

# Translating Vortex Pairs with Prescribed Profiles

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**Abstract.** We generate translating vortex pairs with smooth or more arbitrary profiles that reflect modern vortex pairs being generated using prescribed domains of vorticity. Instead of prescribing the domain, we fix the area of the vorticity and iteratively generate the vortex by prescribing the profile function defining the vorticity, which as a side effect produces a domain of vorticity dependent on the profile function and area. A large class of previously developed and new translating vortex pairs is developed for almost arbitrary vortex profiles using this methodology. This is further enhanced to produce solutions for flows on a rotating sphere.

## 1. Introduction

Two-dimensional, incompressible, inviscid fluid flow is generally represented by a stream function. Streamlines represent paths the flow follows, along which the stream function is constant. We follow the methodology in [1], which finds vortical flow past a cylinder (a circle in 2D flow). Vorticity of a flow represents the rotation of the fluid. The vorticity is determined by the Laplacian ( $\Delta$ ) operator applied to the stream function, which in general is a function (the profile) of the value of the stream function at a point. We generalize the approach in [1] in which vorticity is constant inside the vortex region, by making it instead follow a prescribed profile function. In this manner, we can obtain easily more arbitrary vortex regions by finding the profile function, rather than specifying the exact region of vorticity.

The f-plane is the term for flow on a flat surface for which we initially produce results from the method in [1]. In real life though, most flow is on the surface of a rotating sphere, Earth, which is numerically represented by an additional vorticity factor,  $\beta y$ , where  $y$  is the distance from the equator. This surface is called the  $\beta$ -plane, and extending this method to it is also presented.

## 2. Numerical Procedure

As was introduced, we use the methodology from [1].

In general, we make the circular block arbitrarily small so that our solutions represent unobstructed flow, though it can also be applied to flow past a larger cylinder. The general procedure is a two-stage iterative process. In the outer iteration, we solve for a new stream function,  $\psi$ , given the previous stream function, as

$$\Delta\psi_{n+1} = \omega_{n+1} F(\psi_n - \alpha). \quad (1)$$

We solve this in the upper half plane, and the vortex pair is simply the reflection of this solution onto the lower half plane. On the inner iteration,  $\omega_{n+1}$  is varied in order to fix the area of the vortical domain to the prescribed area. The vortical domain is generally given by the region in which  $F(\psi)$  is non-zero. For the profile functions we will be using,  $F(\psi)$  is non-zero in the region where  $\psi < \alpha$ . The stream function is assumed to be zero on the boundary (the bottom edge of the upper half plane), so that when  $\alpha < 0$ , we have a translating vortex dipole that is detached from the center.

To account for the  $\beta$ -plane, we change the form of equation (1) to get similar results by solving

$$\Delta\psi + \beta y = \beta\psi + \omega F(\psi). \quad (2)$$

## 3. Vortex Dipoles

The initial purpose of this research was to duplicate efforts in [2] to create smooth dipoles in the f-plane. By taking  $F(\psi)$  to be twice differentiable, the vorticity will vary smoothly between the rotational and irrotational components of the flow. The general form we take is

$$F(\psi) = H(-\psi) (-\psi)^\lambda.$$

$H(-\psi)$  is the Heaviside function, and will be 1 when  $\psi < 0$ , and zero elsewhere. When  $\lambda > 2$ , we see that  $F$  is twice differentiable. The resulting stream functions are similar to those produced in [2], pictured in figure

1. (The black half ellipse represents the boundary between irrotational and rotational flow, where vorticity is non-zero.)

If we do not require the profile function to be smooth, we can get other results, including the Lamb-Chaplygn dipole, which has a linear profile function corresponding to  $\lambda = 1$ , as shown in figure 2, with a circular boundary.

In general, the vortex region becomes more elongated for smaller  $\lambda$  closer to zero, and is taller for larger  $\lambda$ .

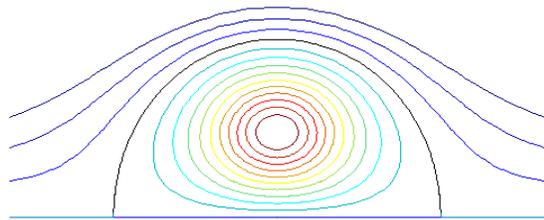


Fig. 1. Streamlines for smooth solution using  $\lambda = 3$

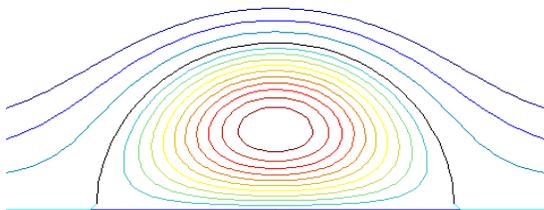


Fig. 2. Circular (Lamb-Chaplygn) dipole using  $\lambda = 1$

Another interesting result comes from matching the steep vortex profile function in [3], shown in figure 3, with a concave vorticity surface in the center using

$$F(\psi) = H(-\psi) (-\delta\psi) e^{\delta\psi}.$$

In the  $\beta$ -plane, the  $\beta\psi$  linear term in smooth solutions creates what is known as a shielded vortex similar to those in [4]. Notably, two opposing rotations inside the vortex flow exist since the vorticity can be both positive and negative inside the vortex region as seen in figure 4 (the black inner circle represents the change in rotation.)

#### 4. Stability and Convergence

Although an arbitrary profile function can be designated in the algorithm, only some of these converge in a reasonable amount of time to a solution. In the examples above, for large  $\lambda$  or  $\delta$ , the convergence will slow. General limits have not been found, however, since the results in general do not become unstable for more dramatic profile functions. Also, the algorithm depends very little on the initial

guess, and will adapt in just a few iterations to almost the right solution even with a bad initial guess for the vorticity domain.

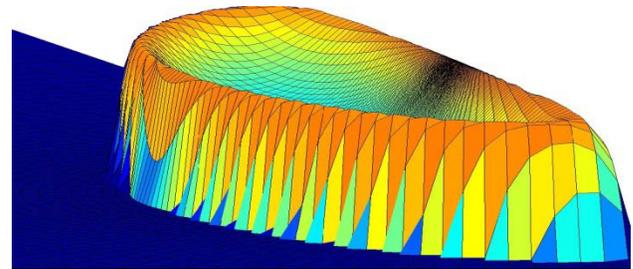


Fig. 3. Surface plot of vorticity similar to [3] for  $\delta = 3$

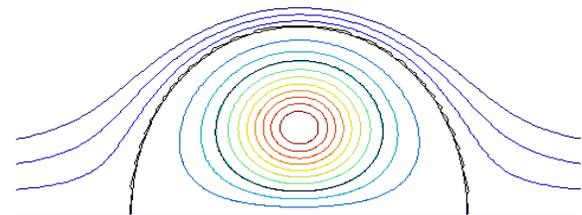


Fig. 4.  $\beta$ -plane shielded vortex solution with  $\beta = 0.2$

The  $\beta$ -plane solutions tend to have less robust solutions, and convergence is not as stable for larger beta, or an unbalanced initial guess; for smaller  $\beta$ , which would reflect flow on a slowly rotating object, such as the Earth, the stability is closer to that of the  $f$ -plane.

#### 5. Conclusion

In short, we have introduced a new method for creating a large class of vortex dipoles given prescribed profiles. Presented were example profile functions that produced dipoles similar to those in recent research. There are more results than those presented here using different profile functions, detaching the vortices from the center, or reintroducing the cylindrical obstacle. The robustness of this method creates a stable algorithm that converges for a wide range of functions, even with little knowledge beforehand of the solution.

[1] Elcrat, A., Fornberg, B., Horn, M., & Miller, K. 2000. Some steady vortex flows past a circular cylinder, *J. Fluid Mech.* 409: 13-27.  
 [2] Kizner, Z., Khvoles, R. 2004. Two variations on the theme of Lamb-Chaplygin: super-smooth dipole and rotating multipoles, *Reg. Chao.Dyn.* 9(4) 509-518.  
 [3] Boyd, J., Ma, H. 1990. Numerical study of elliptical modons using spectral methods, *J.Fluid Mech.*221: 597-611.  
 [4] Khvoles, R., Berson, D., Kizner, Z. 2005. The structure and evolution of elliptical barotropic modons, *J. Fluid Mech.* 530: 1-30.