CRACK DETECTION IN A THREE DIMENSIONAL BODY

A Thesis by

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I have examined the final copy of this Thesis for form and content and recommend that it be accepted in partial fulfillment of the requirements for the degree of Master of Science in Mathematics.

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ABSTRACT

We propose a method of analyzing a crack in a three dimensional body. We treat the problem as an inverse problem and apply Green's Theorem, Trace Theorem, and the Fredholm Alternative. We model the problem using Helmholtz equation.

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CHAPTER 1

INTRODUCTION

An important issue in product reliability and cost control is the identification of material defects from overdetermined data. A source of a physical field is applied to a portion of the boundary and measurements of this field are taken at this portion. The inverse problem is to detect and characterize defects such as cracks in the material. Current work in this area includes the investigation of multiple cracks, detection of a large number of small cracks, and the growth of multiple small cracks into one relatively large crack.

Some work has been completed in this area. For buried cracks, Friedman and Vogelius [6] proved that two special boundary measurements are needed for unique identification. In 1993, Alessandrini, Beretta, and Vessela [1] proved Lipschitz stability for linear cracks. Andrieux and Abda [3] used a special heat flux to solve the two dimensional inverse crack problem using the Laplace equation in the inverse boundary value problem. Eller [5] models a 3D body with a 2D crack using the Laplace equation. Eller proves the crack can be identified uniquely by boundary measurements. The above methods do not work [9] for the Helmholtz equation.

In this paper we will investigate a two-dimensional crack in a three-dimensional body. A body with a simply connected surface will be assumed with some defect (crack) starting at the surface and terminating inside the body. The Helmholtz equation will be used in this inverse boundary value problem. By using the Helmholtz equation, the test function is not limited to static functions but can in fact be a wave function. These wave functions include (but are not limited to) acoustic and electromagnetic fields.

CHAPTER 2

DOMAINS AND SPACES

2.1 Domains

Let $\Omega \subset \mathbb{R}^3$ be a simply connected bounded open domain with boundary $\partial\Omega$. Define $C^1(\Omega)$ the space of all one time continuously differentiable functions. Let $\partial\Omega \in C^1$ be piecewise smooth with nonzero interior angles. The outward normal on $\partial\Omega$ will be designated ν. Let

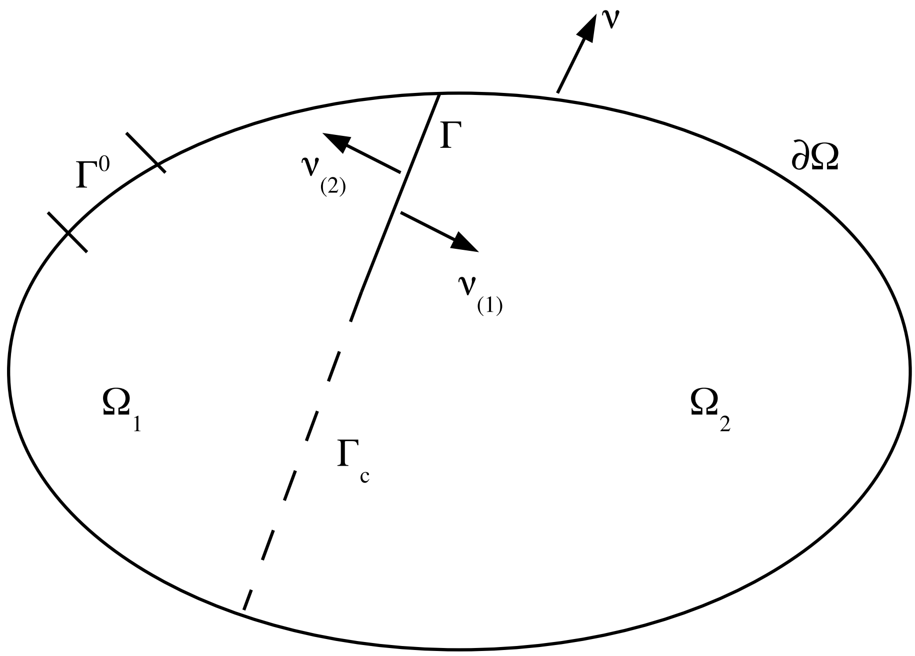

Figure 2.1: Boundary Definition

$\Gamma \in C^1$ [8] be a simply connected surface with no self intersections intersecting $\partial\Omega$ at some non-zero angle α. Γ will represent the crack, and Γ_c will be a complement of the crack.

Define $\Gamma^* = \Gamma \cup \Gamma_c$ and assume that $\Omega_1 \cap \Omega_2 = \emptyset$. Γ divides Ω into two domains designated Ω_1 and Ω_2 so that $\Omega = \Omega_1 \cup \Omega_2 \cup \Gamma^*$ and $\overline{\Omega_1} \cap \overline{\Omega_2} = \Gamma^*$.

The outward normal of the boundary of Ω_1, measured along Γ^*, is defined $\nu(1)$. On Γ^* of Ω_2, the outward normal is defined as $\nu(2)$. Hence along $\Gamma, \nu(1) + \nu(2) = 0$.

2.2 Sobolev Spaces

Let Ω be an open set of $\mathbb{R}^3$. Define $H^1(\Omega)$ the set of all square integrable functions, together with all first derivatives in Ω.

We introduce a Hilbert norm in $H^s_{(\Omega)}$:

$$\|u\|_{H^s_{(\Omega)}} = \left(\sum_{|\alpha| \leq s} \|D^\alpha u\|^2_{L^2_{(\Omega)}} \right)^{\frac{1}{2}} \tag{2.1}$$

for all integers s, such that $s \geq 0$.

The space of smooth functions:

$$C^\infty(\overline{\Omega \backslash \Gamma}) = \left\{ \begin{array}{c} u : u(1) \in C^\infty(\overline{\Omega_1}) \text{ and } u(2) \in C^\infty(\overline{\Omega_2}) \\ \gamma(1)\partial^\alpha u = \gamma(2)\partial^\alpha u \text{ on } \Gamma_c \text{ for all } \alpha \end{array} \right\}.$$

The Sobolev space $H^1(\Omega\backslash\Gamma)$ is defined as the completion of $C^\infty(\overline{\Omega\backslash\Gamma})$ with respect to the norm (2.1).

2.3 Trace Theorem

Let Ω_i have a Lipschitz boundary. Then the mapping

$$\gamma : u \to u|_{\partial\Omega_i}$$

which is defined for all $u \in C^\infty(\overline{\Omega_i})$ has a unique continuous extension as an operator from $H^1(\Omega_i)$ onto $H^{\frac{1}{2}}(\partial\Omega_i)$. This operator has a continuous right inverse. In particular, there exists a constant $C = C(\Omega)$ such that

$$\|u\|_{(0)}(\partial\Omega) \leq C\|u\|_{(1)}(\Omega), u \in H^1(\Omega). \tag{2.2}$$

2.4 Jump Function

Let γ denote the trace operator on $\partial\Omega$. Ω_1 has Lipschitz boundary, [8] therefore the mapping $\gamma(1) : u \to u|_{\partial\Omega_1}$ for $u \in C^\infty(\overline{\Omega_1})$ has a unique continuous extension from $H^1(\Omega_1)$ onto

$H^{\frac{1}{2}}(\partial\Omega_1)$. By similarity, $\gamma(2) : u \to u|_{\partial\Omega_2}$. When the traces $\gamma(1)u$ and $\gamma(2)u$ are well defined, the jump across Γ will be denoted [5] $[u] = \gamma(1)u - \gamma(2)u$.

2.5 Green's Formula

Let Ω have Lipschitz boundary $\partial\Omega$. For all $u, v \in H^1(\Omega)$ we have:

$$\int_\Omega \partial_j uv + \int_\Omega u\partial_j v = \int_{\partial\Omega} \gamma u\gamma v\nu_j.$$

Using this formula we can prove Green's formula. Let Ω be a Lipschitz domain and $u \in H^2(\Omega)$, $v \in H^1(\Omega)$. Then

$$\int_\Omega v\Delta u + \int_\Omega v\nabla u\nabla v = \int_{\partial\Omega} \gamma v\gamma\partial_\nu u.$$

2.6 Fredholm Alternative

We use Theorem 5.3 from Gilbarg and Trudinger [7]. Consider the problem

$$L_\lambda(u) : -\Delta u - \lambda u = f, \text{ in } \Omega \tag{2.3}$$

$$\partial_\nu u = g_1 \text{ on } \partial\Omega.$$

For each $\lambda \in \mathbb{R}$ exactly one of the following statements holds:

EITHER

1) For each $f \in L^2(\Omega)$, $g_1 \in L^2(\partial\Omega)$, there exists a unique weak solution of the boundary value problem (2.3). This is true in particular for all $\lambda < 0$.

OR

2) There exists a weak solution $u_0 \neq 0$ of the homogeneous problem

$$L_\lambda(u) : -\Delta u_0 - \lambda u_0 = 0, \text{ in } \Omega \tag{2.4}$$

$$\partial_\nu u_0 = 0 \text{ on } \partial\Omega.$$

3) A solution to (2.3) exists if and only if:

$$\int_\Omega f u_0 + \int_{\partial\Omega} u_0 g_1 = 0$$

for any u_0 solving (2.4).

For example: Let $\lambda = 0$. The boundary value problem (2.3) becomes $-\Delta u_0 = f$ in $\Omega, \partial_\nu u = g_1$ on $\partial\Omega$.

The Fredholm alternative states:

Either there is a unique solution, or there is a weak solution to the homogeneous equation:

$$-\Delta u_0 = 0 \text{ in } \Omega,\ \partial_\nu u = g_1 \text{ on } \partial\Omega.$$

CHAPTER 3

RESULTS

3.1 The Direct and Inverse Problem

Let us consider the following boundary value problem:

$$Lu \equiv \Delta u + k^2 u = 0 \text{ in } \Omega, \tag{3.1}$$

$$\partial_\nu u = 0 \text{ on } \Gamma, \tag{3.2}$$

$$u = g_0 \text{ on } \partial\Omega. \tag{3.3}$$

Equations (3.1), (3.2), and (3.3) model a crack Γ in the domain $\Omega \subset \mathbb{R}^3$. The inverse problem we are interested in is to find Γ given the additional boundary data

$$\partial_\nu u = g_1 \text{ on } \partial\Omega. \tag{3.4}$$

3.2 Weak Solution

Define a weak solution to the boundary value problem (3.1), (3.2), and (3.4) a function $u \in H^1(\Omega \backslash \Gamma)$ satisfying:

$$0 = \int_{\partial\Omega} g_1 v d\omega - \int_\Omega (\nabla u \nabla v - k^2 uv) dx, \tag{3.5}$$

where $v \in H^1(\Omega \backslash \Gamma)$ is a test function.

Per Ladyzhenskaya and Ural'tseva [10], "the definition that we have given for a generalized solution is indeed an extension of the classical concept of a solution". Rewrite equation (3.5),

$$\int_\Omega (\nabla u \nabla v - k^2 uv) dx = \int_{\partial\Omega} g_1 v d\omega.$$

It is known ([10] p.162) that the Fredholm alternatives hold for (3.1), (3.2), and (3.4). There is a unique solution u to equation (3.5) for sufficiently large $k^2 \neq k_1, \ldots, k_m, \ldots$, where $\lim_{m\to\infty} k_m = \infty$. k_m are eigenvalues.

3.3 Calculations

Now apply Green's theorem to $\int_\Omega (\nabla u \nabla v) dx$. Since Green's formula only applies to [8] bounded Lipschitz domains, use it in Ω_1 and Ω_2. Let $v \in H^2(\Omega)$.

$$\begin{aligned} \int_{\Omega_1} (\nabla u \nabla v) dx &= \int_{\partial\Omega_1} \partial_\nu v u d\omega + \int_{\Omega_1} \Delta v u dx \\ \int_{\Omega_2} (\nabla u \nabla v) dx &= \int_{\partial\Omega_2} \partial_\nu v u d\omega + \int_{\Omega_2} \Delta v u dx \end{aligned} \tag{3.6}$$

Substitute these back into (3.5):

$$\begin{aligned} 0 = &\int_{\partial\Omega_1} (v \partial_\nu u - \partial_\nu v u) d\omega + \int_{\Omega_1} u(\Delta v + k^2 v) dx + \\ &+ \int_{\partial\Omega_2} (v \partial_\nu u - \partial_\nu v u) d\omega + \int_{\Omega_2} u(\Delta v + k^2 v) dx. \end{aligned} \tag{3.7}$$

Expand the $\int_{\partial\Omega_1}$ and $\int_{\partial\Omega_2}$ terms. The three component parts are $\partial\Omega$, Γ, and Γ_c. On $\partial\Omega_1$ and $\partial\Omega_2$ we are left with:

$$\int_{\partial\Omega} (g_1 v - \partial_\nu v u)\, d\omega. \tag{3.8}$$

On the crack boundary Γ, the integral becomes:

$$\int_\Gamma \left(-\partial_{\nu(1)} u \gamma_1 v - \partial_{\nu(2)} u \gamma_2 v\right) d\omega + \int_\Gamma \left(-\partial_{\nu(2)} v \gamma_2 u - \partial_{\nu(1)} v \gamma_1 u\right) d\omega.$$

Since $\partial_{\nu(1)} u = 0$ and $\partial_{\nu(2)} u = 0$ we are left with:

$$\int_\Gamma \left(-\partial_{\nu(2)} v \gamma_2 u - \partial_{\nu(1)} v \gamma_1 u\right) d\omega. \tag{3.9}$$

Simplifying this with the expressions $[u] = \gamma(1)u - \gamma(2)u$, and $\partial_{\nu(2)} = -\partial_{\nu(1)}$ we can write this term using the jump function.

$$\int_\Gamma [u]\, \partial_{\nu_{(1)}} v d\omega \tag{3.10}$$

The final boundary part is Γ_c and it looks like this:

$$\int_{\Gamma_c} \left(\gamma_2 v \partial_{\nu(2)} u + \gamma_1 v \partial_{\nu(1)} u\right) - \left(\gamma_2 u \partial_{\nu(2)} v + \gamma_1 u \partial_{\nu(1)} v\right) d\omega. \tag{3.11}$$

Since $v \in H^1(\Omega)$, and $u \in H^1(\Omega)$, we have the identities $\gamma_2 v = \gamma_1 v$, and $\gamma_2 u = \gamma_1 u$ on Γ_c. By the definition of our trace operator, all terms in the above expression cancel out.

From equation (3.1), $\Delta v + k^2 v = 0$. Substituting expressions (3.8) and (3.10) back into (3.7) leaves us with:

$$0 = \int_{\partial\Omega} (g_1 v - \partial_\nu v u)\, d\omega + \int_{\Gamma} [u]\, \partial_\nu v d\omega, \tag{3.12}$$

provided $\Delta v + k^2 v = 0$ in Ω, and $v \in H^2(\Omega)$.

3.4 Results

Let Γ^0 be that portion of $\partial\Omega$ where the test function is applied and $\partial_\nu u$ is measured. Let

$$\Delta v + k^2 v = 0 \text{ in } \Omega,$$

$$v \in H^2(\Omega),$$

$$\partial_\nu v = 0 \text{ on } \partial\Omega \backslash \Gamma^0.$$

Equation (3.12) becomes:

$$\int_{\Gamma} [u]\, \partial_\nu v d\omega = \int_{\Gamma^0} (\partial_\nu v u - g_1 v) d\omega. \tag{3.13}$$

Let $\Gamma^0 = \partial\Omega$.

Case 1: $v = \sin(k\xi \cdot x)$ where $|\xi|$=1.

We compute the following: $\partial_\nu v = \nabla v \cdot \nu = k \cos(k\xi \cdot x)\xi \cdot \nu$

Equation (3.13) becomes:

$$k \int_{\Gamma} [u] \cos(k\xi \cdot x)\xi \cdot \nu d\omega = \int_{\partial\Omega} [k \cos(k\xi \cdot x)\xi \cdot \nu u - g_1 \sin(k\xi \cdot x)]\, d\omega. \tag{3.14}$$

We see that for a plane crack,

$$\xi \cdot \nu = 0 \text{ when } \xi \text{ is parallel to } \Gamma. \tag{3.15}$$

Case 2: $v = \cos(k\xi \cdot x)$ where $|\xi|$=1.

We compute the following: $\partial_\nu v = \nabla v \cdot \nu = -k \sin(k\xi \cdot x)\xi \cdot \nu$

Equation (3.13) becomes:

$$-k \int_{\Gamma} [u] \sin(k\xi \cdot x)\xi \cdot \nu d\omega = -\int_{\Gamma^0} [k \cos(k\xi \cdot x)\xi \cdot \nu u + g_1 \sin(k\xi \cdot x)]\, d\omega. \tag{3.16}$$

Case 3: $k = 0$ The boundary value problem, equations (3.1), (3.2), (3.4) become:

$$\Delta u = 0,$$

$$\partial_\nu u = 0 \text{ on } \Gamma,$$

$$\partial_\nu u = g_1 \text{ on } \partial\Omega.$$

Use test function v such that $\nabla v = \xi$ where $|\xi| = 1$. Equation (3.13) becomes:

$$\int_\Gamma [u]\,(\xi \cdot \nu) d\omega = \int_{\partial\Omega} [\xi u - g_1(\xi \cdot \nu)]\, d\omega. \tag{3.17}$$

The right hand side is comprised of known functions and can be computed. Again, if Γ is a plane crack and parallel to ξ,

$$\int_{\partial\Omega} [\xi u - g_1(\xi \cdot \nu)]\, d\omega = 0.$$

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