

STABILITY ESTIMATES FOR INVERSE PROBLEMS OF SOME ELLIPTIC EQUATIONS

A Dissertation by

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Submitted to the Department of Mathematics and Statistics
and the faculty of the Graduate School of
Wichita State University
in partial fulfillment of
the requirements for the degree of
Doctor of Philosophy

December 2011

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The following faculty members have examined the final copy of this dissertation for form and content, and recommended that it be accepted in partial fulfillment of the requirement for the degree of Doctor of Philosophy with a major in Applied Mathematics.

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DEDICATION

I would like to dedicate the dissertation to my advisor, Dr. Alexander Bukhgeym, who showed uncommon patience and provided unprecedented support.

ACKNOWLEDGEMENTS

I would like to thank the members of my committee, my friends and my family for the support and encouragement needed to finish my dissertation.

I am sincerely grateful to my advisor, Dr. Alexander Bukhgeym, for his patience, guidance and instruction. I would like to give special thanks Dr. Alan Elcrat for the support and mentoring he provided over the course of many years. I would like to thank Dr. Victor Isakov for his penetrating questions and insights during our Inverse Problems seminar and Dr. Thomas DeLillo for his suggestions. I also want to thank Dr. Elizabeth Behrman for being a special voice during my research; one of practicality, common sense and joy.

My friends and family always supported my return to university and I appreciate all of their corroboration and help. Dr. Prem Bajaj, Dr. Phil Parker, Robert Bowen and my sister Neke Kullman were constant sources of encouragement. Finally thanks to Francis Nguyen, Nathan Thompson, Patric Mitchell and Mrinal Nagrecha for helping me keep my balance. You all helped me keep in mind how much fun math can be.

ABSTRACT

In this dissertation we obtain new Carleman formulas for the solution of the Cauchy problem for equations $Pu = h$, in Ω , $u|_E = f$, where $E \subset \partial\Omega$ and $|E| > 0$. Our elliptic operator is of the form $P = \begin{bmatrix} 2\bar{\partial} & 0 \\ 0 & 2\partial \end{bmatrix} + A(x)$, where A is a 2×2 matrix. We also obtain estimates for the solution of equation $Pu = 0$ when u is given at a finite number of points, and we prove that non-trivial solutions to the equation can not be small on large portions of the boundary, $|E_\delta| \leq \frac{c}{\ln \delta^{-1}}$, $\delta \in (0, 1)$, where $E_\delta = \{z \in \partial\Omega \mid |u(z)| < \delta\}$ and $|E_\delta|$ is the Lebesgue measure of E_δ . Finding the boundary condition from only a finite number of interior measurements of a domain is interesting both theoretically and practically. For example, when the boundary is physically inaccessible, all measurements must be made within the domain itself, and the conditions on the boundary must be reconstructed. We investigate the problem of recovering a boundary condition of the third kind for the Laplace operator defined on a simply connected domain in the complex plane, when the value of the solution and its gradient are known only for a finite number of interior points.

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1 Introduction

1.1 Carleman's Formula and Boundary Behavior

In 1831 Cauchy obtained a formula for solving the following problem bearing his name and based on the Cauchy-Riemann operator, $\bar{\partial} = \frac{1}{2}(\partial_1 + i\partial_2)$. In a domain $\Omega \subset \mathbb{C}$, find a function $u \in C^1(\bar{\Omega})$ such that

$$\begin{aligned}\bar{\partial}u &= 0, & \text{in } \Omega \\ u|_{\partial\Omega} &= f.\end{aligned}\tag{1.1}$$

The solution is the now famous Cauchy formula

$$u(z) = \frac{1}{2\pi i} \int_{\partial\Omega} \frac{f(\xi)}{\xi - z} d\xi = \frac{1}{2\pi} \int_{\partial\Omega} \frac{\nu u(\xi)}{\xi - z} ds.$$

This is true as $d\xi = \tau ds = i\nu ds$, and $\nu = \nu_1 + \nu_2$ is the unit outward normal while $\tau = i\nu$ is a unit tangent vector to $\partial\Omega$. To use this formula one needs to know the trace of $u(z)$ on the whole boundary. It appears Carleman was the first to ask if it is possible to obtain a formula that expresses $u(z)$ in term of the trace of u given only on a subset, $M \subset \partial\Omega$, of the boundary. He was the first to obtain a formula of ‘‘Cauchy’’ type for a domain bounded by lines \overline{OA} and \overline{OB} ; and a simple smooth curve AB lying within the angle $\angle AOB$ (see example 3.9). Carleman published his formula in 1926. However it is inconvenient for computations since it is true only on a bisectrix and the kernel of the integral operator is sufficiently complicated. Theoretically we can determine $u(z)$ in an entire domain by choosing a smaller angle so that $z \in \Omega$ will lie on a bisectrix. However in both cases we lose information since the set which will be used in the formula to reconstruct $u(z)$ is only a subset of the initial data, M . The

main idea of Carleman was to introduce a weight function

$$\Psi = \left(\frac{\zeta - \zeta_0}{z - \zeta_0} \right)^{\frac{1}{\alpha}} - 1$$

and use the Cauchy formula

$$u(z) = \lim_{n \rightarrow \infty} \frac{e^{-n}}{2\pi i} \int_M u(\zeta) e^{n \left(\frac{\zeta - \zeta_0}{z - \zeta_0} \right)^{\frac{1}{\alpha}}} \frac{d\zeta}{\zeta - z}.$$

The main condition of this weight function is that the integral over $\partial\Omega \setminus M$ converges to zero as $n \rightarrow \infty$. It is clear that this idea does not depend on any particular domain. The first generalization for the case of arbitrary bounded, simply connected domains was obtained by Goluzin and Krylov in 1933. For an analytic function Ψ in domain Ω , we apply the Cauchy formula to the function $u(\zeta)e^{\tau\Psi(\zeta)}$ and obtain

$$u(z)e^{\tau\Psi(z)} = \frac{1}{2\pi i} \int_{\partial\Omega \setminus M} \frac{e^{\tau\Psi(\zeta)}u(\zeta)}{\zeta - z} d\zeta + \frac{1}{2\pi i} \int_M \frac{e^{\tau\Psi(\zeta)}u(\zeta)}{\zeta - z} d\zeta.$$

Dividing both sides gives

$$u(z) = \frac{1}{2\pi i} \int_{\partial\Omega \setminus M} \frac{e^{\tau(\Psi(\zeta) - \Psi(z))}u(\zeta)}{\zeta - z} d\zeta + \frac{1}{2\pi i} \int_M \frac{e^{\tau(\Psi(\zeta) - \Psi(z))}u(\zeta)}{\zeta - z} d\zeta.$$

If one lets $\Psi = \varphi + i\chi$ then $|e^{\tau(\Psi(\zeta) - \Psi(z))}| = e^{\tau(\varphi(\zeta) - \varphi(z))}$. Thus we need only one condition to eliminate the integral over $\partial\Omega \setminus M$,

$$\varphi(z) > \varphi(\zeta) \quad (\forall \zeta \in \partial\Omega \setminus M).$$

In this case we use we use the Lebesgue dominated convergence theorem and as $\tau \rightarrow \infty$ we have

$$u(z) = \lim_{\tau \rightarrow \infty} \frac{1}{2\pi i} \int_M \frac{u(\zeta)e^{\tau(\Psi(\zeta) - \Psi(z))}}{\zeta - z} dz.$$

Goluzin and Krylov [14] chose φ to be the solution of the following Dirichlet problem for the Laplace equation,

$$\begin{aligned} \Delta\varphi &= 0 & z \in \Omega \\ \varphi|_{\partial\Omega} &= \begin{cases} 1 & z \in M \\ 0 & z \in \partial\Omega \setminus M, \end{cases} \end{aligned} \tag{1.2}$$

where M is an arbitrary closed subset of $\partial\Omega$ with positive Lebesgue measure. Since Ω is a simply connected domain, we can always find a harmonic conjugate function $\chi(z)$; and hence we obtain an analytic weight function

$$\Psi(z) = \varphi(x) + i\chi(z).$$

The function φ from system (1.2) is called the harmonic measure of M at $z \in \Omega$, which is sometimes denoted $\varphi(z) = w(z, M, \Omega)$.

Lavrent'ev proposed an alternative method based upon the integral representation of a partial differential equation with constant coefficients (see [19], [20] and [18]). Lavrent'ev showed that multidimensional inverse problems for hyperbolic equations reduced to integral geometry problems. In 1975, Muhometov considered an inverse problem of recovering the right hand side of the transport equation along a given family of curves and under natural conditions, an integral geometry problem on the plane, and obtained uniqueness and stability estimates (see [22]).

Recently Arbuzov and Bukhgeym [3] [4] extended stability and uniqueness results to partial differential equations with variable coefficients. In this dissertation, we extend these results to a wider class of partial differential equations with variable coefficients. We obtain Carleman formulas for the solution of the Cauchy problem for equations $Pu = h$, in Ω , $u|_E = f$, where $E \subset \partial\Omega$ and $|E| > 0$. These elliptic operators are of the form $P =$

$\begin{bmatrix} 2\bar{\partial} & 0 \\ 0 & 2\partial \end{bmatrix} + A(x)$, where A is a 2×2 matrix. We also obtain estimates for the solution of equation $\bar{P}u = 0$ when u is given at a finite number of points, and we prove that non-trivial solutions to the equation can not be small on large portions of the boundary.

2 Solutions of Elliptic Equations from Discrete Sets

2.1 Initial Estimates

Estimates of stability for elliptic equations from discrete sets to the plane can be stated as in Bukhgeim [5]:

Problem 2.1. *Let u be the solution of the Cauchy-Riemann system*

$$\partial_{\bar{z}}u = 0, \quad |z| < 1 \quad (2.1)$$

such that

$$|u(z_j)| \leq \varepsilon, \quad j = 1, \dots, n \quad (2.2)$$

We seek an upper bound for the norm of the solution in $L_2(B_r)$ or $C(B_r)$ for $0 < r < 1$, where $B_r = \{z \in \mathbb{C} \mid |z| < r\}$. When the sequence of points, $\{z_j\}$, is infinite, the stability estimates were first obtained by Lavrent'ev [19]. For a finite sequence and $\varepsilon = 0$ stability estimates for an elliptic equation with the Laplace operator in the principal part, and the corresponding parabolic equation were obtained by Shishat'skii [26] using weighted *a priori* estimates. Bukhgeim uses the method of weight estimates to deal with arbitrary sequences and expresses stability estimates in terms of partial sums of the Blaschke series [5]

$$s_n = \sum_{j=1}^n (1 - |z_j|). \quad (2.3)$$

In particular, for problem 2.1 on the unit circle $\Omega = \{z \in \mathbb{C} \mid |z| < 1\}$ Bukhgeim [5] proves the following:

Theorem 2.1. *Let the function $u \in C^1(\bar{\Omega})$ satisfy the inequality*

$$|\partial_{\bar{z}}u|^2 \leq a(z)|u|^2 \tag{2.4}$$

in Ω , where the non-negative function $a(z) \in L_2(\Omega)$. If z_j is a zero of $u(z)$ of multiplicity $m_j > 0$ for each j and

$$\sum_j m_j(1 - |z_j|) = \infty, \tag{2.5}$$

then $u \equiv 0$.

Theorem 2.2. *If $u \in C^2(\bar{\Omega})$ satisfies*

$$|\Delta u|^2 \leq a(|u|^2 + |\partial u|^2) \tag{2.6}$$

in Ω and $a \in L_2(\Omega)$ such that for each $j \in \mathbb{N}$

$$z_j \in \Omega, \quad u(z_j) = 0, \quad \partial u(z_j) = 0$$

and we have equation (2.5), then $u \equiv 0$.

Theorem 2.3. *If $u \in C^2(\bar{\Omega})$ satisfies*

$$|\Delta u|^2 \leq a|\partial u|^2$$

in Ω , where $a \in L_2(\Omega)$, $u(z_0) = 0$, $\partial u(z_j) = 0$ for $j \in \mathbb{N}$, and we have equation (2.5), then $u \equiv 0$.

Theorem 2.4. *If $u \in C^1(\bar{\Omega})$ satisfies*

$$|\partial_{\bar{z}}(u)|^2 \leq a|u|^2 \quad (2.7)$$

in Ω for constant $a \geq 0$, and we have equation (2.2), then for $0 \leq \varepsilon \leq \varepsilon_0 < 1$

$$\|u(z)\|_{L_2(|z|<\lambda)}^2 \leq e^{-\beta_1 s_n} (c \|u\|_{W_2^1(\partial\Omega)}^2 + M_n \varepsilon^{2(1-\beta)}) \quad (2.8)$$

where

$$\beta_1 = \beta \frac{1 - \lambda - h_0}{1 + \lambda + h_0} > 0, \quad h_0 = r_n \varepsilon_0^2 / n, \quad M_n = cn \left(\frac{n}{r_n \mu_n} \right)^\beta \left[1 + \frac{r_n}{n} \|u\|_{C^1(\bar{\Omega})}^2 \right], \quad (2.9)$$

constant c is independent of ε and n , $\beta \in (0, 1)$ is arbitrary, s_n is the Blaschke series defined by condition (2.3), and

$$\mu_n = \min_k \prod_{j \neq k} \left| \frac{z_k - z_j}{1 - z_k \bar{z}_j} \right|, \quad (2.10)$$

$$r_n = \min_{j \neq k} |z_k - z_j|. \quad (2.11)$$

Theorem 2.5. *If $u \in C^2(\bar{\Omega})$ satisfies*

$$|\Delta u|^2 \leq a(|u|^2 + |\partial u|^2)$$

in Ω with constant $a \geq 0$ and for $z_j \in \Omega$, $j = 1, \dots, n$,

$$|u(z_j)| \leq \varepsilon, \quad |\partial u(z_j)| \leq \varepsilon,$$

then for $0 \leq \varepsilon \leq \varepsilon_0 < 1$

$$\|u\|_{W_2^1(|z|<\lambda)}^2 \leq e^{-\beta_1 s_n} \left\{ c \|u\|_{W_2^1(\partial\Omega)}^2 + M_n \varepsilon^{2(1-\beta)} \right\}$$

where $\beta_1, \beta, \lambda, c, s_n$ are defined as in Theorem 2.4 and M_n is determined as in equation (2.9) in which $\|u\|$ is computed in $C^2(\bar{\Omega})$.

2.2 Weight Estimates

In order to prove these and subsequent theorems we must develop an identity and its corollaries for real-valued weight functions $\varphi \in C^2(\bar{\Omega})$ on open bounded set $\Omega \in \mathbb{R}^n$ with class C^1 boundary. In this derivation assume that H is a complex Hilbert space with norm $|\cdot|$ and scalar product (\cdot, \cdot) . Let $\mathcal{L}(H)$ be the algebra of bounded linear operators acting on H , $\alpha = (\alpha_1, \dots, \alpha_n)$ be a multi-index, and A_α be linear operators in H dependent on $x \in \Omega$ and belonging to $C^1(\bar{\Omega}, \mathcal{L}(H))$. We utilize the following

$$\partial = (\partial_1, \partial_2, \dots, \partial_n), \quad \partial_j = \frac{\partial}{\partial x_j}, \quad \partial^\alpha = \partial_1^{\alpha_1} \partial_2^{\alpha_2} \dots \partial_n^{\alpha_n}$$

$$P(x, \partial) = \sum_{|\alpha| \leq 1} A_\alpha(x) \partial^\alpha.$$

Let

$$P^*(x, \partial) = \sum_{|\alpha| \leq 1} (-1)^{|\alpha|} \partial^\alpha A_\alpha^*$$

be an operator which is formally conjugate to P , and let P_1^* be its principal part, that is

$$P_1^* = - \sum_{|\alpha|=1} A_\alpha^* \partial^\alpha$$

Our principal identity will be used to obtain all following estimates:

Lemma 2.1. *Let $\varphi \in C^2(\bar{\Omega})$ be a real function. Then for any function $u \in C^1(\bar{\Omega}; H)$ with values in H we have*

$$\int_{\Omega} |P(x, \partial)u|^2 e^\varphi dx = \int_{\Omega} |P^*(x, \partial + \partial\varphi)u|^2 e^\varphi dx \quad (2.12)$$

$$+ \int_{\Omega} \Re([P^*(x, \partial + \partial\varphi), P(x, \partial)] u, u) e^\varphi dx - \int_{\partial\Omega} (\mathcal{P}u, u) e^\varphi d\sigma$$

where

$$\mathcal{P} = \Re(P_1^*(x, \nu)P(x, \partial) - P(x, \nu)P^*(x, \partial + \partial\varphi)), \quad (2.13)$$

ν is a unit outward normal to $\partial\Omega$, $d\sigma$ is the Euclidean element of the surface $\partial\Omega$, and $[P^*, P] = P^*P - PP^*$ is the commutator of operators P^* and P .

Proof. In this case we use the “weighted” scalar products and their associated norms given by

$$\langle u, v \rangle_{\Omega, \varphi} = \int_{\Omega} (u(x), v(x)) e^{\varphi(x)} dx \quad (2.14)$$

$$\langle u, v \rangle_{\partial\Omega, \varphi} = \int_{\partial\Omega} (u(x), v(x)) e^{\varphi(x)} d\sigma \quad (2.15)$$

$$\|u\|_{\Omega, \varphi}^2 = \langle u, u \rangle_{\Omega, \varphi}$$

$$\|u\|_{\partial\Omega, \varphi}^2 = \langle u, u \rangle_{\partial\Omega, \varphi}.$$

Furthermore we recall the Gauss-Ostrogradski formula

$$\int_{\Omega} \partial_j f(x) dx = \int_{\partial\Omega} \nu_j f_j(x) d\sigma$$

for any smooth vector field $f(x) = (f_1(x), f_2(x))$. First we consider the simple case of

$P = A(x)\partial_j$ for $u, v \in C^1(\bar{\Omega}, H)$ and $\varphi \in C^2(\bar{\Omega}, \mathbb{R})$. Now

$$\begin{aligned}
\partial_j \{(u, A^*v)e^\varphi\} &= \{\partial_j(u, A^*v)\} e^\varphi + (u, A^*v)\partial_j e^\varphi \\
&= [(\partial_j u, A^*v) + (u, \partial_j(A^*v)) + (u, A^*v)\partial_j\varphi] e^\varphi \\
&= [(A(\partial_j u), v) + (u, (\partial_j A^*)v) + (u, A^*(\partial_j v)) + (u, A^*v)\partial_j\varphi] e^\varphi \\
&= (A(\partial_j u), v)e^\varphi + (u, (\partial_j A^*)v + A^*(\partial_j v) + (\partial_j\varphi)A^*v)e^\varphi \\
&= (A(\partial_j u), v)e^\varphi + (u, (\partial_j + \partial_j\varphi)A^*v)e^\varphi.
\end{aligned}$$

Hence

$$\begin{aligned}
\langle Pu, v \rangle &= \int_{\Omega} (Pu(x), v(x)) e^{\varphi(x)} dx \\
&= \int_{\Omega} (A(x)\partial_j u(x), v(x)) e^{\varphi(x)} dx \\
&= \int_{\Omega} [\partial_j \{(u, A^*v)e^\varphi\} dx - (u, (\partial_j + \partial_j\varphi)A^*v)e^\varphi] dx \\
&= \int_{\partial\Omega} \nu_j (Au, v)e^\varphi d\sigma - \int_{\Omega} (u, (\partial_j + \partial_j\varphi)A^*v)e^\varphi dx,
\end{aligned}$$

and for general P we have

$$\begin{aligned}
\langle Pu, v \rangle_{\Omega, \varphi} &= \sum_{|\alpha| \leq 1} \left\{ \int_{\Omega} (u, (-1)^{|\alpha|}(\partial^\alpha + \partial^\alpha\varphi)A_\alpha^*v)e^\varphi dx + \int_{\partial\Omega} (u, A_\alpha^*(\nu_\alpha v))e^\varphi d\sigma \right\} \\
&= \int_{\Omega} \left(u, \sum_{|\alpha| \leq 1} (-1)^{|\alpha|}(\partial^\alpha + \partial^\alpha\varphi)A_\alpha^*v \right) e^\varphi dx + \int_{\partial\Omega} \left(u, \sum_{|\alpha|=1} A_\alpha^*(\nu_\alpha v) \right) e^\varphi d\sigma \\
&= \int_{\Omega} (u, P^*(\partial + \partial\varphi)v)e^\varphi dx - \int_{\partial\Omega} (u, P_1^*(x, \nu)v)e^\varphi d\sigma.
\end{aligned}$$

Therefore if we define

$$P^\sharp := P^*(x, \partial + \partial\varphi) \tag{2.16}$$

one obtains

$$\begin{aligned}
\langle Pu, v \rangle_{\Omega, \varphi} &= \int_{\Omega} (u, P^{\sharp}v) e^{\varphi} dx - \int_{\partial\Omega} (u, P_1^*(x, \nu)v) e^{\varphi} d\sigma. \\
&= \langle u, P^*(x, \partial + \partial\varphi)v \rangle_{\Omega, \varphi} - \langle u, P_1^*(x, \nu)v \rangle_{\partial\Omega, \varphi} \\
u, v &\in C^1(\bar{\Omega}, H).
\end{aligned} \tag{2.17}$$

Thus operator, P^{\sharp} , is the conjugate to operator P in Hilbert Space $L_{2, \varphi}(\Omega; H)$ with scalar product (2.17). Furthermore

$$\begin{aligned}
\langle Pu, Pu \rangle_{\Omega, \varphi} &= \langle u, P^{\sharp}Pu \rangle_{\Omega, \varphi} - \langle u, P_1^*(x, \nu)P(x, \partial)u \rangle_{\partial\Omega, \varphi} u \\
&= \langle P^{\sharp}Pu, u \rangle_{\Omega, \varphi} - \langle P_1^*(x, \nu)P(x, \partial)u, u \rangle_{\partial\Omega, \varphi} \\
\langle P^{\sharp}u, P^{\sharp}u \rangle_{\Omega, \varphi} &= \langle PP^{\sharp}u, u \rangle_{\partial\Omega, \varphi} + \langle P^{\sharp}u, P_1^*(x, \nu)u \rangle_{\partial\Omega, \varphi} \\
&= \langle PP^{\sharp}u, u \rangle_{\Omega, \varphi} + \left\langle P^*(x, \partial + \partial\varphi)u, - \sum_{|\alpha| \leq 1} A_{\alpha}^* \nu_{\alpha} u \right\rangle_{\partial\Omega, \varphi} \\
&= \langle PP^{\sharp}u, u \rangle_{\Omega, \varphi} + \left\langle - \sum_{|\alpha| \leq 1} A_{\alpha} \nu_{\alpha} P^*(x, \partial + \partial\varphi)u, u \right\rangle_{\partial\Omega, \varphi} \\
&= \langle PP^{\sharp}u, u \rangle_{\Omega, \varphi} - \langle P_1(x, \nu)P^*(x, \partial + \partial\varphi)u, u \rangle_{\partial\Omega, \varphi},
\end{aligned}$$

where P_1 is the principal symbol of P . Subtracting these equations gives us

$$\begin{aligned}
\|Pu\|_{\Omega, \varphi}^2 - \|P^{\sharp}u\|_{\Omega, \varphi}^2 &= \langle [P^{\sharp}, P]u, u \rangle_{\Omega, \varphi} \\
&\quad - \langle P_1^*(x, \nu)P(x, \partial)u, u \rangle_{\partial\Omega, \varphi} + \langle P_1(x, \nu)P^*(x, \partial + \partial\varphi)u, u \rangle_{\partial\Omega, \varphi}.
\end{aligned}$$

Since the left hand side of this equation is real valued, the imaginary part of the right hand

side is zero and

$$\begin{aligned}
\|Pu\|_{\Omega,\varphi}^2 - \|P^\sharp u\|_{\Omega,\varphi}^2 &= \langle \Re[P^\sharp, P]u, u \rangle_{\Omega,\varphi} \\
&\quad - \langle \Re(P_1^*(x, \nu)P(x, \partial) - P_1(x, \nu)P^*(x, \partial + \partial\varphi))u, u \rangle_{\partial\Omega,\varphi} \\
&= \langle \Re[P^\sharp, P]u, u \rangle_{\Omega,\varphi} - \langle \mathcal{P}u, u \rangle_{\partial\Omega,\varphi}.
\end{aligned}$$

This equation is equivalent to equation (2.12). \square

Thus we have

$$\|Pu\|_{\Omega,\varphi}^2 + \langle \mathcal{P}u, u \rangle_{\partial\Omega,\varphi} = \|P^\sharp u\|_{\Omega,\varphi}^2 + \langle \Re[P^\sharp, P]u, u \rangle_{\Omega,\varphi} \quad (2.18)$$

or

$$\langle \Re[P^\sharp, P]u, u \rangle_{\Omega,\varphi} \leq \|Pu\|_{\Omega,\varphi}^2 + \langle \mathcal{P}u, u \rangle_{\partial\Omega,\varphi}. \quad (2.19)$$

This gives us the following corollary.

Corollary 2.1. *For any function $u \in C^2(\bar{\Omega}, H)$, then*

$$\Re \int_{\Omega} ([P^*(x, \partial + \partial\varphi), P(x, \partial)]u, u) e^\varphi dx \leq \int_{\Omega} |P(x, \partial)u|^2 e^\varphi dx + \int_{\partial\Omega} (\mathcal{P}u, u) e^\varphi d\sigma \quad (2.20)$$

Example 2.1. *Let us take H to be the complex plane, \mathbb{C} , and $\mathbf{n} = 2$.*

We define

$$P(x, \partial) = \partial_1 + i\partial_2 = 2\partial_{\bar{z}},$$

where $z = x_1 + ix_2$, then one obtains $P^ = -2\partial_z$, $P^\sharp = -2\partial_z - 2\partial_z\varphi$, and*

$$\begin{aligned}
[P^\sharp, P] &= [-2\partial_z - 2\partial_z\varphi, 2\partial_{\bar{z}}] \\
&= (-2\partial_z - 2\partial_z\varphi)2\partial_{\bar{z}} - 2\partial_{\bar{z}}(-2\partial_z - 2\partial_z\varphi) \\
&= -4(\partial_z + \partial_z\varphi)\partial_{\bar{z}} + 4\partial_{\bar{z}}(\partial_z + \partial_z\varphi) \\
&= -4(\partial_z\varphi)\partial_{\bar{z}} + 4\partial_{\bar{z}}\partial_z\varphi + 4(\partial_z\varphi)\partial_{\bar{z}} = \Delta\varphi.
\end{aligned}$$

If we further denote $\partial_\nu = \nu \cdot \partial = \nu_1\partial_1 + \nu_2\partial_2$ and $\nu_\perp = (-\nu_2, \nu_1)$, then we have

$$\begin{aligned}
\mathcal{P} &= \Re [P_1^*(x, \nu)P(x, \partial) - P_1(x, \nu)P^*(x, \partial + \partial\varphi)] \\
&= \Re [(-\nu_1 + i\nu_2)(\partial_1 + i\partial_2) - (\nu_1 + i\nu_2)(-\partial_1 + i\partial_2 + \partial_1\varphi + i\partial_2\varphi)] \\
&= \Re [-\nu_1\partial_1 - \nu_2\partial_2 + i(\nu_2\partial_1 - \nu_1\partial_2) - (-\nu_1\partial_1 + i\nu_1\partial_2 - i\nu_2\partial_1 - \nu_2\partial_2) \\
&\quad + (\nu_1 + i\nu_2) \cdot (-\partial_1\varphi + i\partial_2\varphi)] \\
&= \Re [\nu \cdot \partial\varphi + 2i\nu_2\partial_1 - 2i\nu_1\partial_2] \\
&= \partial_\nu\varphi - 2\Re(i\partial_{\nu_\perp})
\end{aligned}$$

Lemma 2.2. *Let φ be a real valued function of class $C^2(\bar{\Omega})$; then for any $u \in C^1(\bar{\Omega}; \mathbb{C})$*

$$\int_{\Omega} \Delta\varphi |u|^2 e^\varphi dx \leq 4 \int_{\Omega} |\partial_{\bar{z}}u|^2 e^\varphi dx + \int_{\partial\Omega} (\partial_\nu\varphi |u|^2 - 2\Re(i\partial_{\nu_\perp}u \cdot \bar{u})) e^\varphi d\sigma \quad (2.21)$$

Proof. Using $P(x, \partial) = 2\partial_{\bar{z}}$ and substituting this operator into equation (2.18) one obtains

$$\begin{aligned}
\Re \int_{\Omega} ([P^*, P]u, u)e^\varphi dx + \int_{\Omega} |P^*(x, \partial + \partial\varphi)u|^2 e^\varphi dx &= \int_{\Omega} |P(x, \partial)u|^2 dx + \int_{\partial\Omega} (\mathcal{P}u, u)e^\varphi d\sigma \\
\int_{\Omega} \Delta\varphi |u|^2 e^\varphi dx + 4 \int_{\Omega} |(\partial_z + \partial_z\varphi)u|^2 e^\varphi dx &= 4 \int_{\Omega} |\partial_{\bar{z}}u|^2 e^\varphi dx + \int_{\partial\Omega} ([\partial_\nu\varphi - 2\Re(i\partial_{\nu_\perp})]u, u) e^\varphi d\sigma
\end{aligned}$$

$$\int_{\Omega} \Delta\varphi |u|^2 e^{\varphi} dx + 4 \int_{\Omega} |(\partial_z + \partial_{\bar{z}}\varphi)u|^2 e^{\varphi} dx = 4 \int_{\Omega} |\partial_{\bar{z}}u|^2 e^{\varphi} dx + \int_{\partial\Omega} (\partial_{\nu}\varphi |u|^2 - 2\Re(i\partial_{\nu_{\perp}}u \cdot \bar{u})) e^{\varphi} d\sigma. \quad (2.22)$$

Since $|(\partial_z + \partial_{\bar{z}}\varphi)u|^2 e^{\varphi} \geq 0$ we have equation (2.21). \square

Example 2.2. Let $\mathbf{H} = \mathbf{C}$ again and $\mathbf{P}(\mathbf{x}, \partial) = \partial_1 - i\partial_2 = 2\partial_z$

Then we have

$$P^* = -2\partial_{\bar{z}},$$

$$P^{\sharp} = P^*(\partial + \partial\varphi) = -2(\partial_{\bar{z}} + \partial_{\bar{z}}\varphi),$$

$$\begin{aligned} [P^{\sharp}, P] &= [-2\partial_{\bar{z}} - 2\partial_{\bar{z}}\varphi, 2\partial_z] = 4[\partial_z, \partial_{\bar{z}} + \partial_{\bar{z}}\varphi] \\ &= 4[\partial_z(\partial_{\bar{z}} + \partial_{\bar{z}}\varphi) - (\partial_{\bar{z}} + \partial_{\bar{z}}\varphi)\partial_z] \\ &= 4[\partial_z\partial_{\bar{z}} + \partial_z\partial_{\bar{z}}\varphi + (\partial_{\bar{z}}\varphi)\partial_z - \partial_{\bar{z}}\partial_z - (\partial_{\bar{z}}\varphi)\partial_z] \\ &= 4\partial_z\partial_{\bar{z}}\varphi = \Delta\varphi, \end{aligned}$$

and

$$\begin{aligned} \mathcal{P} &= \Re[P_1^*(x, \nu)P(x, \partial) - P_1(x, \nu)P^*(x, \partial + \partial\varphi)] \\ &= \Re[-(\nu_1 + i\nu_2)(\partial_1 - i\partial_2) - (\nu_1 - i\nu_2)(-\partial_1 - i\partial_2 - \partial_1\varphi - i\partial_2\varphi)] \\ &= \Re[-\nu_1\partial_1 - \nu_2\partial_2 + i\nu_1\partial_2 - i\nu_2\partial_1 + \nu_1\partial_1 + \nu_2\partial_2 + i\nu_1\partial_2 - \nu_2\partial_1 \\ &\quad + (\nu_1 - i\nu_2)(\partial_1 + i\partial_2)\varphi] \\ &= \Re[2i\nu_1\partial_2 - 2i\nu_2\partial_1 + \bar{\nu} \cdot \partial\varphi] = \Re[\partial_{\bar{\nu}}\varphi + 2i(-\nu_2, \nu_1) \cdot (\partial_1, \partial_2)] \\ &= \Re[\partial_{\bar{\nu}}\varphi + 2i\nu_{\perp} \cdot \partial] = \partial_{\nu}\varphi + 2\Re(i\partial_{\nu_{\perp}}). \end{aligned}$$

Lemma 2.3. *Let φ be a real valued function of class $C^2(\bar{\Omega})$; then for any $u \in C^1(\bar{\Omega}; \mathbb{C})$*

$$\int_{\Omega} \Delta\varphi |u|^2 e^{\varphi} dx \leq 4 \int_{\Omega} |\partial_z u|^2 e^{\varphi} dx + \int_{\partial\Omega} (\partial_{\nu}\varphi |u|^2 + 2\Re(i\partial_{\nu\perp} u \cdot \bar{u})) e^{\varphi} d\sigma \quad (2.23)$$

Proof. Using $P(x, \partial) = 2\partial_z$ and substituting this operator into equation (2.18) one obtains

$$\begin{aligned} \Re \int_{\Omega} ([P^*, P]u, u) e^{\varphi} dx + \int_{\Omega} |P^*(x, \partial + \partial\varphi)u|^2 e^{\varphi} dx &= \int_{\Omega} |P(x, \partial)u|^2 dx + \int_{\partial\Omega} (\mathcal{P}u, u) e^{\varphi} d\sigma \\ \int_{\Omega} \Delta\varphi |u|^2 e^{\varphi} dx + 4 \int_{\Omega} |(\partial_{\bar{z}} + \partial_{\bar{z}}\varphi)u|^2 e^{\varphi} dx &= 4 \int_{\Omega} |\partial_z u|^2 e^{\varphi} dx + \int_{\partial\Omega} ([\partial_{\bar{\nu}}\varphi + 2\Re(i\partial_{\nu\perp})]u, u) e^{\varphi} d\sigma \\ \int_{\Omega} \Delta\varphi |u|^2 e^{\varphi} dx + 4 \int_{\Omega} |(\partial_{\bar{z}} + \partial_{\bar{z}}\varphi)u|^2 e^{\varphi} dx &= 4 \int_{\Omega} |\partial_z u|^2 e^{\varphi} dx + \int_{\partial\Omega} (\partial_{\bar{\nu}}\varphi |u|^2 + 2\Re(i\partial_{\nu\perp} u \cdot \bar{u})) e^{\varphi} d\sigma. \end{aligned} \quad (2.24)$$

Since φ is real valued and $|\partial_z u|^2 e^{\varphi} \geq 0$ we have equation (2.23). \square

We will need the following identities.

$$\begin{aligned} 4(|\partial_{\bar{z}}u|^2 + |\partial_z u|^2) &= 4(\langle \partial_{\bar{z}}u, \partial_{\bar{z}}u \rangle + \langle \partial_z u, \partial_z u \rangle) \\ &= \langle (\partial_1 + i\partial_2)u, (\partial_1 + i\partial_2)u \rangle + \langle (\partial_1 - i\partial_2)u, (\partial_1 - i\partial_2)u \rangle \\ &= \langle \partial_1 u, \partial_1 u \rangle + \langle i\partial_1, \partial_1 u \rangle + \langle \partial_1 u, i\partial_2 u \rangle + \langle i\partial_2 u, i\partial_2 u \rangle \\ &\quad + \langle \partial_1 u, \partial_1 u \rangle - \langle i\partial_1, \partial_1 u \rangle - \langle \partial_1 u, i\partial_2 u \rangle + \langle i\partial_2 u, i\partial_2 u \rangle \\ &= 2(\langle \partial_1 u, \partial_1 u \rangle - i^2 \langle \partial_2 u, \partial_2 u \rangle) = 2(|\partial_1 u|^2 + |\partial_2 u|^2) \\ &= 2|\partial u|^2 \end{aligned} \quad (2.25)$$

$$|(\partial_{\bar{z}} + \partial_{\bar{z}}\varphi)u|^2 = |e^{-\varphi}(e^{\varphi}\partial_{\bar{z}}u + e^{\varphi}u\partial_{\bar{z}}\varphi)|^2 = e^{-2\varphi}|\partial_{\bar{z}}(e^{\varphi}u)|^2$$

$$|(\partial_z + \partial_z\varphi)u|^2 = |e^{-\varphi}(e^{\varphi}\partial_z u + e^{\varphi}u\partial_z\varphi)|^2 = e^{-2\varphi}|\partial_z(e^{\varphi}u)|^2$$

$$\begin{aligned}
4 \left(|(\partial_{\bar{z}} + \partial_{\bar{z}}\varphi)u|^2 + |(\partial_z + \partial_z\varphi)u|^2 \right) &= e^{-2\varphi} 4 \left(|\partial_{\bar{z}}(e^\varphi u)|^2 + |\partial_z(e^\varphi u)|^2 \right) \\
&= e^{-2\varphi} 2 \left(|\partial_1(e^\varphi u)|^2 + |\partial_2(e^\varphi u)|^2 \right) \\
&= e^{-2\varphi} 2 (\partial(e^\varphi u)) \\
&= 2 |(\partial + \partial\varphi)u|^2
\end{aligned} \tag{2.26}$$

Adding equations (2.22) and (2.24) gives

$$\begin{aligned}
&\int_{\Omega} \Delta\varphi |u|^2 e^\varphi dx + \int_{\Omega} 4 |(\partial_z + \partial_z\varphi)u|^2 e^\varphi dx + \int_{\Omega} \Delta\varphi |u|^2 e^\varphi dx + \int_{\Omega} 4 |(\partial_{\bar{z}} + \partial_{\bar{z}}\varphi)u|^2 e^\varphi dx \\
&= 4 \int_{\Omega} |\partial_{\bar{z}}u|^2 e^\varphi dx + \int_{\partial\Omega} (\partial_\nu\varphi |u|^2 - 2\Re(i\partial_{\nu_\perp} u \cdot \bar{u})) e^\varphi d\sigma \\
&\quad + 4 \int_{\Omega} |\partial_z u|^2 e^\varphi dx + \int_{\partial\Omega} (\partial_\nu\varphi |u|^2 + 2\Re(i\partial_{\nu_\perp} u \cdot \bar{u})) e^\varphi d\sigma,
\end{aligned}$$

or

$$2 \int_{\Omega} \Delta\varphi |u|^2 e^\varphi dx + 2 \int_{\Omega} |(\partial + \partial\varphi)u|^2 e^\varphi dx = 2 \int_{\Omega} |\partial u|^2 e^\varphi dx + 2 \int_{\partial\Omega} \partial_\nu\varphi |u|^2 e^\varphi d\sigma.$$

Since $|(\partial + \partial\varphi)u|^2 e^\varphi \geq 0$ we have

$$\int_{\Omega} \Delta\varphi |u|^2 e^\varphi dx \leq \int_{\Omega} |\partial u|^2 e^\varphi dx + \int_{\partial\Omega} \partial_\nu\varphi |u|^2 e^\varphi d\sigma, \tag{2.27}$$

which may be regarded as a weighted inequality of Poincaré type. Finally we need the following estimates

Lemma 2.4. *Let φ be a real valued function of class $C^2(\bar{\Omega})$; then for any $u \in C^2(\bar{\Omega}; \mathbb{C})$*

$$\int_{\Omega} \Delta\varphi |u|^2 e^\varphi dx \leq \int_{\Omega} |\Delta u|^2 e^\varphi dx + \int_{\partial\Omega} \{ \partial_\nu\varphi |\partial u|^2 + 8\Re(i\partial_{\nu_\perp} \partial_{\bar{z}} \cdot \partial_z u) \} e^\varphi d\sigma \tag{2.28}$$

and

$$\begin{aligned}
& \int_{\Omega} (\Delta\varphi |u|^2 + (\Delta\varphi - 1) |\partial u|^2) e^\varphi dx \\
& \leq \int_{\Omega} |\Delta u|^2 e^\varphi dx + \int_{\partial\Omega} \{ \partial_\nu \varphi (|u|^2 + |\partial u|^2) + 8\Re(i\partial_{\nu\perp} \partial_{\bar{z}} u \cdot \partial_z u) \} e^\varphi d\sigma \quad (2.29)
\end{aligned}$$

Proof. Substituting $\partial_z u$ for u in equation (2.21) one obtains

$$\begin{aligned}
\int_{\Omega} \Delta\varphi |\partial_z u|^2 e^\varphi dx & \leq 4 \int_{\Omega} |\partial_{\bar{z}} \partial_z u|^2 e^\varphi dx + \int_{\partial\Omega} \{ \partial_\nu \varphi |\partial_z u|^2 - 2\Re(i\partial_\nu(\partial_z u) \cdot \overline{\partial_z u}) \} e^\varphi d\sigma \\
& = \frac{1}{4} \int_{\Omega} |\Delta u|^2 e^\varphi dx + \int_{\partial\Omega} \{ \partial_\nu \varphi |\partial_z u|^2 - 2\Re(i\partial_\nu(\partial_z u) \cdot \overline{\partial_z u}) \} e^\varphi d\sigma
\end{aligned}$$

Similarly substituting $\partial_{\bar{z}} u$ for u in equation (2.23) gives

$$\begin{aligned}
\int_{\Omega} \Delta\varphi |\partial_{\bar{z}} u|^2 e^\varphi dx & \leq 4 \int_{\Omega} |\partial_z \partial_{\bar{z}} u|^2 e^\varphi dx + \int_{\partial\Omega} \{ \partial_\nu \varphi |\partial_{\bar{z}} u|^2 + 2\Re(i\partial_\nu(\partial_{\bar{z}} u) \cdot \overline{\partial_{\bar{z}} u}) \} e^\varphi d\sigma \\
& = \frac{1}{4} \int_{\Omega} |\Delta u|^2 e^\varphi dx + \int_{\partial\Omega} \{ \partial_\nu \varphi |\partial_{\bar{z}} u|^2 + 2\Re(i\partial_\nu(\partial_{\bar{z}} u) \cdot \overline{\partial_{\bar{z}} u}) \} e^\varphi d\sigma
\end{aligned}$$

Observe the following identities:

$$\begin{aligned}
|\partial_{\bar{z}} u|^2 & = \frac{1}{4} |\partial_1 u + i\partial_2 u|^2 = \frac{1}{4} (|\partial_1 u|^2 + |\partial_2 u|^2) = \frac{1}{4} |\partial_1 u + i\partial_2 u|^2 = |\partial_z u|^2 \\
-\Re(iw, \bar{z}) & = -\Re(iwz) = \Re(-iwz) = \Re(-i(a + bi)(c + di)) \\
& = \Re(-i(ac - bd + i(ad + bc))) = ad + bc \\
\Re(i\bar{w}z) & = \Re(i\bar{w}\bar{z}) = \Re(i(a - bi)(c - di)) \\
& = \Re(i(ac - bd + i(-ad - bc))) = ad + bc
\end{aligned}$$

Here we have

$$\begin{aligned}
2 \int_{\Omega} \Delta \varphi |\partial_z u|^2 e^\varphi dx &\leq \frac{1}{2} \int_{\Omega} |\Delta u|^2 e^\varphi dx + 2 \int_{\partial\Omega} (\partial_\nu \varphi |\partial_z u|^2 + 2\Re(i\partial_\nu(\overline{\partial_z u}) \cdot \partial_z u)) e^\varphi d\sigma \\
2 \int_{\Omega} \Delta \varphi |\partial_{\bar{z}} u|^2 e^\varphi dx &\leq \frac{1}{2} \int_{\Omega} |\Delta u|^2 e^\varphi dx + 2 \int_{\partial\Omega} (\partial_\nu \varphi |\partial_{\bar{z}} u|^2 + 2\Re(i\partial_\nu(\overline{\partial_{\bar{z}} u}) \cdot \partial_{\bar{z}} u)) e^\varphi d\sigma.
\end{aligned}$$

Adding these two equations gives

$$\begin{aligned}
2 \int_{\Omega} \Delta \varphi (|\partial_z u|^2 + |\partial_{\bar{z}} u|^2) e^\varphi dx &\leq \int_{\Omega} |\Delta u|^2 e^\varphi dx + 2 \int_{\partial\Omega} \partial_\nu \varphi (|\partial_z u|^2 + |\partial_{\bar{z}} u|^2) e^\varphi d\sigma \\
&\quad + \int_{\partial\Omega} 8\Re(i\partial_\nu(\partial_z u) \cdot \partial_z u) e^\varphi d\sigma
\end{aligned}$$

Using equation (2.25) one obtains equation (2.28). Adding equation (2.27) to equation (2.28) gives equation (2.29). \square

2.3 Uniqueness and Stability Theorems

With the estimates obtained in the previous section we can prove the theorems in section 2.1.

Proof. of Theorem 2.1

Applying inequality (2.21) to condition (2.4) yields

$$\int_{\Omega} \Delta \varphi |u|^2 e^\varphi dx \leq 4 \int_{\Omega} a |u|^2 e^\varphi dx + \int_{\partial\Omega} (\partial_\nu \varphi |u|^2 - 2\Re(i\partial_{\nu_\perp} u \cdot \bar{u})) e^\varphi d\sigma. \quad (2.30)$$

We compose $\varphi = \varphi_0 + \varphi_1$ where φ_0 is the solution to the Dirichlet problem

$$\Delta\varphi_0 = 4a + 1, \quad z \in \Omega \quad (2.31)$$

$$\varphi_0|_{\partial\Omega} = 0 \quad (2.32)$$

and φ_1 is the solution of the Dirichlet problem

$$\Delta\varphi_1 = -2\pi \sum_{j=1}^n m_j \delta(z - z_j) \quad (2.33)$$

$$\varphi_1|_{\partial\Omega} = 0; \quad (2.34)$$

where $n \in \mathbb{N}$ and $\delta(z)$ is the Dirac measure. Since $a \in L^2(\Omega)$, the solution to (2.31) and (2.32) has the following properties by Sobolev's Trace Theorem:

$$\varphi_0 \in W_2^2(\Omega) \cap W_2^{3/2}(\partial\Omega).$$

Moreover $\partial\varphi \in W_2^1(\Omega)$, which implies $\partial_\nu\varphi_0 \in W_2^{1/2}(\partial\Omega)$. Therefore Sobolev's imbedding theorems give

$$\varphi_0 \in C(\bar{\Omega}), \quad (2.35)$$

$$\partial_\nu\varphi_0 \in L_1(\partial\Omega). \quad (2.36)$$

The solution to problem solution to (2.33) and (2.32) is given by

$$\varphi_1 = \alpha \sum_{j=1}^n m_j \ln \left| \frac{1 - z\bar{z}_j}{z - z_j} \right|, \quad \alpha \in (0, 2) \quad (2.37)$$

Since z_j is a zero of multiplicity $m_j > 0$ we have

$$|z - z_j|^{-\alpha m_j} |u(z)|^2 = \mathcal{O}(|z - z_j|^{-\alpha m_j}) \rightarrow 0$$

as $z \rightarrow z_j$, and therefore

$$\begin{aligned} \int_{\Omega} \Delta\varphi_1 |u|^2 e^\varphi dx &= -2\pi \int_{\Omega} \sum_{j=1}^n m_j \delta(z - z_j) |u|^2 \exp \left\{ \alpha \sum_{j=1}^n m_j \ln \left| \frac{1 - z\bar{z}_j}{z - z_j} \right| \right\} e^{\varphi_0} dz \\ &= -2\pi \int_{\Omega} \sum_{j=1}^n m_j \delta(z - z_j) |u|^2 \left| \frac{1 - z\bar{z}_j}{z - z_j} \right|^{\alpha m_j} e^{\varphi_0} dz = 0. \end{aligned}$$

Hence we have

$$\begin{aligned} \int_{\Omega} (4a + 1) |u|^2 e^\varphi dx &= \int_{\Omega} \Delta\varphi_0 |u|^2 e^\varphi dx = \int_{\Omega} (\Delta\varphi_0 + \Delta\varphi_1) |u|^2 e^\varphi dx \\ &= \int_{\Omega} \Delta\varphi |u|^2 e^\varphi dx \\ &\leq 4 \int_{\Omega} a |u|^2 e^\varphi dx + \int_{\partial\Omega} (\partial_\nu \varphi |u|^2 - 2\Re(i\partial_{\nu_\perp} u \cdot \bar{u})) d\sigma \end{aligned}$$

since $\varphi = 0$ on $\partial\Omega$ and this implies $e^\varphi = 1$ on $\partial\Omega$. Therefore we have

$$\int_{\Omega} |u|^2 e^\varphi dx \leq \int_{\partial\Omega} (\partial_\nu \varphi |u|^2 - 2\Re(i\partial_{\nu_\perp} u \cdot \bar{u})) d\sigma. \quad (2.38)$$

If we denote $z = x + iy$ and $z_j = x_j + iy_j$ for $z, z_j \in \partial\Omega$, we have the following:

$$\begin{aligned} \partial_x \frac{1}{2} \ln |1 - z\bar{z}_j|^2 &= -\frac{1}{2} \frac{z_j(1 - z\bar{z}_j) + \bar{z}_j(1 - \bar{z}z_j)}{|1 - z\bar{z}_j|^2}, \\ &= -\frac{\Re(z_j(1 - z\bar{z}_j))}{|1 - z\bar{z}_j|^2} = -\frac{\Re(z_j - z |z_j|^2)}{|1 - z\bar{z}_j|^2}, \\ &= \frac{x |z_j|^2 - x_j}{|1 - z\bar{z}_j|^2}, \end{aligned}$$

$$\begin{aligned}
\partial_y \frac{1}{2} \ln |1 - z\bar{z}_j|^2 &= \frac{-1}{2i^2} \frac{\Im(z_j(1 - z\bar{z}_j))}{|1 - z\bar{z}_j|^2} = \frac{y|z_j|^2 - y_j}{|1 - z\bar{z}_j|^2}, \\
\partial_x \frac{1}{2} \ln |z - z_j|^2 &= \frac{1}{2} \frac{\Re(z - z_j)}{|z - z_j|^2} = \frac{x - x_j}{|z - z_j|^2} = \frac{(x - x_j)|z|^2}{|\bar{z} - \bar{z}_j|^2 |z|^2}, \\
&= \frac{(x - x_j)|z|^2}{|z\bar{z} - z\bar{z}_j|^2} = \frac{x - x_j}{|1 - z\bar{z}_j|^2}, \\
\partial_y \frac{1}{2} \ln |z - z_j|^2 &= \frac{-1}{2i^2} \frac{\Im(z - z_j)}{|z - z_j|^2} = \frac{y - y_j}{|1 - z\bar{z}_j|^2}.
\end{aligned}$$

Hence

$$\begin{aligned}
\partial_x \ln \left| \frac{1 - z\bar{z}_j}{z - z_j} \right|^2 &= \frac{x|z_j|^2 - x_j - (x - x_j)}{|1 - z\bar{z}_j|^2} = -\frac{x(1 - |z_j|^2)}{|1 - z\bar{z}_j|^2}, \\
\partial_y \ln \left| \frac{1 - z\bar{z}_j}{z - z_j} \right|^2 &= \frac{y|z_j|^2 - y_j - (y - y_j)}{|1 - z\bar{z}_j|^2} = -\frac{y(1 - |z_j|^2)}{|1 - z\bar{z}_j|^2}.
\end{aligned}$$

Therefore recalling that z is on the unit circle ($\partial\Omega$),

$$\partial_\nu \frac{1}{2} \ln \left| \frac{1 - z\bar{z}_j}{z - z_j} \right|^2 = (x, y) \cdot \left(-\frac{x(1 - |z_j|^2)}{|1 - z\bar{z}_j|^2}, -\frac{y(1 - |z_j|^2)}{|1 - z\bar{z}_j|^2} \right) \quad (2.39)$$

$$= -\left(\frac{(x^2 + y^2)(1 - |z_j|^2)}{|1 - z\bar{z}_j|^2} \right) = -\frac{1 - |z_j|^2}{|1 - z\bar{z}_j|^2}. \quad (2.40)$$

Now since $|z_j|^2 < 1$ one obtains

$$\begin{aligned}
\partial_\nu \varphi_1(z) &= \partial_\nu \sum_{j=1}^n \frac{\alpha m_j}{2} \ln \left| \frac{1 - z\bar{z}_j}{z - z_j} \right|^2 \\
&= -\sum_{j=1}^n \alpha m_j \frac{1 - |z_j|^2}{|1 - z\bar{z}_j|^2} < 0.
\end{aligned}$$

Recalling that $|2\Re(w \cdot z)| \leq |2(w \cdot z)| \leq |w|^2 + |z|^2$, and equation (2.38)

$$\begin{aligned}
\int_{\Omega} |u|^2 e^{\varphi} dx &\leq \int_{\partial\Omega} (\partial_{\nu}(\varphi_0 + \varphi_1) |u|^2 - 2\Re(i\partial_{\nu_{\perp}} u \cdot \bar{u})) d\sigma \\
&\leq \int_{\partial\Omega} \partial_{\nu}\varphi_0 |u|^2 d\sigma + \int_{\partial\Omega} \partial_{\nu}\varphi_1 |u|^2 d\sigma + \int_{\partial\Omega} (|\partial_{\nu_{\perp}} u|^2 + |u|^2) d\sigma \\
&\leq \int_{\partial\Omega} \partial_{\nu}\varphi_0 (|\partial_{\nu_{\perp}} u|^2 + |u|^2) d\sigma + \int_{\partial\Omega} (|\partial_{\nu_{\perp}} u|^2 + |u|^2) d\sigma \\
&\leq \|u\|_{C^1(\partial\Omega)}^2 \left(\int_{\partial\Omega} |\partial_{\nu}\varphi_0| d\sigma + 1 \right) \\
&= \|u\|_{C^1(\partial\Omega)}^2 \left(\|\partial_{\nu}\varphi_0\|_{L^1(\partial\Omega)} + 1 \right). \tag{2.41}
\end{aligned}$$

The Taylor series for $\ln(x^{-1})$ about $x = 1$ on the interval $(0, 2)$ is given by

$$\ln(1/x) = -(x-1) + \frac{1}{2}(x-1)^2 - \frac{1}{3}(x-1)^3 + \frac{1}{4}(x-1)^4 - \frac{1}{5}(x-1)^5 + \mathcal{O}(x-1)^6$$

Note also that for n even and greater than zero we have

$$\begin{aligned}
\frac{1}{n}(x-1)^n - \frac{1}{n+1}(x-1)^{n+1} &= \left(\frac{1}{n} - \frac{x-1}{n+1} \right) (x-1)^n \\
&= \left(\frac{n+1 - n(x-1)}{n(n+1)} \right) (x-1)^n \\
&= \left(\frac{2n - (x-1)}{n(n+1)} \right) (x-1)^n > 0,
\end{aligned}$$

since $x \in (0, 1)$. Thus as $|z| \leq 1$ we have

$$\ln \left| \frac{1 - z\bar{z}_j}{z - z_j} \right| \geq 1 - \left| \frac{z - z_j}{1 - z\bar{z}_j} \right|.$$

Now

$$\begin{aligned}
|1 - z\bar{z}_j|^2 &= (1 - z\bar{z}_j)(1 - \bar{z}z_j) = 1 - z\bar{z}_j - \bar{z}z_j + z\bar{z}z_j\bar{z}_j \\
&= 1 + |z|^2|z_j|^2 - (z\bar{z}_j + \bar{z}z_j) \\
&= 1 + 2|z||z_j| + |z|^2|z_j|^2 - 2|z||z_j| - 2\Re(z\bar{z}_j) \\
&= (1 + |z||z_j|)^2 - 2(|z||z_j| + \Re(z\bar{z}_j)).
\end{aligned}$$

Furthermore $|\Re(z\bar{z}_j)| \leq |z||z_j|$ which implies $|z||z_j| - |\Re(z\bar{z}_j)| \geq 0$. This gives us that $|1 - z\bar{z}_j|^2 \geq (1 + |z||z_j|)^2$. Hence

$$\left| \frac{z - z_j}{1 - z\bar{z}_j} \right| \leq \frac{|z| + |z_j|}{1 + |z||z_j|}, \quad |z_j| < 1, |z| \leq 1.$$

One obtains

$$\begin{aligned}
\ln \left| \frac{1 - z\bar{z}_j}{z - z_j} \right| &\geq 1 - \left| \frac{z - z_j}{1 - z\bar{z}_j} \right| \geq 1 - \frac{|z| + |z_j|}{1 + |z||z_j|} \\
&= \frac{1 + |z||z_j| - |z| - |z_j|}{1 + |z||z_j|} \\
&\geq \frac{(1 - |z|)(1 - |z_j|)}{1 + |z|}
\end{aligned}$$

since $|z_j| < 1$. Hence

$$e^{\varphi_1(z)} = \exp \left\{ \alpha \sum_{j=1}^n m_j \ln \left| \frac{1 - z\bar{z}_j}{z - z_j} \right| \right\} \geq \exp \left\{ \frac{1 - |z|}{1 + |z|} \sum_{j=1}^n \alpha m_j (1 - |z_j|) \right\} \quad (2.42)$$

At this point let us define

$$s_n := \alpha \sum_{j=1}^n m_j (1 - |z_j|) \quad (2.43)$$

$$m_0 := \min_{|z| \leq 1} \varphi_0(z). \quad (2.44)$$

Using equations (2.42) through (2.44) we have for $\lambda < 1$ that

$$\begin{aligned} \int_{\Omega} |u|^2 e^{\varphi} dx &\geq \int_{|z| < \lambda} |u|^2 e^{\varphi} dx = \int_{|z| < \lambda} |u|^2 e^{\varphi_0 + \varphi_1} dx \\ &\geq e^{m_0} \int_{|z| < \lambda} |u|^2 \exp \left\{ \frac{1 - |x|}{1 + |x|} s_n \right\} dx \\ &\geq e^{m_0} \exp \left\{ \frac{1 - \lambda}{1 + \lambda} s_n \right\} \int_{|z| < \lambda} |u|^2 dx \end{aligned} \quad (2.45)$$

Substituting estimate (2.45) into estimate (2.41) results in

$$\int_{|z| < \lambda} |u|^2 dx \leq \|u\|_{C^1(\partial\Omega)}^2 \left(\|\partial_{\nu} \varphi_0\|_{L_1(\partial\Omega)} + 1 \right) \exp \left\{ -\frac{1 - \lambda}{1 + \lambda} s_n - m_0 \right\}.$$

This expression is independent of n , hence letting $n \rightarrow \infty$ we have

$$\lim_{n \rightarrow \infty} s_n = \lim_{n \rightarrow \infty} \alpha \sum_{j=1}^n m_j (1 - |z_j|) = \infty$$

by hypothesis. Thus for $0 < \lambda < 1$ we have

$$\int_{|z| < \lambda} |u|^2 dx \leq 0.$$

Letting $\lambda \rightarrow 1$ gives us that $u \equiv 0$. □

In each of the subsequent two theorems we incrementally relax our hypothesis, while still obtaining $u \equiv 0$.

Proof. of Theorem 2.2

Using estimate (2.29) together with our hypothesis inequality (2.6) gives

$$\begin{aligned} \int_{\Omega} (\Delta\varphi |u|^2 + (\Delta\varphi - 1) |\partial u|^2) e^\varphi dx &\leq \int_{\Omega} a (|u|^2 + |\partial u|^2) e^\varphi dx \\ &+ \int_{\partial\Omega} \{ \partial_\nu (|u|^2 + |\partial u|^2) + 8\Re(i\partial_{\nu_\perp} \partial_{\bar{z}} \cdot \partial_z u) \} e^\varphi d\sigma. \end{aligned}$$

Choosing φ_0 similar to the proof of Theorem 2.1

$$\Delta\varphi_0 = a + 1, \quad z \in \Omega$$

$$\varphi_0|_{\partial\Omega} = 0;$$

and defining φ_1 as in equation (2.33) with $m_j = 1$ for all $j \in \mathbb{N}$, we have for $\alpha \in (0, 2)$,

$$\Delta\varphi_1 = -2\pi\alpha \sum_{j=1}^n \delta(z - z_j)$$

$$\varphi_1|_{\partial\Omega} = 0.$$

Since $u(z_j) = 0$ for $j \in \mathbb{N}$ we still have

$$\varphi_1 = \alpha \sum_{j=1}^n \ln \left| \frac{1 - z\bar{z}_j}{z - z_j} \right|$$

$$|z - z_j|^{-\alpha} |u(z)|^2 = \mathcal{O}(|z - z_j|^{2-\alpha}) \rightarrow 0,$$

and thus

$$\begin{aligned} \int_{\Omega} \Delta\varphi_1 |u|^2 e^\varphi dx &= -2\pi\alpha \int_{\Omega} \sum_{j=1}^n \delta(x - z_j) |u|^2 e^\varphi dx \\ &= -2\pi\alpha \int_{\Omega} \sum_{j=1}^n \delta(x - z_j) |u|^2 \left| \frac{1 - z\bar{z}_j}{z - z_j} \right|^\alpha e^{\varphi_0} dx = 0 \end{aligned}$$

Hence one still obtains

$$\int_{\Omega} \Delta \varphi |u|^2 e^{\varphi} dx = \int_{\Omega} \Delta \varphi_0 |u|^2 e^{\varphi} dx = \int_{\Omega} (a+1) |u|^2 e^{\varphi} dx.$$

This inequality results in the following:

$$\begin{aligned} \int_{\Omega} (\delta \varphi |u|^2 + (\Delta \varphi - 1) |\partial u|^2) e^{\varphi} dx &= \int_{\Omega} (a |u|^2 + |u|^2 + a |\partial u|^2) e^{\varphi} dx \\ &\leq \int_{\Omega} a (|u|^2 |\partial u|^2) e^{\varphi} dx + \int_{\partial \Omega} \{ \partial_{\nu} \varphi (|u|^2 + |\partial u|^2) + 8 \Re(i \partial_{\nu_{\perp}} \partial_{\bar{z}} u \cdot \partial_z u) \} e^{\varphi} d\sigma. \end{aligned}$$

This gives us the estimate

$$\int_{\Omega} |u|^2 e^{\varphi} dx \leq \int_{\partial \Omega} \{ \partial_{\nu} \varphi (|u|^2 + |\partial u|^2) + 8 \Re(i \partial_{\nu_{\perp}} \partial_{\bar{z}} u \cdot \partial_z u) \} d\sigma.$$

Once again $\partial_{\nu} \varphi_1(z) < 0$ so

$$\begin{aligned} \int_{\Omega} |u|^2 e^{\varphi} dx &\leq \int_{\partial \Omega} \{ \partial_{\nu} \varphi (|u|^2 + |\partial u|^2) + 8 \Re(i \partial_{\nu_{\perp}} \partial_{\bar{z}} u \cdot \partial_z u) \} d\sigma \\ &\leq \int_{\partial \Omega} |\partial_{\nu} \varphi_0| (|u|^2 + |\partial u|^2) d\sigma + 4 \int_{\partial \Omega} (|\partial_{\nu_{\perp}} \partial_{\bar{z}} u|^2 + |\partial_z u|^2) d\sigma \\ &\leq \|u\|_{W_2^1(\partial \Omega)}^2 \|\partial_{\nu} \varphi_0\|_{L_1(\partial \Omega)} + 4 \int_{\partial \Omega} (|\partial \partial_{\bar{z}} u|^2 + |\partial_z u|^2) d\sigma \end{aligned}$$

Now for $v \in C^2(\bar{\Omega})$ we have

$$\begin{aligned} |\partial v|^2 &= |\partial_1 v|^2 + |\partial_2 v|^2 \\ &= |\partial_z v + \partial_{\bar{z}} v|^2 + |\partial_z v - \partial_{\bar{z}} v|^2 \\ &= 2 |\partial_z v|^2 + 2 |\partial_{\bar{z}} v|^2. \end{aligned} \tag{2.46}$$

Hence we continue

$$\begin{aligned}
\int_{\Omega} |u|^2 e^{\varphi} dx &\leq \|u\|_{W_2^1(\partial\Omega)}^2 \|\partial_{\nu}\varphi_0\|_{L_1(\partial\Omega)} + 4 \int_{\partial\Omega} (2|\partial_z\partial_{\bar{z}}u|^2 + 2|\partial_{\bar{z}}\partial_zu| + |\partial_zu|^2) d\sigma \\
&\leq \|u\|_{W_2^1(\partial\Omega)}^2 \|\partial_{\nu}\varphi_0\|_{L_1(\partial\Omega)} + \int_{\partial\Omega} \left(\frac{1}{2}|\Delta u|^2 + 8|\partial_{\bar{z}}^2u|^2 + 2|\partial_zu|^2 \right) d\sigma \\
&\leq \|u\|_{W_2^1(\partial\Omega)}^2 \|\partial_{\nu}\varphi_0\|_{L_1(\partial\Omega)} + \int_{\partial\Omega} \frac{|a|}{2}|u|^2 d\sigma + \int_{\partial\Omega} \left(\frac{|a|}{2} + 2 \right) |\partial_zu|^2 d\sigma \\
&\quad + 8 \int_{\partial\Omega} |\partial_{\bar{z}}^2u|^2 d\sigma \\
&\leq \|u\|_{W_2^1(\partial\Omega)}^2 \left(\|\partial_{\nu}\varphi_0\|_{L_1(\partial\Omega)} + \frac{1}{2}\|a\|_{L_2(\partial\Omega)} + 2 \right) + 8\|\partial_{\bar{z}}^2u\|_{L_2(\partial\Omega)}
\end{aligned}$$

If we define

$$\begin{aligned}
c &= \|u\|_{W_2^1(\partial\Omega)}^2 \left(\|\partial_{\nu}\varphi_0\|_{L_1(\partial\Omega)} + \frac{1}{2}\|a\|_{L_2(\partial\Omega)} + 2 \right) + 8\|\partial_{\bar{z}}^2u\|_{L_2(\partial\Omega)} \\
s_n &= \alpha \sum_{j=1}^n m_j (1 - |z_j|) \\
m_0 &= \min_{|z|\leq 1} \varphi_0
\end{aligned}$$

as before in Theorem 2.1, then for $0 < \lambda < 1$ one obtains

$$\exp \left\{ m_0 + s_n \frac{1-\lambda}{1+\lambda} \right\} \int_{|z|<\lambda} |u|^2 dx \leq \int_{\Omega} |u|^2 e^{\varphi} dx \leq c.$$

Therefore

$$\int_{|z|<\lambda} |u|^2 dx \leq c \exp \left\{ -m_0 - s_n \left(\frac{1-\lambda}{1+\lambda} \right) \right\}$$

and this estimate is independent of n . Letting $n \rightarrow \infty$ gives

$$\int_{|z|<\lambda} |u|^2 dx \leq 0$$

for all $0 < \lambda < 1$. Since this estimate is independent of λ , letting $\lambda \rightarrow 1$ gives $u \equiv 0$. \square

Proof. of Theorem 2.3

Using the fact that $\Delta u = 4\bar{\partial}\partial_1 u$, and applying the proof of Theorem 2.1 to ∂u , we obtain $\partial u \equiv 0$. Thus $u(z_0) = 0$ implies $u \equiv 0$. \square

The following two theorems determine a stability estimates based on conditions similar to Theorems 2.1 and 2.2.

Proof. of Theorem 2.4

Following the proof of Theorem 2.1 define for arbitrary $\beta \in (0, 1)$:

$$\begin{aligned}\varphi_0(z) &:= (4a + 1) \frac{|z|^2 + 1}{4}, \text{ and} \\ \varphi_1(z) &:= \beta \sum_{j=1}^n j \ln \left| \frac{1 - z\bar{z}_j}{z - z_j} \right|.\end{aligned}$$

Thus φ_0 and φ_1 have the following properties:

$$\Delta\varphi_0 = 4\partial_z\partial_{\bar{z}}(4a + 1) \frac{z\bar{z} - 1}{4} = 4a + 1 \quad (2.47)$$

$$\Delta\varphi_1 = -2\pi\beta \sum_{j=1}^n \delta(z - z_j) \quad (2.48)$$

$$\varphi_0|_{\partial\Omega} = \varphi_1|_{\partial\Omega} = 0. \quad (2.49)$$

Choose $\psi \in C_0^\infty(\mathbb{R}^2)$ such that $\psi \geq 0$, $\psi(x) \equiv 0$ whenever $|x| \geq 1$, ψ is radially symmetric and

$$\int_{\mathbb{R}^2} \psi(x) dx = 1.$$

Furthermore for $h > 0$ define

$$\begin{aligned}\psi_h &:= h^{-1}\psi\left(\frac{x}{h}\right), \\ \varphi_{1h} &:= \varphi_1 * \varphi_h, \\ \varphi_h &:= \varphi_0 + \varphi_{1h}, \text{ and} \\ \varphi &:= \varphi_0 + \varphi_1.\end{aligned}$$

Computing the Laplace operator of φ_{1h} we have

$$\begin{aligned}\Delta\varphi_{1h}(z) &= \Delta\varphi_1(z) * \psi_1(z) = -2\pi\beta \int_{\mathbb{R}^2} \sum_{j=1}^n \delta(z - z_j - y)\psi_h(y) dy \\ &= -2\pi\beta \sum_{j=1}^n \psi_h(z - z_j).\end{aligned}\tag{2.50}$$

Substituting φ_h into equation (2.21) and utilizing inequality (2.7) one obtains

$$\begin{aligned}\int_{\Omega} \Delta(\varphi_0 + \varphi_{1h}) |u|^2 e^\varphi dx &\leq 4 \int_{\Omega} a |u|^2 e^\varphi dx \\ &\quad + \int_{\partial\Omega} (\partial_\nu \varphi_h |u|^2 - 2\Re(i\partial_{\nu_\perp} \cdot u)) e^\varphi d\sigma \\ \int_{\Omega} (4a + 1) |u|^2 e^\varphi dx &+ \int_{\Omega} -2\pi\beta \sum_{j=1}^n \delta(z - z_j) |u|^2 e^\varphi dx \\ &\leq \int_{\Omega} 4a |u|^2 e^\varphi dx + \int_{\partial\Omega} (\partial_\nu \varphi_h |u|^2 - 2\Re(i\partial_{\nu_\perp} \cdot u)) e^\varphi d\sigma \\ \int_{\Omega} |u|^2 e^\varphi dx &\leq 2\pi\beta \sum_{j=1}^n \left\{ \int_{\Omega} \psi_h(z - z_j) |u|^2 e^\varphi dx \right\} + \int_{\partial\Omega} (\partial_\nu \varphi_h |u|^2 - 2\Re(i\partial_{\nu_\perp} \cdot u)) e^\varphi d\sigma\end{aligned}\tag{2.51}$$

Now $\Delta(-\varphi_{1h}) = 2\pi\beta \sum_{j=1}^n \psi_h(z - z_j) \geq 0$, which implies $-\varphi_{1h}$ is subharmonic and thus ψ_h is

upper semi-continuous as a function of h [16]. Moreover computing the convolution

$$\begin{aligned} \lim_{h \rightarrow 0^+} \varphi_{1h}(z) &= \lim_{h \rightarrow 0^+} \int_{\mathbb{R}^n} \varphi(z-x) \psi_h(x) dx = \lim_{h \rightarrow 0^+} \int_{\mathbb{R}^n} \varphi(z-uh) \psi(u) du \\ &= \int_{|u| \leq 1} \lim_{h \rightarrow 0^+} \varphi(z-uh) \psi(u) du = \varphi(z) \int_{\mathbb{R}^n} \psi(u) du = \varphi(z). \end{aligned}$$

Hence we may take $\varphi_{10} = \varphi_1$ and for $h > 0$ we have $\varphi_{1h} \leq \varphi_{10} = \varphi_1$. Furthermore, as in our earlier arguments, on $\partial\Omega$, $\partial_\nu \varphi \leq 0$ and so we have

$$\partial_\nu \varphi_{1h} = \partial_\nu \varphi * \psi_h \leq 0.$$

on $\partial\Omega$ as well. Hence using boundary conditions (2.49) we have

$$\begin{aligned} \int_{\partial\Omega} (\partial_\nu \varphi_h |u|^2 - 2\Re(i\partial_{\nu_\perp} \cdot u)) e^\varphi d\sigma &\leq \int_{\partial\Omega} \partial_\nu \varphi |u|^2 d\sigma + 4 \int_{\partial\Omega} (|\partial_{\nu_\perp} u|^2 + |\partial_z u|^2) d\sigma \\ &\leq \int_{\partial\Omega} |\partial_\nu \varphi_0| |u|^2 d\sigma + 4 \int_{\partial\Omega} (|\partial u|^2 + |\partial_z u|^2) d\sigma \\ &\leq \|u\|_{W_2^1(\partial\Omega)}^2 \left(\|\varphi_0\|_{L_1(\partial\Omega)} + 4 \right), \end{aligned} \quad (2.52)$$

and the constant $c = \|\varphi_0\|_{L_1(\partial\Omega)} + 4$ is dependent only on a . Recalling that $\psi_h(z - z_j) \equiv 0$ or $|z - z_j| \geq h$, define $\Omega_h := \Omega \cap \{z : |z - z_j| < h\}$. Thus we have

$$2\pi\beta \int_{\Omega} \psi_h(z - z_j) |u|^2 e^\varphi dx = 2\pi\beta \int_{\Omega_h} \psi_h(z - z_j) |u|^2 e^\varphi dx. \quad (2.53)$$

Since $\varphi_0 \leq 0$ and $\varphi_{1h} \leq \varphi_1$ in $\Omega \subset \Omega_h$ we have

$$\varphi \leq \varphi_1 = \beta \ln \left| \frac{1 - z\bar{z}_j}{z - z_j} \right| + \beta \sum_{k \neq j} \ln \left| \frac{1 - z\bar{z}_k}{z - z_k} \right|.$$

Along the line segment joining points z and z_j , the real and imaginary parts of φ_{1h} reduce to a real function of a single variable each. The mean value theorem applies to each separately

[10]. In any case we have

$$|\varphi_{1j}(z) - \varphi_{1j}(z_j)| \leq |z - z_j| \max_{t \in [0,1]} |\partial \varphi_{1j}(z_j + t(z - z_j))|.$$

Recalling equation (2.46) and if we know that $\partial_z v = \partial_{\bar{z}} w = 0$,

$$\begin{aligned} 2 \left| \partial_z \ln \left| \frac{v}{w} \right| \right|^2 &= \frac{1}{2} \left| \partial_z \ln \left| \frac{v\bar{v}}{w\bar{w}} \right| \right|^2 \\ &= \frac{1}{2} \left| \frac{w\bar{w}}{v\bar{v}} \partial_z \ln \left(\frac{v\bar{v}}{w\bar{w}} \right) \right|^2 \\ &= \frac{1}{2} \left| \frac{w\bar{w}}{v\bar{v}} \cdot \frac{(\partial_z v)\bar{v}w\bar{w} - v\bar{v}(\partial_z w)\bar{w}}{(w\bar{w})^2} \right|^2 \\ &= \frac{1}{2} \left| \frac{(\partial_z v)\bar{w} - v(\partial_z w)}{vw} \right|^2. \end{aligned}$$

Applying this identity to $\ln \left| \frac{1 - z\bar{z}_k}{z - z_k} \right|$ gives

$$\begin{aligned} 2 \left| \partial_z \ln \left| \frac{1 - z\bar{z}_k}{z - z_k} \right| \right|^2 &= \frac{1}{2} \left| \frac{-\bar{z}_k(z - z_k) - (1 - z\bar{z}_k) \cdot 1}{(1 - z\bar{z}_k)(z - z_k)} \right|^2 \\ &= \frac{1}{2} \left| \frac{|z|^2 - 1}{(1 - z\bar{z}_k)(z - z_k)} \right|^2 \\ &= \frac{1}{2} \cdot \frac{(|z|^2 - 1)^2}{|1 - z\bar{z}_k|^2 |z - z_k|^2}. \end{aligned}$$

Similarly we have the same for $2 \left| \partial_{\bar{z}} \ln \left| \frac{1 - z\bar{z}_k}{z - z_k} \right| \right|^2$, hence

$$\left| \partial \ln \left| \frac{1 - z\bar{z}_k}{z - z_k} \right| \right|^2 = \frac{(|z|^2 - 1)^2}{|1 - z\bar{z}_k|^2 |z - z_k|^2} = \frac{4}{|z - z_k|^2} \cdot \frac{(|z|^2 - 1)^2}{4|1 - z\bar{z}_k|^2}.$$

Since we have $\frac{(|z|^2 - 1)^2}{4|1 - z\bar{z}_k|^2} \leq 1$, then $\left| \partial \ln \left| \frac{1 - z\bar{z}_k}{z - z_k} \right| \right| \leq \frac{2}{|z - z_k|}$. Now given that $z, z_k \in$

Ω_h , we have that

$$\begin{aligned}
|z_j - z_k| &= |z_j - z_k + t(z - z_j) + t(z - z_j)| \\
&= |z_j + t(z - z_j) - z_k| + |t(z - z_j)| \\
&\leq |z_j + t(z - z_j) - z_k| + h.
\end{aligned}$$

Therefore $|z_j + t(z - z_j) - z_k| \geq |z_j - z_k| - h \geq r_n - h$ in Ω_h , and thus

$$|\partial\varphi_{1j}(z_j + (z - z_j))| \leq \frac{2(n-1)\beta}{r_n - h}, \quad (2.54)$$

for $h < r_n$. Using the definition of μ_n we have

$$e^{\varphi_{1j}(z_j)} \leq \mu_n^{-\beta}.$$

Combining all of these estimates in Ω_h we obtain

$$e^\varphi \leq \left(\frac{2}{\mu_n}\right)^\beta \exp\left\{\frac{2(n-1)\beta h}{r_n - h}\right\} |z - z_j|^{-\beta}. \quad (2.55)$$

Defining $d := \left(\frac{2}{\mu_n}\right)^\beta \exp\left\{\frac{2(n-1)\beta h}{r_n - h}\right\}$ and recalling our hypothesis, one obtains

$$\begin{aligned}
2\pi\beta \int_{\Omega} \psi_h(z - z_j) |u|^2 e^\varphi dx &= 2\pi\beta \left[|u(z_j)|^2 \int_{\Omega_h} \psi_h e^\varphi dx + \int_{\Omega_h} \psi_h e^\varphi (|u(z)|^2 - |u(z_j)|^2) dx \right] \\
&\leq 2\pi\beta d \left[\varepsilon^2 \int_{|z-z_j|<h} \psi_h(z - z_j) |z - z_j|^{-\beta} dx \right. \\
&\quad \left. + \int_{|z-z_j|<h} \psi_h(z - z_j) |z - z_j|^{-\beta+1} \left(\frac{|u(z)|^2 - |u(z_j)|^2}{|z - z_j|} \right) dx \right] \\
&\leq \frac{2\pi\beta d}{2 - \beta} \left[\varepsilon^2 h^{-\beta} + \|u\|_{C^1(\bar{\Omega})}^2 h^{-\beta+1} \right];
\end{aligned}$$

and this follows from the fact that $\int_{x<h} \psi_h(x) |x|^{-\beta} dx = \frac{h^{-\beta}}{2 - \beta}$. If we let $h = \frac{r_n \varepsilon^2}{n}$, then for

ε sufficiently small we have $h < r_n$ and for $n > 1$,

$$\begin{aligned}
d &= \left(\frac{2}{\mu_n}\right)^\beta \exp\left\{\frac{2(n-1)\beta\frac{r_n\varepsilon^2}{n}}{r_n - \frac{r_n\varepsilon^2}{n}}\right\} \\
&= \left(\frac{2}{\mu_n}\right)^\beta \exp\left\{\frac{2(n-1)\beta\frac{\varepsilon^2}{n}}{1 - \frac{\varepsilon^2}{n}}\right\} \\
&= \left(\frac{2}{\mu_n}\right)^\beta \left(\exp\left\{\frac{(n-1)\varepsilon^2}{n - \varepsilon^2}\right\}\right)^{2\beta} \\
&\leq \left(\frac{2}{\mu_n}\right)^\beta e^{2\beta} = (2e^2)^\beta \mu_n^{-\beta}.
\end{aligned}$$

Hence using equation (2.52) we have

$$\begin{aligned}
\int_{\Omega} |u|^2 e^\varphi dx &\leq \|u\|_{W_2^1(\partial\Omega)}^2 \left(\|\varphi_0\|_{L_1(\partial\Omega)} + 4\right) + \frac{2\pi\beta(2e^2)^\beta}{2-\beta} \mu_n^{-\beta} \left(\frac{r_n\varepsilon^2}{n}\right)^{-\beta} \left[\varepsilon^2 + \|u\|_{C^1(\bar{\Omega})}^2 \left(\frac{r_n\varepsilon^2}{n}\right)\right] \\
&= \|u\|_{W_2^1(\partial\Omega)}^2 \left(\|\varphi_0\|_{L_1(\partial\Omega)} + 4\right) + \frac{2\pi\beta(2e^2)^\beta}{2-\beta} \left(\frac{n}{\mu_n r_n}\right)^\beta \left[1 + \frac{\|u\|_{C^1(\bar{\Omega})}^2 r_n}{n}\right] \varepsilon^{2(1-\beta)}.
\end{aligned}$$

Recalling again that $\varphi_{1h} \leq \varphi_1$, and choosing $0 < \lambda < 1 - h \leq 1 - h_0$, where $h_0 = \frac{r_n v e_0^2}{n}$, we have that for $|x| \leq \lambda$,

$$\varphi(x) \geq \varphi_1(x) \geq \beta s_n \frac{1 - \lambda - \lambda_0}{1 + \lambda + \lambda_0} = \beta_1 s_n.$$

Therefore we have

$$\int_{\Omega} |u|^2 e^\varphi dx \geq e^{\beta_1 s_n} \|u\|_{L_2(|z| < \lambda)}^2.$$

Combining the previous two estimates we have

$$\|u\|_{L_2(|z| < \lambda)}^2 \leq$$

$$e^{-\beta_1 s_n} \|u\|_{W_2^1(\partial\Omega)}^2 \left(\|\varphi_0\|_{L_1(\partial\Omega)} + 4\right) + \frac{2\pi\beta(2e^2)^\beta}{2-\beta} \left(\frac{n}{\mu_n r_n}\right)^\beta \left[1 + \frac{\|u\|_{C^1(\bar{\Omega})}^2 r_n}{n}\right] \varepsilon^{2(1-\beta)}. \quad (2.56)$$

□

Proof. of Theorem 2.5

Finally, under the conditions of Theorem 2.5 we choose β_1 , β , λ , and s_n as in Theorem 2.4. Then using steps similar to those used to prove Theorem 2.4, one can prove

$$\|u\|_{W_2^1(|z|<\lambda)}^2 \leq e^{-\beta_1 s_n} \left\{ c \|u\|_{W_2^2(\partial\Omega)}^2 + M_n \varepsilon^{2(1-\beta)} \right\}$$

where M_n is determined as in equation (2.56) using $\|u\|_{C^2(\bar{\Omega})}$ instead of $\|u\|_{C^1(\bar{\Omega})}$. This completes the proof of Theorem 2.5.

□

3 Stability Estimates for the Inverse Problem of Finding the Boundary Condition

3.1 Stability Estimates

Example 3.1. *Analytic continuation for the inverse gravimetry problem (compare with [18] and [20])*

Let $\bar{\partial}$ be the Cauchy-Riemann operator and $\mathbb{D} = \{z \in \mathbb{C} : |z| < 1\}$ be the unit disk. For the inverse gravimetry problem for cylindrical bodies we are asked to find a function h with support in \mathbb{D} from the following conditions (see figure 3.1).

$$\bar{\partial}u = h, \quad z \in \mathbb{C} \tag{3.1}$$

$$u|_{\gamma} = f \tag{3.2}$$

$$\lim_{|z| \rightarrow \infty} u(z) = 0 \tag{3.3}$$

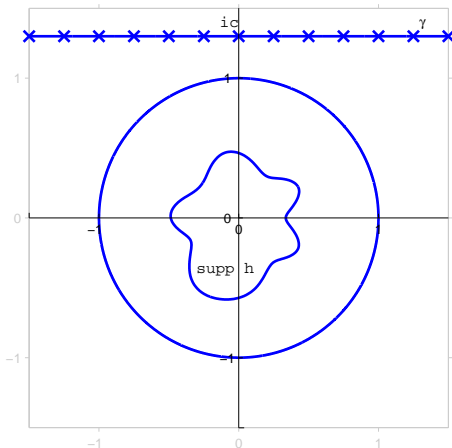


Figure 3.1

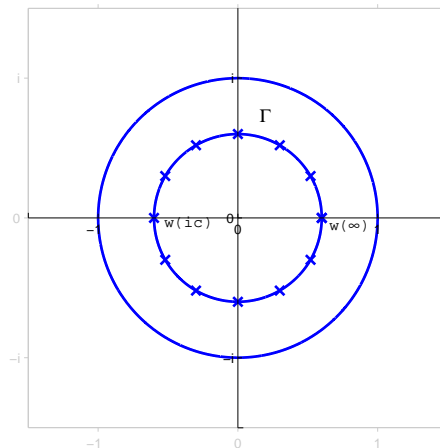


Figure 3.2

Here $c > 1$ is given, $\gamma = \{z \in \mathbb{C} : \Im(z) = c\}$ and f is a given trace function of u on γ . The trace \bar{f} has a very simple physical interpretation: up to a constant factor, it is the gravitational force of attraction due to mass distribution h . In the case where h is the characteristic function of a star shaped region with respect to its center of gravity, P.S. Novikov [24] obtained uniqueness for this inverse problem. To determine this function numerically it is a natural first step to find an analytic continuation of the function f onto the boundary of \mathbb{D} . Using the following conformal map (Möbius transformation)

$$w = \frac{iz + \lambda}{i\lambda z + 1}, \quad \lambda = c + \sqrt{c^2 - 1}$$

we reduce this problem to the following canonical form.

Problem 3.1. Analytic continuation to the boundary

Function u is analytic in some neighborhood of \mathbb{D} with trace on the circle $\Gamma = w(\gamma)$ centered at the origin with radius $r = c - \sqrt{c^2 - 1}$ (see Figure 3.2); and $u|_{\Gamma} = f(w(z))$. Find u on $\partial\mathbb{D}$, the boundary of \mathbb{D} . In practice we know function u only at a finite number of points $z_j \in \gamma$ with accuracy ε :

$$|u(z_j) - a_j| \leq \varepsilon, \quad j = 1, 2, \dots, n - 1. \quad (3.4)$$

Notice the value of u at $z_n = r = w(\infty)$ exactly known from condition (3.3):

$$u(z_n) = 0 \quad (3.5)$$

Suppose $u_n(z)$ is an analytic function in some neighborhood of \mathbb{D} such that $u_n(z_j) = a_j$, $j = 1, \dots, n$ and $a_n = 0$. We know that $|u(z_j) - u_n(z_j)| \leq \varepsilon$ for $j = 1, \dots, n$. The problem is to estimate $\|u(z) - u_n(z)\|$ in some functional space on $\partial\mathbb{D}$, say in $C(\partial\mathbb{D})$ or $H^s(\partial\mathbb{D})$. Here H^s is the Sobolev space of order $s \in \mathbb{R}$. Since $\bar{\partial}$ is a linear operator our problem reduces to the following.

Problem 3.2. Analytic continuation from discrete data

Let

$$\bar{\partial}u = 0,$$

on the set $\mathbb{D}_\rho = \{z \in \mathbb{C} : |z| < \rho\}$, $\rho \geq 1$ and

$$|u(z_j)| \leq \varepsilon,$$

for $|z_j| < 1$, $j = 1, \dots, n$. Find an upper bound for the norm of the solution on the domain \mathbb{D}_ρ or on the boundary $\partial\mathbb{D}_\rho$.

Example 3.2. Free oscillation of a drum

Suppose that we have the Helmholtz equation with an unknown boundary condition of Robin type on the unit disk:

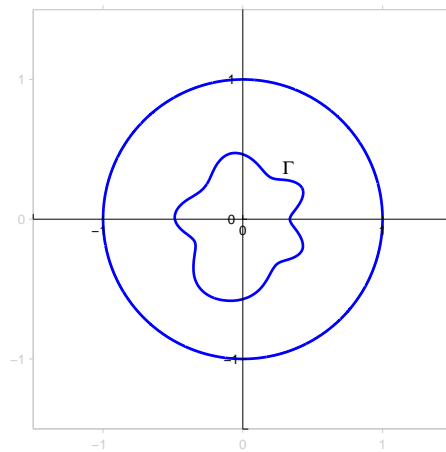


Figure 3.3

$$\Delta u + \lambda u = 0, \quad z \in \mathbb{D} \tag{3.6}$$

$$\frac{\partial u}{\partial \nu} = a(x)u, \quad z \in \partial\mathbb{D} \tag{3.7}$$

We know the Cauchy data of u on some smooth curve Γ :

$$u|_{\Gamma} = f, \quad \frac{\partial u}{\partial \nu}|_{\Gamma} = g \quad (3.8)$$

or at a finite number of points:

$$u(z_j) = f_j, \quad \nabla u(z_j) = g_j, \quad j = 1, \dots, n \quad (3.9)$$

We seek the coefficient $a(x)$ from boundary condition (3.7) in the case (3.8) or an estimate for $a(x)$ in case (3.9). Notice that this is a nonlinear problem. First we solve the linear problem of finding (or estimating) the Cauchy data u and $\frac{\partial u}{\partial \nu}$ on $\partial \mathbb{D}$. Then we determine a from (3.7),

$$a = \frac{\partial u / \partial \nu}{u}, \quad (3.10)$$

on the set $\partial \mathbb{D} \setminus E_0$, where $E_0 = \{z \in \partial \mathbb{D} : u(z) = 0, \frac{\partial u}{\partial \nu} = 0\}$. In order to find a on the whole boundary $\partial \mathbb{D}$ it is needed to show that the Lebesgue measure of E_0 equal zero, $|E_0| = 0$. For stability estimates it is in fact necessary to show a bit more. One must show that for $0 < \delta < 1$ we have

$$|E_{\delta}| \rightarrow 0 \text{ as } \delta \rightarrow 0 \quad (3.11)$$

where

$$E_{\delta} = \left\{ z \in \partial \mathbb{D} : |u(z)| < \delta, \left| \frac{\partial u}{\partial \nu} \right| < \delta \right\} \quad (3.12)$$

Since $E_0 = \bigcap_{\delta > 0} E_{\delta}$, then E_0 is measurable and from (3.12) we have $|E_0| = 0$. In other words it is needed to show that Cauchy data for a nontrivial solution of an elliptic equation can be small only on a small set E_{δ} . This a nontrivial problem, and it is not always the case that $|E_{\delta}| \rightarrow 0$ (see [27]).

Example 3.3. The Steklov eigenvalue problem

$$\Delta u = 0 \quad \text{in } \mathbb{D} \quad (3.13)$$

$$\frac{\partial u}{\partial \nu} = \lambda \rho(x) u \quad \text{on } \partial \mathbb{D} \quad (3.14)$$

There are several physical interpretations of this problem. In particular, if a membrane has negligible mass on its interior the problem describes the vibration of an ideal free membrane with its whole mass M distributed on the boundary $\partial \mathbb{D}$ with density ρ ; thus

$$M = \int_{\partial \mathbb{D}} \rho(s) ds \quad (3.15)$$

The spectrum of the Steklov problem is discrete, and the eigenvalues $\lambda_n \rightarrow \infty$ as $n \rightarrow \infty$,

$$0 = \lambda_0 < \lambda_1 \leq \lambda_2 \leq \lambda_3 \leq \dots \leq \lambda_n \rightarrow \infty.$$

If we know λ or the total mass M and the either the Cauchy data (3.8) or discrete data (3.9) as in example 2, then our inverse problem is to find the density $\rho(x)$ similar to example 2.

Example 3.4. Thermal conductivity of a solid body

Let $\Omega \subset \mathbb{R}^n$ be an open connected set with smooth boundary $\partial \Omega = \gamma \cup \Gamma$ that has two components (see Figure 3.4) We consider Ω as a solid body with the thermal conductivity $K(x) > 0$ and steady-state temperature $u(x)$. Then from Fourier's law of heat conduction and conservation of energy law we have the differential equation

$$\nabla \cdot (K(x) \nabla u(x)) = 0 \quad \text{in } \Omega. \quad (3.16)$$

We assume that on γ , which we consider as an accessible part of the boundary, the temperature is a known function f , thus

$$u|_{\gamma} = f. \quad (3.17)$$

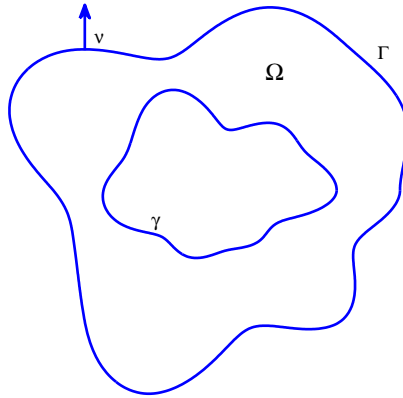


Figure 3.4

We consider Γ an inaccessible part of the boundary of Ω upon which we have the boundary condition

$$K \frac{\partial u}{\partial \nu} + a(x)(u - u_0) = 0, \quad x \in \Gamma. \quad (3.18)$$

This is Newton's law of cooling which states that the heat energy flowing out ($a > 0$) is proportional to the difference between the temperature at the surface $u(x)$ and the temperature outside $u_0(x)$. Since Γ is inaccessible, generally speaking we don't know the function $a(x)$, or the outside temperature $u_0(x)$, and would like to determine them measuring the heat flux on γ ,

$$K \frac{\partial u}{\partial \nu} = g. \quad (3.19)$$

More precisely we have the following inverse problems.

Problem 3.3. Thermal conductivity $K(x) > 0$ **is given and** $u_0 \equiv 0$

Given the Cauchy data f and g on γ , determine the coefficient $a(x)$ on Γ . Similar to example 3.2 we first find the Cauchy data on Γ and then recover $a(x)$ from boundary condition (3.18),

$$a(x) = -\frac{K \frac{\partial u}{\partial \nu}}{u}, \quad x \in \Gamma \setminus E_0 \quad (3.20)$$

$$E_0 = \left\{ x \in \Gamma : u(x) = 0, \frac{\partial u}{\partial \nu}(x) = 0 \right\}. \quad (3.21)$$

Problem 3.4. *Functions $K(x) > 0$ and a are given and $a \neq 0$ on Γ*

We know the Cauchy data f and g on γ and determine the 'outside' temperature u_0 on Γ .

This problem has the obvious solution

$$u_0 = \frac{K \frac{\partial u}{\partial \nu} + au}{a}. \quad (3.22)$$

Problem 3.5. *We have two measurements of Cauchy data on γ*

$$u_j|_{\gamma} = f_j, \quad \frac{\partial u}{\partial \nu}|_{\gamma} = g_j, \quad j = 1, 2 \quad (3.23)$$

We would like to find $a(x)$ and $u_0(x)$ assuming again that $a \neq 0$ on Γ . We recover a and u_0 by solving the linear system

$$\begin{cases} au_1 - b = K \frac{\partial u_1}{\partial \nu} \\ au_2 - b = K \frac{\partial u_2}{\partial \nu} \end{cases} \quad (3.24)$$

where $b = au_0$. This system has only one solution (a, b) for all $x \in \Gamma \setminus E_0$ where now

$$E_0 = \left\{ x \in \Gamma : u_1 - u_2 = 0, \frac{\partial u_1}{\partial \nu} - \frac{\partial u_2}{\partial \nu} = 0 \right\} \quad (3.25)$$

Let

$$E_{\delta} = \left\{ x \in \Gamma : |u_1 - u_2| < \delta, \left| \frac{\partial u_1}{\partial \nu} - \frac{\partial u_2}{\partial \nu} \right| < \delta \right\} \quad (3.26)$$

In the case where $f_1 - f_2$ is not identically zero we will show that $|E_\delta| \rightarrow 0$ as $\delta \rightarrow 0$ in the two-dimensional case. For $n \geq 3$ this is a difficult open problem.

3.2 Methods and Tools for These Inverse Problems

In order to develop methods and tools for these inverse problems it is convenient to study the Cauchy problem

$$Pu = h, \quad \text{in } \Omega \quad (3.27)$$

$$u|_E = f, \quad E \subset \partial\Omega \quad (3.28)$$

for elliptic operator

$$P = \begin{bmatrix} 2\bar{\partial} & 0 \\ 0 & 2\partial \end{bmatrix} + A(x) \quad (3.29)$$

with 2×2 matrix potential $A = [a_{ij}]$ and vector solution

$$u = \begin{bmatrix} u_1 \\ u_2 \end{bmatrix}. \quad (3.30)$$

Here $\Omega \subset \mathbb{R}^2$ is a bounded domain with piece-wise smooth boundary. E is a closed subset of $\partial\Omega$ with positive Lebesgue measure, $|E| > 0$.

Our first goal will be to find an explicit formula for $u(z)$, $z \in \Omega$ based on Cauchy data f on E . We assume that $u \in C^1(\bar{\Omega})$ and $A \in C^1(\bar{\Omega})$. Using the substitution $u = e^{\Phi(z)}v$ it is possible to reduce matrix A to the form

$$A = \begin{bmatrix} 0 & b \\ a & 0 \end{bmatrix} \quad (3.31)$$

We assume for simplicity that A has the form (3.31).

Example 3.5. If $\Delta u + a(x)u = f$, then since $\Delta = 2\partial\bar{\partial}$ we choose $u_1 = u$, $u_2 = 2\bar{\partial}u$,
 $\mathbf{u} = \begin{bmatrix} u_1 \\ u_2 \end{bmatrix}$, $\mathbf{f} = \begin{bmatrix} 0 \\ f \end{bmatrix}$, and $b = -1$. From this we obtain $P\mathbf{u} = \mathbf{f}$.

Theorem 3.1. Let $u \in C^1(\bar{\Omega})$ be a solution of the Cauchy problem (3.27), (3.28), and let us assume that for each $z \in \Omega$ we can find a bounded function $\phi(\xi)$ that depends on z as a parameter such that

$$\phi(z) = 0, \quad (3.32)$$

$$\Re \phi(\xi) < 0 \quad \xi \in \partial\Omega \setminus E, \quad (3.33)$$

$$\bar{\partial}e^{\tau\phi(\xi)} = 0 \quad \xi \in \Omega.$$

Then

$$u_j(z) = \lim_{\tau \rightarrow \infty} \left\{ \int_E \langle e^{\tau\Phi} f, \mathcal{V} v_j \rangle ds - \int_{\Omega} \langle e^{\tau\Phi} h, v_j \rangle ds \right\} \quad (3.34)$$

where

$$\Phi(\xi) = \begin{bmatrix} \phi(\xi) & 0 \\ 0 & \bar{\phi}(\xi) \end{bmatrix}, \quad \mathcal{V} = \begin{bmatrix} \nu & 0 \\ 0 & \bar{\nu} \end{bmatrix}.$$

Here ν is the unit outward normal on the boundary of Ω , e_1 and e_2 are the canonical basis vectors of \mathbb{R}^2 , and v_j , $j = 1, 2$ are solutions to the equations

$$P_{\tau}^t v_j := (-D + A_{\tau}^t) v_j = -\delta(\xi - z) e_j$$

$$A_{\tau}^t = \begin{bmatrix} 0 & a_{\tau} \\ b_{-\tau} & 0 \end{bmatrix}, \quad a_{\pm\tau} = a e^{\pm i 2\tau \Im \phi}.$$

The relevant question is “How does one find a function ϕ with properties (3.32),(3.33).” The answer of course depends on Ω , E and $z \in \Omega$. There are several classic examples of domains with the associated functions ϕ .

Example 3.6. Goluzin and Krylov [14]

If Ω is a simply connected domain we can take $\phi(\xi) = \psi(\xi) - \psi(z)$, where $\Re\psi$ is an harmonic function in Ω such that

$$\Re\psi|_{\partial\Omega} = \begin{cases} 1, & \xi \in E \\ 0, & \xi \in \partial\Omega \setminus E \end{cases}$$

and $\Im\psi$ is an harmonic conjugate function. In other words $\Re\psi$ is the harmonic measure of set E . (About harmonic measure and its properties see [13])

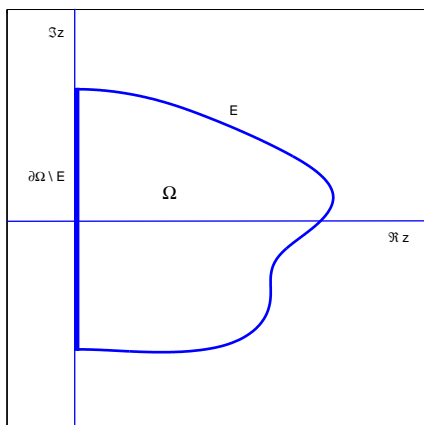


Figure 3.5

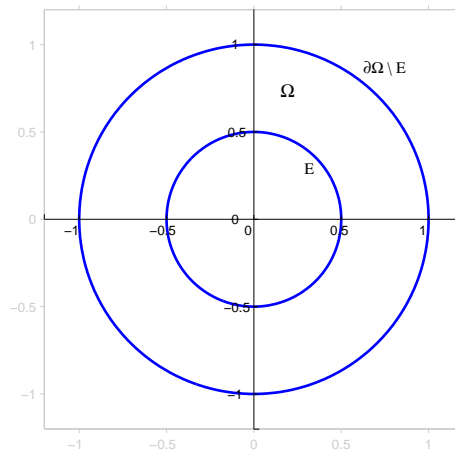


Figure 3.6

Example 3.7. Fok-Kuni [11]

For domain Ω and set E from figure 3.5 we can choose $\phi(\xi) = \xi - z$.

Example 3.8. Annulus Domain

We can choose $\phi(\xi) = \ln \frac{z}{\xi}$, $\tau = n \in \mathbb{N}$ for the annulus in figure 3.6. Notice in the statement and proof of Theorem 3.1, it is sufficient to assume the weaker condition that ϕ satisfy equations (3.32) and (3.33) such that $e^{\tau\phi}$ is analytic and bounded in Ω , to obtain (3.34). In this case $e^{\tau\phi} = \exp \left\{ n \ln \frac{z}{\xi} \right\} = \frac{z^n}{\xi^n}$, which is analytic for $.5 < \xi < 1$.

Example 3.9. Carleman [9]

For domain Ω from figure 3.7 and z lying on the bisectrix we may take $\phi(\xi) = \left(\frac{\xi - \xi_0}{z - \xi_0} \right)^{1/\alpha}$.

Note that $z \neq \xi_0$, the vertex of the interior angle.

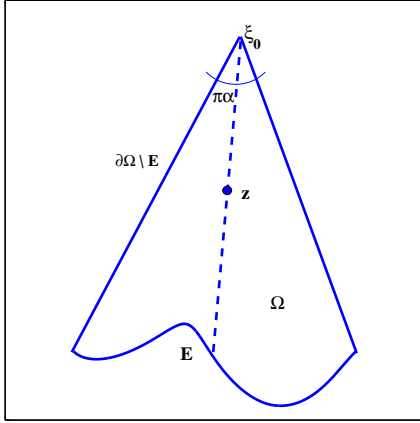


Figure 3.7

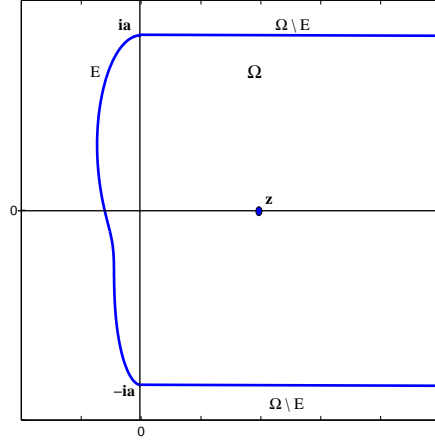


Figure 3.8

Example 3.10. Half-strip

For domain Ω in figure 3.8 and $z \in \mathbb{R} \cap \Omega$ we can choose $\phi(\xi) = \exp \left\{ \frac{\pi(z - \xi)}{2a} \right\}$. This is just the limiting case of example 3.9

Theorem 3.2. Let $Pu = 0$ in \mathbb{D} , u not be identically zero and $E_\delta = \{z \in \partial\mathbb{D} : |u| < \delta\}$, $\delta \in (0, 1)$. Then there exists a constant C such that

$$|E_\delta| \leq \frac{C}{\ln(1/\delta)} .$$

Theorem 3.3. Let $Pu = 0$ in \mathbb{D} and suppose that for $j = 1, \dots, n$ we have $z_j \in \mathbb{D}$ and $|u(z_j)| \leq \varepsilon$. Then there exists an ε_0 , $0 < \varepsilon \leq \varepsilon_0 < 1$, a constant C independent of ε and $n \in \mathbb{N}$, and $\beta \in (0, 1)$ arbitrary such that, for $0 < r < 1$ and \mathbb{D}_r the disk of radius r centered at the origin, one obtains

$$\|u\|_{L_2(\mathbb{D}_r)}^2 \leq e^{-\beta_1 S_n} \left\{ C \|u\|_{H^1(\partial\mathbb{D})}^2 + M_n \varepsilon^{2(1-\beta)} \right\}$$

where

$$\beta_1 = \beta \frac{1 - r - h_0}{1 + r + h_0} > 0, \quad h_0 = r_n \frac{\varepsilon_0^2}{n}$$

$$M_n = Cn \left(\frac{n}{r_n \mu_n} \right)^\beta \left[1 + \frac{r_n}{n} \|u\|_{C^1(\mathbb{D})}^2 \right],$$

Values S_n , μ_n and r_n are given by

$$S_n = \sum_{j=1}^n (1 - |z_j|), \quad \mu_n = \min_k \prod_{j \neq k} \left| \frac{z_k - z_j}{1 - z_k \bar{z}_j} \right|, \quad r_n = \min_{j \neq k} |z_j - z_k|.$$

Corollary 3.1. For $|z| \leq r < 1$, there exists a constant $C = C(r) > 0$, $\beta \in (0, 1)$ and $M_n = M(n, \mu_n, r_n, \beta)$ such that

$$|u(z)| \leq e^{-C\beta S_n} (\|u\|_{H^1(\partial\mathbb{D})} + M_n \varepsilon^{1-\beta})$$

Theorem 3.1 with the function ϕ from example 3.6 was proved in Arbuzov and Bukhgeim [4]. The proof for our general case is similar. Theorem 3.2 follows from Theorem 3.1. We will show details of the proof for a simple operator $P = \bar{\partial}$ in the next section. The proofs for Theorem 3.3 and corollary 3.1 follow from Bukhgeim [5]. For a general reference for Carleman formulas we refer to Aizenberg [2] for operator $\bar{\partial}$. Results from this section are sufficient to obtain stability estimates for all inverse problems we introduced in section 1. In the subsequent section we will show how to do this for example 1.2 with exact data (3.9) for simplicity.

3.3 Stability Estimate for Example 3.2 with Discrete Data

Using the technique based on *a priori* weighted estimates of Carleman type, Bukhgeim [5] determined several uniqueness and stability results for elliptic equations on simply connected domains when the solution and its gradient are known on a discrete subset of the domain. Similarly Bukhgeim and Kardakov [6] found a stability estimate for the boundary coefficient, $a(x)$, of “two-logarithmic” type when the trace of the solution is known on the interior of

the domain. That is, when the solution is known on the boundary, $\partial\Omega'$, of class C^∞ of a simply connected sub-domain, $\Omega' \subset\subset \Omega$.

We combine these results with Carleman's formulas to find a logarithmic stability estimate for the boundary coefficient, $a(x)$, when the value of the solution and its gradient are known for a finite number of interior points bounded uniformly away from the boundary of the domain; i.e. there is some $\alpha \in (0, 1)$ and for each interior point, z_j for which the value of the solution and the gradient of the solution is known, $|z_j| < \alpha$.

Theorem 3.4. *Consider unit circle \mathbb{D} with closure $\bar{\Omega}$, and let u be a normalized eigenfunction of the Laplace operator in the domain Ω generated by the third kind boundary condition:*

$$\begin{aligned} \Delta u &= \lambda u & x \in \Omega \\ \partial_\nu u + a(x)u &= 0, & x \in \partial\Omega \\ \|u\|_{L_2(\Omega)} &= 1. \end{aligned} \tag{3.35}$$

Suppose also that we know the values of u and ∂u at a finite number of interior points with $\alpha \in (0, 1)$ such that for $1 \leq j \leq n$ we have

$$u(z_j) = v_j, \quad \partial u(z_j) = w_j, \quad \text{and} \quad |z_j| < \alpha \tag{3.36}$$

Suppose also that $u_1, u_2 \in C^2(\bar{\Omega})$ and $a_1, a_2 \in C(\partial\Omega)$ are two solutions to (3.35) and agreeing on (3.36) with constant $m > 0$ such that

$$\|a_j\|_{C(\partial\Omega)} \leq m, \quad \|u_j\|_{C^2(\bar{\Omega})} \leq m \quad j = 1, 2. \tag{3.37}$$

Then there exist constants c depending only on Ω , α , m , and $p \in (0, \infty)$ such that

$$\|a_1 - a_2\| \leq c (\ln(n))^{-\frac{1}{p}}. \tag{3.38}$$

3.4 Preliminaries

We need the following basic Carleman Estimate (see Lemma 2.4):

Let φ be a real function of the class $C^2(\bar{\Omega})$. Then for any function u of the class $C^1(\bar{\Omega}; \mathbb{C})$ with values in \mathbb{C}

$$\int_{\Omega} (\Delta\varphi|u|^2 + (\Delta\varphi - 1)|\partial u|^2) e^{\varphi} dx \leq \int_{\Omega} |\Delta u|^2 e^{\varphi} dx + \int_{\partial\Omega} \{ \partial_{\nu}\varphi (|u|^2 + |\partial u|^2) + 8\text{Re}(i\partial_{\nu_{\perp}}\partial_{\bar{z}}u \cdot \partial_z u) \} e^{\varphi} ds,$$

where ν is the unit outward normal to $\partial\mathbb{D}$, $\nu_{\perp} = (-\nu_2, \nu_1)$, and $dx = dx_1 dx_2$ for $x = x_1 + ix_2$.

Now given equations (3.35) - (3.37) define

$$u := u_1 - u_2 \quad \text{and} \quad a := a_1 - a_2. \quad (3.39)$$

This gives

$$\left. \begin{aligned} \Delta u &= \lambda u & x \in \Omega \\ \partial_{\nu} u + a_1 u_1 - a_2 u_2 &= 0 & x \in \partial\Omega \\ u(z_j) &= 0 \\ \partial u(z_j) &= 0. & 1 \leq j \leq n \end{aligned} \right\} \quad (3.40)$$

Adding and subtracting $a_2 u_1$ gives

$$\partial_{\nu} u + a_1 u_1 - a_2 u_2 - a_2 u_1 + a_2 u_1 = 0$$

$$\partial_{\nu} u + (a_1 - a_2) u_1 + a_2 (u_1 - u_2) = 0$$

$$\partial_{\nu} u + a u_1 + a_2 u = 0$$

$$a = -\frac{1}{u_1} [\partial_{\nu} u + a_2 u.] \quad (3.41)$$

Let $E_\delta = \{z \in \partial\Omega \mid |u_1(z)| < \delta\}$. Using Carleman's formula we can provide a new proof for a lemma from [6] which states that $|E_\delta| \leq \frac{c}{\ln \delta^{-1}}$. Let us consider our function u defined on $\bar{\Omega}$ such that $|u(z)| \leq m$ and $u(0) = 1$. For any $\delta \in (0, 1)$ define

$$E_\delta := \{x \in \partial\Omega : |u(z)| < \delta\}. \quad (3.42)$$

For simplicity we will prove the following lemma for a u analytic to demonstrate the technique used to prove Theorem 3.2.

Lemma 3.1. *If u is analytic in Ω such that u is bounded by m on $\bar{\Omega}$ and $|u(0)| = 1$, then for any $\delta > 0$ the Lebesgue measure of E_δ is bounded as follows:*

$$|E_\delta| \leq \frac{2\pi \ln(m)}{\ln\left(\frac{m}{\delta}\right)} \leq \frac{C}{\ln(1/\delta)}$$

Proof. Carleman's formula for u in Ω is given by

$$e^{\tau\phi(z)}u(z) = \frac{1}{2\pi} \int_{\partial\Omega} \frac{e^{\tau\phi(\zeta)}u(\zeta)}{\zeta - z} \nu ds,$$

where ϕ is analytic on Ω and $\tau > 0$. Noting the value of u at the origin and dividing both sides of our equations by $e^{\tau\phi(0)}$ we have

$$\begin{aligned} 1 &= |u(0)| \\ &= \left| \frac{1}{2\pi} \int_{E_\delta} \frac{e^{\tau(\phi(\zeta)-\phi(0))}u(\zeta)}{\zeta} \nu ds + \frac{1}{2\pi} \int_{\partial\Omega \setminus E_\delta} \frac{e^{\tau(\phi(\zeta)-\phi(0))}u(\zeta)}{\zeta} \nu ds \right| \\ &\leq \frac{1}{2\pi} \int_{E_\delta} \left| \frac{e^{\tau(\phi(\zeta)-\phi(0))}u(\zeta)}{\zeta} \right| ds + \frac{1}{2\pi} \int_{\partial\Omega \setminus E_\delta} \left| \frac{e^{\tau(\phi(\zeta)-\phi(0))}u(\zeta)}{\zeta} \right| ds \end{aligned} \quad (3.43)$$

Now we choose our analytic function $\phi(\zeta) = \psi(\zeta) + i\chi(\zeta)$, where we choose $\psi(\zeta)$ to be the harmonic measure of E_δ and χ its harmonic conjugate. That is ψ is an harmonic function

on Ω such that

$$\psi = \begin{cases} 1, & \zeta \in E_\delta \\ 0, & \zeta \in \partial\Omega \setminus E_\delta \end{cases}.$$

Since χ is determined up to a constant and ϕ is holomorphic we may choose χ such that $\phi(0) = \frac{1}{2\pi}|E_\delta|$. Defining $\mu := \frac{|E_\delta|}{2\pi}$ (3.43) becomes

$$\begin{aligned} 1 &\leq \frac{1}{2\pi} \int_{E_\delta} \delta e^{\tau(1-\mu)} ds + \frac{1}{2\pi} \int_{\partial\Omega \setminus E_\delta} m e^{-\tau\mu} ds \\ &\leq \frac{1}{2\pi} |E_\delta| \delta e^{\tau(1-\mu)} + \frac{1}{2\pi} (2\pi - |E_\delta|) m e^{-\tau\mu} \\ &\leq \mu \delta e^{\tau(1-\mu)} + (1-\mu) m e^{-\tau\mu} \quad (\forall \tau > 0). \end{aligned}$$

To calculate the minimum τ define $F(\tau) = \mu \delta e^{\tau(1-\mu)} + (1-\mu) m e^{-\tau\mu}$. Setting the derivative equal to zero we have

$$\tau = \ln\left(\frac{m}{\delta}\right).$$

Hence we have

$$|E_\delta| \leq \frac{2\pi \ln(m)}{\ln\left(\frac{m}{\delta}\right)}$$

□

3.5 Proof of Theorem 3.4

Proof. For $p \in [1, \infty)$, estimating the p -norm of a defined as in equation (3.39) and using Theorem 3.2 we have

$$\begin{aligned} \|a\|_{L^p}^p &= \int_{L_1(\partial\Omega \setminus E_\delta)} |a| \cdot |a|^{p-1} ds + \int_{E_\delta} |a|^p ds \\ &\leq \sqrt{2\pi} (2m)^{p-1} \|a\|_{L_1(\partial\Omega \setminus E_\delta)} + (2m)^p \mu(E_\delta) \\ &\leq \sqrt{2\pi} (2m)^{p-1} \|a\|_{L_1(\partial\Omega \setminus E_\delta)} + (2m)^p \frac{C}{\ln(1/\delta)} \end{aligned} \quad (3.44)$$

Furthermore, since a is bounded on $\partial\Omega$ and $|u_1| \geq \delta$ on $\partial\Omega \setminus E_\delta$ we have

$$\begin{aligned} \|a\|_{L_1(\partial\Omega \setminus E_\delta)} &= \left\| \frac{1}{u_1} (\partial_\nu u + a_2 u) \right\|_{L_1(\partial\Omega \setminus E_\delta)} \\ &\leq \frac{1}{\delta} [m \|u\|_{L_1(\partial\Omega \setminus E_\delta)} + \|\partial_\nu u\|_{L_1(\partial\Omega \setminus E_\delta)}] \\ &\leq \frac{c}{\delta} [\|u\|_{L_1(\partial\Omega \setminus E_\delta)} + \|\partial_\nu u\|_{L_1(\partial\Omega \setminus E_\delta)}] \end{aligned} \quad (3.45)$$

where $c := \max(m, 1)$. Next define

$$\varphi = \varphi_0 + \varphi_1. \quad (3.46)$$

where φ_0 is the solution to the Dirichlet problem

$$\begin{aligned} \Delta \varphi_0 &= |\lambda| + 2 \quad z \in \Omega \\ \varphi_0|_{\partial\Omega} &= 0 \end{aligned} \quad (3.47)$$

and φ_1 is the solution to the Dirichlet problem

$$\begin{aligned} \Delta \varphi_1 &= -2\pi \sum_{j=1}^n \delta(z - z_j) \quad z \in \Omega \\ \varphi_1|_{\partial\Omega} &= 0. \end{aligned} \quad (3.48)$$

Since $u \in C^2(\bar{\Omega})$, and from the last two conditions from (3.40) we have $|u(z_j)|e^{\varphi(z_j)} = 0$ and $|\partial u(z_j)|e^{\varphi(z_j)} = 0$. Simplifying the left hand side of estimate (2.28) based on (3.40), (3.47) and (3.48) we have

$$\begin{aligned}
& \int_{\Omega} (\Delta\varphi|u|^2 + (\Delta\varphi - 1)|\partial u|^2) e^{\varphi} dx \\
&= \int_{\Omega} ((\Delta\varphi_0 + \Delta\varphi_1)|u|^2 + (\Delta\varphi_0 + \Delta\varphi_1 - 1)|\partial u|^2) e^{\varphi} dx \\
&= \int_{\Omega} [(|\lambda| + 2)|u|^2 + (|\lambda| + 1)|\partial u|^2] e^{\varphi} dx - 2\pi \int_{\Omega} \left[\sum_{j=1}^n \delta(z - z_j)(|u|^2 + |\partial u|^2) \right] e^{\varphi} dx \\
&= \int_{\Omega} [|\lambda|(|u|^2 + |\partial u|^2) + 2|u|^2 + |\partial u|^2] e^{\varphi} dx - 2\pi \sum_{j=1}^n (|u(z_j)|^2 + |\partial u(z_j)|^2) e^{\varphi(z_j)} \\
&= \int_{\Omega} |\lambda|(|u|^2 + |\partial u|^2) e^{\varphi} dx + \int_{\Omega} (2|u|^2 + |\partial u|^2) e^{\varphi} dx.
\end{aligned} \tag{3.49}$$

Subtracting $\int_{\Omega} |\lambda|(|u|^2 + |\partial u|^2) e^{\varphi} dx$ from both sides of equations (2.28) and (3.49) gives

$$\int_{\Omega} (2|u|^2 + |\partial u|^2) e^{\varphi} dx \leq \int_{\partial\Omega} \partial_{\nu}(|u|^2 + |\partial u|^2) e^{\varphi} ds + \int_{\partial\Omega} 4(|\partial_{\nu_{\perp}} \partial_{\bar{z}} u|^2 + |\partial_z u|^2) ds.$$

From equations (3.46) - (3.48) we have that $\partial_{\nu} \phi_1 \leq 0$ and $\varphi \equiv 0$ on $\partial\Omega$ so we have

$$\int_{\Omega} (|u|^2 + |\partial u|^2) e^{\varphi} dx \leq \int_{\partial\Omega} \partial_{\nu} \varphi_0 (|u|^2 + |\partial u|^2) ds + 2 \int_{\partial\Omega} (|u|^2 + |\partial u|^2) ds.$$

Using Sobolev's embedding theorems we have by equation (3.47)

$$\varphi_0 \in C(\Omega) \quad \partial_{\nu} \varphi_0 \in L_1(\partial\Omega).$$

Hence we can define $L := \|\partial_{\nu} \varphi_0\|_{L_1(\partial\Omega)} + 2 < +\infty$ and by equation (3.37) we have that

$$\int_{\partial\Omega} (|u|^2 + |\partial u|^2) ds \leq 2\pi \cdot 8m^2 = 16\pi m^2.$$

Thus we have

$$\int_{\Omega} (|u|^2 + |\partial u|^2) e^{\varphi} dx \leq 16\pi m^2 L. \quad (3.50)$$

Now let us find a lower bound for (3.50). Recall that

$$\ln \left(\frac{1}{y} \right) \geq 1 - y, \quad y \in (0, 1)$$

and obtain

$$\ln \left| \frac{1 - z\bar{z}_j}{z - z_j} \right| = \ln \left| \frac{z - z_j}{1 - z\bar{z}_j} \right|^{-1} \geq 1 - \left| \frac{z - z_j}{1 - z\bar{z}_j} \right|. \quad (3.51)$$

Furthermore we have the following inequality from [12]:

$$\left| \frac{z - z_j}{1 - z\bar{z}_j} \right| \leq \frac{|z| + |z_j|}{1 + |z||z_j|}$$

Hence

$$\begin{aligned} \ln \left| \frac{1 - z\bar{z}_j}{z - z_j} \right| &\geq 1 - \frac{|z| + |z_j|}{1 + |z||z_j|} \\ &= \frac{1 + |z||z_j| - |z| - |z_j|}{1 + |z||z_j|} \\ &\geq \frac{(1 - |z|)(1 - |z_j|)}{1 + |z|} \end{aligned}$$

And this implies that

$$e^{\varphi_1(z)} \geq \exp \left\{ \frac{1 - |z|}{1 + |z|} \sum_{j=1}^n (1 - |z_j|) \right\} \quad (3.52)$$

This gives us

$$\begin{aligned} \int_{\Omega} (|u|^2 + |\partial u|^2) e^{\varphi} dx &\geq \int_{|z| \leq \alpha} (|u|^2 + |\partial u|^2) e^{\varphi} dx \\ &\geq e^{m_0} \exp \left\{ \frac{1-\alpha}{1+\alpha} S_n \right\} \int_{|z| \leq \alpha} (|u|^2 + |\partial u|^2) e^{\varphi} dx \end{aligned} \quad (3.53)$$

where $m_0 = \min_{z \in \bar{\Omega}} \varphi_0(x)$. Recall that $S_n = \sum_{j=1}^n (1 - |z_j|)$ and $(1 - \alpha)n \leq S_n \leq n$. Thus by equations (3.50) and (3.53) we have

$$\|u\|_{H^1(|z| \leq \alpha)}^2 \leq c_2 e^{-c_1 n}$$

where $c_1 = \frac{(1-\alpha)^2}{1+\alpha}$ and $c_2 = 16\pi m^2 L e^{-m_0}$.

Using trace theorem for Sobolev space there is a constant, $c_3 \geq 1$, depending only on α , such that

$$\|u\|_{H^{1/2}(|z|=\alpha)}^2 \leq c_3 e^{-c_1 n} \quad (3.54)$$

Thus by using Theorem 1 from Bukhgeĭm and Kardakov [6], there is a constant, C , depending only on Ω , α , m and $p \in [1, \infty)$ such that

$$\|a_1 - a_2\|_{L_p(\partial\Omega)} \leq \left(\ln \ln \left(\frac{1}{c_3 e^{-c_1 n}} \right) \right)^{-1/p}. \quad (3.55)$$

Combining this result with our inequality above allows us to conclude

$$\|a_1 - a_2\|_{L_p(\partial\Omega)} \leq \frac{C}{(\ln n)^{1/p}}. \quad (3.56)$$

□

4 Proofs of Main Results

4.1 Technical Lemmas

Let Ω be the unit disk, $\Omega = \{x \in \mathbb{R}^2 \mid |x| < 1\}$. Generally we can consider be a simply connected bounded domain with smooth boundary conformally equivalent to the unit disk [27]. Let $\varphi \in C^2(\bar{\Omega})$ be a real-valued harmonic function which has only one critical point $x_0 \in \Omega$ satisfies,

$$|\partial\varphi(x)| = |x - x_0|^m \cdot (c + o(1)) \text{ as } x \rightarrow x_0, \quad c \neq 0. \quad (4.1)$$

Our main tools are operators $\bar{\partial} = \partial_{\bar{z}}$, $\partial = \partial_z$, the Teodorescu operator

$$(Tu)(z) = -\frac{1}{\pi} \int_{\Omega} \frac{u(\xi)}{\xi - z} d\xi, \quad (4.2)$$

the Cauchy operator

$$(Cu)(z) = \frac{1}{2\pi i} \int_{\partial\Omega} \frac{u(\xi)}{\xi - z} d\xi, \quad z \notin \partial\Omega, \quad (4.3)$$

and the operator P from equation (3.29)

$$P = \begin{bmatrix} 2\bar{\partial} & 0 \\ 0 & 2\partial \end{bmatrix} + A(x). \quad (4.4)$$

If ψ is analytic on Ω then

$$\bar{\partial}(e^{\tau\psi}) = (e^{\tau\psi})(\tau\bar{\partial}\psi) + e^{\tau\psi}\bar{\partial} = e^{\tau\psi}\bar{\partial} \text{ and}$$

$$\partial(e^{\tau\bar{\psi}}) = (e^{\tau\bar{\psi}})(\tau\partial\bar{\psi}) + e^{\tau\bar{\psi}}\partial = e^{\tau\bar{\psi}}\partial.$$

Thus operators $\bar{\partial}$ and $e^{\tau\psi}$; and ∂ and $e^{\tau\bar{\psi}}$ commute. If we define

$$D = \begin{bmatrix} 2\bar{\partial} & 0 \\ 0 & 2\partial \end{bmatrix} \text{ and } \Phi(x) = \begin{bmatrix} \psi(x) & 0 \\ 0 & \bar{\psi}(x) \end{bmatrix},$$

then for ψ analytic in Ω , matrix operators D and $e^{\tau\Phi}$ commute also. Hence $De^{\tau\Phi} = e^{\tau\Phi}D$, or equivalently $e^{-\tau\Phi}De^{\tau\Phi} = D$. Hence

$$e^{-\tau\Phi}Pe^{\tau\Phi} = e^{-\tau\Phi}(D + A)e^{\tau\Phi} = D + e^{-\tau\Phi}Ae^{\tau\Phi} = D + A_\tau, \text{ where}$$

$$A_\tau = e^{-\tau\Phi}Ae^{\tau\Phi} = \begin{bmatrix} 0 & be^{-\tau(\psi-\bar{\psi})} \\ ae^{\tau(\psi-\bar{\psi})} & 0 \end{bmatrix}.$$

Letting $\varphi = 2\Im\Psi$ gives $\psi - \bar{\psi} = i\varphi$. Further defining $a_{\pm\tau}(x) = a(x)e^{\pm i\tau\varphi(x)}$ implies

$$A_\tau = \begin{bmatrix} 0 & b_{-\tau} \\ a_\tau & 0 \end{bmatrix}.$$

Now for all $v \in C_0^\infty(\mathbb{R}^2)$ we have that as $a \in C^1(\Omega)$ (or $L_p(\Omega)$), then $a(x)v(x) \in L_1(\mathbb{R}^2)$ (or $L_p(\mathbb{R}^2)$) and has compact support so

$$\int_{\mathbb{R}^2} a_{\pm\tau}(x)v(x) dx = \int_{\mathbb{R}^2} a(x)v(x)e^{\pm i\tau\varphi(x)} dx \rightarrow 0 \text{ as } \tau \rightarrow \infty.$$

Thus A_τ converges to 0 as $\tau \rightarrow \infty$ in a weak sense.

Further define the operator

$$\mathcal{S}u = \frac{1}{4}Tb_{-\tau}\bar{T}a_\tau u. \tag{4.5}$$

We have the following well-known identities about the Teodorescu operator

$$T(C^\alpha(\bar{\Omega})) \subseteq C^{1+\alpha}(\bar{\Omega}), \quad \alpha \in (0, 1), \quad (4.6)$$

$$T(L_p)(\bar{\Omega}) \subseteq C^\alpha(\bar{\Omega}), \quad p \geq \frac{2}{1-\alpha}, \quad (4.7)$$

$$T(L_p)(\bar{\Omega}) \subseteq W_p^1(\bar{\Omega}), \quad 1 < p < \infty, \quad (4.8)$$

$$\|Tu\|_{1+\alpha} \leq c \|u\|_\alpha, \quad (4.9)$$

$$\|Tu\|_\alpha \leq c \|u\|_{L_p}, \quad p \geq \frac{2}{1-\alpha}, \quad (4.10)$$

$$\|Tu\|_{W_p^1} \leq c \|u\|_{L_p}, \quad 1 < p < \infty. \quad (4.11)$$

Moreover we know that $\bar{\partial}Tu = u$ and $\partial\bar{T}u = u$ in the distributional sense. The space $C^\alpha(\bar{\Omega})$ is a Hölder space where we define the following norm

$$\|f\|_\alpha = \sup_{x \in \Omega} |f(x)| + \sup_{\substack{x, x' \in \Omega \\ x \neq x'}} \frac{|f(x) - f(x')|}{|x - x'|^\alpha}. \quad (4.12)$$

If $\|f\|_\alpha < \infty$ for some $\alpha \in (0, 1]$ then we say that $f \in C^\alpha$ and C^α endowed with this norm is a Banach space, and we have Cauchy-Schwarz-Bunyakovsky inequality and Hölder-type inequality for all $f, g \in C^\alpha(\bar{\Omega})$,

$$\|fg\|_\alpha \leq \|f\|_\alpha \|g\|_\alpha, \text{ and} \quad (4.13)$$

$$\|fg\|_\alpha \leq \|f\|_\alpha \|g\|_0 + \|f\|_0 \|g\|_\alpha. \quad (4.14)$$

Similarly for $m \in \mathbb{N}$

$$C^{m+\alpha}(\bar{\Omega}) = \{u \in C^m(\bar{\Omega}) | \partial^\gamma u \in C^\alpha(\bar{\Omega}), |\alpha| = \gamma_1 + \gamma_2 + \dots + \gamma_n = m\}$$

is also a Hölder space, and a Banach space when endowed with the norm

$$\|u\|_{m+\alpha} = \|u\|_m + \sum_{|\gamma|=m} \sup_{\substack{x, x' \in \Omega \\ x \neq x'}} \frac{|\partial^\gamma u(x) - \partial^\gamma u(x')|}{|x - x'|^\alpha} \quad (4.15)$$

where

$$\|u\|_m = \|u\|_{C^m(\bar{\Omega})} = \sum_{|\gamma| \leq m} \sup_{x \in \Omega} |\partial^\gamma u(x)|. \quad (4.16)$$

We also have the Cauchy, Schwarz, Bunyakovsky inequality for all $f, g \in C^{m+\alpha}$,

$$\|fg\|_{m+\alpha} \leq \|f\|_{m+\alpha} \|g\|_{m+\alpha}. \quad (4.17)$$

We will need a few standard estimates for the Teodorescu operator. If we denote the Lebesgue measure a set Ω by $|\Omega|$ and allow $\frac{1}{p} + \frac{1}{q} = 1$ for $1 < p < \infty$, then consider

$$\begin{aligned} \int_{\Omega} \frac{dy}{|x-y|^{n-\alpha}} &= \int_{\Omega \cap \{|x-y| < \varepsilon\}} \frac{dy}{|x-y|^{n-\alpha}} + \int_{\Omega \cap \{|x-y| \geq \varepsilon\}} \frac{dy}{|x-y|^{n-\alpha}} \\ &\leq \int_{\Omega \cap \{|x-y| < \varepsilon\}} \frac{dy}{\varepsilon^{n-\alpha}} + \int_{|x-y| < \varepsilon} \frac{dy}{|x-y|^{n-\alpha}} \\ &= \frac{|\Omega|}{\varepsilon^{n-\alpha}} + \int_{|\omega|=1} \int_0^\varepsilon r^{-(n-\alpha)} r^{n-1} dr d\omega \\ &= \frac{|\Omega|}{\varepsilon^{n-\alpha}} + \frac{\omega^\alpha}{\alpha} \varepsilon^\alpha \end{aligned}$$

Choosing ε so that the terms are equal we have

$$\frac{|\Omega|}{\varepsilon^{n-\alpha}} = \frac{\omega^\alpha}{\alpha} \varepsilon^\alpha \quad \varepsilon^\alpha \varepsilon^{n-\alpha} = \frac{\alpha |\Omega|}{\omega_n} \quad \varepsilon = \left(\frac{\alpha |\Omega|}{\omega_n} \right)^{\frac{1}{n}}.$$

This gives us

$$\begin{aligned} \int_{\Omega} \frac{dy}{|x-y|^{n-\alpha}} &\leq \frac{\omega_n}{\alpha} \left(\frac{\alpha |\Omega|}{\omega_n} \right)^{\frac{\alpha}{n}} + \frac{|\Omega|}{\left(\frac{\alpha |\Omega|}{\omega_n} \right)^{\frac{\alpha}{n}}} \\ &= 2 \left[\frac{\omega_n}{\alpha} \right]^{1-\frac{\alpha}{n}} |\Omega|^{\frac{\alpha}{n}}, \end{aligned}$$

an estimate for the Riesz potential in \mathbb{R}^n where $\omega_n = \int_{|\omega|=1} d\omega$, $\alpha > 0$ and $\Omega \subset \mathbb{R}^2$. Here we choose $n = 2$ and $\alpha = 1$. Then writing

$$\frac{|u(y)|}{|x-y|} = \frac{1}{|x-y|^{1/q}} \cdot \frac{|u(y)|}{|x-y|^{1/p}}$$

in the Teodorescu integral and applying our estimate for the Riesz potential when $\Omega' \subset\subset \Omega$ we have

$$\|Tu\|_{L_p(\Omega')} \leq c |\Omega|^{\frac{1}{2q}} \cdot |\Omega'|^{\frac{1}{2p}} \cdot \|u\|_{L_p(\Omega)}, \quad (4.18)$$

and for the case when $\text{supp}(u) \subset \Omega'$ we have

$$\|Tu\|_{L_p(\Omega)} \leq c |\Omega'|^{\frac{1}{2q}} \cdot |\Omega|^{\frac{1}{2p}} \cdot \|u\|_{L_p(\Omega')}. \quad (4.19)$$

Finally we need the integration by parts formula for the Teodorescu operator. This follows from the Cauchy formula

$$T\bar{\partial}u = u - Cu.$$

Using the chain rule and the product rule give

$$\begin{aligned} \bar{\partial}(e^{i\tau\varphi}) &= i\tau\bar{\partial}\varphi e^{i\tau\varphi} \\ e^{i\tau\varphi}u &= \frac{u}{i\tau\bar{\partial}\varphi} \bar{\partial}(e^{i\tau\varphi}) = \bar{\partial} \left[\frac{u}{i\tau\bar{\partial}\varphi} e^{i\tau\varphi} \right] - e^{i\tau\varphi} \bar{\partial} \left[\frac{u}{i\tau\bar{\partial}\varphi} \right] \\ &= \frac{1}{i\tau} \left\{ \bar{\partial} \left(e^{i\tau\varphi} \frac{u}{\bar{\partial}\varphi} \right) - e^{i\tau\varphi} \bar{\partial} \left(\frac{u}{\bar{\partial}\varphi} \right) \right\}. \end{aligned}$$

Applying the Teodorescu operator to both sides gives

$$T\bar{\partial} e^{i\tau\varphi} = \frac{1}{i\tau} \left\{ T\bar{\partial} \left(e^{i\tau\varphi} \frac{u}{\bar{\partial}\varphi} \right) - T \left(e^{i\tau\varphi} \bar{\partial} \left(\frac{u}{\bar{\partial}\varphi} \right) \right) \right\}.$$

Recalling that $\bar{\partial}\varphi(x_0) = 0$ for some interior point $x_0 \in \Omega$ and, noting that if $u(x) = 0$ near x_0 and $u|_{\partial\Omega} = 0$ as well then $Cu = 0$, results in

$$Te^{i\tau\varphi}u = \frac{1}{i\tau} \left[e^{i\tau\varphi} \frac{u}{\bar{\partial}\varphi} - T \left(e^{i\tau\varphi} \bar{\partial} \left(\frac{u}{\bar{\partial}\varphi} \right) \right) \right]. \quad (4.20)$$

Lemma 4.1.

(i) Let $a \in L_p$, $p > 2$, $b \in W_\infty^1$ and . Then there exists $\tau_0 > 0$, such that for all $u \in C^\alpha$, $\tau > \tau_0$ and $\alpha \in (0, 1 - \frac{2}{p})$ we have that for $\beta = \frac{1-\alpha}{2mp+1}$,

$$\| \mathcal{S}u \|_\alpha \leq c\tau^{-\beta} \| u \|_\alpha. \quad (4.21)$$

(ii) Let $a \in L_p$, $p > 2$, $b \in C^\alpha$. Then there exists $\tau_0 > 0$, such that for all $u \in C^\alpha$, $\tau > \tau_0$ and $\alpha \in (0, 1 - \frac{2}{p})$ we have

$$\| \mathcal{S}u \|_{1+\alpha} \leq c\tau^\alpha \| u \|_\alpha. \quad (4.22)$$

(iii) Let $a \in L_\infty$, $b \in W_\infty^1$. Then for all $u \in L_p(\Omega)$, $\beta = \frac{1}{2m+1}$, $\tau > 0$ and $1 < p < \infty$,

$$\| \mathcal{S}u \|_{L_p} \leq c\tau^{-\beta} \| u \|_{L_p}. \quad (4.23)$$

(iv) Let $a, b \in L_\infty$. Then for all $u \in L_p$, $\tau > 0$ and $1 < p < \infty$,

$$\| \mathcal{S}u \|_{W_p^1} \leq c \| u \|_{L_p}. \quad (4.24)$$

Remark 4.1. In estimates (4.21)-(4.24) and in subsequent conclusions we denote by constant c some (in general different) values which are independent of u and important parameters such as τ and δ which appear later in the proof. This constant will absorb other known constant factors and norms of fixed but arbitrary functions such as a or b .

Proof. Estimates (4.22) and (4.24) follow from estimates (4.9) and (4.11), respectively. It is sufficient to prove estimates (4.21) and (4.23) for $x_0 = 0$. For each inequality we need to choose an appropriate $0 < \delta < 1$ and a cut-off function $h \in C_0^\infty(\Omega)$ such that $0 \leq h \leq 1$, $h(x) = 0$ for $|x| < \delta$ or $|x| > 1 - \delta^2$, and $h(x) = 1$ for $|x| > 2\delta$ and $|x| < 1 - 4\delta^2$. Set $\Omega' = \text{supp}(1 - h)$; then $|\Omega'| = \mathcal{O}(\delta^2)$ and for $|x| \leq 1 - 4\delta^2$ we have $\nabla h = \mathcal{O}(\delta^{-1})$. Recalling the definition of \mathcal{S} gives

$$\mathcal{S}u = \frac{1}{4} (Tb_{-\tau}(1 - h)\overline{T}a_\tau u + Tb_{-\tau}h\overline{T}a_\tau). \quad (4.25)$$

Choosing δ such that $\delta^{\frac{1}{p}} = \delta^{-2m}\tau^{\alpha-1}$, and using inequalities (4.10) and (4.18) we have

$$\begin{aligned} \|Tb_{-\tau}(1 - h)\overline{T}a_\tau u\|_\alpha &\leq c \|b_{-\tau}(1 - h)\overline{T}a_\tau u\|_{L_p(\Omega)} \\ &= c \|b\| (1 - h) \|\overline{T}a_\tau u\|_{L_p(\Omega')} \\ &\leq c \|b\| (1 - h) |\Omega|^{\frac{1}{2q}} |\Omega'|^{\frac{1}{2p}} \|a_\tau u\|_{L_p(\Omega)} \\ &\leq c \|b\| (1 - h) |\Omega|^{\frac{1}{2q}} \delta^{\frac{1}{p}} \|a_\tau u\|_{L_p(\Omega)} \\ &\leq \left(c \|b\| (1 - h) |\Omega|^{\frac{1}{2q}} \|a\|_{L_p(\Omega)} \right) \delta^{\frac{1}{p}} \|u\|_{L_\infty(\Omega)} \\ &= c\delta^{\frac{1}{p}} \|u\|_\alpha. \end{aligned} \quad (4.26)$$

Similarly, for the $L_p(\Omega)$ case, using inequalities (4.18) and (4.19), and choosing δ such that $\delta = \delta^{-2m}\tau^{-1}$ we obtain

$$\|Tb_{-\tau}(1 - h)\overline{T}a_\tau u\|_{L_p(\Omega)} \leq c\delta \|u\|_{L_p(\Omega)}. \quad (4.27)$$

In order to estimate the second term of equation (4.25) we use the integration by parts formula (4.20). Assuming first that $b, \bar{T}a_\tau u \in C^1(\Omega)$ and using the usual density arguments we obtain

$$Te^{-i\tau\varphi}bh\bar{T}a_\tau = \frac{1}{\tau} \left[-e^{-i\tau\varphi} \frac{bh\bar{T}a_\tau}{\bar{\partial}\varphi} + Te^{-i\tau\varphi} \bar{\partial} \left(\frac{bh\bar{T}a_\tau}{\bar{\partial}\varphi} \right) \right]$$

Using estimates $\|b\|_\alpha \leq c\|b\|_{W_\infty^1}$, $\left\| \frac{h}{\bar{\partial}\varphi} \right\|_{C(\Omega)} \leq c\delta^{-1}$, $\left\| \frac{h}{\bar{\partial}\varphi} \right\|_\alpha \leq \left\| \frac{h}{\bar{\partial}\varphi} \right\|_{C^1(\Omega)} \leq c\delta^{-2m}$, (4.10), (4.13) and

$$\|e^{i\tau\varphi}\|_\alpha \leq c\tau^\alpha, \quad \tau \geq \tau_0 > 0, \quad (4.28)$$

we obtain

$$\left\| \frac{i}{\tau} \left(e^{-i\tau\varphi} \frac{bh\bar{T}a_\tau u}{\bar{\partial}\varphi} \right) \right\| \leq c\tau^{\alpha-1}\delta^{-2m} \|\bar{T}a_\tau u\|_\alpha \leq c\tau^{\alpha-1}\delta^{-2m} \|a\|_{L_p(\Omega)} \|u\|_\alpha.$$

Using inequalities (4.10) and (4.11) gives

$$\begin{aligned} \left\| Te^{-i\tau\varphi} \bar{\partial} \left(\frac{bh\bar{T}a_\tau u}{\bar{\partial}\varphi} \right) \right\|_\alpha &\leq c\tau^{-1} \left\| \bar{\partial} \left(\frac{bh\bar{T}a_\tau u}{\bar{\partial}\varphi} \right) \right\|_{L_p(\Omega)} \\ &\leq c\tau^{-1}\delta^{-2m} \|\bar{T}a_\tau u\|_{W_p^1} \\ &\leq c\tau^{-1}\delta^{-2m} \|a_\tau u\|_{L_p(\Omega)} \\ &\leq \left(c\|a\|_{L_p(\Omega)} \right) \tau^{-1}\delta^{-2m} \|u\|_{L_\infty(\Omega)} \\ &\leq c\tau^{-1}\delta^{-2m} \|u\|_\alpha. \end{aligned}$$

Hence we have

$$\|Tb_{-\tau}h\bar{T}a_\tau u\|_\alpha \leq c_1\tau^{\alpha-1}\delta^{-2m} \|a\|_\alpha + c_2\tau^{-1}\delta^{-2m} \|u\|_\alpha \leq c\delta^{-2m}\tau^{\alpha-1} \|u\|_\alpha. \quad (4.29)$$

Similarly for the $L_p(\Omega)$ case

$$\begin{aligned} \left\| T e^{-i\tau\varphi} \bar{\partial} \left(\frac{bh\bar{T}a_\tau u}{\bar{\partial}\varphi} \right) \right\|_\alpha &\leq c\tau^{-1}\delta^{-m} \|u\|_{L_p(\Omega)}, \text{ and} \\ \left\| T b_{-\tau} h\bar{T}a_\tau u \right\|_\alpha &\leq c\tau^{-1}\delta^{-2m} \|u\|_{L_p(\Omega)}. \end{aligned}$$

and we obtain

$$\left\| T b_{-\tau} h\bar{T}a_\tau u \right\|_{L_p(\Omega)} \leq c\delta^{-2m}\tau^{-1} \|u\|_{L_p(\Omega)}. \quad (4.30)$$

Choosing $\delta^{\frac{1}{p}} = \delta^{-2m}\tau^{\alpha-1}$ implies $\delta^{\frac{1}{p}} = \tau^{-\beta}$ where $\beta = \frac{1-\alpha}{2mp+1}$. With this δ , equations (4.26) and (4.29) give equation (4.21). Choosing $\delta = \delta^{-2m}\tau^{-1}$ implies $\delta = \tau^{-\frac{1}{2m+1}} = \tau^{-\beta}$, where $\beta = \frac{1}{2m+1}$. Choosing this δ , equations (4.27) and (4.30) give equation (4.23). Now that we have inequalities (4.21) and (4.23), inequalities (4.22) and (4.24) follow from equations (4.10) and (4.11), respectively. \square

Now for the transpose operator $P^t = -D + A^t$, let us define an operator, S^t analogous to operator S for P :

$$S^t u = \frac{1}{4} \bar{T} b_\tau T a_{-\tau} u. \quad (4.31)$$

Operator P^t satisfies Green's formula for operator P given by

$$\int_\Omega \langle Pu, v \rangle dx = \int_\Omega \langle u, P^t v \rangle dx + \int_{\partial\Omega} \langle \mathcal{V}u, v \rangle ds, \quad (4.32)$$

for $u, v \in C^1(\bar{\Omega}, \mathbb{C}^2)$.

Here $u = \begin{bmatrix} u_1 \\ u_2 \end{bmatrix}$, $v = \begin{bmatrix} v_1 \\ v_2 \end{bmatrix}$, $\langle \cdot, \cdot \rangle$ is the scalar product in \mathbb{R}^2 : $\langle u, v \rangle = u_1 v_1 + u_2 v_2$,

$\mathcal{V} = \begin{bmatrix} \nu & 0 \\ 0 & \bar{\nu} \end{bmatrix}$, and $\nu = \nu_1 + \nu_2$ is the unit outward normal to $\partial\Omega$.

Lemma 4.2. *Under the conditions of Lemma 4.1 we can replace operator S by S^t in estimates (4.21)-(4.24).*

Proof. Clearly by following the proof of Lemma 4.1 replacing S^t for S we have the same estimates as for S , mutatis mutandis. \square

Later we will use the fact that there is a sufficiently large τ_0 such that $\|S\| < 1$ and $\|S^t\| < 1$ for all $\tau > \tau_0$ in either $C^\alpha(\bar{\Omega})$ or $L_p(\Omega)$, where Ω is any bounded connected set in \mathbb{C} , such that the boundary, $\partial\Omega$, consists of a finite number of C^1 (or piece-wise C^1) Jordan curves, and φ an arbitrary harmonic function bounded in $\bar{\Omega}$. This fact follows from the following observations.

Using a partition of unity we write $\Omega = \Omega_\delta \cup (\Omega \setminus \Omega_\delta)$, where $\Omega_\delta = \{z \in \Omega \mid \text{dist}(z, \partial\Omega) > \delta\}$. In $\Omega \setminus \Omega_\delta$, the norms of S and S^t will be small for small δ . In Ω_δ , analytic function $\partial\varphi$ has only a finite number of zeros z_1, z_2, \dots, z_n . Hence there is an $\varepsilon > 0$ such that all disks $\mathbb{D}(z_j, \varepsilon) = \{z \in \mathbb{C} \mid |z - z_j| < \varepsilon\}$ are disjoint subsets of Ω_δ . In the set $\Omega_\delta \setminus \cup \mathbb{D}(z_j, \varepsilon)$ we can integrate by parts and obtain a small factor $\frac{1}{\tau}$ for the norms of S and S^t . From Lemmas 4.1 and 4.2, in each disk we have that $\|S\| + \|S^t\| \leq c\tau^{-\beta}$ for some $\beta > 0$.

Lemma 4.3. (equivalent statement given in [3]) *Let $a \in L_\infty(\Omega)$, $b \in W_\infty^1(\Omega)$. Then there is a $\tau_0 > 0$ such that for all $\tau \geq \tau_0$ the equation*

$$P_\tau^t v = -\delta(x - x_0)e_1, \quad e_1 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$$

has a solution, v , of the form $v = \tilde{v} + w$, where

1. If $\tilde{v} = 2\partial E(x - x_0)e_1$ and $E(x) = -\frac{1}{\pi} \ln|x|$, then

$$\begin{aligned} w_1 &= \frac{1}{2} T a_{=\tau} w_2, \\ w_2 &= (I - S^t)^{-1} \frac{1}{2} \bar{T} b_\tau \tilde{v}_1 \quad \text{and} \\ w &\in W_p^1, \quad 1 < p < 2. \end{aligned}$$

2. If $\tilde{v} = 2\partial E(x - x_0)e_1 + b_\tau E(x - x_0)e_2$, then

$$\begin{aligned} g(x) &= E(x - x_0) [abe_1 - (2(\partial b)_\tau + \tau b_\tau 2\partial\varphi) e_2], \\ w_1 &= \frac{1}{2}Tg_1 + \frac{1}{2}Ta_{-\tau}w_2, \text{ and} \\ w_2 &= (1 - S^t)^{-1} \left(\frac{1}{4}\bar{T}b_\tau Tg_1 + \frac{1}{2}\bar{T}g_2 \right). \end{aligned}$$

In this case $w \in W_p^1(\Omega) \cap C^\alpha(\bar{\Omega})$, $p > 2$, and $\alpha \in (0, 1 - 2/p]$.

Proof. The proof follows by direct computation using the fact that $\|S^t\| < 1$ in L_p . Furthermore a similar lemma holds for $P_\tau^t v = -\delta(x - x_0)e_2$, where $e_2 = [0 \ 1]^T$. \square

We can write the Green's formula (4.32) in the weighted form

$$\int_{\Omega} \langle e^{\tau\Phi} P u, v \rangle dx = \int_{\Omega} \langle e^{\tau\Phi} u, P_\tau^t v \rangle dx + \int_{\partial\Omega} \langle e^{\tau\Phi} \mathcal{V} u, v \rangle ds, \quad (4.33)$$

where

$$P_\tau^t = e^{-\tau\Phi} P^t e^{\tau\Phi} = -D + A_\tau^t.$$

4.2 Main Results

Proof. Proof of Theorem 3.1

If we choose a matrix-valued function Φ as in Theorem 3.1, $v_1 = v$ as in Lemma 4.3 and τ tend to ∞ , we obtain Carleman's formula (3.34) for $j = 1$ from equation (4.32). The integral over $\partial\Omega \setminus E$ vanishes due to the factor $e^{\tau\Re\phi}$ since $\Re\phi(\xi) < 0$ for $\xi \in \partial\Omega \setminus E$. The proof for $P_\tau^t v = -\delta(x - x_0)e_2$ follows mutatis mutandis. The proof of the main result is complete. \square

Proof. Proof of Theorem 3.2 The proof of this theorem is similar to the proof of Lemma 3.1 in which we consider the case $P = \bar{\partial}$. Since $Pu = 0$ in \mathbb{D} and $u \neq 0$ we can find $z_0 \in \mathbb{D}$ such that $u(z_0) = c \neq 0$. Without loss of generality we may assume, after rescaling and conformal mapping, that $z_0 = 0$. and $u_1(z_0) = 1$. We use the fact that the form of our operator P is

conformally invariant. Then we use Carleman's formula (3.34) with $\varphi(\xi)$ from example 3.6 with $E := E_\delta$. We repeat the analogous calculations in the proof of Lemma 3.1. If we choose vector functions $v_1 = v$ as in Lemma 4.3, part 1., we obtain the estimate

$$|E_\delta| \leq \frac{c}{\ln \frac{m}{\delta}},$$

for all $\delta \in (0, 1)$ and some constants $c > 0$, $m > 0$. If we choose $v_1 = v$ from Lemma 4.3, part 2., we obtain

$$|E_\delta| \leq \frac{c(1 + \ln \ln \frac{c}{\delta})}{\ln \frac{m}{\delta}}.$$

□

The proofs of Theorem 3.3 and Corollary 3.1 follow completely analogously to the proof of Theorem 2.4.

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