TURBULENT BOUNDARY LAYER MODELS FOR ACOUSTIC ANALYSIS

A Dissertation by

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ABSTRACT

An analysis of the three types of turbulent boundary layer (TBL) models for acoustic analysis is presented because current preferred models over-predict TBL contributions to aircraft interior noise predictions. The mean square pressure is a measure of the total energy due to the pressure fluctuations beneath a turbulent boundary layer. The single point wall pressure spectrum sorts the energy into frequencies. The normalized wavenumber-frequency spectrum sorts the energy into wavenumbers. The pressure fluctuations beneath a turbulent boundary layer are found by solving the Poisson equation. In this work, the Poisson equation is solved both numerically and analytically using data from an LES/DES simulation. The numerical solution uses the point Gauss-Seidel method and has reasonable results. The analytical solution uses an eigenvalue expansion method that is less successful. The empirical mean square pressure models predict a relatively large spread in the pressure fluctuation values. It is difficult to draw any meaningful conclusions on which mean square pressure model is preferred when compared to data from the Spirit AeroSystems 6x6 duct. The single point wall pressure spectrum models are evaluated and the two more modern models of Smol’yakov and Goody seem to perform the best. These models are also compared to data from the Spirit AeroSystems 6x6 duct. The spectrum at low frequencies rolled off similar to the Goody model. This analysis indicates that the Goody model is the appropriate single point wall pressure spectrum model for aircraft applications. Important features of the normalized wavenumber-frequency spectrum models are presented and can be classified as either separable or non-separable. Separable models in the Corcos normalized wavenumber-frequency spectrum model class tend to over-predict the response for a range of cases. Both the non-separable Chase 1 and Smol’yakov-Tkachenko models appear to match the M.I.T. low noise, low turbulence wind tunnel data throughout the range of comparison. The Smol’yakov-Tkachenko model does not lend itself to straight forward Fourier transforms needed by the acoustic models. But the Chase 1 model can be converted from wavenumber-frequency spectrum to the cross spectrum, so it is the preferred model for aircraft applications. Therefore, the preferred turbulent boundary layer models for aircraft interior noise predictions are the single point wall pressure spectrum model of Goody and the normalized wavenumber-frequency spectrum model of Chase 1.
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NOMENCLATURE

\( f \) Frequency

\( k \) Wavenumber vector

\( k_x, k_z \) Wavenumber components

\( M \) Mach number

\( q \) Dynamic pressure

\( Re_x \) Reynolds number with respect to distance

\( Re_\theta \) Reynolds number with respect to momentum thickness

\( Re_\tau \) Reynolds number with respect to shear stress

\( Sh \) Strouhal number

\( T_\infty \) Freestream temperature

\( U_c \) Convective velocity

\( U_{\infty} \) Freestream velocity

\( U_\tau \) Friction velocity

\( W(f) \) Experimental spectral density

\( \alpha_\infty, \alpha_z \) Graham decay coefficients

\( \alpha_{\infty}, \beta_{\infty} \) Mellen decay coefficients

\( \delta \) Boundary layer thickness

\( \delta^* \) Displacement thickness

\( \nu \) Kinematic viscosity

\( \rho \) Freestream density

\( \theta \) Momentum thickness

\( \tau_w \) Wall shear stress
$\omega$ Angular velocity

$\overline{p'^{2}}$ Mean square pressure

$\Phi(\omega)$ Single point wall pressure spectrum

$\Phi_{pp}(k_x,k_z,\omega)$ Wavenumber-frequency spectrum

$\tilde{\Phi}_{pp'}(k_x,k_z,\omega)$ Normalized wavenumber-frequency spectrum

$\Psi_{pp'}(x,z,\omega)$ Space-frequency spectrum

$\hat{\Psi}_{pp'}(k_x,k_z,\omega)$ Fourier transform of space-frequency spectrum
CHAPTER 1
INTRODUCTION

Although the interest in noise and vibration research in airplane industry has always been an area of concern, the level of interest has increased significantly in the past few years. In part, this is caused by a steady increase in airplane engine power and the resulting increase in Mach numbers of a typical cruise flight. The increased interior noise levels have generated a need for better understanding of noise transmission mechanisms into the airplane cabin and for the development of advanced noise reduction techniques in order to maintain acceptable interior noise levels and passenger comfort without increasing weight. (Alujevic, 2008)

The sound pressure levels produced can be intense enough to result in an unacceptable interior noise environment affecting passenger discomfort, interfering with communication, causing crew fatigue, or malfunction of sensitive electronic equipment. Control of the noise environment requires substantial work in both aircraft design and analysis. The sound pressure level control measures usually result in penalties such as added structural weight, reduced cabin volume, reduced performance, or increased cost. Interior noise control therefore requires a continuing search for means to reduce the sound pressure levels without incurring the penalties. (Mixson and Wilby, 1991)

Although there are currently no certification requirements on interior (passenger cabin or flight deck) noise, airlines still require that the manufacturers guarantee noise levels in the passenger cabin. As a minimum, the guarantee is specified at the passenger seats. Often, flight deck, galley, and/or lavatory noise levels are also specified. (Hodge, 1991) Three parameters are in common use to quantify the subjective aspects of interior noise. 1) The overall sound pressure level (OASPL, dB) adds most audible frequency components equally. 2) The A-weighted sound
level reduces the contributions of very low- and high-frequency components and has been found to correlate closely with the subjective response of human laboratory subjects and airplane passengers. 3) Speech interference level (SIL) includes only the frequencies between 500 Hz and 2000 Hz and relates to the quality of voice communication. (Mixson and Wilby, 1991)

At present, the most cost effective method of addressing interior noise is to design the sound attenuation features into the aircraft structure. Without accurate analytical tools, the design process would roughly estimate an appropriate noise reduction package and then assess the effectiveness during flight test. Additional features would be added until acceptable interior sound levels were achieved. It is clear that it would better to be able to analytically predict the interior noise levels early in the design process in order to incorporate noise reducing features into the structural design.

1.1 Sources of Noise

A variety of noise sources and transmission paths contribute to cabin interior noise. Sources such as propellers, inlet and exhaust systems of reciprocating or turbofan engines, turbomachinery, and turbulent airflow over the airplane surfaces generate noise that impinges directly on the exterior of the fuselage and transmits into the cabin. (Mixson and Wilby, 1991) The advent of turbojet powered commercial aircraft with cruise speeds substantially greater than those of propeller driven airplanes, provided the impetus to investigate the generation of noise by turbulent boundary layers. However, turbulent boundary layer (TBL) noise or aerodynamic noise in general is not restricted to jet powered aircraft. It was recognized in the 1940s that aerodynamic noise became important in certain frequency ranges at aircraft speeds above about 200 mph. It has been demonstrated that aerodynamic noise generated by turbulent boundary
layers is a significant contributor to mid- and high frequency sound pressure levels in the cabin. (Wilby, 1996)

Noise from internal sources, such as air conditioning systems and electronic equipment can also be important. Similarly, the increase of engine power has led to an increase in acoustic power generated by the power plant. However, the emergence of modern jet engines with a bypass stream has resulted in lower jet velocities, and thus also in relatively lower acoustic outputs when compared to traditional single stream engines having equal thrusts. This effect puts the emphasis on the increase in the TBL generated noise. In modern commercial airplanes, propelled by jet engines, the TBL noise source is a major contributor to the airplane cabin noise during typical cruise conditions. (Alujevic, 2008)

The interior noise in an aircraft is normally classified according to two transmission paths: airborne and structure borne. The airborne path is characterized by acoustic or aerodynamic excitation of the fuselage skin which then radiates sound to the interior. The structure-borne path is instead characterized by vibration excitation of the fuselage frame structure and skin which radiates sound to the interior of the aircraft. Noise from the power plant and the TBL sources is mainly transmitted via airborne paths. (Alujevic, 2008)

The noise generated by airflow over the airplane surfaces is important for virtually all classes of airplane. For the smaller airplane with less streamlining, more exposed struts, and light structure, airflow noise is important at higher frequencies. For the larger, jet-powered, well-streamlined airplane, high speed flows generate significant levels of turbulent boundary layer noise that usually constitutes one of the most important sources of cabin noise during cruise. (Mixson and Wilby, 1991) This noise source is the subject of this dissertation.
1.2 Contribution of Turbulent Boundary Layers

The pressure fluctuations at the wall in a turbulent boundary layer (TBL) excite the fuselage surface which in turn radiates noise into the cabin. There is a basic assumption that the boundary layer fluctuations excite the structure but the moving structure does not change the boundary layer. This allows us to examine the fluid motion independent of the structure. Figure 1.1 shows how the turbulent boundary layer creates noise in the cabin of an airplane.

![Forcing Function](#)

**Figure 1.1: Interior noise creation due to a turbulent boundary layer.**

TBL noise has gained importance since the introduction of commercial turbojet airplanes with higher cruise speeds. As mentioned above, the airflow over the fuselage surface is characterized by a fluctuating pressure which excites the fuselage skin. The nature of this excitation is random both in frequency and spatial domains. In general, the interior noise generated by the TBL source is important at mid and high frequencies, and it dominates the interior noise field at frequencies which are in the range between 400 Hz and 2 kHz. (Alujevic, 2008)

The boundary layer pressure field is convected in the direction of the airflow. Aerodynamic coincidence occurs when the speed of the turbulence in the TBL is the same as the speed of the wave in the structure due to the induced vibration. The convection speed of the turbulence is proportional to the aircraft speed such that at certain cruise speeds aerodynamic coincidence takes place. In this case, the phase of the boundary layer induced pressure matches
the phase of the bending wave vibration of the fuselage skin. The consequent large vibration amplitudes of the fuselage skin results in large sound pressure levels in the aircraft cabin. (Alujevic, 2008, Gardonio, 2002) Aerodynamic coincidence has been observed on airplane fuselage structures at high subsonic Mach numbers \((M \geq 0.6)\), but not for lower Mach numbers. (Wilby, 1996)

Using data from Wilby (1996), Figure 1.2 shows the interior sound pressure levels measured in a business jet at three Mach numbers. It is known that boundary layer noise dominates the sound pressure levels, as shown in this chart, at frequencies above 400 Hz. As Mach number increases, the effect of boundary layer noise becomes increasingly important.

**Figure 1.2:** Airplane interior sound pressure levels for different flight Mach numbers.

### 1.3 Methods of Predicting Interior Noise

For many years, the analysis of aircraft interior noise relied on architectural acoustics methods and extrapolation from previous aircraft of similar design and performance. Recently,
significant advances have been made in prediction of noise transmission into airplane interiors. Several different approaches have been applied to the analytical modeling of airborne and structure borne noise transmission into airplanes with varying degrees of success, and the application of a particular method is often restricted to a specific frequency range that is determined to some extent by the computational requirements and the validity of the assumptions of the model. The main methods used are finite element analysis (FEA), statistical energy analysis (SEA), dynamic element analysis (DEA), and boundary element analysis (BEA). (Wilby, 1996)

It has been indicated in Section 1.2 that the pressure fluctuations at the fuselage surface under a TBL is a significant source of noise in the cabins of airplanes and the noise levels increase with Mach number. Each one of the sophisticated analysis techniques, such as SEA and DEA, uses a TBL model which consists of a mean square wall pressure model, a single point wall pressure spectrum model and a wavenumber-frequency spectrum model. The mean square pressure gives an estimate of the total energy. The single point wall pressure spectrum gives the frequency distribution at each point. The wavenumber-frequency spectrum model allows for modal analysis, and is used to analyze how the pressure fluctuations drive the structure.

The early TBL models were developed from incompressible, low-speed flow data only. More recent models have incorporated limited numerical and analytical techniques. In this dissertation, analytical, numerical and experimental techniques will be used to study the pressure fluctuations at the wall due to turbulent boundary layers. Existing methods will be examined and evaluated in detail for their strengths and weaknesses. Results will be compared with experimentally obtained data from the Spirit AeroSystems 6x6 duct and from M.I.T. low noise, low turbulence wind tunnel. Therefore, the purpose of the present project is to evaluate various
TBL models which can be incorporated into the SEA and DEA models in order to obtain more accurate interior noise predictions early in the design process in order to influence the product definition.
CHAPTER 2
LITERATURE REVIEW

2.1 Background

Kraichnan (1956) was an early investigator of pressure fluctuations under a turbulent boundary layer due to his interest in the noise levels encountered inside high-speed aircraft. He wanted to investigate theoretically the magnitude and structure of the driving forces behind the fluid structures. Based on a solution of the Poisson equation using similarity arguments, he developed important qualitative features of the pressure fluctuation structure associated with boundary layer flow over a smooth flat surface to predict more accurately the pressure forces. He found an approximate dependence of the mean-square intensity, spatial scale and frequency scale on Mach number and distance from transition point. Lilley and Hodgson (1960) studied incompressible flow over a rigid flat plate focusing on wall turbulence. Their theory was based on work by Kraichnan which was modified and extended to include the separate effects of the large eddy structure and the convection of the eddies. In this method, the intensity of the pressure fluctuations was obtained based on the two-point velocity correlations, mean velocity gradient and the turbulence intensity and scale. Their results compared well to the limited experimental data available at the time.

Willmarth and Wooldridge (1962 and 1963) are credited with performing the first comprehensive measurement of the pressure fluctuations on the wall beneath a turbulent boundary layer by using a pressure transducer flush with the surface. They were interested in studying the detailed structure of the wall pressure fluctuations and wanted to learn more about the structure and scale of the eddies that produce the pressure fluctuations. The data included the mean square pressure, power spectrum of the pressure, space-time correlation of the pressure.
parallel to the stream and spatial correlation of the pressure transverse to the stream. They found that both large and small scale pressure producing eddies decay after travelling a distance proportional to their scale. They also found that the convection of the pressure producing eddies varied from 0.56 to 0.83 times the freestream velocity. This is the range for convective velocity that is still used in analysis today.

In 1964, Corcos studied measurements of the statistical properties of the pressure field at the wall of turbulent attached shear flows. Corcos used previous work by Willmarth and Wooldridge (1962 and 1963) and his own experiments. He showed that measurements of the longitudinal cross spectral densities led to similarity variables for the space time covariance of the pressure and for the corresponding spectra. He postulated that the existence of these similarity variables might be due to the dispersion of the sources of pressure by the mean velocity gradient.

Ffowcs Williams (1965) was the first to study the effects of compressibility which were considered negligible in previous work. It was known that the intensity of the turbulence induced surface pressure varied slowly as a function of Mach number. Previous experimental data showed that even in high Mach number flows, the region of turbulence which caused the dominate part of the surface pressure occurs close to the wall which is at a slower velocity. The most important result of this study was that the correlation area is proportional to the square of mean flow Mach number so it does not vanish in a flow with finite compressibility. Therefore, the wavenumber spectral density does not approach zero at low wavenumbers. It does in incompressible flow but finite compressibility ensures a finite value.

Bull (1967) was the first to discuss how the different parts of the boundary layer could produce different wall pressure fluctuation frequencies. Experimental results were given for
various statistical properties of the fluctuating wall pressure field associated with a subsonic turbulent equilibrium boundary layer. The statistical quantities of the wall pressure field investigated were root mean square (rms) pressure, frequency power spectrum and space time correlations. Space time correlation measurements were made for both broad and narrow frequency bands at Mach numbers of 0.3 and 0.5. The main conclusion was that the wall pressure field has a structure produced by contributions from pressure sources in the boundary layer with a wide range of convection velocities. The first is for high wavenumber components which is associated with turbulent motion in the constant stress layer. These components are longitudinally coherent for times proportional to the convected distances equal to their wavelengths. They were also shown to be somewhat laterally coherent over distances proportional to their wavelengths. The second is for components of wavelength greater than twice the boundary layer thickness. These components lose their coherence somewhat independently of wavelength and are associated with large scale eddy motion in the boundary layer outside the constant stress layer.

The experiments performed by Blake (1970) occurred in the subsonic low turbulence acoustic wind tunnel in the Acoustics and Vibration laboratory at Massachusetts Institute of Technology. Blake was one of the first researchers to be able to use very small flush mounted microphones. In his study, turbulent boundary layer wall pressure measurements were made with pinhole microphones three times smaller than the boundary layer displacement thickness. The improved high frequency resolution permitted examination of the influence of high frequency eddies on the smooth wall pressure statistics. It was found that the space time decay rate was considerably higher than previously reported. The high frequency pressure levels were determined by length and velocity scales characteristic of the constant stress region. The scaling
factor used here was friction velocity, $U_f$. The low frequency pressure levels were scaled on outer flow parameters. The scaling factors used here were the freestream velocity $U_\infty$ and displacement thickness $\delta^*$.  

As stated by Willmarth (1975), the early research on wall pressure fluctuation behavior focused on creating analytical models based on simplified solutions of the Poisson equation or by building empirical models based on experiments in a wind tunnel or flight test. The theoretical work mainly used turbulence models that were based upon experimental measurements. The models were then used to provide approximate results for the statistics of the turbulent pressure field. The experimental investigations of the statistical properties of the fluctuating pressure far outnumbered the theoretical studies.

More recent research has concentrated on conducting computational fluid dynamic simulations. In general, the flow field of a turbulent boundary layer over a complicated surface is so complex that it is impossible to develop a completely analytical solution or to model the entire flow field numerically. It is not surprising that direct numerical simulation (DNS) model is not a practical solution for a problem that requires such a large computational domain at this time. Therefore, current turbulent boundary layer models continue to be developed using a combination of analytical, numerical and experimental techniques.

2.2 Models Required for Acoustic Analysis

The acoustic analysis models use flow field statistics which are described by the mean square pressure, the single point wall pressure spectrum, and the normalized wavenumber-frequency spectrum. The mean square pressure is a measure of the total energy due to the pressure fluctuations beneath a turbulent boundary layer. The single point wall pressure spectrum sorts the energy into frequencies. The normalized wavenumber-frequency spectrum
sorts the energy into wavenumbers. All three models are needed for an acoustic analysis to
determine the structural response at each point on the panel of interest. Since each of these
models has its own unique characteristics, they have typically been investigated in different
research efforts. Therefore, each will be discussed in separate sections that follow.

2.2.1 Mean Square Pressure

Many researchers have developed models that estimate the mean square value of the
pressure fluctuations and there is quite a bit of numerical spread among them. Bull (1996) states
that the root mean square pressure fluctuation value is subject to greater experimental error than
individual frequency spectral magnitudes because it is the integral of the single point wall
pressure spectrum. This means that it is subject to an accumulation of errors over the whole
spectral range. This is particularly severe at high frequencies due to transducer resolution
limitations. In this work, the mean square pressure models of Kraichnan (1956), Lilley and
Hodgson (1960), Willmarth and Wooldridge (1962), Corcos (1964), Bull (1967), Lowson
(1968), Blake (1970), Schewe (1983), Lauchle and Daniels (1987), Farabee and Casarella
(1991), and Lueptow (1995) will be examined in detail.

2.2.2 Single Point Wall Pressure Spectrum

The single point wall pressure spectrum or power spectrum represents the distribution of
the mean square fluctuating pressure with frequency. Much of the work in this field has been
funded by the Navy which means the research has been focused on lower Mach numbers than
what is of interest to aircraft applications. This would include the works of Chase (1980), Howe
(1987) and Goody (2004 and 2007). Other researchers, such as Efimstov (1982 and 1995) and
Rackl and Weston (2005), have focused on aircraft applications at higher Mach numbers. It
should be no surprise that the single point wall pressure spectrum models from these different
areas of interest have somewhat different spectral features. As part of the present research, the single point wall pressure spectrum models from Robertson (1971), Efimtsov, Rackl and Weston, Chase-Howe, Goody, and Smol’yakov (2000) will be scrutinized.

2.2.3 **Normalized Wavenumber-Frequency Spectrum**

The normalized wavenumber-frequency spectrum models are used by the acoustic analysis programs to define the wavenumber distribution. In the acoustic analysis models, it is important that the normalized wavenumber-frequency model can be transformed easily from the wavenumber-frequency domain to the space-time domain. For that reason, the separable Corcos (1964) model has been the preferred choice. Other researchers, such as Mellen (1990 and 1994) and Chase (1980), have developed non-separable models they feel represent the physical domain better. In this work, the models of Corcos, Mellen, Chase, and Smol’yakov-Tkachenko (1991) will be reviewed.

2.3 **Research Contributions**

As indicated earlier, the purpose of the present effort is to identify improved TBL acoustic models that can be used in the SAE and DEA methods, aiming for higher accuracy in prediction of noise levels during the initial design stages. Therefore, the present research aims to make the following contributions:

- For the mean square pressure models, the theoretical derivation of the incompressible mean square pressure will be shown and solved using data from an LES/DES simulation. The theoretical results to previously developed empirical models along with new data from the Spirit AeroSystems 6x6 duct will be compared.

- For the single point wall pressure spectrum models, the current theory of the spectral features will be shown. The existing models will be compared to new data from the Spirit
AeroSystems 6x6 duct. At the completion of this effort, a single point wall pressure spectrum model will be recommended.

- For the normalized wavenumber-frequency spectrum models, a theoretical derivation of the separable and the non-separable models will be offered. These models will be compared to the MIT-Martini data and the requirement for a non-separable model will be confirmed. At the completion of this effort, a normalized wavenumber-frequency spectrum model will be recommended.
CHAPTER 3
MODELING THE TURBULENT BOUNDARY LAYER (TBL)

3.1 Description of a Turbulent Boundary Layer

The flow over an airplane is driven by potential flow outside the boundary layer. However, the pressure fluctuations at the surface of the fuselage are confined to the boundary layer only. The flow in the boundary layer begins as laminar but quickly changes to turbulent; therefore, the largest portion of the boundary layer around a fuselage is turbulent.

The general understanding of a boundary layer has been known since the 1950's and it is composed of chaotic fluid motion which results in pressure fluctuations at the wall surface. Figure 3.1 is a sketch of a boundary layer which shows the irregular division between the turbulent and freestream flow and the flattened shape of the mean velocity as a function of the distance from the wall. (Klebanoff, 1955) It is generally assumed that the velocity and pressure fluctuations disappear outside the edge of the boundary layer.

![Diagram of a turbulent boundary layer](image)

Figure 3.1: Diagram of a turbulent boundary layer (Klebanoff, 1955).

A turbulent boundary layer is commonly divided into several regions where the specific turbulence behavior can be identified as seen in Figure 3.2. The thin area near the wall is called the viscous sublayer. The outer region is the fully turbulent region. The region in between is the
buffer zone. Another way to classify the parts of a turbulent boundary layer is to group the flow into inner and outer regions. The inner region includes the viscous sublayer, buffer zone, and part of the fully turbulent region. The remaining part of the boundary layer is considered to be the outer region or wake. (Hoffmann and Chiang, 2000)

![Diagram of turbulent boundary layer regions](image)

**Figure 3.2: Velocity profile of a turbulent boundary layer (Hoffmann and Chiang, 2000).**

Researchers frequently refer to the regions of a turbulent boundary layer and scale their results based on physical parameters defined in these regions. The physical parameters that influence the fluid behavior near the wall include density ($\rho$), kinematic viscosity ($\nu$), distance to the wall ($y$), and shear stress at the wall ($\tau_w$). The division of a turbulent boundary layer into
regions is commonly identified by the definition of a non-dimensional velocity $u^+$ and a normal spatial surface coordinate $y^+$. These quantities are defined by

$$u^+ = \frac{u}{U_\tau} \quad (3.1)$$

$$y^+ = y \frac{U_\tau}{\nu} \quad (3.2)$$

where the friction velocity is given by

$$U_\tau = \sqrt{\frac{\tau_w}{\rho}} \quad (3.3)$$

The relationship for the near wall region is known as the law of the wall which is empirically determined. Measurements show that the streamwise velocity in the flow near the wall varies logarithmically with distance from the surface. (Wilcox, 2006) The relationship for the outer region is known as the defect law has been established based on dimensional analysis and experimental observations. These two relationships must be merged together smoothly over a finite region which is known as the log or overlap layer.

In the viscous sublayer where $y^+ < 2$ to 8, the viscous shear dominates and the velocity profile is approximately linear. Here the velocity profile is given by

$$u^+ = y^+ \quad (3.4)$$

The velocity profile from the viscous sublayer must merge smoothly with the velocity profile for the outer log layer. A relationship satisfying this requirement is given by

$$u^+ = 5.0 \ln y^+ - 3.05 \quad (3.5)$$
This equation is valid for the buffer zone which is approximately $2 < y^+ < 2$ to 50. The typical range of the log layer is approximately $50 < y^+ < 400$. The velocity profile for this region is given by

$$u^+ = 2.5 \ln y^+ + 5.5$$ (3.6)

These equations are for incompressible flows over smooth flat plates with zero pressure gradients and are also shown in Figure 3.2. Similar relationships are available for compressible flows. (Hoffmann and Chiang, 2000)

### 3.2 Boundary Layer Values

In order to solve for the pressure fluctuations under a turbulent boundary layer, the values for the boundary layer thickness ($\delta$), the mean velocity ($\bar{u}$) throughout the boundary layer, and the velocity fluctuations ($u'$, $v'$) in the boundary layer are needed. White (1991) approximated the boundary layer thickness as

$$\delta = \frac{0.16x}{(Re_x)^{1/7}}$$ (3.7)

For the mean velocity and the velocity fluctuations, the most widely used approximations are from Klebanoff (1955). These measurements were taken in a wind tunnel with turbulence levels of 0.02 percent at 30 feet per second and 0.04 percent at 100 feet per second. The boundary layer was developed on a smooth, flat, aluminum plate with a free-stream speed of 50 feet per second. This corresponds to a Mach number of 0.045 at sea level conditions. The boundary layer was fully developed, smooth wall type having an apparent development length of 14.2 feet of smooth surface. Klebanoff gives the corresponding length Reynolds number based on value of $x$ measured from the virtual origin as $4.2 \times 10^6$. 
Close to the wall, $0 < y < 0.05$ inches, the mean velocity was measured with a hot-wire. A platinum wire 0.0003 inch in diameter and approximately 0.5 inches long was used and operated at low-temperature loadings in order to minimize the influence of the wall on the heat-loss characteristics of the wire. A correction was made for the effect of the turbulence level on the measured hot-wire values. This was done by a graphical method using the mean-velocity-voltage calibration curve and the measured root mean square of the voltage fluctuations. The corrected mean velocity was higher than the observed, with a maximum correction of about 10 percent. The hot-wire values were in good agreement with the pitot static tube values. As shown in Figure 3.3, the Klebanoff mean flow data compares well with the 1/7th rule equation. The mean velocity equation is given by

$$\bar{u} = U_\infty \left( \frac{y}{\delta} \right)^{1/7}$$

(3.8)

Figure 3.3: Mean velocity distribution versus the 1/7th rule.
Klebanoff defined the velocity fluctuations as the root-mean-square values \((\text{rms})\) of the velocity fluctuations. The value of \(u'\) was obtained as close to the wall as 0.004 inch, but because of the comparatively larger size of the probes necessary for the measurement of \(v'\) and \(w'\) it was not possible to measure these closer than about 0.045. The root-mean-square velocity fluctuations were given graphically in Klebanoff and shown here in Figure 3.4. By only showing the rms values of the velocity fluctuations, the time dependent characteristics are lost. Correlated, instantaneous velocity field values are not available from this data.

![Graph showing rms velocity fluctuations](image.png)

**Figure 3.4: Root mean square velocity fluctuations.**

### 3.3 Turbulent Flow Statistics

There are two paths to turbulent flow analysis: 1) a statistical theory of turbulent correlation functions and 2) a semiempirical modeling of turbulent mean quantities. The first studies the statistical properties of the fluid motion fluctuations: their frequency correlations, space-time correlations, and interactions with each other. The latter approach emphasizes the
turbulent properties which are considered to be most important in typical engineering practice: mean velocity profiles, shear-layer thickness parameters, and root mean square (rms) fluctuation profiles. (White, 1991) The acoustic world tends to focus on the first path while the aerodynamics world tends to focus on the second path. For this analysis, an understanding of both approaches and how they are related is needed.

The Taylor-von Kármán definition of turbulent flow is characterized by irregular motion that is typically aperiodic and cannot be described as a straight-forward function of time and space coordinates. Wilcox (2006) noted that since turbulent motion is irregular, it can be described by the laws of probability. Hinze (1975) restated the definition of turbulence as an irregular condition of flow in which the various quantities show a random variation with time and space coordinates so that the statistically distinct average values can be distinguished. This implies that a turbulent flow has a wide variety of scales. These time and length scales of turbulence are represented by frequencies and wavelengths that can be found using Fourier analysis of turbulent flow time histories. (Wilcox, 2006)

Because of the random nature of fully developed turbulent flow fields, statistical methods are usually employed for their description. However, in the statistical averages much of the information that may be relevant to the understanding of the turbulent mechanisms may be lost, especially phase relationships. In recent years, sampling techniques have been employed in experimental investigations for both wall-bounded and free-shear flows, and these have given much new information on the structure of turbulence. (Landahl and Mollo-Christensen, 1992)

3.4 Statistical Tools and Spectral Analysis

A random variable is a function that cannot be predicted from its past. (Landahl and Mollo-Christensen, 1992) These variables do not repeat with any definite sequence but instead
must be described in terms of probability and statistics. (Blake, 1986) Since the turbulent flow found in a boundary layer results in random variables, statistical tools must be developed for use in this study. The central notion involved in the concept of a random process is that not just one time history is described but the whole family or ensemble of possible time histories which might have been the outcome of the same experiment are described. (Crandall and Mark, 1963)

In this study, the concept of an ensemble average which allows one to form averages for time-dependent processes will be used. An example of an ensemble average statistical measure is the autocorrelation function. It gives information about the average time dependence of a process. The Fourier transform of the autocorrelation, in turn, describes the frequency content of the process. The Fourier transform of the cross correlation with respect to a time delay gives the cross spectral density. (Landahl and Mollo-Christensen, 1992)

A stationary process is analogous to the assumption of a steady state forced vibration. A process is said to be stationary if its probability distributions are invariant under a shift of time scale. That means that the family of probability densities applicable now also applies 10 minutes from now or 3 weeks from now. In particular, the probability density $P(a)$ becomes a universal distribution independent of time. (Crandall and Mark, 1963)

The analysis is begun with an arbitrary state variable $a(\bar{x}, t)$ which can be thought of as a velocity component, the pressure, or the temperature. This variable consists of two components, $a = a' + \bar{a}$, where $a'$ is the fluctuating component and $\bar{a}$ is the mean value. The time average of the variable is defined by

$$
\overline{a(\bar{x})} = \langle a(\bar{x}, t) \rangle = \lim_{T \to \infty} \frac{1}{T} \int_{0}^{T} a(\bar{x}, t) dt
$$

(3.9)
The time average just means that the random variable at a fixed location is integrated over a suitably long time interval $T$. The variable $a(\bar{x}, t)$ is said to be statistically stationary if $\overline{a(x)}$ converges to a well defined value under the limiting process. This method of averaging is the most widely used in both turbulence theory and experimental turbulence since the output from a probe held at a fixed spatial location in a flow can be readily averaged. In practice, the averaging time is established by increasing $T$ until the value of $\overline{a}$ is unchanged within an acceptable tolerance. (Libby, 1996)

A variation of time averaging is called ensemble averaging and is used when a continuous history of the statistically stationary variable $a(\bar{x}, t)$ is not available. The average is formed from a large number of discrete measurements of $a$, given by $a(\bar{x}, t_n)$ where $n = 1, 2, ..., N$.

$$E[a(\bar{x}, t)] = \overline{a(\bar{x})} = \lim_{N \to \infty} \frac{1}{N} \sum_{n=1}^{N} a(\bar{x}, t_n)$$

(3.10)

where the symbol $E[ ]$ indicates an ensemble average and is sometimes referred to as the expected value. (Libby, 1996) and (Newland, 1984) The ensemble average is also called the mathematical expectation of $a(\bar{x})$. (Crandall and Mark, 1963)

Given the statistical description of a variable such as velocity or pressure at a fixed location, it is easy to see that these descriptions contain no information on either the length or time scales associated with $a'(\bar{x}, t)$. As defined in Libby (1996), Figure 3.5 shows the fluctuations of a generic variable at discrete times.
Figure 3.5: The sampling of a variable $a'(x, t)$ at discrete times.

In each of the averaging methods, the deviation of an instantaneous value can be identified. The fluctuation can be thought of as a new random variable but with zero mean.

$$
\langle a'(\bar{x}, t) \rangle = \overline{a'(\bar{x}, t)} = 0
$$

(3.11)

The mean square value, $\overline{a'^2}$, is the variance and measures the extent of the variation about the mean value. The square root of this value is known as the root mean square value or rms value. (Libby, 1996)

For the rest of this section, it will be assumed that the variables in this analysis are statistically stationary. Stationary data is defined as the special case where all average values of interest remain constant with changes in the time, $t_1$. For stationary variables, the average values at all times can be computed from the appropriate ensemble averages at a single time $t_1$. For almost all stationary data, the average values computed over the ensemble at time $t_1$ will equal the corresponding time average values computed from a single time history record. The definition of the ergodic theorem from Bendat and Piersol (1980) states that for stationary data, the properties computed from time averages over individual records of the ensemble will be the same from one record to the next and will equal the corresponding properties computed from an ensemble average over the records at any time $t$ if
\[
\lim_{T \to \infty} \frac{1}{T} \int_{0}^{\tau} \left| R_{aa}(\tau) - \overline{a^2} \right| d\tau \to 0
\]  
\tag{3.12}

where \( R_{aa} \) is defined as the autocorrelation function and is given by

\[
R_{aa}(\tau) = \lim_{T \to \infty} \frac{1}{T} \int_{0}^{T} a(t) a(t + \tau) dt = \overline{a(x,t)a(x,t+\tau)} = \overline{a^2}
\tag{3.13}

Figure 3.6 is a graphical representation from Bendat and Piersol (1980) which demonstrates the concept of stationary data and ergodic data.

Figure 3.6: Ensemble of time histories which are used to define stationary and ergodic data.

Now it will be defined what correlated and uncorrelated means. Consider a population of pairs of values of two random variables \( a' \) and \( b' \). Suppose that each pair of values is represented by a point on a graph of \( b' \) against \( a' \) as described by Newland (1984). In Figure 3.7 (a), the values of \( a' \) and \( b' \) in each pair have no apparent pattern but in Figure 3.7 (b), there is a definite pattern. For the latter case, a large value of \( a' \) is associated with a large value of \( b' \). A small value of \( a' \) is associated with a small value of \( b' \). The variables in Figure 3.7 (b) are said to be correlated while the variables in Figure 3.7 (a) are uncorrelated.
(a) $a'$ and $b'$ values are uncorrelated  \quad  (b) $a'$ and $b'$ values exhibit correlation

Figure 3.7: Illustration of correlation between two random variables $a'$ and $b'$.

Frequently, the correlation of the fluctuations of two variables is of interest. Given that

$$a(\bar{x},t)b(\bar{x},t) = \overline{a(\bar{x})b(\bar{x})} + \overline{a'(\bar{x},t)b'(\bar{x},t)},$$

the last term is the term of interest. The correlation coefficient is defined as

$$\rho_{a'b'}(\bar{x}) = \frac{\overline{a'b'(\bar{x})}}{[\overline{a'^2(\bar{x})} \overline{b'^2(\bar{x})}]^{1/2}} \quad (3.14)$$

This function indicates the interdependence of the two variables. A fundamental property of this coefficient is that it is bounded between $+1$ and $-1$. If $|\rho_{a'b'}| << 1$, the two variables are poorly correlated. If $|\rho_{a'b'}| \approx 1, -1$, the two variables are nearly perfectly correlated or anticorrelated, respectively.

A different characterization of the fluctuations of one or more variables is provided by the probability density functions. Consider again a statistically stationary variable $a(\bar{x},t)$ whose values are considered unbounded and a function $P(a, \bar{x})$ with the following properties:
\[ 1 = \int_{-\infty}^{\infty} P(a, \bar{x}) da \]  \hspace{1cm} (3.15a)

\[ \bar{a}(\bar{x}) = \int_{-\infty}^{\infty} a P(a, \bar{x}) da \]  \hspace{1cm} (3.15b)

\[ \bar{a}^2(\bar{x}) = \int_{-\infty}^{\infty} (a - \bar{a})^2 P(a, \bar{x}) da \]  \hspace{1cm} (3.15c)

The physical significance of \( P(a, \bar{x}) \) relates to its determining the fraction of time during which \( a \) is between \( a_1 \) and \( a_2 \), which is the probability that \( a(\bar{x}, t) \) is within this range.

Probability density functions are also defined for more than one variable called multivariate distribution. Consider a statistically stationary system with two variables \( a(\bar{x}, t) \) and \( b(\bar{x}, t) \) whose values are considered unbounded. Assume a bivariate probability density function is given by \( P(a, b, \bar{x}) \) with the following properties: (Libby, 1996)

\[ 1 = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} P(a, b, \bar{x}) da \ db \]  \hspace{1cm} (3.16a)

\[ P(a, \bar{x}) = \int_{-\infty}^{\infty} P(a, b, \bar{x}) \ db \]  \hspace{1cm} (3.16b)

\[ P(b, \bar{x}) = \int_{-\infty}^{\infty} P(a, b, \bar{x}) \ da \]  \hspace{1cm} (3.16c)

\[ \bar{a}'b'(\bar{x}) = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} (a - \bar{a})(b - \bar{b}) P(a, b, \bar{x}) \ da \ db \]  \hspace{1cm} (3.16d)

The previous discussion focused on the statistical description of a generic variable \( a(\bar{x}, t) \).

If this is a principal variable, it is evident that these descriptions contain no information on either the length or time scales associated with \( a(\bar{x}, t) \). Now it will be determined how to obtain the scale information from \( a'(\bar{x}, t) \) itself. In a statistically stationary system, consider the fluctuations of a representative variable \( a'(\bar{x}, t) \) and a second random variable found by the
introduction of a time advance $\tau$ which gives $a'(\bar{x}, t + \tau)$. Figure 3.8 shows the fluctuations of this generic variable and the two values associated with $t$ and $t + \tau$ as defined by Libby (1996).

![Figure 3.8: The distribution of the fluctuations of $a(\bar{x}, t)$ and its treatment as two random variables.](image)

A variation of equation (3.14) leads to the time autocorrelation coefficient

$$\rho_{a',a'}(\bar{x}, \tau) = \frac{\overline{a'(\bar{x}, t)a'(\bar{x}, t + \tau)}}{a'^2(\bar{x})} \tag{3.17}$$

where the bounds of $\pm 1$ still apply. Clearly, $\rho_{a',a'}(\bar{x}, 0) = 1$. When equation (3.17) is applied to turbulence, it generally gives $\rho_{a',a'}(\bar{x}, |\tau| \to \infty) \approx 0$. This just means that with sufficiently long time separation, the two variables become uncorrelated.

This formulation can be extended to two or more variables. Now the correlation between $a'(\bar{x}, t)$ and $b'(\bar{x}, t + \tau)$ will be considered, which is between two variables at the same spatial location but at times differing by $\tau$. Equation (3.17) becomes

$$\rho_{a'b'}(\bar{x}, \tau) = \frac{\overline{a'(\bar{x}, t)b'(\bar{x}, t + \tau)}}{[a'^2(\bar{x}) b'^2(\bar{x})]^{1/2}} \tag{3.18}$$

In this case, $\rho_{a'b'}(\bar{x}, 0) = \rho_{a'b'}(\bar{x})$ and is the same as equation (3.14), which is the one-point, one-time correlation coefficient between two variables. When this is applied to turbulence, then
\( \rho_{a'b}(\bar{x},|\tau| \to \infty) \approx 0 \). This also means that with sufficiently long time separation, the two
variables become uncorrelated. (Libby, 1996)

Returning to the statistically stationary variables, a similar treatment involving time
separation can be applied to spatial separation where two fluctuations, \( a'(\bar{x},t) \) and \( a'(\bar{x}+\bar{\zeta},t) \),
are fluctuations of the same variable. For example, a velocity component \( u_i \), at the same time
but at spatial locations separated by \( \bar{\zeta} \). Now equation (3.17) becomes

\[
\rho_{a'a}(\bar{x},\bar{\zeta}) = \frac{\overline{a'(\bar{x},t)a'(\bar{x}+\bar{\zeta},t)}}{a^2(\bar{x})} \quad (3.19)
\]

Although the time and length scales have been introduced on the basis of separate
considerations of one-point, two-time and two-point, one-time correlations, more general two-
point, two-time correlations are clearly suggested. A combination of the correlation coefficients
of equations (3.17) and (3.19) would give (Libby, 1996)

\[
\rho_{a'a}(\bar{x}, \bar{\zeta}, \tau) = \frac{\overline{a'(\bar{x},t)a'(\bar{x}+\bar{\zeta},t+\tau)}}{\left[a^2(\bar{x})a^2(\bar{x}+\bar{\zeta})\right]^{1/2}} \quad (3.20)
\]

The concept of stationary random variable represents a significant simplification because
an ensemble average like \( \langle a'(\bar{x}) \rangle \) will be independent of time. Time correlations of stationary
random variables should be independent of the choice of time origin, but they will be dependent
on the time difference \( \tau = t - t' \). (Landahl and Mollo-Christensen, 1992)

As seen earlier in this section, it is customary to use the autocorrelation function which is
defined exactly as the autocorrelation coefficient except it is not normalized. (Tennekes and
Lumley, 1972)

\[
R_{a'a}(\tau) = \langle a'(t)a'(t+\tau) \rangle \quad (3.21)
\]
The double subscript indicates that the autocorrelation function is the correlation of \( a' \) with \( a' \); the argument \( \tau \) shows that the samples are taken a time interval \( \tau \) apart. The cross correlation is a function of two random variables - that is, in each realization there are two results - a function \( a'(t) \) and a function \( b'(t) \). These could be two velocity or pressure components at a point in the flow. The cross correlation function is defined as

\[
R_{a'b'}(\tau) = \langle a'(t)b'(t+\tau) \rangle \quad (3.22)
\]

It was assumed that the process is stationary since the cross correlation function was written as dependent on the relative time delay \( \tau \) only. (Landahl and Mollo-Christensen, 1992)

The autocorrelation functions and cross correlation functions will be assumed to fall off for large values of the time delay so that the functions possess Fourier transforms. The Fourier transforms of the time autocorrelation function \( R_{a'a}(\tau) \) is called the spectral density

\[
S_{a'a}(\omega) = \frac{1}{2\pi} \int_{-\infty}^{\infty} e^{i\omega \tau} R_{a'a}(\tau) d\tau \quad (3.23)
\]

Fourier's integral theorem written for a function \( f(t) \) is

\[
f(t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} e^{-i\omega \tau} \int_{-\infty}^{\infty} e^{i\omega \tau} f(t') dt' d\omega \quad (3.24)
\]

Applying this to the spectral density, gives

\[
R_{a'a}(\tau) = \int_{-\infty}^{\infty} e^{-i\omega \tau} S_{a'a}(\omega) d\omega \quad (3.25)
\]

In a similar manner, the cross spectral density for a joint pair of random variables \( a' \) and \( b' \) is defined as

\[
S_{a'b'}(\omega) = \frac{1}{2\pi} \int_{-\infty}^{\infty} e^{i\omega \tau} R_{a'b'}(\tau) d\tau \quad (3.26)
\]
Now a space-time correlation function will be defined. Let $a'(\bar{x}, t)$ be a random variable with a position $\bar{x}$ and time $t$. The space-time correlation function is given by

$$R_{a'a'}(\bar{x}, \bar{x}', t, t') = \langle a'(\bar{x}, t) a'(\bar{x}', t') \rangle$$

(3.27)

If $a'$ is a stationary random variable, $R_{a'a'}$ is independent of the choice of time origin. If $R_{a'a'}$ is also independent of the choice of spatial origin, and the same holds true for the other statistical measures as well, then $a'(\bar{x}, t)$ is a homogeneous function of $\bar{x}$. So for a stationary and homogeneous random variable, the space-time correlation function is

$$R_{a'a'}(\bar{x}, \tau) = \langle a'(\bar{x}, t) a'(\bar{x} + \bar{x}, t + \tau) \rangle$$

(3.28)

One can transform both with respect to space and time, and this transformation will be described step by step. First, the spectral density of $a'$ is the Fourier transform of the time autocorrelation function $R_{a'a'}(0, \tau)$ and is equal to the local spectral density

$$S_{a'a'}(\omega) = \frac{1}{2\pi} \int_{-\infty}^{\infty} e^{i\omega \tau} R_{a'a'}(0, \tau) d\tau$$

(3.29)

The wavenumber spectrum $S(\bar{k})$ is defined by

$$S(\bar{k}) = \frac{1}{(2\pi)^3} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \exp[-i(\bar{k} \cdot \bar{x})] R_{a'a'}(\bar{x}, 0) d\bar{x}_1 d\bar{x}_2 d\bar{x}_3$$

(3.30)

Finally, the wavenumber frequency spectrum is defined by

$$S(\bar{k}, \omega) = \frac{1}{(2\pi)^3} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \exp[-i(\bar{k} \cdot \bar{x} - \omega\tau)] R_{a'a'}(\bar{x}, \tau) d\bar{x}_1 d\bar{x}_2 d\bar{x}_3 d\tau$$

(3.31)

For a two dimensional $\bar{x}$ domain, the integration with respect to one of the $\bar{x}$ terms is not included and the power of $1/2\pi$ is 3. (Landahl and Mollo-Christensen, 1992)
3.5 Definition of Terms

Every author in this field uses a slightly different nomenclature, coordinate systems and normalizing constants. In this section, the nomenclature used in this analysis will be defined to try to avoid confusion. First, Figure 3.9 shows the coordinate system that will be used in this analysis. The flow direction is identified as the $x$, the wall normal direction is $y$ and the cross flow direction is $z$.

![Flow Direction]

**Figure 3.9: Definition of flow field for present analysis.**

In this analysis, the pressure fluctuations at the wall are of interest so the coordinates that appear in most of the equations are in terms of $x$ and $z$. In aerodynamics, the geometric space domain $(x, y, z, \tau)$ is generally the typical domain of choice. In the previous section, the space-time correlation function for pressure fluctuations at the wall is given by $R_{pp}(x, z, \tau)$. In the derivation of the wavenumber-frequency spectrum model found in Chapter 6, the Fourier transform from time to angular frequency will be assumed to have already occurred. To avoid confusion, the space-frequency correlation function symbol will be $\psi_{pp}(x, z, \omega)$. The Fourier transform of the space-frequency correlation function gives the wavenumber-frequency spectral density and will be $\hat{\psi}_{pp}(k_x, k_z, \omega)$. In the previous section, the symbol used was $S_{pp}(k_x, k_z, \omega)$. 
This is also commonly called the wavenumber-frequency spectrum and many authors use the terms interchangeably in the same analysis. In this work, the wavenumber-frequency spectrum models defined by Graham (1991) are going to be used which have a specific normalization factor. To designate these models, the symbol $\Phi_{\rho\rho}(k_x,k_z,\omega)$ will be used.

Both the single point wall pressure spectrum model and the normalized wavenumber-frequency spectrum model are needed in the acoustical analysis programs to determine the structural response. The single point wall pressure spectrum model sorts the energy into frequencies at a single point. The normalized wavenumber-frequency spectrum model sorts the energy into wavelengths which gives the spatial distribution.

The single point wall pressure spectrum is defined as

$$\Phi(\omega) = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \Phi_{\rho\rho}(k_x,k_z,\omega) dk_x dk_z$$  \hspace{1cm} (3.32)

Again, this is sometimes called the single point wall pressure spectral density or power spectrum, but this is really the same thing.

It is common for previous authors to define these terms and normalize them slightly differently because they are typically empirically derived. In this work, the normalized wavenumber-frequency spectrum models will be represented by

$$\tilde{\Phi}_{\rho\rho}(k_x,k_z,\omega) = \frac{\omega^2}{U_z^2 \Phi(\omega)} \Phi_{\rho\rho}(k_x,k_z,\omega)$$  \hspace{1cm} (3.33)

It is important to note that the single point wall pressure spectrum model is used to normalize the wavenumber-frequency model.

### 3.6 Conceptual Single Element Wavenumber Filters

Corcos (1963) states that the finite size of a transducer sensing element limits its spatial resolution of a pressure field associated with a local turbulent flow. Such pressure fields are
translated at a speed comparable to the characteristic velocity of the flow. Consequently, a lack of resolution in space causes an apparent inability to resolve the signal in time. Therefore, the size of the transducer is important and it must be small enough to resolve the fine spatial structure of the turbulent flow. Schewe (1983) showed that a transducer with a dimensionless diameter of \( d_+ = 19 \) was small enough to resolve the essential structures of the turbulent pressure fluctuations. The dimensionless diameter scales the transducer diameter, \( d \), with the kinematic viscosity, \( \nu \), and the friction velocity, \( U_\tau \).

\[
d_+ = d U_\tau / \nu \tag{3.34}
\]

Lueptow (1995) described microphones as essentially wavenumber filters because the size of the transducer limits the accuracy of measurements of the fluctuating wall pressure field by the spatial averaging that occurs over the sensing surface. Lueptow studied two types of sensing elements in his analysis. The first type had a uniform sensing area which he called a piston type transducer. Pinhole microphones are considered to have a uniform sensitivity and can be modeled this way. The second was a displacement type transducer, such as a condenser microphone, where the sensitivity would vary over the sensing surface.

The wavenumber filter shape for a microphone of radius \( r_0 \) with uniform sensitivity is:

\[
H(k) = \frac{2 J_1(k r_0)}{k r_0} \tag{3.35}
\]

where:

\[
k = \sqrt{k_x^2 + k_z^2} \tag{3.36}
\]

The wavenumber filter shape for a circular deflection microphone of radius \( r_0 \) is modeled by the first mode of vibration of a membrane and is given by:

\[
H(k) = \frac{a^2 J_1(k r_0)}{a^2 - (k r_0)^2} \tag{3.37}
\]
where $a$ is the location of the first zero of the zero order Bessel function of the first kind:

$$a = 2.40482556043883$$

(A.38)

A square transducer of side $L_0$ with a uniform sensitivity has a normalized wavenumber response function of

$$H(k) = \left( \frac{\sin(k_x L_0 / 2)}{k_x L_0 / 2} \right) \left( \frac{\sin(k_z L_0 / 2)}{k_z L_0 / 2} \right)$$

(A.39)

The measured microphone responses are then the turbulent wall pressure spectrum times the wavenumber filter shape integrated over all wavenumber.

$$\Phi_m(\omega) = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \Phi_{p'p'}(k_x, k_z, \omega) |H(k_x, k_z)|^2 dk_x dk_z$$

(A.40)

Corcos (1963) was the first to develop a method to assess the error introduced in measurements by the finite size of a small transducer. The upper limit for the resolution of a transducer can be calculated from Figure 3.10. For example, a practical limit is when the measured single point wall pressure signal is reduced to half of the theoretical value.

$$\frac{\Phi_m}{\Phi} = 0.50$$

(A.41)

From the chart, the maximum practical limit for the measured frequency response of a transducer can be calculated from the following equation where $r_0$ is the radius of a round transducer, $L_0$ is the length of a square transducer and $U_c$ is the convective velocity. Knowing the radius or length of the transducer, the maximum frequency is

$$\frac{\omega r_0}{U_c} = 0.94$$

(A.42)
Figure 3.10: Attenuation of the single point wall pressure spectrum by a round and square transducer.

Lueptow (1995) described another error in the measured response. The lobed structure of the measured spectrum is a direct result of the zeros in the normalized wavenumber response function for the transducer. At lower frequencies, the measured response will match the theoretical response well. At higher frequencies, the measured response will not match the theoretical response if the first zero is inside the range of frequencies of the transducer. Table 3.1 gives the maximum measurable frequency for different transducers as a function of the convective velocity and the diameter or length of the transducer.
### TABLE 3.1

MAXIMUM MEASURABLE FREQUENCY FOR A TRANSDUCER

<table>
<thead>
<tr>
<th>Type of Sensor</th>
<th>Dimension of Sensor</th>
</tr>
</thead>
<tbody>
<tr>
<td>Square Piston</td>
<td>$\omega_0 = \frac{2\pi U_c}{L_o}$</td>
</tr>
<tr>
<td>Circular Piston or Pinhole</td>
<td>$\omega_0 = \frac{7.7 U_c}{d}$</td>
</tr>
<tr>
<td>Circular Deflection</td>
<td>$\omega_0 = \frac{11.0 U_c}{d}$</td>
</tr>
</tbody>
</table>

#### 3.7 Sample Rate and Aliasing

One of the most important parameters of an analog input system is the rate at which samples are taken from an incoming signal. The sampling rate determines how often an analog to digital conversion takes place. A fast sampling rate results in more points in a given time and can form a better representation of the original signal than a slow sampling rate. Sampling too slowly may result in a poor representation of the analog signal. Figure 3.11 compares an adequate sampling rate with an inadequate sampling rate. Undersampling can result in a signal that appears to have a different frequency than it really does. The misrepresentation of a signal is called an alias. According to the Nyquist theorem, to avoid aliasing one must sample at a rate greater than twice the maximum frequency component in the signal of interest. For a given sampling rate, the maximum frequency that can be represented accurately without aliasing is known as the Nyquist frequency. The Nyquist frequency is one half the sampling frequency. (National Instruments, 1993) The wall pressure fluctuation frequency range can be really large requiring an even larger sampling rate. In addition, it will be seen in Chapter 5 that the wall pressure fluctuation frequency range of interest increases with Mach number. If the sampling...
rate is set high, the space (i.e. memory) allocated for storage will be filled up before the time is long enough for a valid analysis.

Figure 3.11: Aliasing effects of an inadequate sampling rate.

To address the aliasing issue, the LabVIEW data acquisition software uses a low pass filter to be completely sure that the frequency content of the input signal is limited. A low pass filter is a filter that passes low frequencies but attenuates the high frequencies and is called an anti-alias filter because it attenuates the frequencies about the Nyquist preventing the aliasing components from being sampled. This filter is added before the sampler and analog to digital converter. This means that the resulting signal will have frequencies above the Nyquist frequency filtered out. Therefore, picking a sample rate is a tradeoff between analysis accuracy versus data storage.
3.8 Summary

In summary, the structure of a turbulent boundary layer is important in the analysis of wall pressure fluctuations. The regions of interest are the viscous sublayer, the buffer zone, the log layer, and the outer wake region. The wall pressure fluctuations beneath a turbulent boundary layer can be considered to be a random process; therefore, they can be described statistically. Some of these descriptions include space-time statistics which are called the autocorrelation function and the cross correlation function. The Fourier transform of these functions gives the auto spectral density and cross spectral density.

Both the single point wall pressure spectrum model and the normalized wavenumber-frequency spectrum model are used in acoustic analysis. The single point wall pressure spectrum model sorts the energy into frequencies at a single point and is based on the auto spectral density. The normalized wavenumber-frequency spectrum model sorts the energy into wavelengths which gives the spatial distribution and is based on the cross spectral density.

Measurements of the wall pressure fluctuations are made with microphones. The spatial extent of the microphone causes errors in the measurements. The size of the transducer must be small enough to resolve the fine structure of the turbulent flow. The errors induced by the size of the microphone can be characterized by the wavenumber filter shape of the microphone. A theoretical microphone wavenumber filter shape is used to estimate the attenuation of the true signal.

Signals above the Nyquist frequency are attenuated by an anti-aliasing filter in the data acquisition system. As the Mach number increases, the wall pressure fluctuation frequency range increases causing more signal to be attenuated. Picking a sample rate is a tradeoff between analysis accuracy versus data storage.
In light of the previous discussion, each of the models have distinct attributes the will be examined in detail in the following chapters. The mean square pressure is a measure of the total energy due to the pressure fluctuations beneath a turbulent boundary layer and will be discussed in Chapter 4. The single point wall pressure spectrum sorts the energy into frequencies which will be addressed in Chapter 5. The normalized wavenumber-frequency spectrum sorts the energy into wavenumbers and will be focused on in Chapter 6.
4.1 Background

The first model required for an acoustic analysis is that of the mean square pressure fluctuations. This quantity gives an estimate of the overall energy of the wall pressure fluctuations in a turbulent boundary layer. It also can serve as a simple check for a single point wall pressure spectrum model by integrating it over the frequency. These models have typically been empirical and based on experimental data. An estimate for the mean square pressure can also be derived numerically or analytically leading to wall pressure fluctuations. The derivation for incompressible, two dimensional Poisson equation is shown in section 4.1.1. In section 4.1.2, the boundary conditions and analysis assumptions required for the solution of the Poisson equation are given. Two solutions of the Poisson equation for a two dimensional boundary layer flow are given in section 4.1.3. The first solution is numerical, using a point Gauss-Seidel method. The second is an analytical solution using an eigenvalue expansion method.

4.1.1 Derivation of the Poisson Equation

Following the development found in White (1991), a model with incompressible turbulent flow and constant transport properties but with possible significant fluctuations as the sum of the mean value plus the fluctuation value is derived. Therefore, any variable \( Q \) is defined as \( Q = \overline{Q} + Q' \) where \( \overline{Q} \) is the mean value and \( Q' \) is the fluctuating value. The mean value is defined as

\[
\overline{Q} = \frac{1}{T} \int_{0}^{T} Q dt
\]  

(4.1)
where $T$ is large compared to the relevant period of the fluctuations. The fluctuating values in this case are velocity and pressure. Density and viscosity are treated as constants here.

\[
\begin{align*}
    u &= \bar{u} + u' \quad (4.2a) \\
    v &= \bar{v} + v' \quad (4.2b) \\
    p &= \bar{p} + p' \quad (4.2c)
\end{align*}
\]

First, the continuity equation is considered

\[
\nabla \cdot \bar{\mathbf{V}} = 0 \quad (4.3)
\]

In two dimensions, this becomes

\[
\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0 \quad (4.4)
\]

Substituting in the mean and fluctuating terms gives

\[
\frac{\partial}{\partial x} (\bar{u} + u') + \frac{\partial}{\partial y} (\bar{v} + v') = 0 \quad (4.5)
\]

The following rules of averaging will be required:

\[
\frac{\partial \bar{f}}{\partial s} = \frac{\partial \bar{f}}{\partial s} \quad \bar{f} + \bar{g} = \bar{f} + \bar{g} \quad \bar{f} = \bar{f} \quad \bar{f}' = 0 \quad \bar{fg}' = \bar{fg}' = 0
\]

\[
\bar{f}'g' \neq 0 \quad f'^2 \neq 0 \quad (4.6)
\]

Taking the time average of each part of the equation gives

\[
\frac{\partial}{\partial x} (\bar{u} + u') + \frac{\partial}{\partial y} (\bar{v} + v') = 0 \quad (4.7)
\]

which reduces to

\[
\frac{\partial}{\partial x} (\bar{u} + u') + \frac{\partial}{\partial y} (\bar{v} + v') = 0 \quad (4.8)
\]

And

\[
\frac{\partial}{\partial x} (\bar{u} + u') + \frac{\partial}{\partial y} (\bar{v} + v') = 0 \quad (4.9)
\]

Using the averaging rules to eliminate the zero terms gives
\[ \frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0 \]  \hspace{1cm} (4.10)

Subtracting equation (4.10) from equation (4.5) results in

\[ \frac{\partial u'}{\partial x} + \frac{\partial v'}{\partial y} = 0 \]  \hspace{1cm} (4.11)

The mean and fluctuating velocity components each separately satisfy the continuity equation.

The Navier-Stokes equation for incompressible flow with no body forces and constant viscosity is given by

\[ \rho \frac{D\vec{V}}{Dt} = -\nabla\vec{p} + \mu \nabla^2 \vec{V} \]  \hspace{1cm} (4.12)

In two dimensions, this becomes:

**x-term** \[ \rho \left( \frac{\partial u}{\partial t} + u \frac{\partial u}{\partial x} + v \frac{\partial u}{\partial y} \right) = -\frac{\partial p}{\partial x} + \mu \left( \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} \right) \]  \hspace{1cm} (4.13a)

**y-term** \[ \rho \left( \frac{\partial v}{\partial t} + u \frac{\partial v}{\partial x} + v \frac{\partial v}{\partial y} \right) = -\frac{\partial p}{\partial y} + \mu \left( \frac{\partial^2 v}{\partial x^2} + \frac{\partial^2 v}{\partial y^2} \right) \]  \hspace{1cm} (4.13b)

Before substituting for \( u, v \) and \( p \), the following rearrangement of the convective-acceleration term found in White (1991) are used

\[ \vec{V} \cdot \nabla Q = u \frac{\partial Q}{\partial x} + v \frac{\partial Q}{\partial y} = \frac{\partial}{\partial x} (uQ) + \frac{\partial}{\partial y} (vQ) \]  \hspace{1cm} (4.14)

Rewriting the incompressible Navier-Stokes equation using this relationship results in

**x-term** \[ \rho \left( \frac{\partial u}{\partial t} + uu + uv \right) = -\frac{\partial p}{\partial x} + \mu \left( \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} \right) \]  \hspace{1cm} (4.15a)

**y-term** \[ \rho \left( \frac{\partial v}{\partial t} + uv + vv \right) = -\frac{\partial p}{\partial y} + \mu \left( \frac{\partial^2 v}{\partial x^2} + \frac{\partial^2 v}{\partial y^2} \right) \]  \hspace{1cm} (4.15b)

Substituting in the mean and fluctuating terms gives
And multiplying out the terms gives

\[
\begin{align*}
\text{x-term} & \quad \rho \left[ \frac{\partial}{\partial t} (\overline{u} + u') + \frac{\partial}{\partial x} (\overline{u} + u')^2 + \frac{\partial}{\partial y} (\overline{u} + u')(\overline{v} + v') \right] \\
& = -\frac{\partial}{\partial x} (\overline{p} + p') + \mu \left[ \frac{\partial^2}{\partial x^2} (\overline{u} + u') + \frac{\partial^2}{\partial y^2} (\overline{u} + u') \right] \\
& = -\frac{\partial}{\partial y} (\overline{p} + p') + \mu \left[ \frac{\partial^2}{\partial x^2} (\overline{v} + v') + \frac{\partial^2}{\partial y^2} (\overline{v} + v') \right] \\
& (4.16b)
\end{align*}
\]

\[
\begin{align*}
\text{y-term} & \quad \rho \left[ \frac{\partial}{\partial t} (\overline{v} + v') + \frac{\partial}{\partial x} (\overline{u} + u')(\overline{v} + v') + \frac{\partial}{\partial y} (\overline{v} + v')^2 \right] \\
& = -\frac{\partial}{\partial y} (\overline{p} + p') + \mu \left[ \frac{\partial^2}{\partial x^2} (\overline{v} + v') + \frac{\partial^2}{\partial y^2} (\overline{v} + v') \right] \\
& (4.16b)
\end{align*}
\]

Taking the time average of each part of the equation gives

\[
\begin{align*}
\text{x-term} & \quad \rho \left[ \frac{\partial}{\partial t} (\overline{u} + u') + \frac{\partial}{\partial x} (\overline{u} + u' + u'u') + \frac{\partial}{\partial y} (\overline{v} + v' + u'u') \right] \\
& = -\frac{\partial}{\partial x} (\overline{p} + p') + \mu \left[ \frac{\partial^2}{\partial x^2} (\overline{u} + u') + \frac{\partial^2}{\partial y^2} (\overline{u} + u') \right] \\
& = -\frac{\partial}{\partial y} (\overline{p} + p') + \mu \left[ \frac{\partial^2}{\partial x^2} (\overline{v} + v') + \frac{\partial^2}{\partial y^2} (\overline{v} + v') \right] \\
& (4.17b)
\end{align*}
\]

\[
\begin{align*}
\text{y-term} & \quad \rho \left[ \frac{\partial}{\partial t} (\overline{v} + v') + \frac{\partial}{\partial x} (\overline{u} + u' + u'u') + \frac{\partial}{\partial y} (\overline{v} + v' + u'u') \right] \\
& = -\frac{\partial}{\partial y} (\overline{p} + p') + \mu \left[ \frac{\partial^2}{\partial x^2} (\overline{v} + v') + \frac{\partial^2}{\partial y^2} (\overline{v} + v') \right] \\
& = -\frac{\partial}{\partial y} (\overline{p} + p') + \mu \left[ \frac{\partial^2}{\partial x^2} (\overline{v} + v') + \frac{\partial^2}{\partial y^2} (\overline{v} + v') \right] \\
& (4.17b)
\end{align*}
\]

Which is equivalent to
\[ x\text{-term} \quad \rho \left[ \frac{\partial}{\partial t} (\bar{u} + u') + \frac{\partial}{\partial x} (\bar{uu} + 2\bar{u}u' + u'u') + \frac{\partial}{\partial y} (\bar{uv} + \bar{u}v' + v'u' + u'u') \right] \\
= -\frac{\partial}{\partial x} (\bar{p} + p') + \mu \left[ \frac{\partial^2}{\partial x^2} (\bar{u} + u') + \frac{\partial^2}{\partial y^2} (\bar{u} + u') \right] \tag{4.19a} \]

\[ y\text{-term} \quad \rho \left[ \frac{\partial}{\partial t} (\bar{v} + v') + \frac{\partial}{\partial x} (\bar{uv} + \bar{u}v' + v'u' + u'u') + \frac{\partial}{\partial y} (\bar{vv} + 2\bar{v}v' + v'v') \right] \\
= -\frac{\partial}{\partial y} (\bar{p} + p') + \mu \left[ \frac{\partial^2}{\partial x^2} (\bar{v} + v') + \frac{\partial^2}{\partial y^2} (\bar{v} + v') \right] \tag{4.19b} \]

and reduces to

\[ x\text{-term} \quad \rho \left[ \frac{\partial}{\partial t} (\bar{u} + u') + \frac{\partial}{\partial x} (\bar{uu} + 2\bar{u}u' + u'u') + \frac{\partial}{\partial y} (\bar{uv} + \bar{u}v' + v'u' + u'u') \right] \\
= -\frac{\partial}{\partial x} (\bar{p} + p') + \mu \left[ \frac{\partial^2}{\partial x^2} (\bar{u} + u') + \frac{\partial^2}{\partial y^2} (\bar{u} + u') \right] \tag{4.20a} \]

\[ y\text{-term} \quad \rho \left[ \frac{\partial}{\partial t} (\bar{v} + v') + \frac{\partial}{\partial x} (\bar{uv} + \bar{u}v' + v'u' + u'u') + \frac{\partial}{\partial y} (\bar{vv} + 2\bar{v}v' + v'v') \right] \\
= -\frac{\partial}{\partial y} (\bar{p} + p') + \mu \left[ \frac{\partial^2}{\partial x^2} (\bar{v} + v') + \frac{\partial^2}{\partial y^2} (\bar{v} + v') \right] \tag{4.20b} \]

Using the averaging rules to eliminate the zero terms gives

\[ x\text{-term} \quad \rho \left[ \frac{\partial}{\partial t} (\bar{u}) + \frac{\partial}{\partial x} (\bar{uu}) + \frac{\partial}{\partial y} (\bar{uv}) \right] = -\frac{\partial \bar{p}}{\partial x} + \mu \left( \frac{\partial^2 \bar{u}}{\partial x^2} + \frac{\partial^2 \bar{u}}{\partial y^2} \right) \tag{4.21a} \]

\[ y\text{-term} \quad \rho \left[ \frac{\partial}{\partial t} (\bar{v}) + \frac{\partial}{\partial x} (\bar{uv}) + \frac{\partial}{\partial y} (\bar{vv}) \right] = -\frac{\partial \bar{p}}{\partial y} + \mu \left( \frac{\partial^2 \bar{v}}{\partial x^2} + \frac{\partial^2 \bar{v}}{\partial y^2} \right) \tag{4.21b} \]

Using the rearrangement technique of the convective acceleration term from White (1991) gives

\[ x\text{-term} \quad \rho \left[ \frac{\partial \bar{u}}{\partial t} + \frac{\partial \bar{u}}{\partial x} (u'u') + \frac{\partial \bar{u}}{\partial y} (uv' + v'u') \right] + \rho \left[ \frac{\partial (u'u')}{\partial x} + \frac{\partial (uv')}{\partial y} \right] = -\frac{\partial \bar{p}}{\partial x} + \mu \left( \frac{\partial^2 \bar{u}}{\partial x^2} + \frac{\partial^2 \bar{u}}{\partial y^2} \right) \tag{4.22a} \]

\[ y\text{-term} \quad \rho \left[ \frac{\partial \bar{v}}{\partial t} + \frac{\partial \bar{v}}{\partial x} (v'u') + \frac{\partial \bar{v}}{\partial y} (vu' + v'u') \right] + \rho \left[ \frac{\partial (v'u')}{\partial x} + \frac{\partial (vu')}{\partial y} \right] = -\frac{\partial \bar{p}}{\partial y} + \mu \left( \frac{\partial^2 \bar{v}}{\partial x^2} + \frac{\partial^2 \bar{v}}{\partial y^2} \right) \tag{4.22b} \]

If the two dimensional equation is rearranged to display the turbulent inertia terms as if they were stresses, the results are

45
In index notation, equations (4.23a) and (4.23b) reduces to

\[ \rho \frac{D\vec{V}}{Dt} = -\nabla p + \nabla \cdot \tau_{ij} \]  

where

\[ \tau_{ij} = \mu \left( \frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i} \right) - \rho u_i'u_j' \]

Mathematically, the turbulent inertia terms behave as if the total stress on the system were composed of the Newtonian viscous stresses plus an additional turbulent stress tensor \( \rho u_i'u_j' \).

This is also the origin of the Reynolds stress which is given by \( \rho u_i'u_j' \). In a boundary layer, the dominant term \( \rho u_i'u_j' \) is called the turbulent stress.

To develop the Poisson equation for the pressure field in an incompressible turbulent boundary layer, Corcos (1964) begins with the Navier-Stokes equation. Starting with equations (4.13a) and (4.13b), which are repeated here for clarity.

\[ x\text{-term} \quad \rho \left( \frac{\partial u}{\partial t} + \frac{\partial u}{\partial x} u + \frac{\partial u}{\partial y} v \right) = -\frac{\partial p}{\partial x} + \mu \left( \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} \right) \]  

\[ y\text{-term} \quad \rho \left( \frac{\partial v}{\partial t} + \frac{\partial u}{\partial x} v + \frac{\partial v}{\partial y} v \right) = -\frac{\partial p}{\partial y} + \mu \left( \frac{\partial^2 v}{\partial x^2} + \frac{\partial^2 v}{\partial y^2} \right) \]  

Corcos (1964) instructs us to take the divergence of each term of this equation and make use of the continuity equation. Dividing the Navier-Stokes equation up into pieces and taking the divergence results in
x-term \[\rho \frac{\partial u}{\partial t} + \rho \left( \frac{\partial}{\partial x} uu + \frac{\partial}{\partial y} uv \right) = -\frac{\partial p}{\partial x} + \mu \left( \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} \right) \] (4.25a)

y-term \[\rho \frac{\partial v}{\partial t} + \rho \left( \frac{\partial}{\partial x} uv + \frac{\partial}{\partial y} vv \right) = -\frac{\partial p}{\partial y} + \mu \left( \frac{\partial^2 v}{\partial x^2} + \frac{\partial^2 v}{\partial y^2} \right) \] (4.25b)

Taking the divergence of the time derivative term and using the continuity equation gives

\[
\nabla \cdot \rho \left( \frac{\partial u}{\partial t} i + \frac{\partial v}{\partial t} j \right) = \frac{\partial}{\partial x} \left( \rho \frac{\partial u}{\partial t} \right) + \frac{\partial}{\partial y} \left( \rho \frac{\partial v}{\partial t} \right)
\]

\[= \rho \frac{\partial}{\partial t} \left( \frac{\partial u}{\partial x} \right) + \rho \frac{\partial}{\partial t} \left( \frac{\partial v}{\partial y} \right)\]

\[= \rho \frac{\partial}{\partial t} \left( \frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} \right) = 0 \] (4.26)

Taking the divergence of the next term results in

\[
\nabla \cdot \rho \left[ \left( \frac{\partial}{\partial x} uu + \frac{\partial}{\partial y} uv \right) i + \left( \frac{\partial}{\partial x} uv + \frac{\partial}{\partial y} vv \right) j \right] = \frac{\partial}{\partial x} \rho \left( \frac{\partial}{\partial x} uu + \frac{\partial}{\partial y} uv \right) + \frac{\partial}{\partial y} \rho \left( \frac{\partial}{\partial x} uv + \frac{\partial}{\partial y} vv \right)
\]

\[= \rho \frac{\partial^2}{\partial x^2} uu + \rho \frac{\partial}{\partial x \partial y} uv + \rho \frac{\partial}{\partial x \partial y} uv + \rho \frac{\partial^2}{\partial y^2} vv \]

\[= \rho \left[ \frac{\partial^2}{\partial x^2} uu + 2 \frac{\partial^2}{\partial x \partial y} uv + \frac{\partial^2}{\partial y^2} vv \right] \] (4.27)

Taking the divergence of the pressure terms gives

\[
\nabla \cdot \left( -\frac{\partial}{\partial x} p i - \frac{\partial}{\partial y} p j \right) = -\frac{\partial}{\partial x} \left( \frac{\partial p}{\partial x} \right) - \frac{\partial}{\partial y} \left( \frac{\partial p}{\partial y} \right)
\]

\[= -\frac{\partial^2 p}{\partial x^2} - \frac{\partial^2 p}{\partial y^2} \] (4.28)

Taking the divergence of the viscous terms results in

\[
\nabla \cdot \left[ \mu \left( \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} \right) i + \mu \left( \frac{\partial^2 v}{\partial x^2} + \frac{\partial^2 v}{\partial y^2} \right) j \right] = \frac{\partial}{\partial x} \mu \left( \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} \right) + \frac{\partial}{\partial y} \mu \left( \frac{\partial^2 v}{\partial x^2} + \frac{\partial^2 v}{\partial y^2} \right)
\]

\[= \mu \left[ \frac{\partial^3 u}{\partial x^3} + \frac{\partial^3 u}{\partial x \partial y^2} + \frac{\partial^3 v}{\partial x^2 \partial y} + \frac{\partial^3 v}{\partial y^3} \right] \]
\[
\frac{\partial^2 p}{\partial x^2} + \frac{\partial^2 p}{\partial y^2} = -\rho \left[ \frac{\partial^2}{\partial x^2} uu + 2 \frac{\partial^2}{\partial x \partial y} uv + \frac{\partial^2}{\partial y^2} vv \right]
\]  
(4.30)

Summing the nonzero terms results in

\[
\frac{\partial^2 \tilde{p}}{\partial x^2} + \frac{\partial^2 \tilde{p}}{\partial y^2} = -\rho \frac{\partial^2 \bar{u}_i \bar{u}_j}{\partial x_i \partial x_j}
\]  
(4.31)

In index notation, this equation is equivalent to Corcos's equation and can be written as:

(Corcos, 1964)

To model this pressure field with an incompressible turbulent flow, the mean and fluctuating terms from equation (4.2) will be substituted.

\[
\frac{\partial^2 (\bar{p} + p')}{\partial x^2} + \frac{\partial^2 (\bar{p} + p')}{\partial y^2} = -\rho \left[ \frac{\partial^2}{\partial x^2} (\bar{u} + u')(\bar{u} + u') + 2 \frac{\partial^2}{\partial x \partial y} (\bar{u} + u')(\bar{v} + v') 
+ \frac{\partial^2}{\partial y^2} (\bar{v} + v')(\bar{v} + v') \right]
\]  
(4.32)

Multiplying out the terms gives

\[
\frac{\partial^2 (\bar{p} + p')}{\partial x^2} + \frac{\partial^2 (\bar{p} + p')}{\partial y^2} = -\rho \left[ \frac{\partial^2}{\partial x^2} (\bar{u} u + 2 \bar{u} u' + u u') + 2 \frac{\partial^2}{\partial x \partial y} (\bar{u} v + \bar{v} v' + \bar{v} u' + u v') 
+ \frac{\partial^2}{\partial y^2} (\bar{v} v + 2 \bar{v} v' + v v') \right]
\]  
(4.33)

Taking the time average of each part of the equation gives
\[ \frac{\partial^2}{\partial x^2} (\bar{p} + p') + \frac{\partial^2}{\partial y^2} (\bar{p} + p') = -\rho \left[ \frac{\partial^2}{\partial x^2} (\bar{u}\bar{u} + 2\bar{u}u' + u'u') + 2 \frac{\partial^2}{\partial x\partial y} (\bar{u}\bar{v} + \bar{u}v' + v'u' + u'v') \right. \\
\left. \quad + \frac{\partial^2}{\partial y^2} (\bar{v}\bar{v} + 2\bar{v}v' + v'v') \right] \\
\] (4.34)

which is equivalent to

\[ \frac{\partial^2}{\partial x^2} (\bar{p} + p') + \frac{\partial^2}{\partial y^2} (\bar{p} + p') = -\rho \left[ \frac{\partial^2}{\partial x^2} (\bar{u}\bar{u} + 2\bar{u}u' + u'u') + 2 \frac{\partial^2}{\partial x\partial y} (\bar{u}\bar{v} + \bar{u}v' + v'u' + u'v') \right. \\
\left. \quad + \frac{\partial^2}{\partial y^2} (\bar{v}\bar{v} + 2\bar{v}v' + v'v') \right] \\
\] (4.35)

and reduces to

\[ \frac{\partial^2}{\partial x^2} (\bar{p} + p') + \frac{\partial^2}{\partial y^2} (\bar{p} + p') = -\rho \left[ \frac{\partial^2}{\partial x^2} (\bar{u}\bar{u} + 2\bar{u}u' + u'u') + 2 \frac{\partial^2}{\partial x\partial y} (\bar{u}\bar{v} + \bar{u}v' + v'u' + u'v') \right. \\
\left. \quad + \frac{\partial^2}{\partial y^2} (\bar{v}\bar{v} + 2\bar{v}v' + v'v') \right] \\
\] (4.36)

Using the averaging rules to eliminate the zero terms gives

\[ \frac{\partial^2}{\partial x^2} \bar{p} + \frac{\partial^2}{\partial y^2} \bar{p} = -\rho \left[ \frac{\partial^2}{\partial x^2} (\bar{u}\bar{u} + 2\bar{u}u' + u'u') + 2 \frac{\partial^2}{\partial x\partial y} (\bar{u}\bar{v} + \bar{u}v' + v'u' + u'v') \right. \\
\left. \quad + \frac{\partial^2}{\partial y^2} (\bar{v}\bar{v} + 2\bar{v}v' + v'v') \right] \\
\] (4.37)

Subtracting equation (4.37) from equation (4.33) results in

\[ \frac{\partial^2}{\partial x^2} p' + \frac{\partial^2}{\partial y^2} p' = -\rho \left[ \frac{\partial^2}{\partial x^2} (2\bar{u}u' + u'u' - \bar{u}u') + 2 \frac{\partial^2}{\partial x\partial y} (\bar{u}v' + \bar{u}v' + u'v' - \bar{u}v') \right. \\
\left. \quad + \frac{\partial^2}{\partial y^2} (2\bar{v}v' + v'v' - \bar{v}v') \right] \\
\] (4.38)

Now it will be shown that this equation is equivalent to Corcos' solution. (Corcos, 1964) To begin the process, equation (4.38) will be rearranged.
\[
\frac{\partial^2 p'}{\partial x^2} + \frac{\partial^2 p'}{\partial y^2} = -2\rho \left[ \frac{\partial^2}{\partial x^2} (\bar{u} u') + \frac{\partial^2}{\partial x \partial y} (\bar{v} v') + \frac{\partial^2}{\partial y^2} (\bar{v} v') \right]
- \rho \left[ \frac{\partial^2}{\partial x^2} (u' u' - \bar{u} u') + 2 \frac{\partial^2}{\partial x \partial y} (u' v' - \bar{u} v') + \frac{\partial^2}{\partial y^2} (v' v' - \bar{v} v') \right]
\] (4.39)

Next the product rule will be used to expand the terms inside the first brackets on the right side:

\[
\frac{\partial^2}{\partial x \partial y} (fg) = f \cdot \frac{\partial^2 g}{\partial x \partial y} + \frac{\partial f}{\partial x} \frac{\partial g}{\partial y} + \frac{\partial f}{\partial y} \frac{\partial g}{\partial x} + g \cdot \frac{\partial^2 f}{\partial x \partial y}
\] (4.40)

This results in

\[
\frac{\partial^2 p'}{\partial x^2} + \frac{\partial^2 p'}{\partial y^2} = -2\rho \left[ \left( \bar{u} \frac{\partial^2 u'}{\partial x^2} + 2 \frac{\partial \bar{u}}{\partial x} \frac{\partial u'}{\partial x} + u' \frac{\partial^2 \bar{u}}{\partial x^2} \right) + \left( \bar{u} \frac{\partial^2 v'}{\partial x \partial y} + \frac{\partial \bar{u}}{\partial x} \frac{\partial v'}{\partial y} + \frac{\partial \bar{u}}{\partial y} \frac{\partial v'}{\partial x} + v' \frac{\partial^2 \bar{u}}{\partial y \partial x} \right) \right]
+ \left( \bar{v} \frac{\partial^2 u'}{\partial x \partial y} + \frac{\partial \bar{v}}{\partial x} \frac{\partial u'}{\partial y} + \frac{\partial \bar{v}}{\partial y} \frac{\partial u'}{\partial x} + u' \frac{\partial^2 \bar{v}}{\partial x \partial y} \right) + \left( \bar{v} \frac{\partial^2 v'}{\partial y^2} + 2 \frac{\partial \bar{v}}{\partial y} \frac{\partial v'}{\partial y} + v' \frac{\partial^2 \bar{v}}{\partial y^2} \right)
- \rho \left[ \frac{\partial^2}{\partial x^2} (u' u' - \bar{u} u') + 2 \frac{\partial^2}{\partial x \partial y} (u' v' - \bar{u} v') + \frac{\partial^2}{\partial y^2} (v' v' - \bar{v} v') \right]
\] (4.41)

Rearranging the terms again gives

\[
\frac{\partial^2 p'}{\partial x^2} + \frac{\partial^2 p'}{\partial y^2} = -2\rho \left[ \frac{\partial \bar{u}}{\partial x} \frac{\partial u'}{\partial x} + \frac{\partial \bar{v}}{\partial y} \frac{\partial v'}{\partial x} + \frac{\partial \bar{u}}{\partial x} \frac{\partial u'}{\partial y} + \frac{\partial \bar{v}}{\partial y} \frac{\partial v'}{\partial y} \right]
- 2\rho \left[ \left( \bar{u} \frac{\partial^2 u'}{\partial x^2} + \frac{\partial \bar{u}}{\partial x} \frac{\partial u'}{\partial x} + u' \frac{\partial^2 \bar{u}}{\partial x^2} \right) + \left( \bar{u} \frac{\partial^2 v'}{\partial x \partial y} + \frac{\partial \bar{u}}{\partial x} \frac{\partial v'}{\partial y} + \frac{\partial \bar{u}}{\partial y} \frac{\partial v'}{\partial x} + v' \frac{\partial^2 \bar{u}}{\partial y \partial x} \right) \right]
+ \left( \bar{v} \frac{\partial^2 u'}{\partial x \partial y} + \frac{\partial \bar{v}}{\partial x} \frac{\partial u'}{\partial y} + \frac{\partial \bar{v}}{\partial y} \frac{\partial u'}{\partial x} + u' \frac{\partial^2 \bar{v}}{\partial x \partial y} \right) + \left( \bar{v} \frac{\partial^2 v'}{\partial y^2} + 2 \frac{\partial \bar{v}}{\partial y} \frac{\partial v'}{\partial y} + v' \frac{\partial^2 \bar{v}}{\partial y^2} \right)
- \rho \left[ \frac{\partial^2}{\partial x^2} (u' u' - \bar{u} u') + 2 \frac{\partial^2}{\partial x \partial y} (u' v' - \bar{u} v') + \frac{\partial^2}{\partial y^2} (v' v' - \bar{v} v') \right]
\] (4.42)
\[
\frac{\partial^2 p'}{\partial x^2} + \frac{\partial^2 p'}{\partial y^2} = -2 \rho \left[ \frac{\partial \bar{u} \partial u'}{\partial x \partial x} + \frac{\partial \bar{u} \partial v'}{\partial y \partial x} + \frac{\partial \bar{v} \partial u'}{\partial x \partial y} + \frac{\partial \bar{v} \partial v'}{\partial y \partial y} \right] \\
-2 \rho \left[ \bar{u} \left( \frac{\partial^2 u'}{\partial x^2} + \frac{\partial^2 v'}{\partial x \partial y} \right) + \bar{v} \left( \frac{\partial^2 u'}{\partial y \partial x} + \frac{\partial^2 v'}{\partial y^2} \right) \right] + u' \left( \frac{\partial \bar{u}}{\partial x} + \frac{\partial \bar{v}}{\partial y} \right) + v' \left( \frac{\partial \bar{u}}{\partial y} + \frac{\partial \bar{v}}{\partial x} \right) \\
- \rho \left[ \frac{\partial^2}{\partial x^2} (u'u' - \bar{u}\bar{u}') + 2 \frac{\partial^2}{\partial x \partial y} (u'v' - \bar{u}\bar{v}') + \frac{\partial^2}{\partial y^2} (v'v' - \bar{v}\bar{v}') \right] 
\]  

(4.43)

\[
\frac{\partial^2 p'}{\partial x^2} + \frac{\partial^2 p'}{\partial y^2} = -2 \rho \left[ \frac{\partial \bar{u} \partial u'}{\partial x \partial x} + \frac{\partial \bar{u} \partial v'}{\partial y \partial x} + \frac{\partial \bar{v} \partial u'}{\partial x \partial y} + \frac{\partial \bar{v} \partial v'}{\partial y \partial y} \right] \\
-2 \rho \left[ \bar{u} \left( \frac{\partial^2 u'}{\partial x^2} + \frac{\partial^2 v'}{\partial x \partial y} \right) + \bar{v} \left( \frac{\partial^2 u'}{\partial y \partial x} + \frac{\partial^2 v'}{\partial y^2} \right) \right] + u' \left( \frac{\partial \bar{u}}{\partial x} + \frac{\partial \bar{v}}{\partial y} \right) + v' \left( \frac{\partial \bar{u}}{\partial y} + \frac{\partial \bar{v}}{\partial x} \right) \\
- \rho \left[ \frac{\partial^2}{\partial x^2} (u'u' - \bar{u}\bar{u}') + 2 \frac{\partial^2}{\partial x \partial y} (u'v' - \bar{u}\bar{v}') + \frac{\partial^2}{\partial y^2} (v'v' - \bar{v}\bar{v}') \right] 
\]  

(4.44)

Making use of the continuity equation for both the mean and fluctuating velocities given in equations (4.10) and (4.11) simplifies this equation to

\[
\frac{\partial^2 p'}{\partial x^2} + \frac{\partial^2 p'}{\partial y^2} = -2 \rho \left[ \frac{\partial \bar{u} \partial u'}{\partial x \partial x} + \frac{\partial \bar{u} \partial v'}{\partial y \partial x} + \frac{\partial \bar{v} \partial u'}{\partial x \partial y} + \frac{\partial \bar{v} \partial v'}{\partial y \partial y} \right] \\
- \rho \left[ \frac{\partial^2}{\partial x^2} (u'u' - \bar{u}\bar{u}') + 2 \frac{\partial^2}{\partial x \partial y} (u'v' - \bar{u}\bar{v}') + \frac{\partial^2}{\partial y^2} (v'v' - \bar{v}\bar{v}') \right] 
\]  

(4.45)

In index notation, this gives

\[
\nabla^2 p' = \frac{\partial^2 p'}{\partial x_i^2} = -2 \rho \frac{\partial \bar{u}_i \bar{u}_j'}{\partial x_j \partial x_i} - \rho \frac{\partial}{\partial x_i} \frac{\partial^2}{\partial x_j \partial x_j} (u_i \bar{u}_j' - \bar{u}_j u_i') 
\]  

(4.46)
The first set of terms on the right side is referred to as the mean shear (MS) term. Other authors refer to this term as linear or rapid source term. The second set is the turbulence-turbulence (TT) terms. Other authors refer to this set of terms as the nonlinear or slow source terms. (Chang, 1998) Further simplifications can be made by applying the product rule to expand the terms inside the second bracket on the right side of equation (4.45).

\[
\frac{\partial^2 p'}{\partial x^2} + \frac{\partial^2 p'}{\partial y^2} = -2\rho \left[ \frac{\partial u' \partial u'}{\partial x} + \frac{\partial u' \partial v'}{\partial y} + \frac{\partial v' \partial u'}{\partial x} + \frac{\partial v' \partial v'}{\partial y} \right] \\
- \rho \left[ 2u' \frac{\partial^2 u'}{\partial x^2} + 2 \left( \frac{\partial u'}{\partial x} \right)^2 + 2v' \frac{\partial^2 v'}{\partial y^2} + 2 \left( \frac{\partial v'}{\partial y} \right)^2 \\
+ 2u' \frac{\partial^2 v'}{\partial x \partial y} + 2 \frac{\partial u'}{\partial x} \frac{\partial v'}{\partial y} + 2 \frac{\partial v'}{\partial x} \frac{\partial u'}{\partial y} + 2v' \frac{\partial^2 u'}{\partial x \partial y} - \frac{\partial^2}{\partial x^2} (u' u') - 2 \frac{\partial^2}{\partial x \partial y} (u' v') - \frac{\partial^2}{\partial y^2} (v' v') \right]
\]

(4.47)

Rearranging the terms again gives

\[
\frac{\partial^2 p'}{\partial x^2} + \frac{\partial^2 p'}{\partial y^2} = -2\rho \left[ \frac{\partial u' \partial u'}{\partial x} + \frac{\partial u' \partial v'}{\partial y} + \frac{\partial v' \partial u'}{\partial x} + \frac{\partial v' \partial v'}{\partial y} \right] \\
- \rho \left[ \left( \frac{\partial u'}{\partial x} \right)^2 + 2 \frac{\partial v'}{\partial x} \frac{\partial u'}{\partial y} + \left( \frac{\partial v'}{\partial y} \right)^2 \\
+ 2u' \frac{\partial^2 v'}{\partial x \partial y} + 2u' \frac{\partial^2 v'}{\partial x \partial y} + 2v' \frac{\partial^2 u'}{\partial x \partial y} + 2v' \frac{\partial^2 v'}{\partial y^2} \\
+ \left( \frac{\partial u'}{\partial x} \right)^2 + \frac{\partial u' \partial v'}{\partial x} + \left( \frac{\partial u'}{\partial x} \right)^2 + \frac{\partial u' \partial v'}{\partial y} \\
- \frac{\partial^2}{\partial x^2} (u' u') - 2 \frac{\partial^2}{\partial x \partial y} (u' v') - \frac{\partial^2}{\partial y^2} (v' v') \right]
\]

(4.48)
\[
\frac{\partial^2 p'}{\partial x^2} + \frac{\partial^2 p'}{\partial y^2} = -2\rho \left[ \frac{\partial \bar{u} \partial u'}{\partial x \partial x} + \frac{\partial \bar{u} \partial v'}{\partial x \partial y} + \frac{\partial \bar{v} \partial u'}{\partial x \partial y} + \frac{\partial \bar{v} \partial v'}{\partial y \partial y} \right] \\
- \rho \left[ \left( \frac{\partial u'}{\partial x} \right)^2 + 2 \frac{\partial v' \partial u'}{\partial x \partial y} + \left( \frac{\partial v'}{\partial y} \right)^2 \right] \\
+ 2 \frac{\partial u'}{\partial x} \left( \frac{\partial u'}{\partial x} + \frac{\partial v'}{\partial y} \right) + 2 \frac{\partial v'}{\partial y} \left( \frac{\partial u'}{\partial x} + \frac{\partial v'}{\partial y} \right) \\
+ \frac{\partial u'}{\partial x} \left( \frac{\partial u'}{\partial x} + \frac{\partial v'}{\partial y} \right) + \frac{\partial v'}{\partial y} \left( \frac{\partial u'}{\partial x} + \frac{\partial v'}{\partial y} \right) \\
- \frac{\partial^2}{\partial x^2} (u'u') - 2 \frac{\partial^2}{\partial x \partial y} (u'v') - \frac{\partial^2}{\partial y^2} (v'v') \right] 
\] (4.49)

Making use of the continuity equation for fluctuating velocities given in equations (4.11) simplifies this equation to

\[
\frac{\partial^2 p'}{\partial x^2} + \frac{\partial^2 p'}{\partial y^2} = -2\rho \left[ \frac{\partial \bar{u} \partial u'}{\partial x \partial x} + \frac{\partial \bar{u} \partial v'}{\partial x \partial y} + \frac{\partial \bar{v} \partial u'}{\partial x \partial y} + \frac{\partial \bar{v} \partial v'}{\partial y \partial y} \right] \\
- \rho \left[ \left( \frac{\partial u'}{\partial x} \right)^2 + 2 \frac{\partial v' \partial u'}{\partial x \partial y} + \left( \frac{\partial v'}{\partial y} \right)^2 \right] \\
- \frac{\partial^2}{\partial x^2} (u'u') - 2 \frac{\partial^2}{\partial x \partial y} (u'v') - \frac{\partial^2}{\partial y^2} (v'v') \right] 
\] (4.50)

If it is assumed that the mean flow in the region of interest is very nearly parallel to the plate, this allows the assumption that \( \bar{v} = 0 \) and \( \bar{u} / \partial x = 0 \). This simplifies the equation even more.

\[
\frac{\partial^2 p'}{\partial x^2} + \frac{\partial^2 p'}{\partial y^2} = -2\rho \left[ \frac{\partial \bar{u} \partial u'}{\partial y \partial x} \right] \\
- \rho \left[ \left( \frac{\partial u'}{\partial x} \right)^2 + 2 \frac{\partial v' \partial u'}{\partial x \partial y} + \left( \frac{\partial v'}{\partial y} \right)^2 \right] - \frac{\partial^2}{\partial x^2} (u'u') - 2 \frac{\partial^2}{\partial x \partial y} (u'v') - \frac{\partial^2}{\partial y^2} (v'v') \right] 
\] (4.51)

Following the nomenclature used by Chang (1998):

\[
\frac{\partial^2 p'}{\partial x^2} + \frac{\partial^2 p'}{\partial y^2} = -\rho \bar{p}_{MS}^{MS} + T^{TT} 
\] (4.52)

where \( T^{MS} \) represents the mean shear terms and \( T^{TT} \) represents the turbulence-turbulence terms.

The mean shear term is given by
The turbulence-turbulence term is given by

\[ T^{TT} = \sum_{i=1}^{2} \sum_{j=1}^{2} T_{ij} \]  

(4.54)

Table 4.1 lists the turbulence-turbulence terms in two dimensions.

TABLE 4.1

<table>
<thead>
<tr>
<th>( T_{11}^{TT} )</th>
<th>( T_{12}^{TT} )</th>
<th>( T_{21}^{TT} )</th>
<th>( T_{22}^{TT} )</th>
</tr>
</thead>
<tbody>
<tr>
<td>( \left( \frac{\partial u'}{\partial x} \right)^2 - \frac{\partial^2}{\partial x^2} \left( \overline{u'u'} \right) )</td>
<td>( \frac{\partial u'}{\partial y} \frac{\partial v'}{\partial x} - \frac{\partial^2}{\partial x \partial y} \left( \overline{u'v'} \right) )</td>
<td>( \frac{\partial u'}{\partial y} \frac{\partial v'}{\partial x} - \frac{\partial^2}{\partial x \partial y} \left( \overline{u'v'} \right) )</td>
<td>( \left( \frac{\partial v'}{\partial y} \right)^2 - \frac{\partial^2}{\partial y^2} \left( \overline{v'v'} \right) )</td>
</tr>
</tbody>
</table>

4.1.2 Boundary Conditions and Source Terms

As shown in the previous section, the pressure is sometimes split into mean shear and turbulence-turbulence parts by dividing the source term \( f(x, y) \) into two parts, one containing the mean velocity gradient and the other without it. The Poisson equation was derived in section 4.1.1. At this point, the primes that denote the fluctuating nature of the pressure are dropped for convenience.

\[ \nabla^2 p = \frac{\partial^2 p}{\partial x_i^2} = -2\rho \frac{\partial \overline{u_i' \overline{u_j'}}}{\partial x_j} - \rho \frac{\partial^2}{\partial x_i \partial x_j} \left( \overline{u_i \overline{u_j}} - \overline{u_i' u_j'} \right) \]  

(4.55)

The first set of terms on the right side is referred to as the mean shear (MS) interaction and is also known as the rapid term or the linear term. The second set of terms is the turbulence-turbulence (TT) interaction, also known as the slow or nonlinear term. Kim (1989) states that the terms 'rapid' and 'slow' terms refer to the fact that only the rapid part responds immediately to a change imposed on the mean field and the slow part will feel the change through the nonlinear
interactions. Sometimes the rapid and slow pressure are referred to as linear and nonlinear pressure, respectively, because the corresponding source terms are linear in the fluctuating quantities for the rapid part and nonlinear for the slow part. As shown in the previous section, this equation can be simplified where the new MS and TT terms are now

\[ \frac{\partial^2 p}{\partial x^2} + \frac{\partial^2 p}{\partial y^2} = -2\rho \left[ \frac{\partial \bar{u} \partial v'}{\partial y \partial x} \right] - \rho \left[ \left( \frac{\partial u'}{\partial x} \right)^2 + 2 \frac{\partial v'}{\partial x} \frac{\partial u'}{\partial y} + \left( \frac{\partial v'}{\partial y} \right)^2 - \frac{\partial^2 (u'u')}{\partial x^2} - 2 \frac{\partial^2 (u'v')}{\partial x \partial y} - \frac{\partial^2 (v'v')}{\partial y^2} \right] \]  

(4.56)

Following the suggestion of Kim (1989), the components of the Poisson equation can be split into two parts and a third part can be added which represents the Stokes pressure.

\[ \nabla^2 p_{\text{MS}} = -2\rho \frac{\partial \bar{u}}{\partial y} \frac{\partial v'}{\partial x} \]  
\[ \text{where } \frac{\partial p_{\text{MS}}}{\partial y} = 0 \]

\[ \nabla^2 p_{\text{TT}} = -\rho \sum_{i=1}^{2} \sum_{j=1}^{2} T_{ij} \]  
\[ \text{where } \frac{\partial p_{\text{TT}}}{\partial y} = 0 \]

\[ \nabla^2 p_{\text{ST}} = 0 \]  
\[ \text{where } \frac{\partial p_{\text{ST}}}{\partial y} = Re \frac{\partial^2 v'}{\partial y^2} \]  

(4.57)

The Stokes pressure is due solely to the inhomogeneous boundary condition and one can add the Stokes pressure either into the mean shear part or the turbulence-turbulence part. The Reynolds number is defined as \( Re = U \delta / \nu \). The above triple splitting avoids the arbitrary treatment of the boundary conditions. The next step is to decide how many of the three terms need to be carried into the analysis of the pressure fluctuation solution.

Kim (1989) found that much of the earlier work assumed that the contribution from the linear source term in the Poisson equation for pressure, representing the interaction of the turbulence with the mean shear, is much larger than that from the nonlinear source terms representing interactions of the turbulence with itself. Corcos (1964), however, indicated that the nonlinear source terms should be of the same order of importance as the linear source term, after
analyzing the data from Willmarth and Wooldridge (1962). In later DNS studies, Chang (1998) also concluded that both terms are of the same order of magnitude.

The results of the root mean square (rms) pressure fluctuations profiles from the equation (4.55) split into three parts across a turbulent channel flow using a DNS analysis by Kim (1989) are shown in Figure 4.1. Kim (1984) graphed these parts separately in order to examine the relative importance of each term. This chart shows that the Stokes part is small compared to the other two parts across the channel, including the near wall region where the Stokes part has its maximum. This chart also shows that both the rapid and slow terms are important and must be included in the analysis. Kim (1989) also noted that this result does not imply that the presence of the wall is not affecting the pressure fluctuations. However, it indicates that most of the influence from the presence of the wall is already built into the source terms through the no slip boundary conditions and the explicit influence through the boundary condition to the Poisson equation is not significant. Based on this result and the analysis of others, both the rapid and slow terms will be included in this work.

Figure 4.1: Root mean square fluctuations for the pressure profile across a turbulent channel for the three terms (Kim, 1984).
The form of the solution of the Poisson equation is dictated by the choice of boundary conditions. These conditions could be Dirichlet type boundary conditions where the dependent variable along the boundary is specified. They could also be Neumann type boundary conditions where the normal gradient of the dependent variable is specified. Or they could be a combination of both of these boundary layer types. (Hoffmann and Chiang, 1998) However, for uniqueness, at least at one point on the boundary must have a Dirichlet boundary condition. In this analysis, the boundary conditions will be of the mixed type.

The left and right sides of the computational domains will be of the Dirichlet boundary condition type. The pressure fluctuation profile at each side of the computation domain will be specified using data from a LES/DES analysis that was developed as part of a project titled Quiet Interiors Development as described in Hoffmann (2010) and is discussed in more detail in Section 4.2.1. The boundary condition at the lower wall will be a Neumann type. In this analysis, the Stokes pressure will be either defined by a specified value or assumed to be small, \( \partial p / \partial y = h(x) = 0 \). The boundary condition at the top of the computational domain is defined as the top of the boundary layer and the pressure fluctuations are assumed to be zero as shown in Figure 4.2. This assumption can be justified by noting that it was shown in Chapter 3 that the velocity fluctuations vanish outside the boundary layer. Hinze (1975) gave the following relationship for the pressure fluctuation profile inside a turbulent boundary layer.

\[
p_{rms} = \frac{\rho u'_{rms}^2}{\sqrt{2}}
\]  

(4.58)

If \( u' \) vanishes at the edge of the boundary layer, then it can also be assumed the pressure fluctuations due to the turbulence also vanish at the edge of the boundary layer. However, the acoustic noise does not vanish at the edge of the boundary layer, only the pressure fluctuations due to the turbulent boundary vanish. No new noise due to the turbulence is generated outside
the boundary layer. More recent experimental studies by Tsuji et al. (2007) also support this assumption.

4.1.3 Solution of the Poisson Equation

The Poisson equation can be solved either numerically or analytically. In this analysis, the Poisson equation will be solved by two different methods. The first is a numerical solution using a point Gauss-Seidel method. The second is an analytical solution using an eigenvalue expansion method.

4.1.3.1 Numerical Solution

The Poisson equation is an elliptic equation which can be approximated using a central difference formulation in two dimensions. This is given by

\[ \frac{p_{i+1,j} - 2p_{i,j} + p_{i-1,j}}{(\Delta x)^2} + \frac{p_{i,j+1} - 2p_{i,j} + p_{i,j-1}}{(\Delta y)^2} = f(x, y) \]  

(4.59)

This can be written as

\[ p_{i+1,j} - 2P_{i,j} + P_{i-1,j} + \left( \frac{\Delta x}{\Delta y} \right)^2 (p_{i,j+1} - 2p_{i,j} + p_{i,j-1}) = (\Delta x)^2 f(x, y) \]  

(4.60)
\[ p_{i+1,j} + p_{i-1,j} + \left( \frac{\Delta x}{\Delta y} \right)^2 p_{i,j+1} + \left( \frac{\Delta x}{\Delta y} \right)^2 p_{i,j-1} - 2 \left[ 1 + \left( \frac{\Delta x}{\Delta y} \right)^2 \right] p_{i,j} = (\Delta x)^2 f(x, y) \]  \hspace{1cm} (4.61)

In this section, the point Gauss-Seidel method as described in Hoffmann and Chiang (1998) is used to solve the Poisson equation. The finite difference equation is given by

\[ p_{i,j} = \frac{1}{2} \left[ \left( \frac{\Delta x}{\Delta y} \right)^2 \right] \left[ p_{i+1,j} + p_{i-1,j} + \left( \frac{\Delta x}{\Delta y} \right)^2 (p_{i,j+1} + p_{i,j-1}) - (\Delta x)^2 f(x, y) \right] \]  \hspace{1cm} (4.62)

In order to solve for the value of \( p \) at grid point \( i, j \), the values of \( p \) on the right side are either defined by the boundary conditions or provided by the previous iteration. Therefore, the values for \( p \) at the right and left boundary are specified by those generated by an LES/DES analysis. The pressure fluctuations are assumed to vanish at the top of the computational domain. The Stokes pressure at the wall is given by

\[ p_{i,j} = p_{i,j+1} - \Delta y h_i \]  \hspace{1cm} (4.63)

where \( h_i \) is either set to zero or defined by data from the LES/DES analysis.

**4.1.3.2 Analytical Solution**

Poisson's equation for pressure fluctuations is a linear partial differential equation and may have an infinite set of unrelated (linearly independent) solutions. Since Poisson's equation is non-homogeneous when \( f(x, y) \neq 0 \), it cannot be solved directly by the method of separation of variables. In this section, the pressure at the wall is determined using eigenvalue expansions. The pressure fluctuation problem is a combination of both homogeneous and non-homogeneous boundary conditions. Therefore, the solution is the sum of the Laplace equation with the non-zero boundary conditions plus the solution of the Poisson equation with the homogeneous boundary conditions. The final solution is the sum of the two components.
\[ p(x, y) = p_{\text{Poisson}}(x, y) + p_{\text{Laplace}}(x, y) \] (4.64)

For the complete solution of the Poisson equation, the problem is set up in two dimensions with defined boundary conditions on a rectangle. On the right and left sides, the pressure fluctuations are specified as a function of \( y \). At the top of the boundary layer, the pressure fluctuations are assumed to vanish. At the solid surface \( (y = 0) \), the normal gradient of the pressure fluctuation is assumed to be either specified or zero. The assumption of the Stokes pressure value at the wall determines the form of the eigenvalue expansion equations. It should be noted that the primes to denote the fluctuating value have been dropped. The Poisson problem is given by

\[ \nabla^2 p = \frac{\partial^2 p}{\partial x^2} + \frac{\partial^2 p}{\partial y^2} = f(x, y) \] (4.65)

with the following boundary conditions

\[
\begin{align*}
\frac{\partial p}{\partial y}(x, 0) &= h(x) & p(x, H) &= 0 & 0 < x < L \\
p(0, y) &= g_1(y) & p(L, y) &= g_2(y) & 0 < y < H
\end{align*}
\] (4.66) (4.67)

For the case where the Stokes pressure is set to zero, \( \frac{\partial p}{\partial y}(x, 0) = h(x) = 0 \), the complete solution is shown in Figure 4.3.

**Figure 4.3:** The two parts of the analytical Poisson equation problem with Stokes pressure set to zero.
The Poisson equation is solved with zero boundary conditions first. The differential equation is the same but the boundary conditions are now all homogeneous

$$\nabla^2 p_p = \frac{\partial^2 p_p}{\partial x^2} + \frac{\partial^2 p_p}{\partial y^2} = f(x, y)$$

(4.68)

with the following boundary conditions

$$\frac{\partial p_p}{\partial y}(x,0) = 0 \quad p_p(x,H) = 0 \quad 0 < x < L$$

(4.69)

$$p_p(0,y) = 0 \quad p_p(L,y) = 0 \quad 0 < y < H$$

(4.70)

A hint at the form can be found by examining the solutions of the Laplace equation in the $x$-direction and the $y$-direction with similar boundary conditions. In the $x$-direction, the solution is in the form of $\sin\left(\frac{m\pi}{L}x\right)$. In the $y$-direction, the solution is dominated by the zero gradient boundary condition at $y = 0$. The form of the $y$-direction solution is $\cos\left(\frac{n\pi - \pi/2}{H}y\right)$. The solution of Poisson's equation is then a sum of these solutions with a coefficient that incorporates $f(x, y)$.

$$p_p(x,y) = \sum_{n=1}^{\infty} \sum_{m=1}^{\infty} E_{mn} \sin\left(\frac{m\pi}{L}x\right) \cos\left(\frac{n\pi - \pi/2}{H}y\right)$$

(4.71)

where $E_{mn}$ is given by

$$E_{mn} = -\frac{4}{HL\lambda_{mn}} \oiint_{0}^{H} f(x, y) \sin\left(\frac{m\pi}{L}x\right) \cos\left(\frac{n\pi - \pi/2}{H}y\right) dxdy$$

(4.72)

and

$$\lambda_{mn} = \left[\left(\frac{m\pi}{L}\right)^2 + \left(\frac{n\pi - \pi/2}{H}\right)^2\right]$$

(4.73)
Now the Laplace equation is solved with nonzero boundary conditions. The differential equation is now homogeneous

\[ \nabla^2 p_L = \frac{\partial^2 p_L}{\partial x^2} + \frac{\partial^2 p_L}{\partial y^2} = 0 \]  

(4.74)

The non-homogeneous boundary conditions are on the right and left side of the computational domain so the boundary conditions are

\[ \frac{\partial p_L(x,0)}{\partial y} = h(x) = 0 \quad p_L(x,H) = 0 \quad 0 < x < L \]  

(4.75)

\[ p_L(0,y) = g_1(y) \quad p_L(L,y) = g_2(y) \quad 0 < y < H \]  

(4.76)

The Laplace equation problem is broken into two component solutions, one for each non-zero boundary condition. The first part addresses the left boundary condition and the second addresses the right boundary condition as shown in Figure 4.4.

![Figure 4.4: The two parts of the Laplace equation problem.](image)

The first part of the Laplace equation in two dimensions becomes

\[ \nabla^2 p_{L1} = \frac{\partial^2 p_{L1}}{\partial x^2} + \frac{\partial^2 p_{L1}}{\partial y^2} = 0 \]  

(4.77)

The boundary conditions become

\[ \frac{\partial p_{L1}}{\partial y} (x,0) = 0 \quad p_{L1}(x,H) = 0 \quad 0 < x < L \]  

(4.78)

\[ p_{L1}(0,y) = g_1(y) \quad p_{L1}(L,y) = 0 \quad 0 < y < H \]  

(4.79)
Following the concepts of the solution of the Poisson equation, the solution of the first Laplace equation is given by

$$p_{L1}(x, y) = \sum_{n=1}^{\infty} C_n \sinh \left( \frac{n\pi - \pi/2}{H} (L - x) \right) \cos \left( \frac{n\pi - \pi/2}{H} y \right)$$  \hspace{1cm} (4.80)

where the hyperbolic sine term allows for the non-homogeneous boundary condition and $C_n$ is given by

$$C_n = \frac{2}{H \sinh (\frac{n\pi - \pi/2}{H} L)} \int_{0}^{H} g_n(y) \cos \left( \frac{n\pi - \pi/2}{H} y \right) dy$$  \hspace{1cm} (4.81)

Following the same process to solve the second Laplace equation gives

$$\nabla^2 p_{L2} = \frac{\partial^2 p_{L2}}{\partial x^2} + \frac{\partial^2 p_{L2}}{\partial y^2} = 0$$  \hspace{1cm} (4.82)

The boundary conditions become

$$\frac{\partial p_{L2}}{\partial y} (x, 0) = 0 \quad p_{L2}(x, H) = 0 \quad 0 < x < L$$  \hspace{1cm} (4.83)
$$p_{L2}(0, y) = 0 \quad p_{L2}(L, y) = g_2(y) \quad 0 < y < H$$  \hspace{1cm} (4.84)

The solution is given by

$$p_{L2}(x, y) = \sum_{n=1}^{\infty} D_n \sinh \left( \frac{n\pi - \pi/2}{H} x \right) \cos \left( \frac{n\pi - \pi/2}{H} y \right)$$  \hspace{1cm} (4.85)

where the hyperbolic sine term allows for the non-homogeneous boundary condition and $D_n$ is given by

$$D_n = \frac{2}{H \sinh (\frac{n\pi - \pi/2}{H} L)} \int_{0}^{H} g_2(y) \cos \left( \frac{n\pi - \pi/2}{H} y \right) dy$$  \hspace{1cm} (4.86)

The Poisson equation was previously defined as

$$\nabla^2 p = \frac{\partial^2 p}{\partial x^2} + \frac{\partial^2 p}{\partial y^2} = f(x, y)$$  \hspace{1cm} (4.65)
with the following boundary conditions

\[
\frac{\partial p}{\partial y}(x,0) = 0 \quad p(x, H) = 0 \quad 0 < x < L \quad (4.66)
\]
\[
p(0, y) = g_1(y) \quad p(L, y) = g_2(y) \quad 0 < y < H \quad (4.67)
\]

The complete solution is given by

\[
p(x, y) = \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} E_{mn} \sin\left(\frac{m\pi}{L} x\right) \cos\left(\frac{n\pi - \pi/2}{H} y\right) + \sum_{m=1}^{\infty} C_n \sinh\left(\frac{n\pi - \pi/2}{H} (L-x)\right) \cos\left(\frac{n\pi - \pi/2}{H} y\right) + \sum_{m=1}^{\infty} D_n \sinh\left(\frac{n\pi - \pi/2}{H} x\right) \cos\left(\frac{n\pi - \pi/2}{H} y\right) \quad (4.87)
\]

where

\[
E_{mn} = -\frac{4}{HL \lambda_{mn}} \int_0^H \int_0^H f(x, y) \sin\left(\frac{m\pi}{L} x\right) \cos\left(\frac{n\pi - \pi/2}{H} y\right) dx dy \quad (4.88)
\]
\[
\lambda_{mn} = \left[\left(\frac{m\pi}{L}\right)^2 + \left(\frac{n\pi - \pi/2}{H}\right)^2\right] \quad (4.89)
\]
\[
C_n = \frac{2}{H \sinh\left(\frac{n\pi - \pi/2}{H} H\right)} \int_0^H g_1(y) \cos\left(\frac{n\pi - \pi/2}{H} y\right) dy \quad (4.90)
\]
\[
D_n = \frac{2}{H \sinh\left(\frac{n\pi - \pi/2}{H} H\right)} \int_0^H g_2(y) \cos\left(\frac{n\pi - \pi/2}{H} y\right) dy \quad (4.91)
\]

For the case where the Stokes pressure has a specified value, \( \frac{\partial p}{\partial y}(x,0) = h(x) \), The complete solution is shown in Figure 4.5.
Figure 4.5: The two parts of the analytical Poisson equation problem with Stokes pressure set to a specified value.

Again, the Poisson equation will again be solved with zero boundary conditions first with the same differential equation with homogeneous boundary conditions. The differential equation is the same but the boundary conditions are now all homogeneous

\[ \nabla^2 p_p = \frac{\partial^2 p_p}{\partial x^2} + \frac{\partial^2 p_p}{\partial y^2} = f(x, y) \quad (4.92) \]

with the following boundary conditions

\[
\begin{align*}
    p_p(x,0) &= 0 & p_p(x,H) &= 0 & 0 < x < L \\
p_p(0,y) &= 0 & p_p(L,y) &= 0 & 0 < y < H
\end{align*}
\quad (4.93) \quad (4.94)
\]

Again, the form can be found by examining the solutions of the Laplace equation in the \(x\)-direction and the \(y\)-direction with similar boundary conditions. In the \(x\)-direction, the solution is in the form of \( \sin\left(\frac{m\pi}{L} x\right) \). In the \(y\)-direction, the solution is no longer dominated by the zero gradient boundary condition at \( y = 0 \) like the case where the Stokes pressure is set to zero.

Here, the form of the \(y\)-direction solution is \( \sin\left(\frac{n\pi}{H} y\right) \). The solution of Poisson's equation is then a sum of these solutions with a coefficient that incorporates \( f(x, y) \).

\[
p_p(x, y) = \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} E_{mn} \sin\left(\frac{m\pi}{L} x\right) \sin\left(\frac{n\pi}{H} y\right) \quad (4.95)
\]
where $E_{mn}$ is given by

$$E_{mn} = \frac{-4}{HL \lambda_{mn}} \iint_{0}^{H} f(x, y) \sin\left(\frac{m\pi}{L} x\right) \sin\left(\frac{n\pi}{H} y\right) dx dy$$

and

$$\lambda_{mn} = \left[\left(\frac{m\pi}{L}\right)^2 + \left(\frac{n\pi}{H}\right)^2\right]$$

Now the Laplace equation is solved with nonzero boundary conditions. The differential equation is now homogeneous

$$\nabla^2 p_L = \frac{\partial^2 p_L}{\partial x^2} + \frac{\partial^2 p_L}{\partial y^2} = 0$$

The Laplace equation problem must be broken into three component solutions, one for each non-zero boundary condition. The non-homogeneous boundary conditions are on the right side, left side and at the wall of the computational domain so the boundary conditions are

$$\frac{\partial p_L}{\partial y}(x, 0) = h(x) \quad p_L(x, H) = 0 \quad 0 < x < L$$

$$p_L(0, y) = g_1(y) \quad p_L(L, y) = g_2(y) \quad 0 < y < H$$

The first part of the Laplace equation in two dimensions becomes

$$\nabla^2 p_{ll1} = \frac{\partial^2 p_{ll1}}{\partial x^2} + \frac{\partial^2 p_{ll1}}{\partial y^2} = 0$$

The boundary conditions become

$$\frac{\partial p_{ll1}}{\partial y}(x, 0) = 0 \quad p_{ll1}(x, H) = 0 \quad 0 < x < L$$

$$p_{ll1}(0, y) = g_1(y) \quad p_{ll1}(L, y) = 0 \quad 0 < y < H$$

The solution of the first Laplace equation is given by
\[ p_{L1}(x, y) = \sum_{n=1}^{\infty} C_n \sinh \left( \frac{n\pi - \pi/2}{H} x \right) \cos \left( \frac{n\pi - \pi/2}{H} y \right) \]  
\[ (4.104) \]

where the hyperbolic sine term allows for the non-homogeneous boundary condition and \( C_n \) is given by

\[ C_n = \frac{2}{H \sinh(\frac{n\pi - \pi/2}{H} L)} \int_0^H g_1(y) \cos \left( \frac{n\pi - \pi/2}{H} y \right) dy \]  
\[ (4.105) \]

Following the same process to solve the second Laplace equation gives

\[ \nabla^2 p_{L2} = \frac{\partial^2 p_{L2}}{\partial x^2} + \frac{\partial^2 p_{L2}}{\partial y^2} = 0 \]  
\[ (4.106) \]

The boundary conditions become

\[ \frac{\partial p_{L2}}{\partial y}(x, 0) = 0 \quad p_{L2}(x, H) = 0 \quad 0 < x < L \]  
\[ p_{L2}(0, y) = 0 \quad p_{L2}(L, y) = g_2(y) \quad 0 < y < H \]  
\[ (4.107) \]
\[ (4.108) \]

The solution is given by

\[ p_{L2}(x, y) = \sum_{n=1}^{\infty} D_n \sinh \left( \frac{n\pi - \pi/2}{H} x \right) \cos \left( \frac{n\pi - \pi/2}{H} y \right) \]  
\[ (4.109) \]

where the hyperbolic sine term allows for the non-homogeneous boundary condition and \( D_n \) is given by

\[ D_n = \frac{2}{H \sinh(\frac{n\pi - \pi/2}{H} L)} \int_0^H g_2(y) \cos \left( \frac{n\pi - \pi/2}{H} y \right) dy \]  
\[ (4.110) \]

The third Laplace equation gives the solution due to the Stokes pressure at the wall

\[ \nabla^2 p_{L3} = \frac{\partial^2 p_{L3}}{\partial x^2} + \frac{\partial^2 p_{L3}}{\partial y^2} = 0 \]  
\[ (4.111) \]

The boundary conditions become
\[ \frac{\partial p_{L3}}{\partial y} (x,0) = h(x) \quad p_{L3} (x, H) = 0 \quad 0 < x < L \] (4.112)
\[ p_{L3} (0, y) = 0 \quad p_{L3} (L, y) = 0 \quad 0 < y < H \] (4.113)

The solution is given by
\[ p_{L3} (x, y) = \sum_{m=1}^{\infty} B_m \sinh \left( \frac{m\pi}{L} (H - y) \right) \sin \left( \frac{m\pi}{L} x \right) \] (4.114)

where \( B_m \) is given by
\[ B_m = \frac{-2}{mH \cosh \left( \frac{m\pi}{L} H \right)} \left[ h(x) \sin \left( \frac{m\pi}{L} x \right) \right]_0^L dx \] (4.115)

The Poisson equation was previously defined as
\[ \nabla^2 p = \frac{\partial^2 p}{\partial x^2} + \frac{\partial^2 p}{\partial y^2} = f(x, y) \] (4.116)

with the following boundary conditions
\[ \frac{\partial p}{\partial y} (x,0) = h(x) \quad p(x, H) = 0 \quad 0 < x < L \] (4.117)
\[ p(0, y) = g_1 (y) \quad p(L, y) = g_2 (y) \quad 0 < y < H \] (4.118)

The complete solution is given by
\[ p(x, y) = \sum_{n=1}^{\infty} \sum_{m=1}^{\infty} E_{mn} \sin \left( \frac{m\pi}{L} x \right) \sin \left( \frac{n\pi}{H} y \right) \]
\[ + \sum_{m=1}^{\infty} C_n \sinh \left( \frac{n\pi - \pi/2}{H} (L - x) \right) \cos \left( \frac{n\pi - \pi/2}{H} y \right) \]
\[ + \sum_{m=1}^{\infty} D_n \sinh \left( \frac{n\pi - \pi/2}{H} x \right) \cos \left( \frac{n\pi - \pi/2}{H} y \right) \]
\[ + \sum_{m=1}^{\infty} B_m \sinh \left( \frac{m\pi}{L} (H - y) \right) \sin \left( \frac{m\pi}{L} x \right) \] (4.119)
\[ E_{mn} = -\frac{4}{HL2_{mn}} \int_0^H \int_0^L f(x, y) \sin \left( \frac{m\pi}{L} x \right) \sin \left( \frac{n\pi}{H} y \right) dx dy \] (4.120)
\[ \lambda_{mn} = \left[ \left( \frac{m\pi}{L} \right)^2 + \left( \frac{n\pi}{H} \right)^2 \right] \]  
\[ (4.121) \]

\[ C_n = \frac{2}{H \sinh\left( \frac{n\pi - \pi/2}{H} \right)} \int_0^H g_1(y) \cos\left( \frac{n\pi y}{H} \right) dy \]  
\[ (4.122) \]

\[ D_n = \frac{2}{H \sinh\left( \frac{n\pi - \pi/2}{H} \right)} \int_0^H g_2(y) \cos\left( \frac{n\pi y}{H} \right) dy \]  
\[ (4.123) \]

\[ B_m = \frac{-2}{mH \cosh\left( \frac{m\pi}{L} \right)} \int_0^L h(x) \sin\left( \frac{m\pi x}{L} \right) dx \]  
\[ (4.124) \]

### 4.2 Estimating the Mean Square Pressure

The mean square pressure can be calculated either numerically, analytically or estimated using an empirical model. For the numerical solution, a point Gauss-Seidel model is used; and for the analytical solution, an eigenvalue expansion method is used. The forcing function and boundary condition values are calculated using data from a LES/DES model of Hoffmann (2010) at \( M = 0.30 \) under conditions that simulate the Spirit AeroSystems 6x6 duct. These methods are instantaneous solutions with no time dependent behavior. Since no frequency dependent information is needed for the mean square pressure, the numerical and analytical solutions are valid assessments. Next, the mean square pressure is estimated using a variety of empirically derived mean square pressure models which have been developed over the past 50 years.

#### 4.2.1 Description of LES/DES Data

A numerical model using Large Eddy Simulation/Detached Eddy Simulation (LES/DES) has been developed at Wichita State University by Hoffmann (2010) as part of the Quiet Interiors Development project. The model is based on the first two meters of the Spirit 6x6 duct described in Section 4.3.1.1. This analysis uses the conditions recorded during the preliminary testing. These values are listed in Table 4.2. As shown in Section 4.3.1.1, the test section in the
Spirit duct where data can be taken is at 3 meters. Therefore, no Spirit experimental data is available to verify the LES/DES results.

**TABLE 4.2**

**FLOW PARAMETERS FOR THE LES/DES MODEL**

<table>
<thead>
<tr>
<th>Flow Parameters</th>
<th>Values</th>
</tr>
</thead>
<tbody>
<tr>
<td>Mach number</td>
<td>0.30</td>
</tr>
<tr>
<td>temperature, (K)</td>
<td>288.71</td>
</tr>
<tr>
<td>density, (kg/m$^3$)</td>
<td>1.1957</td>
</tr>
<tr>
<td>pressure, (Pa)</td>
<td>94461</td>
</tr>
<tr>
<td>kinematic viscosity, (m$^2$/s)</td>
<td>$1.4826 \times 10^{-5}$</td>
</tr>
</tbody>
</table>

The LES/DES model data was available between 0.8 and 2.0 meters at the centerline of the duct. In the $x$-direction, data was saved at the following locations: 0.8, 1.0, 1.2, 1.4, 1.6, 1.8, and 2.0 meters. In the $y$-direction, data was saved at the following locations: 0.0000, 0.0004, 0.0010, 0.0026, 0.0053, 0.0070, and 0.0093 meters. These grid locations are shown graphically in Figure 4.6. At each location, the velocity fluctuations in all directions, as well as, the pressure fluctuations were available. These variables are required to solve both the numerical and analytical solutions of the Poisson equation. This data is not available from the Spirit experimental data. It should also be noted that the grid point density used for the LES/DES model was much greater than what was saved for data analysis purposes.
Figure 4.6: WSU LES/DES grid point data locations.

Data was recorded at times 0.00740 through 0.02000 seconds at 0.00001 second intervals. Values for the pressure fluctuations and velocity fluctuations in the $x$, $y$ and $z$-directions were collected at each time step. To characterize the data, all the time step data was used in the analysis. For the numerical and analytical formulations discussed in Section 4.2.3 and 4.2.4, the analysis was performed using times 0.00750 through 0.02000 seconds at 0.00050 second intervals for 26 separate datasets. Figures 4.7, 4.8 and 4.9 show the instantaneous velocity fluctuations in the $x$-direction, $y$-direction and $z$-direction for a series of time steps. The velocity fluctuations are clearly correlated and the values for $u'$, $v'$, and $w'$ are coincident.
Figure 4.7: LES/DES instantaneous velocity fluctuation profiles in $x$-direction.

Figure 4.8: LES/DES instantaneous velocity fluctuation profiles in $y$-direction.
Figure 4.9: LES/DES instantaneous velocity fluctuation profiles in z-direction.

The RMS values of the LES/DES velocity fluctuation data is compared to the curves from Klebanoff (1955) in Figures 4.10 a, b, and c. Although the curve shapes approximate Klebanoff’s, the magnitudes are much lower especially in the y and z-direction.

Figure 4.10 a: RMS LES/DES velocity fluctuation profiles in x-direction.
The mean velocity profile for a flat plate was shown by Klebanoff (1955) to follow the $1/7$th rule given in Chapter 3 as

$$\bar{u} = U_\infty \left( \frac{y}{\delta} \right)^{1/7} \quad (4.125)$$

The mean velocity profiles of the LES/DES data are compared to the theoretical in Figure 4.11. Since the LES/DES analysis models the 6x6 duct, it is not surprising that the velocity profiles
exhibit more of duct flow profile. Figure 4.11 shows that the velocity profiles approximate the flat plate turbulent boundary layer $1/7^{th}$ rule.

![Figure 4.11: LES/DES mean velocity profiles.](image)

The instantaneous wall pressure fluctuations from the LES/DES model are shown in Figure 4.12. The RMS wall pressure fluctuations are given in Figure 4.13. These values will be compared with the results of the numerical and analytical analysis given in Sections 4.2.3 and 4.2.4.
Figure 4.12: LES/DES instantaneous wall pressure fluctuation profiles.

Figure 4.13: RMS LES/DES wall pressure fluctuations.

4.2.2 Estimate of 3D to 2D Error

The analytical and numerical models are based on two dimensional analyses but the actual flow and LES/DES data is three dimensional. It is difficult to estimate the error that is induced by simplifying the problem to two dimensions. One way to bound the estimate is to
calculate the coefficient of pressure for both a circular cylinder and a sphere. According to Anderson (1991), the pressure over a sphere is relieved in comparison with the flow over a cylinder because the flow over the former has an extra dimension in which to move out of the way of the solid body. The flow over a sphere can move sideways as well as up and down because of the extra dimension. The flow over a cylinder is restricted to motion that is either up or down only. The surface coefficient of pressure over a circular cylinder is given by

\[
C_p = 1 - 4\sin^2 \theta 
\]  

(4.126)

The pressure distribution on the surface of a sphere is given by

\[
C_p = 1 - \frac{9}{4} \sin^2 \theta 
\]  

(4.127)

Therefore, the pressure over a sphere will be lower than over a cylinder for the same flow conditions as shown in Figure 4.14. Using the LES/DES flow conditions at Mach 0.3 to calculate the pressure at 90 degrees, the difference is less than 12%.

Figure 4.14: Pressure distribution over the surface of sphere and a cylinder.
4.2.3 Results of the Numerical Model

A numerical solution of the Poisson equation has been developed using a two-dimensional Point Gauss-Seidel method (PGS) as described in Section 4.1.3.1. The LES/DES data detailed in the previous section is used to define the boundary conditions. The pressure fluctuations vanish at the top of the boundary layer which defines the top of the computational domain. The pressure fluctuation profiles at the left and right of the computational domain are defined by the LES/DES data. The gradient of the pressure fluctuations (Stokes pressure) at the wall are defined to be either zero or a specified value which is calculated from the LES/DES data. The forcing function \( f(x, y) \) is calculated using the velocity fluctuation data from the LES/DES data. Figure 4.15 shows the PGS method wall pressure profiles with Stokes pressure set to zero and Figure 4.16 shows the results with the Stokes pressure specified.

Figure 4.15: PGS instantaneous wall pressure fluctuation profiles with Stokes pressure set to zero.
Figure 4.16: PGS instantaneous wall pressure fluctuation profiles with Stokes pressure specified.

The wall pressure fluctuations found from the LES/DES model are used to validate the two dimensional PGS solution method. Figure 4.17 shows the RMS wall pressure fluctuation values for both PGS solutions; Stokes pressure set to zero, and Stokes pressure specified from LES/DES data. Setting the Stokes pressure to zero results in wall pressure fluctuation values that are considerably below the predicted values. Using the Stokes pressure values derived from the LES/DES data results in reasonable RMS wall pressure fluctuation values. This is not in agreement with the previous work of Kim (1984) which predicted that the Stokes pressure contribution would be small.
4.2.4 Results of the Analytical Model

A test of the convergence of the eigenvalue expansion algorithm for the Poisson equation with homogeneous boundary condition was performed. The forcing function $f(x, y)$ was set to 1.0 over the computational domain and numerically tested at $x = L/2$ and $y = H/2$. Setting the number of iterations to $m = 120$ and $n = 7$ resulted in an error of 0.1% as shown in Figure 4.18.
A test of the convergence of the eigenvalue expansion algorithm for the Laplace equation with a non-homogeneous boundary condition was also performed. The boundary condition $g(y)$ was set to 1.0 over the entire boundary length and numerically tested at $x = 0$ and $y = H/2$. The number of iterations was limited to 3 due to a hyperbolic sine term. As shown in Figure 4.19, the resulting error was 2.0%.

![Figure 4.19: Convergence of the Laplace equation with a non-homogeneous boundary condition.](image)

Finally, a test of the convergence of the eigenvalue expansion algorithm for the Laplace equation with the Stokes pressure specified at the wall was carried out. The Stokes pressure $h(x)$ was set to 1.0 over the entire length of the wall and numerically tested at $x = L/2$ and $y = 0$. Limiting the number of iterations to $m = 120$ resulted in an error of 0.5% as shown in Figure 4.20. From these convergence tests, it appeared that the EE methodology should be sufficient for the solution of the Poisson equation problem. But it should be noted that the forcing function, $f(x, y)$, and the boundary condition functions, $g(y)$ and $h(x)$, in these examples were set to a uniform, well behaved values which is not what the LES/DES data looks like.
Figure 4.20: Convergence of the Laplace equation with Stokes pressure specified at the wall.

For the EE solution method, the LES/DES data detailed in the previous section was used to define the boundary conditions. The pressure fluctuations vanished at the top of the boundary layer which was also the top of the computational domain. The pressure fluctuation profiles at the left and right of the computational domain were derived from the LES/DES data. The gradient of the pressure fluctuations (Stokes pressure) at the wall was set to be either zero or a specified value which was calculated from the LES/DES data. The forcing function $f(x, y)$ was calculated using the velocity fluctuation data from the LES/DES data. Figure 4.21 shows the EE method wall pressure profiles with Stokes pressure set to zero and Figure 4.22 shows the results with the Stokes pressure specified.
Figure 4.21: EE instantaneous wall pressure fluctuation profiles with Stokes pressure set to zero.

Figure 4.22: PGS instantaneous wall pressure fluctuation profiles with Stokes pressure specified.
The wall pressure fluctuations found from the LES/DES model were again used to validate the two dimensional EE solution method. Figure 4.23 shows the RMS wall pressure fluctuation values for both EE solutions. Setting the Stokes pressure to zero resulted in wall pressure fluctuation values that were higher than the LES/DES prediction. Using the Stokes pressure values derived from the LES/DES data resulted in wall pressure fluctuation values much lower than the LES/DES prediction. This result was the opposite from that found from the point Gauss-Seidel numerical method where adding the Stokes pressure resulted in higher wall pressure fluctuation values similar to the predicted LES/DES results. This indicates that the EE solution method may not be an appropriate technique to solve the Poisson equation in this case.

![Figure 4.23: RMS Eigenvalue expansion wall pressure fluctuations.](image)

**Figure 4.23: RMS Eigenvalue expansion wall pressure fluctuations.**

### 4.2.5 Empirical Models

One of the earliest empirical estimates of the mean square wall pressure fluctuation is given by Kraichnan (1956) for low to moderate Mach numbers. His estimate is a function of wall shear stress $\tau_w$ and is given by

$$\overline{p^2} = (6\tau_w)^2$$  \hspace{1cm} (4.128)
He based this estimate on the assumption that the turbulence-mean shear contribution is dominant in a turbulent boundary layer. A few years later, Lilley and Hodgson (1960) made an estimate based on their own measurements in a wind tunnel as a function of dynamic pressure, \( q \). They found

\[ \bar{p}^{\prime 2} = (0.008 \, q)^2 \]  

(4.129)

Based on measurements made in an experiment described in Section 2.1, Willmarth and Wooldridge (1962) measured the mean square wall pressure at 150 ft/sec and 200 ft/sec over a frequency band of \( 0.14 < \omega \delta^* / U_\infty < 28 \). After correcting for extraneous signals from the transducer, the mean square wall pressure was given by

\[ \bar{p}^{\prime 2} = (2.15 \, \tau_w)^2 \] for 150 ft/sec

(4.130)

\[ \bar{p}^{\prime 2} = (2.19 \, \tau_w)^2 \] for 200 ft/sec

(4.131)

Corcos (1964) reviewed the values of mean square wall pressure found by previous investigators. He found that in general the resolution errors differed from one experiment to another and agreement after the resolution errors had been accounted for was only mediocre. He suggested that a relationship for the mean square wall pressure as a function of wall shear stress for Reynolds numbers of \( U_\infty \delta / \nu \approx 300,000 \) and is given by

\[ \bar{p}^{\prime 2} = ((3.0 \pm 0.5)\tau_w)^2 \]  

(4.132)

Corcos believed that the dependence on Reynolds number appeared to be small.

Bull (1967) found a different result in his investigation. He found that the mean square wall pressure value increased with Reynolds number

\[ \bar{p}^{\prime 2} = (2.11 \tau_w)^2 \] for \( Re_\theta = 6400 \) \( (M = 0.3) \)  

(4.133)
The factor changes at a rate of about the 0.17 power of Reynolds number. For the mean square wall pressure measurements, Bull noted that the bandwidth was determined by the response characteristics of the microphone system. For his experimental setup with pinhole microphones, he used 80 to 100,000 Hz.

Lowson (1968) investigated the basic mechanism underlying the production of surface pressure fluctuations beneath turbulent boundary layers for Mach numbers up to about 3. He found that a meaningful estimate for the mean square wall pressure as a function of Mach number can be found using the dynamic pressure as a normalizing parameter. His empirical relationship is given by

\[ \overline{p'^2} = (2.80 \tau_w)^2 \quad \text{for} \quad Re_0 = 33800 \quad (M = 0.5) \] (4.134)

Blake (1970) noted that previous studies of pressure fluctuations on walls beneath turbulent boundary layers were limited by the finite size of the microphones and by extraneous tunnel disturbances. In this study, Blake made use of a low-noise wind tunnel facility and microphones capable of better resolution. He developed two relationships: one based on the dynamic pressure and the second based on the wall shear stress. He noted that the constants in these relations were considerably higher than those reported by other investigators at the time. He stated that these higher values were consistent with the improved microphone resolution used in his experiment.

\[ \overline{p'^2} = (0.006 q)^2 \] (4.135)

\[ \overline{p'^2} = (3.59 \tau_w)^2 \] (4.136)

\[ \overline{p'^2} = (59.3 \tau_w)^2 \] (4.137)
Schewe (1983) conducted more modern wind tunnel measurements using various sizes of transducers at 6.3 m/sec. He found a corrected mean square wall pressure value of

$$\overline{p'^2} = (0.0102 q)^2$$  \hspace{1cm} (4.138)

which was found by extrapolation to zero transducer size.

Lauchle and Daniels (1987) had a different experimental approach. They studied wall pressure fluctuations beneath turbulent boundary layers using fully developed turbulent flow of glycerin in a long pipe. They found a mean square wall pressure value of

$$\overline{p'^2} = (0.0106 q)^2$$  \hspace{1cm} (4.139)

which is very similar to Schewe's results.

Farabee and Casarella (1991) also noted the difficulty that arises in comparing wall pressure measurements from different investigators because one must account for both the effects of transducer sizes and the variations of the wall pressure fluctuations with Reynolds number. They obtained the mean wall pressure fluctuation values by numerically integrating the spectra from 50 to 20000 Hz. They picked the lower frequency limit because the facility-related noise dominates below 50 Hz. The high frequency integration limit was set by the upper frequency limit of response of the pinhole microphone system used to measure the pressure fluctuations. They obtained a relationship dependent on the Reynolds number, for $Re_\tau = U_\tau \delta / \nu$

$$\overline{p'^2} = 6.5 \tau_w^2 \hspace{1cm} \text{for} \hspace{0.2cm} Re_\tau \leq 333$$  \hspace{1cm} (4.140)

$$\overline{p'^2} = [6.5 + 1.86 \ln(R_\tau / 333)]\tau_w^2 \hspace{0.2cm} \text{for} \hspace{0.2cm} Re_\tau > 333$$  \hspace{1cm} (4.141)

Lueptow (1995) investigated measurement effects and the turbulent wall pressure spectrum. He chose to develop the mean square wall pressure fluctuation value as a function of the dynamic pressure. At the limit of the microphone size, he found
Although Lueptow recognized that Farabee and Casarella (1991) showed that the mean square wall pressure was dependent on Reynolds number, he felt that non-dimensionalizing it with $q$ seemed to reduce the sensitivity to Reynolds number. There is not agreement in the community on this. Bull (1996) stated that similarity relations for the frequency spectrum indicate that the appropriate pressure scale in all three ranges which make the dominant contribution to the mean square pressure the wall shear stress, $\tau_w$. This was the parameter that was most used in early investigations rather than dynamic pressure, $q$, which he noted has been favored in more recent work.

### 4.2.6 Comparison of Mean Square Pressure Models

The required parameters for the empirical models are the dynamic pressure or the wall shear stress. Shear stress has units of force per unit area. The shear stress acts tangentially to the body surface and is caused by friction between the body and the air which occurs in viscous flow. The influence of friction is to create $u = 0$ at the wall. This is called the no-slip condition which dominates viscous flow. Clearly, the region of flow near the surface has velocity gradients, $\partial u / \partial y$, which are due to the frictional force between the surface and the wall. The shear stress is different depending on whether the flow is laminar or turbulent, and incompressible or compressible. For turbulent flow, the velocity is higher immediately above the surface; therefore,

$$\frac{\partial u}{\partial y}_{\text{turbulent}} > \frac{\partial u}{\partial y}_{\text{laminar}}$$

(4.143)

The basic definition of wall shear stress is given by
where $\mu$ is the viscosity of the fluid. Another important parameter is the coefficient of friction which is defined as

$$c_f = \frac{\tau_w}{q}$$

(4.145)

where $q$ is defined as the dynamic pressure $q = \frac{1}{2} \rho U_w^2$.

In this work, the flow over a flat plate is of interest. However, the experimental data from the Spirit 6x6 duct is really a channel flow. It is important to ensure that the boundary layer data from the Spirit 6x6 is behaves like a turbulent flat plate boundary layer. Anderson (1991) states that the theoretical solutions for the wall shear are very different for fully developed channel flow than for a flat plate boundary layer. Fully developed channel flow is driven by a pressure gradient as opposed to than in a flat plate boundary layer. For fully developed flow, the pressure is constant along the flow direction and the velocity varies parabolically across the flow. The wall shear stress is given by

$$\tau_w = -\frac{D}{2} \mu \left( \frac{dp}{dx} \right)$$

(4.146)

Anderson (1989) calculates the skin friction coefficient for compressible laminar flow over a flat plate by using the reference temperature method.

$$c_f = \frac{0.664}{\sqrt{Re_x^*}}$$

(4.147)

where $Re_x^*$ is evaluated at a reference temperature $T^*$. That is,

$$Re_x^* = \frac{\rho^* u_w x}{\mu^*}$$
where $\rho^*$ and $\mu^*$ are evaluated for the reference temperature $T^*$.

\[
\frac{T^*}{T_e} = 1 + 0.032 M_e^2 + 0.58 \left( \frac{T_w}{T_e} - 1 \right) \tag{4.148}
\]

The ratio of the wall temperature to the flow temperature was estimated to be approximately $T_w/T_e = 1.0$ in this case. Therefore,

\[
T^* = T_e (1 + 0.032 M_e^2) \tag{4.149}
\]

Figure 4.24 shows the influence of Mach number on the reference temperature. The flow temperature is assumed to be $T = 288.7$ K.

![Mach Number vs. Reference Temperature](image)

**Figure 4.24: Influence of Mach number on reference temperature.**

Anderson gives the following equation for an incompressible, turbulent flow over a flat plate

\[
c_f = \frac{0.0592}{(Re_\tau)^{0.2}} \tag{4.150}
\]

Carrying over the reference temperature concept to the turbulent case, the compressible turbulent flat plate skin friction coefficient can be approximated as

\[
c_f = \frac{0.0592}{(Re_\tau^*)^{0.2}} \tag{4.151}
\]
where $Re_1^*$ is evaluated at a reference temperature $T^*$ as defined in the previous paragraph. For the Mach numbers used in this work, the reference temperature effect is very small which agrees with the early work of Van Driest (1952). Figure 4.25 illustrates the influence of compressibility on this definition of the skin friction coefficient using the Spirit 6x6 duct conditions at 1.4 meters as a reference point.

![Graph](image)

**Figure 4.25:** Influence of compressibility on skin friction coefficient using Anderson’s equation.

Using the momentum integral analysis technique, White (1991) estimates the flat plate skin friction coefficient for incompressible turbulent flow by a power-law curve-fit approximation

$$c_f = \frac{0.020}{(Re_\delta)^{1/6}}$$  \hspace{1cm} (4.152)

White suggests that the flat plate skin friction coefficient for incompressible turbulent flow can be approximated more easily by
He also states that the flat plate skin friction coefficient for incompressible turbulent flow can also be approximated by

\[ c_f = \frac{0.027}{(Re_c)^{1/7}} \]  

(4.153)

which is similar to Anderson (1989) given in equation (4.151). White notes that his information was developed from very limited low-Reynolds number data and may not be very accurate. He then states that a more exact formula can be found using a method developed by Kestin and Persen (1962) which is given by

\[ c_f = \frac{0.455}{\ln^2 (0.06 Re_c)} \]  

(4.155)

Figure 4.26 compares the incompressible, turbulent skin friction coefficients from Anderson equation (4.150) and White equations (4.153) and (4.155) at the Spirit 6x6 reference conditions at 1.4 meters.

**Figure 4.26: Comparison of skin friction coefficients.**
Following the reference temperature method found in Anderson (1989), an estimate of the compressibility effect can be made using White’s (1991) exact skin friction coefficient equation given in equation (4.155).

\[ c_f = \frac{0.455}{\ln^2(0.06 Re_x^*)} \]  

(4.156)

Figure 4.27 shows the compressibility effect using White’s ‘exact’ equation. As seen in Anderson’s work, the effect is small. Therefore, the wall shear stress equation used in this work is

\[ \tau_w = \frac{0.455 q}{\ln^2(0.06 Re_x^*)} \]  

(4.157)

Figure 4.27: Influence of compressibility on skin friction coefficient using White’s ‘exact’ equation.

The required parameters for the empirical models are found in Table 4.3. Table 4.4 shows the results at \( M = 0.30 \) and 1.4 meters. It is interesting to note that the earliest models are at the limits of all the empirical models. Kraichnan (1956) is at the high end of the predictions. Lowson (1968), Bull (1967) and Willmarth and Wooldridge (1962) are at the low end. Of the two newest models, Lueptow (1995) and Farabee and Casarella (1991) are both near the middle.
of the range of values at all distances down the plate. Both the LES/DES and the PGS solution with the Stokes pressure specified give reasonable results although near the bottom of the range of values. The EE solution with the Stokes pressure specified is much higher than any of the empirical model estimates. Figure 4.28 shows these results in addition to the other locations along the wall.

**TABLE 4.3**

*6X6 TEST CONDITIONS AT M = 0.30*

<table>
<thead>
<tr>
<th></th>
<th>x = 0.8 m</th>
<th>x = 1.0 m</th>
<th>x = 1.2 m</th>
<th>x = 1.4 m</th>
<th>x = 1.6 m</th>
<th>x = 1.8 m</th>
<th>x = 2.0 m</th>
</tr>
</thead>
<tbody>
<tr>
<td><strong>dynamic pressure, q (N/m²)</strong></td>
<td>6239.38</td>
<td>6239.38</td>
<td>6239.38</td>
<td>6239.38</td>
<td>6239.38</td>
<td>6239.38</td>
<td>6239.38</td>
</tr>
<tr>
<td><strong>shear stress at wall, (N/m²)</strong></td>
<td>17.6066</td>
<td>17.0038</td>
<td>16.5339</td>
<td>16.1517</td>
<td>15.8312</td>
<td>15.5563</td>
<td>15.3165</td>
</tr>
</tbody>
</table>
## TABLE 4.4

### MEAN SQUARE WALL PRESSURE VALUES AT $M = 0.30$ AND 1.4 METERS

<table>
<thead>
<tr>
<th>Author, date</th>
<th>Mean Square Pressure ($\text{Pa}^2$)</th>
</tr>
</thead>
<tbody>
<tr>
<td>Kraichnan, 1956</td>
<td>9391.57</td>
</tr>
<tr>
<td>Lilley and Hodgson, 1960</td>
<td>2491.51</td>
</tr>
<tr>
<td>Willmarth and Wooldridge, 1962</td>
<td>1251.19</td>
</tr>
<tr>
<td>Corcos, 1964</td>
<td>2347.89</td>
</tr>
<tr>
<td>Bull, 1967</td>
<td>1251.19</td>
</tr>
<tr>
<td>Lowson, 1968</td>
<td>1397.95</td>
</tr>
<tr>
<td>Blake, 1970 (function of $\tau_w$)</td>
<td>3362.21</td>
</tr>
<tr>
<td>Blake, 1970 (function of $q$)</td>
<td>2987.39</td>
</tr>
<tr>
<td>Schewe, 1983</td>
<td>4050.27</td>
</tr>
<tr>
<td>Lauchle and Daniels, 1987</td>
<td>4374.16</td>
</tr>
<tr>
<td>Farabee and Casarella, 1991</td>
<td>3059.01</td>
</tr>
<tr>
<td>Lueptow, 1995</td>
<td>5605.90</td>
</tr>
<tr>
<td>WSU LES/DES, 2010</td>
<td>1575.09</td>
</tr>
<tr>
<td>Point Gauss-Seidel with Stokes Pressure</td>
<td>1650.03</td>
</tr>
<tr>
<td>Eigenvalue Expansion without Stokes Pressure</td>
<td>63881.09</td>
</tr>
</tbody>
</table>

**Figure 4.28:** Mean square wall pressure values along the plate at $M = 0.30$. 

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4.2.7 Summary of Mean Square Pressure Models

A numerical model using Large Eddy Simulation/Detached Eddy Simulation (LES/DES) was developed by Hoffmann (2010) based on preliminary test conditions in the Spirit 6x6 duct at Mach 0.3. The LES/DES RMS velocity fluctuations were lower than what Klebanoff predicted but the velocity profile approximated a flat plate boundary layer. There was no experimental data to verify LES/DES results; however, the mean square wall pressure values were similar to those predicted by the empirical models. The error due to the simplification to two dimensions was estimated to be small.

Both the point Gauss-Seidel and eigenvalue expansion methods were used in this dissertation to solve the Poisson equation using the WSU LES/DES data to validate the solution methods. The two dimensional Point Gauss-Seidel method with the Stokes pressure specified resulted in wall pressure fluctuations which were in agreement with the three dimensional LES/DES results. This could indicate that the differences between a two dimensional and three dimensional solutions might be small. The eigenvalue expansion method did not produce results that were in agreement with the LES/DES results. The LES/DES data was very sparse and it is possible that more accurate wall pressure fluctuation values from the eigenvalue expansion method could be obtained with a more dense flow field grid. It is important to note that since the pressure gradient at the wall must be calculated from the LES/DES data, methods that require that the Stokes pressure be specified are not useful for predictive activities.
4.3 **Comparison to Experimental Data**

In this section, the mean square pressure is calculated from wall pressure fluctuation data collected in the Spirit 6x6 duct and compared to the empirical models. This effort was part a Spirit AeroSystems Acoustics Lab project to develop methods for predicting interior noise in aircraft which the author was a member. First, the changes to the Spirit 6x6 duct are described which were designed to quiet the convected noise and turbulence that occur before the boundary layer is formed. The pinhole microphones used to collect the wall pressure fluctuation data are described and contrasted to the preliminary, flush-mounted microphones used in the initial phases of the testing. The empirical models described in Section 4.2 are evaluated at conditions that simulate the Spirit 6x6 duct and then compared to the experimental data.

4.3.1 **Experimental Setup**

Numerous changes have been made by the Spirit Acoustics Lab staff to the Spirit 6x6 duct to increase the quality of the wall pressure fluctuation data which will be discussed in this section. The changes to the microphone mounting technique and Kulite arrangement in the test section will also be discussed.

4.3.1.1 **Duct Upgrades**

The existing Spirit AeroSystems 6x6 flow duct is used to study aircraft nacelle flows and acoustics and is described in Gallman et al. (2002) and Drouin et al. (2006). The duct is used to study the effect of sound on wall skin friction, as well as, the effect of sound pressure level on surface acoustic impedance. Sound is provided by up to three EPT 200 10,000 Watt drivers. This results in a maximum of 156 dB of band limited white noise in the test section.

Since the baseline inlet was not designed for wall pressure fluctuation measurements, it had several deficiencies for the low noise and smooth flow required to measure wall pressure
fluctuations. The previous configuration included no provision for absorbing upstream equipment and valve sound. The existing inlet allowed for intense mixing in the settling chamber. There was also a quarter inch mismatch at the attachment to the duct which most likely caused a separation bubble at the beginning of the duct. A separation bubble would not allow for an equilibrium boundary layer to form and would also change the location of transition and the local boundary layer thickness in the test section.

The baseline duct is shown in Figure 4.29. Air is supplied to the settling chamber by a maximum of three inlets which are connected to shop air by a series of valves. Flow in the test section has a maximum Mach number of approximately 0.6. A bell mouth connects the settling chamber to the test section which runs twenty feet to a diffuser and exhausts to the shop. There is a two-axis traverse that runs axially along the duct and vertically along the duct centerline.

![Figure 4.31: Spirit AeroSystems baseline 6x6 flow duct.](image)
Because the baseline 6x6 flow duct was designed to support the nacelle acoustic lining design, there was intense mixing in the settling chamber and an abrupt transition to the test section which resulted in a free stream turbulence intensity of 3.5%. Preliminary testing showed that wall pressure fluctuations measured in the baseline 6x6 duct configuration were dominated by an acoustic wave propagating downstream from the intense mixing in the inlet and settling chamber. In order to obtain low noise turbulent boundary layer wall pressure fluctuation data, a new inlet for the existing six inch square flow duct had to be developed.

The objective for the development of the new inlet was to minimize the facility related noise in the duct in order to facilitate turbulent boundary layer wall pressure measurements. It was a requirement to work within the space constraints of the existing lab, which had a total duct length of 180 inches. To achieve the required mass flow, three supply lines needed to be merged smoothly. Finally a round to square transition was required in the contraction section. Three configurations were studied by the team. The third configuration where the three feed lines are coplanar with the centerline of the duct was chosen. Figure 4.30 shows the coplanar feed line mixer configuration.

Figure 4.30: Coplanar feed line mixer configuration.
The design of the new inlet includes the 3 to 1 mixer, an inline muffler, and a diffuser. Downstream of the diffuser, flow straightening screens are located to minimize the turbulence entering the settling chamber. Next, there is a nozzle that transitions from the round cross section of the inlet to the square cross section of the test section. Finally the 6x6 flow duct ends in a diffuser which exhausts into the lab. Figure 4.31 shows the design of the new inlet. Figure 4.32 shows the key components of the new inlet before installation. Figure 4.33 shows the new inlet assembly. Figure 4.34 shows the new inlet as it is installed in the duct.

Figure 4.31: Design of the new quiet inlet.
Figure 4.32: Key components of the new inlet.

Figure 4.33: New inlet assembly.
A significant source of error found in the preliminary experimental results was due to vibrations that occurred on the microphone mounting plate. Although several different signal processing methods were tried, the effect of the vibration could not be removed because it was correlated with the turbulence signal in the data. Therefore, the one-inch thick Plexiglas mounting plate was required in the final design to remove any vibration. Figure 4.35 shows a schematic of the microphone array position in the test section of the 6x6 duct. Figure 4.36 shows the Spirit 6x6 test section with the new one-inch thick mounting plate.

Figure 4.35: Kulite array location in 6x6 flow duct test section.
In summary, the 6x6 flow duct was updated by the Spirit Acoustics Lab staff to allow turbulent boundary layer wall pressure fluctuation measurements. The updates included a new 3 to 1 mixer, muffler, diffuser, screen assembly, settling chamber and round to square nozzle. The duct with the new inlet is capable of making turbulent boundary layer wall pressure fluctuation measurements up to a Mach number of about 0.6. A new one-inch thick mounting plate replaced the original, thinner plate that had caused significant vibration and extraneous noise in the resulting wall pressure fluctuation measurements.

### 4.3.1.2 Sensors

The wall pressure fluctuation measurements are made with Kulite high intensity microphones model number MIC-093. Figure 4.37 shows a schematic of the Kulite MIC-093 microphone. In the preliminary measurements, the Kulite microphones were flush-mounted to
the Plexiglas plate. As discussed in Chapter 3, the accuracy of the mean square pressure measurement values is dependent on the transducer size. Because the Plexiglas mounting plate had to be made thicker to address the vibration issue, there was an opportunity to improve the Kulite mounting method to a pinhole configuration. Figure 4.38 shows a schematic of the pinhole mounting technique.

Figure 4.37: Kulite high intensity microphone, MIC-093.

Figure 4.38: New pinhole microphone mounting technique.
In section 3.6, the dimensionless diameter was defined as

\[ d_+ = dU_\tau / \nu \]  

(4.158)

where the transducer diameter is \( d \), the kinematic viscosity is \( \nu \), and the friction velocity is \( U_\tau \).

When the diameter of the transducer becomes too large, it can no longer resolve the complete signal spectrum, especially at high frequencies because of spatial resolution. Following the work of Schewe (1983), Keith et al. (1992), and Lueptow (1995), a practical limit of \( d_+ = 160 \) based on the Corcos correction factor and experimental results can be assumed.

The design of the Kulite microphone gives it uniform sensitivity across the sensing surface for both the flush-mount and pinhole configurations. The outside diameter of the Kulite microphone is given in the data sheet as 2.4 mm. The effective diameter used in the calculations for the flush-mounted microphone is estimated as 60 percent of the outside diameter which is 1.44 mm. The pinhole diameter is given as 0.79 mm. Figure 4.39 shows the value of \( d_+ \) for the flush-mounted and pinhole Kulite microphone as a function of Mach number. Using the limit of \( d_+ = 160 \), the flush-mounted microphone used in the preliminary experiments begins to attenuate the signal at Mach numbers above 0.15. The pinhole microphone does not begin to attenuate the signal until Mach numbers above about 0.30.
4.3.1.3 Velocity Profile

One method to analyze how closely the measured boundary layer matches an equilibrium turbulent boundary is to examine the velocity profile. Velocity profile measurements were made using a cylindrical Pitot probe on upper section of the duct. Velocity profiles at $M = 0.15$ were measured for both the baseline and new inlet configurations. Figure 4.40 compares the measured velocity profiles at 3 meters and a theoretical turbulent boundary layer based on the $1/7^{th}$ rule. This graph shows that the velocity profile from the new inlet does a much better job matching the flat plate turbulent boundary layer $1/7^{th}$ rule than the data from the baseline inlet.
Figure 4.40: Turbulent boundary layer velocity profile for $M = 0.15$ at 3 meters.

4.3.1.4 Background and Convected Noise

A significant source of error in the preliminary experimental results was the equipment and valve noise convected down the flow. However, a relatively simple method was found to discriminate between the convected sound and the remaining signal. This method is developed in Simpson et al. (1987) and is described in more detail in Chapter 5. This method works when data from two microphones that are at the same distance downstream and more than one boundary layer length apart axially is used. Since they are more than one boundary layer apart, the convected sound signals are correlated but the rest of the signal (including the turbulence) is not. Figure 4.41 shows the preliminary microphone layout in the Spirit 6x6 duct. The microphone numbers are shown in green. In the preliminary case, data from microphones 0 and 3 is used. For the final measurements, the microphone locations in the array were changed to a “U” shape to allow for more opportunities to use the simple noise reduction technique. Figure
4.42 shows the microphone layout for the final data analysis. To calculate the mean square pressures in the final data analysis, data from microphones 1 and 5 is used.

Figure 4.41: Preliminary microphone layout in the Spirit 6x6 duct.

Figure 4.42: Final microphone layout in the Spirit 6x6 duct.
Figures 4.43, 4.44 and 4.45 show the experimental results for the pressure fluctuations with and without the convected sound at $M = 0.1$, $M = 0.3$ and $M = 0.5$ for both the preliminary and final analyses. The simple noise reduction data analysis technique effectively removes the convected sound at all three Mach numbers. The Spirit 6x6 duct improvements to quiet the convected noise and remove the plate vibration can be seen in the final measurement data at all Mach numbers, especially at Mach 0.5.

Figure 4.43: Raw time history of pressure fluctuations at $M = 0.1$ with and without convected sound.

Figure 4.44: Raw time history of pressure fluctuations at $M = 0.3$ with and without convected sound.
Figure 4.45: Raw time history of pressure fluctuations at $M = 0.5$ with and without convected sound.

4.3.1.5 Spirit 6x6 Pressure Fluctuation Data

The mean square pressure was determined by taking the raw pressure time signal, squaring it and averaging over time.

$$
\overline{p^2} = \frac{1}{T} \int_0^T p^2(t)dt
$$

(4.159)

The preliminary mean square pressure values were calculated from 6.14395 seconds of data. At a 20 KHz sample rate, this resulted in 122,880 individual pressure readings to use. The final mean square pressure values were calculated from 18.4319 seconds worth of data. At a 10 KHz sample rate, this resulted in 184320 individual pressure readings. The final data from the updated duct also includes measurements at higher Mach numbers up to 0.6.

As noted in the previous section, the upgrades to the duct reduced the convected sound and the addition of a thicker mounting plate removed the unwanted vibration. The use of the pinhole microphone also delays attenuation of the signal to above $M = 0.3$. However, a decrease in sample rate from 20 KHz to 10 KHz decreases the Nyquist frequency from 10 KHz to 5 KHz. In Chapter 5, it will be shown that the frequency content of the wall pressure fluctuation spectrum spreads to higher frequencies at higher Mach numbers. This pushes more and more
signal content above the Nyquist frequency as the Mach number increases. By decreasing the Nyquist frequency, the anti-aliasing filter attenuates more of the signal as the Mach number increases.

Figure 4.46 compares the preliminary to the final mean square pressure fluctuation values with and without the convected sound. The simple noise reduction signal processing technique continues to effectively remove the convected sound. It is clear that the improvements to the duct have reduced the convected noise contamination especially at higher Mach numbers. It is also clear that the thicker plate reduces the vibration contamination which cannot be removed by the simple noise reduction technique. The pinhole microphone mounting technique starts to attenuate the signal at Mach numbers above 0.3. By decreasing the sample rate to 10 KHz, signal is lost at the higher Mach numbers due to the anti-aliasing filter in the data acquisition system.

![Figure 4.46: Mean square wall pressure values with and without convected sound as a function of Mach number.](image.png)
4.3.2 Comparison of Spirit 6x6 Data to Empirical Models

As seen in Section 4.2, there is a large range to the results of the empirical mean square pressure models. To determine which model best represents the aircraft environment; these models must be compared with representative data. In this section, the empirical models are compared to data from the updated Spirit 6x6 duct.

In Section 4.3 where the numerical and analytical solutions were discussed, the empirical models were compared at one Mach number and a range of distances along the plate. In this section, the empirical models are compared at one location along the plate and a range of Mach numbers. Figure 4.47 shows these empirical results graphically. It is again interesting to note that the earliest models are at the limits of all the empirical models. Kraichnan (1956) is at the high end of the predictions. Lowson (1968), Bull (1967) and Willmarth and Wooldridge (1962) are at the low end. The two newest models, Lueptow (1995) and Farabee and Casarella (1991) are pretty much in the middle of the pack.

![Figure 4.47: Mean square wall pressure values as a function of Mach number at 3 meters.](image)
The results from this graph show that the experimental mean square pressure fluctuation values agree well with the empirical models at lower Mach numbers. However, the experimental values show a shallower slope than the empirical models predict. There are two issues that can influence this slope: the effective diameter of microphone and the Nyquist frequency. The pinhole microphone begins to attenuate the signal above a Mach number of 0.3. But the anti-aliasing filter appears to have a bigger influence because the slope of the experimental data begins to drop much earlier than $M = 0.3$. From this data, it is difficult to draw any meaningful conclusions on which mean square pressure model is best.

4.4 Summary

The mean square pressure value gives an estimate of the overall energy of the wall pressure fluctuations in a turbulent boundary layer. It can also serve as a simple check for a single point wall pressure spectrum model by integrating over the frequency. The pressure fluctuations beneath a turbulent boundary layer were found by solving the Poisson equation which is derived from the Navier-Stokes equation. The Poisson equation was solved numerically by the point Gauss-Seidel method and analytically using an eigenvalue expansion method. These models were solved using LES/DES data from WSU at $M = 0.3$. The numerical and analytical models were simplified to two dimensions. It was estimated that the error introduced by this simplification was small.

The RMS wall pressure fluctuation values were calculated using the point Gauss-Seidel method with the Stokes pressure set to zero and specified. Setting the Stokes pressure to zero results in wall pressure fluctuation values that were significantly below the predicted values. Using the Stokes pressure values derived from the LES/DES data resulted in reasonable RMS wall pressure fluctuation values.
The RMS wall pressure fluctuation values were also calculated using the eigenvalue expansion method with the Stokes pressure set to zero and specified. Setting the Stokes pressure to zero results in wall pressure fluctuation values that were higher than the LES/DES prediction. Using the Stokes pressure values derived from the LES/DES data resulted in wall pressure fluctuation values much lower than the LES/DES prediction.

There are many empirical models for the mean square pressure value based either on the wall shear stress, $\tau_w$, or the dynamic pressure, $q$. These empirical models gave a wide range of predicted mean square pressure values and it was not clear which model was more accurate.

Experimental data was obtained in the Spirit AeroSystems 6x6 duct using a Kulite array at Mach numbers from 0.1 to 0.6. A new inlet was made for the Spirit 6x6 duct and is discussed in Section 4.3.1. The new inlet produced a boundary layer that behaved more like an equilibrium turbulent boundary layer.

A major source of error for the mean square pressure analysis in the preliminary data was the vibration in the signal. The new mounting method for the microphones using a much thicker plate effectively minimized the vibration. The new pinhole mounting method reduced the diameter of the sensing element which allowed the microphone to resolve the turbulence at higher Mach numbers. However, by reducing the sample rate to 10 KHz, the anti-aliasing filter attenuated the overall mean square pressure values by dropping the high frequency signals which was especially noticeable at higher Mach numbers.

A simple data analysis technique was used to effectively remove the convected sound from the wall pressure fluctuation data. The experimental data showed that the mean square pressures fell within the range of the empirical models at low Mach numbers but were lower than
predicted at higher Mach numbers. Based on this analysis, it was difficult to draw any meaningful conclusions on which mean square pressure model is best.

In this chapter, the mean square pressure models were examined. In Chapter 5, the single point wall pressure spectrum models will be analyzed and compared to the mean square pressure models by integrating over the frequency.
CHAPTER 5

SINGLE POINT WALL PRESSURE SPECTRUM MODELS

5.1 Background

The single point wall pressure spectrum models are used by the acoustic analysis programs to estimate the frequency response of a panel. This section begins with the definition of the experimental spectral density. This is the value that is measured and is what is typically used for analysis comparisons. A relationship between the single point wall pressure spectrum and the experimental spectral density is derived. A simple check for a single point wall pressure spectrum model is to compare it to an estimate of the mean square pressure fluctuation which was discussed in the previous chapter. The scaling factors and spectral features that are important in the development of the single point wall pressure spectrum models are discussed in section 5.1.2.

5.1.1 Definition of Experimental Spectral Density

As was noted previously, each investigator defines the single point wall pressure spectrum slightly differently. According to Crandall and Mark (1963), a physical meaning can be given to $\Phi(\omega)$ by considering the limiting case where $\tau = 0$ by using Parseval’s formula for integrals.

$$R_{pp}(0) = E[p^{T^2}] = \overline{p^{T^2}} = \int_{-\infty}^{\infty} \Phi(\omega) d\omega$$

The mean square of the process equals the sum over all frequencies of $\Phi(\omega)$ so this parameter can be interpreted as the mean square spectral density or power spectrum. The dimensions of $\Phi(\omega)$ are mean square pressure per unit of angular frequency. It should be noted that in equation (5.1) both negative and positive frequencies are counted. This definition of spectral density is
convenient for analytical investigations. However in experimental work, a different unit of spectral density is used. This difference is due to the use of cycles per unit time instead of radians per unit time and the use of only positive frequencies. The experimental spectral density symbol used is $W(f)$ where $f$ is frequency in cycles per unit time. The relationship between $\Phi(\omega)$ and $W(f)$ is

$$W(f) = 4\pi \Phi(\omega) \quad (5.2)$$

The factor $4\pi$ includes a factor of $2\pi$ to account for the change in frequency units and a factor of 2 to account for using positive frequencies only. For the experimental spectral density, equation (5.1) becomes

$$R_{pp}(0) = E[p'^2] = \overline{p'^2} = \int_0^\infty W(f) \, df \quad (5.3)$$

Not all investigators define the single point wall pressure spectrum as two sided. This would change the relationship defined in equation (5.2). In fact, all of the investigators in this section define the spectral density as

$$\int_0^\infty \Phi(\omega) \, d\omega = \overline{p'^2} \quad (5.4)$$

This is different than the double sided theoretical relationship given in equation (5.1) and the symbol $\Phi(\omega)$ versus $\Phi(\omega)$ is used to show the difference in the definition. Therefore, equation (5.2) becomes

$$W(f) = 2\pi \Phi(\omega) \quad (5.5)$$

The mean square value of the pressure fluctuations can be checked against experimental data to ensure that the relationships have been interpreted correctly.

$$\overline{p'^2} = \int_0^\infty \Phi(\omega) \, d\omega = \int_0^\infty W(f) \, df \quad (5.6)$$
5.1.2 **Effect of Scaling Factors and Spectral Features**

As Goody (2004) explains, scaling is a technique that is often used in the analysis of boundary layers. It is based on the concept of self-similarity which uses a function of variables that retain the shape for a wide variety of conditions. If the boundary layer can be identified as self-similar, then a dependent variable is made nondimensional by appropriate length, velocity and pressure scales. The function is called universal because it does not change for a wide variety of conditions.

But the application of self-similarity principles to turbulent boundary layers is complicated because the character of the boundary layer varies with the distance to the wall. The wall pressure field may be considered as generated by sources that are distributed throughout the boundary layer. (Blake, 1970) The different boundary layer scales are related to organized motions or coherent structures that exist in the boundary layer which vary in the regions normal to the wall. (Goody, 2004) Because the structure of the wall pressure field is inevitably complex, there is not a single universal scaling for the boundary layer. (Bull, 1996)

Early investigators divided the boundary layer into two regions. Bull (1967) described these two layers as a constant stress inner region and a wake-like outer region. Panton (1990) describes the specific locations of the two layers. He defines the term outer region as the log layer plus the wake \((y^+ = 40 \text{ to } y/\delta = 1)\) where \(y^+ = yU_*/\nu\). He states that the log layer and the wake layer are put together because their main turbulent motions are inviscid. He defines the term inner region as the viscous sublayer, the buffer layer and the log layer \((y^+ = 0 \text{ to } y/\delta = 0.2)\). He also notes that the Reynolds number \(Re_*=U_*/\nu\) is a direct measurement of the size of the outer region compared to the inner region. It is generally agreed that the inner region is most responsible for the high frequency behavior of the single point wall
pressure spectrum. The outer region rather than the inner region determines the pressure levels at low frequencies. (Blake, 1970) Implications from the scaling of the low frequency region of the spectra are not especially clear except to suggest that the sources are associated with the large scale structures induced by the unsteady potential flow observed above the boundary layer. In contrast, the small scale fluctuations associated with the high frequency region result from turbulence activity in the highly active buffer region of the wall layer. (Farabee and Casarella, 1991) The scaling variables used in the literature are typically defined as either inner, mixed or outer variables and are given in Table 5.1.

**TABLE 5.1**

**TYPICAL SCALING VARIABLES**

<table>
<thead>
<tr>
<th>Scaling Variables</th>
<th>Spectrum Scaling</th>
<th>Frequency Scaling</th>
</tr>
</thead>
<tbody>
<tr>
<td>Inner</td>
<td>$\Phi(\omega)U_\tau^2/\tau_w^2 \nu$</td>
<td>$\omega \nu / U_\tau^2$</td>
</tr>
<tr>
<td>Mixed</td>
<td>$\Phi(\omega)U_\tau / \tau_w^2 \delta$</td>
<td>$\omega \delta / U_\tau$</td>
</tr>
<tr>
<td>Outer</td>
<td>$\Phi(\omega)U_\infty / q^2 \delta^*$</td>
<td>$\omega \delta^* / U_\infty$</td>
</tr>
</tbody>
</table>

More recently, researchers have identified more regions within the boundary layer. Panton and Linebarger (1974) first proposed a three layer boundary layer: a wall region $0 < y_* < 33.2$, a universal or overlap region $y_* = 33.2$ to $y/\delta = 0.2$, and an outer region $y/\delta > 0.2$ where $y_* \equiv yU_\tau / \nu$. The overlap region is essentially the log region and the outer region contains the wake components.

Bull (1996) summarizes characteristics of the scaled frequency spectrum as follows: 1) low frequency range, $\omega \delta^* / U_\infty \leq 0.30$, $\Phi(\omega)U_\infty / q^2 \delta^* = \text{constant}(\omega \delta^* / U_\infty)^2$; 2) mid frequency range (peak location), $5 \leq \omega \delta / U_\tau \leq 100$, $\Phi(\omega)U_\tau / \tau_w^2 \delta = f_2(\omega \delta / U_\tau)$; 3) overlap range,
100 ≤ \omega \delta / U_τ ≤ 0.3U_τ \delta / ν , \Phi(\omega)\omega / \tau_w^2 = \text{constant} ; \text{ and } 4) \text{ high frequency range, } \omega V / U_τ^2 ≥ 0.3 , \\
Φ(\omega)U_τ^2 / \tau_w^2 ν = f_4(\omega V / U_τ^2). \text{ This agrees with the ranges given by Farabee and Casarella (1991). Farabee and Casarella also give the ranges based on mixed scaling variables and inner scaling variables which are summarized in Table 5.2.}

**TABLE 5.2**

**FREQUENCY RANGES BASED ON MIXED AND INNER SCALING VARIABLES**

<table>
<thead>
<tr>
<th>Frequency Range</th>
<th>Mixed Scaling Variable</th>
<th>Inner Scaling Variable</th>
</tr>
</thead>
<tbody>
<tr>
<td>Low</td>
<td>\omega \delta / U_τ ≤ 5</td>
<td></td>
</tr>
<tr>
<td>Mid (Peak location)</td>
<td>5 ≤ \omega \delta / U_τ ≤ 100</td>
<td>\omega V / U_τ^2 ≤ 100 / Re_τ</td>
</tr>
<tr>
<td>Overlap</td>
<td>100 ≤ \omega \delta / U_τ ≤ 0.30 Re_τ</td>
<td>100 / Re_τ ≤ \omega V / U_τ^2 ≤ 0.30</td>
</tr>
<tr>
<td>High</td>
<td>0.30 Re_τ ≤ \omega \delta / U_τ</td>
<td>0.30 ≤ \omega V / U_τ^2</td>
</tr>
</tbody>
</table>

5.1.2.1 Low Frequency Region

Fowcs-Williams (1965) first predicted the slope in the low frequency region to be \omega^2 at low frequencies but suggested that it might become a finite value at zero frequency due to compressibility effects. Panton and Linebarger (1974) also found that the single point wall pressure spectrum would rise as \omega^3 at low frequencies. It is commonly acknowledged that it is difficult to measure the wall pressure spectrum at low frequencies because the pressure fluctuations are of such small magnitude that it is difficult to separate them from disturbances in the freestream. In almost all flow facilities, the sound field at low frequencies has made it impossible to determine accurately the power spectrum of the wall pressure at the lowest frequencies. (Willmarth, 1975) Experimental verification of the low frequency trends of the
spectrum has proved to be a difficult task. Wind tunnel measurements suffer from spurious acoustic noise, vibrations, freestream turbulence and in some cases secondary flows. For these reasons, experimenters have been required to filter the low frequencies from their signals and concentrate on the mid frequency range. (Panton, et al., 1980) Both Goody (2007) and Farabee and Casarella (1991) also found that at low frequencies, the spectral density increases as $\omega^2$.

5.1.2.2 Peak Location

Blake (1970) predicted that the pressure spectra all peak at approximately the same level $\Phi(\omega)U_\omega \sqrt{\tau_s \delta^*} \equiv 0.2$. Farabee and Casarella (1991) predicted that this maximum level occurs in the mid frequency at $\omega \delta / U_\tau \approx 50$. In Table 5.2, this peak location is given as a range. It should be noted that these researchers used a mixed scale as one would expect for the mid frequency range.

5.1.2.3 Overlap Region

Panton and Linebarger (1974) indicated that as the curves drop and come together again, they enter a region of constant slope of minus one (-1). This region is an overlap region which occurs between the outer part scaling and the inner part scaling whose existence is a dimensional requirement. Calculations using any turbulence model having an inner-outer structure will produce a region with a -1 slope. The relationship $\Phi(\omega) \approx \omega^{-1}$ was first proposed by Bradshaw (1967). Experimental verification of the overlap region in the frequency spectrum has proven to be very difficult. (Panton, 1990) Both Farabee and Casarella (1991) and Bull (1996) predicted an overlap region with a slope of $\omega^{-1}$. 
5.1.2.4 High Frequency Region

Schewe (1983) found that high frequency spectrum taken with the smallest microphone shows a decay rate of $\Phi(\omega) \approx \omega^{-7/3}$. Based on more recent experimental data, Goody (2007) predicted the spectrum decays as $\omega^{-5}$ at high frequencies. Table 5.3 summarizes these frequency range slope predictions.

### TABLE 5.3

SUMMARY OF FREQUENCY RANGE SLOPE PREDICTIONS

<table>
<thead>
<tr>
<th></th>
<th>Low Frequency Range</th>
<th>Overlap Region</th>
<th>High Frequency Range</th>
</tr>
</thead>
<tbody>
<tr>
<td>Ffowcs-Williams (1965)</td>
<td>$\omega^2$</td>
<td></td>
<td></td>
</tr>
<tr>
<td>Bradshaw (1967)</td>
<td></td>
<td>$\omega^{-1}$</td>
<td></td>
</tr>
<tr>
<td>Panton and Linebarger (1974)</td>
<td>$\omega^2$</td>
<td>$\omega^{-1}$</td>
<td></td>
</tr>
<tr>
<td>Schewe (1983)</td>
<td>$\omega^{-7/3}$</td>
<td></td>
<td></td>
</tr>
<tr>
<td>Farabee and Casarella (1991)</td>
<td>$\omega^2$</td>
<td>$\omega^{-1}$</td>
<td></td>
</tr>
<tr>
<td>Bull (1996)</td>
<td></td>
<td>$\omega^{-1}$</td>
<td></td>
</tr>
<tr>
<td>Goody (2007)</td>
<td>$\omega^2$</td>
<td></td>
<td>$\omega^{-5}$</td>
</tr>
</tbody>
</table>

5.2 Examples of Models

As discussed earlier, the single point wall pressure spectrum or power spectrum represents the distribution of the mean square fluctuating pressure with frequency. In this section, the single point wall pressure spectrum models from Robertson, Efimtsov, Rackl and Weston, Chase-Howe, Goody, and Smol’yakov will be compared at conditions that simulate the Spirit 6x6 duct at $M = 0.1, 0.3,$ and 0.5. The required parameters are found in Table 5.4. Each
model is then integrated over frequency to find the mean square pressure which is then compared to values found in Chapter 4.

**TABLE 5.4**

**SPIRIT 6X6 DUCT TEST CONDITIONS AT 3 METERS**

<table>
<thead>
<tr>
<th></th>
<th>M = 0.1</th>
<th>M = 0.3</th>
<th>M = 0.5</th>
</tr>
</thead>
<tbody>
<tr>
<td>kinematic viscosity, ( \text{m}^2/\text{s} )</td>
<td>( 1.4991 \times 10^{-5} )</td>
<td>( 1.4991 \times 10^{-5} )</td>
<td>( 1.4991 \times 10^{-5} )</td>
</tr>
<tr>
<td>freestream velocity, ( \text{m/s} )</td>
<td>35.76</td>
<td>99.76</td>
<td>174.69</td>
</tr>
<tr>
<td>dynamic pressure, ( q ) ( \text{N/m}^2 )</td>
<td>764.32</td>
<td>5951.61</td>
<td>18244.58</td>
</tr>
<tr>
<td>shear stress at wall, ( \text{N/m}^2 )</td>
<td>2.0803</td>
<td>13.9125</td>
<td>39.4743</td>
</tr>
<tr>
<td>friction velocity, ( \text{m/s} )</td>
<td>1.3190</td>
<td>3.4111</td>
<td>5.7458</td>
</tr>
<tr>
<td>boundary layer thickness, ( \text{m} )</td>
<td>0.050676</td>
<td>0.043765</td>
<td>0.040400</td>
</tr>
<tr>
<td>displacement thickness, ( \text{m} )</td>
<td>0.006896</td>
<td>0.005520</td>
<td>0.004902</td>
</tr>
<tr>
<td>momentum thickness, ( \theta ) ( \text{m} )</td>
<td>0.005241</td>
<td>0.004292</td>
<td>0.003853</td>
</tr>
</tbody>
</table>

5.2.1 **Robertson**

An early single point wall pressure spectrum model was investigated by Robertson (1971). He based his model on the work of Lowson (1968). In comparing Lowson's formula with data especially measurements at supersonic speeds by NASA-Ames, he noted that the formula appears to underestimate the spectral levels at low Strouhal numbers and also gives too large a roll off at high Strouhal numbers. Therefore, a new formula was presented which appears
to be more representative of experimental findings throughout the Mach number range. In this formula, it should be noted that $\delta^*$ and $U_\infty$ were used as normalizing parameters.

$$
\Phi(\omega) = \frac{\overline{p'^2}}{\omega_0 [1 + (\omega/\omega_0)^{0.9}]^2}
$$

(5.7)

where

$$
\omega_0 = 0.5 \frac{U_\infty}{\delta^*}
$$

(5.8)

Robertson used the definition of the mean square pressure fluctuation from Lowson (1968).

$$
\overline{p'^2} = \left( \frac{0.006 q}{1.0 + 0.14M^2} \right)^2
$$

(5.9)

The experimental spectral density $W(f)$ can be found by using equation (5.5) and substituting $\omega = 2\pi f$ which gives

$$
W(f) = \frac{(2\pi)\overline{p'^2}}{\omega_0 [1.0 + (2\pi f / \omega_0)^{0.9}]^2}
$$

(5.10)

![Figure 5.1: Robertson experimental spectral density at $M = 0.1$, 0.3, and 0.5.](image)
As a check, a numerical integration of the single point wall pressure spectrum and the experimental spectral density were carried out. Choosing the upper integration value was somewhat arbitrary. As discussed above, Farabee and Casarella (1991) chose integration limits of 50 to 20000 Hz which will be used here and for the rest of the analysis in this section. Integrating the single point wall pressure spectrum model at $M = 0.1$ gives

$$\int_{50}^{20000} W(f) \, df = \int_{50}^{20000} \Phi(\omega) \, d\omega = \frac{P^2}{\omega^2} = 20.44 \text{ Pa}^2$$

(5.11)

### 5.2.2 Efimtsov (2 Models)

In this section, two single point wall pressure spectrum models from B.M. Efimtsov of TsAGI (Central Aerohydrodynamic Institute in Moscow, Russia) will be reviewed. The first model which will be referred to as Efimtsov 1 is from his 1982 work. The second model will be referred to as Efimtsov 2 is from his 1995 work which is documented in Rackl and Weston (2005) and Efimtsov et al. (1999).

Efimtsov (1982) stated that a single point wall pressure spectrum model should be dependent on Mach number ($M$), Reynolds number ($Re$) and Strouhal number ($Sh$). In this case, the Strouhal number is defined as $Sh = \omega \delta / U_\tau$. He noted that it is very difficult to obtain experimental data to cover the range of conditions of interest to the aircraft industry. In this study, Efimtsov and others gathered data from a series of flight tests in the range of Mach numbers of $M = 0.41$ to 2.1 with Reynolds numbers of $Re_x = 0.5 \times 10^8$ to $4.85 \times 10^8$. The pressure fluctuations were measured in various zones along the fuselage where the boundary layer was considered fully developed with zero pressure gradient. Efimtsov’s single point wall pressure spectrum is given by

$$\Phi(\omega) = \frac{0.01 \tau_w^2 \delta}{U_\tau [1.0 + 0.02(\omega \delta / U_\tau)^{2/3}]}$$

(5.12)
The experimental spectral density $W(f)$ can be found by using equation (5.5) and substituting $\omega = 2\pi f$ which gives

$$W(f) = \frac{(2\pi)0.01 \tau_w^2 \delta}{U_r [1.0 + 0.02(2\pi / U_r)^{2/3}]}$$  \hspace{1cm} (5.13)$$

Figure 5.2: Efimtsov 1 experimental spectral density at $M = 0.1$, 0.3, and 0.5.

Integrating the Efimtsov 1 single point wall pressure spectrum model at $M = 0.1$ with integration limits of 50 to 20000 Hz gives

$$\int_{50}^{20000} W(f) \, df = \int_{50}^{20000} \Phi(\omega) \, d\omega = p^2 = 55.31 \text{ Pa}^2$$  \hspace{1cm} (5.14)$$

Efimtsov 2 is an updated model using data from the low and high speed TsAGI wind tunnels and flight measurements using a TU-144LL. In this equation, the principal independent variables are the Strouhal number and the Reynolds number.

$$\Phi(\omega) = \frac{2\pi \alpha U_r^3 \rho \delta^2}{\beta} \left[ 1 + 8\alpha \left( \frac{\omega \delta}{U_r} \right)^2 \right]^{1/3} + \alpha \beta Re_f \left[ \left( \frac{\omega \delta}{U_r} \right) / Re_f \right]^{10/3}$$  \hspace{1cm} (5.15)$$
\[ Re_\tau = \frac{\delta U_\tau}{v_w} \quad (5.16) \]

\[ Re_{\tau0} = 3000 \quad (5.17) \]

\[ \beta = \left[ 1 + \left( \frac{Re_{\tau0}}{Re_\tau} \right)^3 \right]^{1/3} \quad (5.18) \]

\[ \alpha = 0.01 \quad (5.19) \]

\[ v_w = v_\infty \frac{\rho_w}{\rho_\infty} \left( \frac{T_w}{T_\infty} \right)^\gamma \quad (5.20) \]

\[ \gamma = 0.905 \quad (5.21) \]

\[ T_w = T_\infty \left( 1 + r \frac{\kappa - 1}{2} M^2 \right) \quad (5.22) \]

\[ r = 0.89 \quad (5.23) \]

\[ \kappa = 1.4 \quad (5.24) \]

\[ \rho_w = \rho_\infty \frac{T_\infty}{T_w} \quad (5.25) \]

It will be assumed that Efimtsov continued to define his single point wall pressure spectrum as one sided similar to equation (5.5). Therefore, the experimental spectral density \( W(f) \) can be found by substituting \( \omega = 2\pi f \) and including the \( 2\pi \) factor which gives

\[ W(f) = \frac{(2\pi)^2 \alpha U_\tau^3 \rho_w^2 \delta \beta}{\left( 1 + 8\alpha^3 \left( \frac{2\pi \delta}{U_\tau} \right)^2 \right)^{1/3} + \alpha \beta Re_\tau \left( \frac{2\pi \delta}{U_\tau} / Re_\tau \right)^{10/3}} \quad (5.26) \]
Integrating the Efimtsov 2 single point wall pressure spectrum model at $M = 0.1$ with integration limits of 50 to 20000 Hz gives

\[
\int_{50}^{20000} W(f) \, df = \int_{50}^{20000} \Phi(\omega) \, d\omega = p^2 = 326.80 \, \text{Pa}^2
\]  
(5.27)

### 5.2.3 Rackl and Weston

Rackl and Weston (2005) compared the measured flight data to the predictions from the second Efimtsov single point wall pressure spectrum model and found two characteristics that the Efimtsov model did not predict. The first was a broadband spectral peak near a Strouhal number of 0.6 where the Strouhal number was calculated by $Sh = 2\pi f \delta^+ / U_\infty$. They suggested that one might expect that certain frequency regions would contribute more strongly to turbulence energies according to the length scales imposed on the flow by the boundary layer thickness. They noted that such behavior had been shown to exist for free shear layers in jet
plumes. The behavior of the measured data supports this reasoning by exhibiting its spectral peak at a constant Strouhal number.

The measured data also showed a steeper roll off at high frequencies above 1000 Hz. They noted that the Efimtsov 2 model predicts a much shallower high frequency negative slope than all other models, as well as the TU-144LL flight test data.

Two analytical functions were used to adjust the Efimtsov 2 model to more accurately reproduce the measured data. The broadband peak was modeled by a log normal distribution equation given by

\[
factor_1 = 2.5 \exp \left[ - \left( \ln \left( \frac{2\pi \delta^*}{U_\infty} \right) - \ln(0.6) \right)^2 \right]
\] (5.28)

To account for the slope of the high frequency roll off, an additional function was used. The Efimtsov 2 model over predicts for flight conditions below Mach 1.65 and under predicts for Mach numbers above 1.65. Since only the high frequency slope needs to be corrected, an addition factor centered at 1000 Hz is included. This correction factor is given by

\[
factor_2 = \frac{1}{4} \left[ \tanh \left( \log_{10} \left( \frac{f}{1000} \right) \right) + 1 \right] (M - 1.65) \log_{10}(f)
\] (5.29)

It should be noted that these correction factors were developed from the experimental data shown in dB (re $2 \times 10^{-5}$ Pa). Figure 5.4 shows the correction factors in the decibel scale for the $M = 0.1$ duct case.
Figure 5.4: Rackl and Weston correction factors at $M = 0.1$.

To see the effect of these factors on the experimental spectral density model, the Efimtsov 2 experimental spectral density model one must first be converted to decibels, then add the Weston and Rackl factors, and finally convert the corrected model back in to the original units. The result is shown in Figure 5.5.

Figure 5.5: Rackl and Weston experimental spectral density correction at $M = 0.1$, $0.3$, and $0.5$ based on Efimtsov 2 model.
Integrating the corrected single point wall pressure spectrum model at $M = 0.1$ with integration limits of 50 to 20000 Hz gives

$$\int_{50}^{20000} W(f) \, df = \int_{50}^{20000} \Phi(\omega) \, d\omega = p^{1/2} = 257.09 \, \text{Pa}^2 \quad (5.30)$$

### 5.2.4 Chase-Howe

Howe (1998) gave the following model which he attributed to Chase (1980). It should be noted that this model is a much simpler than what Chase proposed. Goody (2004) referred to this model as Chase-Howe which will be adopted in this work.

$$\Phi(\omega) = \frac{2(\delta^*/U_\infty)^3(\tau_w \omega)^2}{[(\omega \delta^*/U_\infty)^2 + 0.0144]^{3/2}} \quad (5.31)$$

At low frequencies, the model spectrum is proportional to $\omega^2$ and at higher frequencies it varies with $\omega^{-1}$ which is consistent with theoretical results. This model does not include an $\omega^{-5}$ spectral decay which has been measured and theoretically shown to exist at the highest frequencies. It should be noted that Goody modified Howe’s model to conform to the one sided definition of the single point wall spectrum. Since all of the models reviewed in this section are of the one sided type, this is the definition that will be used. The experimental spectral density $W(f)$ can be found by substituting $\omega = 2\pi f$ and including the $2\pi$ factor which gives

$$W(f) = \frac{4\pi(\delta^*/U_\infty)^3(2\pi f \tau_w)^2}{[(2\pi f \delta^*/U_\infty)^2 + 0.0144]^{3/2}} \quad (5.32)$$

This is shown in Figure 5.6.
Integrating the Chase-Howe single point wall pressure spectrum model at $M = 0.1$ with integration limits of 50 to 20000 Hz gives

$$\int_{50}^{20000} W(f) df = \int_{50}^{20000} \Phi(\omega) d\omega = \overline{p^{\omega}} = 42.99 \text{ Pa}^2$$

(5.33)

### 5.2.5 Goody

Goody (2004) started with the Chase-Howe model described in the previous section because it is able to describe the essential features of the single point wall pressure spectrum with a limited number of variables. He felt that it is more closely related to the observed scaling behavior than a simple polynomial curve fit. Goody compared the Chase-Howe model to data from 19 different experimental spectra that covered a range of Reynolds numbers $1400 < Re_\theta < 23400$. (Goody, 2007) He found that the model was low at low frequencies and did not decay rapidly enough at high frequencies. Goody modified the Chase-Howe with the following changes. First, a term was added to the denominator so that spectral levels would decay as $\omega^{-5}$ as $\omega \to \infty$. The exponents in the denominator were changed to agree better with
the measured single point wall spectrum data at the middle frequencies. A constant was multiplied in the numerator to raise the spectral levels at all frequencies so that they would agree better with the experimental data. The Goody model is given by

$$\Phi(\omega) = \frac{3.0(\delta/U_\infty)^3(\omega \tau_w)^2}{[(\omega \delta/U_\infty)^{0.75} + 0.5]^{3.7} + [(1.1R_T^{-0.57})(\omega \delta/U_\infty)]^7}$$

(5.34)

where

$$R_T = \frac{U_T^2 \delta}{U_\infty V}$$

(5.35)

The experimental spectral density $W(f)$, shown in Figure 5.7, can be found by substituting $\omega = 2\pi f$ and including the $2\pi$ factor which gives

$$W(f) = \frac{6.0\pi(\delta/U_\infty)^3(2\pi f \tau_w)^2}{[(2\pi f \delta/U_\infty)^{0.75} + 0.5]^{3.7} + [(1.1R_T^{-0.57})(2\pi f \delta/U_\infty)]^7}$$

(5.36)

![Figure 5.7: Goody experimental spectral density at $M = 0.1$, 0.3, and 0.5.](image-url)
Integrating the Goody single point wall pressure spectrum model at \( M = 0.1 \) with integration limits of 50 to 20000 Hz gives

\[
\int_{50}^{20000} W(f) \, df = \int_{50}^{20000} \Phi(\omega) \, d\omega = \frac{\rho^2}{2} = 80.76 \, \text{Pa}^2
\]

(5.37)

### 5.2.6 Smol'yakov

Smol'yakov (2000) also found that the single point wall pressure spectrum scales on different variables depending on the frequency as discussed previously. He described the three regions as low frequency, universal, and high frequency. In his model, the first part of each equation describes the main laws governing the behavior of the spectra. The second part in brackets provides a smooth matching between the regions.

\[
\Phi(\omega) = \frac{1.49 \times 10^{-5} \tau_v^2 \nu Re_\theta^{2.74} \omega^{-2}}{U_r^2} \left\{ 1 - 0.117 Re_\theta^{0.44} \omega^{-1.52} \right\} \quad \text{at} \quad \omega < \omega_0
\]

(5.38)

\[
\Phi(\omega) = \frac{2.75 \tau_w^2 \nu}{U_r^2 \omega^{1.11}} \left\{ 1 - 0.82 \exp[-0.51(\omega/\omega_0 - 1)] \right\} \quad \text{at} \quad \omega_0 < \omega < 0.2
\]

(5.39)

\[
\Phi(\omega) = \frac{(38.9e^{-8.35\omega} + 18.6e^{-3.58\omega} + 0.31e^{-2.14\omega}) \tau_w^2 \nu}{U_r^2} \times \left\{ 1 - 0.82 \exp[-0.51(\omega/\omega_0 - 1)] \right\} \quad \text{at} \quad \omega > 0.2
\]

(5.40)

where

\[
\bar{\omega} = \frac{\omega \nu}{U_r^2}
\]

(5.41)

\[
\omega_0 = 49.35 Re_\theta^{-0.88}
\]

(5.42)

\[
Re_\theta = \frac{U_r \theta}{\nu}
\]

(5.43)
Assuming that the Smol’yakov single point wall pressure spectrum is one sided, the experimental spectral density $W(f)$ can be found by substituting $\omega = 2\pi f$ and including the $2\pi$ factor which gives

$$W(f) = \frac{(2\pi)1.49 \times 10^{-5} \nu^3 \Re_{\theta}^{2.74} (2\pi f \tau_w)^2}{U_f^6} \times \left\{ 1 - \frac{0.117 \Re_{\theta}^{0.44} (2\pi f \nu)^{0.5}}{U_f} \right\} \text{ at } f < \overline{f_0} \tag{5.44}$$

$$W(f) = \frac{(2\pi)2.75 \nu^2 \tau_w^2}{U_f^2 (2\pi f \nu / U_f)^{1.11}} \times \left\{ 1 - 0.82 \exp \left[ -0.51 \left( \frac{f}{f_0} - 1 \right) \right] \right\} \text{ at } \overline{f_0} < f < \frac{0.2 U_f^2}{2\pi \nu} \tag{5.45}$$

$$W(f) = \frac{(2\pi)(38.9 e^{-8.35(2\pi f \nu / U_f^2)} + 18.6 e^{-3.58(2\pi f \nu / U_f^2)} + 0.31 e^{-2.14(2\pi f \nu / U_f^2)}) \tau_w^2 \nu}{U_f^2} \times \left\{ 1 - 0.82 \exp \left[ -0.51 \left( \frac{f}{f_0} - 1 \right) \right] \right\} \text{ at } f > \frac{0.2 U_f^2}{2\pi \nu} \tag{5.46}$$

where

$$\overline{f_0} = \frac{49.35 U_f^2}{2\pi \nu \Re_{\theta}^{0.88}} \tag{5.47}$$

This spectral density is shown in Figure 5.8.
Figure 5.8: Smol’yakov experimental spectral density at $M = 0.1$, 0.3, and 0.5.

Integrating the Smol’yakov single point wall pressure spectrum model at $M = 0.1$ with integration limits of 50 to 20000 Hz gives

$$\int_{50}^{20000} W(f) df = \int_{50}^{20000} \Phi(\omega) d\omega = \overline{p^2} = 49.53 \text{ Pa}^2$$

(5.48)

5.3 Summary of Models

The first evaluation of the models is to compare their integrated mean square pressure at $M = 0.1$ and 0.5 to the empirical models of the previous chapter. It would be expected that an acceptable single point wall pressure spectrum model would be in the range of the latest of the mean square pressure models which are Lueptow (1995) and Farabee and Casarella (1991) because these models were based on newer measurements. It is assumed that these measurements would be more accurate due to the use of newer, more accurate microphones and measurement techniques. Figures 5.9 and 5.10 show that the models of Robertson and Chase-Howe under-predict the mean square pressure, while the models of Efimtsov 2 and Rackl and
Weston over predict the means square pressure. The Efimtsov 1, Goody, and Smol’yakov models appear to be approximately within the expected range. Is should be noted that the Efimtsov 1 or Goody models would be preferred because of their mathematical simplicity.

Figure 5.9: Comparison of the single point wall pressure spectrum models at $M = 0.1$ to mean square pressure empirical estimates.

Figure 5.10: Comparison of the single point wall pressure spectrum models at $M = 0.5$ to mean square pressure empirical estimates.
Figures 5.11 and 5.12 show all the single point wall pressure spectrum models at $M = 0.1$ and $M = 0.5$. From section 5.1.2, a slope of $\omega^2$ at low frequencies with a peak of $\Phi(\omega)U_\infty / (\tau_\infty \delta^*) \approx 0.2$ at a mid frequency location of $\omega \delta / U_\tau \approx 50$ would be expected. The slope in the overlap region is generally predicted to be $\omega^{-1}$. The slope in the high frequency range is predicted to decrease at a rate of between $\omega^{-7/3}$ and $\omega^{-5}$. Based on these published predictions, only the Goody and Smol’yakov model seem to be valid for incompressible flow.

Figure 5.11: Comparison of the experimental spectral density of all the models at $M = 0.1$ and 3 meters.

Figure 5.12: Comparison of the experimental spectral density of all the models at $M = 0.5$ and 3 meters.
5.4 Comparison to Experimental Data

Based on the published predictions of the shape of a single point wall pressure spectrum in an incompressible flow, a slope of $\omega^2$ as the frequency goes to zero, a slope of $\omega^{-4}$ in the overlap region, and a slope of between $\omega^{-7/3}$ and $\omega^{-5}$ in the high frequency range is expected. As discussed in the previous section, only the Goody and the Smol’yakov models meet these criteria. The Efimtsov 1 model, the Goody model and the Smol’yakov models seem to be within range of the expected mean square pressure model predictions. The Efimtsov 1 and Goody models are attractive because they are mathematically simple enough to be considered for incorporation into acoustic analysis codes. Data from the updated Spirit 6x6 duct has been used to determine which type single point wall pressure spectrum model is best suited for aircraft acoustic analysis.

5.4.1 Spirit 6x6 Data

5.4.1.1 Filtering

The data was taken at Spirit AeroSystems in the 6x6 duct which was modified for these experiments as described in Chapter 4. The raw data was processed to remove the convected sound, also described in Chapter 4. As was discussed previously, the turbulence and the vibration signals were correlated and could not be separated. The resulting signal was then processed with a Fourier transform to give the frequency distribution. A Hanning window was used along with 50 percent overlap averaging. This method ensured that the final data would be consistent with Parseval's formula as defined by Crandall and Mark (1963) which was discussed in section 5.1.1.
Bendat and Piersol (1980) described the finite Fourier transform of \( x(t) \) as a Fourier transform of an unlimited time history record \( v(t) \) multiplied by a rectangular time window \( u(t) \) where

\[
u(t) = \begin{cases} 
1 & 0 \leq t \leq T \\
0 & \text{otherwise}
\end{cases}
\]  

(5.49)

The sample time history \( x(t) \) can be considered to be the product

\[
x(t) = u(t)v(t)
\]

(5.50)

Therefore the Fourier transform is the convolution of the Fourier transforms of \( u(t) \) and \( v(t) \).

\[
\hat{\hat{f}}(f) = \int_{-\infty}^{\infty} \hat{u}(\alpha)\hat{v}(f-\alpha) \, d\alpha
\]

(5.51)

Bendat and Piersol (1980) gave the Fourier transform of the \( u(t) \) function as

\[
\hat{u}(f) = T \left( \frac{\sin \pi f T}{\pi f T} \right) e^{-j\pi f T}
\]

(5.52)

Figure 5.13 shows the plot the function \( u(t) \), while the plot of the Fourier transform \( \hat{u}(f) \) which represents the spectral window of the analysis is shown in Figure 5.14.

![Figure 5.13: Rectangular analysis window in the time domain.](image-url)
The large side lobes of $|\hat{u}(f)|$ allow leakage of power at frequencies separated from the main lobe of the spectral domain and may introduce significant anomalies in the estimated spectra. To contain this problem, one can use a time window that tapers the data to allow a more gradual entrance to and exit from the time history of the data. In this case, a Hanning window was used which is given by

$$u_h(t) = \frac{1}{2} \left( 1 - \cos \frac{2\pi t}{T} \right) \begin{cases} 0, & 0 \leq t \leq T \text{ \quad \text{otherwise}} \\ 1 - \cos \frac{2\pi t}{T}, & 0 \leq t \leq T \end{cases} \quad (5.53)$$

Bendat and Piersol (1980) give the Fourier transform of this as

$$\hat{u}_h(f) = \frac{1}{2} \hat{u}(f) - \frac{1}{4} \hat{u}(f - f_1) - \frac{1}{4} (f + f_1) \quad (5.54)$$

where

$$f_1 = \frac{1}{T} \quad (5.55)$$

$$\hat{u}(f - f_1) = -T \left[ \frac{\sin \pi (f - f_1)T}{\pi (f - f_1)T} \right] e^{-j\pi f/T} \quad (5.56)$$

$$\hat{u}(f + f_1) = -T \left[ \frac{\sin \pi (f + f_1)T}{\pi (f + f_1)T} \right] e^{-j\pi f/T} \quad (5.57)$$
Figure 5.15 shows the plot of the function $u(t)$, while the plot of the Fourier transform $\hat{u}(f)$ which represents the spectral window of the Hanning analysis is shown in Figure 5.16.

![Figure 5.15: Hanning analysis window in the time domain.](image1)

![Figure 5.16: Hanning analysis window in the spectral domain.](image2)

An additional technique to enhance the quality of the spectral density is to overlap the averaging procedure. This can be done when the length of the data set that is being transformed is larger than necessary to provide the desired frequency resolution. The data is then subdivided into smaller sets and windowed individually. To control the loss at the edges of the window, the individual sets can be overlapped in time. In the present case, the windows were overlapped by 50 percent. Figure 5.17 shows a 50-percent overlap for a Hanning analysis window.
5.4.1.2 Simple Noise Reduction Technique and Signal Processing

As discussed in Chapter 4, the simple noise reduction technique (Simpson et al. 1987) assumes that only axial sound waves can propagate in the duct. In this case, the noise propagating down the 6x6 duct is coherent but the turbulent boundary layer wall pressure fluctuations are not. In addition, the turbulent boundary layer wall pressure fluctuations are incoherent at an axial spacing greater than a boundary layer thickness. This means that the signals from two sensors at least one boundary thickness apart at the same axial location can be subtracted to find the signal due to the turbulent boundary layer alone.

Figure 5.18 shows the microphone layout in the Spirit 6x6 duct. The microphone numbers are shown in blue. In this analysis, microphones 1 and 5 are used because they are located at the same axial distance down the duct. The subscript $a$ refers to the acoustic noise propagating down the duct, the subscript $tbl$ refers to the turbulent boundary layer wall pressure fluctuations, and the subscript $v$ refers to the signal due to the vibrations. Therefore, the total signal at the microphones is given by

$$p l^t (t) = p l^{t\ a}_a (t) + p l^{t\ tbl}_.tbl (t) + p l^{t\ v}_v (t) \quad (5.58)$$

$$p 5^t (t) = p 5^{t\ a}_a (t) + p 5^{t\ tbl}_.tbl (t) + p 5^{t\ v}_v (t) \quad (5.59)$$
The vibration signal is uncorrelated with the turbulent boundary layer signal and it will be assumed that it is included in the turbulent boundary layer signal. The data model assumes that the acoustic noise is propagating straight down the duct; therefore, the signal should be the same for both position 1 and 5. Subtracting $p5'(t)$ from $p1'(t)$ and working with the mean square values gives

$$
\overline{(p1'-p5')^2} = \overline{(p1'_{a}+p1'_{tbl})-(p5'_{a}+p5'_{tbl})}^2
$$

$$
= p1^2_{a} + p1^2_{tbl} + p5^2_{a} + p5^2_{tbl} + 2p1'_{a}p1'_{tbl} + 2p5'_{a}p5'_{tbl}
$$

$$
- 2p1'_{a}p5'_{a} - 2p1'_{a}p5'_{tbl} - 2p1'_{tbl}p5'_{a} - 2p1'_{tbl}p5'_{tbl}
$$

(5.60)

The mean square values of the turbulent boundary layer pressure fluctuations from two different microphones at the same axial location will be the same.

$$
\overline{p1^2_{tbl}} = \overline{p5^2_{tbl}}
$$

(5.61)

The difference between the acoustic signals is uncorrelated and becomes zero.

$$
\overline{(p1'_{a}-p5'_{a})^2} = p1^2_{a} - 2p1'_{a}p5'_{a} + p5^2_{a} = 0
$$

(5.62)

The uncorrelated turbulent contributions are also zero

$$
\overline{p1'_{a}p5'_{a}} = 0
$$

(5.63)

The rest of the acoustic and turbulent contributions are also uncorrelated and become zero

$$
\overline{p1'_{a}p1'_{tbl}} = \overline{p5'_{a}p5'_{tbl}} = \overline{p5'_{a}p5'_{tbl}} = \overline{p1'_{tbl}p5'_{a}} = 0
$$

(5.64)

Therefore

$$
\overline{p1^2_{tbl}} = \frac{1}{2} \overline{(p1'-p5')^2}
$$

(5.65)

To find the single point wall pressure spectrum, a discrete Fourier transform is performed on the $\overline{(p1'-p5')/\sqrt{2}}$ values. Since a Hanning window was used, the result must be normalized by a factor of 8/3 as described by Bendat and Piersol (1980).
5.4.2 Comparison of Models to Spirit 6x6 Data

The data from the updated Spirit 6x6 duct was obtained for a range of Mach numbers from 0.1 to 0.6. The data was limited at low frequencies to 100 Hz. At high frequencies, the data was limited by noise contamination. These frequency limitations are given in Table 5.5. The experimental spectral densities for all the Mach numbers is shown in Figure 5.19. This figure shows that the noise reduction analysis method results in a relatively smooth output. It should also be noted that there appears to be little Mach number effects up to the maximum Mach number of 0.604.
TABLE 5.5

SUMMARY OF FREQUENCY LIMITATIONS

<table>
<thead>
<tr>
<th>Mach Number</th>
<th>Low Frequency Limit (Hz)</th>
<th>High Frequency Limit (Hz)</th>
</tr>
</thead>
<tbody>
<tr>
<td>0.105</td>
<td>100</td>
<td>2578</td>
</tr>
<tr>
<td>0.150</td>
<td>100</td>
<td>2744</td>
</tr>
<tr>
<td>0.207</td>
<td>100</td>
<td>2910</td>
</tr>
<tr>
<td>0.254</td>
<td>100</td>
<td>3076</td>
</tr>
<tr>
<td>0.293</td>
<td>100</td>
<td>3242</td>
</tr>
<tr>
<td>0.341</td>
<td>100</td>
<td>3408</td>
</tr>
<tr>
<td>0.391</td>
<td>100</td>
<td>3574</td>
</tr>
<tr>
<td>0.442</td>
<td>100</td>
<td>3740</td>
</tr>
<tr>
<td>0.476</td>
<td>100</td>
<td>3916</td>
</tr>
<tr>
<td>0.513</td>
<td>100</td>
<td>4072</td>
</tr>
<tr>
<td>0.558</td>
<td>100</td>
<td>4238</td>
</tr>
<tr>
<td>0.604</td>
<td>100</td>
<td>4404</td>
</tr>
</tbody>
</table>

Figure 5.19: Experimental spectral density.
As discussed in Section 5.1.2, scaling is a technique that is often used in the analysis of boundary layers especially in single point wall pressure spectrum models. Since the frequency range of the data from the Spirit 6x6 is limited below the high frequency range, it is expected that either the mixed or outer variables should collapse the spectrums the best. To obtain the single point wall pressure spectrum, the experimental spectral density is divided by $2\pi$. Figure 5.20 shows the single point wall pressure spectrum without any scaling. Figures 5.21 is the single point wall pressure spectrum scaled by inner variables. The spectrum is beginning to collapse at the higher frequencies as expected. Figures 5.22 and 5.23 shows the single point wall pressure spectrum scaled by the mixed and outer variables. These variables do a much better job of scaling the single point wall pressure spectrum from the Spirit 6x6 duct; however, the mixed variables do the best job. Since the spectrum collapses so well, it indicates that the Mach number dependence is small for this range of Mach numbers and reduced frequencies.

![Graph showing single point wall pressure spectrum](image)

**Figure 5.20:** Single point wall pressure spectrum, no scaling.
Figure 5.21: Single point wall pressure spectrum, inner scaling.

Figure 5.22: Single point wall pressure spectrum, mixed scaling.
Figure 5.23: Single point wall pressure spectrum, outer scaling.

Investigating the behavior of the coherence is a good way to check that the data is behaving correctly. Again, it was found that the mixed variables do best job of scaling for the coherence. Figures 5.24, 5.25, and 5.26 show that the coherence drops as the sensor spacing increases as expected.

Figure 5.24: Coherence, mixed scaling, 0.5 inch separation.
Figure 5.25: Coherence, mixed scaling, 1.5 inch separation.

Figure 5.26: Coherence, mixed scaling, 2.0 inch separation.

Figure 5.27 shows the cutoff or corner frequency is the same for both the single point wall pressure spectrum and coherence indicating that this is where the signal begins to drop off.
Figure 5.27: Cutoff/corner frequency, mixed scaling.

Figures 5.28, 5.29, and 5.30 compare the single point wall pressure spectra to the theoretical models of Efimtsov 1 and Goody at $M = 0.1$, $0.3$ and $0.5$. The Spirit 6x6 data falls between the two models. The spectrum at low frequencies rolls off similar to the Goody model. At very low frequencies and as Mach number increases, the signal becomes flat which indicates that the response is most likely system noise only.

Figure 5.28: Single point wall pressure spectrum, mixed scaling, $M = 0.1$. 
Figure 5.29: Single point wall pressure spectrum, mixed scaling, $M = 0.3$.

Figure 5.30: Single point wall pressure spectrum, mixed scaling, $M = 0.5$. 
5.5 Summary

A single point wall pressure spectrum model represents the distribution of the mean square fluctuating pressure with frequency. The mean square pressure can be found by integrating the single point wall pressure spectrum over all the frequencies. An analytically developed single point wall pressure spectrum model is not possible due to the complex nature of the flow field; therefore, typical models have been empirically developed using both high speed and low speed experimental data. The quality of data available in the literature was not uniform resulting in significant scatter. Inner, mixed, and outer boundary layer parameters have been used for single point wall pressure model generation. The more recent models, such as Goody or Smol’yakov, were developed using a mixture of scaling variables.

Three frequency regions have been identified: the low, overlap, and high ranges. The location of each depends on which scaling variables are being used. Various authors have characterized the single point wall pressure spectrum shape based on the frequency ranges. In general, the slope in the low frequency range is predicted to be approximately $\omega^2$. In the overlap region, the slope is predicted to be approximately $\omega^{-1}$. In the high region, the slope is predicted to be either $\omega^{-7/3}$ or $\omega^{-5}$.

Seven models were compared using the Spirit 6x6 operating conditions for comparison purposes: Robertson, Efimtsov 1, Efimtsov 2, Rackl and Weston, Chase-Howe, Goody and Smol’yakov. The models of Robertson, Robertson, Efimtsov 1, Efimtsov 2, and Rackl and Weston were developed for aircraft applications based on high speed data. The models of Chase-Howe, Goody and Smol’yakov were developed primarily for marine applications and were based on low speed, incompressible data. The seven models were then integrated so they could be compared to the mean square pressure models of Chapter 4 resulting in a significant
spread in the values. The models of Efimtsov 1, Goody, and Smol’yakov seemed to have more reasonable values. Based on the slope predictions discussed previously, only the incompressible Goody and Smol’yakov models seem to be valid.

Based on their mathematical simplicity and preliminary analysis, the models of Efimtsov 1 and Goody were compared to the experimental data. Experimental spectral density measurements were made in the Spirit 6x6 duct at Mach numbers ranging from 0.1 to 0.6. Little Mach number dependence was noted. The single point wall pressure spectrums were developed from the experimental data and it was found that the mixed variables did the best job of scaling. The coherence was analyzed at three distances and found to behave as expected. The cutoff/corner frequency was found to occur at the same location for the coherence and single point wall pressure spectrum indicating where the signal begins to drop off.

The single point wall pressure spectrums were compared to the Efimtsov 1 and Goody models. The Spirit 6x6 data falls between the two models. The spectrum at low frequencies rolls off similar to the Goody model. This analysis indicated that the Goody model is the appropriate single point wall pressure spectrum model for aircraft applications.

So far in this analysis, the mean square pressure and single point wall pressure spectrum models have been investigated. In the next chapter, the normalized-wavenumber frequency spectrum models will be characterized.
CHAPTER 6
NORMALIZED WAVENUMBER-FREQUENCY SPECTRUM MODELS

6.1 Background

The normalized wavenumber-frequency spectrum models are used by the acoustic analysis programs to define the wavenumber distribution. This section begins with a discussion of the important Fourier and Hankel transforms required to go from the space-time domain to the wavenumber-frequency domain. Some of the important features of a normalized wavenumber-frequency spectrum model are described in section 6.1.2.

6.1.1 Fourier and Hankel Transforms

Integral transform methods are found in many references. However, each reference uses a slightly different version of the integral transform kernel and each has its own notation. A brief review is included here to ensure that notation used in this analysis is clearly understood.

Although the problems addressed in this research are in the two dimensional domain, several concepts from the one dimensional case will prove to be useful.

The one dimensional Fourier transform of $f(x)$ is given by

$$\hat{f}(k) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(x)e^{-iks}dx \quad (6.1)$$

The inverse is given by

$$f(x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \hat{f}(k)e^{iks}dk \quad (6.2)$$

A summary of some useful one dimensional Fourier transform pairs are given in Table 6.1.
TABLE 6.1
ONE DIMENSIONAL FOURIER TRANSFORM PAIRS

<table>
<thead>
<tr>
<th></th>
<th>$f(x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \hat{f}(k)e^{+i\alpha}dk$</th>
<th>$\hat{f}(k) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(x)e^{-i\alpha}dx$</th>
</tr>
</thead>
<tbody>
<tr>
<td>shift: $e^{i\alpha}f(x)$</td>
<td>$\hat{f}(k - \alpha)$</td>
<td>$\hat{f}(k - \alpha) + \hat{f}(k + \alpha)$</td>
</tr>
<tr>
<td>similarity: $f(\alpha x)$</td>
<td>$\frac{1}{</td>
<td>\alpha</td>
</tr>
<tr>
<td>$\cos(\alpha x)f(x)$</td>
<td>$\hat{f}(k - \alpha) + \hat{f}(k + \alpha)$</td>
<td>$\frac{\sqrt{2}}{\pi} \frac{\alpha}{\alpha^2 + k^2}$</td>
</tr>
<tr>
<td>$e^{-\alpha</td>
<td>x</td>
<td>}$, $\alpha &gt; 0$</td>
</tr>
</tbody>
</table>

In two dimensions, the Fourier transform of $F(x, y)$ can be written as

$$\hat{F}(k_x, k_y) = \frac{1}{2\pi} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} F(x, y)e^{-i(k_x x + k_y y)}dx dy$$

The inverse is given by

$$F(x, y) = \frac{1}{2\pi} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \hat{F}(k_x, k_y)e^{i(k_x x + k_y y)}dk_x dk_y$$

A summary of some useful two dimensional Fourier transform pairs are given in Table 6.2.
TABLE 6.2
TWO DIMENSIONAL FOURIER TRANSFORM PAIRS

<table>
<thead>
<tr>
<th>Transformation</th>
<th>Formula</th>
</tr>
</thead>
<tbody>
<tr>
<td>Shift: $e^{i(\alpha x + \beta y)} F(x, y)$</td>
<td>$\hat{F}(k_x - \alpha, k_y - \beta)$</td>
</tr>
<tr>
<td>Similarity: $F(\alpha x, \beta y)$</td>
<td>$\frac{1}{</td>
</tr>
<tr>
<td>Separable Product: $F(x, y) = f(x)g(y)$</td>
<td>$\hat{F}(k_x, k_y) = \hat{f}(k_x) \hat{g}(k_y)$</td>
</tr>
<tr>
<td>$\cos(\alpha x) F(x, y)$</td>
<td>$\frac{\hat{F}(k_x - \alpha, k_y) + \hat{F}(k_x + \alpha, k_y)}{2}$</td>
</tr>
</tbody>
</table>

If the function $F(x, y)$ can be written as a product of independent functions of $x$ and $y$, $f(x)g(y)$, then the Fourier transform can easily be found by using product of one dimensional Fourier transform pairs.

$$\hat{F}(k_x, k_y) = \hat{f}(k_x) \hat{g}(k_y)$$

$$= \frac{1}{2\pi} \left( \int_{-\infty}^{\infty} f(x) e^{-i k_x x} dx \right) \left( \int_{-\infty}^{\infty} g(y) e^{-i k_y y} dy \right) \quad (6.5)$$

If the function cannot be separated, then the transform process can become more complicated. As Bracewell (1986) states, two dimensional systems may often show circular symmetry. A simplification can be made if one radial variable can be used in place of the two independent variables $x$ and $y$. This generalization results in a zero order Hankel transform, which is a one dimensional transform with a Bessel function kernel. Circular symmetry exists when

$$F(x, y) = H(r) \quad (6.6)$$

where
\[ r = \sqrt{x^2 + y^2} \quad (6.7) \]

The inverse transform is also circularly symmetrical when

\[ \hat{F}(k_x, k_y) = \hat{H}(k) \quad (6.8) \]

where

\[ k = \sqrt{k_x^2 + k_y^2} \quad (6.9) \]

To show the relationship between the two dimensional Fourier transform and the Hankel transform, Bracewell (1986), Davies (2002), and Miles (1971) changed the transform formula to polar coordinates where \( x = r \cos \theta, \ y = r \sin \theta, \ k_x = k \cos \alpha, \) and \( k_y = k \sin \alpha. \) The following derivation is similar to the mentioned references.

The exponential function becomes

\[
\exp[-i(xk_x + yk_y)] = \exp[-i(r \cos \theta k \cos \alpha + r \sin \theta k \sin \alpha)]
\]

\[
= \exp[-ikr(\cos \theta \cos \alpha + \sin \theta \sin \alpha)]
\]

\[
= \exp[-ikr \cos(\theta - \alpha)] \quad (6.10)
\]

Substituting into the two dimensional Fourier transform in polar coordinates gives

\[
\hat{F}(k, \alpha) = \frac{1}{\sqrt{2\pi}} \int_0^{2\pi} \int_0^\infty F(r, \theta) e^{-ikr \cos(\theta - \alpha)} r \, dr \, d\theta \quad (6.11)
\]

\[
F(r, \theta) = \frac{1}{\sqrt{2\pi}} \int_0^\infty \int_k^{2\pi} \hat{F}(k, \alpha) e^{-ikr \cos(\theta - \alpha)} k \, dk \, d\alpha \quad (6.12)
\]

Expanding \( F(r, \theta) \) as a complex Fourier series with an index of \( n \)

\[
F(r, \theta) = H(r)e^{int\theta} \quad (6.13)
\]

and then one can consider each term in the series. Substituting equation (6.13) into (6.11) gives
\[ \hat{F}(k, \alpha) = \frac{1}{2\pi} \int_0^\infty \int_0^{2\pi} H(r) e^{i(n\theta - kr \cos(\theta - \alpha))} r \, dr \, d\theta \]  
(6.14)

The following change of variable, \( \varphi = \theta - \alpha + \frac{1}{2} \pi \), results in

\[ \hat{F}(k, \alpha) = \frac{1}{2\pi} \int_0^\infty \int_{\phi_0}^{\phi_0 + \pi/2} H(r) e^{i(n(\alpha + \varphi - \pi/2) - kr \sin \varphi)} r \, dr \, d\varphi \]  
(6.15)

Rearranging the terms and defining \( \varphi_0 = \frac{\pi}{2} - \alpha \) gives

\[ \hat{F}(k, \alpha) = e^{i(n(\alpha - \pi/2))} \int_0^\infty H(r) \left[ \frac{1}{2\pi} \int_{\phi_0}^{\phi_0 + \pi/2} e^{i(n\varphi - kr \sin \varphi)} d\varphi \right] r \, dr \]  
(6.16)

As shown in Miles (1971), Watson (1945) defined the Bessel function of the first kind integral as

\[ J_n(kr) = \frac{1}{2\pi} \int_{\phi_0}^{\phi_0 + \pi/2} e^{i(n\varphi - kr \sin \varphi)} d\varphi \]

Using a zero order Bessel function sets \( n = 0 \) and the zero order Hankel transform can be defined as

\[ \hat{H}(k) = \int_0^\infty H(r) J_0(kr) \, r \, dr \]  
(6.17)

Following a similar derivation, the inverse can be found and is given by

\[ H(r) = \int_0^\infty \hat{H}(k) J_0(kr) \, k \, dk \]  
(6.18)

An alternate form of the Hankel transform was used by Oberhettinger (1972) and others.

This form is given by

\[ \hat{G}(k) = \int_0^\infty G(r) J_n(kr) (kr)^{1/2} dr \]  
(6.19)

\[ G(r) = \int_0^\infty \hat{G}(k) J_n(kr) (kr)^{1/2} dk \]  
(6.20)

Miles (1971) stated that this can be reconciled with equations (6.17) and (6.18) by setting
\[ G(r) = r^{1/2}H(r) \]
\[ \hat{G}(k) = k^{1/2}\hat{H}(k) \]

A summary of some useful zero order Hankel transform pairs are given in Table 6.3.

**TABLE 6.3**

**ZERO ORDER HANKEL TRANSFORM PAIRS**

<table>
<thead>
<tr>
<th>( H(r) = \int_0^\infty \hat{H}(k)J_0(\alpha r),k,dk )</th>
<th>( \hat{H}(k) = \frac{1}{\alpha^2} \hat{J}\left(\frac{k}{\alpha}\right) )</th>
</tr>
</thead>
<tbody>
<tr>
<td>similarity: ( H(\alpha r) )</td>
<td>( \frac{1}{\alpha^2} \hat{J}\left(\frac{k}{\alpha}\right) )</td>
</tr>
<tr>
<td>( \frac{1}{r} )</td>
<td>( \frac{1}{k} )</td>
</tr>
<tr>
<td>( e^{-\alpha}, \alpha &gt; 0 )</td>
<td>( \frac{\alpha}{(k^2 + \alpha^2)^{3/2}} )</td>
</tr>
<tr>
<td>( \frac{1}{r}e^{-\alpha}, \alpha &gt; 0 )</td>
<td>( \frac{1}{(k^2 + \alpha^2)^{1/2}} )</td>
</tr>
<tr>
<td>( re^{-\alpha}, \alpha &gt; 0 )</td>
<td>( \frac{3\alpha k}{(k^2 + \alpha^2)^{5/2}} )</td>
</tr>
<tr>
<td>( r^2e^{-\alpha}, \alpha &gt; 0 )</td>
<td>( \frac{15\alpha k^2}{(k^2 + \alpha^2)^{7/2}} )</td>
</tr>
<tr>
<td>( r^\nu e^{-\alpha}, \alpha &gt; 0, \nu &gt; -1 )</td>
<td>( \frac{2^{\nu+1}}{\sqrt{\pi}}\Gamma(\nu + \frac{3}{2}) \frac{\alpha k^\nu}{(k^2 + \alpha^2)^{\nu+3/2}} )</td>
</tr>
</tbody>
</table>

from: Oberhettinger (1972)

6.1.2 Features of the Normalized Wavenumber-Frequency Spectrum Models

In a review of wavenumber-frequency spectrum models, Tkachenko et al. (2006) found that many attempts were made to determine the wavenumber-frequency spectrum from experimental measurements. The first attempts were based on measuring the cross-spectrum and
then applying a Fourier transform. However, this did not yield good results, because it could provide reliable data for only the convective region of the wavenumber-frequency spectrum.

The main conclusion that Bull (1967) drew from his early experiments was that the wall pressure field has a structure produced by contributions from pressure sources in the boundary layer with a range of convective velocities and comprises two families of convected wavenumber components. One family is of high wavenumber components and is associated with turbulent motion in the constant stress layer. The components are longitudinally coherent for times proportional to the times taken for them to be convected along distances equal to their wavelengths. They are laterally coherent over distances proportional to their wavelengths. The other family comprises components of wavelength greater than about twice the boundary layer thickness, which lose coherence as a group, more or less independent of wavelength. The latter are associated with large scale eddy motion in the boundary layer, outside the constant stress layer. If the wall pressure field receives significant contributions from pressure sources in both regions of the boundary layer, the various functions associated with it will obviously show Reynolds number dependence when expressed in terms of either the outer or inner layer scales.

Work continued on the identification of the major regions of the wavenumber-frequency spectrum. Bull (1996) classified four characteristic regions: the supersonic or acoustic region, $k < \omega/c$, the subconvective region, $k \approx \omega/c$, the convective region, centered on $\omega/c < k < \omega/U_\epsilon$, and the viscous region, $k >> \omega/U_\epsilon$. The acoustic region is the very low wavenumber region. The subconvective region is known to be very important to underwater applications. Graham (1997) speculated that this region is not as important to aircraft applications. The convective region is where the turbulence is convected with the flow and most of the energy is located. The viscous region is where the very small scale turbulence occurs.
Figure 6.1 is a schematic of the characteristic regions of a wavenumber-frequency spectrum at a constant frequency. As described in Chapter 3, the $x$ dimension is defined in the flow direction and the $z$ dimension is defined in the cross flow direction.

According to Abraham and Keith (1998), the highest levels of energy occur along the convective ridge. The width of the ridge increases with frequency. Manoha (1996) also found that the most energetic region lies around the convective peak which occurs at $k = (k_c,0)$ and corresponds to small vortices convected at the velocity $U_c$ which is known to be slightly slower than the flow outside the boundary layer. The convective peak is much wider in the transverse direction compared to the flow direction.
A significant theoretical result for incompressible flow was derived early in the study of wall pressure fluctuations by Kraichnan (1956) and Phillips (1956). Bull (1996) explained that their analyses show that if the boundary layer turbulence is assumed to be of infinite extent and statistically stationary and homogeneous in planes, the wavenumber-frequency spectrum is zero at zero wavenumber irrespective of frequency. However, the Kraichnan-Phillips theorem must be modified as the Mach number increases from zero requiring the inclusion of compressibility effects. Similar to the discussion in Chapter 5 for the single point wall pressure spectrum models, Ffowcs Williams (1965) showed that the wavenumber-frequency spectral density is zero only if both $k$ and $\omega$ are zero. It does not approach zero as $k \to 0$ unless the frequency is zero. The wavenumber spectrum asymptotically approaches a non-zero low wavenumber limit as $k \to 0$. These results indicate significantly different constraints on the wall pressure spectrum than those implied by Kraichnan-Phillips theorem.

### 6.2 Examples of Models

Smol’yakov and Tkachenko (1991) stated that it is a well known fact that there is no analytically developed wavenumber-frequency spectrum model, so that experiments provide the main source of data on the field of turbulent wall pressures. This field can be regarded as statistically stationary and homogeneous in the majority of practical applications. Accordingly, all necessary and sufficient information about the field is contained in the space-time correlation functions and the related Fourier transforms of the cross spectral function.

According to Mellen (1990), models of convective turbulence have typically been empirically derived based on space-frequency correlation measurements of the pressure fluctuations along the longitudinal and transverse axes. From this space-frequency spectrum model, the wavenumber-frequency spectrum model can be found by Fourier transforms.
Previous experiments show that the flow can be modeled as random fluctuations convecting down the flow as if riding on a steady current. These results show that the mean longitudinal velocity is independent of frequency and that the amplitudes decay in both directions as the exponential of an effective decay wavenumber times the spatial separation. Mellen looked at both the separable Corcos model and a non-separable Elliptical model and found that the Elliptical model gave smoother results. Graham (1997) also compared the Corcos model to several empirically derived wavenumber-frequency spectrum models which show non-separable behavior.

A requirement for all of the normalized wavenumber-frequency spectrum models is that they must integrate to one. This limits the models discussed to Corcos, Mellen Elliptical, Chase 1, and Smol’yakov and Tkachenko. (Graham, 1997)

\[
\frac{1}{(2\pi)^2} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \tilde{\Phi}_{p'p'}(k_x, k_z, \omega) \left( \frac{U_c}{\omega} \right)^2 dk_x dk_z = 1.0
\]

In the next section, it will be shown that each of these models satisfies equation (6.23). To be consistent, it will be assumed that the models are normalized by

\[
\tilde{\Phi}_{p'p'}(k_x, k_z, \omega) = \frac{(2\pi)^2 \omega^2}{U_c^2 \Phi(\omega)} \Phi_{p'p'}(k_x, k_z, \omega)
\]

For this analysis, the Goody single point wall pressure spectrum model from Chapter 5 will be used to normalize the wavenumber-frequency spectrum models of this chapter.

The Corcos and Mellen Elliptical models can be considered convertible models, which allow one to pass from the wavenumber-frequency spectrum to the cross spectrum and back in an analytical way via Fourier transformation. The convertibility of the models is convenient for using different methods in calculating both structural vibrations and flow noise. The Chase 1 model can be converted from wavenumber-frequency spectrum to the cross spectrum using a
method described by Josserand (1986). The Smol’yakov and Tkachenko model does not appear to possess this convenient property.

In this analysis, the wavenumber-frequency spectrum models for both the Corcos and Elliptical models based on Mellen's analysis will be developed. Then these models will be normalized to conform to Graham's development so that they can be compared to the other models such as Chase 1 and Smol'yakov-Tkachenko also found in Graham (1997). Ultimately, these normalized wavenumber-frequency spectrum model will be compared to available data to determine which should be used for aircraft applications.

6.2.1 Corcos

Following the development in Mellen (1990), the Corcos model for pressure fluctuations
in the space-frequency domain is given by the exponential function

$$\Psi_{pp'}(x, z, \omega) = \Phi(\omega) \exp(ik_{\omega}x - |\alpha_{\omega}x| - |\beta_{\omega}z|)$$  \hspace{1cm} (6.25)

which can be written as

$$\Psi_{pp'}(x, z, \omega) = \Phi(\omega) \exp(ik_{\omega}x) \exp(-|\alpha_{\omega}x|) \exp(-|\beta_{\omega}z|)$$  \hspace{1cm} (6.26)

To find the pressure fluctuations in wavenumber-frequency domain, the Fourier transform of
equation (6.26) needs to be found. In this form, it is easy to see that function is separable so that
the Fourier transform can be performed on each axis independently as discussed above. Let

$$\Psi_{pp'} = \Phi(\omega)\Psi_1(x)\Psi_2(z) = \Phi(\omega)\exp(ik_{\omega}x)f(x)g(z)$$  \hspace{1cm} (6.27)

where

$$f(x) = e^{-\alpha_{\omega}|x|}$$

$$g(z) = e^{-\beta_{\omega}|z|}$$

The Fourier transform of the $\Psi_1(x)$ term first requires a shift transform.

$$\Psi_1(x) = \exp(ik_{\omega}x)f(x) \rightarrow \hat{\Psi}_1(k_x - k_{\omega})$$  \hspace{1cm} (6.28)
The second step of transform makes use of an equation from Table 6.1.

\[ f(x) = e^{-\alpha|x|} \rightarrow \hat{f}(k) = \frac{2}{\sqrt{\pi}} \frac{\alpha}{\alpha^2 + k^2} \]  

(6.29)

This gives

\[ \Psi_1(x) = \exp(ik_\omega x)e^{-\alpha_\omega |x|} \rightarrow \hat{\Psi}_1(k_z - k_\omega) = \frac{2}{\sqrt{\pi}} \frac{\alpha_\omega}{\alpha_\omega^2 + (k_z - k_\omega)^2} \]  

(6.30)

The second term, \( \Psi_2(z) \), only requires the last step.

\[ \Psi_2(z) = g(z) = e^{-\beta_\omega z} \rightarrow \hat{\Psi}_2 = \hat{g}(k_z) = \frac{2}{\sqrt{\pi}} \frac{\beta_\omega}{\beta_\omega^2 + k_z^2} \]  

(6.31)

The Corcos wavenumber-frequency spectrum model of the pressure fluctuations is therefore given by

\[ \hat{\Psi}_{pp}(k_x, k_z, \omega) = \hat{\Psi}_1(k_z)\hat{\Psi}_2(k_z)\Phi(\omega) = \frac{2\Phi(\omega)}{\pi} \frac{\alpha_\omega \beta_\omega}{(\alpha_\omega^2 + (k_z - k_\omega)^2) (\beta_\omega^2 + k_z^2)} \]  

(6.32)

To normalize this equation to one, it must be divided by the area. Integrating this equation over \( k_x \) and \( k_z \) gives \( 2\pi \). This gives the equation found in Mellen (1990).

\[ \text{Area} = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \frac{2\Phi(\omega)}{\pi} \frac{\alpha_\omega \beta_\omega}{(\alpha_\omega^2 + (k_z - k_\omega)^2) (\beta_\omega^2 + k_z^2)} dk_x dk_z = 2\pi \]  

(6.33)

\[ \hat{\Psi}_{pp}(k_x, k_z, \omega) = \frac{\Phi(\omega)}{\pi^2} \frac{\alpha_\omega \beta_\omega}{(\alpha_\omega^2 + (k_z - k_\omega)^2) (\beta_\omega^2 + k_z^2)} \]  

(6.34)

To put the Corcos model equation into the form found in Graham (1997), it must be multiplied by his normalizing factor.

\[ \tilde{\Phi}_{pp}(k_x, k_z, \omega) = \frac{\Phi(\omega)}{\pi^2} \frac{(2\pi \alpha_\omega^2 \omega^2)}{U_z^2 \Phi(\omega)} \]  

(6.35)

It should be noted that Mellen (1990) defines his coefficients as
\[ \alpha_{\omega} = k_{\omega} \alpha_x = \frac{\omega}{U_c} \alpha_x \]  
(6.36)

\[ \beta_{\omega} = k_{\omega} \alpha_z = \frac{\omega}{U_c} \alpha_z \]  
(6.37)

These substitutions give the equation found in Graham (1997)

\[ \Phi_{pp}(k_x, k_z, \omega) = \frac{4 \alpha_x \alpha_z}{\left[ \alpha_x^2 + \left( \frac{U_x k_z}{\omega} \right)^2 \right] \left[ \alpha_z^2 + \left( \frac{U_z k_z}{\omega} - 1 \right)^2 \right]} \]  
(6.38)

Graham defines the values for the coefficients as \( \alpha_x = 0.10 \) and \( \alpha_z = 0.77 \). These are the coefficients that will be used in this analysis. Figure 6.2 shows a surface and contour plot of the Corcos model for 15 m/s at 3000 Hz. The separable form causes the shoulders seen in the graphs. Graham (1991) notes that the Corcos model overpredicts the spectrum at low wavenumbers, which could be explained by these shoulders.

![Figure 6.2: Corcos model surface and contour plots for 15 m/s at 3000 Hz.](image)

### 6.2.2 Mellen Elliptical

Mellen's Elliptical model for pressure fluctuations in the space-frequency domain is given by the exponential function
\[ \Psi_{p'p}(x,z,\omega) = \Phi(\omega) \exp[i k_{\omega} x - \sqrt{(\alpha_{\omega} x)^2 + (\beta_{\omega} z)^2}] \] (6.39)

which can be written as

\[ \Psi_{p'p}(x,z,\omega) = \Phi(\omega) \exp(i k_{\omega} x) \exp[-(\alpha_{\omega} x)^2 + (\beta_{\omega} z)^2] \] (6.40)

In this form, it is easy to see that this equation is not separable. The first step in finding the Fourier transform of this equation is to perform a shift transform similar to what was done to the Corcos model. However, the Elliptical model requires the use of the two dimensional Fourier transform from Table 6.2 with the \( y \) coefficient equal to zero.

\[ \exp(i k_{\omega} x) F(x,z) \rightarrow \hat{F}(k_x - k_{\omega}, k_z) \] (6.41)

The next step is to use the similarity transform also from Table 6.2.

\[ F(\alpha x, \beta z) \rightarrow \frac{1}{|\alpha \beta|} \hat{F} \left( \frac{k_x}{\alpha}, \frac{k_z}{\beta} \right) \] (6.42)

Since \( \alpha_{\omega} > 0 \) and \( \beta_{\omega} > 0 \), the transform equation is simplified to

\[ \Psi_{p'p}(\alpha_{\omega} x, \beta_{\omega} z) \rightarrow \frac{1}{\alpha_{\omega} \beta_{\omega}} \hat{\Psi}_{p'p} \left( \frac{k_x - k_{\omega}}{\alpha_{\omega}}, \frac{k_z}{\beta_{\omega}} \right) \] (6.43)

With these simplifications, a Hankel transform from Table 6.3 can be used

\[ e^{-\sigma_x} \rightarrow \frac{\alpha}{(k^2 + \alpha^2)^{3/2}} \] (6.44)

where

\[ k = \sqrt{\left( \frac{k_x - k_{\omega}}{\alpha_{\omega}} \right)^2 + \left( \frac{k_z}{\beta_{\omega}} \right)^2} \] (6.45)

\[ \alpha = 1 \] (6.46)

The transform becomes
\[ \hat{\Psi}_{p'p}(k_x, k_z, \omega) = \frac{\Phi(\omega)}{\alpha_{\omega} \beta_{\omega}} \frac{1}{\sqrt{\left(\frac{k_x - k_{\omega}}{\alpha_{\omega}}\right)^2 + \left(\frac{k_z}{\beta_{\omega}}\right)^2 + 1}} \]  

(6.47)

After a few algebraic steps, this equation becomes

\[ \hat{\Psi}_{p'p}(k_x, k_z, \omega) = \Phi(\omega) \frac{(\alpha_{\omega} \beta_{\omega})^2}{[\beta_{\omega}^2 (k_x - k_{\omega})^2 + \alpha_{\omega}^2 k_z^2 + \alpha_{\omega}^2 \beta_{\omega}^2]^{3/2}} \]  

(6.48)

To normalize this equation to one, it is divided by the area. Integrating this equation over \( k_x \) and \( k_z \) gives \( 2\pi \). This gives the equation found in Mellen (1990).

\[ Area = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \Phi(\omega) \frac{(\alpha_{\omega} \beta_{\omega})^2}{[\beta_{\omega}^2 (k_x - k_{\omega})^2 + \alpha_{\omega}^2 k_z^2 + \alpha_{\omega}^2 \beta_{\omega}^2]^{3/2}} dk_x dk_z = 2\pi \]  

(6.49)

\[ \hat{\Psi}_{p'p}(k_x, k_z, \omega) = \frac{\Phi(\omega)}{2\pi} \frac{(\alpha_{\omega} \beta_{\omega})^2}{[\beta_{\omega}^2 (k_x - k_{\omega})^2 + \alpha_{\omega}^2 k_z^2 + \alpha_{\omega}^2 \beta_{\omega}^2]^{3/2}} \]  

(6.50)

To put the Elliptical model equation into the form which can be compared to other models in Graham (1997), the equation must be multiplied by his normalizing factor.

\[ \tilde{\Phi}_{p'p}(k_x, k_z, \omega) = \hat{\Psi}_{p'p}(k_x, k_z, \omega) \frac{(2\pi)^2 \omega^2}{U_z^2 \Phi(\omega)} \]  

(6.51)

And using Mellen's definition of the coefficients

\[ \alpha_{\omega} = k_{\omega} \alpha_x \]  

(6.52)

\[ \beta_{\omega} = k_{\omega} \alpha_z \]  

(6.53)

These substitutions give the Elliptical equation found in Miller and Moeller (2009).

\[ \tilde{\Phi}_{p'p}(k_x, k_z, \omega) = \frac{2\pi (\alpha_x \alpha_z)^2 k_{\omega}^3}{[\alpha_x \alpha_z k_{\omega}^2 + (\alpha_x k_z)^2 + \alpha_z^2 (k_x - k_{\omega})^2]^{3/2}} \]  

(6.54)

The values for the coefficients as \( \alpha_x = 0.10 \) and \( \alpha_z = 0.77 \) as defined by Graham will be used again. Figure 6.3 shows a surface and contour plot of the Elliptical model for 15 m/s at 3000 Hz.
The non-separable form does not have the shoulders like the Corcos model and the local gradient always points to the convective peak.

\[ \Phi_{pp}(k_x, k_z, \omega) = \frac{(2\pi)^3 \rho^2 \omega^2 U_T^2}{U_c^2 \phi(\omega)} \left[ \frac{C_M k_x^2}{[K_+^2 + (b_T \delta)^{-2}]^{5/2}} + \frac{C_r |k|^2}{[K_+^2 + (b_T \delta)^{-2}]^{5/2}} \right] \]  

(6.55)

Figure 6.3: Elliptical model surface and contour plots for 15 m/s at 3000 Hz.

6.2.3 Chase 1

Chase (1980) developed a model that describes the wavenumber-frequency spectrum over a large part of the domain instead of focusing on the convective ridge region like the Corcos model. He started with the solution of the Poisson equation and transformed it by the introduction of a wall normal component of wavenumber with geometric mean wall distance as a parameter. From there Chase employed a number of heuristic arguments in his derivation which allowed for a number of adjustable constants. In this analysis, Graham’s (1997) form of Chase’s most popular normalized wavenumber-frequency model will be used. It was found that the Chase 1 model can be transformed from the wavenumber-frequency domain to the space-time domain using a method described by Josserand (1986). It should also be noted that there are later models from Chase such as Chase 2 also described by Graham (1997). However, this model does not integrate to 1.0 and was not investigated more in this work.
where

\[ K_x^2 = \frac{(\omega - U_c k_x)^2}{k^2 h^2 U_c^2} + |k|^2 \]  \hspace{1cm} (6.56)

\[ \phi(\omega) = \frac{(2\pi)^2 h \rho^2 U_c^4}{3\omega(1 + \mu^2)} (C_M F_M + C_T F_T) \]  \hspace{1cm} (6.57)

\[ F_M = \frac{1 + \mu^2 \alpha_M^2 + \mu^4 (\alpha_M^2 - 1)}{[\alpha_M^2 + \mu^2 (\alpha_M^2 - 1)]^{3/2}} \]  \hspace{1cm} (6.58)

\[ F_T = \frac{1 + \alpha_T^2 + \mu^2 (3\alpha_T^2 - 1) + 2\mu^4 (\alpha_T^2 - 1)}{[\alpha_T^2 + \mu^2 (\alpha_T^2 - 1)]^{3/2}} \]  \hspace{1cm} (6.59)

\[ \alpha_M^2 = 1 + \left( \frac{U_c}{b_M \omega \delta} \right)^2 \]  \hspace{1cm} (6.60)

\[ \alpha_T^2 = 1 + \left( \frac{U_c}{b_T \omega \delta} \right)^2 \]  \hspace{1cm} (6.61)

\[ \mu = \frac{h U_c}{U_c} \]  \hspace{1cm} (6.62)

\[ C_M = 0.0745 \]  \hspace{1cm} (6.63)

\[ C_T = 0.0475 \]  \hspace{1cm} (6.64)

\[ b_M = 0.756 \]  \hspace{1cm} (6.65)

\[ b_T = 0.378 \]  \hspace{1cm} (6.66)

\[ h = 3.0 \]  \hspace{1cm} (6.67)

Figure 6.4 shows a surface and contour plot of the Chase 1 model for 15 m/s at 3000 Hz.

This graph clearly shows the discontinuity at \( k_x = 0 \) and \( k_z = 0 \) which results in lower levels in the acoustic domain.
6.2.4 Smol'yakov-Tkachenko

Graham (1997) summarized the development of the Smol’yakov and Tkachenko model. He stated that they based their model on measured spatial pressure correlations as a function of spatial separation and boundary layer thickness and fitted exponential curves to their experimental results. Rather than using distinct lateral and longitudinal separations like the Corcos model, they used a combined correlation similar to the Elliptical model and used Fourier transforms from there. They found that the low wavenumber levels improved over the Corcos prediction but were still higher than their experimental data. Therefore, they added a correction factor to bring the levels down to the experimental values without affecting the convective peak. Smol’yakov and Tkachenko (1991) stated that they deliberately ignored the Kraichnan-Phillips condition because it was not supported by the existing experimental data.

\[
\tilde{\Phi}_{\rho'\rho'}(k_x, k_z, \omega) = 0.974 A(\omega) h(\omega)[F(k_x, \omega) - \Delta F(k_x, \omega)]
\]  

(6.68)

where

\[
A(\omega) = 0.124 \left[ 1 - \frac{U_c}{4\omega\delta^*} + \left( \frac{U_c}{4\omega\delta^*} \right)^2 \right]^{1/2}
\]

(6.69)
\[ h(\omega) = \left[ 1 - \frac{m_1 A}{6.515 \sqrt{G}} \right]^{-1} \]  

(6.70)

\[ m_1 = \frac{1 + A^2}{1.025 + A^2} \]  

(6.71)

\[ G = 1 + A^2 - 1.005 m_1 \]  

(6.72)

\[ F(k, \omega) = \left[ A^2 + \left(1 - \frac{k U_c}{\omega} \right)^2 + \left( \frac{k U_c}{6.45 \omega} \right)^2 \right]^{-3/2} \]  

(6.73)

\[ \Delta F(k, \omega) = 0.995 \left[ 1 + A^2 + \frac{1.005}{m_1} \left( m_1 - \frac{k U_c}{\omega} \right)^2 + \left( \frac{k U_c}{6.45 \omega} \right)^2 - m_1^2 \right]^{-3/2} \]  

(6.74)

Figure 6.5 shows surface and contour plot for 15 m/s at 3000 Hz. This graph clearly shows the similarity between the Elliptical model and the Smol’yakov-Tkachenko. The main difference is the correction factor that lowers the values in the acoustic region.

Figure 6.5: Smol’yakov-Tkachenko model surface and contour plots for 15 m/s at 3000 Hz.

6.3 Summary of Models

As was shown in the previous section, the separable Corcos model results in an anomalous behavior because of the overly restricted model. The features that are non-
representative are the ‘shoulders’ on the normalized wavenumber-frequency spectrum which are a result of the separable construct of the mathematical model. This can be seen in Figure 6.6 where the Corcos model is compared to the Elliptical model at 15 m/s at 3000 Hz in a surface plot. The effect of the Corcos shoulders can be more easily seen in Figures 6.7, 6.8 and 6.9 which are cuts through the surface plot at \( k_z = 0, 1000, \) and 2000 respectively.

Figure 6.6: Corcos and Elliptical model surface plots for 15 m/s at 3000 Hz.
Figure 6.7: Corcos and Elliptical models at 15 m/s, 3000 Hz, and $k_z = 0$.

Figure 6.8: Corcos and Elliptical models at 15 m/s, 3000 Hz, and $k_z = 1000$. 
Figure 6.9: Corcos and Elliptical models at 15 m/s, 3000 Hz, and $k_z = 2000$.

All four normalized wavenumber-frequency spectrum models are compared in Figure 6.10 at a frequency of 3122 Hz, a Strouhal number of 248, and $k_z = 0$. The Strouhal number is defined as $Sh = \omega \delta / U_z$. The Elliptical model has a narrower peak but has a similar form to the Corcos model due to the combination of the longitudinal and lateral spatial separation. The Smol’yakov-Tkachenko model shows a similar more narrow peak as compared to the Elliptical model; however, it also has a much smaller value at low wavenumbers due to the correction factor $\Delta F$. The Chase 1 model also results in a narrower peak but it is the only model that satisfies the Kraichnan-Phillips theorem.
6.4 Comparison to Experimental Data

The four models discussed in the previous section will now be compared to experimental data collected by Martini et al. (1984) in the low noise, low turbulence wind tunnel using the equipment of the M.I.T. Acoustics and Vibration Laboratory. The wind tunnel configuration is discussed in more detail in Appendix B. The data used in this analysis was taken at 15 m/s because it had the minimum convective noise contamination. The test conditions used in this analysis are summarized in Table 6.5.
TABLE 6.5
M.I.T. TEST CONDITIONS

<table>
<thead>
<tr>
<th></th>
<th>$U_x = 15.0 \text{ m/s}$</th>
</tr>
</thead>
<tbody>
<tr>
<td>Mach number</td>
<td>0.044</td>
</tr>
<tr>
<td>density (kg/m$^3$)</td>
<td>1.2250</td>
</tr>
<tr>
<td>convective velocity, (m/s)</td>
<td>10.5</td>
</tr>
<tr>
<td>friction velocity, (m/s)</td>
<td>0.55</td>
</tr>
<tr>
<td>boundary layer thickness, (m)</td>
<td>0.043688</td>
</tr>
<tr>
<td>displacement thickness, (m)</td>
<td>0.005461</td>
</tr>
</tbody>
</table>

Two different microphones were used to measure the wavenumber spectrum: a pinhole microphone and a flush mounted B&K microphone. The wavenumber filter shape of a pinhole microphone is described by a piston type transducer. Lueptow (1995) gives the equation for the filter shape of a piston type transducer with radius $r_0$ as

$$H(k) = \frac{2J_1(kr_0)}{kr_0}$$  \hspace{1cm} (6.75)

where

$$k = \sqrt{k_x^2 + k_z^2}$$  \hspace{1cm} (6.76)

$J_1 = \text{Bessel function of order one}$

A pinhole microphone was characterized as a 1/8 inch microphone with a 1/32 inch pinhole cap and is assumed to be uniformly responding across the surface. The pinhole microphone accepts most of the energy and the sensitivity does not drop off like a bigger microphone.
The wavenumber filter shape of a flush mounted B&K microphone is described by a circular deflection type transducer. Lueptow (1995) gave the equation for the filter shape of a circular deflection type transducer with radius \( r_0 \) as

\[
H(k) = \frac{a^2 J_0(kr_0)}{a^2 - (kr_0)^2}
\]  

(6.77)

where

\[ a = 2.4048255604 \ 3883 \]  

(6.78)

\( J_0 = \) Bessel function of order zero

The value of \( a \) is defined as the location of the first zero the Bessel function. The one inch microphone is characterized as an ideal displacement transducer with a 0.36 inch radius where the sensitivity is not uniform across the surface. Figure 6.11 shows the filter shape for the one inch flush mounted B&K microphone.

![Figure 6.11: The wavenumber filter shape for one inch, flush mounted B&K microphone, \( |H(k_x, k_z)|^2 \).](image-url)
The measured microphone response is a function of the turbulent boundary layer wall pressure spectrum and the wavenumber filter shape integrated over all the wavenumbers.

\[
\Phi_M(\omega) = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \Phi_{pp}(k_x, k_z) |H(k_x, k_z)|^2 dk_x dk_z
\]  \hspace{1cm} (6.79)

In Figure 6.12, the Goody single point wall pressure spectrum estimate is compared to MIT data which used a pinhole microphone and a one inch B&K microphone. Since the comparison is made with the experimental values, Goody’s experimental spectral density formula, which was first derived in Chapter 5, is also used here.

\[
W(f) = \frac{6.0\pi(\delta/U_\infty)^3 (2\pi f \tau_w)^2}{[(2\pi f \delta/U_\infty)^{0.75} + 0.5]^3 + [(0.1 \cdot R_T^{0.57})(2\pi f \delta/U_\infty)]^7}
\]  \hspace{1cm} (6.80)

where

\[
R_T = \frac{U_f^2 \delta}{U_\infty V}
\]  \hspace{1cm} (6.81)

As expected, the Goody model closely matches the MIT-Martini pinhole microphone data.
Comparisons of the predicted pinhole microphone response and the predicted one inch B&K flush mounted microphone response for Corcos, Chase 1, Mellen Elliptical and Smol' yakov-Tkachenko models to the MIT-Martini measured data are given in Figures 6.13 and 6.14. The predicted results from the pinhole microphone are all similar. However, the predicted results from the one inch microphone show distinct differences. At frequencies up to approximately 1000 Hz, all the wavenumber-frequency spectrum models appear to have similar results. However at frequencies above 1000 Hz, this graph shows that the both the Corcos and Mellen Elliptical wavenumber-frequency models are incompatible with the data. In fact, the separable Corcos model overpredicts the microphone response by almost 20 dB at 2000 Hz. Above 2500 Hz, the measurement data is contaminated by facility acoustic noise and will be
ignored. From this analysis, both the Chase 1 and Smol’yakov-Tkachenko models appear to match the Martini data throughout the range of comparison.

Figure 6.13: Comparison of models to MIT-Martini pinhole microphone measurements.

Figure 6.14: Comparison of models to MIT-Martini one inch B&K microphone measurements.
Figure 6.15 shows a comparison of the Corcos wavenumber-frequency spectrum model to the wavenumber filter shape for the one inch flush mounted B&K microphone. This graph shows that the shoulder of the Corcos model occurs directly over the most sensitive part of the B&K microphone. This could explain the over-prediction of the Corcos model.

Figure 6.15: Corcos wavenumber-frequency spectrum model compared to the wavenumber filter shape for the B&K microphone.

This analysis shows that the Corcos model over-predicts the microphone response above 1000 Hz and it is postulated that the overprediction is due to the shoulders seen in the surface plot. The non-separable Mellen Elliptical model performs slightly better than the separable Corcos model but also does not agree well with the Martini data set. However, both the non-separable Chase 1 and the Smol' yakov-Tkachenko models seem to match the Martini data set. But the Chase 1 model can be converted from wavenumber-frequency spectrum to the cross spectrum, so it is the preferred model for aircraft applications.
6.5 Summary

The normalized wavenumber-frequency spectrum models are used by the acoustic analysis programs to define the wavenumber distribution of the turbulent boundary layer pressure fluctuations. It is important to the acoustic models that wavenumber-frequency spectra can be transformed from the space-time domain to the wavenumber-frequency and back again easily. Many normalized wavenumber-frequency models exist. They can be classified as either separable or non-separable.

In this work, the separable model of Corcos and the non-separable models of Mellen Elliptical, Chase 1 and the Smol’yakov-Tkachenko were compared. The separable Corcos model over-predicts the Martini microphone response above 1000 Hz and it is postulated that this is due to the shoulders which can be easily seen in the surface plots. Both the non-separable Chase 1 and Smol’yakov-Tkachenko models appear to match the Martini data throughout the range of comparison. The Smol’yakov-Tkachenko model does not lend itself to straightforward Fourier transforms needed by the acoustic models. But the Chase 1 model can be converted from wavenumber-frequency spectrum to the cross spectrum, so it is the preferred model for aircraft applications.

With the conclusion of this chapter, three different types of turbulent boundary layer models required for acoustic analysis have been reviewed and analyzed. In Chapter 7, the results from this analysis will be summarized and final conclusions will be determined.
CHAPTER 7
SUMMARY AND CONCLUSIONS

7.1 Summary

The purpose of this project was to develop improved turbulent boundary layer models which can be incorporated into the SEA and DEA acoustic analysis models in order to obtain more accurate interior noise predictions early in the design process. This study consisted of the definition of the important characteristics of a turbulent boundary layer. A review and analysis of the following models was performed: mean square pressure, single point wall pressure spectrum, and normalized wavenumber-frequency spectrum.

7.1.1 Modeling the Turbulent Boundary Layers

It was shown that the structure of a turbulent boundary layer is important in the analysis of wall pressure fluctuations leading to acoustic emissions. These pressure fluctuations can be considered to be a random process, meaning they can be described statistically. Some of these descriptions include space-time statistics which are called the autocorrelation function and the cross correlation function. The Fourier transform of these functions gives the auto spectral density and cross spectral density. The mean square pressure is a measure of the total energy due to the pressure fluctuations beneath a turbulent boundary layer. The single point wall pressure spectrum model sorts the energy into frequencies at a single point and is based on the auto spectral density. The normalized wavenumber-frequency spectrum model sorts the energy into wavelengths which gives the spatial distribution and is based on the cross spectral density.

Measurements of the wall pressure fluctuations are made with microphones. The spatial extent of the microphone causes errors in the measurements. The size of the transducer must be small enough to resolve the fine structure of the turbulent flow. The errors induced by the size of
the microphone can be characterized by the wavenumber filter shape of the microphone. A theoretical microphone wavenumber filter shape was used to estimate the attenuation of the true signal.

7.1.2 Mean Square Pressure Models

The mean square pressure value gives an estimate of the overall energy of the wall pressure fluctuations in a turbulent boundary layer. It also serves as a simple check for a single point wall pressure spectrum model by integrating over the frequency. The pressure fluctuations beneath a turbulent boundary layer can be found by solving the Poisson equation which is derived from the Navier-Stokes equation. In the present analysis, the Poisson equation was solved numerically using a point Gauss-Seidel method and analytically using an eigenvalue expansion method. These models were solved using the velocity fluctuations and the boundary conditions from LES/DES solutions performed at WSU for $M = 0.3$. Using the point Gauss-Seidel method with the Stokes pressure set to zero resulted in wall pressure fluctuation values that were considerably below the predicted values. Using the point Gauss-Seidel method with the Stokes pressure specified resulted in reasonable RMS wall pressure fluctuation values. The RMS wall pressure fluctuation values were also calculated using the eigenvalue expansion method with the Stokes pressure set to zero and at specified values. Setting the Stokes pressure to zero resulted in wall pressure fluctuation values that were higher than the LES/DES prediction. Using the Stokes pressure values derived from the LES/DES data resulted in wall pressure fluctuation values much lower than the LES/DES prediction.

There are many empirical models for the mean square pressure value based either on the wall shear stress, $\tau_w$, or the dynamic pressure, $q$. These empirical models gave a wide range of predicted mean square pressure values and it was not clear which model was more accurate.
Preliminary data was obtained in the Spirit AeroSystems 6x6 duct using a Kulite array at Mach numbers between 0.1 and 0.6 using an updated inlet designed for this purpose. The velocity profile of boundary layer indicated that the flow was more representative of an equilibrium turbulent boundary layer than the profile found previously.

Two significant sources of error to the means square pressure measurements were identified: 1) structural vibration in the duct and equipment, and 2) valve noise convected with the flow. A one-inch thick mounting plate was used to successfully address the vibration problem. A simple data analysis technique was developed to effectively remove the convected sound from the wall pressure fluctuation data. The experimental data showed the mean square pressure to fall within the range of the empirical models at low Mach numbers but was lower than predicted at higher Mach numbers. By reducing the sample rate to 10 KHz, the anti-aliasing filter attenuated the overall mean square pressure values by dropping the high frequency signals which were especially noticeable at higher Mach numbers. From this analysis, it was difficult to draw any meaningful conclusions on which mean square pressure model was the best.

7.1.3 Single Point Wall Pressure Spectrum Models

A single point wall pressure spectrum model was used to estimate the frequency response of a panel. The mean square pressure was found by integrating the single point wall pressure spectrum over all the frequencies. An analytically developed single point wall pressure spectrum model is not possible due to the complex nature of the flow field. Therefore, typical models have been empirically developed using both high speed and low speed experimental data. The quality of data available in the literature is not consistent resulting in significant scatter. Inner, mixed, and outer boundary layer parameters have been used for single point wall pressure model generation. The more recent models have been developed using a mixture of scaling variables.
Three frequencies regions have been identified; the low, overlap, and high ranges, the location of each depending on whether the inner or outer scaling variables are being used. Various authors have characterized the single point wall pressure spectrum shape based on the frequency ranges. In general, the slope in the low frequency range is predicted to be approximately $\omega^2$. In the overlap region, the slope is predicted to be approximately $\omega^{-1}$. In the high region, the slope is predicted to be either $\omega^{-7/3}$ or $\omega^{-5}$.

Seven models were compared using the Spirit 6x6 operating conditions for comparison purposes; Robertson, Efimstov 1, Efimstov2, Rackl and Weston, Chase-Howe, Goody and Smol’yakov. The models of Robertson, Robertson, Efimstov 1, Efimstov2, and Rackl and Weston were developed for aircraft applications based on high speed data. The models of Chase-Howe, Goody and Smol’yakov were developed primarily for marine applications and were based on low speed, incompressible data. These models were then integrated so they could be compared to the mean square pressure models of Chapter 4 resulting in a significant spread in the values. The models of Efimtsov 1, Goody, and Smol’yakov seemed to have more reasonable values. Based on the slope predictions discussed previously, only the incompressible Goody and Smol’yakov models seemed to be valid. Based on their mathematical simplicity and preliminary analysis, the models of Efimtsov 1 and Goody were compared to the experimental data.

Experimental spectral density measurements were made in the Spirit 6x6 duct with the updated inlet at Mach numbers between 0.1 and 0.6. Little Mach number dependence was noted. The single point wall pressure spectrums were developed and it was found that the mixed variables did the best job of scaling. The coherence of the signal was examined at three locations and was found to be as expected. The single point wall pressure spectrums were compared to the Efimtsov 1 and Goody models. The Spirit 6x6 data fell between the two models. The spectrum
at low frequencies rolled off similar to the Goody model. This analysis indicated that the Goody model is the appropriate single point wall pressure spectrum model for aircraft applications.

7.1.4 Normalized Wavenumber-Frequency Spectrum Models

The normalized wavenumber-frequency spectrum models are used by the acoustic analysis programs to define the wavenumber distribution of the turbulent boundary layer pressure fluctuations. It is important for acoustic analysis that these models can be transformed from the space-time domain to the wavenumber-frequency and back again easily.

Many normalized wavenumber-frequency models exist and they can be classified as either separable or non-separable. In the present work, the separable model of Corcos and the non-separable models of Mellen Elliptical, Chase 1 and the Smol'yakov-Tkachenko were compared. The separable Corcos model overpredicted the Martini microphone response above 1000 Hz and it was postulated that this was due to the shoulders which can be easily seen in its surface plots. Both the non-separable Chase 1 and Smol’yakov-Tkachenko models appeared to match the Martini data throughout the range of comparison. The Smol’yakov-Tkachenko model did not lend itself to straight forward Fourier transforms needed by the acoustic models. But the Chase 1 model could be converted from wavenumber-frequency spectrum to the cross spectrum, so it was the preferred model for aircraft applications.

7.2 Conclusions

During the preliminary data analysis, three sources of error were identified. The first issue was that it appeared from the velocity profiles that there was not an equilibrium turbulent boundary layer in the Spirit 6x6 duct. It was postulated that this was due to instabilities in the flow introduced by the previous inlet. A new inlet was designed and fabricated which allows for a turbulent boundary layer to form that is closer to equilibrium in nature than before.
Two additional, significant sources of error to the means square pressure measurements were identified; structural vibration in the duct and equipment and valve noise convected with the flow. A new microphone mounting method was developed to address the structural vibration error using a one-inch Plexiglas panel instead of the 1/4\textsuperscript{th} inch panel that was used in the baseline measurements. A simple data analysis technique was developed to effectively remove the convected sound from the wall pressure fluctuation data.

For the mean square pressure models, the empirical models were compared to new data from the Spirit AeroSystems 6x6 duct. From this comparison, it was difficult to draw any meaningful conclusions on which mean square pressure model was the best. For the single point wall pressure spectrum models, the existing models were compared to the new data from the Spirit AeroSystems 6x6 duct. This analysis indicated that the Goody model was the appropriate single point wall pressure spectrum model for aircraft applications. For the normalized wavenumber-frequency spectrum models, the empirical models were compared to the MIT-Martini data and the requirement for a non-separable model was confirmed. At the completion of this effort, the Chase 1 model was the preferred model for aircraft applications.
REFERENCES
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MIT EXPERIMENTAL SETUP

This analysis will make use of two different sets of experimental data. The first set is from an MIT research project carried out under the Office of Naval Research, ACSAS Project under Contract N00014-82-K-0728. The data used from this project will be referred to by the name of the principal investigator of the project, Martini.

The experiments were conducted in the M.I.T. low noise, low turbulence wind tunnel using the equipment of the M.I.T. Acoustics and Vibration Laboratory. The wind tunnel is shown in Figure A.1. The wind tunnel was an open circuit wind tunnel and the boundary layer was allowed to form naturally on the bottom wall of the wind tunnel. The tunnel consisted of an intake, a flow straightening section, a test section enclosed in an air tight concrete blockhouse, a muffler diffuser, and a variable speed centrifugal blower. The blockhouse had a semi-anechoic treatment that consisted of a 4 inch blanket of urethane foam covering the walls, floor and ceiling of the blockhouse. A set of 4 inch foam blocks was then draped at random on the walls, floor, and ceiling, with a set of 2 inch foam blocks draped randomly over the 4 inch blocks. (Martini et al., 1984)
For the data used in this work, the wind tunnel was operated in the free jet mode at 15 m/s. This test configuration is shown in Figure A.2. The free jet mode was used to isolate the wavenumber filters in the test sections from noise propagating upstream from the blower. The isolation was provided by allowing the noise to propagate into the blockhouse and be absorbed by the semi-anechoic treatment. (Martini, 1984) The background noise level in the blockhouse was determined using B&K 4144 free standing microphones. The background noise was found to be comparable to those measured by Martin (1976) for the same test configuration. The acoustic levels at the wavenumber filter location were more than 10 dB louder than the blockhouse background noise levels. (Miller and Moeller, 2009) There were several measurement methods used in this study: a clamped plate, a pinhole microphone, a 6-element microphone array, 12-element collinear array, a 12-element staggered array, and a 12-element cross array.
Figure A.2: Hard-walled duct configuration. (Martini et al., 1984)