Design of Output Feedback using Reduced-Order Observer

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1. Introduction

The basic idea of feedback control is to exploit information from the system and use it to regulate the system. Under ideal conditions, the designer has complete information on all state variables to construct the control input for the system. However, in most cases, not all states are available for measurement. Only the output variable is available and correlated to other states of the system. In this case, instead of a state-feedback design, an output-feedback design must be employed to regulate the system. The objective of this paper is to improve the feedback design using the output from the open-loop system. This paper takes advantage of the idea that a set of properly selected eigenvectors with 90° angles between them should make the eigenvalues less sensitive to perturbation of the system's parameters [1]. The problem of the missing states can be overcome by designing an observer to estimate them by using the available output as its inputs, along with the control input for the plant.

2. Result

Consider a continuous-time plant represented as an open-loop system that is completely controllable and observable described by its state-space model

$$\dot{x}(t) = Ax(t) + Bu(t), \quad y(t) = Cx(t) \tag{1}$$

with control input $u(t) \in \mathbb{R}^m$, state vector $x(t) \in \mathbb{R}^n$, output vector $y(t) \in \mathbb{R}^r$, and matrices $A \in \mathbb{R}^{n \times n}$, $B \in \mathbb{R}^{n \times m}$, and $C \in \mathbb{R}^{r \times n}$. The control input $u(t)$ is generated from the available output $y(t)$ using the output-feedback gain $F \in \mathbb{R}^{m \times r}$ as in the following relationship

$$u(t) = Fy(t) \tag{2}$$

Case 1: $r + m > n$

Under the condition of full controllability and observability for the case where $r + m > n$, the gain F can be obtained [2, 3]. The m poles can be set at arbitrary locations based on the relationship

$$W_m(A + BFC) = \Lambda_m W_m \tag{3}$$

where $\Lambda_m \in \mathbb{R}^{m \times m}$ contains the m desired eigenvalues on the diagonal, and $W_m \in \mathbb{R}^{m \times n}$ contains the left eigenvectors. The remaining $(n - m)$ poles are determined from the relationship

$$(A + BFC)\, V_{n-m} = V_{n-m}\, \Lambda_{n-m} \tag{4}$$

where $\Lambda_{n-m} \in \mathbb{R}^{(n-m) \times (n-m)}$ contains the $n - m$ desired eigenvalues on the diagonal, and $V_{n-m} \in \mathbb{R}^{n \times (n-m)}$ contains the right eigenvectors. The output feedback gain is

$$F = (W_m B)^{-1}(\Lambda_m W_m - W_m A) V_{Cr} (U_C \Sigma_C)^{-1} \tag{5}$$

where the matrix C is factorized by its Singular Value Decomposition

$$C = U_C [\Sigma_C \quad 0] \begin{bmatrix} V_{Cr}^T \\ V_{C(n-r)}^T \end{bmatrix} \tag{6}$$

In the process in finding the left eigenvectors in W_m and the right eigenvectors in V_{n-m}, each vector must be selected so that the angles between the eigenvectors as close as possible to 90°.

Case 2: $r + m \leq n$

In this case the total number of inputs and outputs is not enough to obtain the output feedback F by $q = n - m - r + 1$. Then a reduced-order q observer can be designed to estimate the required additional q outputs with

$$A_{ob} = W_q A S_q + Z_q C S_q, \quad B_{ob} = W_q B \tag{7}$$

where $Z_q \in \mathbb{R}^{q \times r}$ is an arbitrary matrix, and the row vectors in $W_q \in \mathbb{R}^{q \times n}$ and the column vectors in $S_q \in \mathbb{R}^{n \times q}$ are obtained from

$$w_i^T = z_i^T C (\lambda_i I - A)^{-1}, \text{ for each } i = 1, 2, \ldots, q \tag{8}$$

$$W_{n-q}^T \in \text{Ker}(W_q) \tag{9}$$

$$\begin{bmatrix} W_q \\ W_{n-q} \end{bmatrix}^{-1} = [S_q \quad S_{n-q}] \tag{10}$$

The row vectors in W_q are selected that the angle between the vectors as close as possible to 90°. The additional required output is

$$y_q(t) = W_q x(t) \tag{11}$$

The output feedback gain F is obtained using equations (3) – (6). The composite system is shown in figure 1
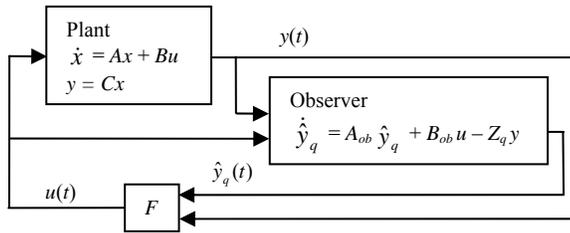
Figure 1. Composite system: reduced-order q observer and output-feedback design.

The condition number of the eigenvalue indicates the magnitude of the eigenvalue shift due to perturbation in any of the matrix in $A+BFC$, and is used to evaluate the design performance in the s-domain

$$\Psi_{max} = \max_k \text{Cond}(\lambda_k) = 1/|w_k^T v_k| \qquad (12)$$

The cost function indicates the magnitude of the system's zero-input response overshoot, and is used to evaluate the design performance in the time domain

$$J_k = \int_0^\infty x_k^2(t)\ dt, \quad \text{for each } k = 1, 2, \ldots, n$$

$$J_{max} = \max_k J_k \qquad (13)$$

Example
Consider the linearized model of a Jet Transport during cruise flight system (taken from MATLAB example)

$$A = \begin{bmatrix} -0.0558 & -0.9966 & 0.0802 & 0.0415 \\ 0.5980 & -0.1150 & -0.0318 & 0 \\ -3.0500 & 0.3880 & -0.4650 & 0 \\ 0 & 0.0805 & 1 & 0 \end{bmatrix}$$

$$B = \begin{bmatrix} 0.0729 & 0.0001 \\ -4.7500 & 1.2300 \\ 1.5300 & 10.6300 \\ 0 & 0 \end{bmatrix} \quad C = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

The states are the sideslip angle, yaw rate, the roll rate, and the bank angle respectively. The desired closed-loop poles are $\lambda = -1.0, -1.50, -0.7 \pm 0.3i$. Since the model has $n = 4$, $m = 2$ and $r = 2$, then in order to realize the output-feedback gain F, an additional $q = 1$ is required with $\lambda_{ob} = -6.0$. Choosing

$$Z_q = [1 \quad 0]$$

the output feedback gain and the observer matrices are

$$F = \begin{bmatrix} -3.0105 & -0.1907 & 21.8886 \\ 2.1206 & -0.0413 & -12.8994 \end{bmatrix}$$

$$A_{ob} = -6.0, \quad B_{ob} = [0.7931 \quad -0.2180]$$

Table 1 shows the tabulated evaluations in the time- and s-domain for various design schemes. Indicated by J_{max}, design 4 has overshoot as high as 180°, which makes the design with arbitrary eigenvectors not acceptable for a real application. Design 5 has the advantage that it has the lowest order observer compared with designs 3 and 4, but it gives better performance. Design 5 has larger Ψ_{max} compared to designs 1 and 2, but design 5 requires less information about the plant's states.

Table 1. Performance Evaluation

	Ψ_{max} [a]	J_{max} [b]
1. State-feedback design	4.98	4.69
2. State-feedback & full-order observer	12.64	1.07
3. State-feedback & reduced-order (n–r) obs.	85.43	31.00
4. Output-feedback & reduced-order q obs.:		
Arbitrary eigenvector	14745	106980
5. Output-feedback & reduced-order q obs.:		
Properly assigned eigenvector	64.09	1.22

[a] $2\Delta F2_{max} = 0.1$
[b] Initial condition $x(0) = [1\ \ 1\ \ 1\ \ 1]^T$

3. Conclusion

Under the condition that the open-loop system is completely controllable and observable, the output feedback gain can be computed when $r + m > n$. When $r + m \leq n$, an observer must be incorporated in the feedback system to estimate the missing $q = n - m - r + 1$ outputs. In designing the closed-loop system and the observer, the eigenvectors must also be selected so that the angles between the eigenvectors are as close as possible to 90°. When the eigenvectors are properly selected, the closed-loop system poles and the observer poles are not easily shifted by any perturbances in the system's parameters. The selection of eigenvectors is quite significant in the design process because if they are not chosen properly, then in certain cases, the design is not an acceptable application.

4. Acknowledgement

5. References

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