A STUDY OF PARTICULAR COORDINATE SYSTEMS

A Thesis by

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The following faculty members have examined the final copy of this thesis for form and content, and recommend that it be accepted in partial fulfillment of the requirement for the degree of Masters of Science with a major in Applied Mathematics.

Kirk Lancaster, Committee Chair

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DEDICATION

To my husband, parents, sister, and my wonderful family and friends

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ABSTRACT

This thesis is a study of the properties and relationships between isothermal, harmonic and characteristic coordinate systems. The proof of the existence of isothermal and characteristic coordinates on a manifold which is a graph is given using the Uniformization theorem. Equations of prescribed mean curvature are discussed and the relationship between equations of minimal surface type and mean curvature type are shown. It is also proven that the map from a domain parameterized by characteristic coordinates to the domain parameterized by isothermal coordinates is quasiconformal.

PREFACE

Let $\Omega \subset \mathbb{R}^2$ be a connected, open set and consider the prescribed mean curvature problem in a cylinder, which consists of finding a solution f of the equation

$$Nf = H(\cdot, f(\cdot)) \text{ in } \Omega, \tag{1}$$

which satisfies one of the following boundary conditions:

(i) $f = \phi$ (a.e.) on $\partial\Omega$,

(ii) $Tf \cdot \nu = \cos\gamma$ (a.e.) on $\partial\Omega$;

here $Tf = \frac{\nabla f}{\sqrt{1+|\nabla f|^2}}$, $Nf = \nabla \cdot Tf$, ν is the exterior unit normal on $\partial\Omega$, $H(x,t)$ is a weakly increasing function of t for each $x \in \Omega$, $\phi : \partial\Omega \to \mathbb{R}$ and $\gamma : \partial\Omega \to [0, \pi]$.

Starting with [23], Lancaster and his coauthors used isothermal parameterizations to investigate the existence and behavior of the radial limits of bounded solutions of (1) which satisfy the Dirichlet boundary condition (i). In [25] and [24], Lancaster and Siegel used isothermal parameterizations to investigate the existence and behavior of the radial limits of bounded solutions of (1) which satisfy the contact angle boundary condition (ii) (e.g. capillary surfaces.)

The goal of this thesis is to begin to create the infrastructure required to further investigate the behavior at corners of solutions of boundary value problems for equations of mean curvature type. The focus here is on the role of special types of local (and global) coordinates. Chapter 1 will consist of definitions and concepts from differential geometry. In chapter 2, we introduce isothermal coordinates followed by discussion of harmonic coordinates and then consider the relationships between isothermal and harmonic coordinates. In chapter 3, we will focus on characteristic coordinates. Chapter 4 will consist of looking at the structure conditions of mean curvature type equations. In chapter 5, we prove the global existence of isothermal and characteristic coordinates on a surface utilizing the Uniformization theorem. We will then prove that the map between a domain parameterized by characteristic coordinates and a domain parametrized by isothermal coordinates is quasiconformal. Chapter 6 discusses the applications of this thesis to Lancaster's work and provides a theorem and conjecture for further study.

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CHAPTER 1

DIFFERENTIAL GEOMETRY AND RIEMANNIAN MANIFOLDS

1.1 Notation

Here we provide a list of notation used throughout this thesis.

- $\mathbf{x} : U \subset \mathbb{R}^n \to M$ is a parametrization of the manifold M
- (u, v) represent isothermal coordinates
- (σ, ρ) represent characteristic coordinates
- Δ_g represents the Laplace-Beltrami Operator for a manifold M with metric g.
- Δ_o is the n-dimensional Laplace Operator in n-dimension Euclidean space
- If g is a Riemannian metric defined as a family of inner products, we will use $\langle w_1, w_2 \rangle_p$ to represent the inner product of w_1, w_2 on T_pM at p.
- If $w_1, w_2 \in T_pM \subset R^3$, then $\langle w_1, w_2 \rangle_o$ will be used to represent the inner product of w_1, w_2 as vectors in R^3.

1.2 Riemannian Manifolds

We will begin with a discussion on differential geometry mostly using [6] and [13] as references. In n-dimensions, we define a **manifold of class** C^k to be a set M and a family of injective mappings $\mathbf{x} : U_\alpha \subset \mathbb{R}^n \to M$ of open sets U_α of $\mathbb{R}^n$ into M such that

(a) $\bigcup_\alpha \mathbf{x}_\alpha(U_\alpha) = M$

(b) for any pair α and β with $\mathbf{x}_\alpha(U_\alpha) \cap \mathbf{x}_\beta(U_\beta) = W \neq \emptyset$, the sets $\mathbf{x}_\alpha^{-1}(W)$ and $\mathbf{x}_\beta^{-1}(W)$ are open sets in $\mathbb{R}^n$ and the mappings $\mathbf{x}_\beta^{-1} \circ \mathbf{x}_\alpha$ are C^k.

(c) The family $\{U_\alpha, \mathbf{x}_\alpha\}$ is maximal relative to (a) and (b).

The mapping $\mathbf{x}_\alpha$ with $p \in \mathbf{x}_\alpha(U_\alpha)$ is called a **parameterization** of M at p and $\mathbf{x}_\alpha(U_\alpha)$ is called a **coordinate neighborhood** or coordinate system at p.

Let the smooth mapping $\gamma : (-\epsilon, \epsilon) \to M$ be a **curve** with $\gamma(0) = p$ with $p \in M$. For a parameterization $\mathbf{x}$ at p, the **tangent vector** to the curve is $w = \frac{\partial}{\partial t}(\mathbf{x}^{-1} \circ \gamma)(0)$. The set of all tangent vectors at p is called the **tangent space**, denoted T_pM.

The choice of parameterization determines a basis $\{\frac{\partial \mathbf{x}}{\partial u_1}, ..., \frac{\partial \mathbf{x}}{\partial u_n}\}$ of T_pM. In two dimensions we find it convenient to write $\mathbf{x}_u = \frac{\partial \mathbf{x}}{\partial u}$ and $\mathbf{x}_v = \frac{\partial \mathbf{x}}{\partial v}$ to represent the basis. If w is a tangent vector at p and f a C^k function defined near p, then differentiating f along any curve gives the **directional derivative** along w to be $df_p(w) : T_pM \to \mathbb{R}$ where

$$df_p(w) = \frac{d}{dt}(f \circ \gamma)(t)$$

if $w = \frac{\partial}{\partial t}(\mathbf{x}^{-1} \circ \gamma)(0)$. Note that this is independent of the chosen path γ.

Returning to the case of n-dimensions, consider a parameterization $\mathbf{x} : U \to M$ and the mapping $a_i : U \to \mathbb{R}$. A **vector field** X on M is a relation that associates to each point $p \in M$ a vector $w(p) \in T_pM$. We write

$$X(p) = \sum_{i=1}^{n} a_i(p) \frac{\partial}{\partial x_i}$$

where $\{\frac{\partial}{\partial x_1}, ..., \frac{\partial}{\partial x_n}\}$ is a local basis for the tangent space at p.

A C^k manifold M equipped with a Riemannian metric g is called a Riemannian manifold, which we denote (M, g). The metric $g : T_pM \times T_pM \to \mathbb{R}$ is a family of inner products $\langle , \rangle_p$ on the tangent space which vary C^k smoothly from point to point. In other words $< w, v >_{p(t)}$ is a C^k function for all $w, v \in \mathbb{R}^n$ where $\{p(t) : 0 \leq t \leq 1\}$ is a smooth curve.

Now consider a 2-dimensional manifold $M \subset \mathbb{R}^3$ (i.e. a surface). The natural inner product of $\mathbb{R}^3$ induces on each tangent plane T_pM an inner product; if $w_1, w_2 \in T_pM \subset \mathbb{R}^3$, then $\langle w_1, w_2 \rangle_o$ is equal to the inner product of w_1 and w_2 as vectors in $\mathbb{R}^3$ ([12]). The **first fundamental form** of a surface in $\mathbb{R}^3$, expressed in the basis $\{\mathbf{x}_u, \mathbf{x}_v\}$ associated to a parameterization $\mathbf{x}(u,v)$ at p, is

$$g_{11}du^2 + 2g_{12}dudv + g_{22}dv^2 \tag{1.1}$$

where

$$g_{11} = \langle \mathbf{x}_u, \mathbf{x}_u \rangle_o, \qquad g_{12} = g_{21} = \langle \mathbf{x}_u, \mathbf{x}_v \rangle_o, \qquad g_{22} = \langle \mathbf{x}_v, \mathbf{x}_v \rangle_o. \tag{1.2}$$

The positive definite, symmetric matrix

$$[g_{ij}] = \begin{bmatrix} E & F \\ F & G \end{bmatrix}$$

provides alternate notation for (1.1) given by Gauss.

We can find the induced metric for a manifold immersed in R^3. As an example, consider a graph $M = \{(x, y, f(x,y) : x, y \in U\} \subset \mathbb{R}^3$. M is a manifold of class C^k if and only if $f \in C^k$ and we can find the Riemannian metric $[g_{ij}]$ induced by $\mathbb{R}^3$. Consider a parameterization $\mathbf{x} : \mathbb{R}^2_{(u,v)} \to \mathbb{R}^3_{(x,y,z)}$ such that $\mathbf{x}(u,v) = (x(u,v), y(u,v), z(u,v))$.

Set

$$x(u,v) = u, \; y(u,v) = v, \; z = f(u,v). \tag{1.3}$$

We find $d\mathbf{x}\left(\frac{\partial}{\partial u}\right)$ and $d\mathbf{x}\left(\frac{\partial}{\partial v}\right)$ to be:

$$\begin{aligned} d\mathbf{x}\left(\frac{\partial}{\partial u}\right) &= \frac{\partial x}{\partial u}\frac{\partial}{\partial x} + \frac{\partial y}{\partial u}\frac{\partial}{\partial y} + \frac{\partial z}{\partial u}\frac{\partial}{\partial z} \\ &= 1\frac{\partial}{\partial x} + 0\frac{\partial}{\partial y} + f_u\frac{\partial}{\partial z} \end{aligned}$$

$$\begin{aligned} d\mathbf{x}\left(\frac{\partial}{\partial v}\right) &= \frac{\partial x}{\partial v}\frac{\partial}{\partial x} + \frac{\partial y}{\partial v}\frac{\partial}{\partial y} + \frac{\partial z}{\partial v}\frac{\partial}{\partial z} \\ &= 0\frac{\partial}{\partial x} + 1\frac{\partial}{\partial y} + f_v\frac{\partial}{\partial z.}. \end{aligned}$$

Now find each component g_{ij} by taking the inner product of the basis vectors

$$\begin{aligned} g_{11} &= \left\langle d\mathbf{x}\left(\frac{\partial}{\partial u}\right), d\mathbf{x}\left(\frac{\partial}{\partial u}\right)\right\rangle_o = 1 + f_u^2 \\ g_{12} = g_{21} &= \left\langle d\mathbf{x}\left(\frac{\partial}{\partial u}\right), d\mathbf{x}\left(\frac{\partial}{\partial v}\right)\right\rangle_o = f_u f_v \\ g_{22} &= \left\langle d\mathbf{x}\left(\frac{\partial}{\partial v}\right), d\mathbf{x}\left(\frac{\partial}{\partial v}\right)\right\rangle_o = 1 + f_v^2. \end{aligned}$$

Thus on the graph M, the local representation of the **metric induced by** $\mathbb{R}^3$ with respect to the given parametrization (1.3) is

$$[g_{ij}] = \begin{bmatrix} 1 + f_u^2 & f_u f_v \\ f_u f_v & 1 + f_v^2 \end{bmatrix}. \tag{1.4}$$

The **inverse** of $[g_{ij}]$ is written $[g^{ij}]$ and equals

$$[g^{ij}] = \frac{1}{1 + f_u^2 + f_v^2} \begin{bmatrix} 1 + f_v^2 & -f_u f_v \\ -f_u f_v & 1 + f_u^2 \end{bmatrix}; \tag{1.5}$$

then $\delta^i_j = [g^{ij}][g_{ij}]$ where δ^i_j is the **Kronecker delta** defined by

$$\delta^i_j = \begin{cases} 1 & \text{if } i = j \\ 0 & \text{if } i \neq j. \end{cases} \tag{1.6}$$

1.3 Differential Operators

We will now provide some definitions for differential operators on a manifold M. The motivation of this thesis is from Lancaster's work where he focused the local behavior at a corner. To expand on his results, one wishes to consider local coordinates at the corner and therefore we will provide the needed definitions in local coordinates. Global definitions have been given in appendix A.

We start by considering a particular connection that is compatible with the metric, called the Levi-Civita Connection, but for a more general definition of connection refer to appendix A. One interpretation of the connection $\nabla_V W$ is that it allows us to define the directional derivative of a vector field W in the direction of another vector field V.

From [13], given a Riemannian manifold M there exists a unique affine connection ∇ called the **Levi-Civita connection** on M satisfying the conditions:

$$X(\langle Y, Z\rangle) = \langle \nabla_X Y, Z\rangle + \langle Y, \nabla_X Z\rangle \text{ and } \nabla_X Y - \nabla_Y X = [X, Y]. \tag{1.7}$$

Consider a parameterization $\mathbf{x} : U \subset \mathbb{R}^n \to M$ and a curve $\gamma(t) = (x_1(t), ..., x_n(t))$. Consider the vector field V defined by

$$V(t) = \sum_{i=1}^{n} v_i(t) \frac{\partial}{\partial x_i}|_{\gamma(t)}$$

where $v_i = \frac{dx_i}{dt}(t)$. Then the **covariant derivative** of a tangent vector field V on M along a curve γ can be written in local coordinates as follows

$$\frac{DV}{dt} = \sum_{i=1}^{n} \left[\frac{dv_i}{dt} \frac{\partial}{\partial x_i} + \sum_{j=1}^{n} v_i(t) x_i'(t) \nabla_{\frac{\partial}{\partial x_j}} \frac{\partial}{\partial x_i} \right]. \tag{1.8}$$

In appendix A, we provide a invariant definition of the covariant derivative and from that we derive (1.8). We define the Christoffel symbols Γ_{ij}^k to be

$$\nabla_{\frac{\partial}{\partial x_j}} \frac{\partial}{\partial x_i} = \sum_{k=1}^{n} \Gamma_{ij}^k \frac{\partial}{\partial x_k} \tag{1.9}$$

and a consequence of (1.9) is that Christoffel symbols can be written as

$$\Gamma_{ij}^k = \frac{1}{2} \sum_l g^{kl} \left(\frac{\partial}{\partial x_j} g_{il} + \frac{\partial}{\partial x_i} g_{jl} - \frac{\partial}{\partial x_l} g_{ij} \right). \tag{1.10}$$

Since the Levi-Civita connection is metric compatible by (1.7), we can use (1.9) and then by [15], the covariant derivative of the metric tensor $[g^{ij}]$ is

$$g^{ij}_{;k} = \nabla_k g^{ij} = \frac{\partial}{\partial x_k} g^{ij} + \sum_l g^{il} \Gamma_{lk}^j + \sum_l g^{jl} \Gamma_{lk}^i. \tag{1.11}$$

from [6]

The **gradient** of f as a vector field grad$f$ on T_pM is defined to be

$$\langle \text{grad} f(p), v\rangle = df_p(v)$$

for $p \in M$, $v \in T_pM$. In local coordinates $(x_1, ..., x_n)$, the gradient of the function f is

$$\text{grad } f = \sum_{i,j} \left[g^{ij} \frac{\partial f}{\partial x_j} \right] \cdot \frac{\partial}{\partial x_i} \tag{1.12}$$

The **divergence** of X is a function $\text{div}X : M \to \mathbb{R}$, where $\text{div}(p)$ is the trace of the linear mapping of a vector $Y(p)$ to $\nabla_Y X(p)$ with $p \in M$ ([13]). In local coordinates, $(x_1, ..., x_n)$, the divergence of a vector field $X = \sum_k a_k \frac{\partial}{\partial x_k}$ is

$$\text{div } X = \sum_k \left[\frac{\partial a_k}{\partial x_k} + \sum_i a_i \Gamma^k_{ik} \right]. \tag{1.13}$$

The **Laplacian** (or Laplace-Beltrami Operator) is defined to be $\Delta_g f = \text{div}(\text{grad} f)$, where f is a $C^k, k \geq 2$, function on M. Using (1.13)-(1.12) the Laplacian can then be written as

$$\Delta_g f = \sum_{k,j=1}^{n} \left[g^{ki} \frac{\partial}{\partial x_k} \frac{\partial f}{\partial x_j} - \sum_l g^{ij} \frac{\partial f}{\partial x_l} \Gamma^l_{kj} \right] \tag{1.14}$$

in local coordinates $(x_1, ..., x_n)$.

The process of equating two indices of a mixed tensor, one being an upper index and the other a lower index, and then summing with respect to this pair of indices, is called **contraction** ([22]). We use this process in the proof of Theorem 1, section 2.2.

1.4 Beltrami Equation

Now we will introduce the Beltrami equation. Consider the partial derivatives of w with respect to z and $\bar{z}$ to be

$$w_{\bar{z}} = \bar{\partial} w = \frac{1}{2}(w_x + i w_y)$$

and

$$w_z = \partial w = \frac{1}{2}(w_x - i w_y).$$

Let D be a domain in $\mathbf{C}$ and $\mu : D \to \mathbf{C}$ a measurable function where $|\mu| < k < 1$. The **Beltrami equation** for μ is

$$w_{\bar{z}} = \mu(z) w_z. \tag{1.15}$$

Notice that if $\mu \equiv 0$, then (1.15) becomes the Cauchy-Riemann equation.

A known relationship between Beltrami equations and quasiconformal mappings is that every homeomorphic solution of (1.15) is k-quasiconformal, provided μ satisfies $|\mu(z)| \leq \frac{k-1}{k+2}$ and conversely, every k-quasiconformal mapping is a solution of some Beltrami equation satisfying (1.15) with $|\mu(z)| \leq \frac{k-1}{k+2}$ ([4]). We will define and discuss quasiconformal mappings in section 5.5.

Another theorem regarding Beltrami equations is as follows: If $w_1(z)$ is a solution of (1.15) and $f(w)$ an analytic function, then $w_2(z) = f(w_1(z))$ is also a solution of (1.15). Conversely, if $w_1(z)$ and $w_2(z)$ are two solutions of the same Beltrami equation defined in the same domain and $w_1(z)$ is a homeomorphism, then $w_2(z) = f(w_1(z))$ where $f(w)$ is an analytic function ([4]).

One can think of the Beltrami equation as a change of variables from $\mathbb{R}^2 \to \mathbb{R}^2$, while a parameterization $\mathbb{R}^2 \to \mathbb{R}^3$ is a mapping from the parameter domain to the manifold.

Gauss first studied the Beltrami equation in the 1820's while exploring the existence of isothermal parameters on a surface which will be defined in section 2.1. In sections 5.3 and 5.4 we will prove the existence of a single normalizing parameter system for isothermal and characteristic coordinates, where we will use the Beltrami equations corresponding to each coordinate system. In the late 1930's Morrey studied the complex Beltrami equation and established the existence of homeomorphic solutions for measurable μ, but it took another 20 years before Bers found that quasiconformal maps can be regarded as homeomorphic solutions of the Beltrami equation ([19]).

We can associate μ with the Riemannian metric

$$Edx^2 + 2Fdxdy + Gdy^2 = |dz + \mu d\bar{z}|^2.$$

Then in real coordinates (1.15) has the form,

$$Wu_x = Fv_x + Gv_y \tag{1.16}$$

$$-Wu_y = Ev_x + Fv_y \tag{1.17}$$

with $W^2 = EG - F^2$. Beltrami equations are found in the study of differential geometry, complex function theory and differential equations ([3]). We will use different parameterizations to find local representations of the metric and the corresponding Beltrami equations.

CHAPTER 2

ISOTHERMAL AND HARMONIC COORDINATES

2.1 Isothermal Coordinates

Consider (M, g) with a parameterization $\mathbf{x} : U \subset R^n \to M$. Local coordinates that make the metric a multiple of the identity are called **Isothermal Coordinates**, also called conformal coordinates. If $(x_1, ..., x_n)$ are local isothermal coordinates, then the local representation of the metric in these coordinates is of the form

$$g = e^{\phi}(dx_1^2 + \cdots + dx_n^2), \tag{2.1}$$

where ϕ is a smooth function.

Now consider a surface (M, g) in R^3 with the local parameterization $\mathbf{x}(u, v)$. The local representation of the metric in isothermal coordinates on a surface is

$$[g_{ij}] = \begin{bmatrix} \lambda^2 & 0 \\ 0 & \lambda^2 \end{bmatrix} \tag{2.2}$$

where $E = G = \lambda^2 = \langle \mathbf{x}_u, \mathbf{x}_u \rangle_o = \langle \mathbf{x}_v, \mathbf{x}_v \rangle_o$ and $F = 0$, where $\langle , \rangle_o$ is the standard inner product from $\mathbb{R}^3$. In other words, in isothermal coordinates the tangential coordinate vectors at each point are perpendicular and have the same length.

Starting with [23], Lancaster and his coauthors used an isothermal parametrization to investigate the existence and behavior of the radial limits of bounded solutions of (1) which satisfy the Dirichlet boundary condition mentioned in the preface. In [25] and [24], Lancaster and Siegel used an isothermal parametrization to investigate the existence and behavior of the radial limits of bounded solutions of (1) which satisfy the contact angle boundary condition. Isothermal coordinates exist in a neighborhood of any point on a two-dimensional Riemannian manifold as long as the metric has certain regularity assumptions as given in [7]. Isothermal coordinates do not exist in higher dimensions unless the manifold is conformally flat ([28]). A proof of the global existence of isothermal coordinates on a surface defined by a graph of a function will be given in Theorem 6 found in section 5.4. A proof with a different approach can be found in [7].

2.2 Harmonic Coordinates

As mentioned in the previous section, isothermal coordinates do not exist in higher dimensions unless the manifold is conformally flat. Therefore, we now consider harmonic coordinates. For a manifold (M, g) with a parameterization $\mathbf{x} : U \to M$ then local coordinates $(x_1, ..., x_n)$ are called **harmonic coordinates** if each coordinate function x_i is harmonic. In other words,

$$\Delta_g x_i = 0, \tag{2.3}$$

for $i = 1, ..., n$ where Δ_g represents the Laplace-Beltrami operator, as given by (1.14). The existence of harmonic coordinates is a consequence of existence theory for elliptic partial differential equations ([8, p. 91]).

Harmonic coordinates in higher dimensions were first used by Einstein in 1916 in the study of general relativity ([2]). Moreover, DeTurck and Kazdan proved in 1981, that "a metric has optimal regularity in any harmonic chart, i.e., that it is no smoother in any other coordinates" ([11]). In [11, p. 252] is the following

Theorem 1. *If a [local representation of a] metric $g \in C^{k,\alpha}, 1 \le k \le \infty$ (or C^ω) in some coordinate chart, then it is also of class $C^{k,\alpha}$ (or C^ω) in harmonic coordinates, while it is of at least class $C^{k-2,\alpha}$ (or C^ω) in geodesic normal coordinates.*

As an example provided in [11], consider $p(x, y) > 0, p \in C^{k,\alpha}(\Omega)$ in an open set $\Omega \subset \mathbb{R}^2$. The metric $h = p(x, y)(dx^2 + dy^2)$ is then of class $C^{k,\alpha}$ in that set. Note that this metric h is of the form (2.1) and therefore is isothermal. We claim that the "metric's "differentiability cannot be increased by changing coordinates" ([11]). This claim is verified because isothermal coordinates are harmonic in two dimensions and utilizing the above theorem h must be of at least class $C^{k,\alpha}$. We will prove isothermal coordinates are harmonic in two-dimensions in Theorem 3.

Returning to the definition of harmonic coordinates, we elaborate on (2.3) using (1.14).

$$\begin{aligned} 0 = \Delta_g x_i &= \sum_{k,j=1}^{n} \left[g^{jk} \frac{\partial}{\partial x_j} \frac{\partial x_i}{\partial x_k} - \sum_l g^{jk} \frac{\partial x_i}{\partial x_l} \Gamma^l_{jk} \right] \\ &= (0) g^{jk} - \sum_{l,j,k} \delta^i_l g^{jk} \Gamma^l_{jk} \\ &= -\sum_{jk} g^{jk} \Gamma^i_{jk} \end{aligned}$$

Therefore,

$$\Delta_g x_i = -\sum_{jk} g^{jk} \Gamma^i_{jk} = 0. \tag{2.4}$$

An alternate (yet, equivalent) definition for harmonic coordinates is given in [18]: "A system of coordinates x_i in an arbitrary fixed n-dimensional Riemannian manifold will be called **harmonic** if in these coordinates the components of the metric tensor of the space satisfy the equations for $i = 1, 2, ..., n$,

$$\sum_k \frac{\partial}{\partial x_k} (\sqrt{|g|} g^{ik}) = 0, \tag{2.5}$$

where $|g|$ represents the determinant of $[g_{ij}]$." We will show that (2.4) and (2.5) are equivalent, but first we provide the following lemma.

Lemma 1. *In two dimensions, if $|g|$ is the determinant of the metric, then* $\Gamma^i_{li} = \sum_l \frac{1}{2|g|} \frac{\partial |g|}{\partial x_l}$.

Proof. Let $|g|$ be the determinant of the metric $[g_{ij}]$.

$$\frac{\partial |g|}{\partial x_l} = \frac{\partial}{\partial x_l} (g_{11} g_{22} - g_{12}^2) = g_{22} \frac{\partial g_{11}}{\partial x_l} + g_{11} \frac{\partial g_{22}}{\partial x_l} - 2 g_{12} \frac{\partial g_{12}}{\partial x_l}.$$

Using $g^{11} = \frac{g_{22}}{|g|}$, $g^{12} = g^{21} = -\frac{g_{12}}{|g|}$, $g^{22} = \frac{g_{11}}{|g|}$, we can write

$$\begin{aligned}
\frac{\partial |g|}{\partial x_l} &= |g| \left(g^{11} \frac{\partial g_{11}}{\partial x_l} + g^{22} \frac{\partial g_{22}}{\partial x_l} + 2 g^{12} \frac{\partial g_{12}}{\partial x_l} \right) \\
&= \sum_{ijl} |g| g^{ij} \frac{\partial g_{ij}}{\partial x_l} \\
&= \sum_{ijl} |g| (\Gamma^i_{il} + \Gamma^j_{jl}).
\end{aligned}$$

Replacing j by i yields

$$\frac{\partial |g|}{\partial x_l} = \sum_l 2|g|\Gamma^i_{il}$$

which implies

$$\Gamma^i_{il} = \sum_l \frac{1}{2|g|} \frac{\partial |g|}{\partial x_l} = \frac{\partial}{\partial x_l} (\log \sqrt{|g|}).$$

□

Theorem 2. *The definitions (2.4) and (2.5) are equivalent.*

Proof. Recall (1.11):

$$g^{ij}_{;k} = \frac{\partial g^{ij}}{\partial x^k} + g^{il}\Gamma^j_{lk} + g^{jl}\Gamma^i_{lk} = 0.$$

Contracting the indices k, j in the above equation gives,

$$\frac{\partial g^{ij}}{\partial x_j} + g^{il}\Gamma^j_{ij} + g^{ij}\Gamma^i_{lj} = 0. \tag{2.6}$$

Consider (2.5), with $|g|$ representing the determinant of the metric

$$\begin{aligned}
0 &= \frac{\partial}{\partial x_j} \left(|g|^{\frac{1}{2}} g^{ij} \right) \\
&= |g|^{\frac{1}{2}} \left[\frac{\partial g^{ij}}{\partial x_j} + \frac{1}{2|g|} g^{ij} \frac{\partial |g|}{\partial x_j} \right] \\
&= |g|^{\frac{1}{2}} \left[\frac{\partial g^{ij}}{\partial x_j} + \frac{1}{2|g|} \frac{\partial |g|}{\partial x_j} g^{ij} \right] \\
&= |g|^{\frac{1}{2}} \left[\frac{\partial g^{ij}}{\partial x_j} + g^{il}\Gamma^j_{lj} \right] && \text{[by Lemma 1]} \\
&= \left[-g^{il}\Gamma^i_{lj} \right] |g|^{\frac{1}{2}} && \text{[by (2.6)]} \\
&= |g|^{\frac{1}{2}} \Delta_g x_j.
\end{aligned}$$

Recall that $\frac{\partial}{\partial x_k}(\sqrt{|g|}g^{ik}) = 0$ and the previous steps showed that $\frac{\partial}{\partial x_k}(\sqrt{|g|}g^{ik}) = |g|^{\frac{1}{2}}\Delta_g x_j$, therefore $|g|^{\frac{1}{2}}\Delta_g x_j = 0$. This implies that $\Delta_g x_j = 0$. □

2.3 Relationships between Isothermal and Harmonic Coordinates

Theorem 3. *Isothermal coordinates are harmonic on a 2-dimensional manifold, M^2.*

Proof. Let $\mathbf{x}(u, v)$ be a parametrization of a surface M, where u and v are local isothermal coordinates. By the definition of isothermal coordinates,

$$g_{ij} = \lambda^2 \delta_{ij} \text{ and } g^{ij} = \lambda^{-2}\delta^{ij},$$

where $\lambda^2 = \langle \mathbf{x}_u, \mathbf{x}_u \rangle_o = \langle \mathbf{x}_v, \mathbf{x}_v \rangle_o$. To show that u is harmonic, let $x_1 = u$ and $x_2 = v$. Now use (2.4) to calculate $\Delta_g u$.

$$\begin{aligned}
\Delta_g u = \Delta_g x_1 &= \sum_{j,k=1}^{2} g^{jk}\Gamma^1_{jk} \\
&= \sum_{j,k=1}^{2} \lambda^{-2}\delta^{jk}\Gamma^1_{jk} \\
&= \lambda^{-2}(\Gamma^1_{11} + \Gamma^1_{22}) \\
&= \frac{1}{2}\lambda^{-2}\left[g^{11}\left(\frac{\partial g_{11}}{\partial u} + \frac{\partial g_{11}}{\partial u} - \frac{\partial g_{11}}{\partial u}\right) + g^{11}\left(\frac{\partial g_{21}}{\partial v} + \frac{\partial g_{21}}{\partial v} - \frac{\partial g_{22}}{\partial u}\right)\right] \\
&= \frac{1}{2}(\lambda^{-2})^2\left[\frac{\partial g_{11}}{\partial u} - \frac{\partial g_{22}}{\partial u}\right] \\
&= \frac{1}{2}(\lambda^{-2})^2\left[\frac{\partial}{\partial u}(\lambda^2) - \frac{\partial}{\partial u}(\lambda^2)\right] = 0.
\end{aligned}$$

Similarly, we can find Δv.

$$\begin{aligned}
\Delta_g v = \Delta_g x_2 &= \sum_{j,k=1}^{2} g^{jk}\Gamma^i_{jk} \\
&= \lambda^{-2}(\Gamma^2_{11} + \Gamma^2_{22}) \\
&= \frac{1}{2}\lambda^{-2}\left[g^{22}\left(\frac{\partial g_{12}}{\partial u} + \frac{\partial g_{12}}{\partial u} - \frac{\partial g_{11}}{\partial v}\right) + g^{22}\left(\frac{\partial g_{22}}{\partial v} + \frac{\partial g_{22}}{\partial v} - \frac{\partial g_{22}}{\partial v}\right)\right] \\
&= \frac{1}{2}(\lambda^{-2})^2\left[\frac{\partial g_{11}}{\partial v} + \frac{\partial g_{22}}{\partial v}\right] \\
&= \frac{1}{2}(\lambda^{-2})^2\left[-\frac{\partial}{\partial v}(\lambda^2) + \frac{\partial}{\partial v}(\lambda^2)\right] \\
&= 0.
\end{aligned}$$

Since $\Delta_g x_i = 0$ for $i = 1, 2$, an isothermal parameterization of M^2 is harmonic. $\square$

Now using (2.5) instead of (2.4), we find the following alternate proof of Theorem 3.

Proof. Let (x_1, x_2) be isothermal coordinates. Consider (2.5) with $i = 1$:

$$\sum_{k=1}^{2} \frac{\partial}{\partial x_k}\left(\sqrt{|g|}g^{2k}\right) = \frac{\partial}{\partial x_1}\left(\sqrt{|g|}g^{11}\right) + \frac{\partial}{\partial x_2}\left(\sqrt{|g|}g^{12}\right).$$

Since (x_1, x_2) is an isothermal coordinate system, then $g^{12} = g^{21} = 0$ and $g^{11} = g^{22} = \lambda^{-2}$. Thus,

$$\begin{aligned}\sum_{k=1}^{2} \frac{\partial}{\partial x_k}\left(\sqrt{|g|}g^{1k}\right) &= \frac{\partial}{\partial x_1}(\lambda^2(\lambda^{-2})) \\ &= \frac{\partial}{\partial x_1}(1) = 0.\end{aligned}$$

Similarly, for $i = 2$:

$$\begin{aligned}\sum_{k=1}^{2} \frac{\partial}{\partial x_k}\left(\sqrt{|g|}g^{2k}\right) &= \frac{\partial}{\partial x_1}\left(\sqrt{|g|}g^{21}\right) + \frac{\partial}{\partial x_2}\left(\sqrt{|g|}g^{22}\right) \\ &= \frac{\partial}{\partial x_2}(\lambda^2(\lambda^{-2})) \\ &= \frac{\partial}{\partial x_2}(1) = 0.\end{aligned}$$

We have found that the system of coordinates (x_1, x_2) is harmonic, since $\sum_{k=1}^{2} \frac{\partial}{\partial x_k}\left(\sqrt{|g|}g^{ik}\right) = 0$ for $i = 1, 2$. $\square$

Next we will consider the converse: Is a harmonic coordinate system isothermal? We find that a harmonic coordinate system is not necessarily isothermal. One can see that the metric

$$[g_{ij}] = \begin{bmatrix} \sqrt{2}\lambda^2 & \lambda^2 \\ \lambda^2 & \sqrt{2}\lambda^2 \end{bmatrix}$$

represents a harmonic coordinate system using the requirement of (2.5). To see this, first we find the determinant of $[g_{ij}]$ is $|g| = (\sqrt{2}\lambda^2)(\sqrt{2}\lambda^2) - (\lambda^2)^2 = 2\lambda^4 - \lambda^4 = \lambda^4$. So, $\sqrt{|g|} = \lambda^2$ Then,

$$[g^{ij}] = \begin{bmatrix} \sqrt{2}\lambda^{-2} & -\lambda^{-2} \\ -\lambda^{-2} & \sqrt{2}\lambda^{-2} \end{bmatrix}$$

and so,

$$\sqrt{|g|}g^{ij} = \begin{cases} \sqrt{2} & \text{for } i \neq j \\ -1 & \text{for } i = j \end{cases}.$$

Obviously, the derivative of $\sqrt{|g|}g^{ij}$ for either case is 0. So, we have a harmonic coordinate system. The metric is not isothermal, since $g_{12} = g_{21} \neq 0$.

In higher dimensions, isothermal coordinates do not have to be harmonic. Consider the metric g with local representation

$$[g_{ij}] = \begin{bmatrix} \lambda^2 & 0 & 0 \\ 0 & \lambda^2 & 0 \\ 0 & 0 & \lambda^2 \end{bmatrix}.$$

Since it is a multiple of the identity matrix, it represents an isothermal coordinate system. The inverse metric will be,

$$[g^{ij}] = \frac{1}{\lambda^6}\begin{bmatrix} \lambda^2 & 0 & 0 \\ 0 & \lambda^2 & 0 \\ 0 & 0 & \lambda^2 \end{bmatrix} = \begin{bmatrix} \lambda^{-2} & 0 & 0 \\ 0 & \lambda^{-2} & 0 \\ 0 & 0 & \lambda^{-2} \end{bmatrix}.$$

Now use (2.5) to see this is not a harmonic coordinate system. Notice that

$$\sqrt{|g|}g^{ij} = \begin{cases} \lambda & \text{for } i = j \\ 0 & \text{for } i \neq j \end{cases}$$

and therefore this is not a harmonic coordinate system provided $\dfrac{\partial \lambda}{\partial x_k} \neq 0$.

CHAPTER 3

CHARACTERISTIC COORDINATES

In this chapter we will introduce a third type of coordinates, characteristic coordinates, that are often used in the study of partial differential equations. We will then derive and discuss the normal form given by these coordinates using [10]. Next the Beltrami equations are derived for characteristic coordinates. In section 3.3, we will find the local representation of the metric of a surface in characteristic coordinates in terms of Cartesian coordinates. Then with an example we show that characteristic coordinates are not harmonic, unless we are working with a minimal surface.

3.1 Quasilinear Operators

We now look at characteristic coordinates, also called characteristic parameters. Characteristic coordinates allow us to write a partial differential equation in a canonical form. The normal form given by characteristic coordinates may provide insight into expanding the results in [25] and [23].

We consider a partial differential operator of the form

$$Q[f] = af_{xx} + 2bf_{xy} + cf_{yy} + d \tag{3.1}$$

with $p = f_x$, and $q = f_y$ and coefficients a, b, c, d being functions of the quantities x, y, p, q. This is called a **quasilinear differential operator** because it is linear in the derivatives of highest order ([10]). We will be focusing on the **elliptic case** which occurs when $b^2 - ac < 0$. Utilizing [10] and [5] we transform the elliptic partial differential equation

$$af_{xx} + 2bf_{xy} + cf_{yy} + d = 0 \tag{3.2}$$

into a canonical form

$$\Delta f + \cdots = f_{\rho\rho} + f_{\sigma\sigma} + \cdots = 0. \tag{3.3}$$

Introducing local coordinates $\sigma = \phi(x, y)$ and $\rho = \psi(x, y)$ where ϕ and ψ are C^2 and $\phi_x\psi_y - \phi_y\psi_x \neq 0$. By the chain rule, we find the following:

$$\begin{aligned}
&p = f_x = f_\sigma\sigma_x + f_\rho\rho_x \\
&q = f_y = f_\sigma\sigma_y + f_\rho\rho_y \\
&p_x = f_{xx} = f_{\sigma\sigma}\sigma_x^2 + 2f_{\rho\sigma}\sigma_x\rho_x + f_{\rho\rho}\rho_x^2 + f_\sigma\sigma_{xx} + f_\sigma\sigma_{xx} \\
&q_y = f_{yy} = f_{\sigma\sigma}\sigma_y^2 + 2f_{\rho\sigma}\sigma_y\rho_y + f_{\rho\rho}\rho_y^2 + f_\sigma\sigma_{yy} + f_\sigma\sigma_{yy} \\
&p_y = q_x = f_{\sigma\sigma}\sigma_x\sigma_y + f_{\sigma\rho}(\sigma_x + \rho_y + \rho_x\sigma_y) + f_{\rho\rho}\rho_x\rho_y + u_\sigma\sigma_{xy} + u_\rho\rho_{xy}.
\end{aligned} \tag{3.4}$$

Substituting (3.4) into (3.2) we get

$$a^*(\sigma, \rho)f_{\sigma\sigma} + b^*(\sigma, \rho)f_{\sigma\rho} + c^*(\sigma, \rho)f_{\rho\rho} = \Phi^*(\sigma, \rho, f, f_\rho, f_\sigma) \tag{3.5}$$

where $\Phi^*(\sigma, \rho, f, f_\rho, f_\sigma)$ includes terms involving d from (3.2) as well as other terms and

$$a^* = a\sigma_x^2 + 2b\sigma_x\sigma_y + c\sigma_y^2 \qquad c^* = a\rho_x^2 + 2b\rho_x\rho_y + c\rho_y^2 \tag{3.6}$$

$$b^* = a\rho_x\sigma_x + b(\sigma_x\rho_y + \sigma_y\rho_x) + c\sigma_y\rho_y.$$

Now we stipulate that $a^* = c^*$ and $b^* = 0$, which can be written explicitly using (3.6) as

$$a\rho_x^2 + 2b\rho_x\rho_y + c\rho_y^2 = a\sigma_x^2 + 2b\sigma_x\sigma_y + c\sigma_y^2, \tag{3.7}$$

$$a\rho_x\sigma_x + b(\rho_x\sigma_y + \rho_y\sigma_x) + c\rho_y\sigma_y = 0. \tag{3.8}$$

Using these conditions and dividing (3.5) by a^*, we obtain the canonical form for an elliptic equation:

$$f_{\sigma\sigma} + f_{\rho\rho} = \Psi(\sigma, \rho, f, f_\sigma, f_\rho) \tag{3.9}$$

where a, b, c and d are specified real-valued functions of σ and ρ. Notice that if $\Psi = 0$, then we obtain Laplace's equation

$$f_{\sigma\sigma} + f_{\rho\rho} = 0.$$

From here we can find the Beltrami equation for characteristic coordinates. Recall σ and ρ transformed (3.2) into the canonical form (3.9), if σ and ρ satisfy (3.7) and (3.8). In order to solve for $\sigma(x, y)$ or $\rho(x, y)$, we multiply equation (3.8) by $2i$ and then add the result to (3.7) to get

$$a(\sigma_x + i\rho_x)^2 + 2b(\sigma_x + i\rho_x)(\sigma_y + i\rho_y) + c(\sigma_y + i\rho_y)^2 = 0.$$

We solve this equation for $\frac{\sigma_x + i\rho_x}{\sigma_y + i\rho_y}$ and recall that since we are considering the elliptic case then $b^2 - ac < 0$ which yields

$$\frac{\sigma_x + i\rho_x}{\sigma_y + i\rho_y} = \frac{-b \pm i\sqrt{b^2 - ac}}{a}.$$

Multiplying by $\sigma_y + i\rho$ and then solving for the real and imaginary parts we get

$$\sigma_x = \frac{1}{a}(-b\sigma_y - \pm\rho_y\sqrt{b^2 - ac})$$
$$\rho_x = \frac{1}{a}(-b\rho_y - \pm\sigma_y\sqrt{b^2 - ac})$$

which can be written

$$a\sigma_x + b\sigma_y = \pm\rho_y\sqrt{b^2 - ac}$$
$$a\rho_x + b\rho_y = \pm\sigma_y\sqrt{b^2 - ac}$$

Then we find the Beltrami equation for characteristic coordinates to be

$$\sigma_x = \frac{b\rho_x + c\rho_y}{\omega}, \qquad -\sigma_y = \frac{a\rho_x + b\rho_y}{\omega}, \tag{3.10}$$
$$\text{where } \omega^2 = ac - b^2. \tag{3.11}$$

The problem of locally reducing $L[u] + \cdots = 0$ to normal form (3.3) by a transformation $\sigma(x, y)$ and $\rho(x, y)$ is equivalent to finding a solution of the Beltrami equation (3.10) for which $\sigma_x\rho_y - \sigma_y\rho_x \neq 0$ ([10]).

3.2 Normal Form

Consider a certain neighborhood on a surface with an equation of elliptic form. As shown previously, we can then find a normal form of the elliptic differential equation. We now want to find a system of three differential equations for x, y, f in terms of σ and ρ. To find these equations we perform a change of variables. Let $\phi(x, y) = \sigma$ and $\psi(x, y) = \rho$. Then we have

$$\begin{aligned}
1 &= \frac{\partial \phi}{\partial \sigma} = \frac{\partial \phi}{\partial x}\frac{\partial x}{\partial \sigma} + \frac{\partial \phi}{\partial y}\frac{\partial y}{\partial \sigma} \\
0 &= \frac{\partial \phi}{\partial \rho} = \frac{\partial \phi}{\partial x}\frac{\partial x}{\partial \rho} + \frac{\partial \phi}{\partial y}\frac{\partial y}{\partial \rho} \\
0 &= \frac{\partial \psi}{\partial \sigma} = \frac{\partial \psi}{\partial x}\frac{\partial x}{\partial \sigma} + \frac{\partial \psi}{\partial y}\frac{\partial y}{\partial \sigma} \\
1 &= \frac{\partial \psi}{\partial \rho} = \frac{\partial \psi}{\partial x}\frac{\partial x}{\partial \rho} + \frac{\partial \psi}{\partial y}\frac{\partial y}{\partial \rho}.
\end{aligned}$$

In matrix notation (with $\sigma_x = \frac{\partial \phi}{\partial x}$, etc) we have

$$\begin{bmatrix} \sigma_x & \sigma_y \\ \rho_x & \rho_y \end{bmatrix} \begin{bmatrix} x_\sigma & x_\rho \\ y_\sigma & y_\rho \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}.$$

Setting $\beta = \sigma_x \rho_y - \rho_x \sigma_y$ and solving the matrix equation to complete a change of variables, we find the following

$$\begin{bmatrix} x_\sigma & x_\rho \\ y_\sigma & y_\rho \end{bmatrix} = \frac{1}{\beta} \begin{bmatrix} \rho_y & -\sigma_y \\ -\rho_x & \sigma_x \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}.$$

This yields

$$\begin{aligned}
x_\sigma &= \frac{1}{\beta}\rho_y & x_\rho &= -\frac{1}{\beta}\sigma_y \\
y_\sigma &= -\frac{1}{\beta}\rho_x & y_\rho &= \frac{1}{\beta}\sigma_x.
\end{aligned}$$

We can then write (3.7)-(3.8) using the inverse relations we just found as the following system of three differential equations for the quantities x, y, f or the position vector $\mathbf{x}$ as function of the parameters σ and ρ

$$a y_\sigma^2 - 2b y_\sigma x_\sigma + c x_\sigma^2 = a y_\rho^2 - 2b y_\rho x_\rho + c x_\rho^2, \tag{3.12}$$

$$a y_\sigma y_\rho - b(y_\sigma x_\rho + y_\rho x_\sigma) + c x_\sigma x_\rho = 0. \tag{3.13}$$

It follows that we have equation (34a) of [10], which is

$$\langle \Delta_o \mathbf{x}, (\mathbf{x}_\sigma \times \mathbf{x}_\rho) \rangle_o = \begin{vmatrix} \Delta_o x & \Delta_o y & \Delta_o z \\ x_\sigma & y_\sigma & z_\sigma \\ x_\rho & y_\rho & z_\rho \end{vmatrix} = (x_\sigma y_\rho - x_\rho y_\sigma)^2 \frac{d}{\sqrt{ac-b^2}}, \tag{3.14}$$

where $\Delta_o \mathbf{x} = \mathbf{x}_{\sigma\sigma} + \mathbf{x}_{\rho\rho}$ denotes the Laplace operator on the vector $\mathbf{x}$. Let $\omega = \sqrt{ac-b^2}$ and set

$$M = \left(\frac{a}{\omega}\right)_x + \left(\frac{b}{\omega}\right)_y \qquad \text{and} \qquad N = \left(\frac{b}{\omega}\right)_x + \left(\frac{c}{\omega}\right)_y.$$

A (tedious) calculation by Professor Lancaster concludes that

$$\Delta_o x = (My_\sigma - Nx_\sigma)x_\rho + (Nx_\rho - My_\rho)x_\sigma \tag{3.15}$$

$$\Delta_o y = (My_\sigma - Nx_\sigma)y_\rho + (Nx_\rho - My_\rho)y_\sigma \tag{3.16}$$

$$\Delta_o z = (My_\sigma - Nx_\sigma)z_\rho + (Nx_\rho - My_\rho)z_\sigma + (x_\sigma y_\rho - x_\rho y_\sigma)\frac{d}{\omega}. \tag{3.17}$$

Notice that the first two equations simplify to

$$\Delta_o x = M(x_\rho y_\sigma - x_\sigma y_\rho)$$

$$\Delta_o y = N(x_\rho y_\sigma - x_\sigma y_\rho).$$

Now suppose the vector $\mathbf{x}$ is a function of isothermal coordinates (u, v) and $d = 0$ in (3.1); then $\Delta_o \mathbf{x} = \alpha \mathbf{x}_u + \beta \mathbf{x}_v$. By the definition of isothermal coordinates

$$\langle \mathbf{x}_u, \mathbf{x}_u \rangle_o = \langle \mathbf{x}_v, \mathbf{x}_v \rangle_o \tag{3.18}$$

$$\langle \mathbf{x}_u, \mathbf{x}_v \rangle_o = 0. \tag{3.19}$$

Now differentiate (3.18) with respect to u and then differentiate (3.19) with respect to v which yields

$$\langle \mathbf{x}_{uu}, \mathbf{x}_u \rangle_o = \langle \mathbf{x}_{vu}, \mathbf{x}_v \rangle_o$$

and

$$\langle \mathbf{x}_{uv}, \mathbf{x}_v \rangle_o + \langle \mathbf{x}_{vv}, \mathbf{x}_u \rangle_o = 0.$$

Combining these we get

$$\langle \mathbf{x}_{uu}, \mathbf{x}_u \rangle_o = \langle \mathbf{x}_{vu}, \mathbf{x}_v \rangle_o = - \langle \mathbf{x}_{vv}, \mathbf{x}_u \rangle_o$$

and algebra produces

$$\langle (\mathbf{x}_{uu} + \mathbf{x}_{vv}), \mathbf{x}_u \rangle_o = \langle \mathbf{x}_{vu}, \mathbf{x}_v \rangle_o .$$

This can be rewritten as

$$\langle \Delta \mathbf{x}, \mathbf{x}_u \rangle_o = 0 \tag{3.20}$$

which implies that $\alpha = 0$. Similarly, if we differentiate (3.18) with respect to v and (3.19) with respect to u we find

$$\langle \Delta \mathbf{x}, \mathbf{x}_v \rangle_o = 0 \tag{3.21}$$

and thus $\beta = 0$. Then from (3.20) and (3.21) we conclude

$$\Delta_o \mathbf{x} = 0.$$

In general in isothermal coordinates when $d \neq 0$ in (3.1), the previous paragraph shows that (3.20) and (3.21) still hold and so

$$\Delta_o \mathbf{x} = \gamma \mathbf{x}_u \times \mathbf{x}_v.$$

It can be shown ([27]) that γ is in fact equal to twice the mean curvature of the surface; that is

$$\Delta \mathbf{x} = 2H(\mathbf{x}) \mathbf{x}_u \times \mathbf{x}_v \tag{3.22}$$

where $H(\mathbf{x})$ is the mean curvature of the surface as defined in section 4.1 and is discussed in more detail in appendix A.2. Notice that if $H(x) = 0$ then this implies that $\Delta_o \mathbf{x} = 0$. In other words, if the mean curvature is zero, (i.e., we have a minimal surface) then the coordinates $\mathbf{x}(\sigma, \rho) = (x(\sigma, \rho), y(\sigma, \rho), z(\sigma, \rho))$ are harmonic in the sense that $\mathbf{x}_{\sigma\sigma} + \mathbf{x}_{\rho\rho} = 0$.

Now, if $d = 0$ in (3.14), then it is independent of a, b, c and it has the form

$$\langle \Delta_o \mathbf{x}, (\mathbf{x}_\sigma \times \mathbf{x}_\rho) \rangle_o = 0 \tag{3.23}$$

with $\Delta_o \mathbf{x} = \mathbf{x}_{\sigma\sigma} + \mathbf{x}_{\rho\rho}$ and so $\Delta_o \mathbf{x} = \alpha \mathbf{x}_\sigma + \beta \mathbf{x}_\rho$. Professor Lancaster's calculation shows that

$$\Delta_o \mathbf{x} = (M y_\sigma - N x_\sigma)\mathbf{x}_\sigma + (N x_\rho - M y_\rho)\mathbf{x}_\rho.$$

3.3 Coordinate Changes

Here we want to find a local representation of the metric in characteristic coordinates. Consider a given function $f : U \subset \mathbb{R}^2 \to \mathbb{R}$ and a given operator Q such that $Qf = 0$. Let $\mathbf{x} : U \to \mathbb{R}^3$ be a parameterization of the graph of f. We want to find a parametrization where (ρ, σ) are characteristic coordinates. Now give $\mathbf{x}(U)$ the metric induced by that of $\mathbb{R}^3$. Since $\mathbf{x}(U)$ is most easily parameterized in terms of (x, y), we can calculate the local components of the metric $\bar{g}_{ij}(x, y)$ to be

$$\bar{g}_{ij} = \begin{bmatrix} 1 + f_x^2 & f_x f_y \\ f_x f_y & 1 + f_y^2 \end{bmatrix}. \tag{3.24}$$

The following calculation allows us to express the local representation of the metric (3.24) in terms of (ρ, σ). Now utilizing $\bar{g}_{\alpha\beta} = g_{ij} \dfrac{\partial u^i}{\partial \bar{u}^\alpha} \dfrac{\partial u^j}{\partial \bar{u}^\beta}$ from ([22]) we can find the following system of equations by letting $u^1 = \sigma$, $u^2 = \rho$, $\bar{u}^1 = x$ and $\bar{u}^2 = y$. Thus we can write

$$\bar{g}_{11} = 1 + f_x^2 = g_{11}\sigma_x^2 + 2g_{12}\sigma_x\rho_x + g_{22}\rho_x^2 \tag{3.25}$$

$$\bar{g}_{12} = \bar{g}_{21} = f_x f_y = g_{11}\sigma_x\sigma_y + g_{12}(\sigma_x\rho_y + \sigma_y\rho_x) + g_{22}\rho_x\rho_y \tag{3.26}$$

$$\bar{g}_{22} = 1 + f_y^2 = g_{11}\sigma_y^2 + 2g_{12}\sigma_y\rho_y + g_{22}\rho_y^2. \tag{3.27}$$

Solving first for g_{11}, we begin by eliminating g_{22}. Combining equations (3.25) and (3.27) to get

$$\rho_y^2 + f_x^2\rho_y^2 = g_{11}\sigma_x^2\rho_y^2 + 2g_{12}\sigma_x\rho_x\rho_y^2 + g_{22}\rho_x^2\rho_y^2$$

$$-\rho_x^2 - f_y^2\rho_x^2 = -g_{11}\sigma_y^2\rho_x^2 - 2g_{12}\sigma_y\rho_y\rho_x^2 - g_{22}\rho_x^2\rho_y^2.$$

Adding these we have

$$\rho_y^2 + f_x^2\rho_y^2 - \rho_x^2 - f_y^2\rho_x^2 \quad = \quad g_{11}(\sigma_x^2\rho_y^2 - \sigma_y^2\rho_x^2) + 2g_{12}(\sigma_x\rho_x\rho_y^2 - \sigma_y\rho_y\rho_x^2). \tag{3.28}$$

Then we multiply (3.26) by ρ_y and (3.27) by $-\rho_x$ to obtain

$$\begin{aligned} f_xf_y\rho_y &= g_{11}\sigma_x\sigma_y\rho_y + g_{12}\rho_y(\sigma_x\rho_y + \rho_x\sigma_y) + g_{22}\rho_y^2\rho_x \\ -\rho_x - f_y^2\rho_x &= -g_{11}\sigma_y^2\rho_x - 2g_{12}\sigma_y\rho_y\rho_x - g_{22}\rho_x\rho_y^2. \end{aligned}$$

Then adding these then gives

$$f_xf_y\rho_y - \rho_x - \rho_xf_y^2 = g_{11}(\sigma_x\sigma_y\rho_y - \sigma_y^2\rho_x) + g_{12}(\sigma_x\rho_y^2 - \sigma_y\rho_y\rho_x). \tag{3.29}$$

We can then multiply (3.29) by $-2\rho_x$ to continue solving for g_{11} by eliminating g_{12}:

$$\rho_y^2 + f_x^2\rho_y^2 - \rho_x^2 - f_y^2\rho_x^2 = g_{11}(\sigma_x^2\rho_y^2 - \sigma_y^2\rho_x^2) + 2g_{12}\rho_x(\sigma_x\rho_y^2 - \sigma_y\rho_y\rho_x)$$

$$-2f_xf_y\rho_y\rho_x + 2\rho_x^2 + 2\rho_x^2f_y^2 = -2g_{11}\rho_x\sigma_y(\sigma_x\rho_y - \sigma_y\rho_x) - 2g_{12}\rho_x(\sigma_x\rho_y^2 - \sigma_y\rho_x\rho_y).$$

Now adding the two above equations, we have

$$\rho_y^2 + \rho_y^2f_x^2 - 2\rho_x\rho_yf_xf_y + \rho_x^2 + \rho_x^2f_y^2 = g_{11}(\sigma_x^2\rho_y^2 - 2\rho_x\rho_y\sigma_x\sigma_y + \rho_x^2\sigma_y^2).$$

Therefore,

$$g_{11} = \frac{\rho_x^2 + \rho_x^2f_y^2 - 2\rho_x\rho_yf_xf_y + \rho_y^2 + \rho_y^2f_x^2}{(\sigma_x\rho_y - \rho_x\sigma_y)^2} = \frac{(1+f_y^2)\rho_x^2 - 2f_xf_y(\rho_x\rho_y) + (1+f_x^2)\rho_y^2}{(\sigma_x\rho_y - \rho_x\sigma_y)^2}.$$

Similarly, we solve for g_{22} by multiplying (3.25) by σ_y^2 and multiplying (3.27) by $-\sigma_x^2$ and then adding them to eliminate g_{11}.

$$\begin{aligned} g_{11}\sigma_x^2\sigma_y^2 + 2g_{12}\sigma_x\rho_x\sigma_y^2 + g_{22}\rho_x^2\sigma_y^2 &= \sigma_y^2 + \sigma_y^2f_x^2 \\ -g_{11}\sigma_x^2\sigma_y^2 - 2g_{12}\sigma_x^2\rho_y\sigma_y - g_{22}\rho_y^2\sigma_x^2 &= -\sigma_x^2 + \sigma_x^2f_y^2. \end{aligned}$$

Addition produces

$$2g_{12}\sigma_x\sigma_y(\rho_x\sigma_y - \sigma_x\rho_y) + g_{22}(\sigma_y^2\rho_x^2 - \rho_y^2\sigma_x^2) = \sigma_y^2 + f_x^2\sigma_y^2 - \sigma_x^2 - f_y^2\sigma_x^2. \tag{3.30}$$

Continue eliminating g_{11} by multiplying (3.25) by σ_y and (3.26) by $-\sigma_x$

$$\begin{aligned} g_{11}\sigma_x^2\sigma_y + 2g_{12}\sigma_x\rho_x\sigma_y + g_{22}\rho_x^2\sigma_y &= \sigma_y + \sigma_y f_x^2 \\ -g_{11}\sigma_x^2\sigma_y - g_{12}\sigma_x(\sigma_x\rho_y + \sigma_y\rho_x) - g_{22}\rho_y\rho_x\sigma_x &= -\sigma_x f_y f_x. \end{aligned}$$

Simplifying we now have

$$g_{12}\sigma_x(\sigma_y\rho_x - \sigma_x\rho_y) + g_{22}\rho_x(\rho_x\sigma_y - \rho_y\sigma_x) = \sigma_y + f_x^2\sigma_y - \sigma_x f_x f_y. \tag{3.31}$$

Next, we eliminate g_{12} by multiplying (3.30) by $-2\sigma_y$ and then add this (3.31),

$$\begin{aligned} 2g_{12}\sigma_x\sigma_y(\rho_x\sigma_y - \sigma_x\rho_y) + g_{22}(\sigma_y^2\rho_x^2 - \rho_y^2\sigma_x^2) &= \sigma_y^2 + f_x^2\sigma_y^2 - \sigma_x^2 - f_y^2\sigma_x^2 \\ -2g_{12}\sigma_x\sigma_y(\sigma_y\rho_x - \sigma_x\rho_y) - 2g_{22}\sigma_y\rho_x(\rho_x\sigma_y - \rho_y\sigma_x) &= -2\sigma_y^2 - 2f_x^2\sigma_y^2 + 2\sigma_x\sigma_y f_x f_y \end{aligned}$$

to get

$$g_{22}(-\rho_x^2\sigma_y^2 + 2\rho_y\rho_x\sigma_y\sigma_x - \rho_y^2\sigma_x^2) = -\sigma_y^2 - f_x^2\sigma_y^2 + 2\sigma_x\sigma_y f_x f_y - \sigma_x^2 - f_y^2\sigma_x^2.$$

Thus we find

$$g_{22} = \frac{\sigma_x^2 + \sigma_x^2 f_y^2 - 2\sigma_x\sigma_y f_x f_y + \sigma_y^2 + \sigma_y^2 f_x^2}{(\sigma_y\rho_x - \rho_y\sigma_x)^2} = \frac{(1+f_y^2)\sigma_x^2 - 2f_x f_y(\sigma_x\sigma_y) + (1+f_x^2)\sigma_y^2}{(\sigma_x\rho_y - \rho_x\sigma_y)^2}.$$

Then solve for g_{12} to obtain

$$\begin{aligned} g_{12} &= \frac{-\sigma_x\rho_x - \sigma_x\rho_x f_y^2 + f_x f_y(\sigma_x\rho_y + \sigma_y\rho_x) - \sigma_y\rho_y - \sigma_y\rho_y f_x^2}{(\sigma_x\rho_y - \rho_x\sigma_y)^2} \\ &= -\left[\frac{(1+f_y^2)\sigma_x\rho_x - f_x f_y(\sigma_x\rho_y + \sigma_y\rho_x) + (1+f_x^2)\sigma_y\rho_y}{(\sigma_x\rho_y - \rho_x\sigma_y)^2}\right]. \end{aligned}$$

Thus, in characteristic coordinates (σ, ρ) the components of the local representation of the metric for a surface in $\mathbb{R}^3$ are

$$g_{11} = \frac{(1+f_y^2)\rho_x^2 - 2f_xf_y\rho_x\rho_y + (1+f_x^2)\rho_y^2}{(\sigma_x\rho_y - \rho_x\sigma_y)^2}$$

$$g_{12} = g_{21} = -\left[\frac{(1+f_y^2)\sigma_x\rho_x - f_xf_y(\sigma_x\rho_y + \sigma_y\rho_x) + (1+f_x^2)\sigma_y\rho_y}{(\sigma_x\rho_y - \rho_x\sigma_y)^2}\right] \tag{3.32}$$

$$g_{22} = \frac{(1+f_y^2)\sigma_x^2 - 2f_xf_y\sigma_x\sigma_y + (1+f_x^2)\sigma_y^2}{(\sigma_x\rho_y - \rho_x\sigma_y)^2}.$$

Now if f is given, we can define an operator Q by setting $a = 1 + f_y^2$, $b = -f_xf_y$, $c = 1 + f_x^2$ and $d = 0$. Then suppose that (ρ, σ) were characteristic coordinates for this f, Q, now depending only on f. Then we get

$$[g_{ij}] = \frac{1}{\beta}\begin{bmatrix} a\rho_x^2 + 2b(\rho_x\rho_y) + c\rho_y^2 & -[a\sigma_x\rho_x + b(\sigma_x\rho_y + \sigma_y\rho_x) + c\sigma_y\rho_y] \\ -[a\sigma_x\rho_x + b(\sigma_x\rho_y + \sigma_y\rho_x) + c\sigma_y\rho_y] & a\sigma_x^2 + 2b(\sigma_x\sigma_y) + c\sigma_y^2 \end{bmatrix} \tag{3.33}$$

where $\beta = (\sigma_x\rho_y - \rho_x\sigma_y)^2$.

Now if the graph of f is a minimal surface, then (3.7)-(3.7), which we recall to be

$$a\sigma_x^2 + 2b\sigma_x\sigma_y + c\sigma_y^2 = a\rho_x^2 + 2b\rho_x\rho_y + c\rho_y^2$$

$$a\sigma_x\rho_x + b(\sigma_x\rho_y + \sigma_y\rho_x) + c\sigma_y\rho_y = 0,$$

would hold. Then if we define Q as in (3.33), the expression for the metric shows that the same coordinates (ρ, σ), which were characteristic for Q, are also isothermal for the metric. We can easily see that $g_{11} = g_{22}$ and $g_{12} = g_{21} = 0$. Thus, characteristic coordinates are isothermal if the surface is minimal, i.e., $1 + f_y^2 = a$, $-f_xf_y = b$ and $1 + f_x^2 = c$.

Next one might ask "Is (σ, ρ) a system of harmonic coordinates?" Now since we previously showed the isothermal coordinates are a harmonic system, we know that if the surface is minimal then (σ, ρ) are harmonic. So now we consider the case where the function f is such that the surface formed by its graph is not minimal.

3.4 An Example

We will use the following example to show that characteristic coordinates need not be harmonic if the surface formed by the graph of the function f is not minimal.

Let $f(y,x) = y^2 - x^2$. First notice that $f_{yy} + f_{xx} = 0$. Since this is in canonical form, we have a characteristic parameterization. Alternatively, we can write the Beltrami equations (1.16) as

$$\sigma_x = \rho_y \text{ and } \sigma_y = -\rho_x.$$

Notice these are in the form of the Cauchy-Riemann equation and one can obtain $\sigma = x$ and $\rho = y$ as solutions. Moreover, we again conclude that σ and ρ are characteristic parameters.

We want to consider the example $f(x,y) = y^2 - x^2$ for the case where the surface is not minimal, so first we must show that the graph of f is not a minimal surface. Recall the second fundamental form for the case where $\Sigma \subset \mathbb{R}^3$ is the graph of f

$$S = \frac{1}{(1+f_x^2+f_y^2)^{\frac{3}{2}}} \begin{bmatrix} (1+f_y^2)f_{xx} - f_x f_y f_{xy} & (1+f_y^2)f_{xy} - f_x f_y f_{yy} \\ (1+f_x^2)f_{xy} - f_x f_y f_{xx} & (1+f_x^2)f_{yy} - f_x f_y f_{xy} \end{bmatrix}. \tag{3.34}$$

For more information on the second fundamental from see appendix A.2. The mean curvature is the trace of (3.34) ([26]). Using $f(x,y) = y^2 - x^2$, we find

$$f_x = -2x,\ f_y = 2y$$

$$f_{xx} = -2\ f_{yy} = 2$$

$$f_{xy} = f_{yx} = 0.$$

Thus

$$\frac{1}{(1+4x^2+4y^2)^{\frac{3}{2}}} \begin{bmatrix} -2(1+4y^2) & 8xy \\ -8xy & 2(1+4x^2) \end{bmatrix}$$
$$= \frac{2}{(1+4x^2+4y^2)^{\frac{3}{2}}} \begin{bmatrix} -(1+4y^2) & 4xy \\ -4xy & (1+4x^2) \end{bmatrix}. \tag{3.35}$$

Now find the trace of the matrix (3.35) to be

$$Tr(S) = \frac{2}{(1+4x^2+4y^2)^{\frac{3}{2}}}(-4y^2+4x^2) = \frac{8(x^2-y^2)}{(1+4x^2+4y^2)^{\frac{3}{2}}} \neq 0.$$

Since the trace of S is not equal to zero thus the mean curvature is not zero. Thus this is not a minimal surface. Now that we have established that the surface defined by the graph of f is not minimal, we will show that the characteristic coordinates are not harmonic.

Let $\sigma = x$ and $\rho = y$ and we find each of the components of the metric (3.33)

$$g_{11} = \frac{(1+f_y^2)(0)^2 - 2f_y f_x(0)(1) + (1+f_x)^2(1)^2}{1} = 1 + f_x^2.$$

Similarly, we can calculate

$$g_{22} = 1 + f_y^2 \quad \text{and} \quad g_{12} = f_y f_x.$$

Letting $|g|$ represent the determinant of $[g_{ij}]$ such that $|g| = 1 + f_y^2 + f_x^2$ we find the inverse components to be

$$g^{11} = \frac{1+f_y^2}{1+f_y^2+f_x^2} \quad g^{12} = \frac{-f_y f_x}{1+f_y^2+f_x^2} \quad g^{22} = \frac{1+f_x^2}{1+f_y^2+f_x^2}$$

Note since $g_{11} \neq g_{22}$ and $g_{12} \neq 0$ this is not an isothermal coordinate system.

Recall if (2.5) is satisfied, then we have a harmonic coordinate system. We first consider $i = 1$. Substituting $\sqrt{|g|}$, the components of $[g^{ij}]$ into (2.5) and letting $x_1 = x$ and $x_2 = y$ we find

$$\frac{\partial}{\partial x}\left(\sqrt{1+f_y^2+f_x^2}\left(\frac{1+f_x^2}{1+f_y^2+f_x^2}\right)\right) + \frac{\partial}{\partial y}\left(\sqrt{1+f_y^2+f_x^2}\;\left(\frac{-f_y f_x}{1+f_y^2+f_x^2}\right)\right).$$

Now substitute $f_y = 2y$ and $f_x = -2x$. Applying the product rule to take the derivative and then simplification yields

$$\begin{aligned}
\frac{\partial}{\partial x}\left(\frac{1+4y^2}{\sqrt{1+4y^2+4x^2}}\right) \;+\; \frac{\partial}{\partial y}\left(\frac{4yx}{\sqrt{1+4y^2+4x^2}}\right) &= \\
&= (1+4y^2+4x^2)^{-\frac{1}{2}}\frac{\partial}{\partial x}\left(1+4y^2\right) + (1+4y^2)\frac{\partial}{\partial x}\left(1+4y^2+4x^2\right)^{-\frac{1}{2}} \\
&\quad +(1+4y^2+4x^2)^{-\frac{1}{2}}\frac{\partial}{\partial y}(4yx) + (4yx)\frac{\partial}{\partial y}\left(1+4y^2+4x^2\right)^{-\frac{1}{2}} \\
&= 0 + -4x(1+4y^2+4x^2)^{-\frac{3}{2}}(1+4y^2) + 4x(1+4y^2+4x^2)^{-\frac{1}{2}} \\
&\quad -4y(4xy)(1+4y^2+4x^2)^{-\frac{3}{2}} \\
&= (1+4y^2+4x^2)^{-\frac{3}{2}}\left[4x(1+4y^2) + 4x(1+4y^2+4x^2) - 16xy^2)\right] \\
&= (1+4y^2+4x^2)^{-\frac{3}{2}}\left[-16xy^2 + 16x^3\right] \neq 0.
\end{aligned}$$

Similarly

$$\frac{\partial}{\partial x}\left(\frac{-f_y f_x}{\sqrt{1+f_y^2+f_x^2}}\right)+\frac{\partial}{\partial y}\left(\frac{1+f_x^2}{\sqrt{1+f_y^2+f_x^2}}\right)=\frac{\partial}{\partial x}\left(\frac{4yx}{\sqrt{1+4y^2+4x^2}}\right)+\frac{\partial}{\partial y}\left(\frac{1+4x^2}{\sqrt{1+4y^2+4x^2}}\right)$$

$$\begin{aligned}
&=(1+4y^2+4x^2)^{-\frac{1}{2}}\frac{\partial}{\partial x}(4yx)+(4yx)\frac{\partial}{\partial x}\left(1+4y^2+4x^2\right)^{-\frac{1}{2}}\\
&\quad+(1+4y^2+4x^2)^{-\frac{1}{2}}\frac{\partial}{\partial y}\left(1+4x^2\right)+(1+4x^2)\frac{\partial}{\partial y}\left(1+4y^2+4x^2\right)^{-\frac{1}{2}}\\
&=4y(1+4y^2+4x^2)^{-\frac{1}{2}}-16x^2y(1+4y^2+4x^2)^{-\frac{3}{2}}\\
&\quad+0-4y(1+4x^2)(1+4y^2+4x^2)^{-\frac{3}{2}}\\
&=(1+4y^2+4x^2)^{-\frac{3}{2}}\left[4y(1+4y^2+4x^2)-16x^2y-4y(1+4x^2)\right]\\
&=(1+4y^2+4x^2)^{-\frac{3}{2}}\left[16y^3-16x^2y\right]\neq 0.
\end{aligned}$$

Therefore, since $\sum_k \frac{\partial}{\partial x_k}(\sqrt{|g|}g^{ik}) \neq 0$ for $i=1,2$ this is not a harmonic coordinate system.

CHAPTER 4

STRUCTURE CONDITIONS FOR MEAN CURVATURE TYPE

4.1 Prescribed Mean Curvature

The **mean curvature** H of the surface at the point under consideration is the arithmetic mean of the principal curvatures κ_i,

$$H = \frac{1}{2}(\kappa_1 + \kappa_2),$$

where κ_1 and κ_2 are the maximum and minimum of the normal curvature at a given point on a surface. Note that the principal curvatures are also the eigenvalues of the shape operator ([26]). A **minimal surface** is a surface for which the mean curvature is zero. A nonparametric minimal surface (i.e. $S = \{(x, y, z) : z = f(x, y)\}$) satisfies

$$(1 + f_y^2)f_{xx} + 2f_x f_y f_{xy} + (1 + f_x^2)f_{yy} = 0. \tag{4.1}$$

Examples of minimal surfaces include planes, catenoids and helicoids. Prescribed mean curvature occurs when you set the left-hand side of (4.1) equal to a prescribed function as follows

$$(1 + f_y^2)f_{xx} - 2f_x f_y f_{xy} + (1 + f_x^2)f_{yy} = 2H(x, y, f(x, y))(1 + |Df|^2)^{\frac{3}{2}}. \tag{4.2}$$

For dimension $n \geq 2$, the prescribed mean curvature is given by

$$\text{div}(Tf) = nH(x, f(x))$$

with $x = (x_1, ..., x_n)$, where the vector field Tf is defined to be $Tf = \frac{\nabla f}{\sqrt{1+|\nabla f|^2}}$. We can then write

$$\sum_{i=1}^{n} \frac{\partial}{\partial x_i} \left(\frac{\frac{\partial f}{\partial x_i}}{\sqrt{1 + |Df|^2}} \right) = nH.$$

The equations (4.1) and (4.2) serve as a prototypes for equations of minimal surface type and equation of mean curvature type respectively which we will define here.

Let Ω be a bounded domain in $\mathbb{R}^2$. An equation of the form

$$a(p,q)f_{xx} + 2b(p,q)f_{xy} + c(p,q)f_{yy} = 0$$
$$p = f_x,\ q = f_y,\ \text{with } f \in C^2(\Omega)$$

is called an **equation of minimal surface type**, if and only if, $a, b, c \in C^{0,\tau}_{loc}(\mathbb{R}^2)$ for some $\tau \in (0,1)$, $ac - b^2 = 1$ and there is a positive number $\epsilon \geq 1$ such that

$$a(p,q)\frac{1+p^2}{W} + c(p,q)\frac{1+q^2}{W} + 2b(p,q)\frac{pq}{W} \leq 2\epsilon, \qquad W = \sqrt{1+p^2+q^2} \tag{4.3}$$

for all $p, q \in \mathbb{R}$ ([14]).

An equation of the form

$$\sum_{i,j=1}^{2} a_{ij}(x,f,\nabla f)D_{ij}f = h(x,f,\nabla f), \qquad f \in C^2(\Omega), \tag{4.4}$$

is called an **equation of mean curvature type**, if and only if, there exists constants γ and μ such that a_{ij}, with $i,j = 1,2$ are real valued functions on $\Omega \times \mathbb{R} \times \mathbb{R}^2$ which satisfy

$$|\xi|^2 - \frac{(p\cdot\xi)^2}{1+|p|^2} \leq a_{ij}(x,z,p)\xi_i\xi_j \leq \gamma\left[|\xi|^2 - \frac{(p\cdot\xi)^2}{1+|p|^2}\right]$$

for all $(x,z,p) \in \Omega \times \mathbb{R} \times \mathbb{R}^2$ and $\xi = (\xi_1, \xi_2) \in \mathbb{R}^2$ and

$$|h(x,z,p)| \leq \mu\sqrt{1+|p|^2}$$

for all $(x,z,p) \in \Omega \times \mathbb{R} \times \mathbb{R}^2$.

4.2 Structure Conditions

Quoting from [17], "The pioneering work on two dimensional equations of mean curvature type was done by Finn who treated the case $a^{ij}(x,z,p) \equiv a^{ij}(p)$ and $b \equiv 0$. Finn called his equations 'equations of minimal surface type' and stated the structure conditions for the coefficients somewhat differently (but equivalently to)

$$|\xi|^2 - \frac{(p\cdot\xi)^2}{1+|p|^2} \leq a_{ij}(x,z,p)\xi_i\xi_j \leq \gamma\left[|\xi|^2 - \frac{(p\cdot\xi)^2}{1+|p|^2}\right] \tag{4.5}$$

where a_{ij}, with $i,j = 1,2$, for all $(x,z,p) \in \Omega \times \mathbb{R} \times \mathbb{R}^2$ and $\xi = (\xi_1, \xi_2) \in \mathbb{R}^2$." This definition is given by Simon in [29]. In the following proof, we will show the second implies the first by finding the particular epsilon in (4.3)

Theorem 4. *Given* $\sum_{i,j=1}^{2} a_{ij}(x,u,Du)D_{ij}u = h(x,u,Du)$ *an equation of mean curvature type (4.5) and setting* $\omega^2 = a_{11}a_{22} - a_{12}^2$ *and* $a = \frac{a_{11}}{\omega}$, $b = \frac{a_{12}}{\omega}$ *and* $c = \frac{a_{22}}{\omega}$ *then* a, b, c *satisfy*

$$a(x,y,z,p,q)\frac{1+p^2}{W} + c(x,y,z,p,q)\frac{1+q^2}{W} + 2b(x,y,z,p,q)\frac{pq}{W} \leq 2\epsilon,$$
$$W = \sqrt{1+p^2+q^2} \tag{4.6}$$

for all $p, q \in \mathbb{R}$ *and where* $\epsilon = \sqrt{2}\gamma^2$.

Proof. Let $W^2 = 1 + p^2 + q^2$. We can rewrite (4.5) as

$$\frac{(1+q^2)\xi_1^2 - 2pq\xi_1\xi_2 + (1+p^2)\xi_2^2}{W^2} \leq a_{11}\xi_1^2 + 2a_{12}\xi_1\xi_2 + a_{22}\xi_2^2 \leq \gamma\left[\frac{(1+q^2)\xi_1^2 - 2pq\xi_1\xi_2 + (1+p^2)\xi_2^2}{W^2}\right].$$

Notice in (4.5) if $\xi_1 = 1$ and $\xi_2 = 0$ then

$$\frac{1+q^2}{W^2} \leq a_{11} \leq \gamma\frac{1+q^2}{W^2}. \tag{4.7}$$

If $\xi_1 = 0$ and $\xi_2 = 1$ in (4.5) then

$$\frac{1+p^2}{W^2} \leq a_{22} \leq \gamma\frac{1+p^2}{W^2}. \tag{4.8}$$

Now let $\xi_1 = \sqrt{a_{22}}$ and $\xi_2 = \sqrt{a_{11}}$ in (4.5) then

$$\begin{aligned}\frac{(1+q^2)a_{22} - 2pq\sqrt{a_{11}a_{22}} + (1+p^2)a_{11}}{W^2} &\leq a_{11}a_{22} + 2a_{12}\sqrt{a_{11}a_{22}} + a_{22}a_{11} \\ &\leq \gamma\left[\frac{(1+q^2)a_{22} - 2pq\sqrt{a_{11}a_{22}} + (1+p^2)a_{11}}{W^2}\right].\end{aligned} \tag{4.9}$$

Similarly, we let $\xi_1 = \sqrt{a_{22}}$ and $\xi_2 = -\sqrt{a_{11}}$ in (4.5) then

$$\begin{aligned}\frac{(1+q^2)a_{22} + 2pq\sqrt{a_{11}a_{22}} + (1+p^2)a_{11}}{W^2} &\leq a_{11}a_{22} - 2a_{12}\sqrt{a_{11}a_{22}} + a_{22}a_{11} \\ &\leq \gamma\left[\frac{(1+q^2)a_{22} + 2pq\sqrt{a_{11}a_{22}} + (1+p^2)a_{11}}{W^2}\right].\end{aligned} \tag{4.10}$$

Now we want to multiply (4.9) and (4.10) together. To simplify this we introduce the following notation. Let

$$G = \frac{(1+q^2)a_{22} + (1+p^2)a_{11}}{W^2} \quad \text{and} \quad J = \frac{2pq\sqrt{a_{11}a_{22}}}{W^2}.$$

Notice that in order to multiply (4.9) and (4.10), we must have $G - J \geq 0$ and $G + J \geq 0$ otherwise the inequality signs would change. We consider the left side of (4.9) and factoring yields

$$G - J = \frac{(1+q^2)a_{22} - 2pq\sqrt{a_{11}a_{22}} + (1+p^2)a_{11}}{W^2} = \frac{a_{11} + a_{22} + (p\sqrt{a_{11}} - q\sqrt{a_{22}})^2}{W^2} \geq 0.$$

Similarly,

$$G + J = \frac{(1+q^2)a_{22} + 2pq\sqrt{a_{11}a_{22}} + (1+p^2)a_{11}}{W^2} = \frac{a_{11} + a_{22} + (p\sqrt{a_{11}} + q\sqrt{a_{22}})^2}{W^2} \geq 0.$$

Therefore $(G-J)(G+J) \geq 0$. Implementing the new notation, we multiply (4.9) and (4.10) to obtain

$$(G-J)(G+J) \leq 4(a_{11}^2 a_{22}^2 - a_{12}^2 a_{11}a_{22}) \leq \gamma(G-J)\gamma(G+J)$$

which simplifies to

$$G^2 - J^2 \leq 4a_{11}a_{22}(a_{11}a_{22} - a_{12}^2) \leq \gamma^2(G^2 - J^2).$$

Next dividing by $4a_{11}a_{22}$ and substituting $\omega^2 = a_{11}a_{22} - a_{12}^2$ gives

$$\frac{G^2 - J^2}{4a_{11}a_{22}} \leq \omega^2 \leq \gamma^2 \frac{G^2 - J^2}{4a_{11}a_{22}}.$$

Multiplying through by W^2 we find

$$\frac{G^2 - J^2}{4a_{11}a_{22}} W^2 \leq \omega^2 W^2 \leq \gamma^2 \frac{G^2 - J^2}{4a_{11}a_{22}} W^2. \tag{4.11}$$

Now utilizing (4.7) - (4.8) we observe that

$$\frac{4(1+q^2)(1+p^2)}{W^4} \leq 4a_{11}a_{22} \leq \frac{4\gamma^2(1+q^2)(1+p^2)}{W^4}. \tag{4.12}$$

Recall $G^2 - J^2$ from (4.11):

$$\begin{aligned}
G^2 - J^2 &= \frac{1}{W^4}\left[(1+2q^2+q^4)a_{22}^2 + 2(1+p^2)(1+q^2)a_{11}a_{22} + (1+2p^2+p^4)a_{11}^2 - 4p^2q^2a_{11}a_{22}\right] \\
&= \frac{1}{W^4}\left[(p^2a_{11} - q^2a_{22})^2 + (a_{11}+a_{22})^2 + 2p^2(a_{11}^2 + a_{11}a_{22}) + 2q^2(a_{22}^2 + a_{11}a_{22})\right] \\
&= \frac{1}{W^4}(p^2a_{11} - q^2a_{22})^2 + \frac{1}{W^4}(a_{11}+a_{22})\left[a_{11}+a_{22}+2p^2a_{11}+2q^2a_{22}\right] \\
&\geq \frac{a_{11}+a_{22}}{W^4}\left[(1+2p^2)a_{11} + (1+2q^2)a_{22}\right]. \qquad (4.13)
\end{aligned}$$

Considering (4.11), and then using (4.7) - (4.8) and (4.12) -(4.13),

$$\begin{aligned}
(\omega^2W^2) &\geq \frac{G^2-J^2}{4a_{11}a_{22}}W^2 \geq \frac{\frac{a_{11}+a_{22}}{W^4}((1+2p^2)a_{11}+(1+2q^2)a_{22})W^2}{\frac{4\gamma^2(1+p^2)(1+q^2)}{W^4}} \\
&= \frac{(a_{11}W^2 + a_{22}W^2)((1+2p^2)a_{11}+(1+2q^2)a_{22})}{4\gamma^2(1+p^2+q^2+p^2q^2)} \\
&\geq \frac{(1+q^2+1+p^2)\left[(1+2p^2)\frac{1+q^2}{W^2} + (1+2q^2)\frac{1+p^2}{W^2}\right]}{4\gamma^2(1+p^2+q^2+p^2q^2)} \\
&= \frac{2+p^2+q^2}{1+p^2+q^2}\cdot\frac{1}{4\gamma^2}\cdot\frac{2+3p^2+3q^2+4p^2q^2}{1+p^2+q^2+p^2q^2} \\
&= \left(1+\frac{1}{W^2}\right)\frac{1}{4\gamma^2}\left(4 - \frac{p^2+q^2+2}{1+p^2+q^2+p^2q^2}\right) \\
&\geq (1)\left(\frac{1}{4\gamma^2}\right)\left(4 - \left[\frac{p^2+1}{(1+p^2)(1+q^2)} + \frac{1+q^2}{(1+p^2)(1+q^2)}\right]\right) \\
&= \frac{1}{4\gamma^2}\left(4 - \frac{1}{1+q^2} - \frac{1}{1+p^2}\right) \\
&\geq \frac{2}{4\gamma^2} = \frac{1}{2\gamma^2}.
\end{aligned}$$

Therefore,

$$\omega W \geq \frac{1}{\gamma\sqrt{2}} = c_1. \qquad (4.14)$$

Let $\xi_1 = p$ and $\xi_2 = q$ then (4.5) becomes,

$$\frac{p^2+q^2}{W^2} \le a_{11}p^2 + a_{22}q^2 + 2a_{12}pq \le \gamma\left[\frac{p^2+q^2}{W^2}\right].$$

Now use (4.7)-(4.8) which yields

$$\frac{1+q^2}{W^2} + \frac{1+p^2}{W^2} + \frac{p^2+q^2}{W^2} \le a_{11} + a_{22} + a_{11}p^2 + a_{22}q^2 + 2a_{12}pq \le \gamma\left[\frac{1+q^2+1+p^2+p^2+q^2}{W^2}\right].$$

From the hypothesis $a = \frac{a_{11}}{\omega}$, $b = \frac{a_{12}}{\omega}$, and $c = \frac{a_{22}}{\omega}$. Dividing by $W\omega$,

$$\frac{1+q^2+1+p^2+p^2+q^2}{W^3\omega} \le \frac{a(1+p^2)+(1+q^2)c+2bpq}{W} \le \gamma\left[\frac{1+q^2+1+p^2+p^2+q^2}{W^3\omega}\right].$$

Notice that $1+q^2+1+p^2+p^2+q^2 = 2W^2$ and so

$$\frac{2}{\omega W} \le \frac{a(1+p^2)+(1+q^2)c+2bpq}{W} \le \frac{2\gamma}{\omega W}.$$

From (4.14),

$$\omega W \ge c_1 = \frac{1}{\gamma\sqrt{2}} > 0.$$

Then

$$\frac{2\gamma}{\omega W} \le \frac{2\gamma}{c_1} = 2\gamma^2\sqrt{2} = c_2 \le \infty$$

and therefore

$$\frac{2\gamma}{\omega W} \le 2\sqrt{2}\gamma^2.$$

We would like 2ϵ to equal $2\sqrt{2}\gamma^2$ and so we set

$$\epsilon = \sqrt{2}\gamma^2.$$

Therefore an equation of mean curvature type (4.5) implies (4.6). □

CHAPTER 5

CHARACTERISTIC COORDINATES ARE QUASICONFORMAL FUNCTIONS OF ISOTHERMAL COORDINATES

5.1 Structure Class

In order to show that we have global solvablity (or the global existence of) characteristic coordinates and later isothermal coordinates, we will introduce the following definitions from [1].

Let S be a connected Hausdorff space and Φ be a family of local homeomorphisms such that

(i) Each $h \in \Phi$ is a topological mapping of an open set $V \subset S$ onto an open set in $\mathbf{C}$.

(ii) If $h \in \Phi$ has domain V, then the restriction of h to any open set $V' \subset V$ is also in Φ.

(iii) Let h be a mapping of an open set $V \subset S$ onto an open set in the complex plane and suppose that V is covered by open subsets V'. If the restriction of h to each $V' \in \Phi$, then the same will be true of h with domain V.

(iv) The domains of all $h \in \Phi$ form a covering of S.

A family Ψ of local homeomorphisms with domain and range on $\mathbf{C}$ is called a **structure class** if the following conditions are satisfied:

(i) The identity mapping belongs to Ψ,

(ii) If $g \in \Psi$, then $g^{-1}\Psi$

(iii) If $g_1, g_2 \in \Psi$, then $g_1 \circ g_2 \in \Psi$ provided it is defined.

Now we say Φ defines a structure of class of Ψ if it satisfies the following:

(i) If $h^{(1)}, h^{(2)} \in \Phi$ have the same domain V then $\left(h^{(1)} \circ (h^{(2)})^{-1}\right) \in \Psi$.

(ii) If $h \in \Phi$ and $g \in \Psi$, then $g \circ h \in \Phi$ provided that it is defined.

A structure class which is formed by all analytic mappings will be called a **conformal structure**. A **Riemann surface** is defined by [1] to be a connected Hausdorff space W together with a conformal structure defined by a family Φ of local homeomorphisms on W. A complex valued function $h = s + it$ is said to be **analytic** on the Riemann surface (W, Φ) if and only if $h \circ f^{-1}$ is analytic on $f(V)$ for every $f \in \Phi$ with domain V.

5.2 Uniformization Theorem

The **Uniformization Theorem** as given by [1] is "the universal covering surface of any Riemann surface is conformally equivalent to a disk, to the complex plane or to the sphere."

The following results are found in [16] and we will state them without proof here. For each Riemann surface R, there is a covering map $\phi : S \to R$ of a simply connected Riemann surface S onto R. The surface S is called the **universal covering surface of R**. The only Riemann surface having the Riemann sphere as its universal covering surface is the sphere itself. The only Riemann surfaces having the complex plane as universal covering surface are the complex plane, the punctured complex plane and tori. All other connected, simply connected Riemann surfaces have the open unit disk as universal covering surface.

5.3 Global Existence of Characteristic Coordinates

In the following section we will use a footnote in [10, p. 160] as motivation; note that much of the terminology is from [1]. We will show the existence of a single normalizing system of characteristic coordinates, $(\sigma(x, y), \rho(x, y))$ in the entire domain. We will first show that the change of coordinates is conformal and therefore $\mathscr{S}$ is a Riemann surface and apply the Uniformization theorem. We then conclude that the characteristic coordinates are global because the Riemann surface is conformally equivalent to the unit disk.

Theorem 5. *Given a surface* $\mathscr{S} = \{(x, y, f(x, y)) : (x, y) \in \Omega\} \subset \mathbb{R}^3$ *with an equation of* *elliptic form (3.1) then there exists a single normalizing parameter system of characteristic coordinates with the mapping* $(x, y) \to (\sigma, \rho)$.

Proof. Consider two solutions $(\sigma^{(1)}, \rho^{(1)})$ and $(\sigma^{(2)}, \rho^{(2)})$ of the Beltrami equations,

$$\sigma_x^{(k)} = \frac{b\rho_x^{(k)} + c\rho_y^{(k)}}{\omega} \quad \text{and} \quad -\sigma_y^{(k)} = \frac{a\rho_x^{(k)} + b\rho_y^{(k)}}{\omega} \qquad (5.1)$$
$$\omega^2 = ac - b^2$$

that are defined in a small parametrized neighborhood on the surface $\mathscr{S}$.

We set $\omega^2 = ac - b^2 = 1$ by replacing a by $\frac{a}{\sqrt{ac-b^2}}$, b by $\frac{b}{\sqrt{ac-b^2}}$, c by $\frac{c}{\sqrt{ac-b^2}}$ and d by $\frac{d}{\sqrt{ac-b^2}}$. We assume the coefficients $a(x, y, z, p, q)$, $b(x, y, z, p, q)$, and $c(x, y, z, p, q)$ are Holder continuous. Therefore, we have local solvability of the Beltrami equations ([4]).

Locally we are working in a plane with the induced metric, so we will use the following

$$h^{(1)}(x, y) = \sigma^{(1)}(x, y) + i\rho^{(1)}(x, y) \;\text{ and }\; h^{(2)}(x, y) = \sigma^{(2)}(x, y) + i\rho^{(2)}(x, y).$$

Let $H(s+it) = \sigma^{(1)}\left((h^{(2)})^{-1}(s+it)\right) + i\rho^{(1)}\left((h^{(2)})^{-1}(s+it)\right) = m(s+it) + in(s+it)$. Notice that $m(s+it) = \sigma^{(1)}(x, y)$ if and only if $s + it = h^{(2)}(x, y)$.
We will show that $\dfrac{\partial H}{\partial \bar{z}} = 0$ which implies

$$\frac{\partial m}{\partial s} - \frac{\partial n}{\partial t} + i\left[\frac{\partial m}{\partial t} + \frac{\partial n}{\partial s}\right] = 0.$$

Here we will give a brief outline the proof. First, we will to show the Cauchy-Riemann equations

$$\frac{\partial m}{\partial s} = \frac{\partial n}{\partial t} \quad \text{and} \quad \frac{\partial m}{\partial t} = -\frac{\partial n}{\partial s}. \qquad (5.2)$$

are satisfied. Then $H(s + it)$ will be analytic in a neighborhood on the surface. Moreover, the mapping $H(s+it) = m(s+it)+in(s+it)$ defined in any two neighborhoods is conformal in the intersection of these two neighborhoods by (i) in the definition of structure of class Ψ and by (ii) from the definition of Φ. Lastly, we will apply the Uniformization Theorem to get global existence of characteristic coordinates.

Let $m(s+it) = \sigma^{(1)}(x, y)$ and $n(s+it) = \rho^{(1)}(x, y)$. We begin by using the chain rule:

$$\begin{aligned} \frac{\partial m}{\partial s} &= \frac{\partial \sigma^{(1)}}{\partial x}\frac{\partial x}{\partial s} + \frac{\partial \sigma^{(1)}}{\partial y}\frac{\partial y}{\partial s} \\ &= \sigma_x^{(1)} x_s + \sigma_y^{(1)} y_s. \end{aligned} \tag{5.3}$$

Similarly,

$$\begin{aligned} \frac{\partial m}{\partial t} &= \sigma_x^{(1)} x_t + \sigma_y^{(1)} y_t \\ \frac{\partial n}{\partial s} &= \rho_x^{(1)} x_s + \rho_y^{(1)} y_s \\ \frac{\partial n}{\partial t} &= \rho_x^{(1)} x_t + \rho_y^{(1)} y_t. \end{aligned} \tag{5.4}$$

Notice that

$$1 = \frac{\partial \sigma^{(2)}}{\partial s} = \frac{\partial \sigma^{(2)}}{\partial x}\frac{\partial x}{\partial s} + \frac{\partial \sigma^{(2)}}{\partial y}\frac{\partial y}{\partial s} \qquad 0 = \frac{\partial \sigma^{(2)}}{\partial t} = \frac{\partial \sigma^{(2)}}{\partial x}\frac{\partial x}{\partial t} + \frac{\partial \sigma^{(2)}}{\partial y}\frac{\partial y}{\partial t}$$

$$0 = \frac{\partial \rho^{(2)}}{\partial s} = \frac{\partial \rho^{(2)}}{\partial x}\frac{\partial x}{\partial s} + \frac{\partial \rho^{(2)}}{\partial y}\frac{\partial y}{\partial s} \qquad 1 = \frac{\partial \rho^{(2)}}{\partial t} = \frac{\partial \rho^{(2)}}{\partial x}\frac{\partial x}{\partial t} + \frac{\partial \rho^{(2)}}{\partial y}\frac{\partial y}{\partial t}.$$

Writing the equations above in matrix form gives

$$\begin{bmatrix} \sigma_x^{(2)} & \sigma_y^{(2)} \\ \rho_x^{(2)} & \rho_y^{(2)} \end{bmatrix} \begin{bmatrix} x_s & x_t \\ y_s & y_t \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}.$$

Solving the above equation and setting $\beta = \sigma_x^{(2)}\rho_y^{(2)} - \sigma_y^{(2)}\rho_x^{(2)}$ yields

$$\begin{bmatrix} x_s & x_t \\ y_s & y_t \end{bmatrix} = \frac{1}{\beta} \begin{bmatrix} \rho_y^{(2)} & -\sigma_y^{(2)} \\ -\rho_x^{(2)} & \sigma_x^{(2)} \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

and therefore

$$x_s = \tfrac{1}{\beta}\rho_y^{(2)} \qquad x_t = -\tfrac{1}{\beta}\sigma_y^{(2)}$$

$$y_s = -\tfrac{1}{\beta}\rho_x^{(2)} \qquad y_t = \tfrac{1}{\beta}\sigma_x^{(2)}.$$

Consider the first Cauchy-Riemann Equation (5.2). Using (5.4) and substituting the Beltrami equations (5.1) for $\sigma_x^{(k)}$ and $\sigma_y^{(k)}$ we find

$$\begin{aligned}\beta\omega\left[\frac{\partial m}{\partial s}-\frac{\partial n}{\partial t}\right] &= \beta\omega\left[\sigma_x^{(1)}x_s+\sigma_y^{(1)}y_s-\left(\rho_x^{(1)}x_t+\rho_y^{(1)}y_t\right)\right]\\ &= \omega\left[\sigma_x^{(1)}\rho_y^{(2)}-\sigma_y^{(1)}\rho_x^{(2)}+\rho_x^{(1)}\sigma_y^{(2)}-\rho_y^{(1)}\sigma_x^{(2)}\right]\\ &= \rho_y^{(2)}\left[b\rho_x^{(1)}+c\rho_y^{(1)}\right]+\rho_x^{(2)}\left[a\rho_x^{(1)}+b\rho_y^{(1)}\right]+\rho_x^{(1)}\left[-a\rho_x^{(2)}-b\rho_y^{(2)}\right]\\ &\quad -\rho_y^{(1)}\left[b\rho_x^{(2)}+c\rho_y^{(2)}\right]\\ &= \rho_x^{(1)}\rho_x^{(2)}(a-a)+\rho_x^{(1)}\rho_y^{(2)}(b-b)+\rho_y^{(1)}\rho_x^{(2)}(b-b)+\rho_y^{(1)}\rho_y^{(2)}(c-c)\\ &= 0.\end{aligned}$$

Therefore, $\dfrac{\partial m}{\partial s}=\dfrac{\partial n}{\partial t}$.

Now consider the second Cauchy-Riemann equation. Substituting the Beltrami equations (5.1) for $\sigma_x^{(k)}$ and $\sigma_y^{(k)}$ yeilds

$$\begin{aligned}\beta\omega^2\left[\frac{\partial m}{\partial t}+\frac{\partial n}{\partial s}\right] &= \beta\omega^2\left[\sigma_x^{(1)}x_t+\sigma_y^{(1)}y_t+\left(\rho_x^{(1)}x_s+\rho_y^{(1)}y_s\right)\right]\\ &= \omega^2\left[-\sigma_x^{(1)}\sigma_y^{(2)}+\sigma_y^{(1)}\sigma_x^{(2)}+\rho_x^{(1)}\rho_y^{(2)}-\rho_y^{(1)}\rho_x^{(2)}\right]\\ &= (b\rho_x^{(1)}+c\rho_y^{(1)})(a\rho_x^{(2)}+b\rho_y^{(2)})+(-a\rho_x^{(1)}-b\rho_y^{(1)})(b\rho_x^{(2)}+c\rho_y^{(2)})\\ &\quad +\omega^2\rho_x^{(1)}\rho_y^{(2)}-\omega^2\rho_y^{(1)}\rho_x^{(2)}\\ &= \rho_x^{(1)}\rho_x^{(2)}(ba-ab)+\rho_x^{(1)}\rho_y^{(2)}(b^2-ac+\omega^2)+\rho_y^{(1)}\rho_x^{(2)}(ca-b^2-\omega^2)\\ &\quad +\rho_y^{(1)}\rho_y^{(2)}(cb-bc)\\ &= 0\end{aligned}$$

Therefore $\dfrac{\partial m}{\partial t}=-\dfrac{\partial n}{\partial s}$ and hence the Cauchy-Riemann equations are satisfied. So we have shown $H(s+it)=\sigma^{(1)}\left((h^{(2)})^{-1}(s+it)\right)+i\rho^{(1)}\left((h^{(2)})^{-1}(s+it)\right)=m(s+it)+in(s+it)$ is analytic. Therefore $\left(h^{(1)}\circ(h^{(2)})^{-1}\right)\in\Psi$. Thus the transformation given by the local normalizing parameters σ and ρ defined in two neighborhoods is conformal in the intersection of these neighborhoods. These neighborhoods covering $\mathscr{S}$ form a Riemann Surface. Using the Uniformization theorem, $\mathscr{S}$ is a simply connected surface that is conformally equivalent to a disk and thus we have that characteristic coordinates are global parameters. □

5.4 Global Existence of Isothermal Coordinates

Using the same method as in Theorem 5, we will now prove the global existence of isothermal coordinates represented locally by (u, v). From [7], we see that in a domain of the (x, y) plane, if the functions E, F, G satisfy a Holder condition then isothermal coordinates exist.

Theorem 6. *Given a smooth surface $\mathscr{S}$ such that the functions E, F, G are Holder continuous, then there exists a parameter system of isothermal coordinates, $(u(x, y), v(x, y))$.*

Proof. Every smooth surface can be written locally as the graph over some plane and we recall the local representation of the metric induced by $\mathbb{R}^3$ to be

$$[g_{ij}] = \begin{bmatrix} 1 + f_x^2 & f_x f_y \\ f_x f_y & 1 + f_y^2 \end{bmatrix}.$$

Consider two solutions $(u^{(1)}, v^{(1)})$ and $(u^{(2)}, v^{(2)})$ of the Beltrami equations. In order to have local solvability we assume that the functions E, F, G must be Holder continuous [7].

As before, Ψ is a conformal structure class such that if $g^{(1)}, g^{(2)} \in \Phi$ have the same domain V then $\left(g^{(1)} \circ (g^{(2)})^{-1}\right) \in \Psi$. So let

$$g^{(1)}(x, y) = u^{(1)}(x, y) + iv^{(1)}(x, y) \text{ and } g^{(2)}(x, y) = u^{(2)}(x, y) + iv^{(1)}(x, y).$$

Furthermore, let

$$G(s + it) = u^{(1)} \left(g^{(2)}\right)^{-1} (s + it) + v^{(1)} \left(g^{(2)}\right)^{-1} (s + it) = m(s + it) + in(s + it)$$

We will show that

$$\frac{\partial m}{\partial s} = \frac{\partial n}{\partial t} \text{ and } \frac{\partial m}{\partial t} = -\frac{\partial n}{\partial s}.$$

Let $m(s + it) = u^{(1)}(x, y)$ and $n(x, y) = v^{(1)}(x, y)$. Using the chain rule,

$$\begin{aligned} \frac{\partial m}{\partial s} &= u_x^{(1)} x_s + u_y^{(1)} y_s \\ \frac{\partial m}{\partial t} &= u_x^{(1)} x_t + u_y^{(1)} y_t \\ \frac{\partial n}{\partial s} &= v_x^{(1)} x_s + v_y^{(1)} y_s \\ \frac{\partial n}{\partial t} &= v_x^{(1)} x_t + v_y^{(1)} y_t. \end{aligned}$$

Notice that

$$1 = \frac{\partial u^{(2)}}{\partial s} = \frac{\partial u^{(2)}}{\partial x}\frac{\partial x}{\partial s} + \frac{\partial u^{(2)}}{\partial y}\frac{\partial y}{\partial s} \qquad 0 = \frac{\partial u^{(2)}}{\partial t} = \frac{\partial u^{(2)}}{\partial x}\frac{\partial x}{\partial t} + \frac{\partial u^{(2)}}{\partial y}\frac{\partial y}{\partial t}$$

$$0 = \frac{\partial v^{(2)}}{\partial s} = \frac{\partial v^{(2)}}{\partial x}\frac{\partial x}{\partial s} + \frac{\partial v^{(2)}}{\partial y}\frac{\partial y}{\partial s} \qquad 1 = \frac{\partial v^{(2)}}{\partial t} = \frac{\partial v^{(2)}}{\partial x}\frac{\partial x}{\partial t} + \frac{\partial v^{(2)}}{\partial y}\frac{\partial y}{\partial t}.$$

Writing the previous equations in matrix form yields

$$\begin{bmatrix} u_x^{(2)} & u_y^{(2)} \\ v_x^{(2)} & v_y^{(2)} \end{bmatrix} \begin{bmatrix} x_s & x_t \\ y_s & y_t \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}.$$

Solving this matrix equation and letting $\delta = u_x^{(2)} v_y^{(2)} - v_x^{(2)} u_y^{(2)}$,

$$\begin{aligned} x_s &= \frac{1}{\delta} v_y^{(2)} \qquad x_t = -\frac{1}{\delta} u_y^{(2)} \\ y_s &= -\frac{1}{\delta} v_x^{(2)} \qquad y_t = \frac{1}{\delta} u_x^{(2)}. \end{aligned} \tag{5.5}$$

Now we find the Beltrami equations for isothermal coordinates

$$u_x^{(k)} = \frac{-pq}{W} v_x^{(k)} + \frac{1+p^2}{W} v_y^{(k)} \tag{5.6}$$

$$-u_y^{(k)} = \frac{1+q^2}{W} v_x^{(k)} + \frac{-pq}{W} v_y^{(k)} \tag{5.7}$$

where $W^2 = 1 + p^2 + q^2$. Substituting (5.5) and (5.6) into the Cauchy-Riemann equation $\frac{\partial m}{\partial s} = \frac{\partial n}{\partial t}$ we have

$$\begin{aligned} \delta W \left[\frac{\partial m}{\partial s} - \frac{\partial n}{\partial t} \right] &= \delta W \left[u_x^{(1)} x_s + u_y^{(1)} y_s - v_x^{(1)} x_t - v_x^{(1)} y_t \right] \\ &= W \left[u_x^{(1)} v_y^{(2)} - u_y^{(1)} v_x^{(2)} + v_x^{(1)} u_y^{(2)} - v_y^{(1)} u_x^{(2)} \right] \\ &= (-pq v_x^{(1)} + (1+p^2) v_y^{(1)}) v_y^{(2)} + ((1+q^2) v_x^{(1)} - pq v_y^{(1)}) v_x^{(2)} \\ &\quad - ((1+q^2) v_x^{(2)} - pq v_y^{(2)}) v_x^{(1)} - (-pq v_x^{(2)} + (1+p^2) v_y^{(2)}) v_y^{(1)} \\ &= v_x^{(1)} v_y^{(2)} [-pq + pq] + v_y^{(1)} v_y^{(2)} [(1+p^2) - (1+p^2)] \\ &\quad + v_x^{(1)} v_x^{(2)} [(1+q^2) - (1+q^2)] + v_y^{(1)} v_x^{(2)} [-pq + pq] \\ &= 0. \end{aligned}$$

Now consider $\frac{\partial m}{\partial t} + \frac{\partial n}{\partial s}$, recall that $W^2 = 1 + p^2 + q^2$ and use (5.5)-(5.6), to simplify

$$\begin{aligned}
\delta W^2 \left[\frac{\partial m}{\partial t} + \frac{\partial n}{\partial s}\right] &= \delta W^2 \left[u_x^{(1)} x_t + u_y^{(1)} y_t + v_x^{(1)} x_s + v_y^{(1)} y_s\right] \\
&= W^2 \left[-u_x^{(1)} u_y^{(2)} + u_y^{(1)} u_x^{(2)} + v_x^{(1)} v_y^{(2)} - v_y^{(1)} v_x^{(2)}\right] \\
&= (-pq v_x^{(1)} + (1+p^2) v_y^{(1)})((1+q^2) v_x^{(2)} - pq v_y^{(2)}) \\
&\quad + (-(1+q^2) v_x^{(1)} + pq v_y^{(1)})(-pq v_x^{(2)} + (1+p^2) v_y^{(2)}) + W^2 v_x^{(1)} v_y^{(2)} - W^2 v_y^{(1)} v_x^{(2)} \\
&= -pq(1+q^2) v_x^{(1)} v_x^{(2)} + p^2 q^2 v_x^{(1)} v_y^{(2)} + (1+p^2)(1+q^2) v_y^{(1)} v_x^{(2)} \\
&\quad - (1+p^2) pq v_y^{(1)} v_y^{(2)} + (1+q^2) pq v_x^{(1)} v_x^{(2)} - (1+q^2)(1+p^2) v_x^{(1)} v_y^{(2)} \\
&\quad - p^2 q^2 v_y^{(1)} v_x^{(2)} + pq(1+p^2) v_y^{(1)} v_y^{(2)} + v_x^{(1)} v_y^{(2)} - v_y^{(1)} v_x^{(2)} \\
&= v_x^{(1)} v_x^{(1)} [-pq(1+q^2) + pq(1+q^2)] + v_x^{(1)} v_y^{(2)} [p^2 q^2 - (1+q^2)(1+p^2) + W^2] \\
&\quad + v_y^{(1)} v_x^{(2)} [(1+p^2)(1+q^2) - p^2 q^2 - W^2] + v_y^{(1)} v_y^{(2)} [pq(1+p^2) - pq(1+p^2)] \\
&= 0.
\end{aligned}$$

Thus the Cauchy-Riemann equations are satisfied. Therefore the transformation $G : G(s + it) = u^{(1)} \left(g^{(2)}\right)^{-1} (s + it) + v^{(1)} \left(g^{(2)}\right)^{-1} (s + it) = m(s + it) + in(s + it)$ defined in two neighborhoods is conformal in the intersection of these neighborhoods. These neighborhoods form a Riemann Surface and we can now apply the Uniformization theorem to find that isothermal coordinates are global parameters, as we did with characteristic coordinates.

□

5.5 Quasiconformal Mappings

Informally, one can think of a conformal map as one that maps infinitesimal circles to infinitesimal circles. A quasiconformal map takes infinitesimal circles to infinitesimal ellipses. In the following proof, we will show that we have a quasiconformal mapping from the isothermal parameter domain to the characteristic parameter domain. For a mapping $w(z) = u(x, y) + iv(x, y)$ having continuous partial derivatives and a non-vanishing Jacobian then the informal description of a quasiconformal mapping can be expressed by any of these three equivalent differential equations as given in [10]

$$\max_{0\le\theta\le 2\pi}|w_x\cos\theta + w_y\sin\theta|^2 \le Q(u_xv_y - u_yv_x)$$
$$u_x^2+u_y^2+v_x^2+v_y^2 \le 2K(u_xv_y - u_yv_x), \tag{5.8}$$
$$|w_{\bar z}| \le k|w_z|.$$

Here $Q \ge 1, K \ge 1, 0 \le k < 1$ and the constants are linked by the following:

$$K = \frac{1}{2}\left(Q + \frac{1}{Q}\right), \qquad k = \frac{Q-1}{Q+1}.$$

From ([1]), we note that if T is k-quasiconformal, then T^{-1} is also k-quasiconformal. Consider the following equation from [14] of the form

$$a(x,y,z,p,q)\phi_{xx} + 2b(x,y,z,p,q)\phi_{xy} + c(x,y,z,p,q)\phi_{yy} = 0 \tag{5.9}$$

with $ac - b^2 = 1$ with $p = \phi_x$ and $q = \phi_y$. It is shown in lemma 8 of [14] that

$$\frac{(1+p^2)a}{W} + \frac{2bpq}{W} + \frac{(1+q^2)c}{W} \le 2\epsilon \tag{5.10}$$

for each $(x,y) \in \Omega$, where $\epsilon \ge 1$ with $\epsilon = 1$ if and only if (5.9) is the minimal surface equation.

Theorem 7. *Let E be the unit disk parametrized by characteristic coordinates (σ,ρ) and isothermal coordinates (u,v). Let $K = E \to \Omega$ where K is a one-to-one and onto mapping such that $K(u,v) = (x(u,v), y(u,v))$. Let $\widetilde{K} = E \to \Omega$ be a one-to-one and onto mapping such that $\widetilde{K}(\sigma,\rho) = (\tilde{x}(\sigma,\rho), \tilde{y}(\sigma,\rho))$. If $T = \tilde{K}^{-1} \circ K$, then T is quasiconformal.*

Proof. We will outline the proof before we begin. In Step 1, we perform a change of variables for σ_x, σ_y, ρ_x and ρ_y. Step 2 will then consist of finding the Beltrami equations in characteristic coordinates in terms of isothermal coordinates. Step 3 will be algebraic calculations to show in Step 4 that those Beltrami equations satisfy the definition of quasiconformal coordinate systems from [10].

Step 1: We begin by using the chain rule to write

$$\begin{aligned}\frac{\partial\sigma}{\partial x} &= \frac{\partial\sigma}{\partial u}\frac{\partial u}{\partial x} + \frac{\partial\sigma}{\partial v}\frac{\partial v}{\partial x}\\ &= \sigma_u u_x + \sigma_v v_x\end{aligned}$$

$$\begin{aligned}\frac{\partial\sigma}{\partial y} &= \frac{\partial\sigma}{\partial u}\frac{\partial u}{\partial y} + \frac{\partial\sigma}{\partial v}\frac{\partial v}{\partial y}\\ &= \sigma_u u_y + \sigma_v v_y.\end{aligned}$$

In matrix form

$$\begin{bmatrix}\sigma_x\\ \sigma_y\end{bmatrix} = \begin{bmatrix}u_x & v_x\\ u_y & v_y\end{bmatrix}\begin{bmatrix}\sigma_u\\ \sigma_v\end{bmatrix}.$$

We solve the matrix equation and complete a change of variables

$$\begin{bmatrix}\sigma_u\\ \sigma_v\end{bmatrix} = \frac{1}{u_x v_y - u_y v_x}\begin{bmatrix}v_y & -v_x\\ -u_y & u_x\end{bmatrix}\begin{bmatrix}\sigma_x\\ \sigma_y\end{bmatrix}.$$

Rewriting and let $\delta = u_x v_y - u_y v_x$ to obtain

$$\begin{aligned}\sigma_u &= \frac{v_y}{\delta}\sigma_x - \frac{v_x}{\delta}\sigma_y\\ \sigma_v &= \frac{-u_y}{\delta}\sigma_x + \frac{u_x}{\delta}\sigma_y.\end{aligned} \tag{5.11}$$

Similarly we can perform a change of variables for ρ_x and ρ_y:

$$\begin{aligned}\frac{\partial\rho}{\partial x} &= \frac{\partial\rho}{\partial u}\frac{\partial u}{\partial x} + \frac{\partial\rho}{\partial v}\frac{\partial v}{\partial x}\\ &= u_x\rho_u + v_x\rho_v\\ \frac{\partial\rho}{\partial y} &= \frac{\partial\rho}{\partial u}\frac{\partial u}{\partial y} + \frac{\partial\rho}{\partial v}\frac{\partial v}{\partial y}\\ &= u_y\rho_u + v_y\rho_v.\end{aligned} \tag{5.12}$$

Then using a similar matrix equation as above, we can find corresponding equations for ρ_u and ρ_v to be

$$\begin{aligned}\rho_u &= \frac{v_y}{\delta}\rho_x - \frac{v_x}{\delta}\rho_y\\ \rho_v &= \frac{-u_y}{\delta}\rho_x + \frac{u_x}{\delta}\rho_y.\end{aligned} \tag{5.13}$$

Step 2: We recall the Beltrami equations (3.10) for characteristic coordinates (ρ, σ) in terms of Cartesian coordinates (x, y) to be

$$\sigma_x = \frac{b\rho_x + c\rho_y}{\omega} \quad \text{and} \quad \sigma_y = -\frac{a\rho_x + b\rho_y}{\omega}$$

where $\omega^2 = ac - b^2$. We can substitute the Beltrami equations for σ_x and σ_y into (5.11) and then replace ρ_x and ρ_y with (5.12) to obtain

$$\begin{aligned}
\sigma_u &= \frac{v_y}{\delta}\left(\frac{b\rho_x + c\rho_y}{\omega}\right) - \frac{v_x}{\delta}\left(-\frac{a\rho_x + b\rho_y}{\omega}\right) \\
&= \frac{v_y b}{\omega\delta}\rho_x + \frac{v_y c}{\omega\delta}\rho_y + \frac{v_x a}{\omega\delta}\rho_x + \frac{v_x b}{\omega\delta}\rho_y \\
&= \frac{v_y b}{\omega\delta}(u_x\rho_u + v_x\rho_v) + \frac{v_y c}{\omega\delta}(u_y\rho_u + v_y\rho_v) + \frac{v_x a}{\omega\delta}(u_x\rho_u + v_x\rho_v) + \frac{v_x b}{\omega\delta}(u_y\rho_u + v_y\rho_v) \\
&= \left[\frac{v_y u_x b + v_y u_y c + v_x u_x a + v_x u_y b}{\omega\delta}\right]\rho_u + \left[\frac{v_x v_y b + v_y v_y c + v_x v_x a + v_x v_y b}{\omega\delta}\right]\rho_v \\
&= \left[\frac{v_x u_x a + (v_y u_x + v_x u_y)b + v_y u_y c}{\omega\delta}\right]\rho_u + \left[\frac{v_x^2 a + 2v_x v_y b + v_y^2 c}{\omega\delta}\right]\rho_v.
\end{aligned}$$

Similarly for σ_v

$$\begin{aligned}
\sigma_v &= \frac{-u_y}{\delta}\sigma_x + \frac{u_x}{\delta}\sigma_y \\
&= \frac{-u_y}{\delta}\left(\frac{b\rho_x + c\rho_y}{\omega}\right) + \frac{u_x}{\delta}\left(-\frac{a\rho_x + b\rho_y}{\omega}\right) \\
&= -\left[\frac{u_y b}{\omega\delta}(u_x\rho_u + v_x\rho_v) + \frac{u_y c}{\omega\delta}(u_y\rho_u + v_y\rho_v) + \frac{u_x a}{\omega\delta}(u_x\rho_u + v_x\rho_v) + \frac{u_x b}{\omega\delta}(u_y\rho_u + v_y\rho_v)\right] \\
&= -\left[\frac{u_y u_x b + u_y u_y c + u_x u_x a + u_x u_y b}{\omega\delta}\right]\rho_u - \left[\frac{v_x u_y b + v_y u_y c + v_x u_x a + v_y u_x b}{\omega\delta}\right]\rho_v \\
&= -\left[\frac{u_x^2 a + 2u_x u_y b + u_y^2 c}{\omega\delta}\right]\rho_u - \left[\frac{v_x u_x a + (u_y v_x + u_x v_y)b + u_y v_y c}{\omega\delta}\right]\rho_v.
\end{aligned}$$

Thus the Beltrami equations in characteristic coordinates (σ, ρ) in terms of isothermal coordinates (u, v) are:

$$\sigma_u = \left[\frac{v_x u_x a + (v_y u_x + v_x u_y)b + v_y u_y c}{\omega\delta}\right]\rho_u + \left[\frac{v_x^2 a + 2v_x v_y b + v_y^2 c}{\omega\delta}\right]\rho_v \tag{5.14}$$

$$-\sigma_v = \left[\frac{u_x^2 a + 2u_x u_y b + u_y^2 c}{\omega\delta}\right]\rho_u + \left[\frac{v_x u_x a + (u_y v_x + u_x v_y)b + u_y v_y c}{\omega\delta}\right]\rho_v, \tag{5.15}$$

where $\omega^2 = ac - b^2$ and $\delta = u_x v_y - u_y v_x$.

Step 3: From [10], every solution $w = \phi + i\psi$ of an elliptic system

$$\sigma_u = \gamma_{11}\rho_u + \gamma_{12}\rho_v$$

$$-\sigma_v = \gamma_{21}\rho_u + \gamma_{22}\rho_v$$

will satisfy (5.8) if the system is uniformly elliptic, that is if a_{12}, and

$$0 < \frac{(\gamma_{12} + \gamma_{21})^2}{4\gamma_{12}\gamma_{21} - (\gamma_{11} + \gamma_{22})^2} < \text{ constant.} \tag{5.16}$$

One can then conclude that the coordinate systems are quasiconformal and the constant Q of quasiconformality depends only on the constant in (5.16). We need to show that the system (5.14) satisfies (5.16). Utilizing (5.14)-(5.15), we find the γ_{ij}'s in (5.16)

$$\gamma_{11} = \frac{v_x u_x a + (v_y u_x + v_x u_y)b + v_y u_y c}{\omega\delta},$$

$$\gamma_{12} = \frac{v_x^2 a + 2v_x v_y b + v_y^2 c}{\omega\delta}, \tag{5.17}$$

$$\gamma_{21} = \frac{u_x^2 a + 2u_x u_y b + u_y^2 c}{\omega\delta},$$

$$\gamma_{22} = \frac{v_x u_x a + (u_y v_x + u_x v_y)b + u_y v_y c}{\omega\delta}.$$

Now we substitute (5.17) into (5.16) and simplify

$$
\begin{aligned}
(\gamma_{12}+\gamma_{21})^2 &= \frac{\left[(v_x^2+u_x^2)a+2b(v_xv_y+u_xu_y)+(u_y^2+v_y^2)c\right]^2}{\omega^2\delta^2} \\
4\gamma_{12}\gamma_{21} &= \frac{4v_x^2u_x^2a^2+8v_x^2u_yu_xab+4v_x^2u_y^2ac+8v_xv_yu_x^2ab+16u_xu_yv_xv_yb^2}{\omega^2\delta^2} \\
&\quad + \frac{8v_xv_yu_y^2bc+4v_y^2u_x^2ac+8u_xu_yv_y^2bc+4v_y^2u_y^2c^2}{\omega^2\delta^2}
\end{aligned} \tag{5.18}
$$

$$
\begin{aligned}
(\gamma_{11}+\gamma_{22})^2 &= \frac{[2u_xv_xa+2b(u_yv_x+u_xv_y)+2u_yv_yc]^2}{\omega^2\delta^2} \\
&= \frac{4u_x^2v_x^2a^2+4b^2(u_yv_x+u_xv_y)^2+4u_y^2v_y^2c^2+8abu_xv_x(u_yv_x+u_xv_y)}{\omega^2\delta^2} \\
&\quad + \frac{8acu_xu_yv_xv_y+8bcu_yv_y(u_yv_x+u_xv_y)}{\omega^2\delta^2}.
\end{aligned} \tag{5.19}
$$

Using (5.18), we simplify to find the denominator in (5.16)

$$
\begin{aligned}
4\gamma_{12}\gamma_{21}-(\gamma_{11}+\gamma_{22})^2 &= \frac{4(ac-b^2)(u_xv_y-u_yv_x)^2}{\omega^2\delta^2} \\
&= \frac{4\omega^2\delta^2}{\omega^2\delta^2} = 4.
\end{aligned} \tag{5.20}
$$

Since (u, v) represent an isothermal coordinate system, we can utilize the Beltrami equations

$$
u_x = \frac{-pq}{W}v_x + \frac{1+p^2}{W}v_y \tag{5.21}
$$

$$
-u_y = \frac{1+q^2}{W}v_x + \frac{-pq}{W}v_y \tag{5.22}
$$

where $W = \sqrt{1+p^2+q^2}$.

Substituting (5.21)-(5.22), we can further simplify parts of (5.16).

$$\begin{aligned}
\delta &= u_x v_y - u_y v_x \\
&= \left(\frac{-pq}{W}v_x + \frac{1+p^2}{W}v_y\right)v_y + \left(\frac{1+q^2}{W}v_x + \frac{-pq}{W}v_y\right)v_x \\
&= \frac{(pv_y - qv_x)^2 + v_x^2 + v_y^2}{W}
\end{aligned}$$

$$\begin{aligned}
u_x u_y + v_x v_y &= \left(\frac{-pq}{W}v_x + \frac{1+p^2}{W}v_y\right)\left(-\frac{1+q^2}{W}v_x + \frac{pq}{W}v_y\right) + v_x v_y \\
&= \frac{pq}{W^2}\left[(qv_x - pv_y)^2 + v_x^2 + v_y^2\right] \\
v_x^2 + u_x^2 &= v_x^2 + \left(\frac{-pq}{W}v_x + \frac{1+p^2}{W}v_y\right)^2 \\
&= \frac{1+p^2}{W^2}\left[(qv_x - pv_y)^2 + v_x^2 + v_y^2\right] \\
u_y^2 + v_y^2 &= \left[\frac{1+q^2}{W}v_x + \frac{-pq}{W}v_y\right]^2 + v_y^2 \\
&= \frac{1+q^2}{W^2}\left[(qv_x - pv_y)^2 + v_x^2 + v_y^2\right].
\end{aligned} \tag{5.23}$$

Using (5.23), we can find

$$\begin{aligned}
(\gamma_{12}+\gamma_{21})^2 &= \frac{\left[(v_x^2+u_x^2)a + 2b(v_x v_y + u_x u_y) + (u_y^2+v_y^2)c\right]^2}{\omega^2\delta^2} \\
&= \frac{1+p^2}{\omega^2\delta^2 W^4}\left[(qv_x - pv_y)^2 + v_x^2 + v_y^2\right]a + 2b\frac{pq}{\omega^2\delta^2 W^4}\left[(qv_x - pv_y)^2 + v_x^2 + v_y^2\right] \\
&\quad + \frac{1+q^2}{\omega^2\delta^2 W^4}\left[(qv_x - pv_y)^2 + v_x^2 + v_y^2\right]c \\
&= \frac{1}{W^4\omega^2\delta^2}\left[\left((qv_x - pv_y)^2 + v_x^2 + v_y^2\right)^2\left((1+p^2)a + 2bpq + (1+q^2)c\right)^2\right] \qquad (5.24) \\
&= \frac{1}{W^4\omega^2}\left[(1+p^2)a + 2bpq + (1+q^2)c\right]^2. \qquad (5.25)
\end{aligned}$$

Substituting (5.20) and (5.25) into (5.16), and using (5.23) to simplify, we obtain

$$\begin{aligned}
\frac{(\gamma_{12}+\gamma_{21})^2}{4\gamma_{12}\gamma_{21} - (\gamma_{11}+\gamma_{22})^2} &= \frac{\frac{1}{W^4\omega^2}\left[(1+p^2)a + 2bpq + (1+q^2)c\right]^2.}{4} \qquad (5.26) \\
&= \frac{1}{4\omega^2 W^2}\left[(1+p^2)a + 2bpq + (1+q^2)c\right]^2. \qquad (5.27)
\end{aligned}$$

Step 4: Using the hypothesis given in the statement of Theorem 7 and given $\epsilon \geq 1$, then

$$\frac{(\gamma_{12}+\gamma_{21})^2}{4\gamma_{12}\gamma_{21}-(\gamma_{11}+\gamma_{22})^2} = \frac{1}{4(ac-b^2)}\left[\frac{(1+p^2)a+2bpq+(1+q^2)c}{W}\right]^2 \leq \frac{4\epsilon^2}{4} = \epsilon^2.$$

Lemma 8 in [14], states if

$$\alpha = \frac{1+q^2}{W},\ \beta = \frac{-pq}{W},\ \gamma = \frac{1+p^2}{W}$$

and $\alpha\gamma - \beta^2 = 1$ with $a\gamma + c\alpha - 2b\beta = 2\epsilon$ where $\epsilon \geq 1$ then,

$$2 \leq \frac{(1+p^2)a}{W} + \frac{2bpq}{W} + \frac{(1+q^2)c}{W}.$$

Therefore,

$$2 \leq \frac{(\gamma_{12}+\gamma_{21})^2}{4\gamma_{12}\gamma_{21}-(\gamma_{11}+\gamma_{22})^2} \leq \frac{4\epsilon^2}{4} = \epsilon^2.$$

Recall from [10] that we require $Q \geq 1$ and "the constant Q of quasiconformality depends only on the constant" in (5.16) which is equal to ϵ^2 as shown. Therefore, we conclude that the mapping T from the parameter domain in isothermal coordinates (u, v) to the domain parameterized by characteristic coordinates (σ, ρ) is quasiconformal. □

Note that the inverse mapping of T which maps from the domain in characteristic coordinates into the domain parameterized by isothermal coordinates is also quasiconformal ([1]).

CHAPTER 6

APPLICATIONS

Let $\alpha \in (0,\pi)$ and Ω be a bounded, locally Lipschitz domain in $\mathbb{R}^2$ with $O = (0,0) \in \partial\Omega$ such that $\theta = \pm\alpha$ are the tangent rays to $\partial\Omega$ at O and $\{r(\cos(\theta), \sin(\theta)) : r \geq 0, -\alpha \leq \theta \leq \alpha\}$ is the cone obtained by blowing up $\overline{\Omega}$ about O. Let

$$\sum_{i,j=1}^{2} a_{ij}(x, f, \nabla f) D_{ij} f = h(x, f, \nabla f), \qquad f \in C^2(\Omega), \tag{6.1}$$

be a specific equation of mean curvature type on Ω and let ϕ be a piecewise continuous function from $\partial\Omega$ to $\mathbb{R}$; we will consider f to be a fixed bounded solution of (6.1) which satisfies $f = \phi$ (a.e.) on $\partial\Omega$. Since our interest is in the limiting behavior of f at O, we shall assume f is discontinuous at O.

We may parametrize the graph of f in isothermal coordinates using the techniques in [23] and [25]. The idea in these papers is to estimate the modulus of continuity of the isothermal parametrization $X : E \to \mathbb{R}^3$ ($E = \{(u,v) : u^2 + v^2 < 1\}$) of

$$\mathcal{S}_f = \{(x, f(x)) \in \mathbb{R}^3 \ : \ x \in \Omega\}$$

using Courant's Lemma 3.1 ([9]) and the comparison principle and to prove that this parametrization extends continuously to $\overline{E}$. In the case here, these techniques yield an isothermal, parametric description $Y : \overline{E} \to \mathbb{R}^3$ of the closure S of $S_f = \{(x, f(x)) : x \in \Omega\}$,

$$Y(u,v) = (x(u,v), y(u,v), z(u,v)), \quad (u,v) \in \overline{E},$$

such that

(i) $Y \in C^2(E : \mathbb{R}^3) \cap W^{1,2}(E : \mathbb{R}^3)$.

(ii) Y is a homeomorphism of E onto S_f.

(iii) Y maps ∂E onto $\{(x, f(x)) : x \in \partial\Omega\} \cup (\{O\} \times [z_1, z_2])$, where $z_1 = \liminf_{\overline{\Omega} \ni x \to O} f(x)$ and $z_2 = \limsup_{\overline{\Omega} \ni x \to O} f(x)$.

(iv) Y is conformal on E: $Y_u \cdot Y_v = 0, |Y_u| = |Y_v|$ on E.

(v) Let $\tilde{H}(u,v) = H(Y(u,v))$ denote the prescribed mean curvature of $\mathcal{S}_f$ at $Y(u,v)$. Then $\triangle Y := Y_{uu} + Y_{vv} = \tilde{H}\ Y_u \times Y_v$.

(vi) $Y \in C^0(\overline{E})$.

(vii) Writing $G(u,v) = (x(u,v), y(u,v)), G(\cos(t), \sin(t))$ moves clockwise about $\partial\Omega$ as t increases, $0 \le t \le 2\pi$, and G is an orientation reversing homeomorphism from E onto Ω; G maps $\overline{E}$ onto $\overline{\Omega}$ and, if f is continuous at O, then G is a homeomorphism from $\overline{E}$ onto $\overline{\Omega}$.

We note that when $h \neq 0$, the proof of (vi) depends on an extension of Theorem 2 of [21] whose correctness Professor Jin has verified. Since we assumed f is discontinuous at O, the map G in (vii) cannot be a homeomorphism from $\overline{E}$ onto $\overline{\Omega}$. In fact, if $G^{-1}(O)$ is a single point of ∂E, then f must be continuous at O; since this conclusion is false, there must be a nonempty, open arc σ_0 of ∂E whose closure σ is mapped by G to O. Now set $B = \{(u,v) : u^2 + v^2 < 1, v > 0\}$, $\partial' B = \{(u,v) : u^2 + v^2 = 1, v > 0\} \cup \{(\pm 1, 0)\}$ and $\partial'' B = \{(u,v) : u^2 + v^2 < 1, v = 0\}$ and then let ψ be the conformal map from B to E which maps $\partial'\Omega$ to $\partial\Omega \setminus \sigma_0$ and $\partial''\Omega$ to σ_0. Finally, define $X = Y \circ \psi$. Notice that $X \in C^2(B : \mathbb{R}^3) \cap W^{1,2}(B : \mathbb{R}^3)$ and satisfies (i)-(vii) above (with E replaced by B and ∂E by ∂B.)

If (6.1) is the prescribed mean curvature equation with bounded mean curvature, then one can use (v) to show that $X \in C^{1,\delta}(E \cup \partial'' B : \mathbb{R}^3)$ and the gradient of X has a specific first-order expansion at any point of $\partial'' B$ at which X_u and X_v vanish; this information allows one to conclude that the radial limits of f at O,

$$Rf(\theta) = \lim_{r \downarrow 0} f(r\cos(\theta), r\sin(\theta)),$$

exist for all (or almost all) $\theta \in (-\alpha, \alpha)$. In the general case where (6.1) is an equation of mean curvature type, we have no specific information about the mean curvature of $\mathcal{S}_f$ and, in particular, cannot expect the mean curvature of $\mathcal{S}_f$ to be bounded. Therefore, we cannot use exactly the same techniques used in [25] and [23] to obtain the regularity of $X(u,v)$ on $\partial'' B$

which is required to establish the existence of the radial limits of f at O. An alternative is to reparametrize $\mathcal{S}_f$ in characteristic coordinates corresponding to equation (6.1) and attempt to use the results of section 3.2 to obtain the necessary regularity.

Let ω be the quasiconformal homeomorphism from $\overline{B}$ to $\overline{B}$ which maps $\partial' B$ to $\partial' B$ and $\partial'' B$ to $\partial'' B$ such that $X \circ \omega$ is a characteristic coordinate parametrization of $\mathcal{S}_f$. (Here the results of chapters 4 and 5 are used to establish the existence of the parametrizations and Theorem 5 and properties of quasiconformal maps are used to show that ω is a homeomorphism from ∂B to ∂B.) Now define $Z : \overline{B} \to \mathcal{S}$ by

$$Z(\sigma,\rho) = X(\omega(\sigma,\rho)), \qquad (\sigma,\rho) \in \overline{B},$$

where $\mathcal{S}$ is the closure in $\mathbb{R}^3$ of $\mathcal{S}_f$. Now the results of section 3.2 hold for Z; and we obtain the following result.

Theorem 8. *Suppose (6.1) is an equation of mean curvature type whose normal form (3.15) -(3.17) in characteristic coordinates* (σ,ρ) *satisfies* $|\Delta \boldsymbol{x}| \le C|D\boldsymbol{x}|^2$ *for some constant* C*; that is*

$$|x_{\sigma\sigma} + x_{\rho\rho}| \le C(|x_\sigma|^2 + |x_\rho|^2 + |y_\sigma|^2 + |y_\rho|^2 + |z_\sigma|^2 + |z_\rho|^2)$$
$$|y_{\sigma\sigma} + y_{\rho\rho}| \le C(|x_\sigma|^2 + |x_\rho|^2 + |y_\sigma|^2 + |y_\rho|^2 + |z_\sigma|^2 + |z_\rho|^2)$$
$$|z_{\sigma\sigma} + z_{\rho\rho}| \le C(|x_\sigma|^2 + |x_\rho|^2 + |y_\sigma|^2 + |y_\rho|^2 + |z_\sigma|^2 + |z_\rho|^2).$$

Then the radial limits $Rf(\theta)$ *of* f *at* 0 *exist for almost all* $\theta \in [-\alpha,\alpha]$.

Conjecture 1. $Z \in C^{1,\delta}(E \cup \partial'' B : \mathbb{R}^3)$ *and* $Rf(\theta)$ *exists for almost every* $\theta \in (-\alpha,\alpha)$.

CHAPTER 7

CONCLUSION

The motivating force of this thesis was to create some of the infrastructure required to investigate the behavior at corners of solutions of boundary value problems for equation of mean curvature type. Further research can utilize the connections that were established between isothermal and harmonic coordinates as well as the relationship between isothermal and characteristic coordinates. We showed that for a two-dimensional manifold, isothermal coordinates are harmonic, though this does not hold in higher dimensions unless the manifold is conformally flat.

Next, we examined characteristic coordinates which arise in partial differential equations and focused on elliptic type. We derived the normal form for an elliptic partial differential equation and the corresponding Beltrami equation. Dr. Lancaster calculated and simplified the Laplacian of the components of the position vector $\mathbf{x}$ with respect to the variables ρ and σ and plans to use this for a future paper. In section 3.3, given a function f we found the local representation for the metric in characteristic coordinates with respect to Cartesian coordinates by solving a system of equations. This representation of the metric in characteristic coordinates allowed us to reach the conclusion that if the graph of the function was minimal, then the characteristic coordinates are isothermal. We then used the local representation of the metric and an example to show that if the surface is not minimal then the characteristic coordinates are not a harmonic coordinate system.

In chapter 4, we introduced equations of minimal surface type and equations of mean curvature type. We then showed in the proof of Theorem 4 that the form for the two dimensional equation of mean curvature type given by Simon in [29] is equivalent to the structure conditions as given by Finn in [14]. We begin chapter 5 by introducing a structure class, Riemann surfaces and the Uniformization theorem. We use these ideas to show the global existence of characteristic and isothermal coordinates on a surface, based on a footnote in [10]. Following the existence proof, we then define quaisconformal maps and note that if a mapping is quasiconformal, then its inverse is also quasiconformal. That allowed us to prove that given a domain parameterized by characteristic coordinates and a domain parameterized by isothermal coordinates, the composition map between them is quasiconformal.

The application section shows how the results of this paper can be utilized to further Dr. Lancaster's studies of the behavior of the radial limits when examining equations of mean curvature type. We were able to use the existence of isothermal and characteristic parameterizations and then the properties of quasiconformal maps to gain the result of Theorem 8.

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APPENDIX

APPENDIX A

In this appendix we provide some of the global definitions and local derivations of terms given throughout the thesis.

A.1 Connection and Covariant Derivative

Let $\mathscr{X}(M)$ be the set of all vector fields of class C^k on M. We defined a connection that is compatible with the metric in (1.7) to be the Levi Civita connection. Now we give the definition for an **affine connection** ∇. On a differentiable manifold M as a mapping we define an affine connection $\nabla : \mathscr{X}(M) \times \mathscr{X}(M) \to \mathscr{X}(M)$ which satisfies:

(i) $\nabla_{fX+gY} Z = f\nabla_X Z + g\nabla_Y Z$,

(ii) $\nabla_X(Y + Z) = \nabla_X Y + \nabla_X Z$,

(iii) $\nabla_X(fY) = f\nabla_X Y + X(f)Y$, in which $X, Y, Z \in \mathscr{X}(M)$ and f, g are real valued functions of class C^∞ defined on M.

Let M be a differentiable manifold with an affine connection ∇. If γ is a curve and V a vector field along γ, then there exists a **covariant derivative** which is a vector field $\frac{DV}{dt}$ along γ with the following properties:

(i) Linear over addition: $\frac{D}{dt}(V + W) = \frac{DV}{dt} + \frac{DW}{dt}$

(ii) Product rule: $\frac{D}{dt}(fV) = V\frac{Df}{dt} + f\frac{DV}{dt}$,

(iii) If V is the restriction to γ of a vector field defined on M, (i.e., there exists Y such that $V(t) = X(c(t))$) then $\frac{DV}{dt} = \nabla_{\gamma'(t)} X$.

In chapter 1, we found a local coordinate expression (1.8) for the covariant derivative. To derive this local expression we first choose a parameterization $\mathbf{x} : U \subset \mathbb{R}^n \to M$ where the curve $\gamma(t) = (x_1(t), ..., x_n(t))$. Writing the vector field v as $v(t) = \sum_{i=1}^{n} v_i(t) \frac{\partial}{\partial x_i}|_{\gamma(t)}$, we find $\frac{DV}{dt}$ in local coordinates as given in (1.8) using the definition of covariant derivative as given above and [20]:

$$\begin{aligned}
\frac{DV}{dt} &= \sum_{i=1}^{n} \frac{D}{dt}(v_i(t)\frac{\partial}{\partial x_i}) && \text{[by (i)]} \\
&= \sum_{i=1}^{n} \left[\frac{dv_i}{dt}\frac{\partial}{\partial x_i} + v_i(t)\frac{D}{dt}\frac{\partial}{\partial x_i} \right] && \text{[by (ii)]} \\
&= \sum_{i=1}^{n} \left[\frac{dv_i}{dt}\frac{\partial}{\partial x_i} + v_i(t)\nabla_{\gamma'(t)}\frac{\partial}{\partial x_i} \right] && \text{[by (iii)]} \\
&= \sum_{i=1}^{n} \left[\frac{dv_i}{dt}\frac{\partial}{\partial x_i} + v_i(t)\nabla_{\sum_{j=1}^{n} x_j'(t)\frac{\partial}{\partial x_j}}\frac{\partial}{\partial x_i} \right] \\
&= \sum_{i=1}^{n} \left[\frac{dv_i}{dt}\frac{\partial}{\partial x_i} + \sum_{j=1}^{n} v_i x_i'(t)\nabla_{\frac{\partial}{\partial x_j}} \partial\partial x_i \right] \\
&= \sum_{i=1}^{n} \left[\frac{dv_i}{dt}\frac{\partial}{\partial x_i} + \sum_{j,k=1}^{n} v_i \frac{dx_i}{dt}\Gamma_{ij}^{k}\frac{\partial}{\partial x_k} \right] && \text{[by (1.9)].}
\end{aligned}$$

A.2 Mean Curvature

The **mean curvature** H of a surface at the point under consideration is the arithmetic mean of the principal curvatures k_i,

$$H = \frac{1}{n}(\kappa_1 + \kappa_2).$$

The maximum and minimum of the normal curvature at a given point on a surface are called the principal curvatures κ_1, κ_2. Note that the principal curvatures at a given point of a surface are the eigenvalues of the shape operator at the point. [26] A surface with zero mean curvature is called a **minimal surface.**

Let w be a unit tangent vector of a regular surface $M \subset \mathbb{R}^3$ at p . Then the normal curvature of M in the direction w is

$$\kappa(w) = S(w) \cdot w$$

where S is the shape operator. Consider a parameterization $p = \mathbf{x}(u, v)$ with $v = a\mathbf{x}(u, v) + b\mathbf{x}(u, v)$. The normal curvature in the direction v is

$$\kappa(v) = \frac{ea^2 + 2fab + gb^2}{Ea^2 + 2Fab + Gb^2}$$

where E, F, G are the coefficients of the first fundamental form and e, f, g and are the coefficients of the second fundamental form. The second fundamental form can be written explicitly as

$$edu^2 + 2fdudv + gdv^2$$

where

$$e = \sum_i X_i \frac{\partial^2 x_i}{\partial u^2}$$
$$f = \sum_i X_i \frac{\partial^2 x_i}{\partial u \partial v}$$
$$g = \sum_i X_i \frac{\partial^2 x_i}{\partial v^2}$$

and X_i are the direction cosines of the surface normal.