CONFORMAL DEFORMATION OF A CONIC METRIC

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The following faculty members have examined the final copy of this thesis for form and content, and recommend that it be accepted in partial fulfillment of the requirement for the degree of Master of Science with a major in Mathematics.

Thalia Jeffres, Committee Chair

Zhiren Jin, Committee Member

Preethika Kumar, Committee Member

DEDICATION

To my husband, family, and friends. Without your support, this would not have been possible.

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ABSTRACT

The problem studied here focuses on a compact manifold M without boundary in which the Riemannian metric g is on $\Lambda = M - \{p_1, p_2, ..., p_k\}$. Near the p_i's, g has a particular type of singularity in which locally $M = (0, \delta)_x \times Y$ where Y is a Riemannian manifold with metric h. Calculation techniques involving Christoffel symbols, scalar curvature, and the Lapalacian of the manifold are used to reduce the Yamabe equation to a system of partial differential equations. After assuming that a function $u > 0$ satisfying the Yamabe equation exists, the most singular partial differential equation is solved using integration techniques to find necessary conditions on Y and h. Also studied in this paper are the conditions on Y and h for which M is already a manifold with constant scalar curvature.

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CHAPTER 1

INTRODUCTION

The main problem in this paper is a special case of the Yamabe Problem. Before beginning work on the main problem, this section will provide an overview of the Yamabe problem and the special case to be considered. The section will be completed with a collection of the theorems and definitions utilized throughout the paper.

1.1 The Yamabe Problem

As stated in [6] Riemannian differential geometry was developed to generalize compact surface theory. In surface theory conformal changes of a metric, multiplication of the metric by a positive function, are very important. For instance, it is known that every surface has a conformal metric of constant (Gaussian) curvature by consequence of the uniformization theorem of complex analysis.

It would be highly desirable for this concept to be applied to n-dimensional manifolds. However, this is not the case. Instead, it has been shown that for every compact smooth Riemannian manifold with dimension n $\geq$ 3 there exists a conformal change of the metric to one having constant scalar curvature.

According to [6], in 1960 Hidehiko Yamabe published a paper [11] containing a proof of his assertion which stated that for every compact smooth Riemannian manifold M with metric g and dimension $n \geq 3$ there exists a metric $\tilde{g}$ conformal to g with constant scalar curvature. This conformal change is achieved by $\tilde{g} = u^{\frac{4}{n-2}}g$. In order that $\tilde{g}$ also be a metric, the function u must be strictly positive. Additionally, the scalar curvature $S(g)$ of the manifold M with metric g is related to that of $\tilde{g}$ conformal to g having scalar curvature $S(\tilde{g})$ by the equation

$$\Delta_g u - \frac{n-2}{4(n-1)}S(g)u + \frac{n-2}{4(n-1)}S(\tilde{g})u^{\frac{n+2}{n-2}} = 0. \tag{1.1}$$

These are known as the Yamabe problem and Yamabe equation, respectively. The new scalar curvature $S(\tilde{g})$ is a prescribed constant. The standard differences are the signs of the constant, namely positive, negative, or zero. The historical account presented next follows [6].

The proof of Yamabe's problem was not entirely correct. Eight years later in 1968 Niel Trudinger [10] modified the assertion and proved it. He restricted the manifold M to be one such that given an $\epsilon > 0$ that depends on g and $S(g)$, if $S(g) < \epsilon$ then there exists a positive smooth solution u that satisfies equation (1.1). Thus the general Yamabe problem was still not proven.

In 1976 Thierry Aubin proved that if $dim(M) \geq 6$ and is not locally conformally flat, then there exists a positive smooth solution u that satisfies equation (1.1). Lee explained that this was proved by looking for test functions f that only depended on the local geometries of the manifold M.

In order to obtain results pertaining to M having dimensions three, four, or five, Schoen [8] considered global test functions. He found that if M has dimensions three, four, five, or if M is locally conformally flat then we can find a metric $\tilde{g}$ conformal to g having constant scalar curvature. This final result proved Yamabe's original problem.

1.2 Conformally Deforming a Conic Metric

In this paper, we are going to study a special case of the Yamabe problem. Consider a manifold M with the following defining properties. M is a compact topological manifold without boundary. The Riemannian metric g is on $\Lambda = M - \{p_1, p_2, ..., p_k\}$ where the p_i's are singular points in M. Near the p_i's, g has a particular type of singularity in which we can find local coordinates for which $M = (0, \delta)_x \times Y$. Here Y is an m-dimensional compact Riemannian manifold without boundary.

Topologically, $(0, \delta)_x \times Y$ is a cylinder. Inclusion of the boundary at $x = 0$ produces a manifold with boundary, $[0, \delta) \times Y$. For local coordinate calculations, the x be a function that vanishes simply on the boundary. Not that if the choice $Y = S^1$ this is just polar coordinates.

By the Gauss lemma, we can write the local expression for the metric g near the $p_i's$. The metric has a singularity each of the $p_i's$. In the coordinates described in the previous paragraph, if $h = \big(h_{ij}\big)$ denotes the Riemannian metric of Y, g takes the form

$$\big(g_{ij}\big) = \begin{pmatrix} 1 & 0 & \dots & 0 \\ 0 & x^2 h_{11}(y) & \dots & x^2 h_{m1}(y) \\ \vdots & \vdots & \ddots & \vdots \\ 0 & x^2 h_{1m}(y) & \dots & x^2 h_{mm}(y) \end{pmatrix}. \tag{1.2}$$

Here the entries denoted $h_{ij}(y)$ refer to i, jth entry of the local expression for the Riemannian metric of the cross-section Y. The notation used here implies that the metric h of Y only depends on the coordinates of Y. It does not depend on x.

Because of the block form of the matrix, its inverse, $\big(g^{ij}\big)$, is then just

$$\big(g^{ij}\big) = \begin{pmatrix} 1 & 0 & \dots & 0 \\ 0 & x^{-2} h^{11}(y) & \dots & x^{-2} h^{m1}(y) \\ \vdots & \vdots & \ddots & \vdots \\ 0 & x^{-2} h^{1m}(y) & \dots & x^{-2} h^{mm}(y) \end{pmatrix}. \tag{1.3}$$

In equation (1.3) $h^{ij}(y)$ denotes the i, jth entry of the inverse of the local expression of the metric $h_{ij}(y)$.

In future work, as noted in Section 3.4, we could make a variation to this problem by allowing h to depend on both x and the coordinates of Y.

Additionally, in a neighborhood around each p_i the manifold M is a cone over the base Y. It is also important to note here that this metric is geodesically incomplete. This means that there exists a point $p \in M$ such that the exponential map, exp_p, is undefined for some $v \in T_pM$. In other words, there exists a geodesic $\gamma(t)$ starting from p that is not defined for some values of the parameter $t \in \mathbb{R}$ [2]. It is because the missing point p_j is reached in finite time that upon introduction of polar coordinates it is natural to take the closure by considering locally, the manifold with boundary $[0, \delta)_x \times Y$.

In making a conformal change $\tilde{g} = u^{\frac{4}{n-2}} g$, we want $\tilde{g}$ to have similar properties. $\tilde{g}$ should be a Riemannian metric on the interior $\Lambda = M - \{p_1, p_2, ..., p_k\}$. In particular, $u > 0$. It should have the same type of singularity as g and u should be smooth on $[0, \delta)_x \times Y$ in the sense of a manifold with boundary [1]. Finally, at $x = 0$, $u^{\frac{4}{n-2}} h$ should restrict to a metric on the boundary Y. We will refer to the following definition throughout the paper.

Definition 1. *An admissible u is a positive function with the following properties:*

1. *$u > 0$ on the interior $\Lambda = M - \{p_1, p_2, ..., p_k\}$.*

2. *u is smooth on $[0, \delta)_x \times Y$ in the sense of a manifold with boundary.*

3. *At $x = 0$, $u^{\frac{4}{n-2}} h$ restricts to a metric on the boundary Y.*

We will see that this leads to fairly restrictive conditions. However, because of the incompleteness of the metric, it seems to be a natural starting point. Allowing u to lie in a more general function space is outside the scope of this paper.

We want to choose u within the admissible class in order that $S(\tilde{g})$ be identically equal to -1 on $\Lambda = M - \{p_1, p_2, ...p_k\}$. In this paper, we discover and obstruction to the solution of this problem. Assuming that $S(g) \leq 0$ we discover the following.

Main Theorem: (Theorem 4, Section 2.4) Suppose $S(g) \leq 0$ on Λ as defined above. If there exists an admissible function u (in the sense of definition 1) for which $S(\tilde{g}) \equiv -1$, then $S(Y, h) \equiv m(m-1)$.

Since g depends on h, it makes sense that these conditions will in turn depend on the metric h and the cross-section Y. In order to show this, we will be utilizing calculations of scalar curvature and the Laplacian on a manifold. Then we will to change equation (1.1) to a partial differential equation involving the most singular terms, the x^{-2} terms. The equations will then be solved using integration techniques. This problem is also solved in [5] using alternative techniques. We will now turn to a description of the methods used in this paper.

1.3 Theorems and Definitions

All of the work done is this paper is on Riemannian manifolds. We will define all of the operators and objects necessary before beginning the calculations. In doing so, we will start with the definition of a differentiable manifold and introduce the tangent space, metric, and connections that define a Riemannian manifold. Following this, we will develop the local expressions for the Christoffel symbols. After doing so, we will introduce Riemannian, Ricci, and scalar curvature. Then we will define the Laplacian, Kronecker delta, and Green's formula that will all be used to solve the partial differential equation.

The following definition can be found in [2] and [1].

Definition 2. *A differentiable manifold of dimension $m+1$ is a set M that is the image of injective mappings $x_\alpha : U_\alpha \subset \mathbb{R}^n \to M$ where U_α are open sets in $\mathbb{R}^n$. We call $\mathcal{U} = \{x_\alpha, x_\alpha(U_\alpha)\}$ a family of coordinate neighborhoods. These coordinate neighborhoods have the following properties:*

1. $\bigcup_\alpha x_\alpha(U_\alpha) = M$.
2. *for any* α, β *such that* $x_\alpha(U_\alpha) \bigcap x_\beta(U_\beta) = V \neq \emptyset$, $x_\alpha^{-1}(V)$ *and* $x_\beta^{-1}(V)$ *are open in* $\mathbb{R}^n$ *and the mappings* $x_\beta^{-1} \circ x_\alpha$ *are differentiable.*
3. *if there exists a coordinate neighborhood* $\{y_\gamma, y_\gamma(W_\gamma)\}$ *for which (2) holds for all* $\{x_\alpha, x_\alpha(U_\alpha)\} \subset \mathcal{U}$, *then* $\{y_\gamma, y_\gamma(W_\gamma)\} \subset \mathcal{U}$.

Another way to define a differentiable manifold is to say that the family of coordinate neighborhoods $\mathcal{U} = \{x_\alpha, x_\alpha(U_\alpha)\}$ that satisfy (1) and (2) of the definition is called a differentiable structure. A differentiable manifold is then a topological manifold with this differentiable structure.

The next definition is adapted from [1].

Definition 3. *A tangent space $T_p(M)$ of M at point p is the set of all mappings $X_p : C^\infty(p) \to \mathbb{R}$ satisfying for all $\alpha, \beta \in \mathbb{R}$ and $f, g \in C^\infty(p)$ the two conditions*

1. $X_p(\alpha f + \beta g) = \alpha(X_p f) + \beta(X_p g)$,

2. $X_p(fg) = X_p f(g(p)) + f(p)(X_p g)$.

A mapping $X_p \in T_p(M)$ is called a tangent vector of M at point p. This tangent vector can also be considered as a directional derivative on C^∞ functions at the point p.

Now that we have introduced the tangent space, we can define a Riemannian metric. The next definition is taken from [2].

Definition 4. *A Riemannian metric g on a differentiable manifold M is a correspondence which associates to each point p of M an inner product $<,>_p$ on the tangent space $T_p(M)$. The metric is symmetric, bilinear, and positive definite. It varies differentiably in the following sense: If $x : U \subset \mathbb{R} \to M$ is a system of coordinates around p with $x(x_1, x_2, ..., x_n) = q \in x(U)$ and $\frac{\partial}{\partial x_i}(q) = dx_q(0, ..., 1, ..., 0)$, then $< \frac{\partial}{\partial x_i}(q), \frac{\partial}{\partial x_j}(q) >_q = g_{ij}(x_1, x_2, ..., x_n)$ is a differentiable function on U.*

It is important to note that g is the Riemannian metric and its local representation in the coordinate system $x : U \subset \mathcal{R}^n \to M$ is denoted g_{ij}. Additionally, since inner products are symmetric, $< \frac{\partial}{\partial x_i}(q), \frac{\partial}{\partial x_j}(q) >_q = < \frac{\partial}{\partial x_j}(q), \frac{\partial}{\partial x_i}(q) >_q$, g_{ij} is also symmetric hence $g_{ij} = g_{ji}$. Furthermore because the metric g is positive definite it is invertible. The inverse of $\left(g_{ij}\right)$ is denoted $\left(g^{ij}\right)$.

The following definition can be found in [2] and [1].

Definition 5. *A differentiable manifold with a Riemannian metric is a called a Riemannian manifold.*

Now that we have developed the necessary background for Riemannian manifolds we will move forward to defining the special operators that we will be using. Definition 5 and its supporting material were adapted from [2].

Definition 6. *A connection ∇ on a differentiable manifold M is a mapping*

$$\nabla : \mathcal{X}(M) \times \mathcal{X}(M) \to \mathcal{X}(M)$$

which is denoted by $\nabla : (X, Y) \to \nabla_X Y$ and which satisfies the following properties:

1. $\nabla_{fX+gY} Z = f\nabla_X Z + g\nabla_Y Z.$

2. $\nabla_X (Y + Z) = \nabla_X Y + \nabla_X Z.$

3. $\nabla_X (fY) = f\nabla_X Y + X(f)Y,$

where $X, Y, Z \in \mathcal{X}(M)$, the set of all vector fields of class C^∞ on M, and $f, g \in C^\infty(M)$, real-valued functions of class C^∞ defined on M.

The connection is a directional derivative of a vector field in the direction of another vector field. $\nabla_X Y$ denotes the directional derivative of Y in the direction of X. A connection is itself independent of coordinates, but we can apply the operator to local coordinates. We can choose a system of coordinates $(x_1, x_2, ..., x_n)$ about a point p. Let $X = \sum_{i=1}^{m+1} x_i \frac{\partial}{\partial x_i}$ and $Y = \sum_{j=1}^{m+1} y_j \frac{\partial}{\partial x_j}$. Then

$$\begin{aligned}
\nabla_X Y &= \sum_{i=1}^{m+1} x_i \nabla_{\frac{\partial}{\partial x_i}} \left(\sum_{j=1}^{m+1} y_j \frac{\partial}{\partial x_j} \right) \\
&= \sum_{i,j=1}^{m+1} x_i y_j \nabla_{\frac{\partial}{\partial x_i}} \frac{\partial}{\partial x_j} + \sum_{i,j=1}^{m+1} x_i \frac{\partial}{\partial x_i}(y_j) \frac{\partial}{\partial x_j}.
\end{aligned}$$

Next write $\nabla_{\frac{\partial}{\partial x_i}} \frac{\partial}{\partial x_j} = \sum_{k=1}^{m+1} \Gamma_{ij}^k \frac{\partial}{\partial x_k}$. By making the substitution, we obtain

$$\begin{aligned}
\nabla_X Y &= \sum_{i,j=1}^{m+1} x_i y_j \sum_{k=1}^{m+1} \Gamma_{ij}^k + \sum_{i,j=1}^{m+1} x_i \frac{\partial}{\partial x_i}(y_j) \frac{\partial}{\partial x_j} \\
&= \sum_{i,j=1}^{m+1} x_i y_j \sum_{k=1}^{m+1} \Gamma_{ij}^k + \sum_{i,j=1}^{m+1} X(y_j) \frac{\partial}{\partial x_j} \\
&= \sum_{i,j,k=1}^{m+1} x_i y_j \Gamma_{ij}^k \frac{\partial}{\partial x_k} + X(y_k) \frac{\partial}{\partial x_k} \\
&= \sum_{k=1}^{m+1} \left(\sum_{i,j=1}^{m+1} x_i y_j \Gamma_{ij}^k + X(y_k) \right) \frac{\partial}{\partial x_k}.
\end{aligned}$$

The symbol Γ_{ij}^k will be very useful in our later calculations. The expression for the Γ_{ij}^k will be found after naming a few more definitions because it is directly related to a special type of connection, the Riemannian connection. Before introducing this, we will need to introduce one more operator. This definition is taken from [2].

Definition 7. *The Lie bracket of two vector fields X and Y, denoted by $[X, Y]$, is the vector field corresponding to the derivation $XY - YX$.*

The Lie bracket is used in defining the Riemannian connection which exists by Theorem 1. The Riemannian connection has the same properties as a general connection does. In addition to those properties, it has additional properties of symmetry and compatibility with the Riemannian metric. These are very significant because we will use them later when obtaining local expressions for curvature.

Theorem 1. *Given a Riemannian manifold M, there exists a unique connection ∇ on M that satisfies:*

1. $\nabla_X Y - \nabla_Y X = [X, Y]$ *(symmetric),*

2. $X < Y, Z >=< \nabla_X Y, Z > + < Y, \nabla_X Z >$ *(compatible with the Riemannian metric),*

where $X, Y, Z \in \mathcal{X}(M)$. This connection is the Riemannian (or Levi-Civita) connection on M.

This theorem and its proof can be found in [2].

The symmetry property of the Riemannian connection is applied to local coordinates as follows. Choose a system of coordinates $(U, x(U))$ where $X = \frac{\partial}{\partial x_i}$ and $Y = \frac{\partial}{\partial x_j}$. Then

$$\nabla_X Y - \nabla_Y X = \nabla_{\frac{\partial}{\partial x_i}} \frac{\partial}{\partial x_j} - \nabla_{\frac{\partial}{\partial x_j}} \frac{\partial}{\partial x_i} = \left[\frac{\partial}{\partial x_i}, \frac{\partial}{\partial x_j} \right] = 0.$$

Since $\nabla_{\frac{\partial}{\partial x_i}} \frac{\partial}{\partial x_j} = \sum_{k=1}^{m+1} \Gamma_{ij}^k \frac{\partial}{\partial x_k}$, we obtain

$$\nabla_X Y - \nabla_Y X = \sum_{k=1}^{m+1} \Gamma_{ij}^k \frac{\partial}{\partial x_k} - \sum_{k=1}^{m+1} \Gamma_{ji}^k \frac{\partial}{\partial x_k} = 0.$$

It follows that $\Gamma^k_{ij} = \Gamma^k_{ji}$.

Additionally, we define Γ^k_{ij} by $\nabla_{x_i} X_j = \sum\limits_{k=1}^{m+1} \Gamma^k_{ij} X_k$. These are the coefficients of the Riemannian connection, and they are called the Christoffel symbols [2]. The Christoffel symbols are related to the local coordinate expression of the metric g_{ij}. They will be useful in later calculations. Thus we will derive that expression next. In order to do so, we will use the expression for the Riemannian connection found in the proof of Theorem 1 and a property of the Lie bracket $[X_j, X_k] = 0$.

$$
\begin{aligned}
< X_n, \sum_{l=1}^{m+1} \Gamma^l_{ij} X_l > =& < X_n, \nabla_{X_i} X_j > \\
&= \frac{1}{2}\Big(X_j < X_i, X_n > + X_i < X_n, X_j > - X_n < X_j, X_i > \\
&- < [X_j, X_n], X_i > - < [X_i, X_n], X_j > - < [X_j, X_i], X_n > \Big) \\
&= \frac{1}{2}\Big(X_j < X_i, X_n > + X_i < X_n, X_j > - X_n < X_j, X_i > \Big).
\end{aligned}
$$

Then we can write each inner product as as the local expression for the metric.

$$
\sum_{l=1}^{m+1} \Gamma^l_{ij} g_{nl} = \frac{1}{2}\left(\frac{\partial g_{nj}}{\partial x_i} + \frac{\partial g_{in}}{\partial x_j} - \frac{\partial g_{ji}}{\partial x_n} \right).
$$

Now by applying the inverse g^{nl} to both sides we obtain the local expression for the Christoffel symbols.

$$
\Gamma^k_{ij} = \frac{1}{2} \sum_{l=1}^{m+1} g^{kl} \left(\frac{\partial g_{lj}}{\partial x_i} + \frac{\partial g_{il}}{\partial x_j} - \frac{\partial g_{ji}}{\partial x_l} \right).
$$

The next mapping we will introduce is necessary in understanding the relationships between the curvatures that we will be using later. In definition 7, T_pM is the tangent space at point p and T_p^*M is the dual space of the tangent space at p. The next definition can be found in [2].

Definition 8. *Let $r, s \geq 0$ where $r, s \in \mathbb{Z}$ and not both equal to zero. A multilinear map $A : (T_p^*M)^r \times (T_pM)^s \to \mathbb{R}$ is called a tensor of type (r,s) on the manifold M. A tensor of type $(0,0)$ is an element of $\mathbb{R}$. The set of all such tensors is denoted $\mathcal{T}^r_s(M)$.*

Definition 9 and some of the supporting material can be found in [2].

Definition 9. *The curvature R of a Riemannian manifold M is a correspondence that associates to every pair $X, Y \in \mathcal{X}(M)$ a mapping $R(X,Y)Z : \mathcal{X}(M) \to \mathcal{X}(M)$ given by*

$$R(X,Y)Z = \nabla_Y \nabla_X Z - \nabla_X \nabla_Y Z + \nabla_{[X,Y]} Z, \qquad Z \in \mathcal{X}(M),$$

where ∇ is the Riemannian connection of M.

We will use this definition to develop an expression using local coordinates. Choose a coordinate neighborhood $(x, x(U))$ and let $X_i = \frac{\partial}{\partial x_i}$. Then set

$$R(X_i, X_k)X_j = \sum_{l=1}^{m+1} R^l_{ikj} X_l.$$

The R^l_{ikj} are the components of the curvature R in the coordinate neighborhood $(x, x(U))$. Next we will find the expression for R^l_{ikj} in terms of the Christoffel symbols.

$$\begin{aligned} R(X_i, X_k)X_j &= \nabla_{X_k} \nabla_{X_i} X_j - \nabla_{X_i} \nabla_{X_k} X_j + \nabla_{[X_i, X_k]} X_j \\ &= \nabla_{X_k} \left(\sum_{l=1}^{m+1} \Gamma^l_{ij} X_l \right) - \nabla_{X_i} \left(\sum_{l=1}^{m+1} \Gamma^l_{kj} X_l \right) + \nabla_0 X_j \end{aligned}$$

Then we obtain

$$R^l_{ikj} = \sum_{l=1}^{m+1} \Gamma^l_{ij} \Gamma^a_{kl} - \sum_{l=1}^{m+1} \Gamma^l_{kj} \Gamma^a_{il} + \frac{\partial \Gamma^a_{ij}}{\partial x_k} - \frac{\partial \Gamma^a_{kj}}{\partial x_i}.$$

The R^l_{ikj} is referred to as the curvature tensor. More specifically, it is a type $(1,3)$ tensor. Next we will use the previous results to develop a local expression for a special type of curvature called scalar curvature. Before we can do this, we will need to introduce another operator and more global definitions. The definition used here can be found in [7], and the notation used in that text appears to be unique.

Definition 10. *A contraction* C_s^r *is defined by the mapping*

$$C_s^r : \mathcal{T}_s^r(M) \rightarrow \mathcal{T}_{s-1}^{r-1}(M)$$

where $\mathcal{T}_s^r$ *is a tensor of type* (r, s)*.*

The indices on the contraction C_s^r denote which one forms and vector fields the contraction is being applied to. Contraction mappings always contract in pairs thus r and s must be greater than or equal to 1 for a contraction to be defined. The contraction is done by equating the indices of the r and s one forms and vector fields and summing over the common index. The contraction can also be interpreted as the trace of a mapping. We will be using the contraction technique when defining the Ricci curvature.

Definition 11. *The Ricci Curvature,* $R(X, Y)$*, of a Riemannian manifold is defined by the mapping*

$$R(X, Y) = C_2^1(R(X, V)Y)$$

where $R(X, V)Y$ *is the Riemannian curvature.*

Note here that since we have contracted the curvature tensor of type (1,3), the Ricci curvature tensor is of type (0,2).

As before, this will be most useful in our application if we have a local coordinate expression. Again choose a coordinate neighborhood $(x, x(U))$ and let $X = \frac{\partial}{\partial x_i}$, $V = \frac{\partial}{\partial x_a}$, and $Y = \frac{\partial}{\partial x_j}$. Then

$$\begin{aligned} R\Big(\frac{\partial}{\partial x_i}, \frac{\partial}{\partial x_j}\Big) &= C_2^1\bigg(R\Big(\frac{\partial}{\partial x_i}, \frac{\partial}{\partial x_k}\Big)\frac{\partial}{\partial x_j}\bigg) \\ &= C_2^1\bigg(\sum_{l=1}^{m+1} R_{ikj}^l X_l\bigg) \\ &= \sum_{a=1}^{m+1} R_{iaj}^a. \end{aligned}$$

Definition 12. *The scalar curvature S(M, g) for a Riemannian manifold* (M, g) *is the trace of the Ricci curvature.*

In local coordinates, the scalar curvature is as follows

$$S(M,g) = \sum_{i,j=1}^{m+1} \sum_{a=1}^{m+1} g^{ij} R^{a}_{iaj}.$$

The scalar curvature is a tensor of type (0,0). Recall from the definition of a tensor that this type of tensor is a function, thus it maps to a real number.

The Yamabe equation also depends on the Laplacian of the function u. Definitions 13 and 14 lay the foundation for developing a local expression for the Laplacian on a manifold M. These two definitions along with definition 15 are adapted from [4] and [9].

Definition 13. *The gradient of a function, grad u, on a Riemannian manifold M is the unique vector field such that*

$$< grad\, u, X >= du(X) = Xu$$

for all vector fields X on M.

Choosing a system of coordinates $(x_1, ..., x_{m+1})$ on M where $X = \sum_{i=1}^{m+1} \frac{\partial}{\partial x_i}$ we obtain an expression for $grad\, u$ in local coordinates

$$grad\, u = \sum_{i=1}^{m+1} \left(\sum_{j=1}^{m+1} g^{ij} \frac{\partial u}{\partial x_j} \right) \frac{\partial}{\partial x_i}. \tag{1.4}$$

Definition 14. *The divergence of a vector field X on a Riemannian manifold M with metric g is given by*

$$div\, X = trDX$$

where D is the covariant derivative.

Now using the coordinate system $(x_1, ..., x_{m+1})$ the divergence of $X = \sum_{i=1}^{m+1} y_i \frac{\partial}{\partial x_i}$ is given by

$$div\, X = \sum_{i=1}^{m+1} \left(\frac{\partial y_i}{\partial x_i} + \sum_{j=1}^{m+1} y_j \Gamma^{i}_{ji} \right). \tag{1.5}$$

Definition 15. *The Laplacian Δ_g of a function u on a manifold M with metric g is given by*

$$\Delta_g u = div(grad\, u).$$

In order to obtain an expression for the Laplacian in local coordinates, we apply equations (1.4) and (1.5)

$$\Delta_g u = \sum_{i,j=1}^{m+1} g^{ij} \frac{\partial^2 u}{\partial w_i \partial w_j} - \sum_{i,j,k=1}^{m+1} g^{ij} \Gamma_{ij}^k \frac{\partial u}{\partial w_k}.$$

Definition 16. *The Kronecker delta is a function of two integers i and j that is defined by*

$$\delta_i^j = \begin{cases} 1 & i = j \\ 0 & i \neq j. \end{cases}$$

In particular $\sum\limits_{i,j=1}^{n} g^{ij} g_{ij} = \sum\limits_{i,j=1}^{n} g^{ij} g_{ji} = \delta_i^i$. The product of a metric and it's inverse will show up in a few of our calculations in chapter 2. The Kronecker delta will be useful in simplifying these situations. The next definition can be found in [3].

Theorem 2. *(Green's formula). Let $u \in C^2(M)$. Then*

$$\int_M \Delta_g u \, d\mu = \int_{\partial M} < \nabla u, \vec{v} > d\sigma$$

where $d\mu$ is a volume form, $d\sigma$ is the induced volume form on the boundary of M, and ∂M denotes the boundary of M.

CHAPTER 2

SOLVING THE PROBLEM

Within all further calculations, we will associate x with the coordinate w_1 and $y_1, ...y_m$ with the coordinates $w_2, ...w_{m+1}$.

2.1 Initial Calculations

We will begin by considering the Yamabe equation. Note that since we are using coordinates $w_1, w_2, ..., w_{m+1}$, the dimension of the manifold is $m+1$. Thus the constants in the Yamabe equation will differ slightly from our original equation.

$$\Delta_g u - \frac{m-1}{4(m)} S(g)u + \frac{m-1}{4(m)} S(\tilde{g})u^{\frac{m+3}{m+1}} = 0. \tag{2.1}$$

Likewise, the exponent of u changes in $\tilde{g} = u^{\frac{4}{m-1}} g$.

Because the metric of the manifold M depends on the metric of the manifold Y, the scalar curvature of M, $S(M, g)$, will depend on the scalar curvature of the manifold Y, $S(Y, h)$. Therefore, in this section we will be using calculation techniques to find necessary conditions on $S(M, g)$ and $S(Y, h)$ for which there exists a function $u > 0$ that are solutions to the Yamabe equation.

Since the Yamabe equation involves the scalar curvature of the manifold and also the Laplacian of the function u, we will calculate the Christoffel symbols and scalar curvature followed by the Laplacian of u. Again, because the metric of the manifold M depends on the metric of the cross-section Y, we expect to see that the Christoffel symbols, scalar curvature and the Laplacian of u on M will also depend on the Christoffel symbols, scalar curvature, and Laplacian of u on Y. As we perform the necessary calculations, we will look for these relationships.

2.1.1 The Christoffel Symbols

Recall the general formula for the Christoffel symbol given by definition 7. Since we are associating x and $y_1, ..., y_m$ with $w_1, ..., w_{m+1}$, we will rewrite the formula as

$$\Gamma^k_{ij} = \frac{1}{2}\sum_{l=1}^{m+1} g^{kl}\left(\frac{\partial g_{il}}{\partial w_j} + \frac{\partial g_{jl}}{\partial w_i} - \frac{\partial g_{ij}}{\partial w_l}\right).$$

Additionally, because x is associated with the coordinate w_1 and $y_1, ...y_m$ on Y are paired with $w_2, ...w_{m+1}$, it will be useful to calculate the Christoffel symbols by selecting indices that fall into two categories. The first type are those that are identically equal to one and the others that are strictly greater than one. By doing this, we will have calculated all the general Christoffel symbols that depend on w_1 and $w_2, ..., w_{m+1}$.

First consider, Γ^1_{11}. We will begin by summing out the term in which $l = 1$. In the first group of parentheses in the second line, notice that all of the partial derivatives are applied to g_{11}. By equation (1.2), $g_{11} = 1$. Thus the term in which $l = 1$ is equal to zero. Next, consider the summation in the second line. Every term in the summation will be multiplied by g^{1l} for which $l > 1$. We see in equation (1.3) $g^{1l} = 0$ for all $l > 1$. Therefore $\Gamma^1_{11} = 0$.

$$\begin{aligned}
\Gamma^1_{11} &= \frac{1}{2}\sum_{l=1}^{m+1} g^{1l}\left(\frac{\partial g_{1l}}{\partial w_1} + \frac{\partial g_{1l}}{\partial w_1} - \frac{\partial g_{11}}{\partial w_l}\right)\\
&= \frac{1}{2}g^{11}\left(\frac{\partial g_{11}}{\partial w_1} + \frac{\partial g_{11}}{\partial w_1} - \frac{\partial g_{11}}{\partial w_1}\right) + \frac{1}{2}\sum_{l=2}^{m+1} g^{1l}\left(\frac{\partial g_{1l}}{\partial w_1} + \frac{\partial g_{1l}}{\partial w_1} - \frac{\partial g_{11}}{\partial w_l}\right)\\
&= \frac{1}{2}g^{11}\left(0 + 0 - 0\right) + \frac{1}{2}\sum_{l=2}^{m+1} 0\left(\frac{\partial g_{1l}}{\partial w_1} + \frac{\partial g_{1l}}{\partial w_1} - \frac{\partial g_{11}}{\partial w_l}\right)\\
&= 0.
\end{aligned} \tag{2.2}$$

Now consider Γ^k_{11} and Γ^1_{1j} where $k > 1$ and $j > 1$. As before, we will expand out the $l = 1$ term. Similar situations will occur in which the partial derivatives equal zero and a factor of $g^{1l} = 0$ appear. Just as in equation (2.1), both $\Gamma^k_{11} = 0$ and $\Gamma^1_{1j} = 0$ where $k > 1$

and $j > 1$.

$$\begin{aligned}
\Gamma^k_{11} &= \frac{1}{2}\sum_{l=1}^{m+1} g^{kl}\left(\frac{\partial g_{1l}}{\partial w_1} + \frac{\partial g_{1l}}{\partial w_1} - \frac{\partial g_{11}}{\partial w_l}\right) \\
&= \frac{1}{2}g^{k1}\left(\frac{\partial g_{11}}{\partial w_1} + \frac{\partial g_{11}}{\partial w_1} - \frac{\partial g_{11}}{\partial w_1}\right) + \frac{1}{2}\sum_{l=2}^{m+1} g^{kl}\left(\frac{\partial g_{1l}}{\partial w_1} + \frac{\partial g_{1l}}{\partial w_1} - \frac{\partial g_{11}}{\partial w_l}\right) \\
&= \frac{1}{2}(0)\left(\frac{\partial g_{11}}{\partial w_1} + \frac{\partial g_{11}}{\partial w_1} - \frac{\partial g_{11}}{\partial w_1}\right) + \frac{1}{2}\sum_{l=2}^{m+1} g^{kl}\Big(0 + 0 - 0\Big) \\
&= 0 \qquad (2.3)
\end{aligned}$$

$$\begin{aligned}
\Gamma^1_{1j} &= \frac{1}{2}\sum_{l=1}^{m+1} g^{1l}\left(\frac{\partial g_{1l}}{\partial w_j} + \frac{\partial g_{jl}}{\partial w_1} - \frac{\partial g_{1j}}{\partial w_l}\right) \\
&= \frac{1}{2}g^{11}\left(\frac{\partial g_{11}}{\partial w_j} + \frac{\partial g_{j1}}{\partial w_1} - \frac{\partial g_{1j}}{\partial w_1}\right) + \frac{1}{2}\sum_{l=2}^{m+1} g^{1l}\left(\frac{\partial g_{1l}}{\partial w_j} + \frac{\partial g_{jl}}{\partial w_1} - \frac{\partial g_{1j}}{\partial w_l}\right) \\
&= \frac{1}{2}g^{11}\Big(0 + 0 - 0\Big) + \frac{1}{2}\sum_{l=2}^{m+1} 0\left(\frac{\partial g_{1l}}{\partial w_j} + \frac{\partial g_{jl}}{\partial w_1} - \frac{\partial g_{1j}}{\partial w_l}\right) \\
&= 0 \qquad (2.4)
\end{aligned}$$

Now, consider Γ^1_{ij} where $i > 1$ and $j > 1$. Expanding the first term of the series we see a g^{11} term. From equation (1.3) $g^{11} = 1$. In addition g_{i1} and g_{j1} appear. Recall, these are both equal to zero by equation (1.2). Thus their partial derivatives are also equal to zero. Finally, the only expression remaining in the first term is $\frac{\partial g_{ij}}{\partial w_l}$. By equation (1.2), $g_{ij} = x^2 h_{ij}(y)$. After taking the partial derivative with respect to w_1 and simplifying we obtain our first nonzero term.

Before we are finished, we must look at the leftover sum. We see that it has a factor of g^{1l} where $l > 1$, which is equal to zero. Therefore the sum over l from 2 to $m + 1$ sum is equal to zero. Thus we obtain our first nonzero Christoffel symbol $\Gamma^1_{ij} = -xh_{ij}(y)$ for $i > 1$

and $j > 1$.

$$\begin{aligned}
\Gamma^1_{ij} &= \frac{1}{2}\sum_{l=1}^{m+1} g^{1l}\left(\frac{\partial g_{il}}{\partial w_j} + \frac{\partial g_{jl}}{\partial w_i} - \frac{\partial g_{ij}}{\partial w_l}\right) \\
&= \frac{1}{2}g^{11}\left(\frac{\partial g_{i1}}{\partial w_j} + \frac{\partial g_{j1}}{\partial w_i} - \frac{\partial g_{ij}}{\partial w_1}\right) + \frac{1}{2}\sum_{l=2}^{m+1} g^{1l}\left(\frac{\partial g_{il}}{\partial w_j} + \frac{\partial g_{jl}}{\partial w_i} - \frac{\partial g_{ij}}{\partial w_l}\right) \\
&= \frac{1}{2}\left(0 + 0 - \frac{\partial}{\partial w_1}\big(x^2 h_{ij}(y)\big)\right) + \frac{1}{2}\sum_{l=2}^{m+1} 0\left(\frac{\partial g_{il}}{\partial w_j} + \frac{\partial g_{jl}}{\partial w_i} - \frac{\partial g_{ij}}{\partial w_l}\right) \\
&= -\frac{1}{2}\left(\frac{\partial}{\partial w_1}\big(x^2 h_{ij}(y)\big)\right) + 0 \\
&= -\frac{1}{2}\big(2x h_{ij}(y)\big) \\
&= -x\, h_{ij}(y). \qquad (2.5)
\end{aligned}$$

Next, examine Γ^k_{1j} where $j > 1$ and $k > 1$. The same technique is used to expand this summation. Notice that in the first term there is a factor of g^{k1} where $k > 1$. By equation (1.3) $g^{k1} = 0$. Thus the first term is equal to zero. Additionally, two terms in the leftover sum are equal to zero; they are g_{1l} and g_{1j}. After simplifying the expression, it is reduced to taking the partial derivative of g_{jl} with respect to w_1 and multiplying the result by $g^{kl} = x^{-2}h^{kl}(y)$ by equation (1.3). After doing so, a product of $h^{kl}(y)$ and $h_{lj}(y)$ emerges. Since metrics are symmetric, $h_{jl}(y) = h_{lj}(y)$. We make the substitution and we see that their product is equal to the Kronecker delta δ^j_k. We finally obtain $\Gamma^k_{1j} = x^{-1}\delta^j_k$ where $j > 1$

and $k > 1$.

$$\begin{aligned}
\Gamma^k_{1j} &= \frac{1}{2}\sum_{l=1}^{m+1} g^{kl}\left(\frac{\partial g_{1l}}{\partial w_j} + \frac{\partial g_{jl}}{\partial w_1} - \frac{\partial g_{1j}}{\partial w_l}\right) \\
&= \frac{1}{2}g^{k1}\left(\frac{\partial g_{11}}{\partial w_j} + \frac{\partial g_{j1}}{\partial w_1} - \frac{\partial g_{1j}}{\partial w_1}\right) + \frac{1}{2}\sum_{l=2}^{m+1} g^{kl}\left(\frac{\partial g_{1l}}{\partial w_j} + \frac{\partial g_{jl}}{\partial w_1} - \frac{\partial g_{1j}}{\partial w_l}\right) \\
&= \frac{1}{2}(0)\left(\frac{\partial g_{11}}{\partial w_j} + \frac{\partial g_{j1}}{\partial w_1} - \frac{\partial g_{1j}}{\partial w_1}\right) + \frac{1}{2}\sum_{l=1}^{m+1} x^{-2}h^{kl}(y)\left(0 + \frac{\partial}{\partial w_1}\big(x^2 h_{jl}(y)\big) - 0\right) \\
&= 0 + \frac{1}{2}\sum_{l=1}^{m+1} x^{-2}h^{kl}(y)\big(2xh_{jl}(y)\big) \\
&= \sum_{l=1}^{m+1} x^{-1}h^{kl}(y)h_{jl}(y) \\
&= \sum_{l=1}^{m+1} x^{-1}h^{kl}(y)h_{lj}(y) \\
&= x^{-1}\delta^j_k. && (2.6)
\end{aligned}$$

Finally, take a look at Γ^k_{ij} where $i > 1$, $j > 1$, and $k > 1$. After expanding out the first term of the sum we see a similar expression as the previous calculation. In this first term, we end up with another factor of g^{k1} where $k > 1$. Again $g^{k1} = 0$ by equation (1.3). Therefore this beginning term is equal to zero. Thus, the only portion left is the sum over l from two to $m+1$. Notice here that every indice in the sum is strictly greater than one. These indices correspond to $w_2 ... w_{m+1}$, the coordinates of the cross-section Y. Therefore this summation is the Christoffel symbol of the cross-section Y. This will be denoted by ${}^Y\Gamma^k_{ij}$.

$$\begin{aligned}
\Gamma^k_{ij} &= \frac{1}{2}\sum_{l=1}^{m+1} g^{kl}\left(\frac{\partial g_{il}}{\partial w_j} + \frac{\partial g_{jl}}{\partial w_i} - \frac{\partial g_{ij}}{\partial w_l}\right) \\
&= \frac{1}{2}g^{k1}\left(\frac{\partial g_{i1}}{\partial w_j} + \frac{\partial g_{j1}}{\partial w_i} - \frac{\partial g_{ij}}{\partial w_1}\right) + \frac{1}{2}\sum_{l=2}^{m+1} g^{kl}\left(\frac{\partial g_{il}}{\partial w_j} + \frac{\partial g_{jl}}{\partial w_i} - \frac{\partial g_{ij}}{\partial w_l}\right) \\
&= \frac{1}{2}(0)g^{k1}\left(\frac{\partial g_{i1}}{\partial w_j} + \frac{\partial g_{j1}}{\partial w_i} - \frac{\partial g_{ij}}{\partial w_1}\right) + \frac{1}{2}\sum_{l=2}^{m+1} g^{kl}\left(\frac{\partial g_{il}}{\partial w_j} + \frac{\partial g_{jl}}{\partial w_i} - \frac{\partial g_{ij}}{\partial w_l}\right) \\
&= 0 + \frac{1}{2}\sum_{l=2}^{m+1} g^{kl}\left(\frac{\partial g_{il}}{\partial w_j} + \frac{\partial g_{jl}}{\partial w_i} - \frac{\partial g_{ij}}{\partial w_l}\right) \\
&= {}^Y\Gamma^k_{ij}. && (2.7)
\end{aligned}$$

As expected, we can see that at least one of the Christoffel symbols on M is dependent on a Christoffel symbol on Y.

2.1.2 The Scalar Curvature

Theorem 3. *Given a manifold $M = Y \times (0, \delta)_x$ with metric g where Y is a Riemannian manifold with metric h, the scalar curvature of M, $S(M, g)$, satisfies $S(M, g) = x^{-2}(S(Y, h) - m(m-1))$ where m is the dimension of Y.*

Proof. We will use the calculations of the Christoffel symbols from the previous section within this proof. They will be substituted into the formula for scalar curvature. The scalar curvature of an arbitrary manifold is given by definition 12.

It will be helpful to break up the formula into four terms because of the nature of the metric and the way in which the w_i's were associated with the variables x and y. Similar to calculating the Christoffel symbols we will be expanding out the terms in which the indices equal one. The first term will be that of which both i and j equal one. The second will have $i = 1$ and $j > 1$. Similarly the third term will have $i > 1$ and $j = 1$. The fourth and final term will be that of which $i > 1$ and $j > 1$. The formula is rewritten as

$$S(M, g) = \sum_{a=1}^{m+1} g^{11} R^a_{1a1} + \sum_{j=2}^{m+1} \sum_{a=1}^{m+1} g^{1j} R^a_{1aj} + \sum_{i=2}^{m+1} \sum_{a=1}^{m+1} g^{i1} R^a_{ia1} + \sum_{i,j=2}^{m+1} \sum_{a=1}^{m+1} g^{ij} R^a_{iaj}.$$

Consider first, the term $\sum_{a=1}^{m+1} g^{11} R^a_{1a1}$. We will utilize a similar strategy for simplifying this as we did for the Christoffel symbols. The first step is to expand out any terms that contain an index of one. Since i and j are already equal to one, the only sum that we need to expand in is the sum over a. This first term contains a factor of R^1_{111}. We will proceed by first rewriting this curvature term using the form depending on the Christoffel symbols.

First recall that $g^{11} = 1$ by equation (1.3). Then, notice that the two sums over b are equivalent and the partial derivatives are equivalent. Thus, their difference are equal to zero. Consequently, the first term in which $a = 1$, $i = 1$, and $j = 1$ is zero.

Next, consider the remaining summation $\sum_{a=2}^{m+1} g^{11} R^a_{1a1}$. Again we see a g^{11} term which is equal to one. In addition, recall $\Gamma^b_{11} = \Gamma^a_{11} = 0$ by equation (2.2). Also $\Gamma^a_{a1} = x^{-1}\delta^a_a$,

$\Gamma^b_{a1} = x^{-1}\delta^b_a$, and $\Gamma^a_{b1} = x^{-1}\delta^a_b$ by equation (2.5). These can then be substituted into the formula. After taking the partial derivative of $x^{-1}\delta^a_a$, multiplying, and simplifying we obtain an expression involving three Kronecker deltas in which their upper and lower indices are both a. Thus after summing over a we obtain a factor of m for each Kronecker delta. The final steps in the calculation are rearranging the terms using basic algebra. We find that $\sum_{a=1}^{m+1} g^{11}R^a_{1a1} = -x^{-2}(m)(m-1)$ where m is the dimension of Y.

$$\begin{aligned}
\sum_{a=1}^{m+1} g^{11}R^a_{1a1} &= g^{11}R^1_{111} + \sum_{a=2}^{m+1} g^{11}R^a_{1a1} \\
&= g^{11}\left(\sum_{b=1}^{m+1}\Gamma^b_{11}\Gamma^1_{b1} - \sum_{b=1}^{m+1}\Gamma^b_{11}\Gamma^1_{b1} + \frac{\partial\Gamma^1_{11}}{\partial w_1} - \frac{\partial\Gamma^1_{11}}{\partial w_1}\right) + \sum_{a=2}^{m+1} g^{11}R^a_{1a1} \\
&= 1(0+0) + \sum_{a=2}^{m+1} g^{11}R^a_{1a1} \\
&= \sum_{a=2}^{m+1} g^{11}\left(\sum_{b=1}^{m+1}\Gamma^b_{11}\Gamma^a_{ba} - \sum_{b=1}^{m+1}\Gamma^b_{a1}\Gamma^a_{b1} + \frac{\partial\Gamma^a_{11}}{\partial w_a} - \frac{\partial\Gamma^a_{a1}}{\partial w_1}\right) \\
&= \sum_{a=2}^{m+1} 1\left(0 - \sum_{b=1}^{m+1} x^{-1}\delta^b_a x^{-1}\delta^a_b + 0 - \frac{\partial}{\partial w_1}\left(x^{-1}\delta^a_a\right)\right) \\
&= \sum_{a=1}^{m+1}\left(-\sum_{b=2}^{m+1} x^{-1}\delta^b_a x^{-1}\delta^a_b - \frac{\partial}{\partial w_1}\left(x^{-1}\delta^a_a\right)\right) \\
&= \sum_{a=2}^{m+1}\left(-\sum_{b=2}^{m+1} x^{-2}\delta^b_a\delta^a_b - \left(-x^{-2}\delta^a_a\right)\right) \\
&= \sum_{a=2}^{m+1}\left(-x^{-2}\delta^a_a\delta^a_a + x^{-2}\delta^a_a\right) \\
&= -x^{-2}(m)(m) + x^{-2}(m) \\
&= -x^{-2}(m^2) + x^{-2}(m) \\
&= -x^{-2}(m^2 - m) \\
&= -x^{-2}(m)(m-1). \qquad (2.8)
\end{aligned}$$

Next, take a look at both the second and third terms of the scalar curvature formula $\sum_{j=2}^{m+1}\sum_{a=1}^{m+1} g^{1j}R^a_{1aj}$ and $\sum_{i=2}^{m+1}\sum_{a=1}^{m+1} g^{i1}R^a_{ia1}$. Notice that the terms contain a factor of g^{1j} and g^{i1}

respectively. By equation (1.3) $g^{1j} = g^{i1} = 0$. Therefore, both of these summations are equal to zero.

$$\sum_{j=2}^{m+1}\sum_{a=1}^{m+1} g^{1j} R^a_{1aj} = \sum_{j=2}^{m+1}\sum_{a=1}^{m+1} (0) R^a_{1aj} = 0 \tag{2.9}$$

$$\sum_{i=2}^{m+1}\sum_{a=1}^{m+1} g^{i1} R^a_{ia1} = \sum_{i=2}^{m+1}\sum_{a=1}^{m+1} (0) R^a_{ia1} = 0 \tag{2.10}$$

The final term in the formula, $\sum_{i,j=2}^{m+1}\sum_{a=1}^{m+1} g^{ij} R^a_{iaj}$, will similarly be expanded by summing out the first term where $a = 1$. Again, we will proceed by replacing the R^1_{j1j} term with its formula in terms of the Christoffel symbols. First note $g^{ij} = x^{-2}\, h^{ij}(y)$ for $i, j > 1$ by equation (1.3). Then recall from equation (2.3) that Γ^1_{b1} and Γ^1_{1j} are both equal to zero. By equations (2.5) and (2.4), we can also substitute $x^{-1}\delta^j_b$ for Γ^b_{1j}, $-x\, h_{bi}(y)$ for Γ^1_{bi}, and $-x\, h_{ij}(y)$ for Γ^1_{ij}. After taking the partial derivatives and summing over b notice the δ^j_j term. This is always equal to one, so we no longer need to write it in the next line. Upon further simplification, we end up with the difference of two equivalent values. Thus, the first half of the sum is equal to zero.

Considering the second half, notice that all of the indices range from two to $m + 1$. These indices correspond to the coordinates $w_2, ... w_{m+1}$, the coordinates of the cross-section Y. Also, note that this term is the formula for scalar curvature. Therefore, this term is the scalar curvature of the cross-section with metric h denoted by $S(Y, h)$.

$$
\begin{aligned}
\sum_{i,j=2}^{m+1}\sum_{a=1}^{m+1} g^{ij}R^a_{iaj} &= \sum_{i,j=2}^{m+1} g^{ij}R^1_{i1j} + \sum_{i,j,a=2}^{m+1} g^{ij}R^a_{iaj} \\
&= \sum_{i,j=2}^{m+1} x^{-2}h^{ij}(y)\left(\sum_{b=1}^{m+1}\Gamma^b_{ij}\Gamma^1_{b1} - \sum_{b=1}^{m+1}\Gamma^b_{1j}\Gamma^1_{bi} + \frac{\partial\Gamma^1_{ij}}{\partial w_1} - \frac{\partial\Gamma^1_{1j}}{\partial w_i}\right) + \sum_{i,j,a=2}^{m+1} g^{ij}R^a_{iaj} \\
&= \sum_{i,j=2}^{m+1} x^{-2}h^{ij}(y)\left(\sum_{b=1}^{m+1}\big((\Gamma^b_{ij})(0)\big) - \sum_{b=1}^{m+1}\Gamma^b_{1j}\Gamma^1_{bi} + \frac{\partial\Gamma^1_{ij}}{\partial w_i} - 0\right) + \sum_{i,j,a=2}^{m+1} g^{ij}R^a_{iaj} \\
&= \sum_{i,j=2}^{m+1} x^{-2}h^{ij}(y)\left(0 - \sum_{b=2}^{m+1}\Gamma^b_{1j}\Gamma^1_{bi} + \frac{\partial\Gamma^1_{ij}}{\partial w_i}\right) + \sum_{i,j,a=2}^{m+1} x^{-2}h^{ij}(y)R^a_{iaj} \\
&= \sum_{i,j=2}^{m+1} x^{-2}h^{ij}(y)\left(- \sum_{b=2}^{m+1}\Gamma^b_{1j}\Gamma^1_{bi} + \frac{\partial\Gamma^1_{ij}}{\partial w_i}\right) + x^{-2}\sum_{i,j,a=2}^{m+1} h^{ij}(y)R^a_{iaj} \\
&= \sum_{i,j=2}^{m+1} x^{-2}h^{ij}(y)\left(- \sum_{b=2}^{m+1} x^{-1}\delta^j_b\big(-xh_{bi}(y)\big) + \frac{\partial}{\partial w_1}\big(-xh_{ij}(y)\big)\right) + x^{-2}S(Y,h) \\
&= \sum_{i,j=2}^{m+1} x^{-2}h^{ij}(y)\left(\sum_{b=2}^{m+1} x^{-1}x\delta^j_b h_{bi}(y) - h_{ij}(y)\right) + x^{-2}S(Y,h) \\
&= \sum_{i,j=2}^{m+1} x^{-2}h^{ij}(y)\big(\delta^j_j h_{ij}(y) - h_{ij}(y)\big) + x^{-2}S(Y,h) \\
&= \sum_{i,j=2}^{m+1} x^{-2}h^{ij}(y)h_{ij}(y) - x^{-2}h^{ij}(y)h_{ji}(y) + x^{-2}S(Y,h) \\
&= \sum_{i=2}^{m+1} x^{-2}\delta^i_i - x^{-2}\delta^i_i + x^{-2}S(Y,h) \\
&= 0 + x^{-2}S(Y,h) \\
&= x^{-2}S(Y,h). \qquad (2.11)
\end{aligned}
$$

Now that we have calculated all of the pieces, we will next put them together to find the expression for the scalar curvature of the manifold.

$$\begin{aligned} S(M,g) &= \sum_{i,j,a=1}^{m+1} g^{ij} R^a_{iaj} \\ &= \sum_{a=1}^{m+1} g^{11} R^a_{1a1} + \sum_{j=2}^{m+1}\sum_{a=1}^{m+1} g^{1j} R^a_{1aj} + \sum_{i=2}^{m+1}\sum_{a=1}^{m+1} g^{i1} R^a_{ia1} + \sum_{i,j=2}^{m+1}\sum_{a=1}^{m+1} g^{ij} R^a_{iaj} \\ &= -x^{-2}(m)(m-1) + 0 + 0 + x^{-2} S(Y,h) \\ &= x^{-2}\big(S(Y,h) - (m)(m-1)\big). \end{aligned} \tag{2.12}$$

Therefore the scalar curvature of the manifold M is dependent on that of the cross-section Y and a dimensional constant, in particular, $S(M,g) = x^{-2}(S(Y,h) - m(m-1))$. □

This is what we had previously expected because of how the metric of M was defined relative to the metric of Y. I also makes sense that as $x \to 0$, $S(M,g) \to \infty$ since the manifold is locally a cone. In addition to the previous theorem, the following corollary arises directly from the calculations.

Corollary 1. *Given a manifold M where $M = Y \times (0,\delta)_x$ and Y is a Riemannian manifold of dimension m, there is only one way $S(M,g)$ can be constant and that is if $S(Y,h) \equiv (m)(m-1)$. Furthermore, if $S(M,g)$ is constant, it follows that $S(M,g) \equiv 0$*

Proof. It was shown in the previous theorem that $S(M,g) = x^{-2}(S(Y,h) - m(m-1))$. Since x is a variable, the only way in which $S(M,g)$ is a constant is if $S(Y,h) - m(m-1) \equiv 0$. □

Now a natural question to ask is, what if $S(g)$ is not already constant? Can the metric g still be deformed to a constant scalar curvature metric? What we will find is that the function u must have the same behavior in its most singular term. Before we can show this, we will need to do another calculation that is involved in the Yamabe equation, the Laplacian of the positive function u.

2.1.3 The Laplacian

Since we have the scalar curvature calculated, we can proceed to calculating the Laplacian of the function u with respect to the metric g. Recall that in general, the Laplacian is given by the formula

$$\Delta_g u = \sum_{i,j=1}^{m+1} g^{ij} \frac{\partial^2 u}{\partial w_i \partial w_j} - \sum_{i,j,k=1}^{m+1} g^{ij} \Gamma_{ij}^k \frac{\partial u}{\partial w_k}.$$

Notice here that the Laplacian depends on the local expression for the metric g_{ij}. Since g_{ij} in turn depends on the metric of the cross-section h_{ij}, we will expect to obtain an expression the depends on the cross section. Due to the fact that we have made this observation, we will calculate in a similar fashion as we did with the scalar curvature and Christoffel symbols. Each time we simplify expressions we will sum out any terms with an index of one. We will calculate by first summing and simplifying the first term, doing the same to the second term, and taking the difference at the end. Throughout the work, we will rewrite the terms having coordinates $w_1, w_2, ..., w_{m+1}$ to have coordinates x and $y_1, ..., y_m$.

We begin by first bringing out the term in which i and j are equal to one. Follow this by letting i equal one in the first sum, and let j run from two to $m+1$. Similarly, let j equal one in the second sum, and let i run from two to $m+1$. The term left over is then a sum in which both i and j run from two to $m+1$. The next step is simply to substitute x for w_1 and $y_1, ..., y_m$ for $w_2, ..., w_{m+1}$.

Recall from definition (1.2) that $g^{11} = 1$ and $g^{1j} = g^{i1} = 0$. This very quickly reduces the first term of the Laplacian to the second partial derivative of u with respect to x and the product of x^{-2} with the sum of the product of the inverse metric of the cross-section Y, and a mixed second partial derivative of u.

$$\begin{aligned}
\sum_{i,j=1}^{m+1} g^{ij}\frac{\partial^2 u}{\partial w_i \partial w_j} &= g^{11}\frac{\partial^2 u}{\partial w_1^2} + \sum_{j=2}^{m+1} g^{1j}\frac{\partial^2 u}{\partial w_1 \partial w_j} + \sum_{i=2}^{m+1} g^{i1}\frac{\partial^2 u}{\partial w_i \partial w_1} + \sum_{i,j=2}^{m+1} g^{ij}\frac{\partial^2 u}{\partial w_i \partial w_j} \\
&= g^{11}\frac{\partial^2 u}{\partial x^2} + \sum_{j=2}^{m} g^{1j}\frac{\partial^2 u}{\partial x \partial y_j} + \sum_{i=2}^{m} g^{i1}\frac{\partial^2 u}{\partial y_i \partial x} + \sum_{i,j=2}^{m} g^{ij}\frac{\partial^2 u}{\partial y_i \partial y_j} \\
&= \frac{\partial^2 u}{\partial x^2} + 0 + 0 + \sum_{i,j=2}^{m} x^{-2}h^{ij}(y)\frac{\partial^2 u}{\partial y_i \partial y_j} \\
&= \frac{\partial^2 u}{\partial x^2} + x^{-2}\sum_{i,j=2}^{m} h^{ij}(y)\frac{\partial^2 u}{\partial y_i \partial y_j}. \qquad (2.13)
\end{aligned}$$

Now looking at the second term we will expand the sum in the same way as before. The first term will be when $i = j = 1$ and k goes from one to $m+1$. The second term will be that of which $j = 1$, i runs from two to $m+1$, and k runs from one to $m+1$. In the third term we will let $i = 1$, j run from two to $m+1$, and k from one to $m+1$. The fourth term will have $k = 1$ and i and j going from two to $m+1$. In the final term all indices will be from two to $m+1$. As before, we will next substitute x for w_1 and $y_1, ..., y_m$ for $w_2, ..., w_{m+1}$.

Beginning with the first and second terms, recall from equations (2.1) and (2.2) that $\Gamma^1_{11} = 0$ and $\Gamma^k_{11} = 0$ respectively. Thus the first and second terms equal zero. Additionally, in the third, fourth, fifth, and sixth terms, we have $g^{i1} = g^{1j} = 0$ by definition (1.3). Hence these four terms are equal to zero. In the final two terms substitute $x^{-2}h^{ij}(y)$ for g^{ij}. Then replace $-x\, h_{ij}(y)$ for Γ^1_{ij} in term seven by equation (2.4). Notice in the last term all indices are two to m which again refers to all coordinates on the cross-section Y. Thus we can replace ${}^{Y}\Gamma^k_{ij}$ for the Γ^k_{ij} in the fifth term by equation (2.6).

$$\begin{aligned}
\sum_{i,j,k=1}^{m+1} g^{ij}\Gamma^k_{ij}\frac{\partial u}{\partial w_k} =& \sum_{k=1}^{m+1} g^{11}\Gamma^k_{11}\frac{\partial u}{\partial w_k} + \sum_{i=2}^{m+1}\sum_{k=1}^{m+1} g^{i1}\Gamma^k_{1i}\frac{\partial u}{\partial w_k} + \sum_{j=2}^{m+1}\sum_{k=1}^{m+1} g^{1j}\Gamma^k_{1j}\frac{\partial u}{\partial w_k} \\
&+ \sum_{i,j=2}^{m+1} g^{ij}\Gamma^1_{ij}\frac{\partial u}{\partial w_1} + \sum_{i,j,k=2}^{m+1} g^{ij}\Gamma^k_{ij}\frac{\partial u}{\partial w_k} \\
=& g^{11}\Gamma^1_{11}\frac{\partial u}{\partial x} + \sum_{k=2}^{m} g^{11}\Gamma^k_{11}\frac{\partial u}{\partial y_k} + \sum_{i=2}^{m} g^{i1}\Gamma^1_{i1}\frac{\partial u}{\partial x} + \sum_{i,k=2}^{m} g^{i1}\Gamma^k_{1i}\frac{\partial u}{\partial y_k} \\
&+ \sum_{j=2}^{m} g^{1j}\Gamma^1_{1j}\frac{\partial u}{\partial x} + \sum_{j,k=2}^{m} g^{1j}\Gamma^k_{1j}\frac{\partial u}{\partial y_k} + \sum_{i,j=2}^{m} g^{ij}\Gamma^1_{ij}\frac{\partial u}{\partial x} + \sum_{i,j,k=2}^{m} g^{ij}\Gamma^k_{ij}\frac{\partial u}{\partial y_k} \\
=& 0+0+0+0+0+0+\sum_{i,j=2}^{m} g^{ij}\Gamma^1_{ij}\frac{\partial u}{\partial y_1} + \sum_{i,j,k=2}^{m} g^{ij}\Gamma^k_{ij}\frac{\partial u}{\partial y_k} \\
=& \sum_{i,j=2}^{m} x^{-2}h^{ij}(y)(-xh_{ij}(y))\frac{\partial u}{\partial y_1} + \sum_{i,j,k=2}^{m} x^{-2}h^{ij}(y)^{Y}\Gamma^k_{ij}\frac{\partial u}{\partial y_k} \\
=& \sum_{i,j=2}^{m} x^{-1}\delta^i_i\frac{\partial u}{\partial x} + x^{-2}\sum_{i,j,k=2}^{m} h^{ij}(y)^{Y}\Gamma^k_{ij}\frac{\partial u}{\partial y_k} \\
=& -x^{-1}(m)\frac{\partial u}{\partial x} + x^{-2}\sum_{i,j,k=2}^{m} h^{ij}(y)^{Y}\Gamma^k_{ij}\frac{\partial u}{\partial y_k}. \qquad (2.14)
\end{aligned}$$

Finally taking the differences of the sums, we obtain an expression for the Laplacian. Note that the x^{-2} terms combine to be the Laplacian of the cross-section Y. We will denote this Laplacian by Δ_h. Consequently, the Laplacian with respect to the metric g of the function u depends on the Laplacian with respect to the metric h of the function u as we had expected.

$$
\begin{aligned}
\Delta_g u &= \sum_{i,j=1}^{m+1} g^{ij} \frac{\partial^2 u}{\partial w_i \partial w_j} - \sum_{i,j,k=1}^{m+1} g^{ij} \Gamma_{ij}^k \frac{\partial u}{\partial w_k} \\
&= \frac{\partial^2 u}{\partial x^2} + x^{-2} \sum_{i,j=2}^{m} h^{ij}(y) \frac{\partial^2 u}{\partial y_i \partial y_j} - \left(-x^{-1}(m) \frac{\partial}{\partial x} + x^{-2} \sum_{i,j,k=2}^{m} h^{ij}(y)^Y \Gamma_{ij}^k \frac{\partial u}{\partial y_k} \right) \\
&= \frac{\partial^2 u}{\partial x^2} + x^{-2} \sum_{i,j=2}^{m} h^{ij}(y) \frac{\partial^2 u}{\partial y_i \partial y_j} + x^{-1}(m) \frac{\partial}{\partial x} - x^{-2} \sum_{i,j,k=2}^{m} h^{ij}(y)^Y \Gamma_{ij}^k \frac{\partial u}{\partial y_k} \\
&= \frac{\partial^2 u}{\partial x^2} + x^{-1}(m) \frac{\partial u}{\partial x} + x^{-2} \sum_{i,j=2}^{m} h^{ij}(y) \frac{\partial^2 u}{\partial y_i \partial y_j} - \sum_{i,j,k=2}^{m} x^{-2} h^{ij}(y)^Y \Gamma_{ij}^k \frac{\partial u}{\partial y_k} \\
&= \frac{\partial^2 u}{\partial y^2} + x^{-1}(m) \frac{\partial u}{\partial x} + x^{-2} \left(\sum_{i,j=2}^{m} h^{ij}(y) \frac{\partial^2 u}{\partial y_i \partial y_j} - \sum_{i,j,k=2}^{m} h^{ij}(y)^Y \Gamma_{ij}^k \frac{\partial u}{\partial y_k} \right) \\
&= \frac{\partial^2 u}{\partial x^2} + x^{-1}(m) \frac{\partial u}{\partial x} + x^{-2} \Delta_h u. \qquad (2.15)
\end{aligned}
$$

It should also be noted here that this expression for the Laplacian is a generalization of the Laplacian in polar coordinates. The Laplacian in one dimension is given by $\Delta u = \frac{\partial^2 u}{\partial y^2}$; in polar coordinates it is given by $\Delta u = \frac{1}{x}\frac{\partial}{\partial x}\left(x \frac{\partial u}{\partial x}\right) + \frac{1}{x^2}\frac{\partial^2 u}{\partial y^2}$. Using these expressions for the Laplacian we can calculate as below. It is clear that the simplified version in the last line is the Laplacian for which m, the dimension of the cross-section, is equal to 1. Furthermore, in the x^{-2} term, we have the expression for the Laplacian in one dimension.

$$
\begin{aligned}
\Delta u &= \frac{1}{x}\frac{\partial}{\partial x}\left(x \frac{\partial u}{\partial x}\right) + \frac{1}{x^2}\frac{\partial^2 u}{\partial y^2} \\
&= \frac{1}{x} \cdot 1 \frac{\partial u}{\partial x} + 1 \frac{\partial^2 u}{\partial x^2} + \frac{1}{x^2}\frac{\partial^2 u}{\partial y^2} \\
&= \frac{1}{x}\frac{\partial u}{\partial x} + \frac{\partial^2 u}{\partial x^2} + \frac{1}{x^2}\frac{\partial^2 u}{\partial y^2} \\
&= \frac{\partial^2 u}{\partial x^2} + \frac{1}{x}\frac{\partial u}{\partial x} + \frac{1}{x^2}\frac{\partial^2 u}{\partial y^2} \\
&= \frac{\partial^2 u}{\partial x^2} + x^{-1}\frac{\partial u}{\partial x} + x^{-2}\frac{\partial^2 u}{\partial y^2}.
\end{aligned}
$$

2.2 A Few Eigenvalue Problems

Before solving the main problem, we will take a look at some standard eigenvalue problems. The techniques used in solving these standard problems will be directly applied in solving the main problem. These examples are simply used to illustrate the techniques. If a former understanding of techniques used in eigenvalue problems has been attained, the reader can skip this section.

2.2.1 Variation One

In this variation we will combine two similar problems. First consider a compact manifold M and a function u in which $u = 0$ on ∂M. Second, consider a compact manifold M without boundary. For the first case we will show that all eigenvalues, λ, are less than zero. In the second case, we will show that the eigenvalues are less than or equal to zero.

We will begin by writing the standard eigenvalue equation $\Delta u = \lambda u$. Next, we multiply both sides of the equation by the function u. Since the two expressions are equal, their integrals must also be equal. Thus we will take the integral of both sides over the domain M. Then we can bring λ out of the integral because it is a constant.

$$\Delta_g u = \lambda u$$
$$u\Delta_g u = \lambda u^2$$
$$\int_M u\Delta_g u \, \mathrm{d}\mu = \int_M \lambda u^2 \, \mathrm{d}\mu$$
$$\int_M u\Delta_g u \, \mathrm{d}\mu = \lambda \int_M u^2 \, \mathrm{d}\mu$$

Now consider the left hand side. Integration by parts asserts

$$\int_M u\Delta_g u \, \mathrm{d}\mu = \int_{\partial M} u < \nabla u, \vec{v} > \, \mathrm{d}\sigma - \int_M < \nabla u, \nabla u > \, \mathrm{d}\mu.$$

Then our integral becomes

$$\int_{\partial M} u < \nabla u, \vec{v} > \, \mathrm{d}\sigma - \int_M < \nabla u, \nabla u > \, \mathrm{d}\mu = \lambda \int_M u^2 \, \mathrm{d}\mu.$$

In the first case of this variation, since $u = 0$ on ∂M, the first term is identically equal to zero. For the second case, since $\partial M = \emptyset$ the first term is identically equal to zero. Additionally, note $< \nabla u, \nabla u >= ||\nabla u||^2$. Thus in either case we have

$$-\int_M ||\nabla u||^2 \, \mathrm{d}\mu = \lambda \int_M u^2 \, \mathrm{d}\mu.$$

Now if we consider the first case, $\int_M u^2 > 0$ since u cannot be identically equal to zero. Thus its integral must be greater than zero. It is also true that $||\nabla u||^2 \geq 0$. The only way in which $||\nabla u||^2 \equiv 0$ is if u is a constant function. But if u is constant, then $u \equiv 0$ by our initial boundary conditions. Thus, u cannot be a constant function. Therefore $||\nabla u||^2 > 0$, and it follows that its integral must be greater than zero. Consequently, $\lambda < 0$. Therefore given a manifold M and a function u in which $u = 0$ on ∂M. All eigenvalues, λ, are less than zero.

Considering the second case, we also know that $\int_M u^2 > 0$ since u cannot be identically equal to zero. Thus its integral must be greater than zero. It is also true that $||\nabla u||^2 \geq 0$. The only way in which $||\nabla u||^2 \equiv 0$ is if u is a constant function. In this case, u can be a constant function since we do not have the same condition on the boundary as the first case. We also have the possibility of which $||\nabla u||^2 > 0$. If this holds, it follows that $\int_M ||\nabla u||^2 > 0$; when this occurs, $\lambda < 0$. Therefore in the case in which we have a compact manifold M with $\partial M = \emptyset$, the eigenvalues, λ, are such that $\lambda \leq 0$.

Similarly, it can be shown for M, a manifold, and a function u in which $\nabla u \cdot \vec{v} = 0$, the Neumann boundary condition, the eigenvalues, λ, are less than or equal to zero. We will not show this variation here.

2.2.2 Variation Two

Now we will consider the problem $\Delta_g u = f(y)u$, where M is a smooth compact manifold without boundary. In this case, we want to find the conditions in which there are solutions u to the given equation. We will apply similar techniques as we did for the previous standard eigenvalue problems. The technique used here is the same used in the proof of the next theorem.

Since we cannot bring the $f(y)$ term out of the integral as we did with the eigenvalue λ before, we will not multiply both sides of the equation by the function u. Instead we will go directly to integrating both sides over the manifold M. Then, by Green's Theorem, $\int_M \Delta_g u d\mu = \int_{\partial M} < \nabla u, \vec{v} > d\sigma$. Finally, since M is a manifold without boundary, $\int_{\partial M} < \nabla u, \vec{v} > d\sigma = 0$.

$$\begin{aligned} \Delta_g u &= f(y)u \\ \int_M \Delta_g u d\mu &= \int_M f(y)u\, d\mu \\ \int_{\partial M} < \nabla u, \vec{v} > d\sigma &= \int_M f(y)u\, d\mu \\ 0 &= \int_M f(y)u\, d\mu \end{aligned}$$

Without further restrictions on u we cannot draw any further conclusions. However, this calculation will be very useful when solving the main problem.

2.3 A System of Partial Differential Equations

First recall the Yamabe equation

$$\Delta_g u - \frac{m-1}{4m} S(g)u + \frac{m-1}{4m} S(\tilde{g})u^{\frac{m+3}{m-1}} = 0.$$

Our goal is to find a function $u > 0$ that satisfies this equation for which $S(\tilde{g}) = K$, where $K = -1$. We will substitute in our earlier findings and let $S(\tilde{g})$ equal to the constant K. By equation (2.11) and equation (2.14) the Yamabe equation becomes

$$x^{-2}\Big[\Delta_h u - \frac{m-1}{4m}(S(Y,h) - m(m-1))u\Big] + x^{-1}\Big[m\frac{\partial u}{\partial x}\Big] + \frac{\partial^2 u}{\partial x^2} + \frac{m-1}{4m}Ku^{\frac{m+3}{m-1}} = 0. \quad (2.16)$$

If we assume such a u exists, then we will know the conditions on the cross-section Y and its metric h for which there is a metric conformal to our metric g that has constant scalar curvature. In order to find these conditions, we will need to simplify the equation.

To simplify the equation, set $f(y) = \frac{m-1}{4m}(S(Y,h) - m(m-1))$. Then f is a function of $y_1, y_2, ..., y_m$, the coordinates of the cross-section Y. In the main theorem we will be assuming that u is smooth up to the boundary $Y \times \{x = 0\}$. As a consequence of this, u will have a partial expansion in powers x^k, $k \geq 0$, $k \in \mathbb{Z}$ by [1]. Thus we can use Taylor's Theorem to write an expansion of the function u in the variable x. Thus $u = u(x,y) = u_0(y) + u_1(y)x + R_1(x,y)$. Since u is smooth up to the boundary, we must be able to expand u in the variable x. Here the u_i's are functions of y and $R_1(x,y)$ is the Taylor remainder of order x^2.

We will proceed by finding the most singular terms of the equation. These terms will give us the most information about the geometry of the cross-section Y with metric h. Beginning with the x^{-2} term, we will substitute in $f(y)$ for $\frac{m-1}{4m}(S(Y,h) - (m)(m-1))$ and $u = u(x,y) = u_0(y) + u_1(y)x + R_1(x,y)$ for u.

$$\begin{aligned} x^{-2}\Big[\Delta_h u(x,y) - f(y)u(x,y)\Big] =& x^{-2}\Big[\Delta_h\Big(u_0(y) + u_1(y)x + R_1(x,y)\Big) \\ & - f(y)\Big(u_0(y) + u_1(y)x + R_1(x,y)\Big)\Big] \end{aligned} \tag{2.17}$$

Notice in equation (2.16) that the only way in which a term will have a factor of x^{-2} is when the Laplacian is applied to $u_0(y)$ and the term that is the product $f(y)u_0(y)$. Thus, the only x^{-2} terms from this part of the equation will be $x^{-2}\Delta_h u_0(y)$ and $-x^{-2}f(y)u_0(y)$.

Using the same substitutions for the x^{-1} term of equation (2.15), we can see that there will not be any x^{-2} coming from this term. Hence, we have obtained all of our most singular terms from the equation.

Recall that a condition of the Yamabe equation is that the expanded terms must sum to zero. Thus we can now set the sum of our x^{-2} terms obtained from equation (2.17) equal to zero. This will be the partial differential equation that we will need to solve.

$$x^{-2}\Big[\Delta_h u_0(y) - f(y)u_0(y)\Big] = 0 \tag{2.18}$$

2.4 Obstruction to Existence of Solutions

Now that we have a partial differential equation, we will use equation (2.18) to find a necessary condition on the metric h and the scalar curvature of Y for such a u to exist. Earlier in the paper it was stated that we will only be considering functions u that are smooth up to the boundary $Y \times \{x = 0\}$. In particular, we want $u^{\frac{4}{m-1}}(h_{ij})$ to be a metric on $Y \times \{x = 0\}$. The following theorem is the main theorem of the paper.

Theorem 4. *Suppose $S(g) \leq 0$ on Λ as defined in Section 1.2. If there exists an admissible function u (in the sense of definition 1) for which $S(\tilde{g}) \equiv -1$, then $S(Y, h) \equiv m(m-1)$.*

Proof. First consider the first partial differential equation (2.18).

$$x^{-2}\Big[\Delta_h u_0(y) - f(y)u_0(y)\Big] = 0$$

If the left hand side is equal to zero, then it is clear that $\Delta_h u_0(y) - f(y)u_0(y) = 0$ since $x^{-2} \neq 0$. Also, we assumed that an admissible function u exists. Since it is strictly positive and the metrics g and $\tilde{g}$ must restrict to metrics on the boundary of M, the first term in its expansion, $u_0(y)$ must also be greater than zero. This is because u has to be positive even when $x = 0$. We will keep this in mind as we solve for $u_0(y)$.

Next, since we found that $\Delta_h u_0(y) - f(y)u_0(y) = 0$, we can write $\Delta_h u_0(y) = f(y)u_0(y)$ and integrate over M. Recall here that $d\mu$ is an m-form. Following this we will apply Green's Theorem to the left-hand side. Here our $d\mu$ becomes $d\sigma$, an $m-1$-form. Since $\partial M = \emptyset$, the integral on the left is equal to zero. Finally, we can substitute $S(Y, h) = m(m-1)$ for $f(y)$

to obtain equation (2.19).

$$\begin{aligned}
\Delta_h u_0(y) - f(y)u_0(y) &= 0 \\
\Delta_h u_0(y) &= f(y)u_0(y) \\
\int_M \Delta_h u_0(y) d\mu &= \int_M f(y)u_0(y)d\mu \\
\int_{\partial M} < \nabla u_0(y), \vec{v} > d\sigma &= \int_M f(y)u_0(y)d\mu \\
0 &= \int_M f(y)u_0(y)d\mu \\
0 &= \int_M \frac{m-1}{4m}\big(S(Y,h) - m(m-1)\big)u_0(y)d\mu \\
0 &= \frac{m-1}{4m}\int_M \big(S(Y,h) - m(m-1)\big)u_0(y)d\mu \\
0 &= \int_M \big(S(Y,h) - m(m-1)\big)u_0(y)d\mu \qquad (2.19)
\end{aligned}$$

Now, consider the final line in equation (2.19). Notice as well that this is the same calculation as performed in Variation 2 of the eigenvalue problems. Since we have conditions on $f(y)$ and u, we can make some conclusions. The only way in which this will integrate to zero is if $u_0(y) \equiv 0$, the product $\big(S(Y,h) - m(m-1)\big)u_0(y)$ changes sign in a way that makes the integral over regions of the domain sum to zero, or if $S(Y,h) - m(m-1) \equiv 0$.

Recall the assumption that there exists a function $u > 0$ and that g and g_0 must restrict to metrics on the boundary of M where $x = 0$. Remember that the expansion of the function u is given by $u = u(x,y) = u_0(y) + u_1(y)x + R_1(x,y)$. If u must be positive everywhere, even when $x = 0$, then the first term in its expansion, $u_0(y)$ must also be greater than zero everywhere. Thus $u_0(y)$ cannot be identically equal to zero.

Since $u_0(y)$ must be strictly positive, either $S(Y,h) - m(m-1)$ changes sign or it is identically equal to zero. We have previously shown that under these conditions that $S(M,g) = x^{-2}(S(Y,h) - m(m-1))$, and recall that we assumed $S(M,g) \leq 0$. Thus $S(Y,h) - m(m-1) \leq 0$. Hence $S(Y,h) - m(m-1)$ cannot change sign. Therefore the only way in which this can integrate to zero is if $S(Y,h) - m(m-1) \equiv 0$. This occurs when $S(Y,h) \equiv m(m-1)$ for all points $y \in Y$. □

CHAPTER 3

CONCLUSION

3.1 Final Remarks

After studying the special case of the Yamabe equation for which the manifold $M = Y \times (0, \delta)_x$ and Y is a Riemannian manifold of dimension m we obtain related results. The first, Theorem 3, states that if $S(Y, h) \equiv (m)(m-1)$, then M has constant scalar curvature. The second, Corollary 1, states that if there exists a function $u > 0$ that satisfies the Yamabe equation then $S(Y, h) \equiv (m)(m-1)$. Combining these two theorems we find that if a $\tilde{g}$ exists such that $S(M, \tilde{g})$ = -1, then the scalar curvature of Y is already a manifold with constant scalar curvature.

3.2 Future Work

There are still more results to be found for this problem. To find an additional necessary condition on the metric h and the scalar curvature of Y the next most-singular terms, the x^{-1} terms, could be used to set up an additional partial differential equation. It can even be taken further by expanding the function u to more terms. For instance $u(x, y) = u_0(y) + u_1(y)x + u_2(y)x^2 + R_2(x, y)$ could be used to obtain an additional partial differential equation to solve.

Another problem that could be solved in a similar way would be that of which the metric h of the cross-section Y depends on x and y. It should be possible to calculate the Christoffel symbols, the scalar curvature, and the Laplacian in the same manner as done in this paper. Then using an expansion of u, a partial differential equation could be derived for the most singular terms. These could be solved to find necessary conditions on h and $S(Y, h)$ for their to exists a $u > 0$ that satisfies the Yamabe equation.

In addition to these, we could also consider looking for a weaker condition on $S(M, g)$. Here we have assumed that $S(M, g) \leq 0$, but it may not be necessary. It has been shown that in some cases it is allowable for the scalar curvature to have a different sign in some areas. We could work on trying to find the weakest condition on the sign of the scalar curvature of the initial metric.

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