TYPICAL POINTS, INVARIANT MEASURES, AND DIMENSION FOR RATIONAL MAPS ON THE RIEMANN SPHERE

A Thesis by

Jacie L. Kaufmann

Bachelor of Science in Mathematics Education, Greenville College, 2001

Submitted to the

Department of Mathematics and Statistics

and the faculty of the Graduate School of

Wichita State University

in partial fulfillment of

the requirements for the degree of

Master of Science

May 2006

TYPICAL POINTS, INVARIANT MEASURES, AND DIMENSION FOR RATIONAL MAPS ON THE RIEMANN SPHERE

I have examined the final copy of this thesis for form and content and recommend that it be accepted in partial fulfillment of the requirements for the degree of Master of Science in Mathematics.

____

Christian Wolf, Committee Chair

We have read this thesis and recommend its acceptance:

____

Andrew Acker, Committee Member

____

Elizabeth Behrman, Committee Member

____

Thomas DeLillo, Committee Member

DEDICATION

To Mom and Dad

ACKNOWLEDGEMENTS

Many thanks to my advisor Christian Wolf for challenging me to do more than I thought I could and for sharing with me his confidence and enthusiasm when my own was waning.

Thanks also to Bill Ingle for his contributions to this research.

To my heaven-sent friends Tina and Heidi, I am grateful for your patient endurance of my not-so-shining moments throughout the writing of this thesis and for always being willing to listen . . . "I thank my God upon every remembrance of you."

Finally, without the love and support of my family I would never have even begun this journey, so to *all* of you I offer my deepest thanks. When I count my blessings, you are at the top of the list!

ABSTRACT

We present three original results for the dynamics of rational maps on the Riemann sphere. Using methods from dimension and ergodic theory, we discuss generalized physical measures and prove their existence for hyperbolic and some parabolic rational maps. This shows that there are sets of typical points for these maps having maximal dimension. We then show that for any NCP rational map, the set of non-typical, or divergence, points is also of maximal dimension. Finally, we examine holomorphic families of stable rational maps and show that the dimension of the maps depends plurisubharmonically on the parameters.

TABLE OF CONTENTS

CHAPTER 1

INTRODUCTION

As this thesis pertains to the mathematical study of dynamical systems, we begin by describing such systems. Three ingredients are essential in any deterministic, discrete, dynamical system:

(i) Phase space. Mathematically, the phase space is simply a set containing all possible states of the system.

(ii) Time. For a discrete system, time is given by the set $\mathbb{N}$ of natural numbers or the integers $\mathbb{Z}$. (Note that there are also continuous dynamical systems for which time is represented by the reals $\mathbb{R}$. We will not consider these continuous systems.)

(iii) Time evolution law. This is a fixed rule that determines what state at time t will follow from the previous states.

Let X be a set and $f : X \to X$ be a function. Then f determines a dynamical system by defining f^n iteratively as $f^1 = f$, $f^n = f \circ f^{n-1}$. Assuming the system is at state x when time $t = 0$, then the system is at state $f^n(x)$ at time $t = n$.

Physical examples of dynamical systems vary widely in complexity, ranging from a simple swinging pendulum to the more complicated systems associated with the weather, population growth, or even neural circuitry. Surprisingly perhaps, even simple dynamical systems can exhibit chaotic, unpredictable behavior. The goal of studying such systems is to determine under what conditions we can assume some measure of stability and predictability over time.

In this thesis, our focus is on one-dimensional complex dynamics, an area which has been of particular interest for the past twenty years. We examine specifically the dynamics of rational maps on the Riemann sphere by applying techniques from smooth ergodic theory. As we iterate a given rational map $f : \overline{\mathbb{C}} \to \overline{\mathbb{C}}$, it is impossible to calculate the exact asymptotic behavior of every orbit largely due to the fact that errors occur at every step. Thus to determine the dynamics of every orbit is out of the question. Instead, we examine the dynamics of a so-called "typical" orbit; that is, we restrict our attention to a set of points having full measure with respect to an invariant probability measure. Here the question of which invariant measure to use must be addressed since the set of invariant measures is quite large. As we describe in more detail later, the answer lies in the set of physical measures and, more specifically, generalized physical measures.

We now discuss the main results of this thesis. Roughly speaking, a generalized physical measure is a measure for which the set of typical orbits is as large as possible. We show that for hyperbolic and some parabolic rational maps on the Riemann sphere such measures exist. More precisely, our first two results are as follows:

Theorem 1.0.1. *Let $f : \overline{\mathbb{C}} \to \overline{\mathbb{C}}$ be a hyperbolic rational map. Then f has a unique generalized physical measure.*

Theorem 1.0.2. *There exist parabolic rational maps having a generalized physical measure as well as parabolic rational maps without a generalized physical measure.*

However, even in the best case scenario, when existence of such a measure is known, we show that for some rational maps there could still be a very large set of points whose orbits do not converge to any measure. In other words, the dynamics of the map are still largely unknown.

Our second result pertains to NCP maps, rational maps for which all critical points in the Julia set, where the interesting dynamics occur, are non-recurrent. In particular, all parabolic and hyperbolic maps are NCP.

Theorem 1.0.3. *Let f be an NCP rational map on $\overline{\mathbb{C}}$ with Julia set J. Then the set of points in J which are not typical for any invariant measure has the same Hausdorff dimension as the Julia set itself.*

After defining and verifying the existence of the right measures and corresponding dimension, we turn our attention to families of rational maps and analyze how dimension varies with the parameters of those families. In particular, we consider stable rational maps since it is an immediate consequence of a well-known result of Mané, Sullivan, and Sad that such maps form an open and dense subset of the space of all rational maps [MSS]. In the following theorem, $d(f_a)$ denotes the dimension of f_a as defined in chapter 2.

Theorem 1.0.4. *Let $(f_a)_{a \in A}$ be a holomorphic family of stable rational maps. Then the map $a \mapsto d(f_a)$ is plurisubharmonic in A.*

To the best of our knowledge, all four theorems above are original and have not yet appeared in the literature. They are a part of a forthcoming paper [IKW] with William Ingle and Christian Wolf.

CHAPTER 2

RATIONAL MAPS, COMPLEX DYNAMICS, DIMENSION, AND HYPERBOLIC SYSTEMS

In this chapter, we introduce the key definitions and concepts used in this thesis.

2.1 Rational maps

When studying dynamical systems in this paper our focus will be on rational maps. A *rational map* defined on the Riemann sphere $\overline{\mathbb{C}}$ is a holomorphic map of the form $f(z) = P(z)/Q(z)$, where P and Q are polynomials with no common factors. To avoid trivialities, the degree of the map f defined by $\deg(f) = \max(\deg(P), \deg(Q))$ will be always greater than or equal to 2. Rational maps play a key role in the study of complex dynamics since it is a well-known result that any holomorphic function on a compact region of the Riemann sphere can be approximated arbitrarily closely by rational maps.

Critical points of a rational map f consist of the solutions to the equations $f'(z) = 0$ and of poles of f with order at least two. Therefore the critical points are precisely those at which the map is not locally injective. Recall that a rational map has a finite number of critical points. The points in the domain of f that are not critical points are called *regular* points.

2.2 Periodic points

We define the *forward orbit* of a point z_0 in the domain of f as the sequence of points $\{z_n\}_{n\in\mathbb{N}}$ where $z_n = f(z_{n-1})$. A point z_0 is called *periodic* if there is an $n \in \mathbb{N}$ such that $z_0 = z_n$. For each periodic point z_0, the minimum $n \in \mathbb{N}$ such that $z_0 = z_n$ is called the *period* of z_0, and the sequence $\{z_1, ..., z_n = z_0\}$ is called the *cycle* of the periodic point z_0 of period n. Let z_0 be a periodic point with period n. If $|(f^n)'(z_0)| \neq 1$, then we call z_0 a *hyperbolic* periodic point of f. If $|(f^n)'(z_0)| < 1$, then we call z_0 an *attracting* periodic point. If $|(f^n)'(z_0)| > 1$

then the periodic point is called a *repelling* periodic point.

2.3 Julia and Fatou sets

There are several equivalent definitions of the *Julia set* of a rational map on $\overline{\mathbb{C}}$ (see for example [CG]). Here we use the definition that the Julia set J of a map is the closure of the set of all repelling periodic points. It can be shown that the Julia set of a rational map of degree ≥ 2 is a non-empty, totally invariant, compact set.

For completeness, we also define the *Fatou set* as the complement of the Julia set. Clearly the Fatou set of a rational map of degree ≥ 2 is open. Equivalently, the Fatou set is the set of points $z_0 \in \overline{\mathbb{C}}$ such that the family of functions $\{f^n\}_{n\in\mathbb{N}}$ is a normal family in a neighborhood of z_0.

In this work we will mostly focus on the dynamics of f restricted to its Julia set.

2.4 Measures and ergodicity

A measurable space is a set X together with a σ-algebra $\mathcal{B}$ of subsets of X. A measure on a measurable space $(X, \mathcal{B})$ is a non-negative set function $\mu : \mathcal{B} \to \mathbb{R} \bigcup \{\infty\}$ such that

(*i*) $\mu(\varnothing) = 0$

(*ii*) $\mu(\bigcup_{n=1}^{\infty} B_n) = \sum_{n=1}^{\infty} \mu(B_n)$ for any pairwise disjoint collection $\{B_n\}_{n=1}^{\infty}$ of members of $\mathcal{B}$.

If $\mu(X) = 1$ we call μ a *probability* measure. We call a map $f : \overline{\mathbb{C}} \to \overline{\mathbb{C}}$ *measurable* if the inverse images of measurable sets are measurable, that is $f^{-1}(\mathcal{B}) \subset \mathcal{B}$. We say that μ is an *invariant measure* for a measurable map f if $\mu(f^{-1}(B)) = \mu(B)$ for every $B \in \mathcal{B}$. For our purposes, we consider the σ-algebra generated by the open sets of the topology of $\overline{\mathbb{C}}$. We call this σ-algebra the *Borel sets*, and we call any measure defined on the σ-algebra of Borel sets a *Borel measure*. We denote by $\mathcal{M}$ the collection of all f-invariant, Borel, probability measures equipped with the weak* topology; that is, $\mu_n \to \mu$ as $n \to \infty \iff \int \varphi d\mu_n \to \int \varphi d\mu$ as $n \to \infty$ for all $\varphi \in C^0(\overline{\mathbb{C}})$. Since $\overline{\mathbb{C}}$ is compact, it follows from standard arguments that $\mathcal{M}$ is a compact, metrizable, convex space. One can show that every invariant measure, with the exception of certain point measures, is supported on the Julia set.

If a measure $\mu \in \mathcal{M}$ satisfies the condition that either $\mu(B) = 0$ or $\mu(B) = 1$ for any sets $B \in \mathcal{B}$ for which $f^{-1}(B) = B$, we call the measure *ergodic*. The set of ergodic measures will be denoted by $\mathcal{M}_E$. The well-known ergodic decomposition theorem shows that the ergodic measures are the building blocks of all f-invariant, Borel, probability measures in the sense that integrating with respect to a measure in $\mathcal{M}$ can always be expressed as integrating with respect to a measure in $\mathcal{M}_E$ (see [DGS] for more details). For this reason, we often work with ergodic measures with the understanding that all our results can be easily extended to all invariant probability measures.

2.5 Dimension

For discussing the size of sets in this thesis, we will make use of many different dimensions. We first define the *t-dimensional Hausdorff measure* of a set E in a locally compact metric space X as

$$H^t(E) := \sup_{\delta>0} \inf \left\{ \sum_i (\operatorname{diam}(A_i))^t : \{A_i : i \geq 1\} \text{ is a } \delta\text{-cover of } E \right\}$$

where t is a positive real number. To say $\{A_i : i \geq 1\}$ is a δ-cover of E means that $E \subset \bigcup_i A_i$ and $\operatorname{diam} A_i \leq \delta$.

From this measure we can derive a definition of *Hausdorff dimension* as

$$\dim_H(E) := \inf \left\{ t : H^t(E) = 0 \right\} = \sup \left\{ t : H^t(E) = \infty \right\}.$$

In this thesis, we will also be interested in the Hausdorff dimension of measures because it is a natural dynamical invariant, coinciding with essentially all other dimension-like characteristics of invariant measures (see [P]). For $\mu \in \mathcal{M}$ we define the Hausdorff dimension of

μ by

$$\dim_H \mu = \inf\{\dim_H(Y) : \mu(Y) = 1\}. \tag{2.1}$$

Many of the results we use apply to ergodic measures that are also hyperbolic. To define this set of hyperbolic, ergodic measures, we first briefly recall that the *Lyapunov exponent* of a measure $\mu \in \mathcal{M}_E$ is given by

$$\chi(\mu) = \int \log|f'|d\mu. \tag{2.2}$$

It is a well known result of Oseledec that the *pointwise* Lyapunov exponent, defined by

$$\chi(z) = \lim_{n\to\infty} \frac{1}{n} \log|(f^n)'(z)|, \tag{2.3}$$

exists for μ a.e. $z \in J$ and coincides with $\chi(\mu)$. The Lyapunov exponent provides a means of quantifying the exponential growth or divergence along the orbit of a map. We say that μ is a *hyperbolic measure* if $\chi(\mu) > 0$. We denote by $\mathcal{M}_{E,\text{hyp}} \subset \mathcal{M}_E$ the subset of hyperbolic measures in $\mathcal{M}_E$.

We define a new dimension, the so-called *dynamical dimension*, by

$$DD(f) = \sup\{\dim_H \mu : \mu \in \mathcal{M}_{E,\text{hyp}}\}.$$

This definition is due to Denker and Urbanski [DU].

It follows naturally from a result of Mané [Ma] and Ruelle's inequality (see, for example, [HK]) that an equivalent characterization of the dynamical dimension is given by

$$DD(f) = \sup\{\dim_H \mu : \mu \in \mathcal{M}_E^+\}$$

where $\mathcal{M}_E^+$ denotes the subset of ergodic measures having positive measure theoretic entropy. (See chapter 3 for more details on entropy.)

We now introduce the hyperbolic dimension. We say that a compact set $\Lambda \subset \overline{\mathbb{C}}$ is a *hyperbolic set* of f if the following holds:

(i) Λ is a forward invariant set for f, i.e. $f(\Lambda) \subset \Lambda$;

(ii) $f|_\Lambda$ is topologically conjugate to a subshift of finite type;

(iii) there exist constants $c > 0$, $\alpha > 1$ such that $|(f^n)'(z)| \geq c\alpha^n$ for all $z \in \Lambda$ and all $n \in \mathbb{N}$.

For more details on subshifts of finite type, we refer to [R]. In the case when property ii) is omitted we say that Λ is expanding. It is well-known that every hyperbolic set of f is contained in the Julia set J. Since much is known about the ergodic theory and dynamics of subshifts of finite type, it seems a logical approach in our objective of understanding the dynamics of J to compare the Hausdorff dimension of the hyperbolic sets of f to the Hausdorff dimension of J itself. To that end, we follow [U] (see [Sh] for the original paper) in defining the *hyperbolic dimension* of f by

$$\text{hypdim}(f) = \sup\{\dim_H \Lambda\},$$

where the supremum is taken over all hyperbolic sets Λ of f.

Next we discuss the conformal dimension of f. A measure μ on the Julia set J of f is $\alpha(f)$-conformal if $\mu(J) = 1$ and $\mu(f(E)) = \int_E \mid f'(z) \mid^{\alpha(f)} d\mu(z)$ for every Borel set $E \subset J$ such that $f|_E$ is injective. We denote by $\delta(f)$ the smallest exponent $\alpha(f)$ for which an $\alpha(f)$-conformal measure exists and call this $\delta(f)$ the *conformal dimension* of f.

Let $h(f)$ denote the smallest positive real zero of the pressure function Π in chapter three. The following result is known (see [U]):

Theorem 2.5.1. *Let $f \in Rat_d$. Then $DD(f) = \text{hypdim}(f) = \delta(f) = h(f)$.*

We denote the joint value of the quantities in Theorem 2.5.1 by $d(f)$ and simply call it the dimension of f.

2.6 Hyperbolic, parabolic, and NCP rational maps

Of all subclasses of rational maps, hyperbolic rational maps are the best understood and serve as a starting point for research on less understood maps. Here we introduce two equivalent representations of hyperbolic maps, both of which coincide with the definition of hyperbolic sets stated in the previous section. First, we say a rational map $f : \overline{\mathbb{C}} \to \mathbb{C}$ with Julia set J is *hyperbolic* if there exists $n \geq 1$ such that $\inf\{|(f^n)'(z)| : z \in J\} > 1$; that is, the map expands uniformly on its Julia set. Equivalently, but in more topological terms, a rational function $f : J \to J$ is *hyperbolic* if and only if $\overline{\bigcup_{n\geq 1} f^n(\text{Crit}(f))} \bigcap J = \varnothing$ where $\text{Crit}(f) = \{z \in \mathbb{C} : f'(z) = 0\}$.

Another class of rational maps that has been studied in great detail is that of parabolic maps. A *parabolic point* is a periodic point z of f, with period $p \geq 1$ such that $(f^p)'(z)$ is a root of unity. Then one characterization of a *parabolic map* is a rational function $f : \overline{\mathbb{C}} \to \mathbb{C}$ for which the Julia set J contains at least one parabolic point but contains no critical points of f.

A *non-recurrent (NCP)* map is a rational map $f : \overline{\mathbb{C}} \to \overline{\mathbb{C}}$ for which all critical points of f in the Julia set J are non-recurrent; that is, for each critical point $z_0 \in J$ there is a neighborhood U_{z_0} such that $f^n(z_0) \bigcap U_{z_0} = \varnothing$ for all $n \in \mathbb{N}$. The class of NCP maps contains all hyperbolic and parabolic maps, but there are NCP maps which are neither hyperbolic nor parabolic.

CHAPTER 3

EXISTENCE OF GENERALIZED PHYSICAL MEASURES

In this chapter, we derive theorems for the existence and uniqueness of generalized physical measures for hyperbolic and parabolic maps. Before stating and proving our theorems, we begin with some definitions, background results, and the proof of an important formula for the dimension of the basin of a measure.

3.1 Definitions

3.1.1 Physical measures and the basin of a measure

For $\mu \in \mathcal{M}_E$ we define the *basin* of μ by

$$\mathcal{B}(\mu) = \left\{ z \in J : \frac{1}{n} \sum_{k=0}^{n-1} \delta_{f^k(z)} \;\rightarrow\; \mu \text{ as } n \rightarrow \infty \right\},$$

where $\delta_{f^k(z)}$ denotes the Dirac measure supported on $f^k(z)$ and the convergence is in the weak* topology. The basin of μ is sometimes also called the set of future generic points of μ (see [DGS]).

For the Julia set J of a rational map $f : \overline{\mathbb{C}} \rightarrow \overline{\mathbb{C}}$ with $\deg f \geq 2$, a natural question that arises is that of finding the "right" measure to use on J. In applications, we would like to have Lebesgue measure, or at least a measure which is absolutely continuous with respect to Lebesgue measure, but this is in general not possible. Instead we look for a measure whose size in the sense of Hausdorff dimension is as large as possible; that is, we would like to have

$$\dim_H \mathcal{B}(\mu) = \dim_H J.$$

The question of finding the best measure for a given set gives rise to the discussion of physical and generalized physical measures. We say $\mu \in \mathcal{M}_E$ is a *physical measure* if $\text{vol}(\mathcal{B}(\mu)) > 0$. We say that $\mu \in \mathcal{M}_E$ is a *generalized physical measure* if $\dim_H \mathcal{B}(\mu) = \dim_H J$.

3.1.2 Measure-theoretic entropy

A closely related quantity to Lyapunov exponents (see (2.2) and (2.3)) is entropy. Entropy is a measure of the inherent amount of uncertainty in obtaining a desired result in a dynamical system. For a complete definition we refer to Walters [W].

3.2 Topological entropy for noncompact sets

In this section, we extend the notion of entropy to noncompact subsets of topological spaces. This extension is accomplished by developing entropy in a way which resembles Hausdorff dimension. For this perspective, we rely on the work of Rufus Bowen [B].

We start with a compact metric space X and let $f : X \to X$ be continuous. Now let $Y \subset X$ and let $\mathcal{A}$ be a finite open cover of X. If a set $E \subset X$ is entirely contained in some member of $\mathcal{A}$, we write $E \prec \mathcal{A}$, and if we have $E_i \prec \mathcal{A}$ for every E_i in a collection of sets $\{E_i\}$ we write $\{E_i\} \prec \mathcal{A}$. Now we define $n_{f,\mathcal{A}}(E)$ to be the largest nonnegative integer such that

$$f^k E \prec \mathcal{A} \text{ for } 0 \leq k < n_{f,\mathcal{A}}(E)$$

with $n_{f,\mathcal{A}}(E) = 0$ if $E \not\prec \mathcal{A}$ and $n_{f,\mathcal{A}}(E) = +\infty$ if $f^k E \prec \mathcal{A}$ for all $k \geq 0$. Thus, $n_{f,\mathcal{A}}(E)$ gives us the smallest iterate of E which makes it "too large" to fit into some member of $\mathcal{A}$, the finite open cover.

Now, for $\mathcal{E} = \{E_i\}_{i=1}^{\infty}$ and $\lambda \in \mathbb{R}$, we write

$$D_{\mathcal{A}}(E) = \exp(-n_{f,\mathcal{A}}(E)) \text{ and } D_{\mathcal{A}}(\mathcal{E}, \lambda) = \sum_{i=1}^{\infty} D_{\mathcal{A}}(E_i)^{\lambda}. \tag{3.1}$$

Using this quantity $D_{\mathcal{A}}(\mathcal{E}, \lambda)$, we can define a measure by

$$m_{\mathcal{A},\lambda}(Y) = \lim_{\varepsilon \to \infty} \inf \left\{ D_{\mathcal{A}}(\mathcal{E}, \lambda) : \bigcup E_i \supset Y \text{ and } D_{\mathcal{A}}(E_i) < \varepsilon \right\}$$

Notice the similarities to the construction of Hausdorff measure; here instead of looking at the diameter of a set, we consider how long the iterates of the set remain inside the members of a given cover. As with Hausdorff measure, as λ increases, the corresponding measure $m_{\mathcal{A},\lambda}$ decreases and there is at most one λ for which $m_{\mathcal{A},\lambda} \notin \{0, +\infty\}$.

We are now prepared to define entropy for arbitrary $Y \subset X$. First, for a given cover $\mathcal{A}$ we set

$$h_{\mathcal{A}}(f, Y) = \inf\{\lambda : m_{\mathcal{A},\lambda}(Y) = 0\}$$

in a manner reminiscent of Hausdorff dimension. Finally, allowing $\mathcal{A}$ to range over all finite open covers of X we define

$$h(f, Y) = \sup_{\mathcal{A}} h_{\mathcal{A}}(f, Y).$$

Thus we have a notion of topological entropy for subsets of a compact X that are not necessarily themselves compact. Of course, in order for this definition to be useful it should equal the usual topological entropy when applied to a set that is compact. It can be shown that this is in fact the case (see [B]).

3.3 Previous results

3.3.1 Birkhoff's Ergodic Theorem

In 1931, G.D. Birkhoff proved a crucial theorem in ergodic theory. This result, sometimes referred to as the Ergodic Theorem, will be essential for our concept of generalized physical measures.

Theorem 3.3.1. *Let $(X, \mathcal{B}, \mu)$ be a probability space and let $f : X \to X$ be a measure-preserving transformation on X and $\varphi \in L^1(\mu)$. Then*

$$\lim_{n\to\infty} \frac{1}{n} \sum_{k=0}^{n-1} \varphi(f^k(x)) := \varphi_f(x)$$

exists μ-a.e.

Moreover, if μ is ergodic, then φ_f is constant a.e. and $\varphi_f = \int \varphi d\mu$ a.e. Thus, for all

$\varphi \in L^1(\mu)$ we have that

$$\lim_{n\to\infty} \frac{1}{n} \sum_{k=0}^{n-1} \varphi(f^k(x)) = \int \varphi d\mu$$

μ a.e.

This is the classic "time average = space average" result.

Our interest in Theorem 3.3.1 is that from it we can easily deduce that $\mu(\mathcal{B}(\mu)) = 1$ and consequently conclude that $\dim_H \mu \leq \dim_H \mathcal{B}(\mu)$ using the definition of $\dim_H \mu$ in (2.1).

3.3.2 Mané's formula

A key component in the proof of Theorem 3.4.1 is the result of Mané that relates entropy, Lyapunov exponents, and the Hausdorff dimension of a measure. We restrict our attention to hyperbolic measures. We have

$$\dim_H \mu = \frac{h_\mu(f)}{\chi(\mu)} \text{ for all } \mu \in \mathcal{M}_{E,hyp}.$$

Recall that here $\chi(\mu)$ denotes the Lyapunov exponent of μ (see section 2.5).

3.3.3 Bowen's theorem

In "Topological Entropy for Noncompact Sets" [B], Bowen proved that for a continuous map f on a compact metric space and a measure $\mu \in \mathcal{M}_E$, we have the equality

$$h(f, \mathcal{B}(\mu)) = h_\mu(f).$$

This is the link that ties the measure-theoretic entropy of the map f with the specially constructed topological entropy of the possibly noncompact set of future generic points of f.

3.4 Preliminary result

To prove our existence theorems for generalized physical measures, we will need the following original result.

Theorem 3.4.1. *Let f be a hyperbolic or parabolic rational map on $\overline{\mathbb{C}}$ with Julia set J. Then* $\dim_H \mathcal{B}(\mu) = \frac{h_\mu(f)}{\chi(\mu)}$ *for all* $\mu \in \mathcal{M}_E^+$.

Recalling the definition of the Hausdorff dimension of a measure, the inequality $\dim_H \mathcal{B}(\mu) \geq \frac{h_\mu(f)}{\chi(\mu)}$ is immediate from Birkhoff's Ergodic Theorem which gives $\mu(\mathcal{B}(\mu)) = 1$ and thus

$$\dim_H \mu \leq \dim_H \mathcal{B}(\mu)$$

and Mané's formula

$$\dim_H \mu = \frac{h_\mu(f)}{\chi(\mu)}.$$

For the other direction, we refer the reader to our paper [IKW] and for simplicity present here the proof of a similar result by Manning already present in the literature [Mn].

As Manning's theorem pertains to Axiom A diffeomorphisms, we briefly define these maps now: an Axiom A diffeomorphism is a diffeomorphism on a smooth manifold for which the nonwandering set is hyperbolic and the periodic points are dense in the nonwandering set. We also need to introduce the reader to the concept of stable and unstable manifolds. The local unstable $W^u_{loc}(x)$ or stable $W^s_{loc}(x)$ manifold of a point x and a map f is the set of points in a suitably small neighborhood of x that get closer to x under backward, respectively forward, iteration. The global unstable manifold $W^u(x)$ is obtained by taking the union of all forward iterations of the local unstable manifold; similarly, the global stable manifold $W^s(x)$ is obtained by taking the union of all backward iterations of the local stable manifold. The global stable manifold of a set is the union of all such stable manifolds for points in the set. For details and more precise definitions of stable and unstable manifolds, see [R].

With that background, we state and prove Manning's theorem.

Theorem 3.4.2. *Let μ be an ergodic Borel probabilty measure of a surface M and let $f : M \to M$ be a C^1 Axiom A diffeomorphism that preserves μ. Let $\mathcal{B}(\mu)$ denote the basin of μ and let Ω_1 be the basic set of f for which $\mathcal{B}(\mu) \subset W^s(\Omega_1)$. Then the Hausdorff dimension of $\mathcal{B}(\mu) \bigcap W^u_{loc}(x)$ does not depend on the choice of $x \in \Omega_1$ and the entropy $h_\mu(f)$ is given by $h_\mu(f) = \dim_H \mu \;\ \chi(\mu)$, for the positive Lyapunov exponent $\chi(\mu)$.*

Proof. We begin by defining a continuous function $\phi^{(u)} : W^u \to \mathbb{R}$ by $\phi^{(u)}(y) = -\log \|Df_y|T_yW^u(y)\|$ and noting that $\chi(\mu) = -\mu(\phi^{(u)})$. Applying the so-called "In-Phase" result [R] for the compact hyperbolic invariant set Ω_1 we can state

$$W^u_\varepsilon(\Omega_1) = \bigcup_{x\in\Omega_1} W^u(x,\varepsilon)$$

where $W^u(x,\varepsilon) = \{y \in M : d(f^jy, f^ix) \le \varepsilon \ \forall \ j \le 0\}$. The function $\phi^{(u)}$ is continuous on this compact set $W^u_\varepsilon(x)$. Now given $\varepsilon > 0$ we choose a cover $\mathcal{A}$ of the basic set Ω_1 by rectangles so small that $\phi^{(u)}$ varies by at most ε in each rectangle. Here we apply (3.1) from our construction of topological entropy to claim that for any $\alpha > 0$ there is a finite open cover $\mathcal{F} = \{F_i\}_{i=1}^k$ of $\mathcal{B}(\mu)$ such that

$$D_{\mathcal{A}}(\mathcal{F}, h+\varepsilon) = \sum_{i=1}^{k} D_{\mathcal{A}}(F_i)^{h+\varepsilon} = \sum_{i=1}^{k} \exp(-n_{f,\mathcal{A}}(F_i)(h+\varepsilon)) < \alpha$$

where $h = h_\mu = h(f, \mathcal{B}(\mu))$ by Bowen's result [B].

We want to consider an arbitrary closed interval W on any $W^u(x)$. Clearly W is compact and, for ε small enough, will cross each member of the cover $\mathcal{A}$ no more than once. Now we would like to divide the basin in such a way that it can be considered as a countable union of smaller sets on which the averages of the function $\phi^{(u)}$ applied to the images under f converge to the Lyapunov exponent at specific rates. That is, we group the points in the basin according to the rates at which this convergence occurs. Thus we define

$$\mathcal{B}_r(\mu) = \left\{ x \in \mathcal{B}(\mu) : \left| \frac{1}{m} \sum_{i=0}^{m-1} \phi^{(u)}(f^jx) + \chi(\mu) \right| \le \varepsilon, \forall \ m \ge r \right\}.$$

Recall the definition of the Lyapunov exponent $\chi(\mu)$ and note that it is in fact equal to $-\mu(\phi^{(u)})$. For ease, we denote $\chi = \chi(\mu)$. Therefore, since μ is ergodic, the so-called Birkhoff sums are constant almost everywhere and we see that $\mathcal{B}(\mu) = \bigcup_{r=0}^{\infty} \mathcal{B}_r(\mu)$.

Since Hausdorff dimension is stable under countable unions, it is sufficient for us to obtain our inequality for a set $\mathcal{B}_r(\mu)$ corresponding to a fixed r and then to extend our result over the entire basin by taking a countable union over all values of r. Thus, we now fix $r > 0$.

We let α be small enough to ensure that $n_{f,\mathcal{A}}(F_i) \geq r$ for each F_i in $\mathcal{F}$. In other words, the images of the elements of the finite open cover $\mathcal{F}$ stay inside the elements of the open cover $\mathcal{A}$ for at least r iterations.

We would now like to apply the mean value theorem to the diameter of the portion of the interval W that lies within the intersection of the subset $\mathcal{B}_r(\mu)$ of the basin with the cover element F_i for some $i = 1, 2, ...k$. To that end, we consider a point $y \in W$ that lies in the convex hull of $W \bigcap \mathcal{B}_r(\mu) \bigcap F_i$. Now we obtain

$$\operatorname{diam}\left(W \bigcap \mathcal{B}_r(\mu) \bigcap F_i\right) = \frac{\operatorname{diam} f^{n_{f,\mathcal{A}}(F_i)}(W \bigcap \mathcal{B}_r(\mu) \bigcap F_i)}{\|Df_y^{n_{f,\mathcal{A}}(F_i)}|E_y^u\|}.$$

We need to establish an upper bound for the diameter of $f^{n_{f,\mathcal{A}}(F_i)}$. Such a bound is obtained by considering the longest interval of an unstable manifold in a member of $\mathcal{A}$; we denote this value by $\operatorname{mesh}_u \mathcal{A}$. Thus we can state

$$\frac{\operatorname{diam} f^{n_{f,\mathcal{A}}(F_i)}(W \bigcap \mathcal{B}_r(\mu) \bigcap F_i)}{\|Df_y^{n_{f,\mathcal{A}}(F_i)}|E_y^u\|} \leq \frac{\operatorname{mesh}_u \mathcal{A}}{\|Df_y^{n_{f,\mathcal{A}}(F_i)}|E_y^u\|}.$$

From time 0 to n the orbit of y will remain in the same element of $\mathcal{A}$ as that of a point of $B_r(\mu)$. Thus we have

$$\|Df_y^{n_{f,\mathcal{A}}(F_i)}|E_y^u\| \geq \exp[(\chi - 2\varepsilon)n_{f,\mathcal{A}}(F_i)].$$

Now we can define a cover $\mathcal{F}'$ containing sets for which we can nicely control the diameter. Namely,

$$\mathcal{F}' = \left\{W \bigcap \mathcal{B}_r(\mu) \bigcap F_i : F_i \in \mathcal{F}\right\}.$$

For this cover we obtain

$$\begin{aligned}
\sum_{F_i' \in \mathcal{F}} (\operatorname{diam} F_i')^{(h+\varepsilon)/(\chi-2\varepsilon)} &\leq \sum_{F_i' \in \mathcal{F}} \frac{(\operatorname{mesh}_u \mathcal{A})^{(h+\varepsilon)/(\chi-2\varepsilon)}}{\|Df_y^{n_{f,\mathcal{A}}(F_i)}|E_y^u\|^{(h+\varepsilon)/(\chi-2\varepsilon)}} \\
&\leq \sum_{F_i' \in \mathcal{F}} \frac{(\operatorname{mesh}_u \mathcal{A})^{(h+\varepsilon)/(\chi-2\varepsilon)}}{[\exp[(\chi-2\varepsilon)n_{f,\mathcal{A}}(F_i)]]^{(h+\varepsilon)/(\chi-2\varepsilon)}} \\
&= \sum_{F_i' \in \mathcal{F}} \frac{(\operatorname{mesh}_u \mathcal{A})^{(h+\varepsilon)/(\chi-2\varepsilon)}}{\exp[(h+\varepsilon)n_{f,\mathcal{A}}(F_i)]} \\
&= (\operatorname{mesh}_u \mathcal{A})^{(h+\varepsilon)/(\chi-2\varepsilon)} \\
&\quad \cdot \sum_{F_i' \in \mathcal{F}} \exp[-(h+\varepsilon)n_{f,\mathcal{A}}(F_i)] \\
&< \alpha(\operatorname{mesh}_u \mathcal{A})^{(h+\varepsilon)/(\chi-2\varepsilon)}
\end{aligned}$$

By making $\mathcal{F}$ fine, we can make F_i' as fine as necessary. Thus $(h+\varepsilon)/(\chi-2\varepsilon)$ provides an upper bound for the Hausdorff dimension of $W \bigcap \mathcal{B}_r(\mu)$. Now we again consider the entire basin by taking a countable union over r, and we let $\varepsilon \to 0$ to obtain

$$\dim_H \left(W \bigcap B(\mu) \right) \leq \frac{h}{\chi} = \frac{h_\mu(f)}{\chi(\mu)}. \qquad \square$$

3.5 Main results

In this section, we discuss in detail generalized physical measures for rational maps. In particular, we derive existence and uniqueness results for hyperbolic and for parabolic maps. In the following $f : \overline{\mathbb{C}} \to \overline{\mathbb{C}}$ will always be a rational map of degree $d \geq 2$ and J its Julia set.

Following [Wo] we make the following definition.

Definition 3.5.1. *We say that* $\mu \in \mathcal{M}_E$ *is a generalized physical measure if* μ *has positive measure-theoretic entropy and* $\dim_H \mathcal{B}(\mu) = \dim_H J$*.*

We note here that since μ must have positive measure-theoretic entropy, Ruelle's inequality implies that every generalized physical measure is hyperbolic. This in particular rules out the possibility of a point measure concentrated on a parabolic periodic point for which the basin has maximal possible Hausdorff dimension 2.

From now on we will additionally assume that f is *expansive* on its Julia set. By expansive, we mean that there is a $\delta > 0$ such that for all distinct pairs of points $z_1, z_2 \in J$ there exists $n \geq 0$ such that $d(f^n(z_1), f^n(z_2)) \geq \delta$ where d is the spherical metric on $\overline{\mathbb{C}}$. This implies that f is either a hyperbolic or a parabolic rational map. In particular, f has no critical points on J.

We define $\phi : J \to \mathbb{R}$ by

$$\phi(z) = \log |f'(z)|$$

and the pressure function $\Pi : \mathbb{R} \to \mathbb{R}$ by

$$\Pi(t) = P(-t\phi),$$

where $P : C(J, \mathbb{R}) \to \mathbb{R}$ denotes the topological pressure of the dynamical system $f|_J$ (see [W] for the definition of the topological pressure). The variational principle gives the following

$$\Pi(t) = \sup_{\mu \in \mathcal{M}} (h_\mu(f) - t \int \phi d\mu) = \sup_{\mu \in \mathcal{M}} (h_\mu(f) - t\chi(\mu)) \tag{3.2}$$

Recall that $\mu \in \mathcal{M}$ is an equilibrium state of the potential $-t\phi$ if the supremum of the right hand side of (3.2) is attained by $h_\mu(f) - t\chi(\mu)$. Moreover, since $f|_J$ is expansive, we always have at least one equilibrium state. This follows from the fact that the entropy map $\mathcal{M} \ni \mu \mapsto h_\mu(f)$ is upper semi-continuous.

We present our main result on generalized physical measures for hyperbolic maps.

Theorem 3.5.2. *Let $f : \overline{\mathbb{C}} \to \overline{\mathbb{C}}$ be a hyperbolic rational map. Then f has a unique generalized physical measure.*

Proof. It was proven by Ruelle [Ru] that the pressure function Π has a unique zero t_0 and $t_0 = \dim_H J$. Moreover, for each $t \in \mathbb{R}$ the potential $-t\phi$ has a unique equilibrium state μ_t. Hence,

$$h_{\mu_{t_0}}(f) - t_0\chi(\mu_{t_0}) = \Pi(t_0) = 0 \tag{3.3}$$

and

$$h_\nu - t_0\chi(\nu) < 0 \tag{3.4}$$

for all $\nu \in \mathcal{M} \setminus \{\mu_{t_0}\}$. By using Theorem 3.4.1, we may conclude from (3.3) that

$$\dim_H \mathcal{B}(\mu_{t_0}) = \frac{h_{\mu_{t_0}}(f)}{\chi(\mu_{t_0})} = t_0 = \dim_H J.$$

Therefore, μ_{t_0} is a generalized physical measure for f. On the other hand, (3.4) implies that if $\mu \in \mathcal{M}_E \setminus \{\mu_{t_0}\}$ with $h_\mu(f) > 0$ then

$$\dim_H \mathcal{B}(\mu) = \frac{h_\mu(f)}{\chi(\mu)} < t_0 = \dim_H J.$$

Thus, μ is not a generalized physical measure, which completes the proof. □

We now discuss the case when f is parabolic. In this case, we have the following (see [U]). Let $t_0 = \dim_H J$. Then

$$\Pi(t_0) = 0,$$

$$\Pi(t) > 0 \text{ for all } t < t_0 \text{ and } \Pi(t) = 0 \text{ for all } t \geq t_0,$$

and

$$\Pi|_{[0,t_0)} \text{ is analytic.}$$

Theorem 3.5.3. *There exist parabolic rational maps having a generalized physical measure as well as parabolic rational maps without a generalized physical measure.*

Proof. Let f be a parabolic rational map. Let $t_0 = \dim_H J$. We claim that μ is a generalized physical measure of f if and only if μ is ergodic, has positive entropy, and is an equilibrium state of the potential $-t_0\phi$. For μ to be a hyperbolic equilibrium state of $-t_0\phi$ means that it satisfies

$$h_\mu(f) - t_0\chi(\mu) = 0. \tag{3.5}$$

which can be true if and and only if

$$\dim_H \mathcal{B}(\mu) = \frac{h_\mu(f)}{\chi(\mu)} = \dim_H J = t_0. \tag{3.6}$$

Thus, our claim is verified. Note that the identity on the left is Theorem 3.4.1.

It is shown in [DU] that there exist parabolic rational maps which have such an equilibrium state and also some which haven't such an equilibrium state. This proves the theorem. □

CHAPTER 4

THE SIZE OF THE SET OF DIVERGENCE POINTS

We have seen in the previous chapter that for certain rational maps there is a generalized physical measure, implying that there is a set of typical points of maximal size. In this chapter, we now explore the size of the set of points which are non-typical for any measure. We call this set the set of divergence points. In particular, we will show that in the case of NCP maps, the set of divergence points has maximal Hausdorff dimension.

Let $\varphi \in C(J, \mathbb{R})$ and denote by $\Delta_f(\varphi)$ the set of divergence points for f with respect to φ defined by

$$\Delta_f(\varphi) = \left\{ z \in J : \lim_{n\to\infty} \frac{1}{n} \sum_{k=0}^{n-1} \varphi(f^k(z)) \text{ does not exist} \right\}. \tag{4.1}$$

It follows immediately from Birkhoff's Ergodic Theorem that $\mu(\Delta_f(\varphi)) = 0$ for every $\mu \in \mathcal{M}$, thus $\Delta_f(\varphi)$ is a small set in the sense of measure. However, we show that the Hausdorff dimension is actually as large as possible. We then infer from this result that the set of points which do not lie in the basin of any measure is as large as possible.

We will need the following theorem from [FLW].

Theorem 4.0.4. *Let K be a repeller of an expanding, $C^{1+\delta}$-conformal toplogical mixing map g. Let $\Phi : K \to \mathbb{R}^d$ be a continuous function. Then either*

(i) all points $x \in K$ have the same ergodics limit; or

(ii) the set of points x such that the limit defining $\alpha(x)$ does not exist is of the same Hausdorff dimension as that of K.

Now we can state our first result.

Theorem 4.0.5. *Let Λ be a hyperbolic set of a rational map f such that $f^n|_\Lambda$ is topologically mixing for some $n \in \mathbb{N}$. Let $\varphi \in C(\Lambda, \mathbb{R})$ such that $\{\int \varphi d\mu : \mu \in \mathcal{M}(\Lambda, f)\}$ is not a singleton. Then* $\dim_H \Delta_f(\varphi) \geq \dim_H \Lambda$.

Proof. Note that since f is a rational hyperbolic map, it is C^∞-conformal; that is, the derivatives of f are the same in every direction. Then f^n is also C^∞-conformal. The result then follows from Theorem 4.0.4 with $K = \Lambda$ and $g = f^n$. □

We now denote by Δ the set of points which are not typical for any invariant measure defined by

$$\Delta = \{z \in J : z \notin \mathcal{B}(\mu) \text{ for all } \mu \in \mathcal{M}\}.$$

We have the following result concerning the dimension of Δ.

Theorem 4.0.6. *Let f be a rational map on $\overline{\mathbb{C}}$ with Julia set J. Then $\dim_H \Delta \geq d(f)$.*

Proof. In [PU] it is shown that $d(f)$ can be approximated by the Hausdorff dimension of hyperbolic sets contained in J such that for each hyperbolic set a certain iterate is conjugate to a full shift. Let $\varepsilon > 0$. Then there exist a hyperbolic set Λ of f and $n \in \mathbb{N}$ such that $f^n|_\Lambda$ is conjugate to a full shift and

$$\dim_H \Lambda \geq d(f) - \varepsilon.$$

Note that Λ is also a hyperbolic set of f^n. Let z_1, z_2 be distinct fixed points of $f^n|_\Lambda$. The existence of such points is due to the fact that a full shift always has at least 2 fixed points. Let $\varphi \in C(J, \mathbb{R})$ such that $\varphi(z_1) \neq \varphi(z_2)$. Let μ_i, $i = 1, 2$, be the δ-Dirac measures on z_i. Then the measures μ_i are invariant probability measures with respect to f and $\int \varphi d\mu_1 \neq \int \varphi d\mu_2$. Applying Theorem 4.0.5 to f^n gives that

$$\dim_H \Delta_{f^n}(\varphi) \geq \operatorname{hypdim}(f^n) - \varepsilon. \tag{4.2}$$

We know

$$\dim_H \Delta_f(\varphi) \geq \dim_H \Delta_{f^n}(\varphi) \tag{4.3}$$

since if the sums in (4.1) do not converge for f^n they will not converge for f. In addition, since every hyperbolic set of f is a hyperbolic set of f^n we obtain

$$\operatorname{hypdim}(f^n) - \varepsilon \geq \operatorname{hypdim}(f) - \varepsilon = d(f) - \varepsilon. \tag{4.4}$$

Now combining (4.2), (4.3), and (4.4) and noting that $\Delta_f(\varphi) \subset \Delta$ gives the desired result. □

Finally, we state the following corollary.

Corollary 4.0.7. *Let f be an NCP rational map on $\overline{\mathbb{C}}$ with Julia set J. Then* $\dim_H \Delta = \dim_H J$.

Proof. This follows immediately from Theorem 4.0.6 and the fact that for NCP maps $\mathrm{hypdim}(f) = \dim_H J$ (see [U]). □

Thus we see that there is still a large set of points whose dynamics we do not know. While this result is somewhat negative, the technique employed here of exhausting the Julia set from the inside by hyperbolic sets could be potentially very helpful in further research.

CHAPTER 5

DEPENDENCE ON PARAMETERS

In this chapter we consider stable rational maps and study the dependence of the dimension on parameters. We begin by introducing some necessary notation.

Let $(f_a)_{a\in A}$ be a family of rational maps where A is an open connected subset of $\mathbb{C}^{2d+1}$. We say that $(f_a)_{a\in A}$ is a holomorphic family of rational maps if there is a holomorphic map $F : A \times \overline{\mathbb{C}} \to \overline{\mathbb{C}}$ such that $f_a = F(a, \cdot)$ for all $a \in A$. Since A is connected it follows that $\deg f_a$ is constant in A. We will denote by J_a the Julia set of the rational map f_a. We say that $(f_a)_{a\in A}$ is stable (or also J-stable) if for each $a_0 \in A$ there is $r > 0$ such that if $|a - a_0| < r$ then $f_a|J_a$ and $f_{a_0}|J_{a_0}$ are topologically conjugate. Recall that a family of maps is structurally stable if nearby maps are topologically conjugate. Clearly, if $(f_a)_{a\in A}$ is a structurally stable holomorphic family of rational maps then $(f_a)_{a\in A}$ is also stable. We can thus conclude immediately from [MSS] that stable rational maps form an open and dense subset of the space of all rational maps. When dealing with a family $(f_a)_{a\in A}$ of rational maps we will write $\mathcal{M}(a)$ for the space $\mathcal{M}(J_a, f_a|J_a)$.

5.1 Preliminary definitions and propositions

We now introduce the concept of holomorphic motions.

Definition 5.1.1. *Let $K \subset \overline{\mathbb{C}}$ with $\left|\overline{\mathbb{C}} \setminus K\right| \geq 3$, and $D = D(0, 1)$ be the unit disk in $\overline{\mathbb{C}}$. A holomorphic motion of K is a mapping $h : D \times K \to \overline{\mathbb{C}}$ such that the following holds:*

(i) $h(0, \cdot) = id_K$;

(ii) $h(t, \cdot)$ *is one-to-one for all* $t \in D$;

(iii) $h(\cdot, z)$ *is holomorphic for all* $z \in K$.

For our purposes, we consider t to be a complex time parameter.

Let X, Y be metric spaces. We say that a bijective map $f : X \to Y$ is an *α-Hölder homeomorphism* if both f and f^{-1} are Hölder-continuous with Hölder exponent α.

We are now prepared to state our first proposition, making use of many previously known results in our proof.

Proposition 5.1.2. *Let $h : D \times K \to \overline{\mathbb{C}}$ be a holomorphic motion of K and let $t \in D$ with $|t| < \frac{1}{3}$. Then $h(t, \cdot)$ is an α-Hölder homeomorphism from K to $h(t, K)$ with $1 \geq \alpha \geq (1 - 3|t|)/(1 + 3|t|)$. Moreover, for all $F \subset K$ we have*

$$|\dim_H F - \dim_H h(t, F)| \leq \frac{6|t|}{1 - 3|t|}. \tag{5.1}$$

Proof. The holomorphic motion h can be extended to a holomorphic motion of $\mathbb{C}$ (see [S]). On the other hand, it follows from the λ-lemma [MSS] that $h(t, \cdot)$ is a $\beta(|t|)$-quasiconformal homeomorphism. The result of [BR] implies that if $|t| < \frac{1}{3}$ then

$$1 \leq \beta(|t|) \leq \frac{1 + 3|t|}{1 - 3|t|}.$$

Now we can apply the Mori inequality (see [A]) to infer that $h(t, \cdot)$ is a $\beta(|t|)^{-1}$-Hölder homeomorphism. Thus our inequality (5.1) follows immediately from the change of the Hausdorff dimension under Hölder continuous maps. □

Our second proposition will be essential to our study of the dependence of dimension on parameters.

Proposition 5.1.3. *Let $D = D(0, 1)$ and let $(f_a)_{a \in D}$ be a holomorphic family of stable rational maps. Then there is a family of mappings $(T_a)_{a \in D}$, where each T_a is a bijection from $\mathcal{M}_E^+(0)$ to $\mathcal{M}_E^+(a)$, such that:*

(1) $(f_0 | J_0, \mu_0)$ and $(f_a | J_a, T_a(\mu_0))$ are measure-theoretically isomorphic for all $\mu_0 \in \mathcal{M}_E^+(0)$ and all $a \in D$,

(2) For all $\mu_0 \in \mathcal{M}_E^+(0)$ the map $a \mapsto \chi(T_a(\mu_0))$ is harmonic in A.

Proof. It follows from the result of [MSS] that there exists a holomorphic motion $h : D \times J_0 \to \overline{\mathbb{C}}$ that preserves the dynamics of $f_a|J_a$. In particular, we know:

(i) $h(0, \cdot) = id_{J_0}$;

(ii) for all $a \in D$ the map $h(a, \cdot)$ is a homeomorphism from J_0 to J_a such that $f_a|J_a \circ h(a, \cdot) = h(a, \cdot) \circ f_0|J_0$;

(iii) $h(\cdot, z)$ is holomorphic for all $z \in J_0$.

For $a \in D$ we define

$$T_a(\mu_0) = h(a, \cdot)_* \mu_0,$$

where $h(a, \cdot)_* \mu_0(B) = \mu_0(h(a, \cdot)^{-1}(B))$ for all Borel sets $B \subset J_a$. Obviously, $T_a(\mu_0) \in \mathcal{M}_E(a)$ for all $\mu_0 \in \mathcal{M}_E^+(0)$ and all $a \in D$. Moreover, it follows directly from the definition that $(f_0|J_0, \mu_0)$ and $(f_a|J_a, T_a(\mu_0))$ are measure-theoretically isomorphic for all $\mu_0 \in \mathcal{M}_E^+(0)$ and all $a \in D$ so we have property (1). Thus,

$$h_{\mu_0}(f_0) = h_{T_a(\mu_0)}(f_a)$$

for all $\mu_0 \in \mathcal{M}_E^+(0)$ and all $a \in D$. In particular, $T_a(\mathcal{M}_E^+(0)) \subset \mathcal{M}_E^+(a)$, and T_a is well-defined. We can now conclude from the definition that T_a is a bijection from $\mathcal{M}_E^+(0)$ to $\mathcal{M}_E^+(a)$.

To show property (2), consider $\mu_0 \in \mathcal{M}_E^+(0)$. For $a \in D(0,1)$ we have

$$\chi(T_a(\mu_0)) = \int \log |f_a' \circ h(a, \cdot)| d\mu_0 \qquad (5.2)$$

(see, for example, [M]). By property (iii) of the holomorphic motion h if $z \in J_0$ and z is not a critical point of f_0 then the map $a \mapsto f_a'(h(a, z))$ is a non-zero holomorphic function. Therefore, the map $a \mapsto \log |f_a'(h(a, z))|$ is harmonic. Note that c_0 is a critical point of f_0 if and only if $h(a, c_0)$ is a critical point of f_a. Since the maps f_a have only finitely many points and since they form a set of $\mu_0(T_a)$ measure zero, we conclude from (5.2) that $a \mapsto \chi(T_a(\mu_0))$ is harmonic. □

Applying Proposition 5.1.3 and the definition of the basin of a measure gives the following corollary.

Corollary 5.1.4. *Let $(f_a)_{a\in A}$ and $(T_a)_{a\in A}$ be as in Proposition 5.1.3. Then $h(a, \mathcal{B}(\mu_0)) = \mathcal{B}(T_a(\mu_0))$ for all $\mu_0 \in \mathcal{M}_E^+(0)$ and all $a \in D$.*

Let μ_f denote the unique measure of maximal entropy of $f \in Rat_d$ (see [L]). It was shown in [D] by using methods from pluripotential theory that if $(f_a)_{a\in A}$ is a holomorphic family of rational maps then $a \mapsto \chi(\mu_f)$ is pluriharmonic if and only if $(f_a)_{a\in A}$ is stable. Proposition 5.1.3 provides a new proof for one of the inclusions of this result.

Corollary 5.1.5. *Let $(f_a)_{a\in A}$ be a holomorphic family of stable rational maps. Then the map $a \mapsto \chi(\mu_{f_a})$ is pluriharmonic in A.*

Proof. We use the notation of Proposition 5.1.3. Consider a fixed rational map $f_0 \in \{f_a : a \in A\}$ and let L be a complex line in $\mathbb{C}^{2d+1}$ containing f_0. By Proposition 5.1.3 the map $a \mapsto \chi(\mu_{f_0})$ is harmonic in a neighborhood of 0 in L. Since μ_{f_0} is the unique measure of maximal entropy of f_0 it follows that $h_{\mu_{f_0}}(f_0) = \log(\deg f_0)$. Note that the topological entropy is constant in a stable holomorphic family of rational maps. Therefore, Proposition 5.1.3 combined with the uniqueness of the measure of maximal entropy implies that $T_a(\mu_{f_0}) = \mu_{f_a}$. This completes the proof. □

5.2 Dependence on parameters

We now employ 5.1.3 to show that the value $d(f_a)$ identified in Theorem 2.5.1 and the Hausdorff dimension of the Julia set depend continuously on the parameter a.

Theorem 5.2.1. *Let $(f_a)_{a\in A}$ be a holomorphic family of stable rational maps. Then the maps $a \mapsto d(f_a)$ and $a \mapsto \dim_H J_a$ are continuous in A.*

Proof. Let $f_{a_0} \in \{f_a : a \in A\}$ and let $r > 0$ such that $f_{a_0} \in B(0, r) \subset A$. Without loss of generality we can rescale and translate to assume $a_0 = 0$ and $r = 1$. Consider a complex line

L in $\mathbb{C}^{2d+1}$ containing f_0. Then $L \cap A$ can be identified with $D = D(0,1)$. We first prove the statement for the map $a \mapsto d(f_a)$ by using the identity $d(f) = DD(f)$. From Proposition 5.1.3 and [MSS] it follows that there exists a holomorphic motion $h : D \times J_0 \to \overline{\mathbb{C}}$. Let $(T_a)_{a \in D}$ be the family of mappings associated with $(f_a)_{a \in D}$, let $\mu_0 \in \mathcal{M}_E^+(0)$, and let $|a| < 1/3$. It follows from Proposition 5.1.2, that for every measurable set $F \in J_0$, equation (5.1) holds:

$$|\dim_H F - \dim_H h(t, F)| \leq \frac{6|t|}{1 - 3|t|}.$$

It can be shown there exist sets $G \in J_0$ and $F \in h(a, J_0)$ of full measure that attain the Hausdorff dimension of μ_0 and $T_a(\mu_0)$, respectively. One replaces $\dim_H(G)$ with $\dim_H(\mu_0)$, respectively $\dim_H(F)$ with $\dim_H(T_a(\mu_0))$, in equation (5.1), and notes that

$$T_a(\mu_0)(h(a, G)) = \mu_0(h^{-1}(a, F)) = 1.$$

Since the map $\mu_0 \mapsto T_a(\mu_0)$ is a measure theoretic isomorphism we have

$$|\dim_H \mu_0 - \dim_H T_a(\mu_0)| \leq \frac{6|a|}{1 - 3|a|}. \tag{5.3}$$

Note that the right-hand side of (5.3) only depends on $|a|$ and not on L. Thus, by using the fact that T_a is a bijection from $\mathcal{M}_E^+(0)$ to $\mathcal{M}_E^+(a)$ we may conclude that

$$|DD(f_0) - DD(f_a)| \leq \frac{6|a|}{1 - 3|a|}.$$

Since $d(f_a) = DD(f_a)$ it follows that $a \mapsto d(f_a)$ is continuous. The continuity of the map $a \mapsto \dim_H J_a$ follows from a similar argument using the statements (i),(ii) and (iii) in the proof of Proposition 5.1.3 and (5.1). □

Finally, we present the main result of this section.

Theorem 5.2.2. *Let $(f_a)_{a \in A}$ be a holomorphic family of stable rational maps. Then the map $a \mapsto d(f_a)$ is plurisubharmonic in A.*

Proof. Let $f_0 \in A$ and let L be a complex line in $\mathbb{C}^{2d+1}$ containing f_0. Then there exists a holomorphic family $(f_a)_{a \in D}$, where $D = D(0,1) \subset \mathbb{C}$ such that $\{f_a : a \in D\}$ is a neighborhood of f_0 in $L \cap A$. Let $(T_a)_{a \in D}$ be the family of maps in Proposition 5.1.3 associated with $(f_a)_{a\in D}$. Proposition 5.1.3 implies that $h_{\mu_0}(f_0) = h_{T_a(\mu_0)}(f_a)$ for all $\mu_0 \in \mathcal{M}_E^+(0)$ and all $a \in D$. Moreover, the functions $a \mapsto \chi(T_a(\mu_0))$ are harmonic in D. Note that $x \mapsto x^{-1}$ is a convex function. This implies that the functions $a \mapsto \chi(T_a(\mu))^{-1}$ are subharmonic in D. The continuous function $a \mapsto d(f_a)$ is therefore given by the supremum over a family of subharmonic functions. We conclude that the function $a \mapsto d(f_a)$ is subharmonic in D. This completes the proof. □

This result is particularly significant due to the fact that stable maps are open and dense in the set of all rational maps. Thus, while the techniques applied are quite technical, the final results have several applications in the further study of rational maps.

BIBLIOGRAPHY

BIBLIOGRAPHY

[A] L. Ahlfors, *Lectures on Quasiconformal Mappings*, Van Nostrand, New York, 1966.

[B] R. Bowen, Topological Entropy for Noncompact Sets, *Transactions of the American Mathematical Society*, 184, (1973), 125–136.

[BR] L. Bers and H. L. Royden, Holomorphic Families of Injections, *Acta Math.*, 157, (1986), no. 3-4, 259–286.

[BW] L. Barreira and C. Wolf, Pointwise Dimension and Ergodic Decompositions, *Ergod. Th. and Dynam. Syst.*, (2006), to appear.

[CG] L. Carleson and T. Gamelin, *Complex Dynamics*, Springer, 1993.

[D] L. DeMarco, Dynamics of Rational Maps: Lyapunov Exponents, Bifurcations, and Capacity, *Math. Ann.*, 326, (2003), no. 1, 43–73.

[DGS] M. Denker, C. Grillenberger and K. Sigmund, Ergodic Theory on Compact Spaces, *Lect. Notes in Math*, 527, Springer, 1976.

[DU] M. Denker and M. Urbanski, On Sullivan's Conformal Measures for Rational Maps on the Riemann Sphere, *Nonlinearity*, 4, (1991), 365–384.

[HK] B. Hasselblatt and A. Katok, *Introduction to the Modern Theory of Dynamical Systems*, Cambridge University Press, Cambridge, 1995.

[IKW] W. Ingle, J. Kaufmann and C. Wolf, Physical Measures, Divergence Points and Dynamical Dimension for Rational Maps, in preparation.

[FLW] D. Feng, K. Lau and J. Wu, Ergodic Limits on the Conformal Repellers, *Advances in Mathematics*, 169, 58–91

[K] A. Khinchine, *Mathematical Foundations of Information Theory*, Dover Publications, New York, 1957.

[L] M. Lyubich, Entropy Properties of Rational Endomorphisms of the Riemann Sphere, *Ergod. Th. and Dynam. Syst.*, 3, (1983), 351–385.

[Ma] R. Mané, The Hausdorff Dimension of Invariant Probabilities of Rational Maps, *Dynamical Systems*, Valparaiso, 1986, *Lect. Notes in Math*, 1331, Springer-Verlag, (1988), 86–117.

[Mn] A. Manning, A Relation between Lyapunov Exponents, Hausdorff Dimension and Entropy, *Ergod. Th. and Dynam. Sys.*, 1, (1981), 451–459.

[M] P. Mattila, *Geometry of Sets and Measures in Euclidean Spaces. Fractals and Rectifiability*, Cambridge University Press, Cambridge, U.K., 1995.

[Mc] C. McMullen, Frontiers in Complex Dynamics, *Bull. of the AMS*, (2), 31, (1994), 155–172.

[MSS] R. Mané, P. Sad and D. Sullivan, On the Dynamics of Rational Maps, *Ann. Sci. ole Norm. Sup.*, (4), 16, (1983), no. 2, 193–217.

[P] Y. Pesin, *Dimension Theory in Dynamical Systems: Contemporary Views and Applications*, University of Chicago Press, 1997.

[PU] F. Przytycki and M. Urbanski, *Fractals in the Plane–the Ergodic Theory Methods*, Cambridge University Press, pre-print.

[R] C. Robinson, *Dynamical Systems: Stability, Symbolic Dynamics, and Chaos*, CRC Press, 1999.

[Ru] D. Ruelle, Repellers for Real Analytic Maps, *Ergod. Th. and Dynam. Sys.*, 2, (1982), no. 1, 99–107.

[Sh] M. Shishikura, The Hausdorff Dimension of the Boundary of the Mandelbrot Set and Julia Sets, *Ann. of Math*, 147, (1998), 225–267.

[S] Z. Slodkowski, Holomorphic Motions and Polynomial Hulls, *Proc. Amer. Math. Soc.*, 111, (1991), no. 2, 347–355.

[U] M. Urbanski, Measures and Dimensions in Conformal Dynamics, preprint.

[W] P. Walters, *An Introduction to Ergodic Theory*, Springer, 1981

[Wo] C. Wolf, Generalized Physical and SRB Measures for Hyperbolic Diffeomorphisms, *Journal of Statistical Physics*, (2006), online.