CARLEMAN ESTIMATES FOR THE GENERAL SECOND ORDER OPERATORS AND APPLICATIONS TO INVERSE PROBLEMS

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DEDICATION

To my husband Sung H.

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ABSTRACT

We derive Carleman estimates with two large parameters for a general partial differential operator of second order under explicit sufficient global conditions of pseudo-convexity on the weight function. We use these estimates to derive the most natural Carleman type estimates for the anisotropic system of elasticity with residual stress. Also, we give applications to uniqueness and stability of the continuation, observability, and identification of the residual stress from boundary measurements.

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CHAPTER 1

INTRODUCTION

There are many results on uniqueness and stability of solutions of the Cauchy problem for general partial differential equations. Carleman type estimates are basic and powerful tools for proofs of uniqueness in the Cauchy problem. Carleman estimates were introduced by the Swedish mathematician Torsio Carleman in 1939. He tried to extend the classical Holmgren uniqueness theorem for the differential operator with nonanalytic coefficients. So he demonstrated the uniqueness results in the Cauchy problem for a two-dimensional elliptic partial differential equation with nonanalytic coefficients. His idea turned out to be very fruitful and until now it has dominated the field. In 1950-80s Carleman type estimates and uniqueness of continuation theorems have been obtained for wide classes of partial differential equations including general elliptic and parabolic equations of second order as well as some hyperbolic equations of second order. For accounts on these results we refer to books [14], [19]. While still there are challenges for scalar partial differential operators, in many cases results are quite complete. The situation with systems is quite different. A useful concept of pseudo-convexity is not available for systems, and Carleman estimates are at present obtained only in particular cases. A general result by Calderón in 1958 is applicable only to some elliptic systems of first order. Only recently was there progress for classical isotropic dynamical Maxwell's and elasticity systems [13]. This progress was achieved by using principal diagonalization of these systems and Carleman estimates for scalar hyperbolic equations. An important system of thermoelasticity can not be principally diagonalized, however it has "triangular" structure which allows one to obtain Carleman estimates and uniqueness of the continuation by exploiting Carleman estimates for second order scalar operators with two large parameters [2], [12], [18]. So far, Carleman estimates with two large parameters, have been obtained only for elliptic, parabolic, and isotropic hyperbolic operators of second order [12]. Carleman estimates are also very useful in control theory (controllability and stabilization for initial boundary value problems) and inverse problems [19]. In particular, they were a main tool in the first proof of uniqueness and stability of all three elastic parameters in the dynamical Lamé system from two sets of boundary data [16]. Until now, anisotropic systems have been studied only in very special cases, like small scalar perturbations of classical elasticity (with residual stress) in [24], [25], [26], where there are Carleman estimates, uniqueness and stability of the continuation, and identification of the elastic coefficients for such systems. Recently in [20], [21], the Carleman estimates with two large parameters have been obtained for general second order equations. Constants in the estimates in [20], [21] depend on partial differential operators.

In this dissertation, we are mainly interested in proving uniqueness of continuation for systems of partial differential equations in an anisotropic case. As an important example, we consider the system of isotropic elasticity with residual stress, R. This system was studied [13], [24], [25] by assuming the smallness of R. We obtain Carleman estimates with two large parameters for the general scalar differential operators of second order, including as a particular case, operators of hyperbolic type. Applying these estimates, we obtain Carleman estimates, global uniqueness, and stability of the continuation results, and the identification of the residual stress, R, without the smallness assumption of R, *i.e.*, globally. We need to assume K-pseudo-convexity with respect to two scalar operators involving residual stress. Also, we improve results of [20], [21], [25], [26] by showing that constants in Carleman estimates depend only on some constants in the pseudo-convexity conditions and bounds on the coefficients of differential operators. So, the constants do not depend on a particular operator. We give explicit conditions of pseudo-convexity with respect to the Euclidean metrics. By using the methods and results of [16], we additionally obtain, for general scalar operators, Carleman estimates with two large parameters in Sobolev spaces of negative order, and using the methods of [25], [26], we derive most natural Carleman estimates for elasticity systems with residual stress. In [21], such estimates are obtained with additional spatial derivatives.

In Chapter 2 we introduce the basic notions of spaces of functions, which are sufficient for understanding this dissertation. We give a discussion on how differential operators with variable constants interact with multiplication by weight functions. A special integration by parts technique is introduced, which is crucial for proving the Carleman estimate for a general operator. Pseudo-convexity, one of the prerequisite notions of Carleman estimates, is introduced and we give several estimates for systems with examples. Finally we discuss the linear elasticity system and energy estimates.

In Chapter 3 we obtain pseudo-convexity conditions for a general second order operator. So we derive Carleman estimates with two large parameters. The known conditions of pseudo-convexity in the anisotropic case are hard to verify, in particular with regard to the hyperbolic operator. We also give explicit sufficient global conditions of pseudo-convexity of the weight function. The main goals of this chapter are to prove strong Carleman estimates for general scalar operators by the technique of differential quadratic forms and Fourier analysis, and to obtain weak Carleman estimates in Sobolev spaces of negative order with special micro-localization arguments.

In Chapter 4, to easily use Carleman estimates for scalar equations, we extend the elasticity system to a new principally triangular system where the leading part is a special lower triangular matrix differential operator with the wave operators in the diagonal. Combining estimates of scalar operators, we prove Carleman estimate for the elasticity system with residual stress, which demonstrates the use of two large parameters.

In Chapter 5 we show uniqueness of continuation results in the Cauchy problem. We prove Hölder and Lipschitz stability estimates for the lateral Cauchy problem.

In Chapter 6 we discuss the inverse problem. We show uniqueness and stability of the identification of six functions defining residual stress from one set of special boundary measurements.

CHAPTER 2

PRELIMINARIES

2.1 Spaces of functions

We introduce the basic spaces of functions, which are sufficient for the understanding of this dissertation. Both data and solutions for problems in partial differential equations are functions defined on certain domains. In order to formulate precise theorems of uniqueness, it is necessary to specify the spaces where these functions lie.

A Sobolev space is a vector space whose elements are functions defined on domains or surfaces in $\mathbb{R}^n$, which is the n-dimensional Euclidean space of points $x = (x_1, x_2, \ldots, x_n)$, and whose partial derivatives satisfy certain integrability conditions. Throughout this dissertation the term *domain*, denoted by the symbol Ω, refers to a nonempty open connected set in $\mathbb{R}^n$. Consider a point $x = (x_1, x_2, \ldots, x_n) \in \mathbb{R}^n$, its norm is given by $|x| = (\sum_{j=1}^n x_j^2)^{1/2}$. The inner product of two points x and y in $\mathbb{R}^n$ is $x \cdot y = \sum_{j=1}^n x_j y_j$.

If $\alpha = (\alpha_1, \ldots, \alpha_n)$ is an n-tuple of nonnegative integers α_j, we call α a multi-index. We denote $x^\alpha = x_1^{\alpha_1} \cdots x_n^{\alpha_n}$, which has degree $|\alpha| = \sum_{j=1}^n \alpha_j$, and also denote the product $\alpha! = \alpha_1! \cdots \alpha_n!$. With $D_j = -i\partial/\partial x_j$, we set

$$D^\alpha = D_1^{\alpha_1} \cdots D_n^{\alpha_n}.$$

Here, i is the imaginary unit. Similarly, $\partial_j = \partial/\partial x_j$, and

$$\partial^\alpha = \partial_1^{\alpha_1} \cdots \partial_n^{\alpha_n}$$

denotes a differential operator of order $|\alpha|$. Note that $D^{(0,\ldots,0)} u = u$.

If α and β are two multi-indices, we define addition and multiplication by

$$\alpha + \beta = (\alpha_1 + \beta_1, \ldots, \alpha_n + \beta_n), \quad k\alpha = (k\alpha_1, \ldots, k\alpha_n).$$

We say that $\beta \le \alpha$ provided $\beta_j \le \alpha_j$ for $1 \le j \le n$. In this case $\alpha - \beta$ is also a multi-index, and $|\alpha - \beta| + |\beta| = |\alpha|$. If $\beta \le \alpha$, we let

$$\binom{\alpha}{\beta} = \prod_{j=1}^{n} \binom{\alpha_j}{\beta_j} = \prod_{j=1}^{n} \frac{\alpha_j!}{\beta_j!(\alpha_j - \beta_j)!} = \frac{\alpha!}{\beta!(\alpha - \beta)!}.$$

Otherwise we set $\binom{\alpha}{\beta} = 0$.

We also recall the Leibniz formula

$$D^\alpha (uv)(x) = \sum_{\beta \le \alpha} \binom{\alpha}{\beta} D^\beta u(x) D^{\alpha - \beta} v(x)$$

which in particular is valid for $u \in C^\infty(\Omega)$ and $v \in \mathcal{D}'(\Omega)$.

We introduce the derivatives of $P(\xi) = \sum a_\alpha \xi^\alpha$:

$$P^{(\beta)}(\xi) = \frac{\partial^{|\beta|}}{\partial \xi^\beta} P(\xi) = \sum_{\beta \le \alpha} \frac{\alpha!}{(\alpha - \beta)!} a_\alpha \xi^{\alpha - \beta}.$$

If a_α is a locally finite family of elements of $\mathcal{D}'(\Omega)$ we can associate with that family the linear differential operator

$$P^{(\beta)}(D) = \sum_{\beta \le \alpha} \frac{\alpha!}{(\alpha - \beta)!} a_\alpha D^{\alpha - \beta}.$$

The Leibniz formula generalizes to

$$P(uv) = \sum \frac{1}{\beta!} D^\beta u P^{(\beta)} v.$$

In general $(D^\beta P)(D)u \ne D^\beta(P(D)u)$, since by the Leibniz formula

$$D^\beta(P(D)u) = \sum_\alpha \sum_{k \le \beta} \frac{\beta!}{k!(\beta - k)!} (D^k a_\alpha)(D^{\alpha + \beta - k} u).$$

If $\Omega \subset \mathbb{R}^n$, we denote the closure of Ω in $\mathbb{R}^n$ by $\bar{\Omega}$. If u is a function defined on Ω, we define the *support* of u to be the set

$$supp(u) = \overline{\{x \in \Omega : u(x) \ne 0\}}.$$

We say that u has *compact support* in Ω if $supp(u) \subset \Omega$ and is compact.

For the rest of this section we define Ω to be an open set in $\mathbb{R}^n$, let m be a nonnegative integer, and let s be a real number.

Spaces of continuous functions

We define the following spaces of continuous functions.

$C^m(\Omega)$ is the space of all functions u with all their partial derivatives $D^\alpha u$ of order $|\alpha| \leq m$ for nonnegative integers, continuous on Ω. Note that $C^\infty(\Omega) = \bigcap_{m=0}^{\infty} C^m(\Omega)$, $C^0(\Omega) \equiv C(\Omega)$. Since Ω is open, functions in $C^m(\Omega)$ need not be bounded on Ω.

$C_b^m(\Omega)$ is the space of all functions which have continuous bounded derivatives up to order m. $C_b^m(\Omega)$ is a Banach space with norm given by

$$\|u\|_{C_b^m(\Omega)} := \sum_{|\alpha| \leq m} \sup_{x \in \Omega} |D^\alpha u(x)|.$$

$C_0^\infty(\Omega)$ (or equivalently $\mathcal{D}(\Omega)$) is the space of infinitely differentiable functions in $C^\infty(\Omega)$ with compact support in Ω. Elements in C_0^∞ are called test functions.

$C^\lambda(\bar{\Omega})$ is the space of Hölder functions of order λ, $0 < \lambda \leq 1$, on $\bar{\Omega}$, *i.e.*, the space of functions u continuous on Ω with the norm

$$\|u\|_{C^\lambda(\bar{\Omega})} = \|u\|_{C(\bar{\Omega})} + \sup_{x \neq y,\, x, y \in \Omega} \frac{|u(x) - u(y)|}{|x - y|^\lambda} < \infty.$$

$C^{m+\lambda}(\bar{\Omega})$ is the space of functions u with finite norm

$$\|u\|_{C^{m+\lambda}(\bar{\Omega})} = \sum_{|\alpha| \leq m} \|\partial^\alpha u\|_{C^\lambda(\Omega)} < \infty.$$

$\mathcal{S}(\mathbb{R}^n)$ is the space of all rapidly decreasing functions on $\mathbb{R}^n$ which are of class C^∞ and such that $|x|^k |D^\alpha u(x)|$ is bounded for every $k \in \mathbb{N}$ and every multi-index α.

Spaces of integrable functions

We define the following spaces of integrable functions.

$L^p(\Omega)$ is the space of all measurable functions u defined on Ω for which

$$\int_\Omega |u(x)|^p dx < \infty$$

where $1 \le p < \infty$. $L^p(\Omega)$ is a Banach space with norm given by

$$\|u\|_{L^p(\Omega)} = \Big(\int_\Omega |u(x)|^p dx \Big)^{1/p} < \infty, \quad 1 \le p < \infty.$$

If $p = 2$, $L^2(\Omega)$ is a Hilbert space, where the scalar product corresponding to the norm is given by

$$(u, v) = \int_\Omega u(x)v(x)dx.$$

$L^\infty(\Omega)$ is the space of all measurable functions u on Ω which are essentially bounded on Ω if there is a constant K such that $|u(x)| \le K$ almost everywhere on Ω. This is a Banach space with the norm given by

$$\|u\|_{L^\infty(\Omega)} = esssup_{x\in\Omega}|u(x)| < \infty.$$

$L^p_{loc}(\Omega)$ is the space of all measurable functions on Ω with $|u(x)|^p$ is locally integrable such that

$$\int_F |u(x)|^p dx < \infty$$

for every compact $F \subset \Omega$.

Space of distributions

We define the following spaces of distributions.

$\mathcal{D}'(\Omega)$ is the space of all distribution (generalized) functions u on Ω. The derivatives of u with respect to x_j are defined as

$$\Big(\frac{\partial u}{\partial x_j}, \phi \Big) = - \Big(u, \frac{\partial \phi}{\partial x_j} \Big).$$

It is easy to see that $\frac{\partial \phi}{\partial x_j}$ is again in $\mathcal{D}'(\Omega)$. For higher derivatives,

$$(D^\alpha u, \phi) = (-1)^{|\alpha|}(u, D^\alpha \phi).$$

Notice that every element of $L^p(\Omega)$ is locally integrable in Ω, so it defines a distribution in Ω. This means that $L^p(\Omega)$ is a linear subspace of $\mathcal{D}'(\Omega)$.

$\mathcal{S}'(\mathbb{R}^n)$ is the space of tempered distributions on $\mathbb{R}^n$, *i.e.*, the set of all continuous linear mappings from $\mathcal{S}(\mathbb{R}^n)$ to $\mathbb{R}$ (or $\mathbb{C}$).

Sobolev spaces

Sobolev spaces are useful subspaces of L^p-spaces equipped with structures of Banach spaces or Hilbert spaces. We define the norms

$$\|u\|_{m,p} = \Big(\sum_{0\le|\alpha|\le m} \|D^\alpha u\|_p^p \Big)^{1/p}, \quad 1 \le p < \infty \tag{2.1}$$

and

$$\|u\|_{m,\infty} = \max_{0\le|\alpha|\le m} \|D^\alpha u\|_\infty. \tag{2.2}$$

$H^{m,p}(\Omega)$ is the completion of $\{u \in C^m(\Omega) : \|u\|_{m,p} < \infty\}$. Notice that for $p = 2$ this is a Hilbert space with an inner product

$$(u, v)_m = \sum_{|\alpha|\le m} \int_\Omega D^\alpha u(x)\overline{D^\alpha v(x)}dx.$$

We use the notation $H^m = H^{m,2}$ with the norm $\|\cdot\|_{(m)} = \|\cdot\|_{m,2}$.

$H_0^m(\Omega)$ is the closure of $\mathcal{D}(\Omega)(= C_0^\infty(\Omega))$ in $H^m(\Omega)$.

$H_0^{-m}(\Omega)$ is the dual space of $H_0^m(\Omega)$, *i.e.*, the set of all continuous linear functionals (mappings) on $H_0^m(\Omega)$.

Define $H^s(\mathbb{R}^n)$ by $\{u \in \mathcal{S}'(\mathbb{R}^n) : \int_{\mathbb{R}^n} (1 + |\xi|^2)^{s/2}\hat{u}(\xi)e^{i2\pi\xi\cdot x}d\xi \in L^2(\mathbb{R}^n)\}, \; s \in \mathbb{R}$,

where $\hat{u}$ is the Fourier transform of u. We define the norm

$$\|u\|^2_{H^s(\mathbb{R}^n)} = (2\pi)^{-n} \int (1 + |\xi|^2)^s |\hat{u}(\xi)|^2 d\xi.$$

We also introduce some weighted norms in this space. For $\tau > 0$, $H^s_\tau(\mathbb{R}^n)$ is the same space as $H^s(\mathbb{R}^n)$ with the norm

$$\|u\|^2_{H^s_\tau(\mathbb{R}^n)} = (2\pi)^{-n} \int (\tau^2 + |\xi|^2)^s |\hat{u}(\xi)|^2 d\xi.$$

Define $W^{m,p}(\Omega)$ by $\{u \in L^p(\Omega) : D^\alpha u \in L^p(\Omega)\,, \; 0 \le |\alpha| \le m\}$.

This is a Banach space with norm defined by (2.1) and (2.2). The case $p = 2$ is most useful. To simplify the writing, we put

$$W^{m,2}(\Omega) = H^m(\Omega),$$

which is a Hilbert space.

$W_0^{m,p}$ is the closure of $C_0^\infty(\Omega)$ in the space of $W^{m,p}(\Omega)$.

Lipschitz domain

Consider an open set Ω and its boundary $\partial\Omega = \Gamma$. The simplest case of a Lipschitz Ω occurs when there is a function $\phi : \mathbb{R}^{n-1} \to \mathbb{R}^n$ such that

$$\Omega = \{x \in \mathbb{R}^{n-1} : x_n < \phi(x') \text{ for all } x' = (x_1, \ldots, x_{n-1}) \in \mathbb{R}^{n-1}\}.$$

If there is a constant M such that

$$|\phi(x') - \phi(y')| \leq M|x' - y'| \text{ for all } x', y' \in \mathbb{R}^n,$$

then we say that ϕ is Lipschitz and Ω is a Lipschitz hypograph.

The open set Ω is a *Lipschitz domain* if its boundary Γ is compact and if there exist finite families $\{\omega_j\}$ and $\{\Omega_j\}$ having the following properties:

1. The family $\{\omega_j\}$ is a finite open cover of Γ, *i.e.*, each ω_j is an open subset of $\mathbb{R}^n$, $\Gamma \subseteq \cup_j \omega_j$.

2. Each Ω_j can be transformed to a Lipschitz hypograph by a rigid motion (rotation and translation).

3. The set Ω satisfies $\omega_j \cap \Omega = \omega_j \cap \Omega_j$ for each j.

Extensions

The proofs of Theorem 2.1 can be found in [31], [35]. It is also observed in [31] that the operator extending u as 0 outside Ω is continuous from $H^s(\Omega)$ into $H^s(\mathbb{R}^n)$ if and only if $0 \leq s < \frac{1}{2}$.

Let $B(0;R)$ be the ball of radius R centered at a point 0, ν be the unit exterior normal to the boundary of a domain, and $diam(\Omega)$ be the diameter of the domain Ω.

Theorem 2.1 *For any set $\Omega \subset \mathbb{R}^n$ there is a linear continuous operator E mapping $C^{m+\lambda}(\bar{\Omega})$ into $C^{m+\lambda}(\mathbb{R}^n)$ such that $Eu = u$ on Ω. Its norm depends on m, λ, and diam(Ω). For any Lipschitz $\Omega \subset \bar{\Omega} \subset B(0;R)$ and m nonnegative integer there is a continuous operator E mapping $H^{m,p}$ into $H_0^{m,p}(B(0;R))$ such that $Eu = u$ on Ω. If $\partial\Omega \in C^m$, then there is a similar continuous extension operator from $H^s(\Omega)$ into $H_0^s(\mathbb{R}^n)$ when $s \leq m$ and a bounded extension operator from $H^{m-1/2}(\partial\Omega) \times \cdots \times H^{1/2}(\partial\Omega)$ into $H^m(\Omega)$ such that the extended function u has the given Cauchy data $(u, \ldots, \partial_\nu^{m-1}u)$ in this product of spaces.*

Embeddings

The embedding for Sobolev spaces are essential in the study of differential and integral operators. The classical results were basically obtained by Sobolev in the 1930s.

The Sobolev embedding theorem states that if $m \geq k$ and $m - n/p \geq k - n/q$ then

$$W^{m,p} \subseteq W^{k,q}$$

where $m,\, k \in \mathbb{R}$.

The following theorem is given in [1].

Theorem 2.2 *For any bounded Lipschitz domain Ω there is a constant $C(p, q, \lambda)$ such that for all functions $u \in H^{m,p}(\Omega)$ we have*

$$\|u\|_q(\Omega) \leq C\|u\|_{m,p}(\Omega), \;\; q \leq np/(n-mp), \; n > mp,$$

$$\|u\|_{k,q}(\Omega) \leq C\|u\|_{m,p}(\Omega), \;\; k \leq m, \; p \leq q, \; n(1/p - 1/q) \leq m - k,$$

$$\|u\|_\lambda(\Omega) \leq C\|u\|_{m,p}(\Omega), \;\; \lambda \leq m - n/p, \; n < mp.$$

Moreover in case of strict inequalities corresponding embedding operators are compact.

Traces

In the boundary value problem for partial differential operators defined in a domain Ω, it is important to determine the space of functions defined on $\partial\Omega$ that contain the traces of functions u in $W^{m,p}(\Omega)$. As shown in the Sobolev embedding theorem functions in $W^{m,p}(\mathbb{R}^n)$, $mp < n$ have traces on $\mathbb{R}^{n-1}$ that belong to $L^q(\mathbb{R}^{n-1})$ for $p \leq q \leq (n-1)p(n-mp)$.

Theorem 2.3 *For any bounded Lipschitz domain* $\Omega \subset \mathbb{R}^n$ *and any* $(n-1)$*-dimensional Lipschitz surface* $S \subset \bar{\Omega}$ *there is a constant* $C(S, m, p, q)$ *such that for all functions* $u \in H^{m,p}(\Omega)$ *we have*

$$\|u\|_q(S) \le C\|u\|_{m,p}(\Omega), \;\; 1 < mp < n, \; q \le p(n-1)/(n-mp), \; S \in C^m,$$

$$\|u\|_{(1/2)}(S) + \|\nabla u\|_{(-1/2)}(S) \le C\|u\|_{(1)}(\Omega).$$

The results for $H^{m,p}(\Omega)$ spaces can be found in [29, chapter 2.2], while the claim about $H^m(\Omega)$ spaces is proven in [31].

Integration by parts

For convenience we recall also the integration by parts formula,

$$\int_\Omega u\partial_j v = \int_{\partial\Omega} uv\nu_j \, d\Gamma - \int_\Omega \partial_j uv, \tag{2.3}$$

which is valid at least for functions $u \in H^{1,p}(\Omega)$, $v \in H^{1,q}(\Omega)$, $1/p + 1/q = 1$, $1 \le p$, and bounded domains Ω with piecewise Lipschitz boundaries $\partial\Omega$.

2.2 Fourier transforms and differential operators

If $u \in L^1(\mathbb{R}^n)$, then the Fourier transform $\hat{u}$ is the bounded continuous function in $\mathbb{R}^n$ defined by

$$\hat{u}(\xi) = \int e^{-i<x,\xi>} u(x)dx, \;\; \xi \in \mathbb{R}^n. \tag{2.4}$$

It yields an isomorphism $\mathcal{S} \to \mathcal{S}$, with Fourier's inversion given by

$$u(x) = (2\pi)^{-n} \int e^{i<x,\xi>} \hat{u}(\xi)d\xi, \;\; x \in \mathbb{R}^n, \tag{2.5}$$

where $< x, \xi >= \sum_{j=1}^n x_j \xi_j$ denotes the inner product between the vector x and the covector ξ.

Now consider a linear partial differential operator

$$P(x, \partial) = \sum_{|\alpha| \le m} b^\alpha(x) \partial^\alpha \tag{2.6}$$

of order m with variable coefficients. If all of the coefficients are independent of x, then P is said to have *constant coefficients*, and if all of the $b^\alpha(x)$'s are real valued, then P is said to have *real coefficients*. The *symbol* of P is given by

$$P(x,\xi) = \sum_{|\alpha|\leq m} b^\alpha(x)(i\xi)^\alpha,$$

where P is a polynomial in ξ of degree m with coefficients depending on x.

We now use the notation $D_j = -i\partial_j$. The differential operator (2.6) can then be represented in another way:

$$P(x,D) = \sum_{|\alpha|\leq m} a^\alpha(x)D^\alpha \tag{2.7}$$

is of order m with variable coefficients $a^\alpha(x) = i^{|\alpha|}b^\alpha(x)$. The differential operator (2.6) is useful when handling real valued functions, while (2.7) is convenient in Fourier analysis.

Lemma 2.4 *The Fourier transformation $u \to \hat{u}$ maps $\mathcal{S}$ continuously into $\mathcal{S}$. The Fourier transform of $x_j u$ is $-D_j\hat{u}$, and the Fourier transform of $D_j u$ is $\xi_j\hat{u}(\xi)$.*

Proof of Lemma 2.4 ([15, page 161])

Differentiate (2.4) with respect to ξ, then we obtain

$$\begin{aligned} D^\alpha\hat{u}(\xi) &= \int e^{-i<x,\xi>}((-i)(-ix))^\alpha\, u(x)dx \\ &= \int e^{-i<x,\xi>}(-x)^\alpha\, u(x)dx, \end{aligned} \tag{2.8}$$

where the integral obtained is uniformly convergent. Hence $\hat{u} \in C^\infty$ and $D^\alpha\hat{u}$ is the Fourier transform of $(-x)^\alpha u$. We also obtain

$$\xi^\beta D_\xi^\alpha\hat{u}(\xi) = \int e^{-i<x,\xi>}D^\beta((-x)^\alpha\, u(x))dx \tag{2.9}$$

using integration by parts. Hence

$$\sup|\xi^\beta D^\alpha\hat{u}(\xi)| \;\leq\; C\sup_x(1+|x|)^{n+1}|D^\beta((-x)^\alpha u(x))|$$

where $C = \int (1+|x|)^{-n-1}dx$. Therefore the Fourier transformation maps $\mathcal{S}$ continuously into $\mathcal{S}$. When $\alpha = 0$ we obtain from (2.9)

$$\xi^\beta \hat{u}(\xi) = \int e^{-i<x,\xi>} D^\beta u(x) dx, \tag{2.10}$$

where $\xi^\beta \hat{u}$ is the Fourier transform of $D^\beta u$. We complete the proof of Lemma 2.4. □

We have thus

$$\sum_{|\alpha|\le m} a^\alpha(x) \widehat{D^\alpha u}(\xi) = \sum_{|\alpha|\le m} a^\alpha(x) \xi^\alpha \hat{u}(\xi). \tag{2.11}$$

The space $\mathcal{S}$ is closed under differentiation, multiplication by polynomials, Fourier transform, and the pointwise product and convolution of elements of $\mathcal{S}$, and so (2.8) and (2.10) hold for functions belonging to $\mathcal{S}$. Also the Fourier transform is a linear isomorphism $\mathcal{S} \to \mathcal{S}$.

Consider the differential operator (2.7), in an open set $\Omega \subset \mathbb{R}^n$, of order m, with variable coefficients $a^\alpha \in C^\infty(\Omega)$ $(\alpha \ne 0)$, $a^0 \in L^2_{loc}(\Omega)$. We denote the *principal part* by

$$P_m(\xi) = P_m(x,\xi) = \sum_{|\alpha|=m} a^\alpha(x)\xi^\alpha, \quad x \in \Omega, \quad \xi \in \mathbb{C}^n. \tag{2.12}$$

We now discuss how the exponential weight functions $w = e^{\tau\varphi}$ interact with the differential operator P of order m. Let $\varphi \in C^m(\bar{\Omega})$ and $\tau \in \mathbb{R}^1$.

Lemma 2.5 *Let $u \in H^m(\Omega)$. The substitution*

$$u = e^{-\tau\varphi} v$$

transforms $P(x,D)$ in (2.7) to $P(x, D + i\tau\nabla\varphi(x))$.

Proof of Lemma 2.5

With a substitution $u = e^{-\tau\varphi}v$ in (2.7)

$$P(x,D)(e^{-\tau\varphi}v) = \sum a^\alpha(x) D^\alpha(e^{-\tau\varphi}v).$$

This implies

$$P(x,D)v = \sum a^\alpha(x) e^{\tau\varphi} D^\alpha(e^{-\tau\varphi}v).$$

We need to show

$$e^{\tau\varphi}D^{\alpha}(e^{-\tau\varphi}v) = (D - \tau D\varphi)^{\alpha}v; \tag{2.13}$$

we prove this by induction. For $|\alpha| = 1$, (2.13) holds since

$$e^{\tau\varphi}D_j(e^{-\tau\varphi}v) = e^{\tau\varphi}(e^{-\tau\varphi}D_jv + e^{-\tau\varphi}(-\tau)D_j\varphi v) = (D_j - \tau D_j\varphi)v.$$

Assume (2.13) holds for $|\alpha| = n$, $n \in \mathbb{N}$. Then with $|\alpha'| = 1$

$$(D - \tau D\varphi)^{\alpha+\alpha'}v$$

$$= (D - \tau D\varphi)^{\alpha'}(D - \tau D\varphi)^{\alpha}v$$

$$= (D_j - \tau D_j\varphi)(e^{\tau\varphi}D^{\alpha}(e^{-\tau\varphi}v))$$

$$= e^{\tau\varphi}D^{\alpha+\alpha'}(e^{-\tau\varphi}v) + \tau D_j\varphi e^{\tau\varphi}D^{\alpha}(e^{-\tau\varphi}v) - \tau D_j\varphi e^{\tau\varphi}D^{\alpha}(e^{-\tau\varphi}v)$$

$$= e^{\tau\varphi}D^{\alpha+\alpha'}(e^{-\tau\varphi}v).$$

Hence (2.13) holds for $|\alpha| = n + 1$. By induction we have (2.13). □

Let $\varphi \in C^1$ be a real valued function defined in a neighborhood of a point x^0 and assume that $\nabla\varphi(x^0) \neq 0$. Then the equation

$$\varphi(x) = \varphi(x^0)$$

defines a C^1-hypersurface in a neighborhood of x^0. The part of a neighborhood of x^0 where $\varphi(x) > \varphi(x^0)$ is called the *positive side of the hypersurface*.

Definition 2.6 *Given $\varphi \in C^1(\Omega)$ with $\nabla\varphi(x^0) \neq 0$, if $P_m(x^0, \nabla\varphi(x^0)) = 0$, the surface $S = \{x \in \Omega : \varphi(x) = \varphi(x^0)\}$ is called a characteristic surface at $x^0 \in \Omega$ with respect to P of order m. If it is possible to find ψ so that $P_m(\nabla(\varphi + \varepsilon\psi))$ is not $O(\varepsilon^2)$ at x^0 when $\varepsilon \to 0$, then the surface is said to be of simple characteristic.*

A surface S is called a *characteristic surface* if it is characteristic at each of its points. Consequently, a surface S is called *noncharacteristic* at x^0 if $P_m(x^0, \nabla\varphi(x^0)) \neq 0$. That is, a surface S is called a *noncharacteristic surface* when it is noncharacteristic at every point.

For the method of integration for the characteristic equation $P_m(\nabla\varphi) = 0$, assume that P_m has real C^2-coefficients in an open set $\Omega \subset \mathbb{R}^n$ and that $\varphi \in C^2(\Omega)$ is a real valued function whose level surfaces are simple characteristic everywhere in Ω. Differentiation of the equation $P_m(x, \nabla\varphi) = 0$ gives

$$\sum_{j,k=1}^{n} \left(\partial_j \partial_k \varphi P_m^{(j)}(x, \nabla\varphi) + P_{m,k}(x, \nabla\varphi)\right) = 0 \tag{2.14}$$

where

$$P_m^{(j)}(x,\xi) = \frac{\partial P_m(x,\xi)}{\partial \xi_j}, \quad P_{m,j}(x,\xi) = \frac{\partial P_m(x,\xi)}{\partial x_j}. \tag{2.15}$$

We now see a necessary condition for a differential equation $P(x,D)u = f$ to have a solution locally for every $f \in C^\infty$. And we shall see that a strengthened form of this condition is also sufficient to imply local existence of solutions for every f, provided that there are no multiple real characteristics. For (2.12) we let

$$\bar{P}_m(x,\xi) = \sum_{|\alpha|=m} \overline{a^\alpha(x)} \xi^\alpha$$

and use the notations (2.15).

Let us consider

$$C_{2m-1}(x,\xi) = \sum_{j=1}^{n} i(P_m^{(j)}(x,\xi)\bar{P}_{m,j}(x,\xi) - \bar{P}_m^{(j)}(x,\xi)P_{m,j}(x,\xi)). \tag{2.16}$$

This is a polynomial in ξ of degree $2m-1$ with real coefficients. If the coefficients of P_m are real or constants, C_{2m-1} is identically zero.

Theorem 2.7 *Suppose that the differential equation*

$$P(x,D)u = f$$

has a solution $u \in \mathcal{D}'(\Omega)$ for every $f \in C_0^\infty(\Omega)$. Then we have

$$C_{2m-1}(x,\xi) = 0 \quad \textit{if} \quad P_m(x,\xi) = 0, \ x \in \Omega, \ \xi \in \mathbb{R}^n. \tag{2.17}$$

We notice that the meaning of (2.17) is the commutator

$$\bar{P}(x,D)P(x,D) - P(x,D)\bar{P}(x,D) = C(x,D)$$

of order $\leq 2m-1$, and $C_{2m-1}(x,D)$ is the sum of the terms of order $2m-1$ exactly in $C(x,D)$ [14, page 157].

The following theorem can be obtained from Theorem 2.7 [14, page 163].

Theorem 2.8 *Suppose that the coefficients of the operator $P(x,D)$ of order m are in $C^\infty(\Omega)$ and $C_{2m-1}(x,\xi) \neq 0$ when $P_m(x,\xi) = 0$, $x \in \Omega$, $\xi \in \mathbb{R}^n$ in any nonvanished open domain $\omega \subset \Omega$. Then there exist functions $f \in \mathcal{S}(\Omega)$ such that $P(x,D)u = f$ does not have any solution $u \in \mathcal{D}'(\omega)$ in $\omega \subset \Omega$.*

Example : (This example is given in [14].)

For the differential operator in $\mathbb{R}^3$

$$P(x,D) = -iD_1 + D_2 - 2(x_1 + ix_2)D_3,$$

we have $C_{2m-1}(x,\xi) = i\big((2i\xi_3 + 2i\xi_3 + 0) - (-2i\xi_3 - 2i\xi_3 + 0)\big) = -8\xi_3$. If we choose $\xi_1 = -2x_2$, $\xi_2 = 2x_1$, and $\xi_3 = 1$, then $P_1(x,\xi) = 0$ but $C_{2m-1}(x,\xi) \neq 0$. Hence the hypotheses of Theorem 2.8 are satisfied for every Ω.

Now we know that existence of solutions in Ω of the differential equation $P(x,D)u = f$ requires that

$$C_{2m-1}(x,\xi) = 2\,\Im \sum_1^n P_{m,j}(x,\xi)\overline{P_m^{(j)}(x,\xi)} = 0 \qquad (2.18)$$
$$\text{if} \quad P_m(x,\xi) = 0\,, \quad \xi \in \mathbb{R}^n, \quad x \in \Omega.$$

Definition 2.9 *We say that $P(x,D)$ is principally normal in $\bar{\Omega}$ if the coefficients of P_m are in $C^1(\bar{\Omega})$ and there exists a differential operator $Q_{m-1}(x,D)$, homogeneous of degree $m-1$ in D with coefficients in $C^1(\bar{\Omega})$, such that*

$$C_{2m-1}(x,\xi) = 2\,\Re\, P_m(x,\xi)\,\overline{Q_{m-1}(x,\xi)}\,, \quad \xi \in \mathbb{R}^n. \qquad (2.19)$$

In particular, P is principally normal if $C_{2m-1}(x,\xi) = 0$ identically, that is, if the commutator of P and its adjoint with respect to the form $\int u\bar{v}dx$ is of order $\le 2m-2$. We can then take $Q = 0$. Note that every operator with constant or real coefficients is principally normal. It is clear that Q_{m-1} is uniquely determined by P_m unless P_m and $\bar{P}_m$ have a common factor, that is, P_m has a real factor.

2.3 Differential quadratic forms

We already introduced differential operators and their symbols in Section 2.2. One of the techniques in the proof of estimates would be an integration by parts in the integral. This technique is based on a concept of a differential quadratic form and its properties. Here we follow [14].

First we introduce a *differential quadratic form*

$$F(D,\bar{D})u\bar{u} = \sum_{\alpha,\,\beta} a_{\alpha\beta} D^\alpha u \overline{D^\beta u}, \quad u \in C_0^\infty(\mathbb{R}^n) \tag{2.20}$$

with constant coefficients $a_{\alpha\beta}$. Here the sum is finite. We associate with $F(D,\bar{D})$ the form

$$F(\zeta,\bar{\zeta}) = \sum a_{\alpha\beta}\zeta^\alpha\bar{\zeta}^\beta, \quad \zeta \in \mathbb{C}^n, \tag{2.21}$$

which we call its symbol. Since

$$D^\alpha e^{i<x,\,\zeta>} = D_1^{\alpha_1}\cdots D_n^{\alpha_n} e^{i<x,\,\zeta>} = (-i\partial_1)\cdots(-i\partial_n)e^{i<x,\,\zeta>}$$

$$= \zeta_1^{\alpha_1}\cdots\zeta_n^{\alpha_n}e^{i<x,\,\zeta>} \; = \; \zeta^\alpha e^{i<x,\,\zeta>}$$

and

$$e^{i<x,\,\zeta>}\overline{e^{i<x,\,\zeta>}} = e^{i<x,\,\zeta-\bar{\zeta}>} = e^{i<x,\,2iIm\zeta>} = e^{-2<x,\,Im\zeta>}$$

for $u = e^{i<x,\,\zeta>}$ we have that

$$\sum_{\alpha,\,\beta} a_{\alpha\beta}D^\alpha u\overline{D^\beta u} \; = \sum_{\alpha,\,\beta} a_{\alpha\beta}\zeta^\alpha u\overline{\zeta^\beta u} \; = \; u\bar{u}\sum_{\alpha,\,\beta} a_{\alpha\beta}\zeta^\alpha\bar{\zeta}^\beta$$

$$= e^{-2<x,\,Im\zeta>}F(\zeta,\,\bar{\zeta}).$$

It is obvious that the correspondence between the differential quadratic form (2.20) and the polynomial (2.21) is one-to-one.

The form (2.20) is said to be of double order $(\mu\,;\,m)$, $m \le \mu \le 2m$, referring to μ as the *total order* and m as the *separate order* of F, if in (2.21) we have $|\alpha| + |\beta| \le \mu$ and $|\alpha| \le m$, $|\beta| \le m$ when $a_{\alpha\beta} \ne 0$.

Now for using the technique of integration by parts, let $F(D, \bar{D})u\bar{u}$ be the vector divergent form with differential quadratic forms $G^k(D, \bar{D})u\bar{u}$, $k = 1, \ldots, n$ as components, that is,

$$F(D, \bar{D})u\bar{u} = \sum_{k=1}^{n} \frac{\partial}{\partial x_k} \big(G^k(D, \bar{D})u\bar{u}\big). \tag{2.22}$$

Since

$$\frac{\partial}{\partial x_k}(u\bar{u}) = iD_k(u\bar{u}) = i[(D_k u)\bar{u} + u(D_k\bar{u})] = i[(D_k u)\bar{u} - u(\overline{D_k u})],$$

the identity (2.22) is equivalent to the algebraic identity

$$F(\zeta,\, \bar{\zeta}) = i\sum_{k=1}^{n}(\zeta_k - \bar{\zeta}_k)G^k(\zeta,\, \bar{\zeta}). \tag{2.23}$$

If F can be represented as (2.22), from (2.23) it follows that

$$F(\xi,\, \xi) = 0, \quad \xi \in \mathbb{R}^n. \tag{2.24}$$

If $u, v \in \mathcal{S}(\mathbb{R}^n)$,

$$\int u\,\bar{v}\,dx \;=\; (2\pi)^{-n} \int \hat{u}\,\bar{\hat{v}}\,dx.$$

This is easy to show: by using the Fourier inversion formula

$$\begin{aligned} \int u(x)\,\overline{v(x)}\,dx \;&=\; \int \Big((2\pi)^{-n} \int \hat{u}(\xi)e^{i<x,\xi>}d\xi\,\Big)\overline{v(x)}\,dx \\ &= (2\pi)^{-n} \int \overline{v(x)} \int \hat{u}(\xi)e^{i<x,\xi>}d\xi\;dx \\ &= (2\pi)^{-n} \int \hat{u}(\xi) \int \overline{v(x)e^{-i<x,\xi>}}dx\;d\xi \\ &= (2\pi)^{-n} \int \hat{u}(\xi)\,\overline{\hat{v}(\xi)}\,d\xi. \end{aligned}$$

Hence by using this Parseval formula

$$\int F(D,\, \bar{D}) u\bar{u} dx = (2\pi)^{-n} \int F(\xi,\, \xi)\, |\hat{u}(\xi)|^2 d\xi, \quad u \in C_0^\infty(\mathbb{R}^n).$$

The following lemma shows that (2.24) is the sufficient condition of (2.22).

Lemma 2.10 *Suppose* (2.24). *Then there exist differential quadratic forms* $G^k(D,\, \bar{D})u\bar{u}$ *satisfying* (2.22). *And we have*

$$G^k(\xi,\, \xi) = -\frac{1}{2}\frac{\partial}{\partial \eta_k} F(\xi + i\eta,\, \xi - i\eta)_{/\eta=0}, \quad \xi \in \mathbb{R}^n. \tag{2.25}$$

Furthermore, if F *is of order* $(\mu\,;\, m)$, *then* G^k *is always chosen of order* $(\mu - 1\,;\, m')$ *where*

$$m' = \begin{cases} m-1 & if \;\; \mu < 2m \quad (1) \\ m & if \;\; \mu = 2m \quad (2). \end{cases} \tag{2.26}$$

Proof of Lemma 2.10 ([14, page 188])

Let $\zeta = \xi + i\eta,\ \xi, \eta \in \mathbb{R}^n$. By the assumption (2.24), the Taylor expansion of the polynomial $F(\xi + i\eta,\, \overline{\xi + i\eta})_{/\eta=0}$ has no independent terms of η. Hence we can find polynomials $g^k(\xi, \eta)$ such that

$$F(\xi + i\eta,\, \overline{\xi + i\eta})_{/\eta=0} \;=\; \sum_{k=1}^{n} \eta_k\, g^k(\xi,\, \eta). \tag{2.27}$$

Using $\xi_k = \frac{1}{2}(\zeta_k + \bar{\zeta}_k)$ and $\eta_k = -\frac{i}{2}(\zeta_k - \bar{\zeta}_k)$ we have

$$\sum_{k=1}^{n} \eta_k\, g^k(\xi,\, \eta) \;=\; i \sum_{k=1}^{n} (\zeta_k - \bar{\zeta}_k)(-\frac{1}{2}) g^k(\xi,\, \eta).$$

Now we set

$$(-\frac{1}{2}) g^k(\xi,\, \eta) = G^k(\zeta,\, \bar{\zeta}) \tag{2.28}$$

to get the equivalent algebraic identity (2.23) of (2.22). Thus we complete the sufficiency of (2.24).

Let us differentiate (2.27) with respect to η; then by the product rule

$$\sum_{k=1}^{n} \frac{\partial}{\partial \eta_k} F(\xi + i\eta,\, \overline{\xi + i\eta})_{/\eta=0} \;=\; \sum_{k=1}^{n} g^k(\xi,\, \eta) \;+\; \sum_{k=1}^{n} \eta_k \frac{\partial}{\partial \eta_k} g^k(\xi,\, \eta).$$

The second term on the right vanishes since we put $\eta = 0$. Hence we have each component

$$\frac{\partial}{\partial \eta_k} F(\xi + i\eta,\, \overline{\xi + i\eta})_{/\eta=0} = g^k(\xi,\, \eta).$$

We get (2.25) from (2.28).

To prove the last statement in this lemma, we need to know about congruence class of order. Consider two polynomials $F_1(\zeta,\, \bar{\zeta})$ and $F_2(\zeta,\, \bar{\zeta})$ of order $(\mu\,;\, m)$. If $F = F_1 - F_2$ can be written in the form (2.23) with G^k of order $(\mu - 1\,;\, m')$ where (2.26), we say these two polynomials are congruent and denote $F_1 \equiv F_2$.

Claim :

Suppose the order of $(\mu\,;\, m)$ such that $|\alpha'| + |\beta'| = |\alpha''| + |\beta''| \leq \mu$ and each length $|\alpha'|, |\beta'|, |\alpha''|$, and $|\beta''| \leq m$. Then $\zeta^{\alpha'}\bar{\zeta}^{\beta'} \equiv \zeta^{\alpha''}\bar{\zeta}^{\beta''}$.

Proof of Claim

(1) Consider the first case $\mu < 2m$. Then either $|\alpha'|$ or $|\beta'|$ is $< m$. Without loss of generality, let $|\alpha'| < m$. We need to show that the congruence class of $\zeta^{\alpha'}\bar{\zeta}^{\beta'}$ does not change even though one factor in $\bar{\zeta}^{\beta'}$ (respectively, $\zeta^{\alpha'}$) is replaced by its complex conjugate. By replacement, the statement

$$\zeta^{\alpha'}\bar{\zeta}^{\beta'} = \zeta^{\alpha'}\bar{\zeta}_1^{\beta'_1}\cdots\bar{\zeta}_j^{\beta'_j}\cdots\bar{\zeta}_m^{\beta'_m}$$

is modified to be

$$\zeta^{\alpha''}\bar{\zeta}^{\beta''} = \zeta^{\alpha'}\bar{\zeta}_1^{\beta'_1}\cdots\zeta_j^{\beta'_j}\cdots\bar{\zeta}_m^{\beta'_m}.$$

Then

$$\zeta^{\alpha'}\bar{\zeta}^{\beta'} - \zeta^{\alpha''}\bar{\zeta}^{\beta''} = \sum(\bar{\zeta}_j^{\beta'_j} - \zeta_j^{\beta'_j})\zeta^{\alpha'}\bar{\zeta}_1^{\beta'_1}\cdots\bar{\zeta}_{j-1}^{\beta'_{j-1}}\bar{\zeta}_{j+1}^{\beta'_{j+1}}\cdots\bar{\zeta}_m^{\beta'_m}.$$

This can be in the form (2.23) with G^k of order $(\mu - 1\,;\, m - 1)$. Hence these two polynomials are congruent, *i.e.*, $\zeta^{\alpha'}\bar{\zeta}^{\beta'} \equiv \zeta^{\alpha''}\bar{\zeta}^{\beta''}$. Letting $|\beta'| < m$, the proof is analogous.

(2) Consider the second case $\mu = 2m$. It is invalid if $|\alpha'| + |\alpha''| = |\beta'| + |\beta''| = \mu$. Because $|\alpha'| = \mu - |\alpha''|$, $|\beta'| = \mu - |\beta''|$ implies $|\alpha'| + |\beta'| = 2\mu - (|\alpha''| + |\beta''|) \leq \mu$. This is a contradiction with $|\alpha''| + |\beta''| \geq \mu$. Hence it is only valid for $|\alpha'| = |\beta'| = |\alpha''| = |\beta''| = m$.

Now we need to show that the congruence class of $\zeta^{\alpha'}\bar{\zeta}^{\beta'}$ does not change even though one factor in $\bar{\zeta}^{\beta'}$ and one in $\zeta^{\alpha'}$ is simultaneously replaced by its complex conjugate, *i.e.*, by replacement, the statement

$$\zeta^{\alpha'}\bar{\zeta}^{\beta'} = \zeta^{\alpha'}\cdots\zeta_k^{\alpha'_k}\cdots\zeta_m^{\alpha'_m}\bar{\zeta}_1^{\beta'_1}\cdots\bar{\zeta}_j^{\beta'_j}\cdots\bar{\zeta}_m^{\beta'_m}$$

is modified to be

$$\zeta^{\alpha''}\bar{\zeta}^{\beta''} = \zeta^{\alpha'}\cdots\bar{\zeta}_k^{\alpha'_k}\cdots\zeta_m^{\alpha'_m}\bar{\zeta}_1^{\beta'_1}\cdots\zeta_j^{\beta'_j}\cdots\bar{\zeta}_m^{\beta'_m}.$$

Using the identity

$$\zeta_k^{\alpha'_k}\bar{\zeta}_j^{\beta'_j} - \bar{\zeta}_k^{\alpha'_k}\zeta_j^{\beta'_j} = (\zeta_k^{\alpha'_k} - \bar{\zeta}_k^{\alpha'_k})\bar{\zeta}_j^{\beta'_j} - (\zeta_j^{\beta'_j} - \bar{\zeta}_j^{\beta'_j})\bar{\zeta}_k^{\alpha'_k}$$

we obtain

$$\begin{aligned}
&\zeta^{\alpha'}\bar{\zeta}^{\beta'} - \zeta^{\alpha''}\bar{\zeta}^{\beta''}\\
&= \big(\zeta_k^{\alpha'_k}\bar{\zeta}_j^{\beta'_j} - \bar{\zeta}_k^{\alpha'_k}\zeta_j^{\beta'_j}\big)\zeta_1^{\alpha'_1}\cdots\zeta_{k-1}^{\alpha'_{k-1}}\zeta_{k+1}^{\alpha'_{k+1}}\cdots\zeta_m^{\alpha'_m}\bar{\zeta}_1^{\beta'_1}\cdots\bar{\zeta}_{j-1}^{\beta'_{j-1}}\bar{\zeta}_{j+1}^{\beta'_{j+1}}\cdots\bar{\zeta}_m^{\beta'_m}\\
&= \big((\zeta_k^{\alpha'_k} - \bar{\zeta}_k^{\alpha'_k})\bar{\zeta}_j^{\beta'_j} - (\zeta_j^{\beta'_j} - \bar{\zeta}_j^{\beta'_j})\bar{\zeta}_k^{\alpha'_k}\big)\zeta_1^{\alpha'_1}\cdots\zeta_{k-1}^{\alpha'_{k-1}}\zeta_{k+1}^{\alpha'_{k+1}}\cdots\zeta_m^{\alpha'_m}\bar{\zeta}_1^{\beta'_1}\cdots\bar{\zeta}_{j-1}^{\beta'_{j-1}}\bar{\zeta}_{j+1}^{\beta'_{j+1}}\cdots\bar{\zeta}_m^{\beta'_m}.
\end{aligned}$$

This can be in the form (2.23) with G^k of order $(\mu - 1\,;\, m)$. Hence these two polynomials are congruent, *i.e.*, $\zeta^{\alpha'}\bar{\zeta}^{\beta'} \equiv \zeta^{\alpha''}\bar{\zeta}^{\beta''}$. This completes the proof of the claim. □

From our claim $\zeta^{\alpha'}\bar{\zeta}^{\beta'} \equiv \zeta^{\alpha''}\bar{\zeta}^{\beta''}$ it follows that every differential quadratic form F_1 of order $(\mu\,;\, m)$ is congruent to a sum of the form

$$F_2(\zeta\,,\,\bar{\zeta}) \;=\; \sum_{|\alpha|+|\beta|\le\mu,\,|\alpha|,|\beta|\le m} a_{\alpha\beta}\zeta^{\alpha}\bar{\zeta}^{\beta}$$

of order $(\mu\,;\, m)$ where there is at most one different non-zero term with the same multi-index sum $\alpha+\beta$ of a differential quadratic form F_1. Notice that $F_1(\xi\,,\,\xi) = 0$ implies $F_2(\xi\,,\,\xi) = 0$. This means all $a_{\alpha\beta}$ must be 0. Hence $F_1 \equiv 0$. This completes the proof of Lemma 2.10. □

We now discuss a *differential quadratic form* with variable coefficients

$$F(x, D, \bar{D})u\bar{u} = \sum_{\alpha,\,\beta} a_{\alpha\beta}(x)D^{\alpha}u\overline{D^{\beta}u}, \;\; u \in C_0^{\infty}(\mathbb{R}^n). \tag{2.29}$$

We associate this form with the polynomial

$$F(x,\zeta,\bar\zeta) = \sum a_{\alpha\beta}(x)\zeta^\alpha\bar\zeta^\beta, \ \ \zeta\in\mathbb{C}^n. \tag{2.30}$$

The following lemma shows the existence of lower total order differential quadratic form.

Lemma 2.11 *Suppose $F(x,D,\bar D)u\bar u$ is a differential quadratic form of order $(\mu\,;\,m)$ with coefficients in $C^\gamma(\Omega)$, $\gamma\geq 1$ and $\Omega\subset\mathbb{R}^n$, and*

$$F(x,\xi,\xi)=0, \ \ x\in\Omega, \ \ \xi\in\mathbb{R}^n. \tag{2.31}$$

Then there exists a differential quadratic form $G(x,D,\ \bar D)u\bar u$ of lower total order with coefficients in $C^{\gamma-1}(\Omega)$ such that

$$\int F(x,D,\bar D)u\bar u dx = \int G(x,D,\bar D)u\bar u dx, \ \ u\in C_0^\infty(\mathbb{R}^n). \tag{2.32}$$

Notice that G can be always chosen of order $(\mu-1\,;\,m')$ where

$$m' = \begin{cases} m-1 & if \ \ \mu<2m \quad (1) \\ m & if \ \ \mu=2m \quad (2). \end{cases} \tag{2.33}$$

Furthermore, we have

$$G(x,\xi,\ \xi)=\frac{1}{2}\sum_{k=1}^{n}\frac{\partial^2}{\partial x_k\partial\eta_k}F(x,\xi+i\eta,\ \xi-i\eta)_{/\eta=0}, \ \ \xi\in\mathbb{R}^n. \tag{2.34}$$

Proof of Lemma 2.11 ([14, page 189])

Let $F_1,F_2,\ldots,F_N$ be a basis in the finite dimensional vector space consisting of all differential quadratic forms of order $(\mu\,;\,m)$ with constant coefficients satisfying (2.24). Then we can find differential quadratic forms G_j^k, $j=1,\ldots,N$, $k=1,\ldots,n$ of order $(\mu-1\,;\,m')$ where (2.33) so that

$$F_j(D,\bar D)u\bar u = \sum_{k=1}^{n}\frac{\partial}{\partial x_k}G_j^k(D,\bar D)u\bar u, \ \ j=1,\ldots,N. \tag{2.35}$$

By assumption (2.31) and notion of basis, coefficients $a_j(x) \in C^\gamma(\Omega)$ are uniquely determined, so

$$F(x, D, \bar{D})u\bar{u} = \sum_{j=1}^{n} a_j(x) F_j(D, \bar{D})u\bar{u}. \tag{2.36}$$

Using (2.35) we have a vector component form

$$F(x, D, \bar{D})u\bar{u} = \sum_{j=1}^{N} a_j(x) \sum_{k=1}^{n} \frac{\partial}{\partial x_k} G_j^k(D, \bar{D})u\bar{u}.$$

Then using integration by parts

$$\int F(x, D, \bar{D})u\bar{u}dx$$

$$= \int_{\partial\Omega} \sum_{j=1}^{N} a_j(x) G_j(D, \bar{D})u\bar{u}dx - \int \sum_{j=1}^{N} \sum_{k=1}^{n} \frac{\partial a_j}{\partial x_k}(x) G_j^k(D, \bar{D})u\bar{u}dx.$$

Since $u \in C_0^\infty(\Omega)$, the boundary term vanishes. Hence we obtain (2.32) with

$$G(x, D, \bar{D})u\bar{u} = -\sum_{j=1}^{N} \sum_{k=1}^{n} \frac{\partial a_j}{\partial x_k}(x) G_j^k(D, \bar{D})u\bar{u}. \tag{2.37}$$

We now need to show (2.34). Equation (2.37) is equivalent to the algebraic identity

$$G(x, \zeta, \bar{\zeta}) = -\sum_{j=1}^{N} \sum_{k=1}^{n} \frac{\partial a_j}{\partial x_k}(x) G_j^k(\zeta, \bar{\zeta}).$$

Take $\zeta = \xi + i\eta$ and put $\eta = 0$. Then

$$G(x, \xi, \xi) = -\sum_{j=1}^{N} \sum_{k=1}^{n} \frac{\partial a_j}{\partial x_k}(x) G_j^k(\xi, \xi).$$

Using (2.25) from Lemma 2.10 we have

$$G(x, \xi, \xi) = \frac{1}{2} \sum_{k=1}^{n} \sum_{j=1}^{N} \frac{\partial a_j}{\partial x_k}(x) \frac{\partial}{\partial \eta_k} F_j(\xi + i\eta,\, \xi - i\eta)_{/\eta=0}. \tag{2.38}$$

From (2.36) we consider the corresponding polynomial

$$F(x, \zeta, \bar{\zeta}) = \sum_{j=1}^{N} a_j(x) F_j(\zeta, \bar{\zeta}).$$

Then we obtain by differentiations

$$\frac{\partial^2}{\partial x_k \partial \eta_k} F(x, \zeta, \bar{\zeta}) = \sum_{j=1}^{N} \frac{\partial a_j}{\partial x_k}(x) \frac{\partial}{\partial \eta_k} F_j(\zeta, \bar{\zeta}). \tag{2.39}$$

Using (2.39) in (2.38) we finally obtain (2.34). $\square$

2.4 Pseudo-convexity and Carleman estimates

In this section, we introduce the concept of pseudo-convexity needed for Carleman estimates. The choice of the weight function in Carleman estimates is obvious for parabolic and elliptic operators but is not for hyperbolic operator, in particular, for anisotropic ones.

Let φ, $\psi \in C^2$ be real valued functions defined in neighborhood U of a point x_0 and V of a point x^0, $\nabla\varphi(x_0) \neq 0$, and $\nabla\psi(x^0) \neq 0$. Then the sets

$$S = \{x \in U \,:\, \varphi(x) = \varphi(x_0)\}, \tag{2.40}$$

$$S' = \{x \in V \,:\, \psi(x) = \psi(x^0)\} \tag{2.41}$$

define non-singular oriented level surfaces in U and V. Throughout this section we consider C^2-surfaces given as level surfaces of a real valued function in $C^2(\bar{\Omega})$. Our purpose is to show that solutions of a differential equation $Pu = 0$ vanishing on the positive side $\{x : \psi(x) > \psi(x^0)\}$ must vanish in a full neighborhood of x^0 when suitable convexity conditions are fulfilled. These must only depend on the surface (2.41) and not on the function ψ used to represent it. If φ is another such function then $\varphi'(x^0) = \gamma\,\psi'(x^0)$ for some $\gamma > 0$, and

$$\sum \partial_j\partial_k\varphi(x^0)y_j z_k = \gamma \sum \partial_j\partial_k\psi(x^0)y_j z_k$$

if $\sum y_j\partial_j\psi(x^0) = \sum z_k\partial_k\psi(x^0) = 0$, but not for all $y, z \in \mathbb{R}^n$. This is the reason the following definition contains only a part of the necessary conditions for Carleman estimates.

Definition 2.12 *Suppose that the coefficients of P_m are real-valued. A function $\varphi \in C^2(\Omega)$ is called strongly pseudo-convex on Ω with respect to the differential operator if the conditions*

$$P_m(x, \zeta) = 0 \tag{2.42}$$

for $x \in \Omega$, $\zeta = \xi + i\tau\nabla\varphi(x)$, $|\zeta| = 1$ with $\xi \in \mathbb{R}^n \backslash \{0\}$, and $\tau \neq 0 \in \mathbb{R}^1$ imply that

$$\sum \big(\partial_j\partial_k\varphi \frac{\partial P_m}{\partial \zeta_j}\overline{\frac{\partial P_m}{\partial \zeta_k}} + \frac{1}{\tau}\Im\, \partial_k P_m \overline{\frac{\partial P_m}{\partial \zeta_k}} \big) > \delta \tag{2.43}$$

in Ω, for some positive number δ.

Definition 2.13 *A function ψ is called pseudo-convex on $\bar{\Omega}$ with respect to P if $\psi \in C^2(\bar{\Omega})$, $P(x, \nabla\psi(x)) \neq 0,\ x \in \bar{\Omega}$, and*

$$\sum \partial_j \partial_k \psi(x) \frac{\partial P}{\partial \zeta_j} \frac{\partial P}{\partial \zeta_k}(x,\xi) + \sum \left(\frac{\partial P}{\partial \zeta_k} \partial_k \frac{\partial P}{\partial \zeta_j} - \partial_k P \frac{\partial^2 P}{\partial \zeta_j \partial \zeta_k}\right) \partial_j \psi(x,\xi) > 0 \tag{2.44}$$

for any $\xi \in \mathbb{R}^n$ and any point x of $\bar{\Omega}$ provided

$$P(x,\xi) = 0,\ \sum \frac{\partial P}{\partial \zeta_j}(x,\xi) \partial_j \psi(x) = 0. \tag{2.45}$$

For uniform pseudo-convexity, under the same assumption (2.45), a function ψ is called K-pseudo-convex with respect to P if

$$\sum \partial_j \partial_k \psi(x) \frac{\partial P}{\partial \zeta_j} \frac{\partial P}{\partial \zeta_k}(x,\xi) + \sum \left(\frac{\partial P}{\partial \zeta_k} \partial_k \frac{\partial P}{\partial \zeta_j} - \partial_k P \frac{\partial^2 P}{\partial \zeta_j \partial \zeta_k}\right) \partial_j \psi(x,\xi) \geq K|\xi|^2 \tag{2.46}$$

for some positive constant K.

Notice that the constant K in pseudo-convexity (2.46) depends only on an operator P. Hence the constant C in the stabilities based on Carleman estimates depends only on some constant K in the condition of pseudo-convexity (2.46).

Surfaces S given by (2.40) are called (strongly) pseudo-convex level surfaces if a function φ is (strongly) pseudo-convex. The following theorem shows the stability of (strongly) pseudo-convex level surfaces.

Theorem 2.14 *Suppose the surface S is (strongly) pseudo-convex with respect to P at x^0. Then there exist a neighborhood ω of x^0 and a positive number ε such that every $\vartheta \in C^2(\omega)$ for which*

$$|D^\alpha(\varphi - \vartheta)| < \varepsilon \quad \text{in } \omega,\ |\alpha| \leq 2 \tag{2.47}$$

has (strongly) pseudo-convex level surfaces with respect to P everywhere in ω.

This result is proven in [14, page 204].

Note that when $m = 1$ there is no difference between pseudo-convexity and strong pseudo-convexity. Since $\Im \sum \partial_k P_m \overline{\frac{\partial P_m}{\partial \zeta_k}} = 0$ when $P_m = 0$, equation (2.43) reduces to (2.44), in view of the definition of a principally normal operator, *i.e.*, every operator whose coefficients are real valued is principally normal.

The following theorem tells us for second order operators pseudo-convexity of ψ implies strong pseudo-convexity of φ given by $\varphi = e^{\gamma\psi}$ for large γ.

Theorem 2.15 *Suppose P is a partial differential second order operator with real-valued principal coefficients. Then for operator P the pseudo-convexity of $\psi \in C^2(\bar{\Omega})$ implies the strong pseudo-convexity of $\varphi \in C^2(\Omega)$ with $\varphi = e^{\gamma\psi}$ for large γ. And if the function $\psi \in C^2(\bar{\Omega})$ is pseudo-convex with respect to P on $\bar{\Omega}$, then there are constants C, $C_0(\gamma)$ such that*

$$\tau^{3-2|\alpha|}\int_{\Omega}|\partial^{\alpha}u|^2 e^{2\tau\varphi} \leq C\Big(\int_{\Omega}|Pu|^2 e^{2\tau\varphi} + \int_{\partial\Omega}(\tau|\nabla u|^2 + \tau^3|u|^2)e^{2\tau\varphi}\Big) \tag{2.48}$$

when $C < \gamma$, $C_0(\gamma) < \tau$, $|\alpha| < 1$, for all functions $u \in H^2(\Omega)$.

This result is proven in [19, page 53].

Example 1 : A partial differential operator P is called *elliptic* on $\bar{\Omega}$ if

$$P(x,\xi) \neq 0 \quad \text{for any } \xi \in \mathbb{R}^n \backslash \{0\} \text{ and any } x \in \bar{\Omega}.$$

One example of an elliptic operator is

$$Pu = -div(a\nabla u) + cu, \quad a > \varepsilon_0 > 0 \text{ in } \Omega \tag{2.49}$$

where $a \in C^1(\bar{\Omega})$, $c \in L^{\infty}(\Omega)$,

$$div(a\nabla u) = a\sum_{j=1}^{n}\partial_j^2 u + \sum_{j=1}^{n}\partial_j a \partial_j u.$$

For the principal part we just take the higher order term of (2.49), so

$$P_m(x,\xi) = a(\xi_1^2 + \xi_2^2 + \cdots + \xi_n^2).$$

Then any function $\psi \in C^2(\Omega)$ with $\nabla\psi \neq 0$ on $\bar{\Omega}$ is pseudo-convex with respect to A on $\bar{\Omega}$. More generally, for every second order elliptic operator, any function $\psi \in C^2(\Omega)$ with $|\nabla\psi| > 0$ on Ω is pseudo-convex.

Example 2 : Consider an isotropic hyperbolic wave operator

$$P = a_0^2\partial_t^2 - \Delta + \sum b_j\partial_j + c, \tag{2.50}$$

$\Omega = G \times (-T, T) \subset \mathbb{R}^{n+1}$, $a_0 \in C(\bar{\Omega})$, $a_0 > 0$, b_j, $c \in L^\infty(\Omega)$. Then

$$P_m(x,\xi) = a_0(x)^2\xi_0^2 - \xi_1^2 - \cdots - \xi_n^2.$$

We need a suitable function $\psi(x,t)$ in $\Omega_\varepsilon = \Omega \cap \{\psi > \varepsilon\}$ satisfying the pseudo-convex condition adjusted to space-time geometry. The following is described in [19, page 66]. Motivated by speed of propagation concept we choose

$$\psi(x,t) = -\theta^2t^2 + |x-\beta_n|^2, \tag{2.51}$$

where θ and β_n are constants.

The conditions (2.45)

$$a_0^2\xi_0^2 = \xi_1^2 + \cdots + \xi_n^2 = 1\,, \quad a_0^2\theta^2\xi_0 t + \xi\cdot(x-\beta_n) = 0$$

yield the left side in (2.44)

$$\begin{aligned}
&(2a_0^2\xi_0)(2a_0^2\xi_0)(-2\theta^2) + 2\sum_{j=1}^{n}(-2\xi_j)^2 \\
&+ (4a_0\partial_t a_0\xi_0)(2a_0^2\xi_0) - (2a_0\partial_t a_0\xi_0^2)(2a_0^2)(-2\theta^2 t) \\
&+ \sum_{k=1}^{n}(4a_0\partial_k a_0\xi_0)(-2\xi_k)(-2\theta^2 t) - \sum_{k=1}^{n}(2a_0\partial_k a_0\xi_0^2)(-2)2(x_k-\beta_n) \\
&= -8\theta^2a_0^2(a_0^2\xi_0^2) + 8(\xi_1^2 + \xi_2^2 + \cdots + \xi_n^2) - 8a_0^3\partial_t a_0 t\theta^2\xi_0^2 \\
&+ 16\theta^2 t a_0\xi_0\nabla a_0\cdot\xi + 8a_0\xi_0^2\nabla a_0\cdot(x-\beta_n) \\
&= 8\big(-\theta^2a_0^2 + 1 - a_0\partial_t a_0 t\theta^2 + 2\theta^2 t a_0\xi_0\nabla a_0\cdot\xi + \frac{1}{a_0}\nabla a_0\cdot(x-\beta_n)\big) \\
&= 8\big(1 + \frac{1}{a_0}\nabla a_0\cdot(x-\beta_n) - \theta^2(a_0^2 + a_0\partial_t a_0 t - 2ta_0\xi_0\nabla a_0\cdot\xi)\big) \\
&\geq 8\big(1 + \frac{1}{a_0}\nabla a_0\cdot(x-\beta_n) - \theta^2(a_0^2 + a_0\partial_t a_0 t + 2|a_0\xi_0||t\nabla a_0||\xi|)\big).
\end{aligned}$$

Hence the inequality

$$a_0^2\theta^2(a_0\,+\,\partial_t a_0 t\,+\,2a_0^2|t\nabla a_0|)\,<\,a_0\,+\,\nabla a_0\cdot(x-\beta_n)$$

guarantees pseudo-convexity of the function ψ in (2.51) with respect to P in (2.50) on $\bar{\Omega}$.

For the condition of noncharacteristic $\nabla\psi$, we have

$$P(x,\nabla\psi(x)) = a_0^2(-2\theta^2 t)^2 - 4(x_1^2+x_2^2+\cdots+(x_n-\beta_n)^2)$$

$$= 4(a_0^2\theta^2-1)|x_n-\beta_n|^2.$$

Hence

$$a_0^2\theta^2 \neq 1$$

guarantees that $\nabla\psi$ is noncharacteristic on $\bar{\Omega}_0$.

Special estimates of the Carleman type were obtained in some papers for second order hyperbolic equations, and stability estimates were derived from them for a solution of the Cauchy problem with data on a lateral surface.

We consider the linear differential operator

$$A = -\sum_{j,k=1}^{n} a^{jk}(x,t)\frac{\partial^2}{\partial x_j \partial x_k}\,, \quad a^{jk}=a^{kj}$$

which satisfies the uniform ellipticity condition

$$\varepsilon_0|\xi|^2 \leq \sum_{j,k=1}^{n} a^{jk}(x,t)\xi_j\xi_k \quad \text{for } \ \xi\in\mathbb{R}^n, \ \ (x,t)\in\Omega = G\times(-T,T),$$

and the conditions $\|a^{jk}\|_1(\Omega) = \sum_{j,k=1}^{m}|a^{jk}| \leq 1/\varepsilon_0$.

We introduce a theorem which deals with the stability of the solution $(\mathbf{u},\mathbf{q})$ of the following inverse problem:

$$((\frac{\partial^2}{\partial t^2}+A)E+\mathcal{A}_1)\mathbf{u} = M\mathbf{q}+\mathbf{f}\,, \ \ \partial_t\mathbf{q}=\mathbf{0} \quad \text{on } \ \Omega,$$

$$\mathbf{u}=\mathbf{g}_0\,, \ \partial_\nu\mathbf{u}=\mathbf{g}_1 \quad \text{on } \ \Gamma, \quad \mathbf{u}=\mathbf{g}_2 \ \text{ on } \ G\times\{0\}, \tag{2.52}$$

where $\mathbf{u}$ and $\mathbf{q}$ are vector valued functions with components $(u_1, \ldots, u_m)$ and $(q_1, \ldots, q_m)$, E and M are the $m \times m$ unit matrix and the weighted matrix function, respectively, and $\mathcal{A}_1$ is a first order matrix linear differential operator whose coefficients are bounded in modulus by $1/\varepsilon_0$. In the theorem below the domain G is assumed to lie in the lamina $\{-h < x_n < 0\}$, $h > 0$, while $\Gamma = \partial G \backslash \{x_n = -h\} \in C^3$. Set $\Omega_\varepsilon = \Omega \cap \{\varphi > \varepsilon\}$.

The condition

$$\varepsilon_1 < \det M \quad \text{on } \; G \times (-\varepsilon_1, \varepsilon_1),$$

$$\|M\|_2(\Omega) + \|\partial_t M\|_2(\Omega) \leq 1/\varepsilon_1,$$

where $\|M\|_2 = (\sum_{j,k=1}^m |m^{jk}|^2)^{1/2}$, is imposed on the weighted matrix function M.

Denote by C and κ positive constants that depend on G and $\varepsilon_0, \varepsilon_1, \varepsilon_2$.

The following theorem is given in [28] without proof.

Theorem 2.16 *Let a^{jk} be independent of t, and let*

$$\varepsilon_2 |\xi|^2 \leq \sum_{j,k=1}^n \frac{\partial a^{jk}(x)}{\partial x_n} \xi_j \xi_k \quad \textit{for } \; \xi \in \mathbb{R}^n, \; x \in \Omega.$$

Then there exist constants $C(\varepsilon)$ and $\kappa(\varepsilon)$, $0 < \kappa < 1$, such that if

$$\mathcal{M} h \varepsilon_2^{-1} \varepsilon_0^{-2} < T$$

and $\psi(x,t) = |x - \beta|^2 - \theta^2 t^2$, $\beta = (0, \ldots, 0, \beta_n)$, $\mathcal{M}\varepsilon_0^2 < 1$, $\mathcal{M} = \mathcal{M}(\|a^{jk}\|_1(\Omega))$, then the following estimate holds for the solution $(\mathbf{u}, \mathbf{q})$ of problem (2.52)

$$\|\mathbf{u}\|_{(2)}(\Omega_\varepsilon) + \|\mathbf{q}\|_{(0)}(\Omega_\varepsilon) \leq C(\varepsilon) F^{\kappa(\varepsilon)} \|\mathbf{u}\|_{(3)}^{1-\kappa(\varepsilon)}(\Omega)$$

where

$$F = \|\mathbf{f}\|_{(1)}(\Omega) + \|\mathbf{g}_0\|_{(\frac{5}{2})}(\Gamma) + \|\mathbf{g}_1\|_{(\frac{3}{2})}(\Gamma) + \|\mathbf{g}_2\|_{(\frac{5}{2})}(G \times \{0\}).$$

Note that in the case where $a^{jj} = 1/c$ and $a^{jk} = 0$ for $j \neq k$, the condition of Theorem 2.16 for a^{jk} is identical to the known condition for monotonicity of medium density with respect to depth.

Now we state the interior Schauder type estimate for Hölder stability of the Cauchy problem in Section 5.1. The following theorem is given in [29, chapter 3.2].

Theorem 2.17 *Consider an elliptic operator*

$$Pu = \sum_{|\alpha| \le m} a_\alpha(x) D^\alpha u$$

with $a_\alpha(x) \in C^\infty(\Omega)$, for all $x \in \Omega$,

$$\sum_{|\alpha|=m} a_\alpha(x)\xi^\alpha \neq 0, \quad 0 \neq \xi \in \mathbb{R}^n.$$

For nonnegative integer k and $0 < \lambda < 1$ one has

$$\|u\|_{C^{m+k,\lambda}(\Omega_1)} \le C(\|Pu\|_{C^{k,\lambda}(\Omega_2)} + \|u\|_{C^0(\Omega_2)}), \quad \Omega_1 \subset\subset \Omega_2 \subset\subset \Omega \tag{2.53}$$

where

$$\|u\|_{C^k(\Omega)} = \sup_{|\alpha| \le k,\, x \in \Omega} |D^\alpha u(x)|,$$

$$\|u\|_{C^{k,\lambda}(\Omega)} = \|u\|_{C^k(\Omega)} + \sup_{|\alpha|=k,\, x,y \in \Omega} \frac{|D^\alpha u(x) - D^\alpha u(y)|}{|x-y|^\lambda}.$$

Note that we write $\Omega_1 \subset\subset \Omega_2$ if $\Omega_1 \subset \bar{\Omega}_1 \subset \Omega_2$ and $\bar{\Omega}_1$ is compact, and say Ω_1 is compactly contained in Ω_2.

2.5 Elasticity system

In this section we discuss systems of differential operators.

Consider the system of differential equations

$$\sum_{j=1}^n P_{ij}(D) u_j = f_i, \quad i = 1, \dots, n. \tag{2.54}$$

Let $\mathbf{u} = (u_1, \dots, u_n)^T$ and $\mathbf{P}(D) = (P_{ij}(D))$. Then (2.54) can be written as

$$\mathbf{P}(D)\mathbf{u} = \mathbf{f}.$$

If $\det \mathbf{P}(\xi) \equiv 0$, there are polynomials $Q_1(\xi), \dots, Q_n(\xi)$ and $R_1(\xi), \dots, R_n(\xi)$, where they are not all identically zero, such that

$$\sum_{j=1}^n P_{ij}(\xi) Q_j(\xi) = 0, \quad \sum_{i=1}^n R_i(\xi) P_{ij}(\xi) = 0, \quad i, j = 1, \dots, n. \tag{2.55}$$

It follows that a necessary condition for the existence of a solution of systems of differential equations (2.54) is

$$\sum_{i=1}^{n} R_i(D) f_i = 0.$$

Also it follows from (2.55) that

$$\mathbf{P}(D)\mathbf{u} = \mathbf{0} \quad \text{if} \;\; \mathbf{u} = (Q_1(D)\varphi, \ldots, Q_n(D)\varphi), \quad \varphi \in \mathcal{D}'.$$

Some results can only be obtained for systems of differential operators such that $\det \mathbf{P}(\xi) \neq 0$, which we shall assume from now on.

Here we introduce the system of equations of linear elasticity which is not necessarily isotropic. We need to describe this system in some detail to prove uniqueness by using a Carleman estimate. For simplicity we formulate the system of linear elasticity in $\mathbb{R}^3$. Let Ω be a domain in $\mathbb{R}^3$. We introduce the elastic displacement vector

$$\mathbf{u} : \Omega \to \mathbb{R}^3.$$

We begin with a constituent law (Hooke's law) expressing a linear relation between force (stress) and deformation (strain). The stress tensor is $\sigma_{ij}(=\sigma_{ji})$ and strain tensor is $\varepsilon_{ij}(\mathbf{u})(=\varepsilon_{ji}(\mathbf{u}))$. We recall that

$$\varepsilon_{ij}(\mathbf{u}) = \frac{1}{2}(\partial_i u_j + \partial_j u_i).$$

The equation for the constituent law is

$$\sigma_{ij} = a_{ijkh}\, \varepsilon_{kh}(\mathbf{u}). \tag{2.56}$$

Notice that we made use of the summation convention concerning repeated indices. Here a_{ijkh} are coefficients of the elasticity tensor, independent of the strain tensor ε_{ij}. Hence there are 6 independent equations relating stresses and strains provided symmetric properties of the coefficients of elasticity hold, *i.e.*,

$$a_{ijkh} = a_{jikh} = a_{ijhk} = a_{khij}.$$

The coefficients of elasticity are also assumed to have properties of ellipticity, that is,

$$a_{ijkh}\varepsilon_{ij}\varepsilon_{kh} \geq \alpha_1 \varepsilon_{ij}\varepsilon_{ij}, \quad \alpha_1 \text{ a constant} > 0, \quad \forall\, \varepsilon_{ij}. \tag{2.57}$$

There are twenty-one elastic constants since we are in $\mathbb{R}^3$. Equation (2.57) implies the invertibility of (2.56) and we have

$$\varepsilon_{ij}(\mathbf{u}) = A_{ijkh}\sigma_{kh}, \tag{2.58}$$

where coefficients of compliance A_{ijkh} have the same properties as the a_{ijkh}, *i.e.*,

$$A_{ijkh} = A_{jikh} = A_{ijhk} = A_{khij}$$

and

$$A_{ijkh}\sigma_{ij}\sigma_{kh} \geq \alpha_2 \sigma_{ij}\sigma_{ij}, \quad \alpha_2 \text{ a constant} > 0, \quad \forall\, \sigma_{ij}. \tag{2.59}$$

Setting

$$\alpha = \min(\alpha_1, \alpha_2)$$

we replace the relations (2.57) and (2.59) by

$$\begin{cases} a_{ijkh}\varepsilon_{ij}\varepsilon_{kh} \geq \alpha\varepsilon_{ij}\varepsilon_{ij}, \\ A_{ijkh}\sigma_{ij}\sigma_{kh} \geq \alpha\sigma_{ij}\sigma_{ij}, \end{cases} \quad \alpha > 0. \tag{2.60}$$

In the isotropic case the elasticity tensor has no preferred direction; an applied force (stress) gives the same displacements (strains) no matter the direction in which the force is applied. The coefficients a_{ijkh} are given by

$$a_{ijkh} = \lambda\, \delta_{ij}\, \delta_{kh} + \mu\, (\delta_{ik}\delta_{jh} + \delta_{ih}\delta_{jk}),$$

where the scalars λ and μ are the *Lamé constants*. Then the constituent equation (2.56) is

$$\begin{aligned} \sigma_{ij} &= \lambda\delta_{ij}\varepsilon_{kk} + 2\mu\varepsilon_{ij} \\ &= \lambda\delta_{ij}\nabla_k u_k + \mu\left(\nabla_i u_j + \nabla_j u_i\right). \end{aligned}$$

It follows that

$$\sigma_{kh} = (3\lambda + 2\mu)\varepsilon_{kk}$$

such that the relations inverse to the constituent equation become

$$\varepsilon_{ij} = \frac{1}{2\mu}\big(\sigma_{ij} - \frac{\lambda}{3\lambda + 2\mu}\sigma_{kh}\delta_{ij}\big). \tag{2.61}$$

Hence $3\lambda + 2\mu \geq 0$ with $\mu \geq 0$ implies that

$$\sigma_{ij}\varepsilon_{ij} \geq 0, \tag{2.62}$$

since σ_{ij} and ε_{ij} are linked by the constituent law.

In the nonisotropic case, inequality (2.62) implies

$$\sigma_{ij}\varepsilon_{ij} = a_{ijkh}\varepsilon_{ij}\varepsilon_{kh} = A_{ijkh}\sigma_{ij}\sigma_{kh} \geq 0.$$

Consider the coefficients with the residual stress term

$$a_{ijkh} = \lambda\,\delta_{ij}\,\delta_{kh} + \mu\,(\delta_{ik}\delta_{jh} + \delta_{ih}\delta_{jk}) + r_{jh}\delta_{ik}.$$

Notice that

$$\begin{cases} i \;:\; \text{the index of equation,} \\ j \;:\; \text{the index of differentiation,} \\ k \;:\; \text{the index of function,} \\ h \;:\; \text{the index of differentiation.} \end{cases}$$

Then

$$a_{ijkh}\varepsilon_{ij}\varepsilon_{kh} = \lambda\,\delta_{ij}\,\delta_{kh}\varepsilon_{ij}\varepsilon_{kh} + \mu\,(\delta_{ik}\delta_{jh} + \delta_{ih}\delta_{jk})\varepsilon_{ij}\varepsilon_{kh} + r_{jh}\delta_{ik}\varepsilon_{ij}\varepsilon_{kh}.$$

From now on we use summation notation. Then

$$\begin{aligned} \sum_{i,j,k,h=1}^{3} a_{ijkh}\varepsilon_{ij}\varepsilon_{kh} &= \lambda\sum\delta_{ij}\delta_{kh}\varepsilon_{ij}\varepsilon_{kh} \;+\; \mu\sum\delta_{ik}\delta_{jh}\varepsilon_{ij}\varepsilon_{kh} \\ &\quad + \mu\sum\delta_{ih}\delta_{jk}\varepsilon_{ij}\varepsilon_{kh} \;+\; \sum r_{jh}\delta_{ik}\varepsilon_{ij}\varepsilon_{kh} \\ &= \lambda\sum_{i,h=1}^{3}\varepsilon_{ii}\varepsilon_{hh} \;+\; 2\mu\sum_{i,j=1}^{3}\varepsilon_{ij}\varepsilon_{ij} \;+\; \sum_{i,j,h=1}^{3} r_{jh}\varepsilon_{ij}\varepsilon_{ih} \\ &= \lambda\,(\sum_{i=1}^{3}\varepsilon_{ii})^2 \;+\; 2\mu\sum_{i,j=1}^{3}\varepsilon_{ij}^2 \;+\; \sum_{i,j,h=1}^{3} r_{jh}\varepsilon_{ij}\varepsilon_{ih}. \end{aligned}$$

We need the property of ellipticity given in (2.60) to write

$$\lambda\,(\sum_{i=1}^{3}\varepsilon_{ii})^2\;+\;2\mu\sum_{i,j=1}^{3}\varepsilon_{ij}^2\;+\;\sum_{i,j,h=1}^{3}r_{jh}\varepsilon_{ij}\varepsilon_{ih}\;\geq\;\alpha\sum_{i,j=1}^{3}\varepsilon_{ij}^2. \tag{2.63}$$

For (2.63) we consider sufficient conditions

$$\lambda\geq 0\quad\text{and}\quad 2\mu|\xi|^2\;+\;\sum r_{ih}\xi_i\xi_h\;\geq\;\frac{\alpha}{3}\,|\xi|^2. \tag{2.64}$$

For the explicit condition of semidefiniteness of $2\mu I+R$ we need to have all nonnegative eigenvalues. The characteristic equation of matrix $2\mu I+R$ is

$$\det(Ix-2\mu I-R)=x^3-(6\mu+r_{11}+r_{22}+r_{33})x^2$$

$$+\,(12\mu^2+4(r_{11}+r_{22}+r_{33})\mu+r_{11}r_{22}+r_{22}r_{33}+r_{33}r_{11}-r_{12}^2-r_{23}^2-r_{31}^2)x$$

$$-\,\big(\,8\mu^3+4(r_{11}+r_{22}+r_{33})\mu^2+2(r_{11}r_{22}+r_{22}r_{33}+r_{33}r_{11}-r_{12}^2-r_{23}^2-r_{31}^2)\mu$$

$$+r_{11}r_{22}r_{33}+2r_{12}r_{23}r_{31}-r_{11}r_{23}^2-r_{22}r_{31}^2-r_{33}r_{12}^2\,\big)=0.$$

For all nonnegative eigenvalues we have

$$6\mu+r_{11}+r_{22}+r_{33}\geq 0, \tag{2.65}$$

$$12\mu^2+4(r_{11}+r_{22}+r_{33})\mu+r_{11}r_{22}+r_{22}r_{33}+r_{33}r_{11}-r_{12}^2-r_{23}^2-r_{31}^2\geq 0, \tag{2.66}$$

and

$$8\mu^3+4(r_{11}+r_{22}+r_{33})\mu^2+2(r_{11}r_{22}+r_{22}r_{33}+r_{33}r_{11}-r_{12}^2-r_{23}^2-r_{31}^2)\mu$$

$$+\,r_{11}r_{22}r_{33}+2r_{12}r_{23}r_{31}-r_{11}r_{23}^2-r_{22}r_{31}^2-r_{33}r_{12}^2\geq 0. \tag{2.67}$$

We need some explicit conditions from (2.66) and (2.67):

Solving for μ by using the discriminant D gives the explicit condition from (2.66) by

$$(r_{11}-r_{22})^2+(r_{22}-r_{33})^2+(r_{33}-r_{11})^2+6r_{12}^2+6r_{23}^2+6r_{31}^2\leq 0.$$

This says nothing. It implies that $r_{11}=r_{22}=r_{33}$ and $r_{12}=r_{23}=r_{31}=0$. This is the isotropic case.

Equation (2.67) is more complicated. Let $f(\mu)=\mu^3+b\mu^2+c\mu+d$, where

$$b=\frac{1}{2}(r_{11}+r_{22}+r_{33}),$$

$$c = \frac{1}{4}(r_{11}r_{22} + r_{22}r_{33} + r_{33}r_{11} - r_{12}^2 - r_{23}^2 - r_{31}^2),$$

and

$$d = \frac{1}{8}(r_{11}r_{22}r_{33} + 2r_{12}r_{23}r_{31} - r_{11}r_{23}^2 - r_{22}r_{31}^2 - r_{33}r_{12}^2).$$

We need $f(\mu) \geq 0$ for all $\mu \geq 0$. Then we have two cases:

(Case 1) $d \geq 0$ and $D = b^2 - 3c \leq 0$ from $f'(\mu) = 3\mu^2 + 2b\mu + c$,

(Case 2) $d \geq 0$, $f(\mu_1) \geq 0$, and $D = b^2 - 3c \geq 0$ where $\mu_1 = \frac{-b+\sqrt{b^2-3c}}{3}$ is a real solution of $f'(\mu) = 0$.

We shall obtain more explicit conditions based on above calculations by using Matlab or Maple.

2.6 Energy estimates

We are interested in the Cauchy problem where Γ is the large part of the lateral boundary data, and for the remaining part we have one classical boundary condition like Neumann or Dirichlet data. Then we can show that the operator mapping the initial data into the lateral Cauchy data is isometric with respect to standard energy norms; this is explained in [7] and [19, chapter 3]. So, under reasonable conditions, the lateral Cauchy problem is as stable as any classical problem of mathematical physics. An n-dimensional inverse problem for a hyperbolic or parabolic equation is called the inverse problem, with the lateral data if both the Dirichlet and Neumann data are given on a part $\Gamma_T \subseteq S_T$ of the surface $S_T = \partial G \times (0,T)$ of the time cylinder $\Omega_T = G \times (0,T)$, where $G \subset \mathbb{R}^n$ is a domain and unknown coefficients of this equation are to be determined.

We consider a solution u to the boundary value problem

$$Pu = f \quad \text{in} \quad \Omega = G \times (-T,T),$$

$$u = 0 \quad \text{on} \quad \partial G \times (-T,T), \quad \partial G \in C^2. \tag{2.68}$$

We define the energy integral for (2.48) as

$$E(t) = 1/2 \int_G \left((\partial_t u)^2 + |\nabla u|^2 + u^2 \right)(\, , t \,).$$

This is the standard energy integral, provided that $u = 0$ on ∂G. This can be proven by multiplying the equation $Pu = 0$ by $e^{\tau t}u^T$ in first order, $e^{\tau t}\partial_t u$ in second order case, integrating over $G \times (0, t)$, and using elementary integral inequalities.

Theorem 2.18 *Let $\Gamma = \partial\Omega$. Let P be a t-hyperbolic partial differential operator of second order. Let ψ be (K)-pseudo-convex with respect to P,*

$$\psi < 0 \;\; on \;\; \bar{G} \times \{-T, T\}, \quad and \;\; 0 < \psi \;\; on \;\; \bar{G} \times \{0\}.$$

Then there is a constant C such that for any solution u to (2.68)

$$E(t) \le C \left(\int_{\Gamma} (\partial_\nu u)^2 + \int_{G\times(-T,T)} f^2 \right) \tag{2.69}$$

when $-T < t < T$.

This theorem is proven in [19, page 73].

In Carleman estimate (2.48) of Theorem 2.15, one does not need to include all boundary terms. The following form of (2.48) is obtained.

Theorem 2.19 *Let P be a t-hyperbolic operator of second order in $\Omega = G \times (-T, T)$. Let a function ψ be (K)-pseudo-convex with respect to P on $\bar{\Omega}$ and*

$$\partial_\nu \psi < 0 \;\;\; on \;\; \Gamma_0$$

Then there are constants $C(\gamma)$, C_1 such that

$$\tau^{3-2|\alpha|} \int_{\Omega} |\partial^\alpha u|^2 e^{2\tau\varphi} \le C \left(\int_{\Omega} |Pu|^2 e^{2\tau\varphi} + \int_{\partial\Omega\setminus\Gamma_0} \tau |\partial_\nu u|^2 e^{2\tau\varphi} \right)$$

when $C_1 < \gamma$, $C < \tau$, for all functions $u \in H^2(\Omega)$ for which $u = 0$ on $\partial\Omega$, $u = \partial_t u = 0$ on $G \times \{-T, T\}$.

This theorem is shown in [19, page 74].

CHAPTER 3

CARLEMAN ESTIMATES FOR A GENERAL SECOND ORDER OPERATOR

We consider the general partial differential operator of second order

$$A = \sum_{j,k=1}^{n} a^{jk}\partial_j\partial_k + \sum b^j\partial_j + c$$

in a bounded domain Ω of the space $\mathbb{R}^n$ with the real-valued coefficients $a^{jk} \in C^1(\bar{\Omega})$, and b^j, $c \in L^\infty(\Omega)$. The principal symbol of this operator is

$$A(x;\zeta) = \sum a^{jk}(x)\zeta_j\zeta_k. \tag{3.1}$$

We use the following convention and notations for the rest of this dissertation. Sums are over repeated indices $j,k,l,m = 1,\dots,n$. Let $\partial = (\partial_1,\dots,\partial_n)$, with $D = -i\partial$, and let α be a multi-index with integer components, $\zeta^\alpha = \zeta_1^{\alpha_1}\cdots\zeta_n^{\alpha_n}$. The operators D^α and ∂^α are defined similarly. The vector ν is the outward normal to the boundary of a domain. We use generic constants C (different at different places) depending only on the upper bound, M, of coefficients in $C^1(\Omega)$, $C^2(\Omega)$-norms, on the constant K, on the function ψ, on the value ε_0, and on the domain Ω; any additional dependence is indicated. We recall that $\|\cdot\|_{(k)}$ is the norm of the Sobolev space $H^k(\Omega)$, and we use the norms $|\cdot|_k(\Omega)$ and $\|\cdot\|_\infty(\Omega)$ in the space $C^k(\bar{\Omega})$ and $L^\infty(\Omega)$, respectively, as defined in Section 2.1.

Define the weight function

$$\varphi = e^{\gamma\psi} \tag{3.2}$$

and let $\sigma = \gamma\tau\varphi$, $\Omega_\varepsilon = \Omega \cap \{\psi(x) > \varepsilon\}$.

In Theorem 3.1 we assume, in addition, that the coefficients of a general operator A admit the following bound

$$|a^{jk}|_2(\Omega) + \|b^j\|_\infty(\Omega) + \|c\|_\infty(\Omega) \;\leq\; M.$$

This assumption is needed to guarantee that constants C as used in the theorem do not depend on a particular A. It is not needed for the definition of K-pseudo-convexity in 2.13 where it suffices that $a^{jk} \in C^1$, and can be relaxed there.

Theorem 3.1 *Let ψ be K-pseudo-convex with respect to A in $\bar{\Omega}$. Then there are constants C, $C_0(\gamma)$ such that*

$$\int_\Omega \sigma^{3-2|\alpha|} e^{2\tau\varphi} |\partial^\alpha u|^2 \leq C \int_\Omega e^{2\tau\varphi} |Au|^2 \tag{3.3}$$

for all $u \in C_0^2(\Omega)$, $|\alpha| \leq 1$, $C < \gamma$, and $C_0(\gamma) < \tau$.

In [10] this result (for $a^{jk} \in C^\infty$) with constants depending on A was stated without proof; in [12] there are proofs for isotropic hyperbolic equations, and in [21] there are proofs with constants depending on A. In [27] it is shown that $\psi(x,t) = |x-a|^2 - \theta^2 t^2$ is pseudo-convex with respect to A if the speed of propagation is monotone in a certain direction. According to [30], [33], $\psi(x,t) = d^2(x,a) - \theta^2 t^2$ (d is the distance in the Riemannian metric determined by the elliptic part of A) is pseudo-convex if sectional curvatures are nonpositive. In [2], [12], [18], Carleman estimates with second large parameter under additional assumptions are used to obtain uniqueness of the continuation and controllability results for thermoelasticity systems.

Now we state a weak form of Theorem 3.1, where we assume, in addition, that the coefficients of A admit the bound

$$|a^{jk}|_2(\Omega) + \|b^j\|_\infty(\Omega) + \|\partial_j b^j\|_\infty(\Omega) + \|c\|_\infty(\Omega) \leq M.$$

Theorem 3.2 *Let A be a linear partial differential operator of second order with the principal coefficients in $C^2(\bar{\Omega})$ and with the coefficients of the first order derivatives in $C^1(\bar{\Omega})$. Let ψ be a K-pseudo-convex $C^3(\bar{\Omega})$-function with respect to A in $\bar{\Omega}$. Let $Av = f_0 + \sum_{j=1}^n \partial_j f_j$ in Ω. Then there are constants C, $C_0(\gamma)$ such that*

$$\int_\Omega \sigma e^{2\tau\varphi} v^2 \leq C \int_\Omega e^{2\tau\varphi} \left(\frac{1}{\sigma^2} f_0^2 + \sum_{j=1}^n f_j^2\right) \quad \textit{for all } v \in H_0^2(\Omega) \tag{3.4}$$

provided $C < \gamma$, $C_0(\gamma) < \tau$.

The weighted energy type estimates with large parameter τ, introduced by Carleman, proved first the uniqueness of the continuation results for elliptic systems on the plane with nonanalytic coefficients. Hörmander [14] linked it to the pseudo-convexity condition for the theory of functions of several complex variables and to energy estimates for general hyperbolic equations. At present there are several interesting (and in some cases complete) results on Carleman estimates and uniqueness of the continuation for second order equations, including elliptic, parabolic, Schrödinger type, and hyperbolic equations [19], [30].

Systems of partial differential equations, however, still remain a serious challenge. The only available general result is the celebrated theorem of Calderón of 1958 which is applicable mainly to some elliptic systems. There have been progress for classical dynamical isotropic Maxwell and elasticity systems [13], [17]. First uniqueness of continuation results for some anisotropic systems (including thermoelasticity system) were obtained by Albano and Tataru [2] and Isakov [18]. It was crucial in these papers to use Carleman type estimates with two large parameters (3.3), an idea first introduced and applied to the classical elasticity system in [17]. In [10] Theorem 3.1 (for C^∞-coefficients) was stated without a proof and in [12] there are not complete proofs for isotropic hyperbolic equations.

This chapter is organized as follows. In Section 3.1 we give a sufficient condition of pseudo-convexity of a function ψ with respect to the anisotropic wave operator $\Box(\mu; R) = \partial_t^2 - \sum_{jk} \frac{\mu\delta_{jk}+r_{jk}}{\rho}\partial_j\partial_k$ when R is small relative to constants ρ and μ, and we describe explicitly this smallness condition. Also, we give explicit sufficient global conditions for a general anisotropic hyperbolic operator A. In Section 3.2 we introduce the differential quadratic form. Sections 3.3 and 3.4 are central. There we prove Theorem 3.1 by using an explicit form of pseudo-convexity conditions for second order operators so that one can trace dependence on a second large parameter γ. The crucial part of the proof is Lemma 3.8, which gives a bound on the symbol of the differential quadratic form. Finding a suitable form of this bound is a decisive step in deriving Theorem 3.1. In the remaining part of Section 3.3 we conclude the proof by standard Fourier analysis methods augmented by proper localization and the use of a large parameter τ. In Section 3.4 we prove estimates of Theorem 3.2 in negative norms. A crucial idea of the proofs is to use pseudo-differential operator in (3.3), to localize estimates, and to freeze coefficients in an appropriate way. This substantially facilitates the use of Fourier analysis.

3.1 Pseudo-convexity condition for a general second order operator

It is not obvious or easy to find functions ψ which are pseudo-convex with respect to a general anisotropic operator, in particular, to the hyperbolic operator $A = \partial_t^2 - \sum_{j,k=1}^n a_{jk}\partial_j\partial_k$. In the isotropic case, explicit and verifiable conditions for $\psi(x,t) = |x-\beta|^2 - \theta^2 t^2$ were found by Isakov in 1980 and their simplifications are given in [19, section 3.4]. In the general anisotropic case Khaidarov [27] showed that under certain conditions the same ψ is pseudo-convex if the speed of the propagation determined by A is monotone in a certain direction. The most suitable choice is $\psi(x,t) = d^2(x,\beta) - \theta^2 t^2$ where d is the distance in the Riemannian metric determined by the spacial part of A. Lasiecka, Triggiani, and Yao [30] showed that this function is indeed pseudo-convex when d is convex in the Riemannian metric. Romanov [34] gave a simple independent proof, and emphasized that negativity of sectional curvatures are sufficient. A disadvantage of this choice of ψ is that, in most inverse problems, A and therefore the corresponding Riemannian metric are not known. In addition the known conditions of pseudo-convexity in the anisotropic case are not so easy to verify. For example, conditions in [30], [34] impose restrictions on second partial derivatives of a_{jk}. In applications, residual stress is relatively small [32]. Motivated by these reasons, we give simple sufficient conditions of pseudo-convexity for the scalar operators involving residual stress, $\Box(\mu;R) = \partial_t^2 - \sum_{jk}\frac{\mu\delta_{jk}+r_{jk}}{\rho}\partial_j\partial_k$, where "smallness" of R is explicit. Moreover, we derive explicit sufficient global conditions for A when $\psi(x,t) = |x-\beta|^2 - \theta^2 t^2$ is K-pseudo-convex.

Let θ and d be any real numbers.

Lemma 3.3 *Let G be a domain in $\mathbb{R}^n$ and $\Omega = G \times (0,T)$. Let μ be constant, the matrix R be symmetric positive at any point of Ω, its coefficients depend only on $x \in \mathbb{R}^n$, and*

$$2\mu\rho\theta^2 \,+\, 3\|R+\mu I\|\|\nabla R\||x| \,<\, 2\mu^2 \quad on \;\; \Omega. \tag{3.5}$$

Let

$$\theta^2 < \frac{\mu}{\rho}. \tag{3.6}$$

Then the function $\psi(x,t) = |x|^2 - \theta^2 t^2 - d^2$ is pseudo-convex with respect to the anisotropic wave operator $\square(\mu;R)$ in $\bar{\Omega} \cap \{|x-\beta|^2 > \theta^2 t^2\}$.

We recall that $\|R\|$ is the norm $(\sum_{j,k=1}^{3} r_{jk}^2)^{\frac{1}{2}}$ of a matrix $R = (r_{jk})$. Let $D = \sup|x-\beta|$ over $x \in G$ and $d = \inf|x-\beta|$ over $x \in G$, where $\Omega = G \times (-T,T)$.

Proof of Lemma 3.3

Due to the definition we need the positivity of the quadratic form

$$\mathcal{H} = \sum_{j,k=0}^{n} \partial_j\partial_k\psi \frac{\partial A}{\partial \xi_j}\frac{\partial A}{\partial \xi_k} \,+\, \sum_{j,k=0}^{n} \big((\partial_k \frac{\partial A}{\partial \xi_j})\frac{\partial A}{\partial \xi_k} - \partial_k A \frac{\partial^2 A}{\partial \xi_j \partial \xi_k}\big)\partial_j \psi.$$

Straightforward calculations with $A(x,\zeta) = \zeta_0^2 - \frac{\mu}{\rho}\zeta\cdot\zeta - \sum_{j,k=1}^{n}\frac{r_{jk}}{\rho}\zeta_j\zeta_k$ give

$$\mathcal{H} = -8\theta^2\xi_0^2 \,+\, 8\sum_{j=1}^{n}\big(\frac{1}{\rho}(\sum_{k=1}^{n} r_{jk}\xi_k + \mu\xi_j)\big)^2$$

$$+\sum_{j,k=1}^{n}\big\{\big(-\frac{2}{\rho}\sum_{l=1}^{n}\partial_k r_{jl}\xi_l\big)\big(-\frac{2}{\rho}(\sum_{m=1}^{n} r_{km}\xi_m + \mu\xi_k)\big)(2(x-\beta)_j)\big\}$$

$$-\sum_{j,k=1}^{n}\big\{\big(-\frac{1}{\rho}\sum_{l,m=1}^{n}\partial_k r_{lm}\xi_l\xi_m\big)\big(-\frac{2}{\rho}(r_{jk}+\mu\delta_{jk})\big)(2(x-\beta)_j)\big\}$$

$$= -\frac{8}{\rho}\mu\theta^2|\xi|^2 - \frac{8}{\rho}\theta^2\sum_{j,k=1}^{n} r_{jk}\xi_j\xi_k \,+\, \frac{8}{\rho^2}\sum_{j=1}^{n}\big((\sum_{k=1}^{n} r_{jk}\xi_k)^2 + 2\mu\xi_j(\sum_{k=1}^{n} r_{jk}\xi_k) + \mu^2\xi_j^2\big)$$

$$+\frac{8}{\rho^2}\sum_{j,k=1}^{n}\big\{\big(\sum_{l=1}^{n}\partial_k r_{jl}\xi_l\big)\big(\sum_{m=1}^{n} r_{km}\xi_m + \mu\xi_k\big)((x-\beta)_j)\big\}$$

$$-\frac{4}{\rho^2}\sum_{j,k=1}^{n}\big\{\big(\sum_{l,m=1}^{n}\partial_k r_{lm}\xi_l\xi_m\big)\big(r_{jk}+\mu\delta_{jk}\big)((x-\beta)_j)\big\}.$$

Hence

$$\mathcal{H} \geq -\frac{8}{\rho}\mu\theta^2|\xi|^2 - \frac{8}{\rho}\theta^2 \sum_{j,k=1}^n r_{jk}\xi_j\xi_k + \frac{8}{\rho^2}\sum_{j=1}^n(\sum_{k=1}^n r_{jk}\xi_k)^2$$

$$+ \frac{16}{\rho^2}\mu(\sum_{j,k=1}^n r_{jk}\xi_j\xi_k) + \frac{8}{\rho^2}\mu^2|\xi|^2$$

$$-\frac{8}{\rho^2}|\sum_{j,k=1}^n \{(\sum_{l=1}^n \partial_k r_{jl}\xi_l)(\sum_{m=1}^n r_{km}\xi_m + \mu\xi_k)((x-\beta)_j)\}|$$

$$-\frac{4}{\rho^2}|\sum_{j,k=1}^n \{(\sum_{l,m=1}^n \partial_k r_{lm}\xi_l\xi_m)(r_{jk}+\mu\delta_{jk})((x-\beta)_j)\}|$$

$$\geq \frac{8}{\rho}(\frac{\mu^2}{\rho} - \mu\theta^2)|\xi|^2 + \frac{8}{\rho}(\frac{2\mu}{\rho} - \theta^2)\sum_{j,k=1}^n r_{jk}\xi_k\xi_k$$

$$-\frac{8}{\rho^2}\sum_{k=1}^n \|\partial_k R\||\xi||x-\beta|\sum_{m=1}^n |r_{km}+\mu\delta_{km}||\xi_m|$$

$$-\frac{4}{\rho^2}\sum_{j,k=1}^n \|\partial_k R\||\xi|^2|r_{jk}+\mu\delta_{jk}|\|(x-\beta)_j\|,$$

where we used the relation

$$|\sum_{j,k=1}^n r_{jk}\xi_j\eta_k| \leq \|R\||\xi||\eta|$$

which follows from the Cauchy-Schwartz inequality. Using this inequality again we conclude that

$$\mathcal{H} \geq \frac{8}{\rho}(\frac{\mu^2}{\rho} - \mu\theta^2)|\xi|^2 - \frac{12}{\rho^2}\|R+\mu I\|\|\nabla R\||x-\beta||\xi|^2.$$

Hence the positivity of $\mathcal{H}$ follows from (3.5).

Since $|x-\beta|^2 > \theta^2 t^2$ we have

$$A(x, \nabla\psi(x)) = 4\theta^4 t^2 - \frac{\mu}{\rho}4|x-\beta|^2 - \sum_{j,k=1}^n \frac{r_{jk}}{\rho}4(x-\beta)_j(x-\beta)_k$$

$$< 4((\theta^2 - \frac{\mu}{\rho})|x-\beta|^2 - \sum_{j,k=1}^n \frac{r_{jk}}{\rho}(x-\beta)_j(x-\beta)_k) < 0$$

on $\bar{\Omega} \cap \{\, |x-\beta|^2 > \theta^2 t^2\}$ due to the condition (3.6) and the definition of Ω_0. So $\nabla\psi$ is not characteristic on this set. □

In Lemma 3.4 for a general anisotropic hyperbolic operator we give the condition of K-pseudo-convexity of $\psi(x,t) = |x-\beta|^2 - \theta^2 t^2$, x, $\beta = (0,\dots,0,\beta_n) \in \mathbb{R}^n$ with β_n large enough.

Lemma 3.4 *Let*

$$A = \partial_t^2 - \sum_{j,k=1}^{n} a_{jk}\partial_j\partial_k, \quad a_{jk} = a_{kj},$$

where $a_{jk} \in C^1$ satisfies the uniform ellipticity condition

$$\sum_{j,k=1}^{n} a_{jk}\xi_j\xi_k \ge \varepsilon_0|\xi|^2, \quad \xi \in \mathbb{R}^n, \quad \varepsilon_0 > 0. \tag{3.7}$$

Let

$$\psi(x,t) = |x-\beta|^2 - \theta^2 t^2, \quad \beta = (0,\dots,0,\beta_n).$$

Assume that

$$\sum_{j,l=1}^{n}\Big(\sum_{k=1}^{n} a_{nk}\partial_k a_{jl} - 2\sum_{k=1}^{n-1} a_{lk}\partial_k a_{jn}\Big)\xi_j\xi_l \ \ge\ \varepsilon_1|\xi|^2, \quad \xi \in \mathbb{R}^n \tag{3.8}$$

for some $\varepsilon_1 > 0$. Then there is large β_n such that the function ψ is K-pseudo-convex with respect to A in $\bar{\Omega}$.

Proof of Lemma 3.4

Denoting the left side in (2.46) by $\mathcal{H}$ we have $\mathcal{H} = \mathcal{H}_1 + \mathcal{H}_2$ where

$$\mathcal{H}_1 = \sum_{j,k=0}^{n} \partial_j\partial_k\psi \frac{\partial A}{\partial\xi_j}\frac{\partial A}{\partial\xi_k}$$

and

$$\mathcal{H}_2 = \sum_{j,k=0}^{n} \Big(\big(\partial_k \frac{\partial A}{\partial\xi_j}\big)\frac{\partial A}{\partial\xi_k} - \partial_k A \frac{\partial^2 A}{\partial\xi_j\partial\xi_k}\Big)\partial_j\psi.$$

Using the first equality of (2.45) we yield

$$\mathcal{H}_1 = -8\theta^2\sum_{j,k=1}^{n} a_{jk}\xi_j\xi_k + 8\sum_{j=1}^{n}\Big(\sum_{k=1}^{n} a_{jk}\xi_k\Big)^2.$$

Using that $\partial_0 A$, $\partial_0 \frac{\partial A}{\partial \xi_0}$, $\frac{\partial^2 A}{\partial \xi_j \partial \xi_0}$, $\partial_k \frac{\partial A}{\partial \xi_0}$, $\frac{\partial^2 A}{\partial \xi_0 \partial \xi_k}$, $j, k = 1, \ldots, n$ are all zeros, and that $\partial_j \psi = 2(x_j - \beta_j)$ and $\beta_j = 0$ for $j = 1, \ldots, n-1$,

$$\mathcal{H}_2 = \sum_{j,k=1}^{n} \{(-2\sum_{l=1}^{n} \partial_k a_{jl}\xi_l)(-2\sum_{m=1}^{n} a_{mk}\xi_m)$$

$$-(-\sum_{p,q=1}^{n} \partial_k a_{pq}\xi_p\xi_q)(-2a_{jk})\}2(x_j - \beta_j).$$

Hence

$$\begin{aligned} &\mathcal{H}_1 + \mathcal{H}_2 \\ &= -8\theta^2 \sum_{j,k=1}^{n} a_{jk}\xi_j\xi_k + 8\sum_{j=1}^{n}(\sum_{k=1}^{n} a_{jk}\xi_k)^2 \\ &+4\sum_{j,k=1}^{n} \{2(\sum_{l=1}^{n} \partial_k a_{jl}\xi_l)(\sum_{m=1}^{n} a_{mk}\xi_m) - a_{jk}(\sum_{p,q=1}^{n} \partial_k a_{pq}\xi_p\xi_q)\}x_j \\ &+4\beta_n \sum_{k=1}^{n} \{a_{nk}(\sum_{p,q=1}^{n} \partial_k a_{pq}\xi_p\xi_q) - 2(\sum_{l=1}^{n} \partial_k a_{nl}\xi_l)(\sum_{m=1}^{n} a_{mk}\xi_m)\}. \end{aligned}$$

Let

$$\mathcal{H}_3 = \sum_{j,l=1}^{n}\sum_{k=1}^{n} \{a_{nk}\partial_k a_{jl} - 2a_{lk}\partial_k a_{jn}\}\xi_j\xi_l. \tag{3.9}$$

Then with large β_n the positivity of (3.9) guarantees the positivity of $\mathcal{H}_1 + \mathcal{H}_2$, and hence (2.46).

From the second condition (2.45) of K-pseudo-convexity in Definition 2.13

$$-4\theta^2 t\xi_0 + 2\sum_{j,k=1}^{n} a_{jk}\xi_k x_j - 2\sum_{k=1}^{n} a_{nk}\xi_k\beta_n = 0,$$

hence

$$\sum_{k=1}^{n} a_{nk}\xi_k = \frac{1}{\beta_n}O(|\xi|) \tag{3.10}$$

where $|O(|\xi|)| \le C|\xi|$, $|\xi| = (\xi_1^2 + \cdots + \xi_n^2)^{\frac{1}{2}}$.

Therefore

$$\mathcal{H}_3(\xi) = \sum_{j,l=1}^{n}(\sum_{k=1}^{n} a_{nk}\partial_k a_{jl} - 2\sum_{k=1}^{n-1} a_{lk}(\partial_k a_{jn})\xi_j\xi_l) + \frac{1}{\beta_n}O(|\xi|^2). \tag{3.11}$$

We have

$$A(x,t;\nabla\psi(x,t)) \;=\; 4\theta^4 t^2 - 4\sum_{j,k=1}^{n} a_{jk}(x)(x_j-\beta_j)(x_k-\beta_k) \;=\; -a_{nn}(x)\beta_n^2 \,+\, \cdots$$

where $\cdots$ denotes the terms bounded by $C\beta_n$. So $\nabla\psi$ is not characteristic in Ω for large β_n.

□

A version of this lemma for a different weight function ψ is given in [3].

Corollary 3.5 *Let us assume the monotonicity of the speed of the propagation with respect to* A :

$$\sum_{j,k,l=1}^{n} a_{nk}\partial_k a_{jl}\xi_j\xi_l \geq \varepsilon_1|\xi|^2, \quad \xi\in\mathbb{R}^n, \tag{3.12}$$

for some $\varepsilon_1 > 0$ *and that the symmetrization of the matrix* $(\sum_{k=1}^{n-1} a_{lk}\partial_k a_{jn})$ *is nonpositive. Then there is large* β_n *such that the function* ψ *is* K*-pseudo-convex with respect to* A *in* $\bar{\Omega}$.

One can give more precise sufficient conditions for (3.8). For example, by using the (Frobenius) operator norm of a matrix in $L^2(\mathbb{R}^n)$ one of these conditions is

$$4\sum_{j,k=1}^{n}(\sum_{k=1}^{n-1} a_{lk}\partial_k a_{jn})^2\xi_j\xi_l \;<\; \varepsilon_1^2.$$

This corollary gives more general pseudo-convexity conditions than in [27] where it was assumed that $a_{jn}=0$ when $j<n$.

A useful matrix notation for the main terms of (3.11) is given by

$$\xi^\top(A_n\cdot\nabla)A\xi \,-\, 2\xi^\top(A\cdot\nabla)'A_n^\top\xi \tag{3.13}$$

where A_j is the j^{th} row of the matrix

$$A = \begin{pmatrix} a_{11} & a_{12} & \dots & a_{1n} \\ a_{21} & a_{22} & \dots & a_{2n} \\ \dots & \dots & \dots & \dots \\ a_{n1} & a_{n2} & \dots & a_{nn} \end{pmatrix},$$

with $(A_n\cdot\nabla) = a_{n1}\partial_1 + a_{n2}\partial_2 + \cdots + a_{nn}\partial_n$ and $(A\cdot\nabla)' = A_1\partial_1 + A_2\partial_2 + \cdots + A_{n-1}\partial_{n-1}$.

Now we obtain sufficient conditions for the positivity of the main terms in (3.11). Suppose the monotonicity of speed propagation $(A_n \cdot \nabla)A$ satisfies the uniform ellipticity condition such that

$$\sum_{j,k,l=1}^{n} a_{nk}\partial_k a_{jl}\xi_j\xi_l \geq \varepsilon_1|\xi|^2, \quad \xi \in \mathbb{R}^n, \quad \varepsilon_1 > 2\|A'\|\|\nabla' A_n\|.$$

Then the main terms in (3.11) have the positivity

$$\sum_{j,k,l=1}^{n} a_{nk}\partial_k a_{jl}\xi_j\xi_l \, - \, 2\sum_{k=1}^{n-1}\sum_{j,l=1}^{n} a_{lk}\xi_l(\partial_k a_{jn}\xi_j)$$

$$\geq \varepsilon_1|\xi|^2 \, - \, 2\,\Big|\sum_{k=1}^{n-1}\sum_{l=1}^{n} a_{lk}\xi_l\Big|\Big|\sum_{k=1}^{n-1}\sum_{j=1}^{n} \partial_k a_{jn}\xi_j\Big|$$

$$\geq \varepsilon_1|\xi|^2 \, - \, 2\|A'\|\|\nabla' A_n\||\xi|^2$$

by using Cauchy-Schwartz inequality and the matrix norm $\|A\| = (\sum_{j,k=1}^{n} a_{jk}^2)^{\frac{1}{2}}$.

Now we consider in more detail the cases of two and three dimensions.

Example 1 : $(n = 2)$

From (3.10) we have $\xi_2 = -\frac{a_{21}}{a_{22}}\xi_1 + \beta_2^{-1}O(|\xi|)$, so by routine calculations

$$\sum_{j,l=1}^{2} a_{l1}\partial_1 a_{j2}\xi_j\xi_l$$

$$= \, (a_{11}a_{22} - a_{12}^2)\frac{a_{22}\partial_1 a_{12} - a_{12}\partial_1 a_{22}}{a_{22}^2}\xi_1^2 \, + \, \beta_2^{-1}O(\xi_1).$$

Due to positivity of the matrix (a_{jk}), we have $a_{11}a_{22} - a_{12}^2 > 0$ and $a_{22} > 0$, so the nonpositivity of the principal term (with respect to large β_2) is

$$\partial_1\frac{a_{12}}{a_{22}} \leq 0,$$

which is therefore a sufficient condition for K-pseudo-convexity of ψ for large β_2, provided we have monotonicity of the speed of the propagation in the x_2-direction.

Using $\xi_2 = -\frac{a_{21}}{a_{22}}\xi_1 + \beta_2^{-1}O(|\xi|)$ from (3.10) we have

$$\mathcal{H}_3 + \beta_2^{-1}O(|\xi|) = \sum_{j,k,l=1}^{2} a_{2k}\partial_k a_{jl}\xi_j\xi_l \, - \, 2\sum_{j,l=1}^{2} a_{l1}\partial_1 a_{j2}\xi_j\xi_l$$

$$= \Big((a_{21}\partial_1 a_{11} + a_{22}\partial_2 a_{11}) - 2a_{11}a_{22}\frac{a_{22}\partial_1 a_{12} - a_{12}\partial_1 a_{22}}{a_{22}^2}$$
$$- a_{12}^2(\frac{\partial_2 a_{12}}{a_{12}} - \frac{\partial_2 a_{22}}{a_{22}}) - \frac{a_{12}^2}{a_{22}^2}(a_{12}\partial_1 a_{22} + a_{22}^2\frac{\partial_2 a_{22}}{a_{12}})\Big)\xi_1^2.$$

Since $\partial_2 \frac{a_{12}}{a_{22}} \leq 0$ implies that $\frac{\partial_2 a_{12}}{a_{12}} - \frac{\partial_2 a_{22}}{a_{22}} \leq 0$, we have new sufficient conditions for positivity of $\mathcal{H}_3$:

$$0 < a_{12}\partial_1 a_{11} + a_{22}\partial_2 a_{11}, \quad \partial_1 \frac{a_{12}}{a_{22}} \leq 0, \quad \partial_2 \frac{a_{12}}{a_{22}} \leq 0, \quad \text{and} \quad a_{12}(a_{12}^2\partial_1 a_{22} + a_{22}^2\partial_2 a_{22}) \leq 0.$$

These conditions imply the positivity or the negativity of some derivatives or conormal derivatives in the x_2-direction.

Example 2 : $(n = 3)$

By using $\xi_3 = -\frac{a_{31}}{a_{33}}\xi_1 - \frac{a_{32}}{a_{33}}\xi_2 + \cdots$ from (3.10) we obtain

$$\mathcal{H}_3 + \beta_3^{-1}O(|\xi|) = \sum_{j,k,l=1}^{3} a_{3k}\partial_k a_{jl}\xi_j\xi_l \; - \; 2\sum_{k=1}^{2}\sum_{j,l=1}^{3} a_{lk}\partial_k a_{j3}\xi_j\xi_l$$
$$= A_1\xi_1^2 + 2A_2\xi_1\xi_2 + A_3\xi_2^2$$

where

$$A_1 = (a_{13}\partial_1 a_{11} + a_{23}\partial_2 a_{11} + a_{33}\partial_3 a_{11}) - \frac{a_{13}^2}{a_{33}^2}(a_{13}\partial_1 a_{33} + a_{23}\partial_2 a_{33} + a_{33}\partial_3 a_{33})$$
$$- 2a_{33}(a_{11}\partial_1 \frac{a_{13}}{a_{33}} + a_{23}\partial_2 \frac{a_{13}}{a_{33}} + a_{13}\partial_3 \frac{a_{13}}{a_{33}}),$$
$$A_2 = (a_{13}\partial_1 a_{12} + a_{23}\partial_2 a_{12} + a_{33}\partial_3 a_{12}) - a_{33}(a_{21}\partial_1 \frac{a_{13}}{a_{33}} + a_{22}\partial_2 \frac{a_{13}}{a_{33}} + a_{23}\partial_3 \frac{a_{13}}{a_{33}})$$
$$- a_{33}(a_{11}\partial_1 \frac{a_{23}}{a_{33}} + a_{12}\partial_2 \frac{a_{23}}{a_{33}} + a_{13}\partial_3 \frac{a_{23}}{a_{33}}) - \frac{a_{13}a_{23}}{a_{33}^2}(a_{13}\partial_1 a_{33} + a_{23}\partial_2 a_{33} + a_{33}\partial_3 a_{33}),$$

and

$$A_3 = (a_{13}\partial_1 a_{22} + a_{23}\partial_2 a_{22} + a_{33}\partial_3 a_{22}) - \frac{a_{23}^2}{a_{33}^2}(a_{13}\partial_1 a_{33} + a_{23}\partial_2 a_{33} + a_{33}\partial_3 a_{33})$$
$$- 2a_{33}(a_{12}\partial_1 \frac{a_{23}}{a_{33}} + a_{22}\partial_2 \frac{a_{23}}{a_{33}} + a_{23}\partial_3 \frac{a_{23}}{a_{33}}).$$

The positivity of $\mathcal{H}_3$ follows from the inequalities $0 < A_1$ and $A_2^2 < A_1 A_3$. The formulas for A_1, A_2, and A_3 contain several simple and meaningful blocks which have geometrical or physical interpretations.

Using the well-known inequality $\frac{2A_1A_3}{A_1+A_3} \le \sqrt{A_1A_3}$ with $A_1, A_3 > 0$, we have that $A_2 < \frac{2A_1A_3}{A_1+A_3}$ implies $A_2 < \sqrt{A_1A_3}$. Using $0 \le k \le 1$,

$$\frac{2A_1A_3}{A_1+A_3} = \frac{A_3}{A_1+A_3}A_1 + \frac{A_1}{A_1+A_3}A_3 = kA_1 + (1-k)A_3.$$

Hence $A_2 < kA_1 + (1-k)A_3,\ 0 \le k \le 1$, implies $A_2 < \sqrt{A_1A_3}$.

The positivity of $\mathcal{H}_3$ follows from the inequalities $A_1 > 0$, $A_3 > 0$, and $A_2 < kA_1 + (1-k)A_3$ where $0 \le k \le 1$. Here $j = 1, 2, 3$.

For $A_1 > 0$ and $A_3 > 0$ we have

$$a_{3j} \cdot \partial_j a_{11} > 0, \quad a_{3j} \cdot \partial_j a_{22} > 0, \quad a_{3j} \cdot \partial_j a_{33} < 0,$$

and

$$a_{1j} \cdot \partial_j \frac{a_{13}}{a_{33}} < 0, \quad a_{2j} \cdot \partial_j \frac{a_{23}}{a_{33}} < 0.$$

For $A_2 < kA_1 + (1-k)A_3$ with $0 \le k \le 1$ we have

$$a_{3j} \cdot \partial_j (a_{12} - ka_{11} - (1-k)a_{22}) < 0,$$

$$a_{13}a_{23} - ka_{13}^2 - (1-k)a_{23}^2 < 0,$$

and

$$(2ka_{1j} - a_{2j}) \cdot \partial_j \frac{a_{13}}{a_{33}} + (2(1-k)a_{2j} - a_{1j}) \cdot \partial_j \frac{a_{23}}{a_{33}} < 0.$$

3.2 Divergent form $\mathcal{F}$

For the following, set $\zeta(\varphi)(x) = \xi + i\tau\nabla\varphi(x)$. We introduce the differential quadratic form

$$\mathcal{F}(x, \tau, D, \bar{D})v\bar{v}$$
$$= |A(x, D + i\tau\nabla\varphi(x))v|^2 - |A(x, D - i\tau\nabla\varphi(x))v|^2. \tag{3.14}$$

This differential quadratic form is of order $(3\,;2)$, since the coefficients of the principal part of A are real valued. By Lemma 2.11 there exists differential quadratic form $\mathcal{G}(x, \tau, D, \bar{D})$ of order $(2\,;1)$ such that

$$\int_\Omega \mathcal{G}(x, D, \bar{D})v\bar{v} = \int_\Omega \mathcal{F}(x, D, \bar{D})v\bar{v} \tag{3.15}$$

and its symbol

$$\mathcal{G}(x,\tau,\xi,\xi) = \frac{1}{2}\sum \frac{\partial^2}{\partial x_k \partial \eta_k}\mathcal{F}(x,\tau,\zeta,\bar{\zeta}), \quad \zeta = \xi + i\eta, \quad \text{at} \quad \eta = 0$$

where

$$\mathcal{F}(x,\tau,\zeta,\bar{\zeta}) = A(x,\zeta + i\tau\nabla\varphi)A(x,\bar{\zeta} - i\tau\nabla\varphi) - A(x,\zeta - i\tau\nabla\varphi)A(x,\bar{\zeta} + i\tau\nabla\varphi).$$

Lemma 3.6 *We have*

$$\mathcal{G}(x,\tau,\xi,\xi)$$
$$= 2\tau\sum \partial_j\partial_k\varphi \frac{\partial A}{\partial \zeta_j}\overline{\frac{\partial A}{\partial \zeta_k}} + 2\Im\sum \partial_k A\,\overline{\frac{\partial A}{\partial \zeta_k}} + 2\Im\sum A\,\Big(\overline{\frac{\partial^2 A}{\partial \zeta_k \partial x_k}} - i\tau\partial_j\partial_k\varphi\,\overline{\frac{\partial^2 A}{\partial \zeta_j \partial \zeta_k}}\Big) \quad (3.16)$$

where A, $\partial_k A, \ldots$ are taken at $(x,\zeta(\varphi)(x))$.

Proof of Lemma 3.6

Indeed, at $\eta = 0$

$$\frac{1}{2}\sum \frac{\partial^2}{\partial x_k \partial \eta_k}\mathcal{F}(x,\xi + i\eta,\xi - i\eta)$$
$$= \frac{1}{2}\sum \partial_k\Big(i\frac{\partial A}{\partial \zeta_k}(x,\xi + i\tau\nabla\varphi)A(x,\xi - i\tau\nabla\varphi) - iA(x,\xi + i\tau\nabla\varphi)\frac{\partial A}{\partial \zeta_k}(x,\xi - i\tau\nabla\varphi)$$
$$-i\,\frac{\partial A}{\partial \zeta_k}(x,\xi - i\tau\nabla\varphi)A(x,\xi + i\tau\nabla\varphi) + iA(x,\xi - i\tau\nabla\varphi)\frac{\partial A}{\partial \zeta_k}(x,\xi + i\tau\nabla\varphi)\Big)$$
$$= i\sum \partial_k\Big(\frac{\partial A}{\partial \zeta_k}(x,\zeta(\varphi))A(x,\bar{\zeta}(\varphi)) - \frac{\partial A}{\partial \zeta_k}(x,\bar{\zeta}(\varphi))A(x,\zeta(\varphi))\Big).$$

Using that $i(z\bar{w} - \bar{z}w) = -2\Im(z\bar{w})$, we yield

$$\mathcal{G}(x,\tau,\xi,\xi)$$
$$= -2\Im\sum \partial_k\Big(\frac{\partial A}{\partial \zeta_k}(x,\zeta(\varphi)(x))A(x,\bar{\zeta}(\varphi)(x))\Big)$$
$$= -2\Im\sum\Big(\Big(\frac{\partial^2 A}{\partial x_k \partial \zeta_k}(x,\zeta(\varphi)) + i\tau\partial_j\partial_k\varphi\frac{\partial^2 A}{\partial \zeta_j \partial \zeta_k}(x,\zeta(\varphi))\Big)A(x,\bar{\zeta}(\varphi))$$
$$+\frac{\partial A}{\partial \zeta_k}(x,\zeta(\varphi))\frac{\partial A}{\partial x_k}(x,\bar{\zeta}(\varphi)) - i\tau\partial_j\partial_k\varphi\frac{\partial A}{\partial \zeta_k}(x,\zeta(\varphi))\frac{\partial A}{\partial \zeta_j}(x,\bar{\zeta}(\varphi))\Big)$$
$$= 2\tau\sum \partial_j\partial_k\varphi\frac{\partial A}{\partial \zeta_k}\overline{\frac{\partial A}{\partial \zeta_j}} - 2\Im\sum \partial_k\bar{A}\,\frac{\partial A}{\partial \zeta_k} - 2\Im\sum \bar{A}\,\Big(\frac{\partial^2 A}{\partial x_k \partial \zeta_k} + i\tau\partial_j\partial_k\varphi\frac{\partial^2 A}{\partial \zeta_j \partial \zeta_k}\Big)$$

$$= 2\tau \sum \partial_j\partial_k\varphi \overline{\frac{\partial A}{\partial \zeta_k}} \frac{\partial A}{\partial \zeta_j} + 2\Im \sum \partial_k A \overline{\frac{\partial A}{\partial \zeta_k}} + 2\Im \sum A \, (\overline{\frac{\partial^2 A}{\partial x_k \partial \zeta_k} + i\tau\partial_j\partial_k\varphi \frac{\partial^2 A}{\partial \zeta_j \partial \zeta_k}})$$

$$= 2\tau \sum \partial_j\partial_k\varphi \frac{\partial A}{\partial \zeta_j} \overline{\frac{\partial A}{\partial \zeta_k}} + 2\Im \sum \partial_k A \overline{\frac{\partial A}{\partial \zeta_k}} + 2\Im \sum A \, (\overline{\frac{\partial^2 A}{\partial \zeta_k \partial x_k}} - i\tau\partial_j\partial_k\varphi \, \overline{\frac{\partial^2 A}{\partial \zeta_j \partial \zeta_k}})$$

by the chain rule and the fact that $\frac{\partial \zeta_j}{\partial x_k} = i\tau\partial_j\partial_k\varphi$. Observing that $A(\, , \bar{\zeta}(\varphi)) = \bar{A}$, that $-\Im(z\bar{w}) = \Im(\bar{z}w)$, and noting that the coefficients of A are real-valued and hence

$$\sum \partial_j\partial_k\varphi \frac{\partial A}{\partial \zeta_k} \overline{\frac{\partial A}{\partial \zeta_j}}$$

is real valued, equation (3.16) is obtained. □

The differentiation formulas

$$\partial_j\varphi = \gamma\varphi\partial_j\psi\,, \quad \partial_j\partial_k\varphi = \gamma\varphi\partial_j\partial_k\psi + \gamma^2\varphi\partial_j\psi\partial_k\psi \tag{3.17}$$

follow from the defined weight function (3.2) and are used in our proofs.

Lemma 3.7 *Using formulas (3.17), from Lemma 3.6 we yield*

$$\tau^{-1}\mathcal{G}(x,\tau,\xi,\xi) = \mathcal{G}_1(x,\tau,\xi,\xi) + \mathcal{G}_2(x,\tau,\xi,\xi) + \mathcal{G}_3(x,\tau,\xi,\xi) + \mathcal{G}_4(x,\tau,\xi,\xi) \tag{3.18}$$

where

$$\mathcal{G}_1(x,\tau,\xi,\xi) = 8\gamma\varphi \sum a^{jm}a^{kl}(\xi_m\xi_l + \sigma^2\partial_m\psi\partial_l\psi)\partial_j\partial_k\psi,$$
$$\mathcal{G}_2(x,\tau,\xi,\xi) = 4\gamma\varphi \sum a^{lk}\partial_k a^{jm}(\sigma^2\partial_j\psi\partial_m\psi\partial_l\psi + 2\xi_m\xi_l\partial_j\psi - \xi_j\xi_m\partial_l\psi),$$
$$\mathcal{G}_3(x,\tau,\xi,\xi) = 4\gamma\varphi \big(2\sum a^{km}\partial_j a^{lj}\partial_k\psi\xi_l\xi_m$$
$$- \sum a^{jk}(\partial_m a^{lm}\partial_l\psi + a^{lm}\partial_l\partial_m\psi)(\xi_j\xi_k - \sigma^2\partial_j\psi\partial_k\psi)\big),$$

and

$$\mathcal{G}_4(x,\tau,\xi,\xi) = 4\gamma^2\varphi \big((2\sum a^{jm}\xi_m\partial_j\psi)^2 + 2\sigma^2(\sum a^{jm}\partial_j\psi\partial_m\psi)^2$$
$$- (\sum a^{lm}(\xi_l\xi_m - \sigma^2\partial_l\psi\partial_m\psi))(\sum a^{jk}\partial_j\psi\partial_k\psi)\big).$$

Observe that the terms of $\tau^{-1}\mathcal{G}$ with the highest powers of γ are collected in $\mathcal{G}_4$.

Proof of Lemma 3.7

From (3.16) the expression $\tau^{-1}\mathcal{G}(x,\tau,\xi,\xi)$ yields

$$2\sum \partial_j\partial_k\varphi\frac{\partial A}{\partial\zeta_j}\overline{\frac{\partial A}{\partial\zeta_k}} + 2\tau^{-1}\Im\sum\partial_k A\,\overline{\frac{\partial A}{\partial\zeta_k}} + 2\tau^{-1}\Im\sum A\,\big(\overline{\frac{\partial^2 A}{\partial\zeta_k\partial x_k}} - i\tau\partial_j\partial_k\varphi\,\overline{\frac{\partial^2 A}{\partial\zeta_j\partial\zeta_k}}\big). \quad (3.19)$$

The first term of (3.19) yields

$$2\sum \partial_j\partial_k\varphi\frac{\partial A}{\partial\zeta_j}\overline{\frac{\partial A}{\partial\zeta_k}}$$

$$= 8\sum a^{jm}a^{kl}(\xi_m + i\gamma\tau\varphi\partial_m\psi)(\xi_l - i\gamma\tau\varphi\partial_l\psi)(\gamma\varphi\partial_j\partial_k\psi + \gamma^2\varphi\partial_j\psi\partial_k\psi)$$

$$= 8\sum a^{jm}a^{kl}\big((\xi_m\xi_l + \gamma^2\tau^2\varphi^2\partial_m\psi\partial_l\psi) + i\gamma\tau\varphi(\xi_l\partial_m\psi - \xi_m\partial_l\psi)\big)(\gamma\varphi\partial_j\partial_k\psi + \gamma^2\varphi\partial_j\psi\partial_k\psi)$$

$$= 8\sum\big(a^{jm}a^{kl}\xi_m\xi_l(\gamma\varphi\partial_j\partial_k\psi+\gamma^2\varphi\partial_j\psi\partial_k\psi) + a^{jm}a^{kl}\gamma^2\tau^2\varphi^2\partial_m\psi\partial_l\psi(\gamma\varphi\partial_j\partial_k\psi+\gamma^2\varphi\partial_j\psi\partial_k\psi)\big)$$

$$= 8\gamma\varphi\sum a^{jm}a^{kl}(\xi_m\xi_l+\sigma^2\partial_m\psi\partial_l\psi)\partial_j\partial_k\psi + 8\gamma^2\varphi\big((\sum a^{jm}\xi_m\partial_j\psi)^2 + \sigma^2(\sum a^{jm}\partial_j\psi\partial_m\psi)^2\big). \quad (3.20)$$

Note that since $(\frac{\partial A}{\partial\zeta_j}\overline{\frac{\partial A}{\partial\zeta_k}})$ is a symmetric matrix, it has a real value (imaginary part = 0).

The second term of (3.19) yields

$$2\tau^{-1}\Im\sum\partial_k A\,\overline{\frac{\partial A}{\partial\zeta_k}}$$

$$= 4\tau^{-1}\Im\sum\partial_k a^{jm}\zeta_j\zeta_m a^{lk}\bar{\zeta}_l$$

$$= 4\tau^{-1}\tau\sum a^{lk}\partial_k a^{jm}(\tau^2\partial_j\varphi\partial_m\varphi\partial_l\varphi + \xi_m\xi_l\partial_j\varphi + \xi_j\xi_l\partial_m\varphi - \xi_j\xi_m\partial_l\varphi)$$

$$= 4\sum a^{lk}\partial_k a^{jm}\big(\tau^2\gamma^3\varphi^3\partial_j\psi\partial_m\psi\partial_l\psi + \gamma\varphi(\xi_m\xi_l\partial_j\psi + \xi_j\xi_l\partial_m\psi - \xi_j\xi_m\partial_l\psi)\big)$$

$$= 4\gamma\varphi\sum a^{lk}\partial_k a^{jm}(\sigma^2\partial_j\psi\partial_m\psi\partial_l\psi + 2\xi_m\xi_l\partial_j\psi - \xi_j\xi_m\partial_l\psi). \quad (3.21)$$

The last term of (3.19) yields

$$2\tau^{-1}\Im\sum A\,\big(\overline{\frac{\partial^2 A}{\partial\zeta_k\partial x_k}} - i\tau\partial_j\partial_k\varphi\,\overline{\frac{\partial^2 A}{\partial\zeta_j\partial\zeta_k}}\big)$$

$$= 4\tau^{-1}\Im\sum a^{lm}\zeta_l\zeta_m\,\big(\partial_k a^{jk}\bar{\zeta}_j - i\tau\partial_j\partial_k\varphi a^{jk}\big)$$

$$= 4\tau^{-1}\Im\sum a^{lm}\big((\xi_l\xi_m-\tau^2\partial_l\varphi\partial_m\varphi)+i\tau(\xi_l\partial_m\varphi+\xi_m\partial_l\varphi)\big)\big(\partial_k a^{jk}\xi_j - i\tau(\partial_k a^{jk}\partial_j\varphi + a^{jk}\partial_j\partial_k\varphi)\big)$$

$$= 4\tau^{-1}\tau \sum a^{lm}\big(\partial_k a^{jk}\xi_j(\xi_l\partial_m\varphi + \xi_m\partial_l\varphi) \,-\, (\xi_l\xi_m - \tau^2\partial_l\varphi\partial_m\varphi)(\partial_k a^{jk}\partial_j\varphi + a^{jk}\partial_j\partial_k\varphi)\big)$$

$$= 4\sum a^{lm}\big(2\partial_k a^{jk}\xi_j\xi_m\partial_l\varphi \,-\, \partial_k a^{jk}\xi_l\xi_m\partial_j\varphi \,+\, \tau^2\partial_k a^{jk}\partial_j\varphi\partial_l\varphi\partial_m\varphi \,-\, a^{jk}\xi_l\xi_m\partial_j\partial_k\varphi$$
$$+\,\tau^2 a^{jk}\partial_l\varphi\partial_m\varphi\partial_j\partial_k\varphi\,\big)$$

$$= 4\gamma\varphi\sum\big(2a^{lm}\partial_k a^{jk}\xi_j\xi_m\partial_l\psi \,-\, a^{lm}\partial_k a^{jk}\xi_l\xi_m\partial_j\psi \,+\, \sigma^2 a^{lm}\partial_k a^{jk}\partial_j\psi\partial_l\psi\partial_m\psi$$
$$-\,a^{lm}a^{jk}\xi_l\xi_m(\partial_j\partial_k\psi + \gamma\partial_j\psi\partial_k\psi) \,+\, \sigma^2 a^{lm}a^{jk}\partial_l\psi\partial_m\psi(\partial_j\partial_k\psi \,+\, \gamma\partial_j\psi\partial_k\psi)\big)$$

$$= 4\gamma\varphi\big(2\sum a^{lm}\partial_k a^{jk}\xi_j\xi_m\partial_l\psi \,-\sum a^{lm}(\partial_k a^{jk}\partial_j\psi \,+\, a^{jk}\partial_j\partial_k\psi)(\xi_l\xi_m \,-\, \sigma^2\partial_l\psi\partial_m\psi)\big)$$
$$-\, 4\gamma^2\varphi\sum a^{lm}(\xi_l\xi_m - \sigma^2\partial_l\psi\partial_m\psi)\big(\sum a^{jk}\partial_j\psi\partial_k\psi\big). \tag{3.22}$$

Using equations (3.20) through (3.22) and collecting the highest power of γ in $\mathcal{G}_4$ yields (3.18). $\square$

3.3 Strong Carleman estimates for scalar operators

In this section, we prove Carleman estimates for general scalar operators (Theorem 3.1) with the technique of differential quadratic forms.

Proof of Theorem 3.1

First, using Lemma 2.5 in Section 2.2, make the substitution $u = e^{-\tau\varphi}v$. Obviously $D_k(e^{-\tau\varphi}v) = e^{-\tau\varphi}(D_k + i\tau\partial_k\varphi)v$. Hence

$$\sum a^{jk}D_jD_k(e^{-\tau\varphi}v) = \sum a^{jk}e^{-\tau\varphi}(D_j + i\tau\partial_j\varphi)(D_k + i\tau\partial_k\varphi)v.$$

Accordingly the bound (3.3) is transformed into

$$\sum\int_\Omega \sigma^{3-2|\alpha|}|\partial^\alpha v|^2 \,\le\, C\int_\Omega |A(\,,D + i\tau\nabla\varphi)v|^2. \tag{3.23}$$

Lemma 3.8 *Under the conditions of Theorem 3.1 for any ε_0 there is a constant C such that*

$$\gamma\varphi(x)(2K - \varepsilon_0)|\zeta(\varphi)(x)|^2 \,\le\, \tau^{-1}\mathcal{G}(x,\tau,\xi,\xi) \,+\, \gamma\varphi(x)C\gamma^2\frac{|A(x,\zeta(\varphi)(x))|^2}{|\zeta(\varphi)(x)|^2} \tag{3.24}$$

for all $C < \gamma$, $\xi \in \mathbb{R}^n$, and $x \in \bar\Omega$.

Proof of Lemma 3.8

By homogeneity we can assume $|\zeta(\varphi)|(x) = 1$. In the proof we use that

$$A(x, \zeta(\varphi)(x)) = \sum_{j,k=1}^{n} a^{jk}(\xi_j\xi_k - \sigma^2\partial_j\psi\partial_k\psi) + 2i\sum_{j,k=1}^{n} a^{jk}\sigma\xi_j\partial_k\psi$$

$$= A(x, \xi) - \sigma^2 A(x, \nabla\psi(x)) + 2i\sigma\sum \frac{\partial A}{\partial\zeta_j}(x, \xi)\partial_j\psi(x). \tag{3.25}$$

We assume that $\gamma \geq 1$. To derive (3.24) we use K-pseudo-convexity of ψ and consider four possible cases.

Case 1 :

$$\sigma = 0,\ \ A(x, \xi) = 0,\ \ \sum \frac{\partial A}{\partial\zeta_j}(x, \xi)\partial_j\psi(x) = 0. \tag{3.26}$$

Then

$$\sigma = 0,\ \sum a^{jk}\xi_j\xi_k = 0,\ \sum a^{jk}\xi_j\partial_k\psi = 0$$

and from (3.18) we yield

$$\tau^{-1}\mathcal{G}(x, 0, \xi, \xi)$$

$$= 2\gamma\varphi\sum \partial_j\partial_k\psi 2a^{jm}\xi_m 2a^{kl}\xi_l\ +\ 4\gamma\varphi\sum a^{lk}\partial_k a^{jm}(2\xi_l\xi_m\partial_j\psi - \xi_j\xi_m\partial_l\psi)$$

$$= 2\gamma\varphi\sum \partial_j\partial_k\psi\frac{\partial A}{\partial\zeta_j}\frac{\partial A}{\partial\zeta_k}\ +\ 2\gamma\varphi\sum((\partial_k\frac{\partial A}{\partial\zeta_j})\frac{\partial A}{\partial\zeta_k} - (\partial_k A)\frac{\partial^2 A}{\partial\zeta_j\partial\zeta_k})\partial_j\psi(x, \xi)$$

$$\geq 2\gamma\varphi K \tag{3.27}$$

by K-pseudo-convexity of ψ with respect to A of (2.46).

Case 2 :

$$\sigma < \delta,\ \ |\gamma A(x, \xi)| < \delta,\ \ |\sum \frac{\partial A}{\partial\zeta_j}(x, \xi)\partial_j\psi(x)| < \delta < 1, \tag{3.28}$$

where δ is a (small) positive number to be chosen later.

Using (3.18) as in Case 1, bounding the terms with σ^2 by $-C\gamma\varphi\delta^2$ and dropping the first two and the last (positive) terms in $\mathcal{G}_4$ we obtain

$$\tau^{-1}\mathcal{G}(x,\tau,\xi,\xi)$$

$$\geq\ 2\gamma\varphi\sum\partial_j\partial_k\psi 2a^{jm}\xi_m 2a^{kl}\xi_l - C\gamma\varphi\delta^2\ +\ 4\gamma\varphi\sum a^{lk}\partial_k a^{jm}(2\xi_l\xi_m\partial_j\psi - \xi_j\xi_m\partial_l\psi)$$

$$+\,4\gamma\varphi\big(2\sum a^{km}\partial_k\psi\xi_m\sum\partial_j a^{lj}\xi_l - \sum a^{jk}\xi_j\xi_k\sum(\partial_m a^{lm}\partial_l\psi + a^{lm}\partial_l\partial_m\psi)\big)$$

$$-\,4\gamma\varphi\sum\gamma a^{jk}\xi_j\xi_k\sum a^{lm}\partial_l\psi\partial_m\psi$$

$$\geq\ \gamma\varphi\big(2\sum\partial_j\partial_k\psi 2a^{jm}\xi_m 2a^{kl}\xi_l + 4\sum a^{lk}\partial_k a^{jm}(2\xi_l\xi_m\partial_j\psi - \xi_j\xi_m\partial_l\psi) - C\delta\big)$$

$$\geq\ \gamma\varphi(2K - \varepsilon(\delta))$$

where $\varepsilon(\delta)\to 0$ as $\delta\to 0$.

Indeed, let us assume the opposite. Then there are $\varepsilon_0 > 0$ and sequences $\xi(p)$, $x(p)\in\bar{\Omega}$, $\sigma(p)$, $A(p)$, $p = 1,2,\dots$ with $|\xi(p)|^2 + \sigma(p)^2|\nabla\psi(x(p))|^2 = 1$ and $A(p)$ with coefficients bounded by M in $C^2(\Omega)$ such that

$$\sigma(p) < p^{-1},\ \ |\gamma(p)A(p)(x(p),\xi(p))| < p^{-1},\ \ |\sum\frac{\partial A(p)}{\partial\zeta_j}(x(p),\xi(p))\partial_j\psi(x(p))| < p^{-1},$$

but

$$2\sum\partial_j\partial_k\psi 2a^{jm}(p)(x(p))\xi_m(p)2a^{kl}(p)(x(p))\xi_l(p)$$

$$+\,4\sum a^{lk}(p)\partial_k a^{jm}(p)(x(p))\big(2\xi_l(p)\xi_m(p)\partial_j\psi(x(p)) - \xi_j(p)\xi_m(p)\partial_l\psi(x(p))\big) - Cp^{-1}$$

$$\leq\ 2K - \varepsilon_0.$$

Using compactness and subtracting subsequences, we assume that $x(p)\to x\in\bar{\Omega}$, $\xi(p)\to\xi$ in $\mathbb{R}^n$ and $a^{jk}(p)\to a^{jk}$ in $C^1(\Omega)$. Passing to the limits and using $\gamma(p)\geq 1$ we arrive at Case 1 and the inequality

$$2\sum\partial_j\partial_k\psi(x)2a^{jm}(x)\xi_m 2a^{kl}(x)\xi_l\ +\ 4\sum a^{lk}\partial_k a^{jm}(x)\big(2\xi_l\xi_m\partial_j\psi(x) - \xi_j\xi_m\partial_l\psi(x)\big)$$

$$\leq\ 2K - \varepsilon_0$$

which contradicts (3.27).

From now on we fix δ such that $\varepsilon(\delta) < \varepsilon_0$ and denote it by δ_0. Observe that we choose δ_0 to be dependent on the same parameters as C.

Case 3 : $|\gamma\tau\varphi(x)| > \delta_0, \;\; |\gamma A(x, \zeta(\varphi)(x))| < \delta_0.$

Using (3.18) as above yields

$$\tau^{-1}\mathcal{G}(x,\tau,\xi,\xi) \;\geq\; -C\gamma\varphi(x) + 8C^{-1}\gamma^2\varphi\delta_0^2 \;\geq\; 2\gamma\varphi(x)K$$

when we choose $\gamma > C^2$.

Case 4 : $|\gamma A(x, \zeta(\varphi)(x))| > \delta_0.$

From (3.18) we have

$$\tau^{-1}\mathcal{G}(x,\tau,\xi,\xi) + \gamma\varphi(x)C_1|\gamma A(x,\zeta(\varphi)(x))|^2$$

$$\geq -C\gamma\varphi(x) - C\gamma^2\varphi|A(x,\zeta(\varphi)(x))| + \gamma\varphi C_1|\gamma A(x,\zeta(\varphi)(x))|^2$$

$$\geq -C\gamma\varphi(x) - C\gamma\varphi(x)|\gamma A(x,\zeta(\varphi)(x))| + \gamma\varphi C_1|\gamma A(x,\zeta(\varphi)(x))|^2$$

$$\geq -C\gamma\varphi(x) + C\gamma\varphi(x)|\gamma A(x,\zeta(\varphi)(x))|\big(\frac{C_1}{2C}|\gamma A(x,\zeta(\varphi)(x))|-1\big) + \gamma\varphi(x)\frac{C_1}{2}|\gamma A(x,\zeta(\varphi)(x))|^2$$

$$\geq\; -C\gamma\varphi(x) \;+\; C\gamma\varphi(x)|\gamma A(x,\zeta(\varphi)(x))|\big(\frac{C_1\delta_0}{2C} - 1\big) \;+\; \gamma\varphi(x)\frac{C_1}{2}\delta_0^2$$

$$\geq\; K\gamma\varphi(x)$$

when $C_1 > \frac{2C}{\delta_0} + \frac{C+K}{\delta_0^2}$. This proves Lemma 3.8. $\square$

Later on we need the norm

$$|||\cdot|||_k = (\int |\zeta|^{2k}|\hat{v}(\xi)|^2 d\xi)^{1/2} \tag{3.29}$$

where $\zeta = \xi + i\tau\nabla\varphi(x_0)$, $\xi \in \mathbb{R}^{n+1}$, and x_0 is a fixed point of $\bar{\Omega}$. Here $\hat{v}$ is the Fourier transform of a function v. Then

$$|||v|||_{-1} = \Big(\int \frac{|\hat{v}(\xi)|^2}{|\xi|^2 + \tau^2|\nabla\varphi(x_0)|^2} d\xi\,\Big)^{1/2}$$

$$\leq \Big(\int \frac{|\hat{v}(\xi)|^2}{\tau^2|\nabla\varphi(x_0)|^2} d\xi\,\Big)^{1/2}$$

$$\leq C\big(\tau^{-2}\int |v|^2 dx\,\big)^{1/2} \;=\; C\tau^{-1}\|v\|_2.$$

We observe that

$$|||v|||_{-1} \;\leq\; C\tau^{-1}\|v\|_2. \tag{3.30}$$

Moreover

$$\begin{aligned} |||\, P(x, D+i\tau\nabla\varphi(x_0))v\,|||_{-1}^2 &= \int \frac{|P(x, D+i\tau\nabla\varphi(x_0))|^2}{|\xi|^2+\tau^2|\nabla\varphi(x_0)|^2}|\hat{v}(\xi)|^2 d\xi \\ &\leq \int \frac{|P(x, D+i\tau\nabla\varphi(x_0))|^2}{\tau^2|\nabla\varphi(x_0)|^2}|\hat{v}(\xi)|^2 d\xi \\ &\leq C(\gamma)\tau^{-2}\int |P(x, D+i\tau\nabla\varphi(x_0))|^2|\hat{v}(\xi)|^2 d\xi \\ &= C(\gamma)\tau^{-2}\|P(x, D+i\tau\nabla\varphi(x_0))v\|_{(0)}^2. \end{aligned} \tag{3.31}$$

The following lemma is given in [14].

Lemma 3.9 *Let $a(x)$ be Lipschitz continuous with Lipschitz constant M when $|x| < \delta$, that is, $|a(x) - a(y)| \leq M|x-y|$ if $\max(|x|, |y|) < \delta$. Let $\Omega_\delta = \{x : x \in \Omega,\ |x| < \delta\}$. If $a(0) = 0$ it then follows that*

$$|||\, a(D_j + i\tau\nabla\varphi(x_0))v\,|||_{-1} \;\leq\; M(\delta + |\tau\nabla\varphi(x_0)|^{-1})\|v\|_2, \tag{3.32}$$

where $v \in C_0^\infty(\Omega_\delta)$ and $\|v\|_2$ is the L^2-norm of v.

Proof of Lemma 3.9

By using the following identity

$$a(D_j + i\tau\nabla\varphi_j)v = (D_j + i\tau\nabla\varphi_j)(av) - (D_j a)v$$

and the trivial estimates

$$|||v|||_{-1}^2 \leq |\tau\nabla\varphi(x_0)|^{-2}\|v\|_2^2, \;\; |||\,(D_j + i\tau\nabla\varphi(x_0))v\,|||_{-1}^2 \leq \|v\|_2^2, \;\; v \in L^2(\mathbb{R}^n),$$

we have

$$|||\, a(D_j + i\tau\nabla\varphi(x_0))v\,|||_{-1} \;\leq\; \|av\|_2 \;+\; |\tau\nabla\varphi(x_0)|^{-2}\|(D_j a)v\|_2.$$

Since $|a| < \delta M$ in Ω_δ and $|(D_j a)| \leq M$, the inequality (3.32) follows. □

Lemma 3.10 *There are a function $\varepsilon(\delta;\gamma)$ convergent to 0 as $\delta \to 0$ for fixed γ and a constant $C(\gamma)$ such that*

$$\tau^{-1}|(\mathcal{G}(x_0,\tau,D,\bar{D}) - \mathcal{G}(\ ,\tau,D,\bar{D}))v\bar{v}| \le \varepsilon(\delta;\gamma)\sum_{|\alpha|\le 1}\tau^{2-2|\alpha|}|\partial^\alpha v|^2 \tag{3.33}$$

and

$$\begin{aligned} &|||\, A(x_0, D + i\tau\nabla\varphi(x_0))v \,-\, A(\ , D + i\tau\nabla\varphi)v\,|||_{-1}^2 \\ &\le (\varepsilon(\delta;\gamma) + C(\gamma)\tau^{-1})\sum_{|\alpha|\le 1}(\gamma\tau\varphi(x_0))^{2-2|\alpha|}\int|\partial^\alpha v|^2 \end{aligned} \tag{3.34}$$

for all $v \in C_0^2(B(x_0;\delta))$.

Proof of Lemma 3.10

Due to (3.18)

$$\tau^{-1}(\mathcal{G}(x_0,\tau,D,\bar{D}) - \mathcal{G}(\ ,\tau,D,\bar{D}))v\bar{v}$$

$$= \sum(\gamma(\varphi(x_0)a_1^{jk}(x_0) - \varphi(x)a_1^{jk}(x))\partial_j v(x)\partial_k v(x)) + \gamma^3\tau^2(\varphi(x_0)^2 a_2^{jk}(x_0) - \varphi(x)^2 a_2^{jk}(x))v(x)v(x)$$

$$+\gamma^2\sum((\varphi(x_0)a_3^{jk}(x_0) - \varphi(x)a_3^{jk}(x))\partial_j v(x)\partial_k v(x)) + \gamma^4\tau^2(\varphi(x_0)^2 a_4^{jk}(x_0) - \varphi(x)^2 a_4^{jk}(x))v(x)v(x),$$

where $a_1^{jk},\dots,a_4^{jk}$ are continuous functions determined only by A and ψ. (3.33) follows by the triangle inequality since $|\varphi(x_0)^m a_l^{jk}(x_0) - \varphi(x)^m a_l^{jk}(x)| \le \varepsilon(\delta;\gamma)$ when $|x - x_0| < \delta$.

We have

$$|||\, A(x_0, D + i\tau\nabla\varphi(x_0))v - A(\ , D + i\tau\nabla\varphi)v\,|||_{-1}$$

$$\le |||\sum(a^{jk}(x_0) - a^{jk})(\partial_j - \tau\partial_j\varphi(x_0))(\partial_k - \tau\partial_k\varphi(x_0))v\,|||_{-1}$$

$$\le C(\delta + \tau^{-1})\|(\partial - \tau\partial\varphi)v\|_2 \le (\varepsilon(\delta;\gamma) + C(\gamma)\tau^{-1})\sum_{|\alpha|\le 1}\tau^{1-|\alpha|}\|\partial^\alpha v\|^2 \tag{3.35}$$

by Lemma 3.9.

Furthermore

$$|||\, A(\ , D + i\tau\nabla\varphi(x_0))v - A(\ , D + i\tau\nabla\varphi)v\,|||_{-1}$$

$$= |||\sum a^{jk}\big((\partial_j - \tau\partial_j\varphi(x_0))(\partial_k - \tau\partial_k\varphi(x_0)) - (\partial_j - \tau\partial_j\varphi)(\partial_k - \tau\partial_k\varphi)\big)v\,|||_{-1}$$

$$\leq \||\sum a^{jk}\big(\tau^2(\partial_j\varphi(x_0)\partial_k\varphi(x_0) - \partial_j\varphi\partial_k\varphi) + 2\tau(\partial_j\varphi - \partial_j\varphi(x_0))\partial_k + \tau(\partial_j\partial_k\varphi)\big)v\,\||_{-1}$$

$$\leq \sum \tau^2 \||\, a^{jk}(\partial_j\varphi\partial_k\varphi - \partial_j\varphi(x_0)\partial_k\varphi(x_0))v\,\||_{-1}$$

$$+2\sum \tau \||\,(\partial_j\varphi - \partial_j\varphi(x_0))\partial_k v\,\||_{-1} \,+\, \tau\sum \||\,\partial_j\partial_k\varphi v\,\||_{-1}.$$

By using the property of the norm (3.29) we yield

$$\||\, A(\,,D+i\tau\nabla\varphi)v - A(\,,D+i\tau\nabla\varphi(x_0))v\,\||_{-1}$$

$$\leq C\tau\sum \|a^{jk}(\partial_j\varphi\partial_k\varphi - \partial_j\varphi(x_0)\partial_k\varphi(x_0))v\|_2$$

$$+2\sum \|(\partial_j\varphi - \partial_j\varphi(x_0))\partial_k v\|_2 \,+\, \sum \|\partial_j\partial_k\varphi v\|_2$$

$$\leq\, \tau\varepsilon(\delta;\gamma)\|v\|_2 \,+\, \varepsilon(\delta;\gamma)\sum \|\partial_k v\|_2 + C(\gamma)\|v\|_2$$

$$\leq (\varepsilon(\delta;\gamma) + C(\gamma)\tau^{-1}) \sum_{|\alpha|\leq 1} \tau^{1-|\alpha|}\|\partial^\alpha v\|_2.$$

Hence by using (3.35) we have

$$\||\, A(x_0,D+i\tau\nabla\varphi(x_0))v - A(x,D+i\tau\nabla\varphi(x))v\,\||_{-1}$$

$$\leq\, \||\, A(x_0,D+i\tau\nabla\varphi(x_0))v - A(x,D+i\tau\nabla\varphi(x_0))v\,\||_{-1}$$

$$+\||\, A(x,D+i\tau\nabla\varphi(x_0))v - A(x,D+i\tau\nabla\varphi(x))v\,\||_{-1}$$

$$\leq\, (\varepsilon(\delta;\gamma) + C(\gamma)\tau^{-1}) \sum_{|\alpha|\leq 1} \tau^{1-|\alpha|}\|\partial^\alpha v\|_2.$$

This proves Lemma 3.10. □

Now we continue the proof of Theorem 3.1. By using the Parseval identity,

$$(\tau^2|\nabla\varphi(x_0)|^2)^{m-|\alpha|}\int |\partial^\alpha v|^2 dx \,\leq\, (2\pi)^{-n}\int |\zeta|^{2m}(\varphi)(x_0)\,|\hat{v}(\xi)|^2 d\xi.$$

Multiplying the inequality (3.24) by $|\hat{v}(\xi)|^2$, $v \in C_0^2(\Omega_\varepsilon)$, and integrating over $\mathbb{R}^n$ we yield

$$C^{-1}\gamma\varphi(x_0)\sum_{|\alpha|\leq 1}\int (\gamma\tau\varphi(x_0))^{2-2|\alpha|}|\partial^\alpha v|^2$$

$$\leq\, \tau^{-1}\int \mathcal{G}(x_0,\tau,D,\bar{D})v\bar{v} \,+\, \gamma\varphi(x_0)\gamma^2\int \frac{|A(x_0,\zeta(\varphi)(x_0))|^2}{|\zeta(\varphi)(x_0)|^2}|\hat{v}(\xi)|^2 d\xi$$

$$\le \tau^{-1} \int \mathcal{G}(x_0, \tau, D, \bar{D}) v\bar{v} \,+\, \gamma\varphi(x_0)\gamma^2 |||\, A(x_0, D + i\tau\nabla\varphi(x_0))v\,|||^2_{-1}$$

$$\le \tau^{-1} \int \mathcal{G}(x, \tau, D, \bar{D}) v\bar{v} \,+\, \varepsilon(\delta;\gamma) \sum_{|\alpha|\le 1} \tau^{2-2|\alpha|} \int |\partial^\alpha v|^2$$

$$+\, \gamma\varphi(x_0)\gamma^2 |||\, A(\,, D + i\tau\nabla\varphi)v\,|||^2_{-1} \,+\, (\varepsilon(\delta;\gamma) + C(\gamma)\tau^{-2}) \sum_{|\alpha|\le 1} \tau^{3-2|\alpha|} \int |\partial^\alpha v|^2 \qquad (3.36)$$

for $v \in C^2_0(\Omega_\varepsilon \cap B(x_0, \delta))$. Here we used Lemma 3.10 and the elementary inequality $a^2 \le 2b^2 + 2(b-a)^2$. Choosing $\delta > 0$ small and τ large enough so that

$$(2C)^{-1}\gamma\varphi(x_0)(\gamma\tau\varphi(x_0))^{2-2|\alpha|} \,>\, (\varepsilon(\delta;\gamma) + C(\gamma)\tau^{-2})\tau^{2-2|\alpha|},$$

we absorb the second and fourth term on the right side of the inequality (3.36) to the left side to arrive at the inequality

$$\sum_{|\alpha|\le 1} \int (\gamma\tau\varphi(x_0))^{3-2|\alpha|} |\partial^\alpha v|^2$$

$$\le C\,(\int \mathcal{G}(\,, \tau, D, \bar{D}) v\bar{v} \,+\, \tau\gamma\varphi(x_0)\gamma^2 |||\, A(\,, D + i\tau\nabla\varphi)v\,|||^2_{-1}).$$

As above, by choosing large $\tau > C(\gamma)$ one can replace $\varphi(x_0)$ by φ on the left side of this inequality. Using (3.14), (3.15) and the properties (3.30) and (3.31) of the norm $||| \cdot |||_{-1}$ we conclude that

$$\sum_{|\alpha|\le 1} \int (\gamma\tau\varphi)^{3-2|\alpha|} |\partial^\alpha v|^2$$

$$\le C\, \|A(\,, D + i\tau\nabla\varphi)v\|^2_2 \,+\, C(\gamma)\tau^{-1} \|A(\,, D + i\tau\nabla\varphi)v\|^2_2$$

for $v \in C^2_0(B(x_0;\delta))$. Choosing $\tau > C(\gamma)$ we eliminate the second term on the right side. Now the bound (3.23) follows by partition of the unity argument. Since our choice of δ_0 depends on γ, we give this argument in some detail.

The balls $B(x_0;\delta_0)$ form an open covering of the compact set $\bar{\Omega}$. Hence we can find a finite subcovering $B(x_{0j};\delta_0)$ and a special partition of the unity $\chi_j(\,;\gamma)$ subordinated to this subcovering. In particular, $\chi_j \in C^2_0(B(x_{0j};\delta_0))$, $0 \le \chi_j \le 1$, and $\sum \chi_j^2 = 1$ on $\bar{\Omega}$. By the Leibniz formula

$$\partial^\alpha(\chi_j v) \,=\, \chi_j \partial^\alpha v \,+\, (\partial^\alpha \chi_j) v$$

and

$$A(\,,D+i\tau\nabla\varphi)(\chi_j v)\;=\;\chi_j A(\,,D+i\tau\nabla\varphi)v\;+\;\sum_{|\beta|\le 1} a^\beta \tau^{1-|\beta|}\partial^\beta v$$

with $|a^\beta|\le C(\gamma)$. Hence applying the Carleman estimate (3.23) to $\chi_j v$ and using the elementary inequality $|a+b|^2\ge \frac{1}{2}a^2-b^2$ we obtain

$$\frac{1}{2}\sum_{|\alpha|\le 1}\int \sigma^{3-2|\alpha|}|\chi_j\partial^\alpha v|^2\;-\;\sum_{|\alpha|=1}\int \sigma^3|(\partial^\alpha(\chi_j))v|^2$$

$$\le\; C\,\|\chi_j A(\,,D+i\tau\nabla\varphi)v\|_2^2\;+\;C(\gamma)\sum_{|\beta|\le 1}\tau^{2-2|\beta|}\|\partial^\beta v\|_2^2.$$

Summing up over $j=1,\dots,J$ and using that $\sum\chi_j^2=1$ we yield

$$\frac{1}{2}\sum_{|\alpha|\le 1}\int \sigma^{3-2|\alpha|}|\partial^\alpha v|^2\;-\;\sum_{|\alpha|=1,\,j\le J}\int \sigma|(\partial^\alpha(\chi_j))v|^2$$

$$\le\; C\,\|A(\,,D+i\tau\nabla\varphi)v\|_2^2\;+\;C(\gamma)\sum_{|\beta|\le 1}\tau^{2-2|\beta|}\|\partial^\beta v\|_2^2.$$

Since the highest powers of τ are in the first term on the left side, choosing $C(\gamma)<\tau$ we absorb the second term on the left and the right into the first term on the left. This completes the proof of Theorem 3.1. □

3.4 Weak Carleman estimates for scalar operators

Now, we prove Theorem 3.2, the Carleman estimates in negative norms, which is based on Theorem 3.1 and some additional lemmas.

By basic differentiation rules

$$A_\varphi(D)=A(D+i\tau\nabla\varphi)=A(D)+\tau(A_1(D)+a_0)-\tau^2A(\nabla\varphi), \tag{3.37}$$

where A_1 is the first order differential operator with the C^1-coefficients depending on γ and a_0 is some function in L^∞ depending on γ. Moreover $C^1(\Omega)$-norms of the coefficients of A_1 are bounded by $C(\gamma)$ and $\|a_0\|_{(0)}(\Omega)\le C(\gamma)$. We use the notation $<\xi>=(|\xi|^2+1)^{\frac{1}{2}}$ and the pseudo-differential operator $\Lambda_\tau^s f=\mathcal{F}^{-1}(<\xi>+\tau)^s\mathcal{F}f$, where $\mathcal{F}$ is the Fourier transform and $\xi=(\xi_1,\dots,\xi_n)$. Let Ω^* be a bounded domain in $\mathbb{R}^n$ with a smooth boundary such that $\bar{\Omega} \subset \Omega^*$. We can extend all coefficients of the operator A_φ onto $\mathbb{R}^n$, preserving the regularity in such a way that they have support in Ω^*.

Lemma 3.11 *There exists a constant $C(\gamma)$ such that*

$$\|\Lambda_\tau^{-1} A_\varphi u - A_\varphi \Lambda_\tau^{-1} u\|_{(0)}(\Omega^*) \leq C(\gamma)\|u\|_{(0)}(\Omega) \quad \text{for all } u \in H_0^2(\Omega).$$

Proof of Lemma 3.11

Due to (3.37) it suffices to show that

$$\|\Lambda_\tau^{-1} a\partial^\alpha u - a\partial^\alpha \Lambda_\tau^{-1} u\|_{(0)}(\Omega^*) \leq C(\gamma)\|u\|_{(0)}(\Omega), \quad \text{for all } |\alpha| \leq 2, \tag{3.38}$$

with $a \in C^2(\overline{\Omega^*})$, $|a|_2(\bar{\Omega}) < M$, and $\alpha = (\alpha_1, \ldots, \alpha_n)$; that

$$\tau\|\Lambda_\tau^{-1} a\partial^\beta u - a\partial^\beta \Lambda_\tau^{-1} u\|_{(0)}(\Omega^*) \leq C(\gamma)\|u\|_{(0)}(\Omega), \quad \text{for all } |\beta| \leq 1, \tag{3.39}$$

with $\beta = (\beta_1, \ldots, \beta_n)$; that

$$\tau^2\|\Lambda_\tau^{-1} au - a\Lambda_\tau^{-1} u\|_{(0)}(\Omega^*) \leq C(\gamma)\|u\|_{(0)}(\Omega), \tag{3.40}$$

for $a \in C^1(\overline{\Omega^*})$ (possibly depending on γ); and that

$$\tau\|\Lambda_\tau^{-1} a_0 u - a_0\Lambda_\tau^{-1} u\|_{(0)}(\Omega^*) \leq C(\gamma)\|u\|_{(0)}(\Omega). \tag{3.41}$$

Let $\alpha_j > 0$ and $\beta_j = 1$ while other components of β be zero. We introduce $u_1 = \Lambda_\tau^{-1}\partial^{\alpha-\beta} u$. Using also that $\Lambda_\tau = \Lambda_0 + \tau$ we have

$$\Lambda_\tau^{-1} a\partial^\alpha u - a\partial^\alpha \Lambda_\tau^{-1} u = \Lambda_\tau^{-1}(a\Lambda_\tau - \Lambda_\tau a)\partial_j u_1 = \Lambda_\tau^{-1}(a\Lambda_0 - \Lambda_0 a)\partial_j u_1$$

$$= \Lambda_\tau^{-1}(a\partial_j\Lambda_0 - \partial_j(\Lambda_0 a) + \Lambda_0\partial_j a)u_1 = \Lambda_\tau^{-1}(\partial_j(a\Lambda_0 - \Lambda_0 a) + (\Lambda_0\partial_j a - \partial_j a\Lambda_0))u_1.$$

From the Parseval identity $\|u_1\|_{(0)}(\mathbb{R}^n) \leq C\|u\|_{(0)}(\Omega)$. By known estimates, in [6], of commutators of pseudo-differential operators and of multiplication operators

$$\|(a\Lambda_0 - \Lambda_0 a)u_1\|_{(0)}(\Omega^*) \leq C(\gamma)\|u_1\|_{(0)}(\mathbb{R}^n).$$

A similar estimate is valid when we replace a by $\partial_j a$. Using, as above, that $\Lambda_\tau^{-1}\partial_j$ is a bounded operator in L^2 we complete the proof of the bound (3.38).

Proofs of (3.39), (3.40) are similar.

The bound (3.41) is obvious. Indeed

$$\|\tau\Lambda_\tau^{-1}(a_0u) - \tau a_0\Lambda_\tau^{-1}u\|_{(0)} \le \|\tau\Lambda_\tau^{-1}(a_0u)\|_{(0)} + \|\tau a_0\Lambda_\tau^{-1}u\|_{(0)} \le C(\gamma)\|u\|_{(0)},$$

since $\|\tau\Lambda_\tau^{-1}v\|_{(0)} \le \|v\|_{(0)}$. $\square$

In Lemmas 3.12 and 3.13, the variables x and y denote elements of $\mathbb{R}^n$.

Lemma 3.12 *Let $K(x,y;\tau)$ be the Schwartz kernel of the pseudo-differential operator Λ_τ^{-1} with $\tau > 1$. Then*

$$|\partial_x^\alpha K(x,y;\tau)| \le C(d_1)\tau^{-2}|x-y|^{-2n-2}$$

provided $|\alpha| \le 2$ and $0 < d_1 \le |x-y|$.

A proof can be found in [16, lemma 3.4]. This proof is valid in our case when we choose $l+1$ and replace $n+1$ by n.

Let $\sigma^* = \sup\sigma$ and $\sigma_* = \inf\sigma$ over $B(3\delta)$.

Lemma 3.13 *Let ψ be K-pseudo-convex with respect to A on $\bar{\Omega}$. Then for any $x_0 \in \bar{\Omega}$ there are $\delta(\gamma)$ and a constant C such that*

$$\int_{\mathbb{R}^n}(\sigma^3|\Lambda_{\sigma^*}^{-1}v|^2 + \sigma|v|^2) \le C\int_{\mathbb{R}^n}|\Lambda_{\sigma^*}^{-1}A_\varphi v|^2$$

for all $v \in H_0^2(B(x_0;\delta))$ provided $C < \tau$.

Proof of Lemma 3.13

We can assume that $x_0 = 0$ and we let $B(\delta) = B(x_0;\delta)$. For continuity reasons, ψ is $K/2$ pseudo-convex in $B(3\delta)$. By Theorem 3.1 there exists a constant C such that the Carleman estimate

$$\sum_{|\alpha|=0}^{1}\int_{B(3\delta)}\sigma^{3-2|\alpha|}|\partial^\alpha v_0|^2 \le C\int_{B(3\delta)}|A_\varphi v_0|^2 \quad \text{for all } v_0 \in H_0^2(B(3\delta))$$

holds, provided $C < \gamma$, $C_0(\gamma) < \tau$.

Let $\chi \in C_0^\infty(B(3\delta))$, $\chi = 1$ on $B(2\delta)$. Using this Carleman type estimate for $v_0 = \chi\Lambda_{\sigma^*}^{-1}v$, we obtain

$$\int_{B(3\delta)} (\sigma^3\chi^2|\Lambda_{\sigma^*}^{-1}v|^2 + \sigma \sum_{|\alpha|=1} |\chi\partial^\alpha(\Lambda_{\sigma^*}^{-1}v) + \partial^\alpha\chi\Lambda_{\sigma^*}^{-1}v|^2)$$

$$\leq C \int_{B(3\delta)} |A_\varphi(\chi\Lambda_{\sigma^*}^{-1}v)|^2 \tag{3.42}$$

$$\leq C \int_{B(3\delta)} \big(|A_\varphi(\Lambda_{\sigma^*}^{-1}v)|^2 + C(\gamma)(\tau^2|\Lambda_{\sigma^*}^{-1}v|^2 + \sum_{|\alpha|=1} |\partial^\alpha(\Lambda_{\sigma^*}^{-1}v)|^2)\big),$$

where we used (3.37), the Leibniz formulas

$$A(\chi w) = \chi A w + A_1(\,;\chi)w + A(\chi)w, \quad A_1(\chi w;\varphi) = \chi A_1(w;\varphi) + A_1(\chi;\varphi)w,$$

and the triangle inequality.

Due to the Parseval identity, we have

$$\int_{B(3\delta)} |\partial^\alpha\Lambda_{\sigma^*}^{-1}v|^2 \leq \int_{\mathbb{R}^n} |\partial^\alpha\Lambda_{\sigma^*}^{-1}v|^2 \leq \int_{\mathbb{R}^n} |v|^2 = \int_{B(3\delta)} |v|^2 \tag{3.43}$$

when $|\alpha| = 1$. Similarly,

$$\tau^2 \int_{B(3\delta)} |\Lambda_{\sigma^*}^{-1}v|^2 \leq \int_{B(3\delta)} |v|^2. \tag{3.44}$$

Using these inequalities and recalling that $\chi = 1$ on $B(2\delta)$ we derive from the bound (3.42) that

$$\int_{B(2\delta)} (\sigma^3|\Lambda_{\sigma^*}^{-1}v|^2 + \sigma \sum_{|\alpha|=1} |\partial^\alpha(\Lambda_{\sigma^*}^{-1}v)|^2) - C(\gamma) \int_{B(3\delta)} |v|^2$$

$$\leq C \int_{B(3\delta)} (|A_\varphi(\Lambda_{\sigma^*}^{-1}v)|^2 + C(\gamma)|v|^2). \tag{3.45}$$

The Parseval identity and the definition of Λ_τ yield

$$\int_{B(3\delta)} \sigma v^2 \leq \int_{B(3\delta)} \sigma^* v^2$$

$$= \sigma^* \int_{\mathbb{R}^n} \frac{\sigma^{*2} + 1}{< \xi >^2 + \sigma^{*2}} |\hat{v}(\xi)|^2 d\xi + \sigma^* \int_{\mathbb{R}^n} \frac{|\xi|^2}{< \xi >^2 + \sigma^{*2}} |\hat{v}(\xi)|^2 d\xi$$

$$= (\sigma^* + (\sigma^*)^3) \int_{\mathbb{R}^n} |\Lambda_{\sigma^*}^{-1} v|^2 + \sigma^* \sum_{|\alpha|=1} \int_{\mathbb{R}^n} |\partial^\alpha (\Lambda_{\sigma^*}^{-1} v)|^2$$

$$\leq 2 \, (\frac{\sigma^*}{\sigma_*})^3 \int_{B(3\delta)} \sigma^3 |\Lambda_{\sigma^*}^{-1} v|^2 + \frac{\sigma^*}{\sigma_*} \sum_{|\alpha|=1} \int_{B(3\delta)} \sigma |\partial^\alpha (\Lambda_{\sigma^*}^{-1} v)|^2$$

$$+ \int_{\mathbb{R}^n \setminus B(3\delta)} ((\sigma^*)^3 |\Lambda_{\sigma^*}^{-1} v|^2 + \sigma^* \sum_{|\alpha|=1} |\partial^\alpha (\Lambda_{\sigma^*}^{-1} v)|^2).$$

Since $\psi \in C^2$, using (3.2) we choose $\delta(\gamma)$ so that $\frac{1}{2} < \frac{\sigma_*}{\sigma^*}$. Choosing $\tau > C(\gamma)$ and using Lemma 3.11 we have from (3.45) that

$$\int_{\mathbb{R}^n} (\sigma^3 |\Lambda_{\sigma^*}^{-1} v|^2 + \sigma |v|^2)$$

$$\leq C \int_{B(3\delta)} |\Lambda_{\sigma^*}^{-1} A_\varphi v|^2 + C(\gamma) \int_{\mathbb{R}^n \setminus B(2\delta)} (\tau^3 |\Lambda_{\sigma^*}^{-1} v|^2 + \tau \sum_{|\alpha|=1} |\partial^\alpha \Lambda_{\sigma^*}^{-1} v|^2) \tag{3.46}$$

when $\tau > C(\gamma)$. By using Lemma 3.12 we eliminate the last integral in this bound to complete the proof.

Since $supp\, v \subset B(\delta)$, we have

$$|\partial_x^\alpha \Lambda_\tau^{-1} v(x)| \leq \int_{B(\delta)} |v(y)| |\partial_x^\alpha K(x, y; \tau)| dy \leq C(\gamma) \tau^{-2} \int_{B(\delta)} |x - y|^{-2n-2} |v(y)| dy$$

by Lemma 3.12, provided $x \in \mathbb{R}^n \setminus B(2\delta)$. When $y \in B(\delta)$,

$$|x - y| \geq \frac{1}{2} |x - y| + \frac{1}{4} |x - y| \geq \frac{\delta}{2} + \frac{1}{4} |x| - \frac{1}{4} |y| \geq \frac{\delta}{4} + \frac{1}{4} |x| \geq \frac{1 + |x|}{C(\gamma)}.$$

Hence by using the Schwartz inequality

$$|\partial^\alpha \Lambda_\tau^{-1} v|(x) \leq C(\gamma) \tau^{-2} (1 + |x|)^{-2n-2} (\int_{B(\delta)} |v|^2)^{\frac{1}{2}} \quad \text{for all } |\alpha| \leq 1$$

provided $x \in \mathbb{R}^n \setminus B(2\delta)$. Using this estimate we conclude that the last integral in (3.46) is less than $C(\gamma) \int_{B(\delta)} |v|^2$, so choosing $\tau > C(\gamma)$ we eliminate this integral by using the last integral in the left side of (3.46) as an upper bound. □

Proof of Theorem 3.2

We first assume that $supp\, v \subset B(x_0; \delta)$. Using the substitution $v = e^{-\tau\varphi} w$ and the identity $Av = e^{-\tau\varphi} A_\varphi w$ we reduce (3.4) to the bound

$$\int_\Omega \sigma |w|^2 \leq C \int_{B(x_0;\delta)} (\frac{f_{\bullet 0}^2}{\sigma^2} + \sum_{j=1}^n f_{\bullet j}^2) \quad \text{when } \tau > C(\gamma),$$

provided $A_\varphi w = f_{\bullet 0} + \sum_{j=1}^n \partial_j f_{\bullet j}$, with

$$f_{\bullet 0} = e^{\tau\varphi}(f_0 - \sum_{j=1}^n \tau\partial_j\varphi f_j), \ \ f_{\bullet j} = e^{\tau\varphi} f_j.$$

Using Lemma 3.13 we have

$$\int_{B(x_0;\delta)} \sigma|w|^2 \leq C \int_{B(x_0;\delta)} (\, |\Lambda_{\sigma^*}^{-1} f_{\bullet 0}|^2 + \sum_{j=1}^n |\Lambda_{\sigma^*}^{-1}\partial_j f_{\bullet j}|^2\,)$$

$$\leq C \int_{\mathbb{R}^n} (\, \sigma^{-2}|f_{\bullet 0}|^2 + \sum_{j=1}^n |f_{\bullet j}|^2\,)$$

by the Parseval identity. Using the definition of $f_{\bullet j}$ we complete the proof when *supp* $\mathbf{u} \subset B(x_0; \delta(x_0))$.

We now use a special partition of unity argument.

Due to compactness of $\bar{\Omega}$ we can find a finite covering of $\bar{\Omega}$ by balls $B(x(k); \delta(\gamma)(k))$, $k = 1, \ldots, K$. Let $\chi(\,;k)$ be the special C^∞- partition of the unity subordinated to this covering, *i.e.*, *supp* $\chi(\,;k) \subset B(x(k);\delta)$ and $\sum_{k=1}^K \chi^2(\,;k) = 1$ on Ω. By the Leibniz formula

$$A(\chi(\,;k)v) = \chi(\,;k)Av + \sum_{j=1}^n \partial_j(b_j v) + cv$$

$$= \chi(\,;k)f_0 + \sum_{j=1}^n \partial_j(\chi(\,;k)f_j) - \sum_{j=1}^n (\partial_j\chi(\,;k))f_j + \sum_{j=1}^n \partial_j(b_j v) + cv,$$

where b_j and c depend on γ. Applying Theorem 3.2 to $\chi(\,;k)v$ we obtain

$$\int_\Omega \sigma e^{2\tau\varphi}\chi^2(\,;k)v^2$$

$$\leq C \int_\Omega (\, \frac{f_0^2}{\sigma^2} + \sum_{j=1}^n f_j^2 + C(\gamma)\sum_{j=1}^n \frac{f_j^2}{\sigma^2} + C(\gamma)v^2\,)e^{2\tau\varphi}.$$

Summing over $k = 1, \ldots, K$ and choosing $\tau > C(\gamma)$ we absorb the terms containing v in the right side by the left side and complete the proof of Theorem 3.2. □

CHAPTER 4

CARLEMAN ESTIMATES FOR ELASTICITY SYSTEM WITH RESIDUAL STRESS

As an important application of Theorems 3.1 and 3.2, we consider an elasticity system with residual stress, R, [24], [25], [32], [33]. This is an anisotropic system. At present, there are results on the uniqueness of the continuation and identification of its coefficients under the assumption that the residual stress is "small" (without a quantitative bound of how small). In [33], there are uniqueness of the identification theorems for some coefficients of the residual stress under quite complicated conditions and from all possible boundary data. We derive global uniqueness of the continuation results in $\Omega_0 \subset \Omega$ under some pseudo-convexity conditions on a weight function ψ defining Ω_0. In this chapter we let $x \in \mathbb{R}^3$ and $(x,t) \in \Omega \subset \mathbb{R}^4$. The residual stress is modeled by a symmetric second-rank tensor $R(x) = (r_{jk}(x))_{j,k=1}^3 \in C^2(\overline{\Omega})$ which is divergence free, $\nabla \cdot R = 0$. Let $\mathbf{u}(x,t) = (u_1, u_2, u_3)^\top :$ $\Omega \to \mathbb{R}^3$ be the displacement vector in Ω. We recall the operator of linear elasticity with residual stress ; let

$$\mathbf{A}_R \mathbf{u} = \mathbf{f} \tag{4.1}$$

given by

$$\mathbf{A}_R \mathbf{u} = \rho \partial_t^2 \mathbf{u} - \mu \Delta \mathbf{u} - (\lambda + \mu) \nabla(\nabla \cdot \mathbf{u}) - (\nabla \cdot \mathbf{u}) \nabla \lambda - 2\epsilon(\mathbf{u}) \nabla \mu - \nabla \cdot ((\nabla \mathbf{u}) R) \tag{4.2}$$

where $\rho \in C^1(\bar{\Omega})$ and λ, $\mu \in C^2(\bar{\Omega})$ are density and Lamé parameters depending only on x, with $\epsilon(\mathbf{u}) = \frac{1}{2}(\nabla \mathbf{u} + (\nabla \mathbf{u})^\top)$. Let $\Box(\mu; R) = \partial_t^2 - \sum_{jk} \frac{\mu \delta_{jk} + r_{jk}}{\rho} \partial_j \partial_k$.

Theorems 4.1 and 4.2 of Carleman estimates for elasticity systems are basic tools for stability estimates of the lateral Cauchy problem in Chapter 5, and for solving the inverse problem in Chapter 6.

We assume that

$$|\rho^{-1}|_2(\Omega) + |\lambda|_2(\Omega) + |\mu|_2(\Omega) + |r^{jk}|_2(\Omega) \le M.$$

The estimate (4.3) below was obtained in [24] when R is "small" and in [20], [21] without any smallness condition. Now, we obtain the Carleman estimate (4.3) based on a sufficient global K-pseudo-convexity condition of ψ.

Theorem 4.1 *Let ψ be K-pseudo-convex with respect to $\Box(\mu; R)$, $\Box(\lambda+2\mu; R)$ in $\bar{\Omega}$. Then there are constants C, $C_0(\gamma)$ such that*

$$\int_\Omega \big(\sigma(|\nabla_{x,t}\mathbf{u}|^2 + |\nabla_{x,t} div\mathbf{u}|^2 + |\nabla_{x,t} curl\mathbf{u}|^2) + \sigma^3(|\mathbf{u}|^2 + |div\mathbf{u}|^2 + |curl\mathbf{u}|^2)\big)e^{2\tau\varphi}$$

$$\leq C\int_\Omega (|\mathbf{A}_R\mathbf{u}|^2 + |\nabla(\mathbf{A}_R\mathbf{u})|^2)e^{2\tau\varphi} \tag{4.3}$$

for all $\mathbf{u} \in H_0^3(\Omega)$, $C < \gamma$, $C_0(\gamma) < \tau$.

We now have a weak Carleman estimate for elasticity systems. Theorem 4.2 is the simple version of Theorem 4.1 without additional spatial derivatives on the right side of (4.3).

Theorem 4.2 *Let $\psi \in C^3(\bar{\Omega})$ be K-pseudo-convex with respect to $\Box(\mu; R)$, $\Box(\lambda+2\mu; R)$ in $\bar{\Omega}$. Then there are constants C, $C_0(\gamma)$ such that*

$$\int_\Omega \sigma(|\mathbf{u}|^2 + |div\mathbf{u}|^2 + |curl\mathbf{u}|^2)e^{2\tau\varphi} \leq C\int_\Omega |\mathbf{A}_R\mathbf{u}|^2 e^{2\tau\varphi} \tag{4.4}$$

for all $\mathbf{u} \in H_0^2(\Omega)$, $C < \gamma$, $C_0(\gamma) < \tau$.

Using Theorem 4.2, we have better estimates of Hölder stabilities in Chapters 5 and 6 with reduced regularities in data.

4.1 Reduction to extended principally triangular system

The elasticity system we consider here is not isotropic due to the presence of residual stress. Unfortunately, there is no Carleman estimate for such systems. We already obtained Carleman estimates (3.3) and (3.4) for a general scalar operator. To easily use Carleman estimates for scalar equations, we extend this system to a new, principally triangular system. We need to diagonalize the principal part of (4.1). It is impossible, however, to reduce the principal part of (4.1) to uncoupled wave scalar operators. Here we provide two reductions for (4.1); with a standard substitution ($\mathbf{u}$, $v = div\mathbf{u}$, $\mathbf{w} = curl\mathbf{u}$) the system $\mathbf{A}_R\mathbf{u} = \mathbf{f}$ in (4.1) can be reduced to a new system, where the leading part is a special lower triangular matrix differential operator with the wave operators in the diagonal. Using these reduced systems, we are able to prove the stability estimate for (4.1).

Lemma 4.3 *The system (4.1) implies an extended (principally triangular) system of equations by using two auxiliary functions $v = div\mathbf{u}$ and $\mathbf{w} = curl\mathbf{u}$. This system is*

$$
\begin{aligned}
\Box(\mu; R)\mathbf{u} &= \frac{\mathbf{f}}{\rho} + A_{1;1}(\mathbf{u}, v)\\
\Box(\lambda + 2\mu; R)v &= div\frac{\mathbf{f}}{\rho} + \sum_{jk}\nabla(\frac{r_{jk}}{\rho})\cdot\partial_j\partial_k u + A_{2;1}(\mathbf{u}, v, \mathbf{w}) \qquad (4.5)\\
\Box(\mu; R)\mathbf{w} &= curl\frac{\mathbf{f}}{\rho} + \sum_{jk}\nabla(\frac{r_{jk}}{\rho})\times\partial_j\partial_k\mathbf{u} + A_{3;1}(\mathbf{u}, v, \mathbf{w}),
\end{aligned}
$$

where $A_{j;1}$, $j = 1, 2, 3$, are first order differential operators with the coefficients of first order derivatives of v and $\mathbf{w}$ with $C^1(\Omega)$-norms bounded by a constant C and the coefficients of first order derivatives of $\mathbf{u}$ and of zero order terms with $L^\infty(\Omega)$-norms bounded by C.

Proof of Lemma 4.3

Dividing the both sides of (4.1) by ρ yields

$$
\partial_t^2\mathbf{u} - \frac{\mu}{\rho}\Delta\mathbf{u} - \frac{\lambda+\mu}{\rho}\nabla(\nabla\cdot\mathbf{u}) - (\nabla\cdot\mathbf{u})\frac{\nabla\lambda}{\rho} - (\nabla\mathbf{u}+(\nabla\mathbf{u})^\top)\frac{\nabla\mu}{\rho} - \frac{1}{\rho}\nabla\cdot((\nabla\mathbf{u})R) = \frac{\mathbf{f}}{\rho}. \quad (4.6)
$$

Since the last residual stress term is divergence free, we have

$$
\nabla\cdot((\nabla\mathbf{u})R) = \sum_{j,k=1}^{3} r_{jk}\partial_j\partial_k\mathbf{u} + \sum_{j,k=1}^{3}\partial_k r_{jk}\partial_j\mathbf{u} = \sum_{j,k=1}^{3} r_{jk}\partial_j\partial_k\mathbf{u}
$$

due to $\nabla\cdot R = 0$. Then (4.6) implies

$$
\partial_t^2\mathbf{u} - \frac{\mu}{\rho}\Delta\mathbf{u} - \frac{1}{\rho}\sum_{j,k=1}^{3} r_{jk}\partial_j\partial_k\mathbf{u} - \frac{\lambda+\mu}{\rho}\nabla(\nabla\cdot\mathbf{u}) - (\nabla\cdot\mathbf{u})\frac{\nabla\lambda}{\rho} - (\nabla\mathbf{u}+(\nabla\mathbf{u})^\top)\frac{\nabla\mu}{\rho} = \frac{\mathbf{f}}{\rho}. \quad (4.7)
$$

We now use two auxiliary functions $v = div\mathbf{u}$ and $\mathbf{w} = curl\mathbf{u}$, so that (4.7) implies

$$\partial_t^2\mathbf{u} - \frac{\mu}{\rho}\Delta\mathbf{u} - \frac{1}{\rho}\sum r_{jk}\partial_j\partial_k\mathbf{u} - \frac{\lambda+\mu}{\rho}\nabla v - v\frac{\nabla\lambda}{\rho} - (\nabla\mathbf{u} + (\nabla\mathbf{u})^\top)\frac{\nabla\mu}{\rho} = \frac{\mathbf{f}}{\rho}\,.$$

Hence (4.6) yields the first operator of new system (4.5) with wave operators in diagonal

$$\Box(\mu; R)\mathbf{u} = \frac{\mathbf{f}}{\rho} + \mathbf{A}_{1;1}(\mathbf{u}, v), \tag{4.8}$$

where $\mathbf{A}_{1;1}(\mathbf{u}, v) = \dfrac{\lambda+\mu}{\rho}\nabla v + \dfrac{\nabla\lambda}{\rho}v + (\nabla\mathbf{u} + (\nabla\mathbf{u})^\top)\dfrac{\nabla\mu}{\rho}$.

Below we use the following identities

$$\begin{cases} (1)\;\; \Delta\mathbf{u} = \nabla(\nabla\cdot\mathbf{u}) - curl(\nabla\times\mathbf{u}) = \nabla v - curl\mathbf{w}, \\ (2)\;\; \nabla\cdot(\Delta\mathbf{u}) = \Delta(\nabla\cdot\mathbf{u}), \\ (3)\;\; \nabla\times(\Delta\mathbf{u}) = \Delta(\nabla\times\mathbf{u}), \\ (4)\;\; \nabla\times\mathbf{u} = (\partial_2 u_3 - \partial_3 u_2, \partial_3 u_1 - \partial_1 u_3, \partial_1 u_2 - \partial_2 u_1), \\ (5)\;\; \mathbf{v}\times\mathbf{u} = (v_2u_3 - v_3u_2, v_3u_1 - v_1u_3, v_1u_2 - v_2u_1), \\ (6)\;\; \nabla\cdot(f\mathbf{F}) = f(\nabla\cdot\mathbf{F}) + \nabla f\cdot\mathbf{F}, \\ (7)\;\; \nabla\times(f\mathbf{F}) = f(\nabla\times\mathbf{F}) + \nabla f\times\mathbf{F}, \\ (8)\;\; curl(\nabla f) = \nabla\times(\nabla f) = \mathbf{0}. \end{cases} \tag{4.9}$$

Taking the divergence on both sides of (4.7), we obtain

$$\nabla\cdot\Big(\partial_t^2\mathbf{u} - \frac{\mu}{\rho}\Delta\mathbf{u} - \frac{\lambda+\mu}{\rho}\nabla v - \frac{1}{\rho}\sum r_{jk}\partial_j\partial_k\mathbf{u} - v\frac{\nabla\lambda}{\rho} - (\nabla\mathbf{u} + (\nabla\mathbf{u})^\top)\frac{\nabla\mu}{\rho}\Big) = div\frac{\mathbf{f}}{\rho}\,. \tag{4.10}$$

We now consider the divergence of each term of (4.10) by using the identities (4.9). The second term becomes

$$\nabla\cdot(\frac{\mu}{\rho}\Delta\mathbf{u}) = \frac{\mu}{\rho}\nabla\cdot(\Delta\mathbf{u}) + \nabla(\frac{\mu}{\rho})\cdot(\Delta\mathbf{u}) = \frac{\mu}{\rho}\Delta v + \nabla(\frac{\mu}{\rho})\cdot(\nabla v - curl\mathbf{w}) \tag{4.11}$$

by using the product rule and (1), (3) of (4.9), the third term is

$$\nabla\cdot(\frac{\lambda+\mu}{\rho}\nabla v) = \nabla\cdot(\frac{\lambda+\mu}{\rho}\nabla v) = \frac{\lambda+\mu}{\rho}\Delta v + \nabla(\frac{\lambda+\mu}{\rho})\cdot(\nabla v) \tag{4.12}$$

by using the product rule and (1) of (4.9), the fourth term is

$$\nabla\cdot(\frac{1}{\rho}\sum_{j,k} r_{jk}\partial_j\partial_k\mathbf{u}) = \frac{1}{\rho}\sum_{j,k} r_{jk}\partial_j\partial_k(\nabla\cdot\mathbf{u}) + \frac{1}{\rho}\sum_{j,k,l}\partial_l r_{jk}\partial_j\partial_k u_l + \sum_l(\partial_l\frac{1}{\rho}\sum_{j,k} r_{jk}\partial_j\partial_k u_l)$$

$$= \frac{1}{\rho}\sum r_{jk}\partial_j\partial_k v + \frac{1}{\rho}\sum_{j,k}\nabla r_{jk}\cdot\partial_j\partial_k\mathbf{u} + \nabla\frac{1}{\rho}\cdot\sum_{j,k} r_{jk}\partial_j\partial_k\mathbf{u}$$

$$= \frac{1}{\rho}\sum r_{jk}\partial_j\partial_k v + \sum\nabla(\frac{r_{jk}}{\rho})\cdot\partial_j\partial_k\mathbf{u} \tag{4.13}$$

by using the product rule and (6) of (4.9), and similarly for the fifth term,

$$\nabla\cdot(v\frac{\nabla\lambda}{\rho}) = v\nabla\cdot(\frac{\nabla\lambda}{\rho}) + \nabla v\cdot\frac{\nabla\lambda}{\rho}. \tag{4.14}$$

The divergence of the last term of (4.10) consisting of a matrix is more complicated; it yields

$$\begin{aligned}\nabla\cdot((\nabla\mathbf{u}+(\nabla\mathbf{u})^\top)\frac{\nabla\mu}{\rho}) &= \sum_{j,k}\partial_k\partial_j u_k\frac{\partial_j\mu}{\rho} + \sum_{j,k}\partial_k\partial_k u_j\frac{\partial_j\mu}{\rho} + \sum_{j,k}(\partial_j u_k+\partial_k u_j)\partial_k\frac{\partial_j\mu}{\rho}\\ &= \sum_j\partial_j(\nabla\cdot\mathbf{u})\frac{\partial_j\mu}{\rho} + \Delta\mathbf{u}\cdot\frac{\nabla\mu}{\rho} + (\nabla\mathbf{u}+(\nabla\mathbf{u})^\top)\sum_k\partial_k(\frac{\nabla\mu}{\rho}) \\ &= \nabla v\cdot\frac{\nabla\mu}{\rho} + (\nabla v - curl\mathbf{w})\cdot\frac{\nabla\mu}{\rho} + \big((\nabla\mathbf{u}+(\nabla\mathbf{u})^\top)\nabla\big)\cdot(\frac{\nabla\mu}{\rho}).\end{aligned} \tag{4.15}$$

Using from (4.11) to (4.15) yields

$$\partial_t^2 v - \frac{\mu}{\rho}\Delta v - \frac{\lambda+\mu}{\rho}\Delta v - \frac{1}{\rho}\sum r_{jk}\partial_j\partial_k v - \sum\nabla(\frac{r_{jk}}{\rho})\cdot\partial_j\partial_k\mathbf{u} - (\nabla\frac{\mu}{\rho}+\frac{\nabla\mu}{\rho})\cdot(\nabla v - curl\mathbf{w})$$
$$-(\nabla\frac{\lambda+\mu}{\rho}+\frac{\nabla\lambda+\nabla\mu}{\rho})\cdot\nabla v - \big((\nabla\mathbf{u}+(\nabla\mathbf{u})^\top)\nabla\big)\cdot(\frac{\nabla\mu}{\rho}) - (\nabla\cdot\frac{\nabla\lambda}{\rho})v = div\frac{\mathbf{f}}{\rho}.$$

Hence the second operator of new system (4.5) is obtained by

$$\Box(\lambda+2\mu;R)v = div\frac{\mathbf{f}}{\rho} + \sum_{j,k=1}^{3}\nabla(\frac{r_{jk}}{\rho})\cdot\partial_j\partial_k\mathbf{u} + \mathbf{A}_{2;1}(\mathbf{u},v,\mathbf{w}), \tag{4.16}$$

where $\mathbf{A}_{2;1}(\mathbf{u},v,\mathbf{w}) = (\nabla\frac{\mu}{\rho}+\frac{\nabla\mu}{\rho})\cdot(\nabla v - curl\mathbf{w}) + (\nabla\frac{\lambda+\mu}{\rho}+\frac{\nabla\lambda+\nabla\mu}{\rho})\cdot\nabla v$

$$+\big((\nabla\mathbf{u}+(\nabla\mathbf{u})^\top)\nabla\big)\cdot(\frac{\nabla\mu}{\rho}) + (\nabla\cdot\frac{\nabla\lambda}{\rho})v.$$

Taking the curl of both sides of (4.7) yields

$$\nabla\times\Big(\partial_t^2\mathbf{u} - \frac{\mu}{\rho}\Delta\mathbf{u} - \frac{\lambda+\mu}{\rho}\nabla v - \frac{1}{\rho}\sum r_{jk}\partial_j\partial_k\mathbf{u} - (\nabla\cdot\mathbf{u})\frac{\nabla\lambda}{\rho} - (\nabla\mathbf{u}+(\nabla\mathbf{u})^\top)\frac{\nabla\mu}{\rho}\Big) = curl\frac{\mathbf{f}}{\rho}. \tag{4.17}$$

We now consider the curl of each term of (4.17) by using the identities (4.9), that is,

$$\nabla \times \partial_t^2 \mathbf{u} = \partial_t^2 \mathbf{w}, \tag{4.18}$$

$$\nabla \times (\frac{\mu}{\rho}\Delta \mathbf{u}) = \frac{\mu}{\rho}(\nabla \times \Delta \mathbf{u}) + \nabla\frac{\mu}{\rho} \times \Delta \mathbf{u} = \frac{\mu}{\rho}\Delta \mathbf{w} + \nabla\frac{\mu}{\rho} \times (\nabla v - curl\mathbf{w}), \tag{4.19}$$

and

$$\nabla \times (\frac{\lambda+\mu}{\rho}\nabla v) = \nabla\frac{\lambda+\mu}{\rho} \times \nabla v \tag{4.20}$$

due to $\nabla \times \nabla v = curl\nabla v = \mathbf{0}$, we have

$$\nabla \times (\,\frac{1}{\rho}\sum r_{jk}\partial_j\partial_k\mathbf{u}\,) = \frac{1}{\rho}\sum r_{jk}\partial_j\partial_k\mathbf{w} + \sum \nabla(\frac{r_{jk}}{\rho}) \times \partial_j\partial_k\mathbf{u} \tag{4.21}$$

by using (7) of (4.9), and similarly

$$\nabla \times (v\frac{\nabla\lambda}{\rho}) = v\nabla \times \frac{\nabla\lambda}{\rho} + \nabla v \times \frac{\nabla\lambda}{\rho}. \tag{4.22}$$

Using (7), (8) of (4.9) the last term of (4.17) yields

$$\nabla \times ((\nabla\mathbf{u} + (\nabla\mathbf{u})^\top)\frac{\nabla\mu}{\rho}) = \nabla \times \sum_j (\partial_j u_k\frac{\partial_j\mu}{\rho} + \partial_k u_j\frac{\partial_j\mu}{\rho})$$

$$= \sum_j \frac{\partial_j\mu}{\rho}(\nabla \times \partial_j u_k) \,+\, \sum_j \partial_k\frac{\partial_j\mu}{\rho} \times \partial_j u_k \,+\, \sum_j \frac{\partial_j\mu}{\rho}(\nabla \times \partial_k u_j) \,+\, \sum_j \partial_k\frac{\partial_j\mu}{\rho} \times \partial_k u_j$$

$$= \sum_j \frac{\partial_j\mu}{\rho}\partial_j(\nabla \times u_k) \,+\, \sum_j \frac{\partial_j\mu}{\rho}(\nabla \times \nabla u_j) \,+\, \sum_j \partial_k\frac{\partial_j\mu}{\rho} \times (\partial_j u_k + \partial_k u_j)$$

$$= \frac{\nabla\mu}{\rho}\cdot\nabla(\nabla \times \mathbf{u}) \,+\, \sum_j \partial_k\frac{\partial_j\mu}{\rho} \times (\partial_j u_k + \partial_k u_j). \tag{4.23}$$

Using from (4.18) to (4.23) yields

$$\partial_t^2\mathbf{w} - \frac{\mu}{\rho}\Delta\mathbf{w} - \frac{1}{\rho}\sum r_{jk}\partial_j\partial_k\mathbf{w} - \sum\nabla(\frac{r_{jk}}{\rho}) \times \partial_j\partial_k\mathbf{u} - \nabla\frac{\mu}{\rho} \times (\nabla v - curl\mathbf{w})$$

$$-\,\nabla\frac{\lambda+\mu}{\rho} \times \nabla v - v(\nabla \times \frac{\nabla\lambda}{\rho}) - \nabla v \times \frac{\nabla\lambda}{\rho} - \nabla\mathbf{w}\cdot\frac{\nabla\mu}{\rho} - \sum_j \partial_k\frac{\partial_j\mu}{\rho} \times (\partial_j u_k + \partial_k u_j) \;=\; curl\frac{\mathbf{f}}{\rho}.$$

Hence it yields the third operator of new system (4.5)

$$\Box(\mu;R)\mathbf{w} = curl\frac{\mathbf{f}}{\rho} \,+\, \sum\nabla(\frac{r_{jk}}{\rho}) \times \partial_j\partial_k\mathbf{u} \,+\, \mathbf{A}_{3;1}(\mathbf{u},v,\mathbf{w}), \tag{4.24}$$

where $\mathbf{A}_{3;1}(\mathbf{u}, v, \mathbf{w}) = \nabla\frac{\mu}{\rho} \times (\nabla v - curl\mathbf{w}) + \nabla\frac{\lambda+\mu}{\rho} \times \nabla v + v(\nabla \times \frac{\nabla\lambda}{\rho})$

$$+ \nabla v \times \frac{\nabla\lambda}{\rho} + \nabla\mathbf{w} \cdot \frac{\nabla\mu}{\rho} + \sum_j \partial_k \frac{\partial_j \mu}{\rho} \times (\partial_j u_k + \partial_k u_j).$$

Combining (4.8), (4.16), and (4.24) produces the new system of equations. □

4.2 Strong Carleman estimate for a general elasticity system

In this section we prove Theorem 4.1.

Lemma 4.4 *Let $|\nabla\psi| > 0$ on $\bar{\Omega}$. Then, for a second order elliptic operator A, there are constants C, $C_0(\gamma)$ such that*

$$\gamma \int_\Omega \sigma^{4-2|\alpha|} e^{2\tau\varphi} |\partial^\alpha v|^2 \leq C \int_\Omega \sigma e^{2\tau\varphi} |Av|^2 \tag{4.25}$$

for all $v \in C_0^2(\Omega)$, $|\alpha| \leq 2$, $C < \gamma$, and $C_0(\gamma) < \tau$.

Proof of Lemma 4.4

We apply Carleman estimate in [12],

$$\sum_{|\alpha|\leq 2} \sqrt{\gamma}\, \|\sigma^{\frac{3}{2}-|\alpha|} e^{\tau\varphi} \partial^\alpha u\| \leq C\, \|e^{\tau\varphi} A(x, D)u\|, \tag{4.26}$$

to $u = \sigma^{\frac{1}{2}} v$. By the Leibniz formula

$$\partial^\alpha(\sigma^{\frac{1}{2}} v) = \sigma^{\frac{1}{2}} \partial^\alpha v + \tau^{\frac{1}{2}} A_{|\alpha|-1}(x, D)v, \quad |\alpha| = 1, 2$$

and

$$A(x, D)(\sigma^{\frac{1}{2}} v) = \sigma^{\frac{1}{2}} A(x, D)v + \tau^{\frac{1}{2}} A_1(x, D)v,$$

where A_m is a linear partial differential operator of order m with coefficients bounded by $C(\gamma)$. By using these relations with $|\alpha| = 1$ and the triangle inequality from (4.26) we get

$$\sqrt{\gamma}\, \|\sigma e^{\tau\varphi} \nabla v\| - C(\gamma) \|\tau e^{\tau\varphi} v\| \leq C\, \|\sigma^{\frac{1}{2}} e^{\tau\varphi} A(x, D)v\| + C(\gamma) \sum_{|\alpha|\leq 1} \|\tau^{\frac{1}{2}} e^{\tau\varphi} \partial^\alpha v\|.$$

Similarly, when $|\alpha| = 2$,

$$\sqrt{\gamma}\,\|e^{\tau\varphi}\partial^\alpha v\| - C(\gamma)\sum_{|\alpha|\le 1}\|e^{\tau\varphi}\partial^\alpha v\| \;\le\; C\,\|\sigma^{\frac{1}{2}}e^{\tau\varphi}A(x,D)v\| + C(\gamma)\sum_{|\alpha|\le 1}\|\tau^{\frac{1}{2}}e^{\tau\varphi}\partial^\alpha v\|.$$

Summing the inequalities over $|\alpha| \le 2$, we yield

$$\sqrt{\gamma}\sum_{|\alpha|\le 2}\|\sigma^{2-|\alpha|}e^{\tau\varphi}\partial^\alpha v\| - \,C(\gamma)\sum_{|\alpha|\le 1}\tau^{1-|\alpha|}\|e^{\tau\varphi}\partial^\alpha v\|$$

$$\le\; C\,\|\sigma^{\frac{1}{2}}e^{\tau\varphi}Av\| + C(\gamma)\|\tau^{\frac{1}{2}}e^{\tau\varphi}\partial^\alpha v\|.$$

Since $\sigma = \tau\gamma\varphi$ for $1 \le \gamma$ and $1 \le \varphi$, the second terms in the left hand side and the right hand side are absorbed by the first term on the left side by choosing $\tau > C(\gamma)$. □

Proof of Theorem 4.1

Applying Theorem 3.1 to each of seven scalar differential operators forming the extended system (4.5) and summing up seven Carleman estimates, we get

$$\int_\Omega(\sigma|\nabla_{x,t}\mathbf{u}|^2 + \sigma|\nabla_{x,t}v|^2 + \sigma|\nabla_{x,t}\mathbf{w}|^2 + \sigma^3|\mathbf{u}|^2 + \sigma^3|v|^2 + \sigma^3|\mathbf{w}|^2)e^{2\tau\varphi}$$

$$\le C\int_\Omega(|\mathbf{A}_R\mathbf{u}|^2 + |\nabla(\mathbf{A}_R\mathbf{u})|^2)e^{2\tau\varphi} \,+\, C\int_\Omega\sum_{j,k=1}^{3}|\partial_j\partial_k\mathbf{u}|^2e^{2\tau\varphi}$$

$$+\,C\int_\Omega(|\nabla\mathbf{u}|^2 + |\nabla v|^2 + |\nabla\mathbf{w}|^2 + |\mathbf{u}|^2 + v^2 + |\mathbf{w}|^2)e^{2\tau\varphi}.$$

By choosing $\tau > 2C$, we absorb the third integral in the right side by the left side, arriving at the inequality

$$\int_\Omega(\sigma|\nabla_{x,t}\mathbf{u}|^2 + \sigma|\nabla_{x,t}v|^2 + \sigma|\nabla_{x,t}\mathbf{w}|^2 + \sigma^3|\mathbf{u}|^2 + \sigma^3|v|^2 + \sigma^3|\mathbf{w}|^2)e^{2\tau\varphi}$$

$$\le C\int_\Omega(|\mathbf{A}_R\mathbf{u}|^2 + |\nabla(\mathbf{A}_R\mathbf{u})|^2)e^{2\tau\varphi} \,+\, C\int_\Omega\sum_{j,k=1}^{3}|\partial_j\partial_k\mathbf{u}|^2e^{2\tau\varphi}. \tag{4.27}$$

To eliminate the second order derivatives in the right side, we need a second large parameter γ. By Lemma 4.4

$$\gamma\int_\Omega\sum_{j,k=1}^{3}|\partial_j\partial_k\mathbf{u}|^2e^{2\tau\varphi} \,\le\, C\int_\Omega\sigma|\Delta\mathbf{u}|^2e^{2\tau\varphi}$$

$$\leq C\int_\Omega \sigma(|\nabla v|^2+|\nabla \mathbf{w}|^2)e^{2\tau\varphi} \leq C\int_\Omega(|\mathbf{f}|^2+|\nabla\mathbf{f}|^2)e^{2\tau\varphi} + C\int_\Omega |\partial_j\partial_k\mathbf{u}|^2 e^{2\tau\varphi},$$

where we used the known identity $\Delta\mathbf{u} = \nabla v - curl\mathbf{w}$ and (4.27). Choosing $\gamma > 2C$, we can see that the second order derivative term on the right side is absorbed by the left side. This yields

$$\gamma\int_\Omega \sum_{j,k=1}^{3}|\partial_j\partial_k\mathbf{u}|^2 e^{2\tau\varphi} \leq C\int_\Omega(|\mathbf{f}|^2+|\nabla\mathbf{f}|^2)e^{2\tau\varphi}.$$

So, using again (4.27) proves (4.3). □

4.3 Weak Carleman estimate for a general elasticity system

In this section we prove Theorem 4.2.

Lemma 4.5 *There exists a constant $C(\gamma)$ such that*

$$\|\Lambda_{\sigma^*}^{-1}(\partial^\alpha curl)_\varphi\mathbf{v} - (\partial^\alpha curl)_\varphi\Lambda_{\sigma^*}^{-1}\mathbf{v}\|_{(0)}(\Omega^*) \leq C(\gamma)\|\mathbf{v}\|_{(0)}(\Omega)$$

for all $\mathbf{v}\in H_0^2(\Omega)$, $|\alpha|\leq 1$.

The proof is similar to the proof of Lemma 3.11.

Lemma 4.6 *Let $|\nabla\psi| > 0$ on $\bar\Omega$. Then there are constants C, $C_0(\gamma)$ such that*

$$\gamma\int_\Omega(\sigma^2|\mathbf{u}|^2+\sum_{|\alpha|=1}|\partial^\alpha\mathbf{u}|^2)e^{2\tau\varphi} \leq C\int_\Omega\sigma(|curl\mathbf{u}|^2+|div\mathbf{u}|^2)e^{2\tau\varphi}$$

for all $\mathbf{u}\in H_0^1(\Omega)$ provided $C<\gamma$, $C_0(\gamma)<\tau$.

Lemma 4.6 is proven in [11]. To make our exposition more self-contained, we give a proof different from [11]. We expect this proof to be useful when handling more general systems.

Proof of Lemma 4.6

Let $x_0\in\bar\Omega$. We first consider $\mathbf{u}$ supported in $B(x_0;\delta)$. Using the standard substitution $\mathbf{u} = e^{-\tau\varphi}\mathbf{v}$ as above, this lemma follows from the bound

$$\gamma\int_{B(\delta)}(\sigma^4|\Lambda_{\sigma^*}^{-1}\mathbf{v}|^2+\sigma^2|\mathbf{v}|^2+\sum_{|\alpha|=1}|\partial^\alpha\mathbf{v}|^2)$$

$$\leq C\int_{B(\delta)}\sigma(|(curl)_\varphi\mathbf{v}|^2+|(div)_\varphi\mathbf{v}|^2).$$

Since $\Delta\mathbf{v}_0 = curlcurl\mathbf{v}_0 - \nabla div\mathbf{v}_0$, by Lemma 4.4 there exist C and $C(\gamma)$ such that the following Carleman estimate holds

$$\gamma\sum_{|\alpha|\leq 2}\int_{B(3\delta)}\sigma^{4-2|\alpha|}|\partial^\alpha\mathbf{v}_0|^2 \leq C\int_{B(\delta)}\sigma\sum_{|\alpha|=1}(|(\partial^\alpha curl)_\varphi\mathbf{v}_0|^2+|(\partial^\alpha div)_\varphi\mathbf{v}_0|^2)$$

for all $\mathbf{v}_0\in H_0^2(B(3\delta))$ provided $C<\gamma$, $C(\gamma)<\tau$.

Let $\chi\in C_0^\infty(B(3\delta))$, $\chi = 1$ on $B(2\delta)$. Applying this Carleman type estimate to $\mathbf{v}_0 = \chi\Lambda_{\sigma^*}^{-1}\mathbf{v}$, we obtain

$$\gamma\int_{B(3\delta)}(\sigma^4\chi^2|\Lambda_{\sigma^*}^{-1}\mathbf{v}|^2+\sigma^2\sum_{|\alpha|=1}|\chi\partial^\alpha(\Lambda_{\sigma^*}^{-1}\mathbf{v})+(\partial^\alpha\chi)\Lambda_{\sigma^*}^{-1}\mathbf{v}|^2$$

$$+\sum_{|\alpha|=2}|\chi\partial^\alpha(\Lambda_{\sigma^*}^{-1}\mathbf{v})+2\partial^{\alpha'}\chi\partial^{\alpha-\alpha'}(\Lambda_{\sigma^*}^{-1}\mathbf{v})+(\partial^\alpha\chi)\Lambda_{\sigma^*}^{-1}\mathbf{v}|^2)$$

$$\leq C\int_{B(3\delta)}\sum_{|\alpha|=1}\sigma(|(\partial^\alpha curl)_\varphi(\Lambda_{\sigma^*}^{-1}\mathbf{v})|^2+|(\partial^\alpha div)_\varphi(\Lambda_{\sigma^*}^{-1}\mathbf{v})|^2)$$

$$+C(\gamma)(\tau^3|\Lambda_{\sigma^*}^{-1}\mathbf{v}|^2+\tau\sum_{|\alpha|=1}|\partial^\alpha(\Lambda_{\sigma^*}^{-1}\mathbf{v})|^2), \tag{4.28}$$

where we used that due to (3.37), for any second order partial differential operator A with constant coefficients,

$$A_\varphi(\,,D)(\chi\Lambda_{\sigma^*}^{-1}\mathbf{v}) \;=\; \chi A(\,,D+i\tau\nabla\varphi)\Lambda_{\sigma^*}^{-1}\mathbf{v}+(\tau A_0+A_1(\,,D))\Lambda_{\sigma^*}^{-1}\mathbf{v}, \tag{4.29}$$

where A_0 and A_1 are zero and first order operators with bounded coefficients depending on γ and A, we applied the relation (4.29) to components of $curl_\varphi$ and div_φ, and we used the triangle inequality.

Recalling that $\chi = 1$ on $B(2\delta)$, we derive from inequalities (3.43) and (3.44), and the bound (4.28), so that

$$\gamma\int_{B(2\delta)}\sum_{|\alpha|\leq 2}\sigma^{4-2|\alpha|}|\partial^\alpha\Lambda_{\sigma^*}^{-1}\mathbf{v}|^2 - C(\gamma)\int_{B(3\delta)}|\mathbf{v}|^2$$

$$\leq C\int_{B(3\delta)}\big(\sigma\sum_{|\alpha|=1}(|(\partial^\alpha curl)_\varphi(\Lambda_{\sigma^*}^{-1}\mathbf{v})|^2+|(\partial^\alpha div)_\varphi(\Lambda_{\sigma^*}^{-1}\mathbf{v})|^2)+C(\gamma)\tau|\mathbf{v}|^2\big). \tag{4.30}$$

The Parseval identity and the definition of Λ_τ yield

$$\int_{B(3\delta)} \sigma^2|\mathbf{v}|^2 \le \int_{B(3\delta)} (\sigma^*)^2|\mathbf{v}|^2$$

$$= (\sigma^*)^2 \int_{\mathbb{R}^3} \frac{\sigma^{*2}+1}{<\xi>^2+\sigma^{*2}}|\hat{\mathbf{v}}(\xi)|^2 d\xi + (\sigma^*)^2 \int_{\mathbb{R}^3} \frac{|\xi|^2}{<\xi>^2+\sigma^{*2}}|\hat{\mathbf{v}}(\xi)|^2 d\xi$$

$$\le 2(\sigma^*)^4 \int_{\mathbb{R}^3} |\Lambda_{\sigma^*}^{-1}\mathbf{v}|^2 + (\sigma^*)^2 \sum_{|\alpha|=1} \int_{\mathbb{R}^3} |\partial^\alpha(\Lambda_{\sigma^*}^{-1}\mathbf{v})|^2$$

$$\le 2\,(\frac{\sigma^*}{\sigma_*})^4 \int_{B(2\delta)} \sigma^4|\Lambda_{\sigma^*}^{-1}v|^2 + (\frac{\sigma^*}{\sigma_*})^2 \sum_{|\alpha|=1} \int_{B(2\delta)} \sigma^2|\partial^\alpha(\Lambda_{\sigma^*}^{-1}\mathbf{v})|^2$$

$$+ \int_{\mathbb{R}^3\setminus B(2\delta)} ((\sigma^*)^4|\Lambda_{\sigma^*}^{-1}\mathbf{v}|^2 + (\sigma^*)^2 \sum_{|\alpha|=1} |\partial^\alpha\Lambda_{\sigma^*}^{-1}\mathbf{v}|^2).$$

Since $\psi \in C^2$, using (3.2) we choose $\delta(\gamma)$ so that $\frac{1}{2} < \frac{\sigma_*}{\sigma^*}$. Similarly

$$\int_{B(3\delta)} \sum_{|\alpha|=1} |\partial^\alpha \mathbf{v}|^2 = \int_{\mathbb{R}^3} |\xi|^2|\hat{\mathbf{v}}(\xi)|^2 d\xi$$

$$= ((\sigma^*)^2+1) \int_{\mathbb{R}^3} \frac{|\xi|^2}{<\xi>^2+\sigma^{*2}}|\hat{\mathbf{v}}(\xi)|^2 d\xi + \int_{\mathbb{R}^3} \frac{|\xi|^4}{<\xi>^2+\sigma^{*2}}|\hat{\mathbf{v}}(\xi)|^2 d\xi$$

$$\le 2 \int_{\mathbb{R}^3} \sum_{1\le|\alpha|\le 2} (\sigma^*)^{4-2|\alpha|}|\partial^\alpha\Lambda_{\sigma^*}^{-1}\mathbf{v}|^2$$

$$\le 8 \int_{B(2\delta)} \sum_{1\le|\alpha|\le 2} \sigma^{4-2|\alpha|}|\partial^\alpha\Lambda_{\sigma^*}^{-1}\mathbf{v}|^2 + \int_{\mathbb{R}^3\setminus B(2\delta)} \sum_{1\le|\alpha|\le 2} (\sigma^*)^{4-2|\alpha|}|\partial^\alpha\Lambda_{\sigma^*}^{-1}\mathbf{v}|^2,$$

where we used that $\sigma \le 2\sigma_*$ on $B(3\delta)$. Choosing $\tau > C(\gamma)$ and using Lemma 4.5 we have from (4.30) the inequality

$$\gamma \int_{\mathbb{R}^3} (\sigma^4|\Lambda_{\sigma^*}^{-1}\mathbf{v}|^2 + \sigma^2|\mathbf{v}|^2 + \sum_{|\alpha|=1} |\partial^\alpha\mathbf{v}|^2)$$

$$\le C \int_{B(3\delta)} \sigma_* \sum_{|\alpha|=1} (|\partial^\alpha\Lambda_{\sigma^*}^{-1}(curl)_\varphi\mathbf{v}|^2 + |\partial^\alpha\Lambda_{\sigma^*}^{-1}(div)_\varphi\mathbf{v}|^2)$$

$$+ C(\gamma) \int_{\mathbb{R}^3\setminus B(2\delta)} \sum_{|\alpha|\le 2} \tau^{4-2|\alpha|}|\partial^\alpha\Lambda_{\sigma^*}^{-1}\mathbf{v}|^2 \tag{4.31}$$

when $\tau > C(\gamma)$. From (3.43) and (3.44) we have

$$\int_{B(3\delta)} \sigma_* \sum_{|\alpha|=1} (|\partial^\alpha\Lambda_{\sigma^*}^{-1}(curl)_\varphi\mathbf{v}|^2 + |\partial^\alpha\Lambda_{\sigma^*}^{-1}(div)_\varphi\mathbf{v}|^2)$$

$$\leq \int_{B(\delta)} \sigma(|(curl)_\varphi \mathbf{v}|^2 + |(div)_\varphi \mathbf{v}|^2).$$

By using Lemma 3.13, we eliminate the last integral in the bound (4.31). Since $supp\, \mathbf{v} \subset B(\delta)$,

$$|\partial_x^\alpha \Lambda_{\sigma^*}^{-1} \mathbf{v}(x)| \leq \int_{B(\delta)} |\mathbf{v}(y)||\partial_x^\alpha K(x,y;\sigma^*)|dy \leq C(\gamma)\tau^{-2} \int_{B(\delta)} |x-y|^{-8}|\mathbf{v}(y)|dy$$

by Lemma 3.12, provided $x \in \mathbb{R}^3 \setminus B(2\delta)$. When $y \in B(\delta)$, as in the proof of Lemma 3.13, $|x-y| \geq \frac{1+|x|}{C(\gamma)}$. Hence by using the Schwartz inequality

$$|\partial^\alpha \Lambda_{\sigma^*}^{-1} \mathbf{v}|(x) \leq C(\gamma)\tau^{-2}(1+|x|)^{-8}(\int_\Omega |\mathbf{v}|^2)^{\frac{1}{2}} \quad \text{for all } |\alpha| \leq 1$$

provided $x \in \mathbb{R}^3 \setminus B(2\delta)$. Using this estimate, we conclude that the last integral in (4.31) is less than $C(\gamma)\int_{B(\delta)} |\mathbf{v}|^2$, so choosing $\tau > C(\gamma)$, we eliminate this integral by using the last integral in the left side of (4.31) as an upper bound. This completes the proof of Lemma 4.6 when $supp\, \mathbf{u} \subset B(x_0;\delta)$.

Now we complete the proof by using a special partition of unity argument. Due to compactness of $\bar{\Omega}$ we can find a finite covering of $\bar{\Omega}$ by balls $B(x(k);\delta(\gamma)(k))$, $k = 1,\dots,K$. Let $\chi(\,;k)$ be the special C^∞- partition of the unity subordinated to this covering, *i.e.*, $supp\, \chi(\,;k) \subset B(x(k);\delta)$ and $\sum_{k=1}^K \chi^2(\,;k) = 1$ on Ω. By the Leibniz formula

$$curl(\chi(\,;k)\mathbf{u}) = \chi(\,;k) curl \mathbf{u} + \mathbf{A}_{01}\mathbf{u}$$

and

$$div(\chi(\,;k)\mathbf{u}) = \chi(\,;k) div \mathbf{u} + \mathbf{A}_{02}\mathbf{u},$$

where $\mathbf{A}_{01}$ and $\mathbf{A}_{02}$ are bounded matrix-functions depending on γ. Applying Lemma 4.6 to $\chi(\,;k)\mathbf{u}$ and using the elementary inequality $\frac{1}{2}|a|^2 - |b|^2 \leq |a+b|^2$ we obtain

$$\gamma \int_\Omega (\sigma^2 \chi^2(\,;k)|\mathbf{u}|^2 + \sum_{|\alpha|=1} \chi^2(\,;k)|\partial^\alpha \mathbf{u}|^2 - C(\gamma)|\mathbf{u}|^2)e^{2\tau\varphi}$$

$$\leq C \int_\Omega \sigma\big((|curl \mathbf{u}|^2 + |div \mathbf{u}|^2) + C(\gamma)|\mathbf{u}|^2\big)e^{2\tau\varphi}.$$

Summing over $k = 1, \ldots, K$ and choosing $\tau > C(\gamma)$ we absorb the terms containing $C(\gamma)|\mathbf{u}|^2$ by the first term on the left side and complete the proof. $\square$

Proof of Theorem 4.2

Applying Theorem 3.2 to each of seven scalar differential operators forming the extended system (4.5) and summing up seven Carleman estimates, we get

$$\int_\Omega \sigma(|\mathbf{u}|^2 + |v|^2 + |\mathbf{w}|^2)e^{2\tau\varphi}$$

$$\leq C \int_\Omega |\mathbf{f}|^2 e^{2\tau\varphi} + C \int_\Omega \sum_{j=1}^{3} |\partial_j \mathbf{u}|^2 e^{2\tau\varphi} + C \int_\Omega (|\mathbf{u}|^2 + v^2 + |\mathbf{w}|^2)e^{2\tau\varphi}.$$

By choosing $\sigma > 2C$ we can absorb the third integral in the right side by the left side, arriving at the inequality

$$\int_\Omega \sigma(|\mathbf{u}|^2 + |v|^2 + |\mathbf{w}|^2)e^{2\tau\varphi}$$

$$\leq C \int_\Omega |\mathbf{f}|^2 e^{2\tau\varphi} + C \int_\Omega \sum_{j=1}^{3} |\partial_j \mathbf{u}|^2 e^{2\tau\varphi}. \tag{4.32}$$

To eliminate the first order derivatives in the right side we need the second large parameter γ. By Lemma 4.6 we have

$$\gamma \int_\Omega \sum_{j=1}^{3} |\partial_j \mathbf{u}|^2 e^{2\tau\varphi} \leq C \int_\Omega \sigma(|curl\mathbf{u}|^2 + |div\mathbf{u}|^2)e^{2\tau\varphi}$$

$$\leq C \int_\Omega |\mathbf{f}|^2 e^{2\tau\varphi} + C \int_\Omega \sum_{j=1}^{3} |\partial_j \mathbf{u}|^2 e^{2\tau\varphi},$$

where we used (4.32). Choosing $\gamma > 2C$, we can see that the first order derivatives term on the right side is absorbed by the left side. This yields

$$\gamma \int_\Omega \sum_{j}^{3} |\partial_j \mathbf{u}|^2 e^{2\tau\varphi} \leq C \int_\Omega |\mathbf{f}|^2 e^{2\tau\varphi}.$$

So using again (4.32), we complete the proof of estimate (4.4). $\square$

CHAPTER 5

UNIQUENESS OF CONTINUATION FOR SOLUTIONS OF PARTIAL DIFFERENTIAL EQUATIONS

In this chapter we derive local (of Hölder type) and global (of Lipschitz type) stability estimates for the lateral Cauchy problem for system (4.2).

Let us consider the following Cauchy problem:

$$\begin{aligned} \mathbf{A}_R \mathbf{u} = \mathbf{f} \quad \text{in } \Omega, \\ \mathbf{u} = \mathbf{g}_0, \;\; \partial_\nu \mathbf{u} = \mathbf{g}_1 \quad \text{on } \Gamma \subset \partial\Omega \end{aligned} \tag{5.1}$$

where $\Gamma \in C^3$. Let $\Omega_\delta = \Omega \cap \{\psi > \delta\}$.

The Carleman estimate of Theorem 4.2 by standard argument [19, section 3.2] implies the following conditional Hölder stability estimate for (5.1) in Ω_δ (and hence uniqueness in Ω_0).

Theorem 5.1 *Suppose that all coefficients λ, μ, ρ, R are in $C^2(\bar{\Omega})$. Let $\psi \in C^3(\bar{\Omega})$ be K-pseudo-convex with respect to $\square(\mu; R)$, $\square(\lambda + 2\mu; R)$ in $\bar{\Omega}$. Assume that $\bar{\Omega}_0 \subset \Omega \cup \Gamma$. Then there exist constants $C = C(\delta)$, $\kappa = \kappa(\delta) \in (0, 1)$ such that for a solution $\mathbf{u} \in H^2(\Omega)$ to (5.1) one has*

$$\|\mathbf{u}\|_{(0)}(\Omega_\delta) + \|\nabla_x \mathbf{u}\|_{(0)}(\Omega_\delta) \le C(F + M_1^{1-\kappa} F^\kappa), \tag{5.2}$$

where $F = \|\mathbf{f}\|_{(0)}(\Omega_0) + \|\mathbf{g}_0\|_{(\frac{3}{2})}(\Gamma) + \|\mathbf{g}_1\|_{(\frac{1}{2})}(\Gamma)$, $M_1 = \|\mathbf{u}\|_{(1)}(\Omega)$.

In Lipschitz stability, we assume that $\Omega = G \times (-T, T)$ and that the system $\mathbf{A}_R \mathbf{u} = \mathbf{f}$ in (5.1) is uniformly t-hyperbolic. Applying known [7] we showed that a sufficient condition for hyperbolicity of the residual stress system (2.64) in Section 2.5 is, in more detail, that

$$0 \le \lambda, \quad 0 < \varepsilon_0 I_3 < 2\mu I_3 + R \quad \text{on } \bar{G}.$$

We use the conventional energy integral

$$E(t; \mathbf{u}) = \int_G (|\partial_t \mathbf{u}|^2 + |\nabla \mathbf{u}|^2 + |\mathbf{u}|^2)(\,, t).$$

The Carleman estimate of Theorem 4.1 by standard argument implies the following best possible Lipschitz stability estimate for (5.1) in Ω.

Theorem 5.2 *Suppose that λ, μ, ρ, R are in $C^2(\bar{\Omega})$. Let ψ be K-pseudo-convex with respect to $\square(\mu; R)$, $\square(\lambda + 2\mu; R)$ in $\bar{\Omega}$. Assume that*

$$\psi < 0 \quad \text{on} \;\; \bar{G} \times \{-T, T\}, \qquad 0 \leq \psi \quad \text{on} \;\; G \times \{0\}. \tag{5.3}$$

Let $\Gamma = \partial G \times (-T, T)$. Then there exists a constant C such that for a solution $\mathbf{u} \in H^3(\Omega)$ to (5.1) one has

$$E(t; \mathbf{u}) + E(t; \nabla \mathbf{u}) \;\leq\; C\,(\|\mathbf{f}\|_{(1)}(\Omega) + \|\mathbf{g}_0\|_{(\frac{5}{2})}(\Gamma) + \|\mathbf{g}_1\|_{(\frac{3}{2})}(\Gamma)) \tag{5.4}$$

where $-T < t < T$.

5.1 Hölder stability in the Cauchy problem

In this section we prove Theorem 5.1.

Proof of Theorem 5.1

By extension theorems for Sobolev spaces, we can find $\mathbf{u}^* \in H^2(\Omega)$ so that

$$\mathbf{u}^* = \mathbf{g}_0, \; \partial_\nu \mathbf{u}^* = \mathbf{g}_1 \quad \text{on} \;\; \Gamma$$

and

$$\|\mathbf{u}^*\|_{(2)}(\Omega) \leq CF. \tag{5.5}$$

Let

$$\mathbf{v} = \mathbf{u} - \mathbf{u}^*. \tag{5.6}$$

The function $\mathbf{v}$ solves the Cauchy problem

$$\begin{aligned} \mathbf{A}_R \mathbf{v} = \mathbf{f} - \mathbf{A}_R \mathbf{u}^* \quad &\text{in} \;\; \Omega, \\ \mathbf{v} = \mathbf{0}, \;\; \partial_\nu \mathbf{v} = \mathbf{0} \quad &\text{on} \;\; \Gamma \subset \partial\Omega. \end{aligned} \tag{5.7}$$

To apply Carleman estimates of Theorem 4.2, we need zero Cauchy data on the whole boundary. To achieve this condition, we introduce a cut-off function $\chi \in C^\infty(\bar{\Omega})$ so that $\chi = 1$ on $\Omega_{\frac{\delta}{2}}$, $\chi = 0$ on $\Omega \setminus \Omega_0$. By the Leibniz formula

$$\mathbf{A}_R(\chi \mathbf{v}) = \chi \mathbf{A}_R \mathbf{v} + \mathbf{A}_1 \mathbf{v},$$

where $\mathbf{A}_1$ is a matrix linear partial differential operator of order 1 with bounded coefficients depending on χ. Moreover, $\mathbf{A}_1 = \mathbf{0}$ on $\Omega_{\frac{\delta}{2}}$. Using the Cauchy data (5.7) we conclude that $\mathbf{v} \in H_0^2(\Omega)$, hence by Carleman estimate of Theorem 4.2 we have

$$\int_\Omega (|\mathbf{v}| + |div(\chi \mathbf{v})|^2 + |curl(\chi \mathbf{v})|^2) e^{2\tau\varphi}$$

$$\leq C \int_\Omega (|\mathbf{f}|^2 + |\mathbf{A}_R \mathbf{u}^*|^2 + |\mathbf{A}_1 \mathbf{v}|^2) e^{2\tau\varphi}$$

for $C < \gamma$, $C_0 < \tau$. Shrinking integration domain on the left side to $\Omega_{\frac{3\delta}{4}}$ (where $\chi = 1$) and splitting integration domain of $|\mathbf{A}_1 \mathbf{v}|^2$ into $\Omega_{\frac{\delta}{2}}$ and its complement we yield

$$\int_{\Omega_{\frac{3\delta}{4}}} (|\mathbf{v}|^2 + |div\mathbf{v}|^2 + |curl\mathbf{v}|^2) e^{2\tau\varphi}$$

$$\leq C \int_\Omega (|\mathbf{f}|^2 + |\mathbf{A}_R \mathbf{u}^*|^2) e^{2\tau\varphi} + C \int_{\Omega \setminus \Omega_{\frac{\delta}{2}}} |\mathbf{A}_1 \mathbf{v}|^2 e^{2\tau\varphi}$$

$$\leq CF^2 e^{2\tau\Phi} + C\|\mathbf{v}\|^2_{(1)}(\Omega) e^{2\tau\Phi_2},$$

where we used definition (5.2) of F and bound (5.5) with $\Phi = \sup \varphi$ over Ω and $\Phi_2 = \sup \varphi$ over $\Omega \setminus \Omega_{\frac{\delta}{2}}$. Letting $\Phi_1 = \inf \varphi$ over $\Omega_{\frac{3\delta}{4}}$ and replacing φ on the left side of the preceding inequality by Φ_1 we yield

$$(\|\mathbf{v}\|^2_{(0)}(\Omega_{\frac{3\delta}{4}}) + \|div\mathbf{v}\|^2_{(0)}(\Omega_{\frac{3\delta}{4}}) + \|curl\mathbf{v}\|^2_{(0)}(\Omega_{\frac{3\delta}{4}})) e^{2\tau\Phi_1}$$

$$\leq CF^2 e^{2\tau\Phi} + C\|\mathbf{v}\|^2_{(1)}(\Omega) e^{2\tau\Phi_2}. \tag{5.8}$$

Observe that $\Phi_2 < \Phi_1$.

Using interior Schauder type estimates (2.53) in Theorem 2.17 for the elliptic operator $\Delta = \nabla div - curlcurl$, we obtain

$$\|\mathbf{v}\|^2_{(1)}(\Omega_\delta) \leq C(\|\mathbf{v}\|^2_{(0)}(\Omega_{\frac{3\delta}{4}}) + \|div\mathbf{v}\|^2_{(0)}(\Omega_{\frac{3\delta}{4}}) + \|curl\mathbf{v}\|^2_{(0)}(\Omega_{\frac{3\delta}{4}})).$$

Hence (5.8) yields

$$\|\mathbf{v}\|^2_{(0)}(\Omega_\delta) + \|\nabla_x \mathbf{v}\|^2_{(0)}(\Omega_\delta) \leq CF^2 e^{2\tau(\Phi-\Phi_1)} + C\|\mathbf{v}\|^2_{(1)}(\Omega) e^{2\tau(\Phi_2-\Phi_1)}. \tag{5.9}$$

If $\|\mathbf{v}\|_{(1)}(\Omega)F^{-1} < C$, then $\|\mathbf{v}\|_{(1)}(\Omega) \leq CF$. Otherwise we let $\tau = \frac{log(\|\mathbf{v}\|_{(1)}(\Omega)F^{-1})}{(\Phi+\Phi_1-\Phi_2)}$. Then the bound (5.9) implies that

$$\|\mathbf{v}\|_{(1)}(\Omega_\delta) \leq C\|\mathbf{v}\|_{(1)}(\Omega)^{1-\kappa} F^\kappa$$

with $\kappa = \frac{\Phi_1-\Phi_2}{\Phi+\Phi_1-\Phi_2}$. Combining both cases we yield

$$\|\mathbf{v}\|_{(0)}(\Omega_\delta) + \|\nabla_x \mathbf{v}\|_{(0)}(\Omega_\delta) \leq C(F + \|\mathbf{v}\|_{(1)}(\Omega)^{1-\kappa} F^\kappa).$$

Using the above inequality, the relation $\mathbf{u} = \mathbf{v} + \mathbf{u}^*$, the triangle inequality, inequality (5.5), and the elementary inequality $(a+b)^\kappa \leq a^\kappa + b^\kappa$, $0 < \kappa < 1$, we get (5.2). □

5.2 Lipschitz stability in the Cauchy problem

In this section we prove Theorem 5.2.

Proof of Theorem 5.2

As in the proof of Theorem 5.1, we introduce functions $\mathbf{u}^*$ and $\mathbf{v}$. Let $\mathbf{v} = \mathbf{u} - \mathbf{u}^*$. Since the surface $\Gamma \in C^3$ is noncharacteristic for $\mathbf{A}_R$, we can uniquely solve $\mathbf{A}_R\mathbf{u} = \mathbf{f}$ on Γ for $\partial^2_\nu \mathbf{u}$ in terms of $\mathbf{f}$, $\mathbf{g}_0$, $\mathbf{g}_1$, and their tangential derivatives. Moreover

$$\|\partial^2_\nu \mathbf{u}\|_{(\frac{1}{2})}(\Gamma) \leq C\,(\|\mathbf{f}\|_{(\frac{1}{2})}(\Gamma) + \|\mathbf{g}_0\|_{(\frac{5}{2})}(\Gamma) + \|\mathbf{g}_1\|_{(\frac{3}{2})}(\Gamma)). \tag{5.10}$$

Then extension Theorem 2.1 tells us that for $\mathbf{f} \in H^{\frac{1}{2}}(\Gamma)$ we can find $\mathbf{u}^* \in H^3(\Omega)$ so that

$$\mathbf{A}_R\mathbf{u}^* = \mathbf{0},$$

$$\mathbf{u}^* = \mathbf{g}_0\,,\ \partial_\nu \mathbf{u}^* = \mathbf{g}_1\,,\ \partial^2_\nu \mathbf{u}^* = \partial^2_\nu \mathbf{u} \quad \text{on} \ \ \Gamma,$$

and

$$\|\mathbf{u}^*\|_{(3)}(\Omega) \leq C\,(\|\mathbf{f}\|_{(\frac{1}{2})}(\Gamma) + \|\mathbf{g}_0\|_{(\frac{5}{2})}(\Gamma) + \|\mathbf{g}_1\|_{(\frac{3}{2})}(\Gamma)) \tag{5.11}$$

due to (5.10) in the sense of a linear combination.

The function $\mathbf{v}$ solves the Cauchy problem

$$\begin{aligned} \mathbf{A}_R\mathbf{v} = \mathbf{f} - \mathbf{A}_R\mathbf{u}^* \quad \text{in} \quad \Omega, \\ \mathbf{v} = \mathbf{0}, \ \partial_\nu \mathbf{v} = \mathbf{0} \quad \text{on} \quad \Gamma. \end{aligned} \tag{5.12}$$

Moreover, due to our construction of $\mathbf{u}^*$, we have

$$\partial_\nu^2 \mathbf{v} = \mathbf{0} \quad \text{on} \quad \Gamma. \tag{5.13}$$

We introduce the following energy integrals for the hyperbolic system of elasticity with residual stress

$$E(t;\mathbf{u}) = \int_G (|\partial_t \mathbf{u}|^2 + |\nabla \mathbf{u}|^2 + |\mathbf{u}|^2)(\,,t), \quad E(t) = E(t;\mathbf{v}) + E(t;\nabla\mathbf{v}).$$

Dividing the system (5.1) by ρ and differentiating with respect to space variables we obtain the extended system with the same principal part

$$\begin{aligned} \rho^{-1}\mathbf{A}_R\mathbf{v} = \rho^{-1}\mathbf{f}^*, \\ \rho^{-1}\mathbf{A}_R\partial_j\mathbf{v} = \partial_j\rho^{-1}\mathbf{f}^* - (\partial_j\rho^{-1}\mathbf{A}_R)\mathbf{v} \end{aligned} \quad \text{in} \ \ \Omega = G \times (-T,T), \tag{5.14}$$

where $\mathbf{f}^* = \mathbf{f} - \mathbf{A}_R\mathbf{u}^*$, $j = 1,2,3$, with the zero boundary value conditions in the sense of (5.12), *i.e.*,

$$\mathbf{v} = \mathbf{0}, \quad \partial_j \mathbf{v} = \mathbf{0} \quad \text{on} \quad \Gamma = \partial G \times (-T,T). \tag{5.15}$$

By standard energy estimates for t-hyperbolic systems (*i.e.*, [7])

$$C^{-1}\big(E(0) - \|\mathbf{f}^*\|_{(1)}(\Omega)\big) \le E(t) \le C\big(E(0) + \|\mathbf{f}^*\|_{(1)}(\Omega)\big) \ \text{when} \ t \in (-T,T). \tag{5.16}$$

We choose a smooth cut-off function $0 \le \chi_0(t) \le 1$ such that $\chi_0(t) = 1$ for $-T+2\delta < t < T-2\delta$ and $\chi_0(t) = 0$ for $|t| > T - \delta$. It is clear that

$$\mathbf{A}_R(\chi_0\mathbf{v}) = \chi\mathbf{f}^* + 2\rho\partial_t\chi_0\partial_t\mathbf{v} + \rho\partial_t^2\chi_0\mathbf{v}$$

and

$$\nabla\mathbf{A}_R(\chi_0\mathbf{v}) = \chi_0\nabla\mathbf{f}^* + 2\rho\partial_t\chi_0\partial_t\nabla\mathbf{v} + \rho\partial_t^2\chi_0\nabla\mathbf{v}. \tag{5.17}$$

Using the Cauchy data (5.15) we conclude that $\chi_0\mathbf{v} \in H^3_0(\Omega)$, hence

$$\int_\Omega (|\nabla_{x,t}(\chi_0\mathbf{v})|^2+|\nabla_{x,t}div(\chi_0\mathbf{v})|^2+|\nabla_{x,t}curl(\chi_0\mathbf{v})|^2+|\chi_0\mathbf{v}|^2+|div(\chi_0\mathbf{v})|^2+|curl(\chi_0\mathbf{v})|^2)e^{2\tau\varphi}$$

$$\leq C\int_\Omega (|\mathbf{A}_R(\chi_0\mathbf{v})|^2+|\nabla\mathbf{A}_R(\chi_0\mathbf{v})|^2)e^{2\tau\varphi} \tag{5.18}$$

$$\leq C\Big(\int_\Omega (|\mathbf{f}^*|^2+|\nabla\mathbf{f}^*|^2)e^{2\tau\varphi}+\int_{G\times\{T-2\delta<|t|<T\}}(|\partial_t\mathbf{v}|^2+|\mathbf{v}|^2+|\partial_t\nabla\mathbf{v}|^2+|\nabla\mathbf{v}|^2)e^{2\tau\varphi}\Big)$$

by Theorem 4.1 with fixed γ, (5.12), (5.13), (5.14), and using the definition of a cut-off function $0<\chi_0(t)<1$.

Using the known identity $\Delta\mathbf{v}=-curlcurl\mathbf{v}+\nabla div\mathbf{v}$ and the boundary conditions (5.15), from known elliptic estimates in the Dirichlet problem for the Laplace operator in G we have

$$\int_G |\nabla^2\mathbf{v}|^2 \leq C\int_G (|\nabla div\mathbf{v}|^2+|\nabla curl\mathbf{v}|^2)$$

and

$$\int_G |\partial_t\nabla\mathbf{v}|^2 \leq C\int_G (|\partial_t div\mathbf{v}|^2+|\partial_t curl\mathbf{v}|^2).$$

Shrinking the integration domain Ω on the left side of (5.18) to $G\times(0,\delta)$ where $\chi_0=1$ and choosing ψ by $e^{2\tau(1-\delta)}<e^{2\tau\varphi}$ since $1-\delta<\varphi$ on $G\times(0,\delta)$ and $e^{2\tau\varphi}<e^{2\tau(1-2\delta)}$, since $\varphi<1-2\delta$ on $G\times(T-\delta,T)$, gives

$$e^{2\tau(1-\delta)}\int_0^\delta E(t)dt$$

$$\leq C\Big(\int_\Omega (|\mathbf{f}^*|^2+|\nabla\mathbf{f}^*|^2)e^{2\tau\varphi}\ +\ Ce^{2\tau(1-2\delta)}\int_{T-2\delta}^T\int_G (|\partial_t\mathbf{v}|^2+|\mathbf{v}|^2+|\partial_t\nabla\mathbf{v}|^2+|\nabla\mathbf{v}|^2)\Big).$$

Hence

$$e^{2\tau(1-\delta)}\int_0^\delta E(t)dt \leq C\Big(\int_\Omega (|\mathbf{f}^*|^2+|\nabla\mathbf{f}^*|^2)e^{2\tau\varphi}+Ce^{2\tau(1-2\delta)}\int_{T-2\delta}^T E(t)dt\Big).$$

Choosing $\Phi=\sup_\Omega\varphi$ and using the energy bound (5.16) we yield

$$e^{2\tau(1-\delta)}\frac{\delta}{C}E(0)-Ce^{2\tau\Phi}\|\mathbf{f}^*\|^2_{(1)}(\Omega) \leq C\delta e^{2\tau(1-2\delta)}E(0)+Ce^{2\tau\Phi}\|\mathbf{f}^*\|^2_{(1)}(\Omega). \tag{5.19}$$

We now have the bound

$$E(0) \le C\|\mathbf{f}^*\|^2_{(1)}(\Omega) \tag{5.20}$$

by choosing τ (depending on C) so large that $e^{-2\tau\delta} < \frac{1}{C^2}$ in (5.19).

Using energy estimates (5.16) and (5.20), we finally get

$$E(t;\mathbf{v}) + E(t;\nabla\mathbf{v}) \le C\|\mathbf{f}^*\|_{(1)}(\Omega).$$

Similar to the proof of Theorem 5.1, using $\mathbf{u} = \mathbf{v} + \mathbf{u}^*$ and triangle inequality gives

$$E(t;\mathbf{u}) + E(t;\nabla\mathbf{u})$$

$$\le C\left(\|\mathbf{f}^*\|_{(1)}(\Omega) + E(t;\mathbf{u}^*) + E(t;\nabla\mathbf{u}^*)\right)$$

$$\le C\left(\|\mathbf{f}^*\|_{(1)}(\Omega) + \|\mathbf{A}_R\mathbf{u}^*\|_{(1)}(\Omega) + \|\mathbf{u}^*\|_{(\frac{5}{2})}(\Gamma) + \|\partial_\nu\mathbf{u}^*\|_{(\frac{3}{2})}(\Gamma)\right)$$

$$\le C\left(\|\mathbf{f}\|_{(1)}(\Omega) + \|\mathbf{g}_0\|_{(\frac{5}{2})}(\Gamma) + \|\mathbf{g}_1\|_{(\frac{3}{2})}(\Gamma)\right).$$

□

CHAPTER 6

INVERSE PROBLEM

Now, we state results about identification of residual stress from additional boundary data. For Theorems 6.1 and 6.2, let $\Omega = G \times (-T, T)$ where G is a bounded domain in $\mathbb{R}^3$ with C^8-boundary and let $\Gamma \subset \partial G \times (-T, T)$.

Let $\mathbf{u}(\,;1)$ and $\mathbf{u}(\,;2)$ be solutions to

$$\begin{gathered} \mathbf{A}_R \mathbf{u} = \mathbf{0} \quad \text{in } \Omega, \\ \mathbf{u} = \mathbf{u}_0, \;\; \partial_t \mathbf{u} = \mathbf{u}_1 \quad \text{on } G \times \{0\}, \\ \mathbf{u} = \mathbf{g}_0 \quad \text{on } \partial G \times (-T, T), \end{gathered} \tag{6.1}$$

corresponding to sets of coefficients $R(\,;1)$ and $R(\,;2)$, respectively. In this chapter we assume that ρ, λ, μ, $r_{jk}(\,;j)$ do not depend on t, that

$$|\rho^{-1}|_8(\Omega) + |\lambda|_8(\Omega) + |\mu|_8(\Omega) + |r^{jk}|_8(\Omega) \le M,$$

and that

$$\mathbf{u}_0 \in H^9(G), \;\; \mathbf{u}_1 \in H^8(G), \;\; \text{and} \;\; \mathbf{g}_0 \in C^9(\partial G \times [-T, T]).$$

We also impose compatibility conditions of order 7 at $\partial G \times \{0\}$. Then, by known energy estimates and Sobolev embedding theorems (like in [16], [25]),

$$\|\partial_x^\alpha \partial_t^\beta \mathbf{u}\|_\infty(\Omega) \le C, \;\; \text{when } \; |\alpha| \le 2, \; \beta \le 5. \tag{6.2}$$

We can consider the boundary stress data as measurements (observations). We introduce the norm of the difference of the lateral Cauchy data

$$F_c = \textstyle\sum_{\beta=2}^{4} \|\partial_t^\beta \partial_\nu (\mathbf{u}(\,;2) - \mathbf{u}(\,;1))\|_{(\frac{3}{2})}(\Gamma). \tag{6.3}$$

Since $\mathbf{u}(\,;1) = \mathbf{g}_0 = \mathbf{u}(\,;2)$ on Γ, F_c is a norm of the difference of the Cauchy data on the observation set Γ.

By examining the equation (4.2), we can see that since the residual stress tensor is divergence free, it appears in the equation without first derivatives. It turns out that a single set of Cauchy data is sufficient to recover the symmetric (variable) matrix R. To guarantee the uniqueness, we impose a non-degeneracy condition on the initial data $(\mathbf{u}_0, \mathbf{u}_1)$. Let

$$\mathbf{M} = \begin{pmatrix} \partial_1^2 \mathbf{u}_0 & 2\partial_1\partial_2\mathbf{u}_0 & 2\partial_1\partial_3\mathbf{u}_0 & \partial_2^2\mathbf{u}_0 & 2\partial_2\partial_3\mathbf{u}_0 & \partial_3^2\mathbf{u}_0 \\ \partial_1^2 \mathbf{u}_1 & 2\partial_1\partial_2\mathbf{u}_1 & 2\partial_1\partial_3\mathbf{u}_1 & \partial_2^2\mathbf{u}_1 & 2\partial_2\partial_3\mathbf{u}_1 & \partial_3^2\mathbf{u}_1 \end{pmatrix}. \tag{6.4}$$

Note that $\mathbf{M}$ is a 6×6 matrix-valued function. We assume that

$$\det \mathbf{M} > \varepsilon_0 > 0 \quad \text{on} \ \ \Omega. \tag{6.5}$$

For example let $\mathbf{u}_0(x) = (x_1^2, x_2^2, x_3^2)^\top$ and $\mathbf{u}_1(x) = (x_2x_3, x_1x_3, x_1x_2)^\top$; one can check that (6.5) is satisfied with $\varepsilon_0 = 2^6$.

Now, we state the Hölder type estimate of determining coefficients in Ω_δ defined as $\Omega \cap \{\psi > \delta\}$.

Theorem 6.1 *Let the initial data $(\mathbf{u}_0, \mathbf{u}_1)$ satisfy (6.5). Assume that $\bar{\Omega}_0 \subset \Omega \cup \Gamma$. Assume that $\psi \in C^3(\bar{\Omega})$ is K-pseudo-convex with respect to $\square(\mu; R(\,;2))$, $\square(\lambda + 2\mu; R(\,;2))$ in $\bar{\Omega}$. Then there exist constants $C = C(\delta)$, $\kappa = \kappa(\delta) \in (0,1)$ such that*

$$\|R(\,;2) - R(\,;1)\|_{(0)}(\Omega_\delta) \leq C F_c^\kappa. \tag{6.6}$$

If Γ is the whole lateral boundary and T is sufficiently large, then under more restrictive conditions a much stronger (and in a certain sense best possible) Lipschitz stability estimate holds.

We assume that anisotropic system $\mathbf{A}_R\mathbf{u} = \mathbf{0}$ in (6.1) is t-hyperbolic. A sufficient condition is given in Section 2.5 as

$$0 \leq \lambda, \ \ \varepsilon_0 I_3 \leq 2\mu I_3 + R \quad \text{on} \ \ \bar{G}$$

where ε_0 positive. The conditions are satisfied when any eigenvalue of the matrix R is strictly greater than -2μ. This happens when, for example, $\sum_{i,j=1}^3 r_{ij}^2 < 4\mu^2$ on $\bar{G}$. Under these conditions the anisotropic system $\mathbf{A}_R\mathbf{u} = \mathbf{0}$ is time hyperbolic and hence the initial boundary value problem (6.1), for it is well-posed in standard energy spaces. We are interested in recovery of the residual stress from additional boundary data.

Theorem 6.2 *Assume that λ, μ, ρ, R are in $C^2(\bar{\Omega})$. Let ψ be K-pseudo-convex with respect to $\Box(\mu; R(\,;2))$, $\Box(\lambda+2\mu; R(\,;2))$ in $\bar{\Omega}$. Assume that the condition (5.3) is satisfied. Let the initial data $(\mathbf{u}_0, \mathbf{u}_1)$ satisfy (6.5). Let $\Gamma = \partial G \times (-T, T)$. Then there exists a constant C such that*

$$\|R(\,;2) - R(\,;1)\|_{(1)}(\Omega) \leq CF_c. \tag{6.7}$$

The weaker results of Theorems 6.1 and 6.2 with C depending on $R(\,;2)$ are derived in [20], [21].

6.1 Hölder stability for the residual stress

In this section we prove Theorem 6.1.

Proof of Theorem 6.1

Let $\mathbf{u}(\,;1)$ and $\mathbf{u}(\,;2)$ satisfy (6.1) corresponding to $R(\,;1)$ and $R(\,;2)$, respectively. Denote $\mathbf{u} = \mathbf{u}(\,;2) - \mathbf{u}(\,;1)$ and $\mathbf{F} = R(\,;2) - R(\,;1) = (f_{jk})$, $j, k = 1, \ldots, 3$. By subtracting equations (6.1) for $\mathbf{u}(\,;1)$ from the equations for $\mathbf{u}(\,;2)$ we yield

$$\mathbf{A}_{R(\,;2)}\mathbf{u} = \mathcal{A}(\,;\mathbf{u}(\,;1))\mathbf{F} \quad \text{on} \quad \Omega \tag{6.8}$$

where

$$\mathcal{A}(\,;\mathbf{u}(\,;1))\mathbf{F} = \sum_{j,k=1}^{3} f_{jk}\partial_j\partial_k\mathbf{u}(\,;1) + \sum_{j,k=1}^{3} \partial_j f_{jk}\partial_k\mathbf{u}(\,;1). \tag{6.9}$$

Since the residual is divergence free, the second term on the right of (6.9) vanishes and

$$\mathbf{u} = \partial_t\mathbf{u} = \mathbf{0} \quad \text{on} \quad G \times \{0\}, \quad \mathbf{u} = \mathbf{0} \quad \text{on} \quad \Gamma. \tag{6.10}$$

Differentiating (6.8) in t and using time independence of the coefficients of the system, we get

$$\mathbf{A}_{R(\,;2)}\mathbf{U} = \mathcal{A}(\,;\mathbf{U}(\,;1))\mathbf{F} \quad \text{on} \quad \Omega \tag{6.11}$$

where

$$\mathbf{U} = \begin{pmatrix} \partial_t^2 \mathbf{u} \\ \partial_t^3 \mathbf{u} \\ \partial_t^4 \mathbf{u} \end{pmatrix} \quad \text{and} \quad \mathbf{U}(\,;1) = \begin{pmatrix} \partial_t^2 \mathbf{u}(\,;1) \\ \partial_t^3 \mathbf{u}(\,;1) \\ \partial_t^4 \mathbf{u}(\,;1) \end{pmatrix}. \tag{6.12}$$

By extension theorems for Sobolev spaces there exists $\mathbf{U}^* \in H^2(\Omega)$ such that

$$\mathbf{U}^* = \mathbf{0}, \ \ \partial_\nu \mathbf{U}^* = \partial_\nu \mathbf{U} \ \text{ on } \ \Gamma \tag{6.13}$$

and

$$\|\mathbf{U}^*\|_{(2)}(\Omega) \leq C\|\partial_\nu \mathbf{U}\|_{(\frac{1}{2})}(\Gamma) \leq CF_c \tag{6.14}$$

due to the definition (6.3).

We now introduce $\mathbf{V} = \mathbf{U} - \mathbf{U}^*$. Then

$$\mathbf{A}_{R(\,;2)}\mathbf{V} = \mathcal{A}\mathbf{F} - \mathbf{A}_{R(\,;2)}\mathbf{U}^* \ \text{ on } \ \Omega \tag{6.15}$$

and

$$\mathbf{V} = \partial_\nu \mathbf{V} = \mathbf{0} \ \text{ on } \ \Gamma. \tag{6.16}$$

To use the Carleman estimate (4.4), we need zero Cauchy data on $\partial\Omega_0$. To create such data we introduce a cut-off function $\chi \in C^2(\mathbb{R}^4)$ such that $0 \leq \chi \leq 1$, $\chi = 1$ on $\Omega_{\frac{\delta}{2}}$ and $\chi = 0$ on $\Omega \setminus \Omega_0$. By the Leibniz formula

$$\mathbf{A}_{R(\,;2)}(\chi\mathbf{V}) = \chi\mathbf{A}_{R(\,;2)}(\mathbf{V}) + \mathbf{A}_1\mathbf{V} = \chi\mathcal{A}\mathbf{F} - \chi\mathbf{A}_{R(\,;2)}\mathbf{U}^* + \mathbf{A}_1\mathbf{V}$$

due to (6.15). Here (and below) $\mathbf{A}_1$ denotes a first order matrix differential operator with coefficients uniformly bounded by $C(\delta)$. By the choice of χ we have $\mathbf{A}_1\mathbf{V} = 0$ on $\Omega_{\frac{\delta}{2}}$. Because of (6.16) the function $\chi\mathbf{V} \in H_0^2(\Omega)$, so we can apply to it the Carleman estimate (4.4) with fixed γ to get

$$\int_\Omega \tau|\chi\mathbf{V}|^2 e^{2\tau\varphi} \leq C(\delta)\int_\Omega (|\mathbf{F}|^2 + |\mathbf{A}_{R(\,;2)}(\mathbf{U}^*)|^2)e^{2\tau\varphi} + C\int_{\Omega\setminus\Omega_{\frac{\delta}{2}}} |\mathbf{A}_1\mathbf{V}|^2 e^{2\tau\varphi}$$

$$\leq C\,(\int_\Omega |\mathbf{F}|^2 e^{2\tau\varphi} + F_c^2 e^{2\tau\Phi} + C(\delta)e^{2\tau\delta_1}) \tag{6.17}$$

where $\Phi = \sup \varphi$ over Ω and $\delta_1 = e^{\frac{\gamma\delta}{2}}$. To get the last inequality we use the bounds (6.14).

From (6.1), (6.8), and (6.9) we have

$$\mathbf{A}_{R(\,;2)}\mathbf{u}(\,,0) = \rho\partial_t^2\mathbf{u}(\,,0) - \mu\Delta\mathbf{u}(\,,0) - (\lambda+\mu)\nabla(\nabla\cdot\mathbf{u}(\,,0)) - \nabla\cdot(\nabla\mathbf{u}(\,,0)R(\,;2))$$

$$= \sum f_{jk}\partial_j\partial_k\mathbf{u}(\,,0;1).$$

Using (6.10), since $\mathbf{u}(\,,0)=0$, the space derivatives are $\nabla\mathbf{u}(\,,0)=0$ and $\Delta\mathbf{u}(\,,0)=0$. Hence

$$\rho\partial_t^2\mathbf{u}(\,,0) = \sum f_{jk}\partial_j\partial_k\mathbf{u}(\,,0;1)$$

and

$$\rho\partial_t^3\mathbf{u}(\,,0) = \sum f_{jk}\partial_t\partial_j\partial_k\mathbf{u}(\,,0;1)$$

on $G\times\{0\}$. From now on we consider the symmetric matrix-function $\mathbf{F}$ as a vector function with components $(f_{11}, f_{12}, f_{13}, f_{22}, f_{23}, f_{33})$. Using the definition (6.4) of $\mathbf{M}$ we obtain $\rho(\partial_t^2\mathbf{u}, \partial_t^3\mathbf{u}) = \mathbf{MF}$ on $G\times\{0\}$, and from the condition (6.5) we have

$$\mathbf{F} = \mathbf{M}^{-1}(\rho(\partial_t^2\mathbf{u}, \partial_t^3\mathbf{u}))^{-1}$$

on $G\times\{0\}$. Hence, by using (6.5), we get

$$|\mathbf{F}|^2 \leq C\sum_{\beta=2,3}(|\partial_t^\beta\mathbf{u}(\,,0)|^2). \tag{6.18}$$

Since $\chi(\,,T)=0$, we can write

$$\int_G |\chi\partial_t^\beta\mathbf{u}(x,0)|^2 e^{2\tau\varphi(x,0)}dx \;=\; -\int_0^T \partial_t(\int_G |\chi\partial_t^\beta\mathbf{u}(x,t)|^2 e^{2\tau\varphi(x,t)}dx)dt$$

$$\leq \int_\Omega 2\chi^2(|\partial_t^{\beta+1}\mathbf{u}||\partial_t^\beta\mathbf{u}| + \tau|\partial_t\varphi||\partial_t^\beta\mathbf{u}|^2)e^{2\tau\varphi} \;+\; 2\int_{\Omega\setminus\Omega_{\frac{\delta}{2}}} |\partial_t^\beta\mathbf{u}|^2\chi|\partial_t\chi|e^{2\tau\varphi}$$

where $\beta=2,3$. By using (6.12) and the well-known inequality $|a||b|\leq|a|^2+|b|^2$, the right side does not exceed

$$C(\int_\Omega \tau|\chi\mathbf{U}|^2e^{2\tau\varphi} + C(\delta)\int_{\Omega\setminus\Omega_{\frac{\delta}{2}}}|\mathbf{U}|^2e^{2\tau\varphi})$$

$$\leq C\,(\int_\Omega\tau|\chi\mathbf{V}|^2e^{2\tau\varphi} + C(\delta)\int_{\Omega\setminus\Omega_{\frac{\delta}{2}}}|\mathbf{V}|^2e^{2\tau\varphi} + \tau\int_\Omega|\mathbf{U}^*|^2e^{2\tau\varphi}) \tag{6.19}$$

because $\mathbf{U} = \mathbf{V} + \mathbf{U}^*$.

Using that $\chi = 1$ on $\Omega_{\frac{\delta}{2}}$, $\varphi < \delta_1$ on $\Omega \setminus \Omega_{\frac{\delta}{2}}$, and $\varphi < \Phi$ on Ω in (6.19) and from (6.14) and (6.17) we yield

$$\int_G |\partial_t^\beta \mathbf{u}|^2(\,,0)e^{2\tau\varphi(\,,0)} \leq C\,(\int_\Omega |\mathbf{F}|^2 e^{2\tau\varphi} + C(\delta)e^{2\tau\delta_1} + \tau e^{2\tau\Phi}F_c^2). \tag{6.20}$$

First we get this bound with $G_{\frac{\delta}{2}}$ instead of G on the left side and then add to both sides of the inequality the integral over $G \setminus G_{\frac{\delta}{2}}$, which is bounded by $C(\delta)e^{2\tau\delta_1}$ due to the bound (6.2), and the inequality $\varphi < \delta_1$ on $G \setminus G_{\frac{\delta}{2}}$. From (6.18) and (6.20) we obtain

$$\int_G |\mathbf{F}|^2 e^{2\tau\varphi(\,,0)} \leq C\,(\int_\Omega |\mathbf{F}|^2 e^{2\tau\varphi} + \tau e^{2\tau\Phi}F_c^2 + C(\delta)e^{2\tau\delta_1}). \tag{6.21}$$

To eliminate the integral in the right side of (6.21) we observe that

$$\int_\Omega |\mathbf{F}|^2(x)e^{2\tau\varphi(x,t)}dxdt$$

$$= \int_G |\mathbf{F}|^2(x)e^{2\tau\varphi(x,0)}\,(\int_{-T}^{T} e^{2\tau(\varphi(x,t)-\varphi(x,0))}dt)\,dx.$$

Due to our choice of φ we have $\varphi(x,t) - \varphi(x,0) < 0$ when $t \neq 0$. Hence, by the Lebesgue theorem, the inner integral (with respect to t) converges to 0 as τ goes to infinity. By reasons of continuity of φ, this convergence is uniform with respect to $x \in G$. Choosing $\tau > C$ we therefore can absorb the integral over $\Omega_{\frac{\delta}{2}}$ in the right side of (6.21) by the left side arriving at the inequality

$$\int_{\Omega_\delta} |\mathbf{F}|^2 e^{2\tau\varphi(\,,0)} \leq C\,(\tau e^{2\tau\Phi}F_c^2 + C(\delta)e^{2\tau\delta_1}).$$

Letting $\delta_2 = e^{\gamma\delta} \leq \varphi$ on Ω_δ and dividing the both parts by $e^{2\tau\delta_2}$ we yield

$$\int_{\Omega_\delta} |\mathbf{F}|^2 \leq C\,(\tau e^{2\tau(\Phi-\delta_2)}F_c^2 + e^{-2\tau(\delta_2-\delta_1)}) \leq C(\delta)(e^{2\tau\Phi}F_c^2 + e^{-2\tau(\delta_2-\delta_1)}) \tag{6.22}$$

since $\tau e^{-2\tau\delta_2} < C(\delta)$. To prove (6.6) it suffices to assume that $F_c < \frac{1}{C}$. Then $\tau = \frac{-logF_c}{\Phi+\delta_2-\delta_1} > C$ and we can use this τ in (6.22). Due to the choice of τ,

$$e^{-2\tau(\delta_2-\delta_1)} = e^{2\tau\Phi}F_c^2 = F_c^{2\frac{\delta_2-\delta_1}{\Phi+\delta_2-\delta_1}},$$

and from (6.22) we obtain (6.6) with $\kappa = \frac{\delta_2-\delta_1}{\Phi+\delta_2-\delta_1}$. □

6.2 Lipschitz stability for the residual stress

In this section we prove Theorem 6.2.

Proof of Theorem 6.2

In view of Hölder Stability for the residual stress, since $\partial G \times (-T, T)$ is noncharacteristic with respect to $\mathbf{A}_R$ we can uniquely solve the equation $\mathbf{A}_{R(\,;2)}\mathbf{U} = 0$ on $\partial G \times (-T, T)$ for $\partial_\nu^2 \mathbf{U}$ in terms of $\mathbf{U}$ and $\partial_\nu \mathbf{U}$. In particular,

$$\|\partial_\nu^2 \mathbf{U}\|_{(\frac{1}{2})}(\partial G \times (-T, T))$$

$$\leq C\left(\|\mathbf{U}\|_{(\frac{5}{2})}(\partial G \times (-T, T)) + \|\partial_\nu \mathbf{U}\|_{(\frac{3}{2})}(\partial G \times (-T, T))\right) \tag{6.23}$$

due to definitions of $\mathbf{U}$ in (6.11) and (6.12).

By extension theorems for Sobolev spaces there exists $\mathbf{U}^* \in H^3(\Omega)$ such that

$$\begin{aligned} \mathbf{A}_{R(\,;2)}\mathbf{U}^* &= \mathbf{0}, \\ \mathbf{U}^* = \mathbf{0}, \ \ \partial_\nu \mathbf{U}^* = \partial_\nu \mathbf{U}, \ \ \partial_\nu^2 \mathbf{U}^* &= \partial_\nu^2 \mathbf{U} \end{aligned} \qquad \text{on } \ \partial G \times (-T, T), \tag{6.24}$$

and

$$\|\mathbf{U}^*\|_{(3)}(\Omega) \leq C\big(\|\partial_\nu \mathbf{U}\|_{(\frac{3}{2})}(\partial G \times (-T, T)) \ + \ \|\partial_\nu^2 \mathbf{U}\|_{(\frac{1}{2})}(\partial G \times (-T, T))\big) \leq C F_c \tag{6.25}$$

due to (6.23) and the definition of F_c.

We introduce $\mathbf{V} = \mathbf{U} - \mathbf{U}^*$. Then due to (6.11), (6.12), and (6.24), we have

$$\mathbf{A}_{R(\,;2)}\mathbf{V} = \mathcal{A}(\,;\mathbf{u}(\,;1))\mathbf{F} - \mathbf{A}_{R(\,;2)}\mathbf{U}^* \quad \text{on } \ \Omega, \tag{6.26}$$

$$\mathbf{V} = \partial_\nu \mathbf{V} = \partial_\nu^2 \mathbf{V} = \mathbf{0} \quad \text{on } \ \partial G \times (-T, T). \tag{6.27}$$

As in Section 5.2 (Lipschitz statbility for Cauchy problem), since (6.26) is t-hyperbolic, we use the known energy estimates. Relations (6.25), (6.26), and (6.27) give

$$C^{-1}(E(0) - \|\mathbf{F}\|_{(1)}^2(G) - F_c^2) \ \leq \ E(t) \ \leq \ C(E(0) + \|\mathbf{F}\|_{(1)}^2(G) + F_c^2) \tag{6.28}$$

where

$$E(t) = E(t;\mathbf{V}) + E(t;\nabla\mathbf{V}), \ \ E(t;\mathbf{V}) = \int_G (|\partial_t \mathbf{V}|^2 + |\nabla \mathbf{V}|^2 + |\mathbf{V}|^2)(\,,t). \tag{6.29}$$

Here and below, the operator ∇ is ∇_x.

On the other hand, by using the Carleman estimate of Theorem 4.1 and our choice of the weight function φ we bound the right side of (6.28) by a small fraction of $E(0)$ and given quantities.

To use the Carleman estimate (4.3), we need a cut off $\mathbf{V}$ near $t = T$ and $t = -T$. We first observe that from the definition and from the condition (5.3) that

$$1 \le \varphi(x,0), \quad \varphi(x,T) = \varphi(x,-T) < 1 \quad \text{when} \quad x \in \bar{G}.$$

So there exists a $\delta > \frac{1}{C}$ such that

$$1 - \delta < \varphi \quad \text{on} \quad G \times (0,\delta), \qquad \varphi < 1 - 2\delta \quad \text{on} \quad G \times (T - 2\delta, T). \tag{6.30}$$

We now choose a smooth cut-off function $0 \le \chi_0(t) \le 1$ such that $\chi_0(t) = 1$ for $-T + 2\delta < t < T - 2\delta$ and $\chi_0(t) = 0$ for $|t| > T - \delta$.

Because of (6.27) and the sense of Lipschitz stability for Cauchy problem (Section 5.2), $\chi_0 \mathbf{V} \in H_0^3(\Omega)$. By the Leibniz formula

$$\mathbf{A}_{R(\,;2)}(\chi_0\mathbf{V}) \;=\; \chi_0\mathcal{A}(\,;\mathbf{U}(\,;1))\mathbf{F} \;-\; \chi_0\mathbf{A}_{R(\,;2)}\mathbf{U}^* \;+\; 2\rho(\partial_t\chi_0)\partial_t\mathbf{V} \;+\; \rho(\partial_t^2\chi_0)\mathbf{V}$$

and

$$\nabla\mathbf{A}_{R(\,;2)}(\chi_0\mathbf{V}) \;=\; \chi_0\nabla\mathcal{A}(\,;\mathbf{U}(\,;1))\mathbf{F} \;-\; \chi_0\nabla\mathbf{A}_{R(\,;2)}\mathbf{U}^* \;+\; 2\rho(\partial_t\chi_0)\partial_t\nabla\mathbf{V} \;+\; \rho(\partial_t^2\chi_0)\nabla\mathbf{V}.$$

So, we apply the Carleman estimate (4.3) with fixed γ for (6.26). Since $\partial_t\chi_0(t) = 0$ when $-T + 2\delta < t < T - 2\delta$, the domain of integration shrinks to $G \times \{T - 2\delta < |t| < T\}$. Using (6.2) we have

$$\begin{aligned}
&\int_{G\times(0,T)} (|\chi_0\mathbf{V}|^2 + |\nabla_{x,t}(\chi_0\mathbf{V})|^2 + |\nabla_{x,t}div(\chi_0\mathbf{V})|^2 + |\nabla_{x,t}curl(\chi_0\mathbf{V})|^2)e^{2\tau\varphi} \\
&\quad \le C\Big(\int_\Omega (|\mathbf{F}|^2 + |\nabla\mathbf{F}|^2 + |\mathbf{A}_{R(\,;2)}\mathbf{U}^*|^2 + |\nabla(\mathbf{A}_{R(\,;2)}\mathbf{U}^*)|^2)e^{2\tau\varphi} \\
&\quad + \int_{G\times\{T-2\delta<|t|<T\}} (|\mathbf{V}|^2 + |\partial_t\mathbf{V}|^2 + |\nabla\mathbf{V}|^2 + |\partial_t\nabla\mathbf{V}|^2)e^{2\tau\varphi}\Big).
\end{aligned} \tag{6.31}$$

Using the known identity $\Delta \mathbf{V} = -curlcurl\mathbf{V} + \nabla div\mathbf{V}$ and the boundary conditions (6.27), from known elliptic estimates in the Dirichlet problem for the Laplace operator in G we have

$$\int_G |\nabla^2 \mathbf{V}|^2 \,\leq\, C \int_G (|\nabla div\mathbf{V}|^2 + |\nabla curl\mathbf{V}|^2)$$

and

$$\int_G |\partial_t \nabla \mathbf{V}|^2 \,\leq\, C \int_G (|\partial_t div\mathbf{V}|^2 + |\partial_t curl\mathbf{V}|^2).$$

Integration of the energy bound (6.28) over $(0, \delta)$ gives

$$\delta E(0) \,\leq\, C(\int_0^\delta E(t)dt + \|\mathbf{F}\|^2_{(1)}(G) + F_c^2)$$

$$\leq\, C\,(\int_{G\times(0,\delta)} (|\mathbf{V}|^2 + |\nabla_{x,t}\mathbf{V}|^2 + |\nabla_{x,t}div\mathbf{V}|^2 + |\nabla_{x,t}curl\mathbf{V}|^2) + \|\mathbf{F}\|^2_{(1)}(G) + F_c^2\,).$$

Similarly

$$\int_{G\times\{T-2\delta<|t|<T\}} (|\mathbf{V}|^2 + |\partial_t\mathbf{V}|^2 + |\nabla\mathbf{V}|^2 + |\partial_t\nabla\mathbf{V}|^2)e^{2\tau\varphi}$$

$$\leq\, C\,\big(e^{2\tau(1-2\delta)}(E(0) + \|\mathbf{F}\|^2_{(1)}(G)) + CF_c^2 e^{2\tau\Phi}\big)$$

where $\Phi = \sup_\Omega \varphi$.

Hence using (6.30) the bound of the left side in (6.31) gives

$$e^{2\tau(1-\delta)}\delta E(0) \,+\, \int_{G\times(0,T)} (\chi_0|\mathbf{V}|^2 + |\nabla_{x,t}(\chi_0\mathbf{V})|^2)e^{2\tau\varphi}$$

$$\leq\, C\,(\int_\Omega (|\mathbf{F}|^2 + |\nabla\mathbf{F}|^2)e^{2\tau\varphi} \,+\, F_c^2 e^{2\tau\Phi} \,+\, e^{2\tau(1-2\delta)}(E(0) + \int_G (|\mathbf{F}|^2 + |\nabla\mathbf{F}|^2))\big). \tag{6.32}$$

Now we choose τ large enough such that $e^{2\tau(1-\delta)}\delta > 2Ce^{2\tau(1-2\delta)}$. Then we eliminate $E(0)$ from the right of (6.32).

Since $\mathbf{U} = \mathbf{V} + \mathbf{U}^*$, using (6.25) from (6.32) we obtain

$$\int_{G\times(0,T)} \chi_0^2(|\mathbf{U}|^2 + |\nabla\mathbf{U}|^2)e^{2\tau\varphi}$$

$$\leq\, C\,\big(e^{2\tau\Phi}F_c^2 + \int_G (\int_{-T}^T e^{2\tau\varphi(x,t)}dt + e^{2\tau(1-2\delta)})(|\mathbf{F}|^2 + |\nabla\mathbf{F}|^2)(x)dx\big). \tag{6.33}$$

Using (6.5), (6.9), and (6.10) from (6.8) we get that

$$\rho \partial_t^2 \mathbf{u} = \sum f_{jk} \partial_j \partial_k \mathbf{u}(\,;1),$$

$$\rho \partial_t^3 \mathbf{u} = \sum f_{jk} \partial_t \partial_j \partial_k \mathbf{u}(\,;1)$$

on $G \times \{0\}$. So using the definitions of $\mathbf{M}, \mathbf{F}$ we obtain $\rho(\partial_t^2 \mathbf{u}, \partial_t^3 \mathbf{u}) = \mathbf{MF}$ on $G \times \{0\}$, and from the condition (6.5) we have

$$\mathbf{F} = \mathbf{M}^{-1}(\rho(\partial_t^2 \mathbf{u}, \partial_t^3 \mathbf{u})), \;\; \nabla \mathbf{F} = \nabla(\mathbf{M}^{-1}(\rho(\partial_t^2 \mathbf{u}, \partial_t^3 \mathbf{u})))$$

on $G \times \{0\}$. Hence we obtain

$$|\mathbf{F}|^2 + |\nabla \mathbf{F}|^2 \leq C \sum_{|\alpha| \leq 1,\, \beta = 2,3} |\partial_t^\beta \partial_x^\alpha \mathbf{u}(\,,0)|^2. \tag{6.34}$$

Therefore

$$\int_G (|\mathbf{F}|^2 + |\nabla \mathbf{F}|^2) e^{2s\varphi(\,,0)} \leq C \int_G |\partial_t^\beta \partial_x^\alpha \mathbf{u}(\,,0)|^2 e^{2\tau\varphi(\,,0)}$$

$$= -C \int_0^T \partial_t (\int_G |\chi_0 \partial_t^\beta \partial_x^\alpha \mathbf{u}|^2 e^{2\tau\varphi} dx)\, dt$$

$$\leq C \int_\Omega \chi_0^2 \left(|\partial_t^\beta \partial_x^\alpha \mathbf{u}||\partial_t^{\beta+1} \partial_x^\alpha \mathbf{u}| + \tau |\partial_t \varphi||\partial_t^\beta \partial_x^\alpha \mathbf{u}|^2 \right) e^{2\tau\varphi}$$

$$+ C \int_{G \times (T-2\delta, T)} \chi_0 |\partial_t \chi_0||\partial_t^\beta \partial_x^\alpha \mathbf{u}|^2 e^{2\tau\varphi}$$

where $|\alpha| \leq 1$ and $\beta = 2, 3$. Now, as in the proofs of Hölder stability for the residual stress, the right side is less than

$$C \left(\int_\Omega \tau \chi_0^2 (|\mathbf{U}|^2 + |\nabla \mathbf{U}|^2) e^{2\tau\varphi} + \int_{G \times (T-2\delta, T)} (|\mathbf{U}|^2 + |\nabla \mathbf{U}|^2) e^{2\tau\varphi} \right)$$

$$\leq C \left(\int_\Omega \tau \chi_0^2 (|\mathbf{U}|^2 + |\nabla \mathbf{U}|^2) e^{2\tau\varphi} + e^{2\tau(1-2\delta)} (\|\mathbf{F}\|^2_{(1)}(G) + F_c^2) \right)$$

where we used the equality $\mathbf{U} = \mathbf{U}^* + \mathbf{V}$ with (6.25) and (6.28). From two previous bounds we conclude that

$$\int_G (|\mathbf{F}|^2 + |\nabla \mathbf{F}|^2) e^{2\tau\varphi(\,,0)}$$

$$\leq C\left(\tau e^{2\tau\Phi}F_c^2 + \int_G (\int_{-T}^{T} e^{2\tau\varphi(\,,t)}dt + e^{2\tau(1-2\delta)})(|\mathbf{F}|^2 + |\nabla\mathbf{F}|^2)\right). \tag{6.35}$$

Due to our choice of φ, $1 \leq \varphi(\,,0)$ and $\varphi(\,,t) - \varphi(\,,0) < 0$ when $t \neq 0$. Thus, by the Lebesgue theorem we have

$$2C(\int_{-T}^{T} e^{2\tau\varphi(\,,t)}dt + e^{2\tau(1-2\delta)}) \leq e^{2\tau\varphi(\,,0)}$$

uniformly on G when $\tau > C$. Hence choosing and fixing such large τ (depending only on C) we eliminate the second term on the right side of (6.35). □

CHAPTER 7

CONCLUSION

We believe that the Carleman estimates of Theorems 3.1 and 3.2 for general anisotropic operators that we obtained in Chapter 3 can be applied to other important systems of mathematical physics, for example, to transversely isotropic elasticity systems and to some anisotropic Maxwell systems.

It is not clear at the moment how to include the time derivatives of $\mathbf{u}$ in Theorem 4.2 on elasticity systems. If this were possible, then one can obtain proofs of Lipschitz stability in the lateral Cauchy problem and for identification of residual stress in most natural norms. By using additional spatial derivatives such Lipschitz estimates are constructed in [20], [21], [25].

The next realistic goal is to apply the weak form of Carleman estimates to obtain Carleman estimates, uniqueness of continuation, and coefficient identification results for the important system of transversely isotropic elasticity, where currently there are no analytic results. We expect that the developed theory can be extended to Schrödinger type equations, and therefore to anisotropic systems describing elastic plates and shells.

REFERENCES

LIST OF REFERENCES

[1] R. Adams and J. Fournier, *Sobolev spaces*, Pure and Applied Mathematics, **140** (2003).

[2] P. Albano and D. Tataru, *Carleman estimates and boundary observability for a coupled parabolic-hyperbolic system*, Electr. J. Diff. Equat., (2000), 1-15.

[3] A. Amirov and M. Yamamoto, *A timelike Cauchy problem and an inverse problem for general hyperbolic equations*, Appl. Math. Lett., **21** (2008), 885-891.

[4] A.L. Bukhgeim, *Introduction to the theory of inverse problems*, Brill Academic Publishers, (2000).

[5] A.P. Calderón, *Existence and uniqueness theorems for systems of partial differential equations*, Fluid dynamics and applied mathematics, Gordon and Breach, (1961).

[6] R. Coifman and Y. Meyer, *Commutateurs d'integrales singulieres et operateurs multilineaires*, Ann. Inst. Fourier Grenoble, **28** (1978), 177–202.

[7] G. Duvaut and J.L. Lions, *Inequalities in Mechanics and Physics*, Springer-Verlag, (1976).

[8] R. Dautray and J.L. Lions, *Mathematical analysis and numerical methods for science and technology*, Springer-Verlag, **1-4** (1988).

[9] F. John, *Partial differential equations*, Springer, (1982).

[10] M. Eller, *Carleman estimates with a Second large parameter*, J. Math. Anal. Appl., **249** (2000), 491-514.

[11] M. Eller, *Carleman Estimates for some elliptic systems*, J. of Physics Conf. Series, **124** (2008), 012023.

[12] M. Eller and V. Isakov, *Carleman estimates with two large parameters and applications*, Contemp. Math., AMS, **268** (2000), 117-137.

[13] M. Eller, V. Isakov, G. Nakamura, and D. Tataru, *Uniqueness and stability in the Cauchy problem for the Maxwell's and elasticity systems*, College de France Seminar, 14, Studies in Appl. Math., **31** (2002), 329-349.

[14] L. Hörmander, *Linear Partial Differential Operators*, Springer-Verlag, (1963).

[15] L. Hörmander, *The analysis of linear partial differential operators*, Springer-Verlag, **1-4** (1983).

[16] O. Imanuvilov, V. Isakov, and M. Yamamoto, *An inverse problem for the dynamical Lamé system with two sets of boundary data*, Comm. Pure Appl. Math., **56** (2003), 1-17.

[17] V. Isakov, *A Nonhyperbolic Cauchy Problem for $\Box_b, \Box_c$ and its Applications to Elasticity Theory*, Comm. Pure Appl. Math., **39** (1986), 747-769.

[18] V. Isakov, *On the uniqueness of the continuation for a thermoelasticity system*, SIAM J. Math. Anal., **33** (2001), 509-522.

[19] V. Isakov, *Inverse Problems for Partial Differential Equations*, Springer-Verlag, New York, (2005).

[20] V. Isakov and N. Kim, *Carleman estimates with two large parameters for second order operators and applications to elasticity with residual stress*, Applicationes Mathematicae, **35** (2008), 447-465.

[21] V. Isakov and N. Kim, *Carleman estimates with second large parameter for second order operators*, Some application of Sobolev spaces to PDEs, International Math. Ser., Springer-Verlag, **10** (2009), 135-159.

[22] V. Isakov and N. Kim, *Global uniqueness and Lipschitz stability of residual stress from one boundary measurement*, ESAIM: Proc., **26** (2009), 45-54.

[23] V. Isakov and N. Kim, *Weak Carleman estimates with large parameters for second order operators and applications to elasticity with residual stress*, Discrete Cont. Dyn. Systems-A, **27** (2010), 799-825.

[24] V. Isakov, G. Nakamura, and J.-N. Wang, *Uniqueness and stability in the Cauchy problem for the elasticity system with residual stress*, Contemp. Math., AMS, **333** (2003), 99-113.

[25] V. Isakov, J.-N. Wang, and M. Yamamoto, *Uniqueness and stability of determining the residual stress by one measurement*, Comm. Part. Diff. Equat., **23** (2007), 833-848.

[26] V. Isakov, J.-N. Wang, and M. Yamamoto, *An inverse problem for a dynamical Lamé system with residual stress*, SIAM J. Math. Anal, (2007), 1328-1343.

[27] A. Khaidarov, *Carleman estimates and inverse problems for second order hyperbolic equations*, Math. USSR Sbornik, **58** (1987), 267-277.

[28] A. Khaidarov, *On stability estimates in multidimensional inverse problems for differential equations*, Soviet Math. Dokl, **38** (1989), 614-617.

[29] O.A. Ladyzenskaja and N.N. Uraltseva, *Linear and quasilinear elliptic equations*, New York, Academic, (1968).

[30] I. Lasiecka, R. Triggiani, and P.F. Yao, *Inverse/observability estimates for second order hyperbolic equations with variable coefficients*, J. Math. Anal. Appl., **235** (2000), 13-57.

[31] J. Lions and E. Magenes, *Non-Homogeneous boundary value problems*, Springer, (1972).

[32] C.-S. Man, *Hartig's law and linear elasticity with initial stress*, Inverse Problems, **14** (1998), 313-320.

[33] L. Rachelle, *Uniqueness in Inverse Problems for Elastic Media with Residual Stress*, Comm. Part. Diff. Equat., **28** (2003), 1787-1806.

[34] V. Romanov, *Carleman estimates for second-order hyperbolic equations*, Sib. Math. J., **47** (2006), 135-151.

[35] E. Stein, *Singular integrals and Differentiability properties of functions*, Princeton university press, (1970).