Stability Estimates and Explicit Formulas for the Inverse Problem of Determining the Boundary Condition from Discrete Data

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Introduction

To motivate our work we will start with some examples.

Example 1.1 *Analytic continuation for the inverse gravimetry problem(compare with* [10]*)*

Let $\bar{\partial}$ be the Cauchy-Riemann operator and $D = \{x \in C : |z| < 1\}$ be the unit disk. For the inverse gravimetry problem for cylindrical bodies we are asked to find a function h with support in D from the following conditions:.

$$\bar{\partial} u = h, \qquad z \in C \tag{1.1}$$

$$u|_{\gamma} = f \tag{1.2}$$

$$\lim_{|z| \to 0} u(z) = 0 \tag{1.3}$$

The trace $\bar{f}$ has a very simple physical interpretation: up to a constant factor, it is the gravitational force of attraction due to mass distribution h. In the case where h is the characteristic function of a star shaped region with respect to its center of gravity, P.S. Novikov[11] obtained uniqueness for this inverse problem

Example 1.2 *Thermal conductivity of a solid body*

Let $\Omega \subset R^n$ be an open connected set with smooth boundary $\partial\Omega = \gamma \cup \Gamma$ that has two components (see Figure 4). We consider Ω as a solid body with the thermal conductivity $K(x) > 0$ and steady-state temperature $u(x)$. Then from Fourier's law of heat conduction and conservation of energy law we have the differential equation

$$\nabla \cdot (K(x) \nabla u(x)) = 0 \qquad \text{in } \Omega. \tag{1.16}$$

We assume that on γ, considered an accessible part of the boundary, the temperature is a known function f, thus

$$u|_{\gamma} = f. \tag{1.17}$$

We consider Γ an inaccessible part of the boundary of Ω upon which we have the boundary condition

$$K \frac{\partial u}{\partial \nu} + a(x)(u - u_0) = 0, \quad x \in \Gamma \tag{1.18}$$

This is Newton's law of cooling which states that the heat energy flowing out ($a > 0$) is proportional to the difference between the temperature at the surface $u(x)$ and the temperature outside

$u_0(x)$. Since Γ is inaccessible, generally speaking we don't know the function $a(x)$, or the outside temperature $u_0(x)$, and would like to determine them measuring the heat flux on γ,

$$K \frac{\partial u}{\partial \nu} = g. \tag{1.19}$$

More precisely we have the following inverse problem.

Problem 1.3 *Thermal conductivity* $K(x) > 0$ *is given and* $u_0 \equiv 0$

Given the Cauchy data f and g on γ, determine the coefficient $a(x)$ on Γ. Similar to example (1.2) we first find the Cauchy data on Γ and then recover $a(x)$ from boundary condition (1.18),

$$a(x) = -\frac{K \frac{\partial u}{\partial \nu}}{u}, \qquad x \in \Gamma \setminus E_0, \qquad E_0 = \left\{ x \in \Gamma : u(x) = 0, \frac{\partial u}{\partial \nu}(x) = 0 \right\} \tag{1.20}$$

In order to develop methods and tools for these inverse problems it is convenient to study the Cauchy problem

$$\mathrm{P}u = h, \qquad \text{in } \Omega \tag{1.21}$$

$$u|_E = f, \qquad E \subset \partial\Omega \tag{1.22}$$

for elliptic operator

$$\mathrm{P} = \begin{bmatrix} 2\bar{\partial} & 0 \\ 0 & 2\partial \end{bmatrix} + A(x) \tag{1.23}$$

with 2×2 matrix potential $A = [a_{ij}]$ and vector solution $u = \begin{bmatrix} u_1 \\ u_2 \end{bmatrix}$. (1.24)

Here $\Omega \subset R^2$ is a bounded domain with piece-wise smooth boundary. E is a closed subset of $\partial\Omega$ with positive Lebesgue measure, $|E| > 0$.

Our first goal will be to find an explicit formula for $u(z)$, $z \in \Omega$ based on Cauchy data f on E. We assume that $u \in C^1(\overline{\Omega})$ and $A \in C^1(\overline{\Omega})$. Using the substitution $u = e^{\Phi(z)}v$ it is possible to reduce matrix A to the form

$$A = \begin{bmatrix} 0 & b \\ a & 0 \end{bmatrix} \tag{1.25}$$

We assume for simplicity that A has the form (2.5).

Theorem 2.11 *Let $u \in C^1(\overline{\Omega})$ be a solution of the Cauchy problem (2.1),(2.2) such that for some $z \in \Omega$ we can find a analytic function $\phi(\xi)$, bounded in Ω with*

$$\phi(z) = 0 \tag{1.26}$$

$$\Re\phi(\xi) < 0 \tag{1.27}$$

Then

$$u_j(z) = \lim_{\tau\to\infty}\left\{\int_E < e^{\tau\Phi} f, \mathcal{V}\mathrm{w}_j > ds - \int_\Omega < e^{\tau\Phi}h, w_j > ds\right\} \tag{1.28}$$

$$\Phi(\xi) = \begin{bmatrix} \phi(\xi) & 0 \\ 0 & \overline{\phi(\xi)} \end{bmatrix}, \qquad \mathcal{V} = \begin{bmatrix} \nu & 0 \\ 0 & \overline{\nu} \end{bmatrix}.$$

Here ν is the unit outward normal on the boundary of Ω, e_1 and e_2 are the canonical basis vectors of R^2, and w_j, $j = 1,2$ are solutions to the equations

$$\mathrm{P}_\tau^t w_j := (-D + A_\tau^t)w_j = -\delta(\xi - z)e_j$$

$$A_\tau^t = \begin{bmatrix} 0 & a_\tau \\ b_{-\tau} & 0 \end{bmatrix}, \quad a_{\pm\tau} = ae^{\pm i2\tau\Im\phi}.$$

The revelent question is "How does one find a function ϕ with properties (2.6),(2.7)?". The answer of course depends on Ω, E and $z \in \Omega$. There are several classic examples of domains with the associated functions ϕ.

Theorem 2.2 *Let $\mathrm{P}u = 0$ in D, u not be identically zero and $E_\delta = \{z \in \partial D : |u| < \delta\}$. Then there exists a constant C such that*

$$|E_\delta| \le \frac{C}{\ln(1/\delta)}.$$

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