The Size of the Divergence Points for Rational Maps

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1. Introduction

Recently, there has been a great interest in understanding the mathematics behind fractal sets such as the Mandelbrot set and Julia sets of rational maps on the Riemann sphere. The analysis of these complicated sets and structures lies in the interplay of the disciplines complex dynamics and fractal geometry which are important areas of contemporary mathematical research. Major goals are to understand the long-term behavior under iteration of the typical points in the phase space and to calculate the size of these points in terms of their Hausdorff dimension. In this work we present some recent results concerning the size of the set of points for which the Birkhoff images do not converge. These points are usually called divergence points. Our main result is that in the case of expansive rational maps, and more generally for NCP rational maps, the set of divergence points has the same Hausdorff dimension as the Julia set of the rational map itself.

2. Significance

Consider a rational map $f$ on the Riemann sphere. In this paper we study the size of the set of points for which the Birkhoff sums with respect to a non-trivial observable do not converge. We call these points the divergence points of $f$. It follows from Birkhoff's Ergodic Theorem that the divergence points have zero measure for every $f$-invariant Borel probability measure. This indicates that the set of divergence points is a rather small set. On the other hand and somewhat surprisingly, we are able to prove in this paper that the set of divergence points is large in the sense of Hausdorff dimension. Furthermore, we show that under some mild assumptions on the rational map the set of divergence points is as large as possible. Our results extend previous work on hyperbolic maps to the case of non-hyperbolic systems for which the theory of divergence points is poorly understood.

3. Results

Our main results are the following:

**Theorem 1.** Let $f$ be a rational map on the Riemann sphere and let $D$ denote the set of divergence points with respect to a non-trivial observable. Then the Hausdorff dimension of $D$ is greater than or equal to the hyperbolic dimension of $f$. In particular, the Hausdorff dimension of $D$ is strictly positive.

**Corollary 2.** Suppose $f$ is a NCP rational map. Then the Hausdorff dimension of $D$ coincides with the Hausdorff dimension of the Julia set of $f$.

**Theorem 3.** For every holomorphic family of structural stable rational maps, the hyperbolic dimension depends continuously and plurisubharmonically on the parameter of the map.

4. Methods

The proofs of these Theorems involve the application of tools from several areas of mathematics such as smooth ergodic theory, hyperbolic dynamics and the thermodynamic formalism. The main idea in the proof of Theorem 1 is to consider typical points associated with two distinct ergodic $f$-invariant measures and then to glue these points together in such a way that one obtains divergence points of the same Hausdorff dimension as the corresponding typical points. To prove Theorem 3 we construct a one-parameter family of invariant measures such that their entropies are constant while their Lyapunov exponents depend harmonically on the parameter.

5. Acknowledgements

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5. References