gaussian vector time series with long-range dependence.

3.1.1 A Method for Constructing a Covariance Matrix Function

Theorem 3.1. *Let $\alpha_1, \ldots, \alpha_m$ be positive constants. If θ is a constant between 0 and 1, and $\{g(n) : n = 0, 1, 2, \ldots\}$ is a sequence of positive numbers such that*

$$C(n; u) = \begin{cases} e^{-g(0)u}, & n = 0, \\ \theta e^{-g(|n|)u}, & n = \pm 1, \pm 2, \ldots, \end{cases} \tag{3.1}$$

is the covariance function of a univariate stationary Gaussian time series on $\mathbb{Z}$ for each fixed $u \geq 0$, then there exists an m-variate stationary Gaussian time series with direct and cross covariances

$$C_{kl}(n) = \begin{cases} \Gamma(\alpha_k + \alpha_l)\{g(0)\}^{-(\alpha_k+\alpha_l)}, & n = 0, \\ \theta\Gamma(\alpha_k + \alpha_l)\{g(|n|)\}^{-(\alpha_k+\alpha_l)}, & n = \pm 1, \pm 2, \ldots, \qquad k, l = 1, \ldots, m. \end{cases} \tag{3.2}$$

Proof. Consider an $m \times m$ matrix function, $\mathbf{A}(n)$, with entries

$$A_{kl}(n) = C(n; u)u^{\alpha_k+\alpha_l-1}, \qquad n \in \mathbb{Z}, \quad k, l = 1, \dots, m,$$

where $u > 0$ is fixed. Then $\mathbf{A}(n)$ satisfies (2.1) since for every $n \in \mathbb{N}$ and for any $\mathbf{a}_i \in \mathbb{R}^m$, $i = 1, \dots, n$,

$$\begin{aligned}
\sum_{i=1}^{n}\sum_{j=1}^{n} \mathbf{a}_i'\mathbf{A}(i-j)\mathbf{a}_j &= \sum_{i=1}^{n}\sum_{j=1}^{n}\sum_{k=1}^{m}\sum_{l=1}^{m} a_{ik}C(i-j;u)u^{\alpha_k+\alpha_l-1}a_{jl} \\
&= \sum_{i=1}^{n}\sum_{j=1}^{n}\left(\sum_{k=1}^{m} a_{ik}u^{\alpha_k-\frac{1}{2}}\right) C(i-j;u)\left(\sum_{l=1}^{m} a_{jl}u^{\alpha_l-\frac{1}{2}}\right) \\
&\geq 0,
\end{aligned}$$

where the last inequality follows from the assumption that $C(n; u)$ is a covariance function. Thus, $\mathbf{A}(n)$ is a covariance matrix function and for every $n \in \mathbb{N}$ and for any $\mathbf{a}_i \in \mathbb{R}^m$, $i = 1, \dots, n$,

$$e^{-g(0)u}\sum_{i=1}^{n}\sum_{k=1}^{m}\sum_{l=1}^{m} a_{ik}u^{\alpha_k+\alpha_l-1}a_{il} + \theta\sum_{i\neq j}^{n}\sum_{k=1}^{m}\sum_{l=1}^{m} a_{ik}u^{\alpha_k+\alpha_l-1}e^{-g(|i-j|)u}a_{jl} \geq 0. \tag{3.3}$$

Now, notice that (3.2) can be rewritten as

$$C_{kl}(n) = \begin{cases} \int_0^\infty e^{-g(0)u}u^{\alpha_k+\alpha_l-1}du, & n = 0, \\ \theta\int_0^\infty e^{-g(|n|)u}u^{\alpha_k+\alpha_l-1}du, & n = \pm 1, \pm 2, \dots, \quad k, l = 1, \dots, m. \end{cases}$$

Then, for every $n \in \mathbb{N}$ and for any $\mathbf{a}_i \in \mathbb{R}^m$, $i = 1, \dots, n$, from (3.3) we obtain

$$\begin{aligned}
&\sum_{i=1}^{n}\sum_{j=1}^{n} \mathbf{a}_i'\mathbf{C}(i-j)\mathbf{a}_j \\
&= \sum_{i=1}^{n}\sum_{j=1}^{n}\sum_{k=1}^{m}\sum_{l=1}^{m} a_{ik}C_{kl}(i-j)a_{jl} \\
&= \sum_{i=1}^{n}\sum_{k=1}^{m}\sum_{l=1}^{m} a_{ik}C_{kl}(0)a_{il} + \sum_{i\neq j}^{n}\sum_{k=1}^{m}\sum_{l=1}^{m} a_{ik}C_{kl}(i-j)a_{jl} \\
&= \int_0^\infty \left\{ e^{-g(0)u}\sum_{i=1}^{n}\sum_{k=1}^{m}\sum_{l=1}^{m} a_{ik}u^{\alpha_k+\alpha_l-1}a_{il} + \theta\sum_{i\neq j}^{n}\sum_{k=1}^{m}\sum_{l=1}^{m} a_{ik}u^{\alpha_k+\alpha_l-1}e^{-g(|i-j|)u}a_{jl} \right\} du \\
&\geq 0.
\end{aligned}$$

By Theorem 8 of Ma (2011a), there exists an m-variate Gaussian or elliptically contoured time series with (3.2) as its direct and cross covariance functions. $\square$

The covariances of $C(n;u)$ in (3.1) have exponential decay, so long as $g(n)$ is not a logarithmic function. This yields $C_{kl}(n)$ in (3.2), with the direct and cross covariances having polynomial decay. This particular construction is what enables us to construct examples of multivariate stationary Gaussian time series with power-law decaying and log-law decaying covariance structures.

One might wonder whether the $\Gamma(\alpha_k + \alpha_l)$ in Theorem 3.1 can be dropped when $\alpha_1, \ldots, \alpha_m$ are distinct. As a counterexample, consider $g(0) = 1$, $g(|n|) = |n|$, $n = \pm 1$, $\pm 2, \ldots$, and an $m \times m$ matrix function with entries

$$C_{kl}(n) = \begin{cases} 1, & n = 0, \\ \theta |n|^{-(\alpha_k+\alpha_l)}, & n = \pm 1, \pm 2, \ldots, \end{cases} \quad k, l = 1, \ldots, m.$$

This cannot be the covariance matrix function of an m-variate stationary Gaussian time series since (2.1) does not hold. For example, when $n = 3, m = 2, \alpha_1 = 1, \alpha_2 = 2, \theta \in (0, \frac{1}{2}], \mathbf{a}_1 = (0.5, -2)', \mathbf{a}_2 = (0, 0)'$, and $\mathbf{a}_3 = (\frac{20}{\theta}, -\frac{20}{\theta})'$ we have

$$\sum_{i=1}^{3}\sum_{j=1}^{3} \mathbf{a}_i' \mathbf{C}(i-j)\mathbf{a}_j = -0.25.$$

For the constant θ in Theorem 3.1, it is necessary to require that $0 \le \theta \le 1$. This follows by taking a look at the assumption that (3.1) is a univariate stationary time series when $u = 0$, for which it is necessary that $\theta \le 1$ and that

$$\begin{pmatrix} 1 & \ldots & 1 \end{pmatrix} \begin{pmatrix} 1 & \theta & \ldots & \theta \\ \theta & 1 & \ldots & \theta \\ \vdots & \vdots & \ddots & \vdots \\ \theta & \theta & \ldots & 1 \end{pmatrix} \begin{pmatrix} 1 \\ \vdots \\ 1 \end{pmatrix} \ge 0,$$

which implies $n + n(n-1)\theta \geq 0$ or $\theta \geq -\frac{1}{n-1}$ for all $n \in \mathbb{N}$. Letting $n \to \infty$ yields $\theta \geq 0$.

The next corollary, which follows from Herglotz's theorem (see Brockwell and Davis, 1991), provides a sufficent condition for Theorem 3.1 by ensuring that $C(n; u)$ in (3.1) is a covariance function for each fixed $u > 0$. It utilizes the well-known fact that the spectral density function of a univariate stationary time series

$$f(\omega) = \sum_{n=-\infty}^{\infty} C(n) \cos(n\omega), \qquad \omega \in [0, \pi],$$

is always nonnegative.

Corollary 3.1. *Let* $\alpha_1, \ldots, \alpha_m$ *be positive constants,* θ *be a constant between* 0 *and* 1*, and* $\{g(n) : n = 0, 1, 2, \ldots\}$ *be a sequence of positive numbers. If, for every fixed* $u > 0$*, the inequality*

$$e^{-g(0)u} + 2\theta \sum_{n=1}^{\infty} e^{-g(n)u} \cos(n\omega) \geq 0, \qquad \omega \in [0, \pi], \tag{3.4}$$

holds, then (3.1) forms a covariance function and hence, (3.2) forms a covariance matrix function.

3.1.2 Examples

The following examples illustrate how to use Theorem 3.1 and Corollary 3.1 to obtain covariance matrix functions for m-variate stationary Gaussian time series that, in particular, have long memory. One should note that Theorem 3.1 can also be used to construct m-variate stationary Gaussian time series that exhibit short memory. This first example shows how to use the theorem to obtain a stationary Gaussian vector time series with a power-law decaying covariance structure. The latter two examples show how use the theorem to generate a stationary Gaussian vector time series with log-law and log-log-law decaying covariances.

Example 3.1. In (3.2), taking $g(0) = 1$ and $g(n) = n, n \in \mathbb{N}$, and letting $\theta \in [0, \frac{1}{2}]$, yields an m-variate stationary Gaussian time series with power-law decaying direct and cross covariances

$$C_{kl}(n) = \begin{cases} \Gamma(\alpha_k + \alpha_l), & n = 0, \\ \theta\Gamma(\alpha_k + \alpha_l)|n|^{-(\alpha_k+\alpha_l)}, & n = \pm 1, \pm 2, \dots, \quad k, l = 1, \dots, m, \end{cases}$$

each of whose components is a univariate Gaussian time series with power-law decaying covariance investigated in Martin and Walker (1997). To verify inequality (3.4), notice that for $u > 0$,

$$\sum_{n=1}^{\infty} e^{-nu}\cos(n\omega) = \frac{e^{-u}\cos\omega - e^{-2u}}{1 - 2e^{-u}\cos\omega + e^{-2u}}, \qquad \omega \in [0, \pi]. \tag{3.5}$$

Also, it is easy to verify that

$$\frac{\cos\omega - e^{-u}}{1 - 2e^{-u}\cos\omega + e^{-2u}} \geq -(1 + e^{-u})^{-1}, \qquad \omega \in [0, \pi], \quad u > 0. \tag{3.6}$$

Hence, for $\omega \in [0, \pi]$ and $u > 0$, we obtain

$$\begin{aligned} e^{-g(0)u} + 2\theta\sum_{n=1}^{\infty} e^{-g(n)u}\cos(n\omega) &= e^{-u} + 2\theta\sum_{n=1}^{\infty} e^{-nu}\cos(n\omega) \\ &= e^{-u}\left[1 + 2\theta\frac{\cos\omega - e^{-u}}{1 - 2e^{-u}\cos\omega + e^{-2u}}\right] \\ &\geq e^{-u}\left[1 - 2\theta(1 + e^{-u})^{-1}\right] \\ &\geq 0, \end{aligned}$$

where the last inequality holds under the assumption that $0 \leq \theta \leq \frac{1}{2}$.

Example 3.2. Letting $\theta \in [0, \frac{1}{2}]$, $\beta > 0$, $g(0) = \beta$ and $g(n) = \ln n + \beta, n \in \mathbb{N}$, from (3.2) we obtain an m-variate stationary Gaussian time series with log-law decaying direct and cross covariances

$$C_{kl}(n) = \begin{cases} \Gamma(\alpha_k+\alpha_l)\beta^{-(\alpha_k+\alpha_l)}, & n=0, \\ \theta\Gamma(\alpha_k+\alpha_l)(\ln|n|+\beta)^{-(\alpha_k+\alpha_l)}, & n=\pm 1, \pm 2, \dots, \quad k,l=1,\dots,m. \end{cases}$$

Each component of such a vector time series has been studied in Ma (2002). Now, letting $\alpha_1 = \frac{u}{2}$, consider the direct covariance function $C_{11}(n)$ in Example 3.1. From Herglotz's theorem (see Brockwell and Davis (1991)), we obtain $\Gamma(u) + 2\theta\Gamma(u)\sum_{i=1}^{\infty} n^{-u}\cos(n\omega) \geq 0$, which yields $1 + 2\theta\sum_{n=1}^{\infty} n^{-u}\cos(n\omega) \geq 0$. Thus, for $\omega \in [0, \pi]$ and $u > 0$,

$$\begin{aligned} e^{-g(0)u} + 2\theta\sum_{n=1}^{\infty} e^{-g(n)u}\cos(n\omega) &= e^{-\beta u} + 2\theta\sum_{n=1}^{\infty} e^{-(\ln n+\beta)u}\cos(n\omega) \\ &= e^{-\beta u}\left[1 + 2\theta\sum_{n=1}^{\infty} n^{-u}\cos(n\omega)\right] \\ &\geq 0, \end{aligned}$$

when $0 \leq \theta \leq \frac{1}{2}$. This confirms inequality (3.4).

Similar to Example 3.2, one can show that when $\theta \in [0, \frac{1}{2}]$, $\beta \geq 1$, $g(0) = 1$ and $g(n) = \ln n + \beta, n \in \mathbb{N}$, there exists an m-variate stationary Gaussian time series with log-law decaying direct and cross covariances

$$C_{kl}(n) = \begin{cases} \Gamma(\alpha_k+\alpha_l), & n=0, \\ \theta\Gamma(\alpha_k+\alpha_l)(\ln|n|+\beta)^{-(\alpha_k+\alpha_l)}, & n=\pm 1, \pm 2, \dots, \quad k,l=1,\dots,m. \end{cases}$$

Example 3.3. Here we give an example of an m-variate stationary Gaussian time series with log-log-law decaying direct and cross covariances. To this end, let β and γ be positive constants such that $\ln\beta + \gamma > 0$ and let $\theta \in [0, \frac{1}{2}]$. Taking $g(0) = \ln\beta + \gamma$ and $g(n) = \ln(\ln n + \beta) + \gamma, n \in \mathbb{N}$, from (3.2) we obtain

$$C_{kl}(n) = \begin{cases} \Gamma(\alpha_k+\alpha_l)(\ln\beta+\gamma)^{-(\alpha_k+\alpha_l)}, & n=0, \\ \theta\Gamma(\alpha_k+\alpha_l)(\ln(\ln|n|+\beta)+\gamma)^{-(\alpha_k+\alpha_l)}, & n=\pm 1, \pm 2, \dots, \end{cases} \tag{3.7}$$

$$k, l = 1, \ldots, m.$$

To verify inequality (3.4), let $\alpha_1 = \frac{u}{2}$ and consider the direct covariance function $C_{11}(n)$ in Example 3.2. From Herglotz's theorem we obtain $\Gamma(u)\beta^{-u} + 2\theta\Gamma(u) \sum_{i=1}^{\infty} (\ln n + \beta)^{-u} \cos(n\omega) \geq 0$, which yields $\beta^{-u} + 2\theta \sum_{n=1}^{\infty} (\ln n + \beta)^{-u} \cos(n\omega) \geq 0$. Hence, for $\omega \in [0, \pi]$ and $u > 0$, we obtain

$$\begin{aligned} e^{-g(0)u} + 2\theta \sum_{n=1}^{\infty} e^{-g(n)u} \cos(n\omega) &= e^{-(\ln\beta+\gamma)u} + 2\theta \sum_{n=1}^{\infty} e^{-[\ln(\ln n+\beta)+\gamma]u} \cos(n\omega) \\ &= e^{-\gamma u} \left[\beta^{-u} + 2\theta \sum_{n=1}^{\infty} (\ln n + \beta)^{-u} \cos(n\omega) \right] \\ &\geq 0, \end{aligned}$$

when $0 \leq \theta \leq \frac{1}{2}$; that is, inequality (3.2) holds.

Similar to Example 3.3, it can be shown that when β and γ are positive constants with $\beta \geq 1$, $\theta \in [0, \frac{1}{2}]$, $g(0) = \gamma$ and $g(n) = \ln(\ln n + \beta) + \gamma, n \in \mathbb{N}$, there exists an m-variate stationary Gaussian time series with log-log-law decaying direct and cross covariances

$$C_{kl}(n) = \begin{cases} \Gamma(\alpha_k + \alpha_l)\gamma^{-(\alpha_k+\alpha_l)}, & n = 0, \\ \theta\Gamma(\alpha_k + \alpha_l)(\ln(\ln |n| + \beta) + \gamma)^{-(\alpha_k+\alpha_l)}, & n = \pm 1, \pm 2, \ldots. \end{cases} \tag{3.8}$$

$$k, l = 1, \ldots, m.$$

The stationary Gaussian vector time series in Examples 3.1, 3.2, and 3.3 have long memory when at least one of the $\alpha_k \leq \frac{1}{2}, k = 1, \ldots, m$, since for positive constants β and γ with $\ln \beta + \gamma > 0$,

(i) $\sum_{n=1}^{\infty} |C(n)| = \theta\Gamma(2\alpha) \sum_{n=1}^{\infty} \frac{1}{n^{2\alpha}}$ diverges,

(ii) $\sum_{n=1}^{\infty} |C(n)| = \theta\Gamma(2\alpha) \sum_{n=1}^{\infty} \frac{1}{(\ln n+\beta)^{2\alpha}}$ diverges, and

(iii) $\sum_{n=1}^{\infty} |C(n)| = \theta\Gamma(2\alpha) \sum_{n=1}^{\infty} \frac{1}{(\ln(\ln n+\beta)+\gamma)^{2\alpha}}$ diverges when $0 < \alpha \leq \frac{1}{2}$. Otherwise, these time series have short memory.

3.1.3 Extensions to the Method

The following two corollaries provide alternative methods for obtaining a $g(n)$ that satisfies the sufficient condition in Theorem 3.1. The first corollary is especially useful in constructing covariance matrix functions like Examples 3.1, 3.2, and 3.3. It states that for a function $g(n)$ that satisfies the sufficent condition in Theorem 3.1, a linear function of $g(n)$ will also satisfy that condition. It also allows us to construct a covariance matrix function with as many log's as we desire.

Corollary 3.2. *Let $\{g(n) : n = 0, 1, 2, \ldots\}$ be a sequence of positive numbers such that (3.1) is the covariance function of a univariate stationary Gaussian time series for each fixed $u \geq 0$.*

(i) Let $h(n) = ag(n) + b, n = 0, 1, 2, \ldots,$ where a and b are constants with $a > 0$. Then

$$D_u(n) = \begin{cases} e^{-h(0)u}, & n = 0, \\ \theta e^{-h(|n|)u}, & n = \pm 1, \pm 2, \ldots \end{cases} \tag{3.9}$$

is the covariance function of a univariate stationary Gaussian time series for each fixed $u \geq 0$.

(ii) Let $h(n) = \ln g(n), n = 0, 1, 2, \ldots$. Then $D_u(n)$ defined in (3.9) is the covariance function of a univariate stationary Gaussian time series on $\mathbb{Z}$ for each fixed $u \geq 0$.

Proof. (i) Since (3.1) is a covariance function, it follows that for all $n \in \mathbb{N}$ and for all $a_i \in \mathbb{R}, i = 1, \ldots, n$,

$$\sum_{i=1}^{n}\sum_{j=1}^{n} a_i D_u(i-j) a_j = \sum_{i=1}^{n}\sum_{j=1}^{n} a_i e^{-bu} C_{au}(n) a_j \geq 0.$$

Therefore, inequality (2.1) is satisfied for all $u \geq 0$.

(ii) First, observe that $D_u(n) = \frac{1}{\Gamma(u)} \int_0^\infty C_t(n) t^{u-1} dt$. Hence, inequality (2.1) is satisfied for all $u \geq 0$ since

$$\sum_{i=1}^{n} \sum_{j=1}^{n} a_i D_u(i-j) a_j = \frac{1}{\Gamma(u)} \int_0^\infty \left\{ \sum_{i=1}^{n} \sum_{j=1}^{n} a_i C_t(i-j) a_j \right\} t^{u-1} dt \geq 0$$

holds for all $n \in \mathbb{N}$ and for all $a_i \in \mathbb{R}, i = 1, \ldots, n$. □

A classical result of Young (1913) claims that if a sequence of real numbers $C(0), C(1), C(2), \ldots$ is convex and decreasing to zero, then

$$C(0) + 2 \sum_{n=1}^{\infty} C(n) \cos(n\omega) \geq 0, \qquad \omega \in [0, \pi].$$

In other words, the sequence $\{C(|n|) : n \in \mathbb{Z}\}$ can be treated as the covariance function of a univariate stationary time series by Herglotz's theorem (Brockwell and Davis (1991)). The next corollary is a matrix version of Young's theorem (1913).

Corollary 3.3. *Let $\alpha_1, \ldots, \alpha_m$ be positive constants. If a sequence of positive numbers $\{g(n) : n = 0, 1, 2, \ldots\}$ is concave, increasing, and diverges to infinity, then there exists an m-variate stationary Gaussian time series with direct and cross covariances*

$$C_{kl}(n) = \Gamma(\alpha_k + \alpha_l) \{g(|n|)\}^{-(\alpha_k + \alpha_l)}, \qquad n \in \mathbb{Z}, \quad k, l = 1, \ldots, m. \tag{3.10}$$

Proof. Take $\theta = 1$ in (3.1). Since $\{g(n) : n = 0, 1, 2, \ldots\}$ is concave and increasing to infinity, $\{e^{-g(n)u} : n = 0, 1, 2, \ldots\}$ is convex and decreasing to 0. Then by Young's theorem (1913), inequality (3.4) holds. □

3.1.4 Covariance Structures with a Conditionally Negative Definite Matrix

Next we present a covariance matrix structure that is more general than that in Theorem 3.1, with a conditionally negative definite matrix as an important ingredient. Similar to Theorem 3.1, we can use this theorem to construct examples of Gaussian vector time series whose direct and cross covariances have power-law decaying features. First, we define a conditionally negative definite matrix and provide some examples of such matrices.

Definition 3.1. *For an integer $m \geq 2$, an $m \times m$ symmetric matrix $\boldsymbol{\Theta} = (\theta_{kl})$ is said to be conditionally negative definite if the inequality*

$$\sum_{k=1}^{m} \sum_{l=1}^{m} a_k a_l \theta_{kl} \leq 0 \tag{3.11}$$

holds for any real numbers $a_1, \ldots, a_m$ subject to $\sum_{k=1}^{m} a_k = 0$.

A necessary condition for (3.11) is

$$\frac{\theta_{kk} + \theta_{ll}}{2} \leq \theta_{kl}, \tag{3.12}$$

which is derived from (3.11) by letting $a_k = 1, a_l = -1$, and $a_i = 0$ for all $i \neq k, l$. Inequality (3.12) implies that all entries of a conditionally negative definite matrix are positive when its diagonal entries are positive. Some examples of conditionally negative definite matrices, $\boldsymbol{\Theta} = (\theta_{kl})$, are

(i) $\theta_{kl} \equiv \theta$,

(ii) $\theta_{kl} = \theta_k + \theta_l$, $k, l = 1, \ldots, m$,

(iii) $\theta_{kl} = \max(\theta_k, \theta_l)$, $k, l = 1, \ldots, m$,

where θ and θ_k, $k = 1, \ldots, m$, are constants.

Theorem 3.2. *Let $\alpha_1, \ldots, \alpha_m$ be positive constants and $\boldsymbol{\Theta} = (\theta_{kl})$ be an $m \times m$ conditionally negative definite matrix with nonnegative entries. Let θ be a nonnegative constant and $\{g(n) :$*

$n = 0, 1, 2, \ldots\}$ *be a sequence of nonnegative numbers such that (3.1) is the covariance function of a univariate stationary Gaussian time series on* $\mathbb{Z}$ *for each fixed* $u \geq 0$*. Then there exists an* m*-variate stationary Gaussian time series with direct and cross covariances*

$$C_{kl}(n) = \begin{cases} \Gamma(\alpha_k + \alpha_l)\{g(0) + \theta_{kl}\}^{-(\alpha_k+\alpha_l)}, & n = 0, \\ \theta\Gamma(\alpha_k + \alpha_l)\{g(|n|) + \theta_{kl}\}^{-(\alpha_k+\alpha_l)}, & n = \pm 1, \pm 2, \ldots, \end{cases} \tag{3.13}$$

$$k, l = 1, \ldots, m,$$

so long as $g(|n|) + \theta_{kk} > 0$ *for all* $n \in \mathbb{Z}$ *and for all* $k = 1, \ldots, m$*.*

Proof. The matrix $u\boldsymbol{\Theta}$ is conditionally negative definite for any fixed $u \geq 0$ since the matrix $\boldsymbol{\Theta}$ is by assumption. By Theorem 4.1.3 of Bapat and Raghavan (1997), if $\mathbf{A}$ is a conditionally negative definite matrix, then $\exp(-\mathbf{A})$ is a positive semidefinite matrix, where $\exp(\mathbf{A})$ denotes the Hadamard exponential of $\mathbf{A}$ as follows:

$$\exp(\mathbf{A}) = \mathbf{1} + \mathbf{A} + \frac{1}{2!}\mathbf{A} \circ \mathbf{A} + \ldots + \frac{1}{n!}\underbrace{\mathbf{A} \circ \cdots \circ \mathbf{A}}_{n \text{ terms}} + \cdots,$$

with $\mathbf{1}$ being a matrix of the same size as $\mathbf{A}$ whose entries are all equal to 1 and with $\mathbf{A} \circ \mathbf{B}$ being the Hadamard product of matrices $\mathbf{A}$ and $\mathbf{B}$ of the same size. Thus, the matrix $\exp(-u\boldsymbol{\Theta})$ is positive semidefinite. Now, consider an $m \times m$ matrix function, $\mathbf{A}(n)$, with entries

$$A_{kl}(n) = C(n; u)u^{\alpha_k+\alpha_l-1}, \qquad n \in \mathbb{Z}, \quad k, l = 1, \ldots, m,$$

where $u > 0$ is fixed. In the proof of Theorem 3.1, we showed that $\mathbf{A}(n)$ is a covariance matrix function. Hence, by Corollary 3.1 of Ma (2011b), $\mathbf{A}(n) \circ \exp(-\boldsymbol{\Theta}u)$ is a covariance matrix function. So, for every $n \in \mathbb{N}$ and for any $\mathbf{a}_i \in \mathbb{R}^m$, $i = 1, \ldots, n$,

$$e^{-g(0)u} \sum_{i=1}^{n} \sum_{k=1}^{m} \sum_{l=1}^{m} a_{ik} u^{\alpha_k+\alpha_l-1} e^{-\theta_{kl}u} a_{il} + \theta \sum_{i \neq j}^{n} \sum_{k=1}^{m} \sum_{l=1}^{m} a_{ik} u^{\alpha_k+\alpha_l-1} e^{-[g(|i-j|)+\theta_{kl}]u} a_{jl} \geq 0,$$

from which we obtain

$$\begin{aligned}\sum_{i=1}^{n}\sum_{j=1}^{n}\mathbf{a}_i'\mathbf{C}(i-j)\mathbf{a}_j &= \sum_{i=1}^{n}\sum_{k=1}^{m}\sum_{l=1}^{m} a_{ik}C_{kl}(0)a_{il} + \sum_{i\neq j}^{n}\sum_{k=1}^{m}\sum_{l=1}^{m} a_{ik}C_{kl}(i-j)a_{jl} \\ &= \int_0^\infty \left\{ e^{-g(0)u}\sum_{i=1}^{n}\sum_{k=1}^{m}\sum_{l=1}^{m} a_{ik}u^{\alpha_k+\alpha_l-1}e^{-\theta_{kl}u}a_{il} \right. \\ &\left. +\theta\sum_{i\neq j}^{n}\sum_{k=1}^{m}\sum_{l=1}^{m} a_{ik}u^{\alpha_k+\alpha_l-1}e^{-[g(|i-j|)+\theta_{kl}]u}a_{jl} \right\} du \\ &\geq 0\end{aligned}$$

for every $n \in \mathbb{N}$ and for any $\mathbf{a}_i \in \mathbb{R}^m$, $i = 1, \ldots, n$. Therefore, by Theorem 8 of Ma (2011a), there exists an m-variate Gaussian or elliptically contoured time series with (3.13) as direct/cross covariance functions. □

One might wonder whether the $\Gamma(\alpha_k + \alpha_l)$ in Theorem 3.2 can be dropped when $\alpha_1, \ldots, \alpha_m$ are distinct. As a counterexample, consider $g(|n|) = |n|, n \in \mathbb{Z}$, and an $m \times m$ matrix function with entries

$$C_{kl}(n) = \begin{cases} \{\theta_{kl}\}^{-(\alpha_k+\alpha_l)}, & n = 0, \\ \theta\{|n| + \theta_{kl}\}^{-(\alpha_k+\alpha_l)}, & n = \pm 1, \pm 2, \ldots, \end{cases} \quad k, l = 1, \ldots, m.$$

This is not a covariance matrix function since (2.1) does not hold. For example, when $n = 3, m = 2, \alpha_1 = 1, \alpha_2 = \frac{3}{2}, \theta = 1, \boldsymbol{\Theta} = \left(\begin{smallmatrix}1 & 1\\ 1 & 1\end{smallmatrix}\right), \mathbf{a}_1 = (0.5, -2)', \mathbf{a}_2 = (0, 0)'$, and $\mathbf{a}_3 = (50, -50)'$ we have

$$\sum_{i=1}^{3}\sum_{j=1}^{3}\mathbf{a}_i'\mathbf{C}(i-j)\mathbf{a}_j \approx -0.8245.$$

Example 3.4. Letting $\theta = 1, 0 < \beta \leq 2$, and $g(n) = n^\beta, n = 0, 1, 2, \ldots$, from (3.13) we obtain an m-variate stationary Gaussian time series with power-law decaying direct and cross covariances

$$C_{kl}(n) = \Gamma(\alpha_k + \alpha_l)(|n|^\beta + \theta_{kl})^{-(\alpha_k+\alpha_l)}, \qquad n \in \mathbb{Z}, \quad k, l = 1, \ldots, m,$$

since $\{e^{-|n|^\beta u} : n \in \mathbb{Z}\}$ is a stationary covariance sequence when $0 < \beta \le 2$; see, for example, Lemma 1 (ii) of Ma (2003a).

The m-variate Gaussian time series in Example 3.4 has long memory when at least one of the $\alpha_k \le \frac{1}{2\beta}, k = 1, \ldots, m$. To see this, first fix $\beta > 0$. Then for some $N \in \mathbb{N}$, $(2N^\beta)^{2\alpha} \ge (N^\beta + \theta)^{2\alpha}$ and $\sum\limits_{n=1}^{\infty} \frac{1}{n^{2\alpha\beta}}$ diverges when $0 < \alpha < \frac{1}{2\beta}$. Hence, $\sum_{n=1}^{\infty} |C(n)|$ diverges since

$$\sum_{n=1}^{\infty} |C(n)| = \Gamma(2\alpha) \sum_{n=1}^{\infty} \frac{1}{(n^\beta + \theta)^{2\alpha}} \ge \Gamma(2\alpha) \left[\sum_{n=1}^{N-1} \frac{1}{(n^\beta + \theta)^{2\alpha}} + \sum_{n=N}^{\infty} \frac{1}{(2n^\beta)^{2\alpha}} \right]$$

and the right hand side of this equation diverges when $0 < \alpha \le \frac{1}{2\beta}$.

3.2 Non-stationary Time Series

The next theorem presents a covariance matrix structure for a nonstationary Gaussian vector time series. It utilizes the conditionally negative definite matrix and the max function to construct a covariance matrix function with power-law decaying direct and cross covariances. The necessity of the gamma function is also explained.

Theorem 3.3. *If an $m \times m$ matrix $\mathbf{\Theta} = (\theta_{kl})$ is conditionally negative definite with positive diagonal entries and $\alpha_1, \ldots, \alpha_m$ are positive constants, then there exists an m-variate Gaussian time series with direct and cross covariances*

$$C_{kl}(x_1, x_2) = \Gamma(\alpha_k + \alpha_l)(\max(|x_1|, |x_2|) + \theta_{kl})^{-(\alpha_k + \alpha_l)}, \qquad x_1, x_2 \in \mathbb{Z}, \quad k, l = 1, \ldots, m. \quad (3.14)$$

Proof. First we rewrite (3.14) as

$$C_{kl}(x_1, x_2) = \int_0^\infty u^{\alpha_k + \alpha_l - 1} e^{-[\max(|x_1|, |x_2|) + \theta_{kl}]u} du, \qquad x_1, x_2 \in \mathbb{Z}, \quad k, l = 1, \ldots, m. \quad (3.15)$$

By assumption, the matrix $\boldsymbol{\Theta}$ is conditionally negative definite. So is $u\boldsymbol{\Theta}$ for any fixed $u \geq 0$. By Theorem 4.1.3 of Bapat and Raghavan (1997), the matrix $\exp(-u\boldsymbol{\Theta})$ is positive semidefinite. The symmetric matrix function $\mathbf{C}(x_1, x_2) \equiv \left(e^{-\max(|x_1|,|x_2|)}\right)_{m\times m}$ is the covariance matrix of an m-variate Gaussian times series since inequality (2.1) is satisfied with

$$\begin{aligned}
\sum_{i=1}^{n}\sum_{j=1}^{n} \mathbf{a}_i' \mathbf{C}(x_i, x_j)\mathbf{a}_j &= \sum_{i=1}^{n}\sum_{j=1}^{n}\sum_{k=1}^{m}\sum_{l=1}^{m} a_{ik} e^{-\max(|x_i|,|x_j|)} a_{jl} \\
&= \sum_{i=1}^{n}\sum_{j=1}^{n}\sum_{k=1}^{m}\sum_{l=1}^{m} a_{ik} \min(e^{-|x_i|}, e^{-|x_j|}) a_{jl} \\
&= \int_0^{\infty} \left(\sum_{i=1}^{n}\sum_{k=1}^{m} a_{ik} I_{\left[0, e^{-|x_i|}\right]}(u) \right)^2 du \\
&\geq 0,
\end{aligned}$$

for every $n \in \mathbb{N}$, any $\mathbf{a}_i \in \mathbb{R}^m$, and any $x_i \in \mathbb{Z}, i = 1, \ldots, n$. Moreover, there exists a Gaussian vector time series with (3.14) as its direct and cross covariances, by Theorem 4 of Ma (2011b). □

The $\Gamma(\alpha_k + \alpha_l)$ in Theorem 3.3 cannot be dropped when $\alpha_1, \ldots, \alpha_m$ are distinct. As a counterexample, consider (2.1) when $n = 3, m = 2, \alpha_1 = 1, \alpha_2 = \frac{3}{2}, \boldsymbol{\Theta} = \left(\begin{smallmatrix} 1 & 1 \\ 1 & 1 \end{smallmatrix}\right), x_1 = 2, x_2 = 0, n_3 = 0, \mathbf{a}_1 = (0.5, -2)', \mathbf{a}_2 = (0, 0)'$, and $\mathbf{a}_3 = (50, -50)'$. Then,

$$\sum_{i=1}^{3}\sum_{j=1}^{3} \mathbf{a}_i' \mathbf{C}(i-j)\mathbf{a}_j \approx -3.0269.$$

CHAPTER 4

LOG-GAUSSIAN RANDOM FIELDS

This chapter discusses the covariance structures of log-Gaussian random fields. In Section 4.1, the covariance matrix function of a log-Gaussian vector random field is explored. In Section 4.2, research is presented on the covariance matrix function of a univariate log-Gaussian random field.

4.1 Vector Random Fields

In this section we investigate the covariance structure of a log-Gaussian vector random field. Just like the Gaussian vector random field, a log-Gaussian vector random field is characterized by its mean function and covariance matrix function. However, unlike the Gaussian vector random field, the mean function and the covariance matrix function of a log-Gaussian vector random field are interdependent. As a result, an additional requirement for the covariance matrix function is needed.

4.1.1 A Method for Identifying a Covariance Matrix Function

The following theorem provides the necessary and sufficient conditions for a matrix function to be the covariance matrix function of a log-Gaussian vector random field.

Theorem 4.1. *(i) If* $\{\mathbf{Z}(\mathbf{x}) : \mathbf{x} \in \mathcal{D}\}$ *is an* m*-variate log-Gaussian random field with mean function* $\boldsymbol{\mu}(\mathbf{x})$ *and covariance matrix function* $\mathbf{C}(\mathbf{x}_1, \mathbf{x}_2)$, $\mathbf{x}_1, \mathbf{x}_2 \in \mathcal{D}$, *then* $\boldsymbol{\mu}(\mathbf{x})$ *is a positive vector function,* $C_{kl}(\mathbf{x}_1, \mathbf{x}_2) > -\mu_k(\mathbf{x}_1)\mu_l(\mathbf{x}_2)$, $k, l = 1, \dots, m$, *and the matrix function with* kl*th entry* $\ln\{1 + \mu_k^{-1}(\mathbf{x}_1)\mu_l^{-1}(\mathbf{x}_2)C_{kl}(\mathbf{x}_1, \mathbf{x}_2)\}$ *satisfies inequality (2.1).*

(ii) Conversely, if $\boldsymbol{\mu}(\mathbf{x}), \mathbf{x} \in \mathcal{D}$, *is a positive vector function and the matrix with* kl*th entry* $\ln\{1 + \mu_k^{-1}(\mathbf{x}_1)\mu_l^{-1}(\mathbf{x}_2)C_{kl}(\mathbf{x}_1, \mathbf{x}_2)\}$, $k, l = 1, \dots, m$, *satisfies inequality (2.1), then there exists an* m*-variate log-Gaussian random field with mean function* $\boldsymbol{\mu}(\mathbf{x})$ *and covariance matrix function* $\mathbf{C}(\mathbf{x}_1, \mathbf{x}_2)$.

Proof. (i) Since $\{\mathbf{Z}(\mathbf{x}) : \mathbf{x} \in \mathcal{D}\}$ is an m-variate log-Gaussian random field, $\{\mathbf{Y}(\mathbf{x}) = (\ln Z_1(\mathbf{x}), \ldots, \ln Z_m(\mathbf{x})) : \mathbf{x} \in \mathcal{D}\}$ is an m-variate Gaussian vector random field with mean function $\boldsymbol{\mu}_{\mathbf{Y}}(\mathbf{x})$ and covariance matrix function $\mathbf{C}_{\mathbf{Y}}(\mathbf{x}_1, \mathbf{x}_2)$, $\mathbf{x}_1, \mathbf{x}_2 \in \mathcal{D}$. As in Appendix A.8 of Chilès and Delfiner (1999), we obtain

$$\mu_k(\mathbf{x}) = e^{\mu_{Y_k}(\mathbf{x}) + \frac{1}{2} C_{Y_k}(\mathbf{x}, \mathbf{x})}, \quad k = 1, \ldots, m, \ \mathbf{x} \in \mathcal{D},$$

and

$$C_{kl}(\mathbf{x}_1, \mathbf{x}_2) = \mu_k(\mathbf{x}_1)\mu_l(\mathbf{x}_2) \left[e^{C_{Y_k, Y_l}(\mathbf{x}_1, \mathbf{x}_2)} - 1 \right], \quad k, l = 1, \ldots, m, \ \mathbf{x}_1, \mathbf{x}_2 \in \mathcal{D}.$$

Clearly, $\boldsymbol{\mu}(\mathbf{x})$ is a positive vector function and $C_{kl}(\mathbf{x}_1, \mathbf{x}_2) > -\mu_k(\mathbf{x}_1)\mu_l(\mathbf{x}_2)$, $k, l = 1, \ldots, m$. By expressing $\mathbf{C}_{\mathbf{Y}}(\mathbf{x}_1, \mathbf{x}_2)$ in terms of $\mathbf{C}(\mathbf{x}_1, \mathbf{x}_2)$, we obtain

$$C_{Y_k, Y_l}(\mathbf{x}_1, \mathbf{x}_2) = \ln\{1 + \mu_k^{-1}(\mathbf{x}_1)\mu_l^{-1}(\mathbf{x}_2) C_{kl}(\mathbf{x}_1, \mathbf{x}_2)\}, \quad k, l = 1, \ldots, m, \ \mathbf{x}_1, \mathbf{x}_2 \in \mathcal{D}.$$

From properties of a covariance matrix, $\mathbf{C}_{\mathbf{Y}}(\mathbf{x}_1, \mathbf{x}_2)$ must satisfy inequality (2.1).

(ii) Since $\boldsymbol{\mu}(\mathbf{x})$ is a positive function and the matrix function with klth entry $\ln\{1 + \mu_k^{-1}(\mathbf{x}_1)\mu_l^{-1}(\mathbf{x}_2) C_{kl}(\mathbf{x}_1, \mathbf{x}_2)\}$, $k, l = 1, \ldots, m$, satisfies inequality (2.1), there exists a Gaussian vector random field $\{\mathbf{Y}(\mathbf{x}) : \mathbf{x} \in \mathcal{D}\}$ with mean

$$\mu_{Y_k}(\mathbf{x}) = \ln \mu_k(\mathbf{x}) - \frac{1}{2} \ln\{1 + \mu_k^{-2}(\mathbf{x}) C_{kk}(\mathbf{x}, \mathbf{x})\}, \quad k = 1, \ldots, m,$$

and covariance matrix function with klth entry

$$C_{Y_k, Y_l}(\mathbf{x}_1, \mathbf{x}_2) = \ln\{1 + \mu_k^{-1}(\mathbf{x}_1)\mu_l^{-1}(\mathbf{x}_2) C_{kl}(\mathbf{x}_1, \mathbf{x}_2)\}, \quad k, l = 1, \ldots, m.$$

Consequently, $\{e^{\mathbf{Y}(\mathbf{x})} = (e^{Y_1(\mathbf{x})}, \ldots, e^{Y_m(\mathbf{x})}) : \mathbf{x} \in \mathcal{D}\}$ is an m-variate log-Gaussian random field with corresponding mean function $\boldsymbol{\mu}(\mathbf{x})$ and covariance matrix function $\mathbf{C}(\mathbf{x}_1, \mathbf{x}_2)$. □

It is well known that for a given covariance matrix function $\mathbf{C}(\mathbf{x}_1, \mathbf{x}_2)$, a Gaussian vector random field can have any specificed mean function $\boldsymbol{\mu}(\mathbf{x})$. However, this is not the case for a log-Gaussian vector random field, as the next two corollaries show.

Corollary 4.1. *If for every choice of positive constant mean vector, $\boldsymbol{\mu}$, there exists a log-Gaussian vector random field with covariance matrix function $\mathbf{C}(\mathbf{x}_1, \mathbf{x}_2)$, $\mathbf{x}_1, \mathbf{x}_2 \in \mathcal{D}$, then $\mathbf{C}(\mathbf{x}_1, \mathbf{x}_2)$ must have nonnegative entries.*

Proof. To see this, notice that $C_{kl}(\mathbf{x}_1, \mathbf{x}_2) > -\mu_k \mu_l$ for any positive numbers μ_k, μ_l, $k, l = 1, \ldots, m$. Letting $\boldsymbol{\mu}$ approach $\mathbf{0}$ yields $\mathbf{C}(\mathbf{x}_1, \mathbf{x}_2) \geq \mathbf{0}$ for all $\mathbf{x}_1, \mathbf{x}_2 \in \mathcal{D}$. □

Corollary 4.2. *For any positive constant vector $\boldsymbol{\mu}$, a nonnegative $m \times m$ matrix function $\mathbf{C}(\mathbf{x}_1, \mathbf{x}_2)$, $\mathbf{x}_1, \mathbf{x}_2 \in \mathcal{D}$, is the covariance matrix function for an m-variate log-Gaussian random field with mean vector $\boldsymbol{\mu}$ if and only if the matrix with klth entry $\ln\{1+\mu_k \mu_l C_{kl}(x_1, x_2)\}$, $k, l = 1, \ldots, m$, satisfies inequality (2.1) for every positive constant vector $\boldsymbol{\mu}$.*

4.1.2 Entanglement of the Mean and the Covariance Function

From Theorem 4.1 we see the intricate relationship between the mean and the covariance of a log-Gaussian random field. Because of this relationship, a covariance matrix function of a Gaussian vector random field may not necessarily be the covariance matrix function of a log-Gaussian vector random field. We demonstrate this in the next example, first discussing the univariate case and then the vector case.

Example 4.1. For positive constants μ and σ, we are able to obtain a stationary Gaussian process and a stationary log-Gaussian process on $\mathcal{D}$ with mean μ and covariance function

$$C(\mathbf{x}) = \begin{cases} \sigma^2, & \mathbf{x} = \mathbf{0}, \\ 0, & \mathbf{x} \neq \mathbf{0}, \quad \mathbf{x} \in \mathbb{R}^d, \end{cases}$$

since $C(\mathbf{x})$ and $\ln\{1 + \mu^{-2} C(\mathbf{x})\}$ are both positive definite. This process may be called the nugget model or white noise. Now let us consider the matrix function,

$$C_{kl}(\mathbf{x}) = \begin{cases} \sigma_k \sigma_l, & \mathbf{x} = \mathbf{0}, \\ 0, & \mathbf{x} \neq \mathbf{0}, \quad \mathbf{x} \in \mathbb{R}^d, \quad k, l = 1, \ldots, m. \end{cases} \tag{4.1}$$

Clearly this is the covariance matrix function of a stationary Gaussian vector random field since $\mathbf{C}(\mathbf{x})$ satisfies inequality (2.1). When $\mu_k = \lambda\sigma_k, k = 1, \ldots, m$, for any positive constant λ, then (4.1) is the covariance matrix function of a log-Gaussian vector random field with mean $\boldsymbol{\mu}$ since

$$\sum_{i=1}^{n}\sum_{j=1}^{n}\sum_{k=1}^{m}\sum_{l=1}^{m} a_{ik}a_{jl}\ln\{1+\mu_k^{-1}\ \mu_l^{-1}\ C_{kl}(x_i,x_j)\} = \sum_{i=1}^{n}\sum_{k=1}^{m}\sum_{l=1}^{m} a_{ik}a_{il}\ln\{1+\lambda^{-2}\} \geq 0,$$

for every $n \in \mathbb{N}$ and for any $\mathbf{a}_i \in \mathbb{R}^m, i = 1, \ldots, n$. However, in general, (4.1) is not the covariance matrix function of a log-Gaussian vector random field. Let $m = n = 2, \mathbf{a}_1 = (1\ 1)'$, and $\mathbf{a}_2 = (1\ \ -6)'$ in inequality (2.1). If $\boldsymbol{\sigma} = (10\ 1)'$ and $\boldsymbol{\mu} = (1\ 2)'$, then

$$\sum_{i=1}^{2}\sum_{j=1}^{2}\sum_{k=1}^{2}\sum_{l=1}^{2} a_{ik}a_{jl}\ln\{1+\mu_k^{-1}\mu_l^{-1}C_{kl}(\mathbf{x}_i-\mathbf{x}_j)\} = -.4310.$$

In the previous example, we showed how (4.1) may not be the covariance matrix function of a log-Gaussian vector random field when it was not a function of the mean vector. In general, the covariance function needs to be a function of the mean in order to satisfy inequality (2.1). We illustrate this relationship further with the use of the following example.

Example 4.2. For positive m-variate vectors $\boldsymbol{\mu}(x), x \geq 0$, and $\boldsymbol{\sigma}$, there exists an m-variate log-Gaussian random field with mean function $\boldsymbol{\mu}(x)$ and covariance matrix function

$$C_{kl}(x_1,x_2) = \mu_k(x_1)\mu_l(x_2)\min(\sigma_k x_1, \sigma_l x_2), \quad x_1, x_2 \geq 0, \quad k,l = 1,\ldots,m. \tag{4.2}$$

Proof. For every $n \in \mathbb{N}$ and for any $\mathbf{a}_i \in \mathbb{R}^m, i = 1, \ldots, n$, we have

$$\begin{aligned}
&\sum_{i=1}^{n}\sum_{j=1}^{n}\sum_{k=1}^{m}\sum_{l=1}^{m} a_{ik}a_{jl} \ln\{1 + \mu_k^{-1}(x_i)\ \mu_l^{-1}(x_j)\ C_{kl}(x_i, x_j)\} \\
&= \sum_{i=1}^{n}\sum_{j=1}^{n}\sum_{k=1}^{m}\sum_{l=1}^{m} a_{ik}a_{jl} \ln\{1 + \ \min(\sigma_k x_i, \sigma_l x_j)\} \\
&= \sum_{i=1}^{n}\sum_{j=1}^{n}\sum_{k=1}^{m}\sum_{l=1}^{m} a_{ik}a_{jl}\ \min(\ln\{1 + \sigma_k x_i\}, \ln\{1 + \sigma_l x_j\}) \\
&= \sum_{i=1}^{n}\sum_{j=1}^{n}\sum_{k=1}^{m}\sum_{l=1}^{m} a_{ik}a_{jl} \int_0^\infty I_{[0,\ln\{1+\sigma_k x_i\}]}(u) I_{[0,\ln\{1+\sigma_l x_j\}]}(u)\ du \\
&= \int_0^\infty \left(\sum_{i=1}^{n}\sum_{k=1}^{m} I_{[0,\ln\{1+\sigma_k x_i\}]}(u) \right)^2 du \\
&\geq 0,
\end{aligned}$$

and hence, Theorem 4.1 is satisfied. □

Notice that when $\boldsymbol{\mu} = \mu\mathbf{1}$, where μ is a positive constant and $\mathbf{1}$ is an m-variate vector with all entries equal to one, (4.2) can be reduced to

$$C_{kl}(x_1, x_2) = \min(\sigma_k x_1, \sigma_l x_2), \quad x_1, x_2 \geq 0, \quad k, l = 1, \ldots, m. \tag{4.3}$$

In particular, if $m = 1$, then (4.3) is the covariance function of the Wiener process or Brownian motion on $[0, \infty]$.

However, if we allow $\boldsymbol{\mu}$ to be anything other than $\boldsymbol{\mu} = \mu\mathbf{1}$ in (4.2), then we can no longer guarantee that (4.3) is the covariance matrix function of an m-variate log-Gaussian process. Consider $m = n = 2, \mathbf{a}_1 = (15\ \ -20)'$, and $\mathbf{a}_2 = (10\ \ -15)'$ in inequality (2.1). If $\boldsymbol{\sigma} = (2\ 1.75)'$, $\boldsymbol{\mu} = (1\ 2)'$, and $\mathbf{x} = (2\ 3)'$, then

$$\sum_{i=1}^{2}\sum_{j=1}^{2}\sum_{k=1}^{2}\sum_{l=1}^{2} a_{ik}a_{jl} \ln\{1 + \mu_k^{-1}\mu_l^{-1} C_{kl}(x_i, x_j)\} = -30.1830.$$

So, the covariance matrix function $\mathbf{C}(x_1, x_2)$ in (4.2) being a function of the mean $\boldsymbol{\mu}(x)$ is an important requirement.

4.1.3 Preservation Properties

For a Gaussian vector random field there are several nice operation preserving properties in regard to not only the covariance matrix function but also to the random field as a whole. For example, suppose $\{\mathbf{Z}(x) : \mathbf{x} \in \mathcal{D}\}$ and $\{\mathbf{W}(x) : \mathbf{x} \in \mathcal{D}\}$ are independent Gaussian vector random fields with respective mean functions $\boldsymbol{\mu}(\mathbf{x})$ and $\boldsymbol{\mu}_{\mathbf{W}}(\mathbf{x})$ and covariance matrix functions $\mathbf{C}(\mathbf{x}_1, \mathbf{x}_2)$ and $\mathbf{C}_{\mathbf{W}}(\mathbf{x}_1, \mathbf{x}_2)$, $\mathbf{x}_1, \mathbf{x}_2 \in \mathcal{D}$, and let α and β be scalars. Then

(i) $\{\alpha \mathbf{Z}(\mathbf{x}) + \beta : \mathbf{x} \in \mathcal{D}\}$ is a Gaussian vector random field,

(ii) $\{\alpha \mathbf{Z}(\mathbf{x}) + \beta \mathbf{W}(\mathbf{x}) : \mathbf{x} \in \mathcal{D}\}$ is a Gaussian vector random field, and

(iii) $\alpha \mathbf{C}(\mathbf{x}_1, \mathbf{x}_2) + \beta \mathbf{C}_{\mathbf{W}}(\mathbf{x}_1, \mathbf{x}_2)$ is the covariance matrix function of a Gaussian vector random field whenever α and β are nonnegative.

These are just a few of the properties that Gaussian vector random fields exhibit.

In this subsection we investigate some of the operation preserving properties for log-Gaussian vector random fields. We first provide two theorems that describe some simple preservation properties for three basic operations on log-Gaussian vector random fields. Then we provide a couple of examples that illustrate why some preservation properties that hold for Gaussian vector random fields do not hold for log-Gaussian vector random fields.

Theorem 4.2. *Suppose* $\{\mathbf{Z}(\mathbf{x}) : \mathbf{x} \in \mathcal{D}\}$ *is a log-Gaussian vector random field with mean function* $\boldsymbol{\mu}(\mathbf{x})$ *and covariance matrix function* $\mathbf{C}(\mathbf{x}_1, \mathbf{x}_2)$, $\mathbf{x}_1, \mathbf{x}_2 \in \mathcal{D}$. *Then*

(i) $\{\alpha \mathbf{Z}(\mathbf{x}) : \mathbf{x} \in \mathcal{D}\}$ *is a log-Gaussian vector random field with mean function* $\alpha \boldsymbol{\mu}(\mathbf{x})$ *and covariance matrix function* $\alpha^2 \mathbf{C}(\mathbf{x}_1, \mathbf{x}_2)$, *whenever* α *is a positive constant, and*

(ii) for a constant α, $\{\mathbf{Z}^{\alpha}(\mathbf{x}) : \mathbf{x} \in \mathcal{D}\}$ *is a log-Gaussian vector random field with mean function whose* k*th entry is*

$$\mu_k^{\alpha}(\mathbf{x}) \{1 + \mu_k^{-2}(\mathbf{x}) C_{kk}(\mathbf{x}, \mathbf{x})\}^{\frac{\alpha^2 - \alpha}{2}}, \qquad k = 1, \ldots, m,$$

and covariance matrix function whose kl*th entry is*

$$[\mu_k(\mathbf{x}_1)\mu_l(\mathbf{x}_2)]^\alpha \left[\{1+\mu_k^{-2}(\mathbf{x}_1)C_{kk}(\mathbf{x}_1,\mathbf{x}_1)\}\{1+\mu_l^{-2}(\mathbf{x}_2)C_{ll}(\mathbf{x}_2,\mathbf{x}_2)\}\right]^{\frac{\alpha^2-\alpha}{2}}$$
$$\cdot \left[\{1+\mu_k^{-1}(\mathbf{x}_1)\mu_l^{-1}(\mathbf{x}_2)C_{kl}(\mathbf{x}_1,\mathbf{x}_2)\}^{\alpha^2}-1\right], \qquad k,l=1,\dots,m.$$

Proof. (i) Since $\{\ln \mathbf{Z}(\mathbf{x}) : \mathbf{x}\in\mathcal{D}\}$ is a Gaussian vector random field, $\{\alpha\mathbf{Z}(\mathbf{x}) : \mathbf{x}\in\mathcal{D}\} = \{e^{\ln \mathbf{Z}(\mathbf{x})+\ln\alpha} : \mathbf{x}\in\mathcal{D}\}$ is obviously a log-Gaussian vector random field. Define $\mathbf{Y}(\mathbf{x}) = \ln \mathbf{Z}(\mathbf{x})$ and $\mathbf{W}(\mathbf{x}) = \alpha\mathbf{Z}(\mathbf{x})$. Then for $k,l=1,\dots,m$ and $\mathbf{x},\mathbf{x}_1,\mathbf{x}_2\in\mathcal{D}$,

$$\mu_{W_k}(\mathbf{x}) = e^{\mu_{Y_k}(\mathbf{x})+\ln\alpha+\frac{1}{2}C_{Y_k}(\mathbf{x},\mathbf{x})} = \alpha\mu_k(\mathbf{x}),$$

and

$$\begin{aligned}
C_{W_k,W_l}(\mathbf{x}_1,\mathbf{x}_2) &= \mu_{W_k}(\mathbf{x}_1)\mu_{W_l}(\mathbf{x}_2)\left[e^{C_{Y_k}(\mathbf{x}_1,\mathbf{x}_2)}-1\right]\\
&= \alpha^2\mu_k(\mathbf{x}_1)\mu_l(\mathbf{x}_2)\left[e^{C_{Y_k}(\mathbf{x}_1,\mathbf{x}_2)}-1\right]\\
&= \alpha^2 C_{kl}(\mathbf{x}_1,\mathbf{x}_2).
\end{aligned}$$

(ii) Clearly, $\{\mathbf{Z}^\alpha(\mathbf{x}) : \mathbf{x}\in\mathcal{D}\} = \{e^{\alpha\ln\mathbf{Z}(\mathbf{x})} : \mathbf{x}\in\mathcal{D}\}$ is a log-Gaussian vector random field. Define $\mathbf{Y}(\mathbf{x}) = \ln\mathbf{Z}(\mathbf{x})$ and $\mathbf{W}(\mathbf{x}) = \mathbf{Z}^\alpha(\mathbf{x})$. Then for $k,l=1,\dots,m$ and $\mathbf{x},\mathbf{x}_1,\mathbf{x}_2\in\mathcal{D}$,

$$\begin{aligned}
\mu_{W_k}(\mathbf{x}) &= e^{\alpha\mu_{Y_k}(\mathbf{x})+\frac{\alpha^2}{2}C_{Y_k}(\mathbf{x},\mathbf{x})}\\
&= \left(e^{\mu_{Y_k}(\mathbf{x})+\frac{1}{2}C_{Y_k}(\mathbf{x},\mathbf{x})}\right)^\alpha \left(e^{C_{Y_k}(\mathbf{x},\mathbf{x})}\right)^{\frac{\alpha^2-\alpha}{2}}\\
&= \mu_k^\alpha(\mathbf{x})\{1+\mu_k^{-2}(\mathbf{x})C_{kk}(\mathbf{x},\mathbf{x})\}^{\frac{\alpha^2-\alpha}{2}},
\end{aligned}$$

and

$$\begin{aligned}
C_{W_k,W_l}(\mathbf{x}_1,\mathbf{x}_2) &= \mu_{W_k}(\mathbf{x}_1)\mu_{W_l}(\mathbf{x}_2)\left[e^{\alpha^2 C_{Y_k,Y_l}(\mathbf{x}_1,\mathbf{x}_2)}-1\right]\\
&= \mu_k^\alpha(\mathbf{x}_1)\{1+\mu_k^{-2}(\mathbf{x}_1)C_{kk}(\mathbf{x}_1,\mathbf{x}_1)\}^{\frac{\alpha^2-\alpha}{2}}\\
&\ast \mu_l^\alpha(\mathbf{x}_2)\{1+\mu_l^{-2}(\mathbf{x}_2)C_{ll}(\mathbf{x}_2,\mathbf{x}_2)\}^{\frac{\alpha^2-\alpha}{2}}\\
&\ast \left[\{1+\mu_k^{-1}(\mathbf{x}_1)\mu_l^{-1}(\mathbf{x}_2)C_{kl}(\mathbf{x}_1,\mathbf{x}_2)\}^{\alpha^2}-1\right].
\end{aligned}$$

□

The next theorem shows that the Hadamard product of two independent log-Gaussian vector random fields is also a log-Gaussian vector random field.

Theorem 4.3. *Suppose* $\{\mathbf{Z}(\mathbf{x}) : \mathbf{x} \in \mathcal{D}\}$ *and* $\{\mathbf{W}(\mathbf{x}) : \mathbf{x} \in \mathcal{D}\}$ *are independent log-Gaussian vector random fields with respective mean functions* $\boldsymbol{\mu}(\mathbf{x})$ *and* $\boldsymbol{\mu}_{\mathbf{W}}(\mathbf{x})$ *and covariance matrix functions* $\mathbf{C}(\mathbf{x}_1, \mathbf{x}_2)$ *and* $\mathbf{C}_{\mathbf{W}}(\mathbf{x}_1, \mathbf{x}_2)$, $\mathbf{x}_1, \mathbf{x}_2 \in \mathcal{D}$. *Then* $\{\mathbf{Z}(\mathbf{x}) \circ \mathbf{W}(\mathbf{x}) : \mathbf{x} \in \mathcal{D}\}$ *is a log-Gaussian vector random field with mean funtion* $\boldsymbol{\mu}_{\mathbf{Z}}(\mathbf{x}) \circ \boldsymbol{\mu}_{\mathbf{W}}(\mathbf{x})$ *and covariance matrix function whose* kl*th entry is*

$$\mu_k(\mathbf{x}_1)\mu_{W_k}(\mathbf{x}_1)\mu_l(\mathbf{x}_2)\mu_{W_l}(\mathbf{x}_2)$$
$$\cdot \left[\{1 + \mu_k^{-1}(\mathbf{x}_1)\mu_l^{-1}(\mathbf{x}_2)C_{kl}(\mathbf{x}_1, \mathbf{x}_2)\}\{1 + \mu_{W_k}^{-1}(\mathbf{x}_1)\mu_{W_l}^{-1}(\mathbf{x}_2)C_{W_k,W_l}(\mathbf{x}_1, \mathbf{x}_2)\} - 1\right],$$
$$k, l = 1, \ldots, m.$$

Proof. It is easy to see that $\{\mathbf{Z}(\mathbf{x}) \circ \mathbf{W}(\mathbf{x}) : \mathbf{x} \in \mathcal{D}\} = \{e^{\ln \mathbf{Z}(\mathbf{x}) + \ln \mathbf{W}(\mathbf{x})} : \mathbf{x} \in \mathcal{D}\}$ is a log-Gaussian vector random field when $\mathbf{Z}(\mathbf{x})$ and $\mathbf{W}(\mathbf{x})$ are independent. Define $\mathbf{Y}(\mathbf{x}) = \ln \mathbf{Z}(\mathbf{x}), \mathbf{V}(\mathbf{x}) = \ln \mathbf{W}(\mathbf{x})$, and $\mathbf{U}(\mathbf{x}) = \mathbf{Z}(\mathbf{x}) \circ \mathbf{W}(\mathbf{x})$. Then for $k, l = 1, \ldots, m$ and $\mathbf{x}, \mathbf{x}_1, \mathbf{x}_2 \in \mathcal{D}$,

$$\begin{aligned}\mu_{U_k}(\mathbf{x}) &= e^{\mu_{Y_k}(\mathbf{x}) + \mu_{V_k}(\mathbf{x}) + \frac{1}{2}C_{Y_k}(\mathbf{x},\mathbf{x}) + \frac{1}{2}C_{V_k}(\mathbf{x},\mathbf{x})} \\ &= \mu_k(\mathbf{x})\mu_{W_k}(\mathbf{x})\end{aligned}$$

and

$$\begin{aligned}&C_{U_k,U_l}(\mathbf{x}_1, \mathbf{x}_2) \\ &= \mu_{U_k}(\mathbf{x}_1)\mu_{U_l}(\mathbf{x}_2)\left[e^{C_{Y_k,Y_l}(\mathbf{x}_1,\mathbf{x}_2) + C_{V_k,V_l}(\mathbf{x}_1,\mathbf{x}_2)} - 1\right] \\ &= \mu_k(\mathbf{x}_1)\mu_{W_k}(\mathbf{x}_1)\mu_l(\mathbf{x}_2)\mu_{W_l}(\mathbf{x}_2) \\ &\cdot \left[\{1 + \mu_k^{-1}(\mathbf{x}_1)\mu_l^{-1}(\mathbf{x}_2)C_{kl}(\mathbf{x}_1, \mathbf{x}_2)\}\{1 + \mu_{W_k}^{-1}(\mathbf{x}_1)\mu_{W_l}^{-1}(\mathbf{x}_2)C_{W_k,W_l}(\mathbf{x}_1, \mathbf{x}_2)\} - 1\right].\end{aligned}$$

□

It is well known that the sum of two covariance functions of Gaussian random fields is also the covariance function of a Gaussian random field. The next example illustrates that the sum of two covariance functions of log-Gaussian random fields is not necessarily a covariance function associated with a log-Gaussian random field. Theorem 4.6 provides a sufficient condition for when the sum of two covariance functions does result in the covariance function of a log-Gaussian random field.

Example 4.3. Denote by $||\mathbf{x}||$ the usual Euclidean norm, namely, $||\mathbf{x}|| = \left(\sum_{k=1}^{d} x_k^2\right)^{\frac{1}{2}}, \mathbf{x} \in \mathbb{R}^d$. As was shown by Matheron (1989), there exists a positive bound μ_0 such that the function $e^{-||\mathbf{x}||}, \mathbf{x} \in \mathbb{R}^2$, is no longer an admissable covariance function of a stationary log-Gaussian random field with mean μ when $\mu < \mu_0$. If the sum of two covariance functions were a covariance function of a log-Gaussian process, then $4e^{-||\mathbf{x}||}, \mathbf{x} \in \mathbb{R}^2$, would be the covariance function for a stationary log-Gaussian process with mean μ_0, and thus by Theorem 4.1,

$$\ln\{1 + \mu_0^{-2} \cdot 4e^{-||\mathbf{x}||}\} = \ln\{1 + (\mu_0/2)^{-2} e^{-||\mathbf{x}||}\}$$

would be positive definite on $\mathbb{R}^2$. It then follows from Theorem 4.1 that there would exist a log-Gaussian process with mean $\mu_0/2$ and covariance $e^{-||\mathbf{x}||}$, which is a contradiction to Matheron (1989).

In Theorem 4.3 we showed that the product of two independent log-Gaussian random fields is also a log-Gaussian random field. But, this does not imply that the product of two covariances of log-Gaussian random fields is a covariance for a log-Gaussian random field, as the following counter-example shows.

Example 4.4. When α is a positive constant, the function $(1 + \alpha||\mathbf{x}||^2)^{-1}, \mathbf{x} \in \mathbb{R}^d$, is the covariance function for a stationary log-Gaussian process with any positive mean μ, according to Matheron (1989). If the product of these types of covariances was a covariance function of a log-Gaussian process, then for every positive integer n, $(1+||\mathbf{x}||^2/n)^{-n}, \mathbf{x} \in \mathbb{R}^d$, would be the covariance function of a stationary log-Gaussian process, and thus, $\ln\{1+\mu^{-2}(1+||\mathbf{x}||^2/n)^{-n}\}$ would be positive definite on $\mathbb{R}^d$ for a certain positive constant μ. However, this is not true, since

$$\lim_{n\to\infty} \ln\{1 + \mu^{-2}(1 + ||\mathbf{x}||^2/n)^{-n}\} = \ln\{1 + \mu^{-2} \lim_{n\to\infty}(1 + ||\mathbf{x}||^2/n)^{-n}\} = \ln\{1 + \mu^{-2}e^{-||\mathbf{x}||^2}\},$$

is not positive definite on $\mathbb{R}^d$. Matheron (1989) showed that $e^{-||\mathbf{x}||^2}$, $\mathbf{x} \in \mathbb{R}^d$, is not a covariance function for any log-Gaussian random field in $\mathbb{R}^d$.

4.1.4 Verifying a Gaussian Covariance is a Log-Gaussian Covariance

In the next theorem, we discuss some conditions under which a covariance matrix function of a stationary Gaussian vector random field is also the covariance matrix function of a log-Gaussian vector random field.

Theorem 4.4. *For two symmetric $m \times m$ matrices $\mathbf{A} = (a_{kl})$ and $\mathbf{B} = (b_{kl})$, the matrix function*

$$\mathbf{C}(n) = \begin{cases} \mathbf{A}, & n = 0, \\ \mathbf{B}, & n = \pm 1, \\ \mathbf{0}, & n = \pm 2, \pm 3, \ldots, \end{cases} \tag{4.4}$$

is the covariance matrix function of a stationary Gaussian vector random field on $\mathbb{Z}$ if and only if $\mathbf{A} + 2\mathbf{B}$ and $\mathbf{A} - 2\mathbf{B}$ are both positive definite. Furthermore, $\mathbf{C}(n)$ is the covariance matrix function of a stationary log-Gaussian vector random field with mean $\boldsymbol{\mu}$ if and only if $a_{kl}, b_{kl} > -\mu_k \mu_l$ for all $k, l = 1, \ldots, m$, and the matrix with klth entry $\ln\{1 + \mu_k^{-1}\mu_l^{-1}a_{kl}\} + 2\ln\{1 + \mu_k^{-1}\mu_l^{-1}b_{kl}\}$ and the matrix with klth entry $\ln\{1 + \mu_k^{-1}\mu_l^{-1}a_{kl}\} - 2\ln\{1 + \mu_k^{-1}\mu_l^{-1}b_{kl}\}$ are both positive definite.

Proof. Suppose $\mathbf{C}(n)$ is the covariance matrix function of a stationary Gaussian vector random field. By the Cramér-Kolmogorov characterization, its spectral density function

$$\mathbf{f}(\omega) = \sum_{n=-\infty}^{\infty} \mathbf{C}(n) \cos(n\omega) = \mathbf{A} + 2\mathbf{B}\cos(\omega)$$

is positive definite for every $\omega \in [0, \pi]$. Then $\mathbf{A} + 2\mathbf{B}$ and $\mathbf{A} - 2\mathbf{B}$ are both positive definite, by taking $\omega = 0$ and $\omega = \pi$, respectively. Now assume that $\mathbf{A} + 2\mathbf{B}$ and $\mathbf{A} - 2\mathbf{B}$ are both positive definite. Then so is

$$\begin{aligned} \mathbf{f}(\omega) &= (\mathbf{A} + 2\mathbf{B}) \left(\frac{1 + \cos\omega}{2} \right) + (\mathbf{A} - 2\mathbf{B}) \left(\frac{1 - \cos\omega}{2} \right) \\ &= \mathbf{A} + 2\mathbf{B}\cos\omega \end{aligned}$$

for every $\omega \in [0, \pi]$, since $\frac{1+\cos\omega}{2}$ and $\frac{1-\cos\omega}{2}$ are nonnegative for every $\omega \in [0, \pi]$.

To show that $\mathbf{C}(n)$ is the covariance matrix function of a log-Gaussian vector random field, we must show that the matrix function with klth entry,

$$D_{kl}(n) = \ln\{1 + \mu_k^{-1}\mu_l^{-1}C_{kl}(n)\} = \begin{cases} \ln\{1 + \mu_k^{-1}\mu_l^{-1}a_{kl}\}, & n = 0, \\ \ln\{1 + \mu_k^{-1}\mu_l^{-1}b_{kl}\}, & n = \pm 1, \\ 0, & n = \pm 2, \pm 3, \ldots, \end{cases} \quad (4.5)$$

$$k, l = 1, \ldots, m,$$

satisfies inequality (2.1). For $\omega \in [0, \pi]$, the Fourier transform function of $\mathbf{D}(n)$ is

$$\mathbf{f}(\omega) = \sum_{n=-\infty}^{\infty} \mathbf{D}(n)\cos(n\omega) = \mathbf{D}(0) + 2\mathbf{D}(1)\cos(\omega).$$

Similar to above, $\mathbf{f}(\omega)$ is positive definite for every $\omega \in [0, \pi]$ if and only if $\mathbf{D}(0) + 2\mathbf{D}(1)$ and $\mathbf{D}(0) - 2\mathbf{D}(1)$ are both positive definite. □

In the following example, we use Theorem 4.4 to construct the covariance matrix function of a first-order moving average (MA(1)) Gaussian vector random field. See Reinsel (1993) for further information about MA(1) Gaussian vector random fields.

Example 4.5. For an $m \times m$ symmetric matrix $\mathbf{\Theta}$, the matrix function

$$\mathbf{C}(n) = \begin{cases} \mathbf{I} + \mathbf{\Theta}^2, & n = 0, \\ \mathbf{\Theta}, & n = \pm 1, \\ \mathbf{0}, & n = \pm 2, \pm 3, \ldots, \end{cases} \quad (4.6)$$

is the covariance matrix function of a first-order moving average Gaussian vector random field, where $\mathbf{I}$ is the $m \times m$ identity matrix, since $\mathbf{I} + \mathbf{\Theta}^2 + 2\mathbf{\Theta} = (\mathbf{I} + \mathbf{\Theta})'(\mathbf{I} + \mathbf{\Theta})$ and $\mathbf{I} + \mathbf{\Theta}^2 - 2\mathbf{\Theta} = (\mathbf{I} - \mathbf{\Theta})'(\mathbf{I} - \mathbf{\Theta})$ are both positive definite matrices.

4.1.5 Covariance Structures with a Conditionally Negative Definite Matrix

This subsection demonstrates how to construct the covariance matrix function of a log-Gaussian vector random field from the variogram of a univariate Gaussian stochastic process, with the conditionally negative definite matrix playing an important role. Once again, the covariance matrix function is a function of the mean vector.

Definition 4.1. *For a univariate stochastic process* $\{Z(\mathbf{x}) : \mathbf{x} \in \mathcal{D}\}$ *with second-order increments, its variogram is defined by*

$$\gamma(\mathbf{x}_1, \mathbf{x}_2) = \frac{1}{2} var(Z(\mathbf{x}_1) - Z(\mathbf{x}_2)), \qquad \mathbf{x}_1, \mathbf{x}_2 \in \mathcal{D}.$$

Three properties of $\gamma(\mathbf{x}_1, \mathbf{x}_2)$ that follow immediately from its definition are

(i) $\gamma(\mathbf{x}, \mathbf{x}) = 0$,

(ii) $\gamma(\mathbf{x}_1, \mathbf{x}_2)$ is nonnegative, and

(iii) $\gamma(\mathbf{x}_1, \mathbf{x}_2)$ is a conditionally negative definite function.

Theorem 4.5. *If* $\gamma(\mathbf{x}_1, \mathbf{x}_2)$ *is the variogram of a univariate Gaussian stochastic process in* $\mathcal{D}$*,* $\boldsymbol{\Theta}$ *is an* $m \times m$ *conditionally negative definite matrix with positive diagonal entries, and* $\boldsymbol{\mu}(\mathbf{x})$ *is a positive* m*-variate vector, then there exists an* m*-variate log-Gaussian random field with mean function* $\boldsymbol{\mu}(\mathbf{x})$ *and with covariance matrix function whose* kl*th entry is*

$$C_{kl}(\mathbf{x}_1, \mathbf{x}_2) = \frac{\mu_k(\mathbf{x}_1)\mu_l(\mathbf{x}_2)}{\Theta_{kl} + \gamma(\mathbf{x}_1, \mathbf{x}_2)}, \qquad k, l = 1, \ldots, m. \tag{4.7}$$

Proof. By Theorem 2 of Ma (2011c), the matrix $e^{-t\boldsymbol{\Theta}} \circ e^{-t\gamma(\mathbf{x}_1,\mathbf{x}_2)}\mathbf{1}_{m\times m}$ is positive definite for each fixed $t \geq 0$, where $\mathbf{1}_{m\times m}$ is an $m \times m$ matrix with all entries equal to 1. So, for every $n \in \mathbb{N}$ and for any $\mathbf{a}_i \in \mathbb{R}^m, i = 1, \ldots, n$,

$$
\begin{aligned}
&\sum_{i=1}^{n}\sum_{j=1}^{n}\sum_{k=1}^{m}\sum_{l=1}^{m} a_{ik}\ln(1+\mu_k^{-1}(\mathbf{x}_i)\mu_l^{-1}(\mathbf{x}_j)C_{kl}(\mathbf{x}_i,\mathbf{x}_j))a_{jl} \\
&= \sum_{i=1}^{n}\sum_{j=1}^{n}\sum_{k=1}^{m}\sum_{l=1}^{m} a_{ik}a_{jl}\int_0^1 \frac{\mu_k^{-1}(\mathbf{x}_i)\mu_l^{-1}(\mathbf{x}_j)C_{kl}(\mathbf{x}_i,\mathbf{x}_j)}{1+s\mu_k^{-1}(\mathbf{x}_i)\mu_l^{-1}(\mathbf{x}_j)C_{kl}(\mathbf{x}_i,\mathbf{x}_j)}ds \\
&= \sum_{i=1}^{n}\sum_{j=1}^{n}\sum_{k=1}^{m}\sum_{l=1}^{m} a_{ik}a_{jl}\int_0^1 \frac{1}{\Theta_{kl}+\gamma(\mathbf{x}_i,\mathbf{x}_j)+s}ds \\
&= \sum_{i=1}^{n}\sum_{j=1}^{n}\sum_{k=1}^{m}\sum_{l=1}^{m} a_{ik}a_{jl}\int_0^1\int_0^\infty e^{-t(\Theta_{kl}+\gamma(\mathbf{x}_i,\mathbf{x}_j)+s)}dt\ ds \\
&= \int_0^1\int_0^\infty e^{-ts}\left\{\sum_{i=1}^{n}\sum_{j=1}^{n}\mathbf{a}_i'\left(e^{-t\boldsymbol{\Theta}}\circ e^{-t\gamma(\mathbf{x}_i,\mathbf{x}_j)}\mathbf{1}_{m\times m}\right)\mathbf{a}_j\right\}dt\ ds \\
&\geq 0.
\end{aligned}
$$

Then by Theorem 4.1, the result follows. □

One may wonder why we did not replace the univariate variogram in Theorem 4.5 with a variogram matrix $\boldsymbol{\gamma}(\mathbf{x}_1,\mathbf{x}_2)$, $\mathbf{x}_1,\mathbf{x}_2\in\mathcal{D}$. For an m-variate random field, $\{\mathbf{Z}(\mathbf{x}):\mathbf{x}\in\mathcal{D}\}$, with second-order increments, its variogram matrix is defined as

$$
\boldsymbol{\gamma}(\mathbf{x}_1,\mathbf{x}_2)=\frac{1}{2}E\{\mathbf{Z}(\mathbf{x}_1)-\mathbf{Z}(\mathbf{x}_2)-E(\mathbf{Z}(\mathbf{x}_1)-\mathbf{Z}(\mathbf{x}_2))\}\{\mathbf{Z}(\mathbf{x}_1)-\mathbf{Z}(\mathbf{x}_2)-E(\mathbf{Z}(\mathbf{x}_1)-\mathbf{Z}(\mathbf{x}_2))\}',
$$

$$
\mathbf{x}_1,\mathbf{x}_2\in\mathcal{D}.
$$

Hence, it is possible for the variogram matrix to have negative-valued entries since the off-diagonal entries are of the form

$$
\gamma_{kl}(\mathbf{x}_1,\mathbf{x}_2)=\frac{1}{2}\mathrm{cov}(Z_k(\mathbf{x}_1)-Z_k(\mathbf{x}_2),Z_l(\mathbf{x}_1)-Z_l(\mathbf{x}_2)),\quad \mathbf{x}_1,\mathbf{x}_2\in\mathcal{D}.
$$

4.2 Univariate Random Fields

While investigating the covariance structures of log-Gaussian vector random fields, we discovered some interesting results that apply to the univariate log-Gaussian random field. The following theorem, which is a more general result of Criterion 3 in Matheron (1989), discusses the conditions for which a function can be the covariance function of a log-Gaussian random field with any positive constant mean. It also illustrates a special case of a preservation property by permitting the sum of two covariance functions of log-Gaussian random fields to be a covariance function of a log-Gaussian random field.

Theorem 4.6. *If* $C(\mathbf{x}_1, \mathbf{x}_2), \mathbf{x}_1, \mathbf{x}_2 \in \mathcal{D}$, *is a positive function and* $C^{-1}(\mathbf{x}_1, \mathbf{x}_2)$ *is a conditionally negative definite function, then for positive constants* μ *and* α, *there exists a log-Gaussian random field with mean* μ *and covariance function* $\alpha C(\mathbf{x}_1, \mathbf{x}_2)$.

Proof. Since $C^{-1}(\mathbf{x}_1, \mathbf{x}_2)$, $\mathbf{x}_1, \mathbf{x}_2 \in \mathcal{D}$, is a conditionally negative definite function, $e^{-\lambda C^{-1}(\mathbf{x}_1, \mathbf{x}_2)}$ is a positive definite function for any positive constant λ. So, for every $n \in \mathbb{N}$ and for any $a_i \in \mathbb{R}, i = 1, \ldots, n$,

$$
\begin{aligned}
&\sum_{i=1}^{n}\sum_{j=1}^{n} a_i a_j \ln(1 + \mu^{-2}\alpha C(\mathbf{x}_i, \mathbf{x}_j)) \\
&= \sum_{i=1}^{n}\sum_{j=1}^{n} a_i a_j \int_0^{\alpha\mu^{-2}} \frac{C(\mathbf{x}_i, \mathbf{x}_j)}{1 + sC(\mathbf{x}_i, \mathbf{x}_j)} ds \\
&= \sum_{i=1}^{n}\sum_{j=1}^{n} a_i a_j \int_0^{\alpha\mu^{-2}} \int_0^{\infty} e^{-t\left(C^{-1}(\mathbf{x}_i, \mathbf{x}_j) + s\right)} dt\ ds \\
&= \int_0^{\alpha\mu^{-2}} \int_0^{\infty} e^{-ts} \left\{ \sum_{i=1}^{n}\sum_{j=1}^{n} a_i a_j e^{-tC^{-1}(\mathbf{x}_i, \mathbf{x}_j)} \right\} dt\ ds \\
&\geq 0.
\end{aligned}
$$

Then by Theorem 4.1, the result follows. □

The next theorem provides the bounds for the covariance function of a stationary univariate log-Gaussian random field.

Theorem 4.7. *If* $C(\mathbf{x})$ *is the covariance function of a stationary log-Gaussian random field on* $\mathcal{D}$ *with mean* μ, *then*

$$
-\frac{1}{1 + \mu^{-2} C(\mathbf{0})} \leq \frac{C(\mathbf{x})}{C(\mathbf{0})} \leq 1, \quad \mathbf{x} \in \mathcal{D}.
$$

Proof. Clearly, the lower bound above depends on the mean μ. These bounds follow directly from the inequalities

$$|C(\mathbf{x})| \leq C(\mathbf{0}), \quad \text{and} \quad |\ln\{1 + \mu^{-2}C(\mathbf{x})\}| \leq \ln\{1 + \mu^{-2}C(\mathbf{0})\}, \quad \mathbf{x} \in \mathcal{D},$$

which are obtained from the positive definiteness of $C(\mathbf{x})$ and of $\ln\{1 + \mu^{-2}C(\mathbf{x})\}$. □

4.2.1 Power-law Decaying and Long-range Dependent Random Fields

In this subsection we derive a class of univariate log-Gaussian random fields whose covariance functions decay in power-law or have long-range dependence, with the variogram of a Gaussian random field playing a critical role.

Theorem 4.8. *Suppose that $\gamma(\mathbf{x}_1, \mathbf{x}_2)$ is the variogram of a Gaussian random field on $\mathcal{D}$.*

(i) If k is a constant with $0 < k \leq 1$, then

$$C(\mathbf{x}_1, \mathbf{x}_2) = \{1 + \gamma(\mathbf{x}_1, \mathbf{x}_2)\}^{-k}, \quad \mathbf{x}_1, \mathbf{x}_2, \in \mathcal{D}, \tag{4.8}$$

is a covariance function for a log-Gaussian random field on $\mathcal{D}$ with any positive constant mean μ.

(ii) If k is a positive constant, then (4.8) is a covariance function for a Gaussian random field on $\mathcal{D}$.

Proof. (i) Since $k \in (0, 1]$ and $\gamma(\mathbf{x}_1, \mathbf{x}_2)$ is conditionally negative definite, it follows from Corollary 2.10 in Berg et al. (1984) that $\{1+\gamma(\mathbf{x}_1, \mathbf{x}_2)\}^k$ is also conditionally negative definite. Then by Theorem 4.6, $C(\mathbf{x}_1, \mathbf{x}_2)$ is the covariance function of a log-Gaussian random field with mean μ.

(ii) It follows from the argument on page 75 of Berg et al. (1984) that the function $\{1+\gamma(\mathbf{x}_1, \mathbf{x}_2)\}^{-\frac{k}{[k]+1}}$ is positive definite on $\mathcal{D}$, where $[\cdot]$ denotes the greatest integer function. So the function

$$\{1 + \gamma(\mathbf{x}_1, \mathbf{x}_2)\}^{-k} = \left[\{1 + \gamma(\mathbf{x}_1, \mathbf{x}_2)\}^{-\frac{k}{[k]+1}}\right]^{[k]+1}, \quad \mathbf{x}_1, \mathbf{x}_2 \in \mathcal{D},$$

is positive definite for any positive constant k. □

In the particular case where $k = 1$ and $\gamma(\mathbf{x}_1, \mathbf{x}_2)$ is intrinsically stationary, Theorem 4.8 reduces to Criterion 2 of Section 4 of Matheron (1989). Also, (4.8) may not be the covariance function of a log-Gaussian random field on $\mathcal{D}$ when $k > 1$, as we have seen in Example 4.3.

The next two corollaries illustrate how to extend Theorem 4.8 into a class of univariate log-Gaussian random fields with power-law decay or long-range dependent covariance structures. The idea is to use the variogram, $\gamma(\mathbf{x}_1, \mathbf{x}_2)$, to construct other variograms of Gaussian random fields.

Corollary 4.3. *For a constant k between 0 and 1, the function*

$$C(\mathbf{x}_1, \mathbf{x}_2) = \{1 + \ln(1 + \gamma(\mathbf{x}_1, \mathbf{x}_2))\}^{-k}, \quad \mathbf{x}_1, \mathbf{x}_2 \in \mathcal{D}, \tag{4.9}$$

is a covariance function for a log-Gaussian random field on $\mathcal{D}$ with any positive constant mean.

Corollary 4.4. *For positive constants α_1, α_2 and $k \in (0, 1]$, the function*

$$C(\mathbf{x}_1, \mathbf{x}_2) = \left\{ \frac{\alpha_1 + \alpha_2 \gamma(\mathbf{x}_1, \mathbf{x}_2)}{1 + \alpha_1 + \alpha_2 \gamma(\mathbf{x}_1, \mathbf{x}_2)} \right\}^k, \quad \mathbf{x}_1, \mathbf{x}_2 \in \mathcal{D}, \tag{4.10}$$

is a covariance function for a log-Gaussian random field on $\mathcal{D}$ with any positive constant mean.

Corollaries 4.3 and 4.4 follow from the fact that if $\gamma(\mathbf{x}_1, \mathbf{x}_2)$ is a variogram on $\mathcal{D}$, then so are $\ln(1 + \gamma(\mathbf{x}_1, \mathbf{x}_2))$ and $\gamma(\mathbf{x}_1, \mathbf{x}_2)\{\alpha_1 + \alpha_2 \gamma(\mathbf{x}_1, \mathbf{x}_2)\}^{-1}$.

It is well known that a variogram $\gamma(\mathbf{x}_1, \mathbf{x}_2), \mathbf{x}_1, \mathbf{x}_2 \in \mathcal{D}$, behaves at most like the function $||\mathbf{x}_1 - \mathbf{x}_2||^2$. As a result, the functions (4.8), (4.9), and (4.10) all have power-law decaying or long-range dependence properties. Now let us look at some specific examples. Suppose that $\gamma(\mathbf{x}_1, \mathbf{x}_2) = ||\mathbf{x}_1 - \mathbf{x}_2||^\alpha, \mathbf{x}_1, \mathbf{x}_2 \in \mathbb{R}^d$, where $0 < \alpha \leq 2$. From (4.8) we obtain a power-law covariance function $(1 + ||\mathbf{x}_1 - \mathbf{x}_2||^\alpha)^{-k}$, from (4.9) we get a long-range dependent covariance function $\{1 + \ln(1 + ||\mathbf{x}_1 - \mathbf{x}_2||^\alpha)\}^{-k}$, and (4.10) produces a covariance fuction with an asymptotically positive constant tail. Similar examples can be constructed when $\gamma(\mathbf{x}_1, \mathbf{x}_2) = \left(\sum_{k=1}^{d} |x_{1k} - x_{2k}|\right)^\alpha$, $\mathbf{x}_1, \mathbf{x}_2 \in \mathbb{R}^d$, where $0 < \alpha \leq 1$. Also, we are able to attain log-Gaussian space-time covariances from (4.8), (4.9), and (4.10) by allowing $\gamma(\cdot, \cdot)$ to be a space-time variogram.

Suppose that $\rho(\mathbf{x})$ is a correlaton function for a stationary Gaussian random field on $\mathcal{D}$, so that $1-\rho(\mathbf{x})$ is a stationary variogram. It follows from Theorem 4.8 that $\{1-\alpha\rho(\mathbf{x})\}^{-k}$, $\mathbf{x} \in \mathcal{D}$, is a covariance function of a stationary log-Gaussian random field on $\mathcal{D}$, where $\alpha \in (0, 1)$ and $k \in (0, 1]$ are constants.

CHAPTER 5

CONCLUDING REMARKS AND FUTURE RESEARCH

In Chapter 3, we presented methods for constructing covariance matrix functions of multivariate stationary Gaussian time series from covariance functions of univariate stationary Gaussian time series. One primary objective was to use these methods to construct covariance matrix functions with power-law decaying or log-law decaying direct and cross covariances. Another feature we investigated was under which parameter specifications the processes had long memory. A drawback of the examples we presented is that the parameters $\alpha_1, \ldots, \alpha_m$ and θ may be too restrictive to be useful in modeling real data. More research is needed to obtain (i) real applications for these examples and (ii) examples that utilize these methods but have less restrictive parameters. We also presented a method for formimg a more generalized covariance matrix structure that allowed the direct and cross covariances to have entries from a conditionally negative definite matrix. Finally, we provided one covariance matrix structure for a nonstationary Gaussian vector time series; more research should be completed to construct additional covariance structures for nonstationary vector time series.

Also as a concluding remark, the theorems and examples in Chapter 3 are presented in the neat form. More complicated forms can be derived by appropriate mixture procedures. For example, by taking the Hadamard product of the matrix (3.2) and an $m \times m$ positive definite matrix, $\mathbf{B} = (b_{kl})$, we obtain a covariance matrix function with entries

$$C_{kl}(n) = \begin{cases} \Gamma(\alpha_k + \alpha_l) b_{kl} \{g(0)\}^{-(\alpha_k + \alpha_l)}, & n = 0, \\ \theta \Gamma(\alpha_k + \alpha_l) b_{kl} \{g(|n|)\}^{-(\alpha_k + \alpha_l)}, & n = \pm 1, \pm 2, \ldots, \quad k, l = 1, \ldots, m, \end{cases}$$

by Theorem 3 of Ma (2011b). In addition, the sum of these types of matrices is also a covariance matrix function for an m-variate stationary Gaussian time series.

We know that a matrix function, $\mathbf{C}(\mathbf{x}_1, \mathbf{x}_2)$, $\mathbf{x}_1, \mathbf{x}_2 \in \mathcal{D}$, that satisfies inequality (2.1) is the covariance matrix function for some Gaussian vector random field. However, we showed that this is not necessarily true in the log-Gaussian case. In Chapter 4, we discovered under which conditions a matrix function is the covariance matrix function of a log-Gaussian vector random field. Primarily, we found that a matrix function, $\mathbf{C}(\mathbf{x}_1, \mathbf{x}_2)$, is a covariance matrix function for a log-Gaussian vector random field if not only $\mathbf{C}(\mathbf{x}_1, \mathbf{x}_2)$ satisfies inequality (2.1), but also the matrix function with klth entry

$$\ln\{1 + \mu_k^{-1}(\mathbf{x}_1)\mu_l^{-1}(\mathbf{x}_2)C_{kl}(\mathbf{x}_1, \mathbf{x}_2)\}, \quad k, l = 1, \ldots, m,$$

must satisfy inequality (2.1) as well.

We also discussed the delicate nature of the relationship between the mean vector function and the covariance matrix function of a log-Gaussian vector random field, which helps to explain why few developments have been made regarding the log-Gaussian vector random field. We provided examples that illustrated in the univariate case, a function could be the covariance function of a Gaussian random field and of a log-Gaussian random field, but in the multivariate case, a covariance matrix function for a Gaussian random field is not necessarily a covariance matrix function for a log-Gaussian random field. It seems that in most cases, we need the added requirement that the matrix function be a function of the mean vector in order for the matrix function to be a covariance matrix function of a log-Gaussian vector random field.

We also investigated whether some of the properties that hold for Gaussian vector random fields and their covariance matrix functions (like taking sums and products) still hold for log-Gaussian vector random fields and their covariance matrix functions. Although we found a counterexample to prove that the sum of two covariance matrix functions is not necessarily a covariance matrix function for a log-Gaussian vector random field (although it always is for a Gaussian vector random field), further work should be done to discover if there are any necessary conditions for when this is true. Also, we specifically studied univariate log-Gaussian random fields that exhibited power-law decaying or long-range dependent properties. Although we were able to find some interesting results in the univariate case, more research and examples are needed for the vector case.

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