A BRIDGE FROM STABILITY TO ROBUST PERFORMANCE DESIGN OF PID
CONTROLLERS IN THE FREQUENCY DOMAIN

A Dissertation by

Tooran Emami

Master of Science, Wichita State University, 2006
Bachelor of Science, University of Esfahan, 1994

Submitted to the Department Electrical Engineering and Computer Science
and the faculty of the Graduate School of
Wichita State University
in partial fulfillment of
the requirements for the degree of
Doctor of Philosophy

December 2009
A BRIDGE FROM STABILITY TO ROBUST PERFORMANCE DESIGN OF PID CONTROLLERS IN THE FREQUENCY DOMAIN

The following faculty members have examined the final copy of this dissertation for form and content, and recommend that it be accepted in partial fulfillment of the requirement for the degree of Doctor of Philosophy with major in Electrical Engineering.

John M. Watkins, Committee Chair

Mahmoud E. Sawan, Committee Member

Janet M. Twomey, Committee Member

Ward T. Jewell, Committee Member

Preethika Kumar, Committee Member

Accepted for the College of Engineering

Zulma Toro-Ramos, Dean

Accepted for the Graduate School

J. David McDonald, Dean
To my parents and my sweetheart children
Reza and Mona
To teach is to touch a life forever.

-- Anonymous
ACKNOWLEDGMENTS

The writing of this dissertation would be impossible without the support of numerous people who provided me with motivation, confidence, and the support to succeed. I extend my sincere gratitude and appreciation to my esteemed advisor Professor J. M. Watkins from the Department of Electrical Engineering and Computer Science at Wichita State University. This dissertation would not have begun without his supervision and support. Professor Watkins was the Graduate Coordinator and the Chair of the department during my Ph.D. program. Even though his schedule was very full, he was always accessible for professional advice on my program and comments on the papers that we submitted for publication together. His words always encouraged me forward each step of the way. I am forever grateful for his many years of thoughtful and patient guidance, wisdom, and unceasing support.

I extend my sincere appreciation to Professor Emeritus E. M. Sawan, and Professors W. Jewell, and P. Kumar, all from the Department of Electrical Engineering and Computer Science at Wichita State University, for serving as members on my dissertation committee. I convey my special gratitude to Professor J. M. Twomey from the Department of Industrial and Manufacturing Engineering at Wichita State University, for agreeing to serve on my dissertation committee even though her schedule was very full. I express many thanks to Professor B. Bahr from the Department of Mechanical and Aerospace Engineering at California State University, Long Beach, who served on my dissertation committee for my proposal defense. I convey my sincere appreciation to Professor Emeritus L. Paarmann, from the Department of Electrical Engineering and Computer Science at Wichita State University, who also served on my dissertation committee for my proposal defense. I am very grateful to all my committee members for their encouragement, services, and comments on my dissertation.
I deeply appreciate those individuals who provided computer, paperwork, and editorial assistance. I express my thanks to Ken Tedder, who is in charge of desktop support in the College of Engineering, for helping me update my computer and for his continual support in this process. I am very grateful to Judie Dansby and Stephen W. Copeland, senior administrative assistants in the Department of Electrical Engineering and Computer Science at Wichita State University, for their help with my paperwork. I also give special thanks to Kristie Bixby for her expertise and professional input in editing my dissertation.

I am truly very grateful for my children, Reza and Mona, and my husband Jeff Peyravi for their patience and support during my graduate work. My parents and family were integral to my continuation of graduate work in electrical engineering. I found their words always, encouraging, and I am truly grateful to them for giving me confidence, love, and persistent support. I express my gratitude to all my friends for their friendship during these years.

I extend my sincere gratitude and appreciation to the generous donors of the fellowships that I have received during my Ph.D. program. My external fellowships from Spirit Aerosystems, Inc. and Boeing Integrated Defense Systems were administered by the College of Engineering at Wichita State University. Internal fellowships, including the Dr. Michael P. Tilford Graduate Fellowship, E.L. Cord Foundation Fellowship, Ollie A. & Jo Heskett Fellowship, and Heskett Summer Graduate Research Grants, were administered by the Graduate School at Wichita State University. I also received several Student Travel Grants from the American Control Conference, IEEE Conference on Decision and Control, ASME Dynamic Systems and Control Conference, Graduate School, Student Government Association, College of Engineering, and Department of Electrical Engineering and Computer Science.

Thank you very much, one and all.
ABSTRACT

A graphical technique for finding proportional integral derivative (PID) controllers that stabilize a given single-input-single-output (SISO) linear time-invariant (LTI) system of any order system with time-delay has been solved. In this research, a method is introduced for finding all achievable PID controllers that also satisfy an $H_{\infty}$ sensitivity, complementary sensitivity, weighted sensitivity, robust stability, or robust performance constraint. These problems can be solved by finding all PID controllers that simultaneously stabilize the closed-loop characteristic polynomial and satisfy constraints defined by a set of related complex polynomials.

There are several key advantages of this procedure. It does not require the plant transfer function model, but depends only on the frequency response. The ability to include the time-delay in the nominal model of the system will often allow for designs with reduced conservativeness in plant uncertainty and an increase in size of the set of all PID controllers that robustly stabilize the system and meet the performance requirements.
# TABLE OF CONTENTS

<table>
<thead>
<tr>
<th>Chapter</th>
<th>Page</th>
</tr>
</thead>
<tbody>
<tr>
<td>1. MOTIVATION</td>
<td>1</td>
</tr>
<tr>
<td>1.1 Introduction</td>
<td>1</td>
</tr>
<tr>
<td>1.1.1 Mathematical Model</td>
<td>1</td>
</tr>
<tr>
<td>1.1.2 Time-Delay</td>
<td>2</td>
</tr>
<tr>
<td>1.2 Research Objective</td>
<td>2</td>
</tr>
<tr>
<td>2. LITERATURE REVIEW</td>
<td>4</td>
</tr>
<tr>
<td>2.1 Introduction</td>
<td>4</td>
</tr>
<tr>
<td>2.2 Stability and Time-delay</td>
<td>4</td>
</tr>
<tr>
<td>2.3 Performance and Time-delay</td>
<td>5</td>
</tr>
<tr>
<td>2.4 Robust Stability of Systems with No Time-delay</td>
<td>5</td>
</tr>
<tr>
<td>2.5 Robust Stability of Systems with Time-delay</td>
<td>6</td>
</tr>
<tr>
<td>3. SENSITIVITY PROBLEM</td>
<td>8</td>
</tr>
<tr>
<td>3.1 Introduction</td>
<td>8</td>
</tr>
<tr>
<td>3.2 Design Methodology</td>
<td>9</td>
</tr>
<tr>
<td>3.3 Experimental Results</td>
<td>16</td>
</tr>
<tr>
<td>3.4 Conclusion</td>
<td>23</td>
</tr>
<tr>
<td>4. WEIGHTED SENSITIVITY PROBLEM</td>
<td>24</td>
</tr>
<tr>
<td>4.1 Introduction</td>
<td>24</td>
</tr>
<tr>
<td>4.2 Design Methodology</td>
<td>24</td>
</tr>
<tr>
<td>4.3 Selection of Weighting Functions</td>
<td>31</td>
</tr>
<tr>
<td>4.4 Numerical Example</td>
<td>32</td>
</tr>
<tr>
<td>4.5 Conclusion</td>
<td>41</td>
</tr>
<tr>
<td>5. COMPLEMENTARY SENSITIVITY PROBLEM</td>
<td>42</td>
</tr>
<tr>
<td>5.1 Introduction</td>
<td>42</td>
</tr>
<tr>
<td>5.2 Design Methodology</td>
<td>43</td>
</tr>
<tr>
<td>5.3 Numerical Example</td>
<td>49</td>
</tr>
<tr>
<td>5.4 Conclusion</td>
<td>57</td>
</tr>
<tr>
<td>6. ROBUST STABILITY PROBLEM</td>
<td>59</td>
</tr>
<tr>
<td>6.1 Introduction</td>
<td>59</td>
</tr>
<tr>
<td>6.2 Design Methodology</td>
<td>60</td>
</tr>
<tr>
<td>6.3 Selection of Weighting Functions</td>
<td>67</td>
</tr>
</tbody>
</table>
## TABLE OF CONTENTS (continued)

<table>
<thead>
<tr>
<th>Chapter</th>
<th>Page</th>
</tr>
</thead>
<tbody>
<tr>
<td>6.4 Numerical Example</td>
<td>69</td>
</tr>
<tr>
<td>6.5 Conclusion</td>
<td>78</td>
</tr>
<tr>
<td><strong>7. ROBUST PERFORMANCE PROBLEM</strong></td>
<td>80</td>
</tr>
<tr>
<td>7.1 Introduction</td>
<td>80</td>
</tr>
<tr>
<td>7.2 Design Methodology</td>
<td>81</td>
</tr>
<tr>
<td>7.3 Numerical Example</td>
<td>91</td>
</tr>
<tr>
<td>7.4 Conclusion</td>
<td>101</td>
</tr>
<tr>
<td><strong>8. CONCLUSION</strong></td>
<td>102</td>
</tr>
<tr>
<td>8.1 Summary</td>
<td>102</td>
</tr>
<tr>
<td>8.2 Additional Work</td>
<td>103</td>
</tr>
<tr>
<td>REFERENCES</td>
<td>105</td>
</tr>
<tr>
<td>LIST OF REFERENCES</td>
<td>106</td>
</tr>
<tr>
<td>APPENDIX</td>
<td>111</td>
</tr>
<tr>
<td>WEIGHTED SENSITIVITY PROGRAM</td>
<td>112</td>
</tr>
</tbody>
</table>
# LIST OF FIGURES

<table>
<thead>
<tr>
<th>Figure</th>
<th>Description</th>
<th>Page</th>
</tr>
</thead>
<tbody>
<tr>
<td>1.</td>
<td>Block diagram of sensitivity function</td>
<td>10</td>
</tr>
<tr>
<td>2.</td>
<td>Stability boundary and sensitivity region in the ((K_p, K_i)) plane</td>
<td>17</td>
</tr>
<tr>
<td>3.</td>
<td>Magnitude of sensitivity function frequency response</td>
<td>18</td>
</tr>
<tr>
<td>4.</td>
<td>Stability boundary and sensitivity region in the ((K_p, K_d)) plane</td>
<td>19</td>
</tr>
<tr>
<td>5.</td>
<td>Magnitude of sensitivity function frequency response</td>
<td>20</td>
</tr>
<tr>
<td>6.</td>
<td>Plots of (K_p(\omega, \theta_s, \infty)) and (K_p(\omega, \theta_s, \gamma)) for various values of (\theta_s \in [0,2\pi))</td>
<td>21</td>
</tr>
<tr>
<td>7.</td>
<td>Stability boundary and sensitivity region in the ((K_p, K_d)) plane</td>
<td>22</td>
</tr>
<tr>
<td>8.</td>
<td>Magnitude of sensitivity function frequency response</td>
<td>23</td>
</tr>
<tr>
<td>9.</td>
<td>Block diagram of the system with weighted sensitivity</td>
<td>25</td>
</tr>
<tr>
<td>10.</td>
<td>Stability boundary and weighted sensitivity region in the ((K_p, K_i)) plane</td>
<td>34</td>
</tr>
<tr>
<td></td>
<td>for a fixed (\tilde{K}_d = 0.4)</td>
<td></td>
</tr>
<tr>
<td>11.</td>
<td>Magnitude of (W_{sS}^c) for (G_c(s) = 0.49 + \frac{0.1}{s} + 0.4s)</td>
<td>35</td>
</tr>
<tr>
<td>12.</td>
<td>Step response of the closed loop system</td>
<td>36</td>
</tr>
<tr>
<td>13.</td>
<td>Stability boundary and weighted sensitivity region in the ((K_p, K_d)) plane</td>
<td>37</td>
</tr>
<tr>
<td></td>
<td>for a fixed (\tilde{K}_i = 0.2)</td>
<td></td>
</tr>
<tr>
<td>14.</td>
<td>Magnitude of (W_{sS}^c) for (G_c(s) = 0.48 + \frac{0.2}{s} + 0.50s)</td>
<td>38</td>
</tr>
<tr>
<td>15.</td>
<td>Plots of (K_p(\omega, \theta_s, \infty)) and (K_p(\omega, \theta_s, \gamma)) for various values of (\theta_s \in [0,2\pi))</td>
<td>39</td>
</tr>
<tr>
<td>Figure</td>
<td>Description</td>
<td>Page</td>
</tr>
<tr>
<td>--------</td>
<td>-------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------</td>
<td>-------</td>
</tr>
<tr>
<td>16.</td>
<td>Stability boundary and weighted sensitivity region in the ($K_i$, $K_d$) plane for a fixed $K_p = 0.5$.</td>
<td>40</td>
</tr>
<tr>
<td>17.</td>
<td>Magnitude of $W_{s}S$ for $G_c(s) = 0.5 + \frac{0.11}{s} + 0.39s$.</td>
<td>41</td>
</tr>
<tr>
<td>18.</td>
<td>Block diagram of the system.</td>
<td>43</td>
</tr>
<tr>
<td>19.</td>
<td>Stability boundary and complementary sensitivity region in the ($K_p$, $K_i$) plane for a fixed $K_d = 0.1$.</td>
<td>51</td>
</tr>
<tr>
<td>20.</td>
<td>Magnitude of $T(j\omega)$ for $G_c(s) = 0.8 + \frac{0.2}{s} + 0.1s$.</td>
<td>52</td>
</tr>
<tr>
<td>21.</td>
<td>Stability boundary and complementary sensitivity region in the ($K_p$, $K_d$) plane for a fixed $K_i = 0.1$.</td>
<td>53</td>
</tr>
<tr>
<td>22.</td>
<td>Magnitude of $T(j\omega)$ for $G_c(s) = 0.4 + \frac{0.1}{s} + 0.5s$.</td>
<td>54</td>
</tr>
<tr>
<td>23.</td>
<td>Plots of $K_p(\omega, \theta_T, \infty)$ and $K_p(\omega, \theta_T, \gamma)$ for various values of $\theta_T \in [0,2\pi)$.</td>
<td>55</td>
</tr>
<tr>
<td>24.</td>
<td>Stability boundary and complementary sensitivity region in the ($K_i$, $K_d$) plane for a fixed $K_p = 0.4$.</td>
<td>56</td>
</tr>
<tr>
<td>25.</td>
<td>Magnitude of $T(j\omega)$ for $G_c(s) = 0.4 + \frac{0.2}{s} + 0.5s$.</td>
<td>57</td>
</tr>
<tr>
<td>26.</td>
<td>Block diagram of the system with multiplicative uncertainty.</td>
<td>60</td>
</tr>
<tr>
<td>27.</td>
<td>Magnitude of $W_T(j\omega)$.</td>
<td>69</td>
</tr>
<tr>
<td>28.</td>
<td>Multiplicative errors for different communication delays and the multiplicative weight.</td>
<td>70</td>
</tr>
<tr>
<td>Figure</td>
<td>Description</td>
<td>Page</td>
</tr>
<tr>
<td>--------</td>
<td>-----------------------------------------------------------------------------</td>
<td>------</td>
</tr>
<tr>
<td>29</td>
<td>Nominal stability boundary and robust stability region in the ((K_p, K_i)) plane for a fixed (\tilde{K}_d = 0.2)</td>
<td>72</td>
</tr>
<tr>
<td>30</td>
<td>Magnitude of (W_T(j\omega)T(j\omega)) for (G_c(s) = 2.76 + \frac{7.89}{s} + 0.2s)</td>
<td>73</td>
</tr>
<tr>
<td>31</td>
<td>Nominal stability boundary and robust stability region in the ((K_p, K_d)) plane for a fixed (\tilde{K}_i = 1)</td>
<td>74</td>
</tr>
<tr>
<td>32</td>
<td>Magnitude of (W_T(j\omega)T(j\omega)) for (G_c(s) = 1.63 + \frac{1}{s} + 0.07s)</td>
<td>75</td>
</tr>
<tr>
<td>33</td>
<td>Plots of (K_p(\omega, \theta_T, \infty)) and (K_p(\omega, \theta_T, \gamma)) for various values of (\theta_T \in [0, 2\pi])</td>
<td>76</td>
</tr>
<tr>
<td>34</td>
<td>Nominal stability boundary and robust stability region in the ((\tilde{K}_i, K_d)) plane for a fixed (\tilde{K}_p = 0.5)</td>
<td>77</td>
</tr>
<tr>
<td>35</td>
<td>Magnitude of (W_T(j\omega)T(j\omega)) for (G_c(s) = 0.5 + \frac{1.30}{s} + 0.07s)</td>
<td>78</td>
</tr>
<tr>
<td>36</td>
<td>Block diagram of the system with multiplicative uncertainty.</td>
<td>81</td>
</tr>
<tr>
<td>37</td>
<td>Multiplicative errors for different communication delays and the multiplicative weight.</td>
<td>93</td>
</tr>
<tr>
<td>38</td>
<td>Nominal stability boundary and robust performance region in the ((\tilde{K}_p, \tilde{K}_i)) plane for a fixed (\tilde{K}_d = 0.2)</td>
<td>94</td>
</tr>
<tr>
<td>39</td>
<td>Magnitude of (\left</td>
<td>W_S(j\omega)S(j\omega)\right</td>
</tr>
<tr>
<td>40</td>
<td>Nominal stability boundary and robust performance region in the ((\tilde{K}_p, \tilde{K}_d)) plane for a fixed (\tilde{K}_i = 0.1)</td>
<td>96</td>
</tr>
<tr>
<td>Figure</td>
<td>Description</td>
<td>Page</td>
</tr>
<tr>
<td>--------</td>
<td>-------------------------------------------------------------------------------------------------</td>
<td>------</td>
</tr>
<tr>
<td>41.</td>
<td>Magnitude of $</td>
<td>W_s(j\omega)S(j\omega)</td>
</tr>
<tr>
<td>42.</td>
<td>Plots of $K_p(\omega, \theta_s, \theta_T, \infty)$ and $K_p(\omega, \theta_s, \theta_T, \gamma)$ for various values of $\theta_s \in [0, 2\pi)$ and $\theta_T \in [0, 2\pi)$</td>
<td>98</td>
</tr>
<tr>
<td>43.</td>
<td>Nominal stability boundary and robust performance region in the $(K_i, K_d)$ plane for a fixed $\hat{K}_p = 0.4$</td>
<td>99</td>
</tr>
<tr>
<td>44.</td>
<td>Magnitude of $</td>
<td>W_s(j\omega)S(j\omega)</td>
</tr>
<tr>
<td>45.</td>
<td>Step response of the closed loop system for various time-delays</td>
<td>101</td>
</tr>
</tbody>
</table>
# LIST OF ABBREVIATIONS/NOMENCLATURE

<table>
<thead>
<tr>
<th>Abbreviation</th>
<th>Description</th>
</tr>
</thead>
<tbody>
<tr>
<td>dB</td>
<td>Decibel</td>
</tr>
<tr>
<td>$H_\infty$</td>
<td>H-infinity</td>
</tr>
<tr>
<td>LTI</td>
<td>Linear Time Invariant</td>
</tr>
<tr>
<td>P</td>
<td>Proportional</td>
</tr>
<tr>
<td>PD</td>
<td>Proportional Derivative</td>
</tr>
<tr>
<td>PI</td>
<td>Proportional Integral</td>
</tr>
<tr>
<td>PID</td>
<td>Proportional Integral Derivative</td>
</tr>
<tr>
<td>SISO</td>
<td>Single Input Single Output</td>
</tr>
</tbody>
</table>
LIST OF SYMBOLS

α  Alpha
∀  All-value
∈  An element of
∠  Angle
β  Beta
Δ  Delta (Capital)
γ  Gamma
∞  Infinity
∥  Magnitude
[]  Matrix
∥∥  Norm
ω  Omega
π  Pi
\sqrt{}  Square-root
θ  Theta
~  Tilde
τ  Tau
ξ  Zeta
ε  Epsilon
CHAPTER 1
MOTIVATION

1.1 Introduction

The purpose of feedback control is to ensure that the response of a plant be controlled in a way that satisfies required specifications. These specifications are given in terms of the closed-loop system. The feedback control system generates an input to the plant system based on a comparison of the desired and actual closed-loop response. In the feedback control design process, the task for the control system is to use information about the plant system to design a feedback control system and then to achieve the specifications [1], [2], [3], [4], [5], [6], [7], [8], and [9].

This research attempts to solve two main problems in control system design. These two problems concern the mathematical model and time-delay in an industrial control system. The remainder of this chapter is organized as follows. A brief discussion on the problems with the mathematical model and time-delay are presented in Subsections 1.1.1 and 1.1.2, respectively. Section 1.2 presents the objective of this dissertation.

1.1.1 Mathematical Model

In order to ensure good performance, control system designers typically depend on a mathematical model of a system to synthesize a controller. Unfortunately, when the design is simulated, the simulated response often does not match the actual response. Typically, the mathematical model of the system is based on a number of assumptions that are used to simplify the model. Consequently, the mathematical model is not an accurate representation of the real system. There is always some uncertainty in the system model. Real systems must also cope
with disturbances and sensor noise [1], [2], [4], [5], and [8]. These problems cause a gap between the mathematical model and the real system.

In recent years, numerous articles on control systems have focused on designing controllers that make a system robust. Robust control theory is a bridge between the mathematical model of an industrial plant and the actual industrial plant behavior. Robust control methods try to bound the uncertainty. This boundary contains a set of mathematical models that include the actual industrial plant dynamics. Knowing uncertainty boundaries allows the designer to find a controller for the worst case plant behavior. The main idea of robust control theory is to ensure that design specifications are satisfied for the worst case uncertainty [4] and [5].

1.1.2 Time-Delay

Another big problem in industry is the time-delay. Most of the articles in this area focus on the stability of the nominal system with time-delay. Unfortunately, they do not emphasize the robustness of a system with time-delay. A principal goal of this research is to show that it is possible to design a controller that can be applied to an industrial system that not only stabilizes the nominal system with time-delay, but also robustly meets the performance specifications.

1.2 Research Objective

Due to the extensive use of the proportional integral derivative (PID) controller in industrial and bioengineering applications, there have been significant efforts to determine the set of all PID controllers that meet specific design goals. Since the target of this research is to develop design methods that can be applied in industry, these methods should have several key elements. First, they should be applicable to a broad set of plants. In order for the methods to be applicable in the process control industry, it is particularly important that they handle time-
delays. Ideally, the design methods should be simple to understand and easy to implement. Methods that depend only on the frequency response of the system eliminate the need for a plant model, which may not be available in some applications. In biological systems, for example, rational transfer function models often do not exist.

Most of the early work in this area sought to find all PID controllers that stabilized the nominal industrial plant model, where the information of the plant transfer function is required. Beyond stability, based on information from the plant transfer function, investigators have also looked at performance and robustness. Since these controllers must be implemented on real systems, design methods that deal with robustness are of particular importance. Unfortunately, most of the work in the area of robustness did not deal directly with systems with time-delay, which are prevalent in the process control industry.

The objective of this dissertation is to find frequency domain characteristic equations that can be used to characterize all achievable PID controllers. The systems to be controlled are assumed to be single-input single-output (SISO), linear time invariant (LTI), proper, and of arbitrary order with time-delay. The controllers must simultaneously stabilize the closed-loop system and satisfy sensitivity, weighted sensitivity, complementary sensitivity, robust stability, or robust performance constraints.
CHAPTER 2
LITERATURE REVIEW

2.1 Introduction

This chapter provides a brief review of PID controller design. Unfortunately, a web search for the first inventor of the PID controller shows no record of this. Probably in 1922, the first PID controller was used by Minorsky to control a ship steering [7] and [8]. This work in [8] represented the first mathematical method to design a PID controller for an industrial process.

The remainder of this chapter is organized as follows. Literature covering stability and time-delay is presented in Section 2.2. Section 2.3 focuses on performance and time-delay. In Section 2.4, articles are presented for solving the robust stability and robust performance problems when there is no time-delay in the system. Section 2.5 presents articles on robust stability and robust performance design of PID controllers for systems with time-delay.

2.2 Stability and Time-delay

Not surprisingly, most of the early work in this area sought to find all PID controllers that stabilized the nominal industrial plant model, where knowledge of the plant transfer function model was required. Bhattacharyya and colleagues did much of early work in this area [1] and [9]. Many of their results depended on generalizations of the Hermite-Biehler theorem [11]. They developed results based on theorems by Pontryagin and a generalized Nyquist criterion in [12]. The method introduced by Tan [13] broke the numerator and denominator of the plant transfer function into even and odd parts. A new method, which did not involve complex mathematical derivations, was used to solve the problem of stabilizing an arbitrary order transfer function when only the frequency response of the plant transfer function was known [14], [15], and [16]. This work was extended to a unified approach involving delta operators that found the
stability region for discrete-time and continuous-time PID controllers [17]. Saeki [16] introduced a method for finding the number of unstable poles in different regions of the PID controller parameter space. Wang [18], found the stability region of PID controllers based on the proportional gain of the PID controller. Ou et al. [19] introduced analytical solutions for stabilizing an arbitrary SISO LTI plant with time-delay by using Proportional (P), Proportional Integral (PI), and PID controllers.

### 2.3 Performance and Time-delay

Beyond stability, investigators have also looked at performance. Some researchers found regions where the controllers were guaranteed to meet certain gain and phase margin requirements [13], [14], [15], and [17]. PID controllers that also satisfy gain crossover, phase crossover, and bandwidth requirements for double integrator systems with delay were found [20]. Shafiei and Shenton [21] found all PID controllers that placed the closed-loop poles in certain D-partitions. The parameters of PID controller were determined using a metaheuristic algorithm method in [22] and [23]. The metaheuristic algorithm method was used to adjust the PID parameters to meet the performance requirement for a pouring task [22]. The authors in [24] used a fractional PID controller to meet the performance requirement for an active magnetic bearing system. An adaptive genetic algorithm was used to determine the PID controller parameters that optimized a multi-objective cost function. A constrained pole assignment was used to design Proportional Derivative (PD) controllers for a double-integrator plant model with time-delays or time constant [25].

### 2.4 Robust Stability of Systems with No Time-delay

Since these controllers must be implemented on real systems, design methods that deal with robustness are of particular importance. Saeki and colleagues [26], [27], and [28] looked at
different methods for $H_\infty$ controller design of PID controllers. Ho used a generalization of the Hermite-Biehler theorem for $H_\infty$ PID design [29]. Tantaris et al. looked at a similar problem for first-order controllers [30] and [31]. Keel and Bhattacharyya looked at PID design given a weighted sensitivity and weighted complementary sensitivity constraints for plants with no poles or zeros on the $j\omega$ axis [32].

Ho and Lin [33] looked at PID controller design for robust performance for a plant described by a rational transfer function. Other researchers investigated the robust performance problem for first and second-order system transfer functions [34].

2.5 Robust Stability of Systems with Time-delay

Unfortunately, none of the methods that dealt with robustness worked directly with time-delays, which are prevalent in the process control industry. Keel and Bhattacharyya [35], allowed for time-delays in the nominal model when they investigated weighted sensitivity and robust stability problems. However, they did not consider the robust performance problem and assumed that there was no zero on the $j\omega$ axis.

Emami and Watkins [36], [37], [38], [39], [40], and [41] developed techniques for finding all achievable PID controllers that simultaneously stabilized the closed-loop system, or satisfied an $H_\infty$ sensitivity, complementary sensitivity, weighted sensitivity, robust stability, or robust performance constraint, respectively. These techniques are applicable to proper a SISO LTI systems of arbitrary order with time-delay. These methods do not require the rational plant transfer function model, but depend only on the frequency response of the system. If the plant transfer function is known, the same procedures can be applied by first computing the frequency response. The ability to include the time-delay in the nominal model of the system will often allow for designs with reduced conservativeness in plant uncertainty and an increase in size of
the set of all PID controllers that robustly stabilize the system and meet performance requirements.
CHAPTER 3
SENSITIVITY PROBLEM

3.1 Introduction

Feedback control of a proper system with an integration action results in a sensitivity function with a small magnitude at low frequencies and a magnitude of one at high frequencies. The problem is in the intermediate frequency range, where the feedback control degrades performance. At these frequencies, the peak value of the sensitivity function is larger than one, which is a measure of the worst-case performance degradation. In a SISO LTI system, $M_s$ is the maximum magnitude of the sensitivity function in the frequency domain. Typically, $M_s$ should be less than 2 (6 dB) to achieve good performance and robustness. A large value of $M_s$ indicates poor performance as well as poor robustness. A small value of $M_s$ prevents an increase of noise at high frequencies and also introduces a margin of robustness [5]. In conclusion, the smaller the value of $M_s$, the better the design.

In this chapter, a technique is developed for finding all achievable PID controllers that simultaneously stabilize the closed-loop system and satisfy an $H_\infty$ sensitivity constraint [36]. $H_\infty$ sensitivity is defined as the infinity norm of sensitivity function. This method is applicable for SISO proper transfer functions of order with time-delay. Since this work builds upon the straight-forward development in [15] and [16], it does not require the plant transfer function model, but only the frequency response of the system. If the plant transfer function is known, the same procedures can be applied by first computing the frequency response.
The remainder of this chapter is organized as follows. In Section 3.2, the design methodology is introduced. In Section 3.3, an experimental example of a DC motor that has a communication delay in the feedback loop is presented to demonstrate the application of this method. Finally, the results of this chapter are summarized in Section 3.4.

3.2 Design Methodology

Consider the SISO LTI system shown in Figure 1, where \( G_p(s) \) is the plant, and \( G_c(s) \) is the PID controller. The input signal and error signal are \( R(s) \) and \( Z(s) \), respectively, and \( Y(s) \) is the measurement output. The plant transfer function can be written as

\[
G_p(s) = G(s) e^{-\tau s},
\]

where \( G(s) \) is an arbitrary order transfer function, and \( \tau \) is the time-delay. The PID controller is defined as

\[
G_c(s) = K_p + \frac{K_i}{s} + K_d s,
\]

where \( K_p \), \( K_i \), and \( K_d \) are the proportional, integral, and derivative gains, respectively. The error signal is defined as

\[
Z(s) = R(s) - Y(s) = S(s)R(s),
\]
where \( S(s) = \frac{1}{1 + G_p(s)G_c(s)} \) is the sensitivity function.

Figure 1. Block diagram of sensitivity function.

The transfer functions in Figure 1 can be expressed in the frequency domain. The plant transfer function can be written in terms of their real and imaginary parts as

\[
G_p(j\omega) = R_p(\omega) + jI_m(\omega),
\]

for \( \omega \in [0, \infty) \), where \( \omega \neq \omega_j \) and \( \omega_j \) is any frequency, where the plant has a zero on the \( j\omega \) axis. Note that if the plant has a zero on the \( j\omega \) axis, the same procedures can be derived but there is no real value for the PID controller coefficients at this specific frequency \( \omega_j \). The PID controller is defined in the frequency domain as

\[
G_c(j\omega) = K_p + \frac{K_i}{j\omega} + K_d j\omega.
\]
The deterministic values of $K_p$, $K_i$, and $K_d$ for which the closed-loop characteristic polynomial is Hurwitz stable have been found in [15] and [16]. In this research, the problem is to find all PID controllers that stabilize the system and satisfy the $H_\infty$ sensitivity constraint

$$\left\| S(j\omega) \right\|_\infty \leq \gamma,$$

where $S(j\omega) = \frac{1}{1 + G_p(j\omega)G_c(j\omega)}$ is the sensitivity function, and $\gamma$ is a positive real scalar that corresponds to $M_s$ for SISO systems. The complex function in equation (6) for a SISO system for each value of $\omega$ can be written in terms of its magnitude and phase angles as

$$\left| S(j\omega)e^{j\angle S(j\omega)} \right| \leq \gamma, \quad \forall \omega.$$  

If equation (7) holds, then for each value of $\omega$

$$S(j\omega)e^{j\theta_S} \leq \gamma,$$

must be true for some $\theta_S \in [0,2\pi)$, where $\theta_S = -\angle S(j\omega)$. Consequently, all PID controllers that satisfy equation (6) must lie at the intersection of all controllers that satisfy equation (8) for all $\theta_S \in [0,2\pi)$ [36].
To accomplish this, for each value of $\theta_s \in [0, 2\pi)$, all PID controllers are found on the boundary of equation (8). It is easy to show from equation (8), that all PID controllers on the boundary must satisfy

$$P(\omega, \theta_s, \gamma) = 0,$$  \tag{9}$$

where $P(\omega, \theta_s, \gamma) = 1 + G_p(j\omega)G_c(j\omega) - \frac{1}{\gamma} e^{j\theta_s}$. Note that equation (9) reduces to the frequency response of the standard closed-loop characteristic polynomial as $\gamma \to \infty$.

Substituting equations (4), (5), and $e^{j\theta_s} = \cos \theta_s + j \sin \theta_s$ into equation (9), and solving for the real and imaginary parts yields

$$X_{Rp} K_p + X_{Ri} K_i + X_{Rd} K_d = Y_R,$$  \tag{10}$$

and

$$X_{Ip} K_p + X_{Ii} K_i + X_{Id} K_d = Y_I,$$  \tag{11}$$

where

$$X_{Rp} = \omega R_e(\omega),$$

$$X_{Ri} = I_m(\omega),$$

$$X_{Rd} = -\omega^2 I_m(\omega),$$

$$Y_R = \omega \left( \frac{1}{\gamma} \cos \theta_s - 1 \right),$$
\[ X_{Ip} = \omega I_m(\omega), \]
\[ X_{Ir} = -R_e(\omega), \]
\[ X_{Id} = \omega^2 R_e(\omega), \]
\[ Y_I = \omega \left( \frac{1}{\gamma} \sin \theta_S \right). \]

This is a three-dimensional system in terms of the controller parameters \( K_p, K_i, \) and \( K_d \). The boundary of equation (9) can be found in the \((K_p, K_i)\) plane for a fixed value of \( K_d \).

After setting \( K_d \) to the fixed value \( \tilde{K}_d \), equations (10) and (11) can be rewritten as

\[
\begin{bmatrix}
X_{Rp} & X_{Ri} \\
X_{Ip} & X_{Ir}
\end{bmatrix}
\begin{bmatrix}
K_p \\
K_i
\end{bmatrix}
= 
\begin{bmatrix}
Y_{Rr} - Y_{Rd} \tilde{K}_d \\
Y_I - Y_{Id} \tilde{K}_d
\end{bmatrix}. \tag{12}
\]

Solving equation (12), for all \( \omega \neq 0 \) and \( \theta_S \in [0, 2\pi) \), gives:

\[
K_p(\omega, \theta_S, \gamma) = \frac{\left( \frac{1}{\gamma} \cos \theta_S - 1 \right) R_e(\omega) + \frac{1}{\gamma} \sin \theta_S I_m(\omega)}{\left| G_p(j\omega) \right|^2}, \tag{13}
\]

and

\[
K_i(\omega, \theta_S, \gamma) = \omega^2 \tilde{K}_d + \frac{-\omega \left( \frac{1}{\gamma} \sin \theta_S \right) R_e(\omega) + \omega \left( \frac{1}{\gamma} \cos \theta_S - 1 \right) I_m(\omega)}{\left| G_p(j\omega) \right|^2}, \tag{14}
\]
where $\left[ G_p(j\omega) \right]^2 = R_e^2(\omega) + I_m^2(\omega)$. Setting $\omega = 0$ in equation (12), yields

$$\begin{bmatrix} 0 & X_{R_i}(0) \\ 0 & X_{I_i}(0) \end{bmatrix} \begin{bmatrix} K_p \\ K_i \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix},$$

(15)

and it can be concluded that $K_p(0, \theta_S, \gamma)$ is arbitrary and $K_i(0, \theta_S, \gamma) = 0$, unless $I_m(0) = R_e(0) = 0$, which holds only when $G_p(s)$ has a zero at the origin.

This procedure can be repeated in the $(K_p, K_d)$ plane. After setting $K_i$ to a fixed value $\bar{K}_i$, equations (10) and (11) can be rewritten as

$$\begin{bmatrix} X_{R_p} & X_{R_d} \\ X_{I_p} & X_{I_d} \end{bmatrix} \begin{bmatrix} K_p \\ K_d \end{bmatrix} = \begin{bmatrix} Y_r - X_{R_i} \bar{K}_i \\ Y_I - X_{I_i} \bar{K}_i \end{bmatrix},$$

(16)

Solving equation (16) for all $\omega \neq 0$, $\theta_S \in [0, 2\pi)$ gives the same expression as equation (13) for $K_p(\omega, \theta_S, \gamma)$, and the following equation for $K_d(\omega, \theta_S, \gamma)$:

$$K_d(\omega, \theta_S, \gamma) = \frac{\bar{K}_i}{\omega^2} + \frac{\frac{1}{\gamma} \sin \theta_S R_e(\omega) + \left( \frac{-1}{\gamma} \cos \theta_S + 1 \right) I_m(\omega)}{\omega \left| G_p(j\omega) \right|^2}. $$

(17)
At $\omega = 0$, $\tilde{K}_i$ must be equal to zero for a solution to exist. Furthermore, as $I_m(0) = 0$ for all real plants, $K_d(0, \theta_S, \gamma)$ is arbitrary and

$$K_d(0, \theta_S, \gamma) = -\frac{1}{\gamma} \frac{\cos \theta_S - 1}{R_c(0)}.$$ \hspace{1cm} (18)

Last, the solution is found in the $(K_i, K_d)$ plane. After setting $K_p$ to a fixed value of $\tilde{K}_p$, equations (10) and (11) are rewritten as

$$\begin{bmatrix} X_{Ri} & X_{Rd} \\ X_{Li} & X_{Ld} \end{bmatrix} \begin{bmatrix} K_i \\ K_d \end{bmatrix} = \begin{bmatrix} Y_{R} - X_{R\hat{p}} \tilde{K}_p \\ Y_{L} - X_{L\hat{p}} \tilde{K}_p \end{bmatrix}.$$ \hspace{1cm} (19)

Although the coefficient matrix is singular, a solution will exist in two cases. First, at $\omega = 0$, $K_d(0, \theta_S, \gamma)$ is arbitrary and $K_i(0, \theta_S, \gamma) = 0$, unless $I_m(0) = R_c(0) = 0$, which holds only when the plant has a zero at the origin. In such a case, a PID compensator should be avoided, as the PID pole cancels the zero at the origin and the system becomes internally unstable. A second set of solutions occurs at any frequency $\omega$, where $K_p(\omega, \theta_S, \gamma)$, from equation (13), is equal to $\tilde{K}_p$. At these frequencies, $K_d(\omega, \theta_S, \gamma)$ and $K_i(\omega, \theta_S, \gamma)$ must satisfy the following straight line equation:
\[ K_d(\omega_i, \theta_s, \gamma) = \frac{K_i(\omega_i, \theta_s, \gamma)}{\omega_i^2} + \frac{\left( \frac{1}{\gamma} \sin \theta_s \right) R_e(\omega_i) + \left( -\frac{1}{\gamma} \cos \theta_s + 1 \right) I_m(\omega_i)}{\omega_i \left| G_p(j\omega_i) \right|^2}. \]  

(20)

3.3 Experimental Results

In this section, a PID controller is designed to regulate the shaft position of a SRV-02 DC motor from Quanser Consulting, Incorporated. The feedback loop has a communication delay of 0.1 seconds. The PID controller should stabilize the system and satisfy the sensitivity constraint in equation (6) for \( \gamma = 2 \).

To begin, the frequency response is measured experimentally. Equations (13) and (14) are used in the \( \left( K_p, K_i \right) \) plane for a fixed value of \( \tilde{K}_d = 0.2 \). As discussed previously, the PID stability boundary can be found by setting \( \gamma = \infty \). All PID controllers that satisfy the sensitivity constraint in equation (6) are found by setting \( \gamma = 2 \) and finding the intersection of all regions for \( \theta_s \in [0,2\pi) \).

The region that satisfies the sensitivity constraint and the stability boundary is shown in Figure 2. The intersection of all regions inside the stability boundary of the \( \left( K_p, K_i \right) \) plane is the sensitivity region. To verify the results, an arbitrary controller from this region is chosen, giving the PID controller

\[ G_c(s) = 3.4 + \frac{6}{s} + 0.2s. \]  

(21)
The plot of $|S(j\omega)|$ for the controller in equation (21) is shown in Figure 3. As can be seen,

$$\|S(j\omega)\|_{\infty} = 1.3,$$

which is less than $\gamma = 2$.

Figure 2. Stability boundary and sensitivity region in the $(K_p, K_i)$ plane.
The second method uses equations (13) and (17) in the $(K_p, K_d)$ plane for a fixed value of $\tilde{K}_i = 0.75$. The PID controller is designed to satisfy the sensitivity constraint with $\gamma = 2$. The region that satisfies the sensitivity constraint and the stability boundary is shown in Figure 4. The intersection of all regions inside the stability boundary of the $(K_p, K_d)$ plane is the sensitivity region. To verify the results, an arbitrary controller from this region is chosen, giving the PID controller as
The plot of \( |S(j\omega)| \) for the controller in equation (22) is shown in Figure 5. As can be seen, 

\[
\|S(j\omega)\|_\infty = 1.29, \text{ which is less than } \gamma = 2.
\]

Figure 4. Stability boundary and sensitivity region in the \((K_p, K_d)\) plane.
The third method is applied in the \((K_i, K_d)\) plane for a fixed value of \(\tilde{K}_p = 0.5\). Plots of \(K_p(\omega, \theta_S, \infty)\) and \(K_p(\omega, \theta_S, \gamma)\) from equation (13) for various values of \(\theta_S \in [0, 2\pi]\) are shown in Figure 6. For each curve, \(\omega_i\) are the frequencies at which \(K_p(\omega, \theta_S, \gamma) = \tilde{K}_p = 0.5\). Each \(\omega_i\) is substituted into equation (20) to find the required boundaries. In addition, there is the boundary at \(K_i(0, \theta_S, \gamma) = 0\).
Figure 6. Plots of $K_p(\omega, \theta_s, \infty)$ and $K_p(\omega, \theta_s, \gamma)$ for various values of $\theta_s \in [0, 2\pi]$.

The region that satisfies the sensitivity constraint and the stability boundary is shown in Figure 7. The intersection of all regions inside the stability boundary of the $(K_i, K_d)$ plane is the sensitivity region. To verify the results, an arbitrary controller from this region is chosen, giving the PID controller as

$$G_c(s) = 0.5 + \frac{0.48}{s} + 0.2s.$$  \hspace{1cm} (23)
The plot of $|S(j\omega)|$ for the controller in equation (23) is shown in Figure 8. As can be seen, 

$$\|S(j\omega)\|_\infty = 1.43$$

which is less than $\gamma = 2$.

Figure 7. Stability boundary and sensitivity region in the $(K_i, K_d)$ plane.
3.4 Conclusion

In this chapter, a technique was developed for finding all achievable PID controllers that simultaneously stabilize the closed-loop system and satisfy an $H_\infty$ sensitivity constraint [36]. This method is applicable to SISO proper transfer functions of any order with time-delay. Since this work builds upon the straightforward development in [15] and [16], it does not require the plant transfer function model, only the frequency response of the system. Experimental results from a DC motor were used to demonstrate the application of this method.
CHAPTER 4
WEIGHTED SENSITIVITY PROBLEM

4.1 Introduction

In the weighted sensitivity problem, weights are added to the $H_{\infty}$ sensitivity constraint in order to specify performance requirements. These performance specifications could include the bandwidth, maximum tracking error at a particular frequency, maximum steady state error, percent overshoot, settling time, rise time, or $M_s$ [5].

In this chapter, a technique is presented for finding all achievable PID controllers that simultaneously stabilize the closed-loop system and satisfy an $H_{\infty}$ weighted sensitivity constraint. The technique is applicable to SISO linear time invariant systems of arbitrary order with time-delay. This work builds upon the straightforward development in [15], [16], and [36] regarding the stability and the $H_{\infty}$ sensitivity problems, respectively. Consequently, it does not require the plant transfer function, only its frequency response. If the plant transfer function is known, we can apply the same procedures by first computing the frequency response.

The remainder of this chapter is organized as follows. Section 4.2 introduces the design methodology. In Section 4.3, the selection of weighting functions is presented. In Section 4.4, a numerical example is shown to demonstrate the application of this method. Finally, the conclusion of this chapter is summarized in Section 4.5.

4.2 Design Methodology

Consider the SISO and LTI system shown in Figure 9, where $G_p(s)$ is the plant, $G_c(s)$ is the PID controller, and $W_s(s)$ is the sensitivity function weight. The input signal and the weighted error signal are $R(s)$ and $Z(s)$, respectively, and $Y(s)$ is the measurement output.
The plant transfer function can be written as equation (1), and the PID controller is defined as equation (2). The error signal is written as

\[
Z(s) = W_s(s)E(s)
\]
\[
= W_s(s)\left( R(s) - Y(s) \right)
\]
\[
= W_s(s)S(s)R(s).
\]  

(24)

Figure 9. Block diagram of the system with weighted sensitivity.

All transfer functions in Figure 9 can be expressed in the frequency domain. The plant transfer function is defined as equation (4), and the PID controller is defined in the frequency domain as equation (5). The weighted sensitivity function \( W_s(s) \) can be written in terms of its real and imaginary parts as

\[
W_s(j \omega) = A_s(\omega) + jB_s(\omega).
\]  

(25)

In this section, the problem is to find all PID controllers that stabilize the system and satisfy the weighted sensitivity constraint as
\[ \left\| W_s(j\omega)S(j\omega) \right\|_\infty \leq \gamma, \quad (26) \]

where \( S(j\omega) \) is the sensitivity function, and \( \gamma \) is a positive real scalar and ideally is one. The complex functions in equation (26) for a SISO system can be written in terms of magnitudes and phase angles as

\[ \left\| W_s(j\omega)S(j\omega)e^{j\theta_s} \right\| \leq \gamma \quad \forall \omega, \quad (27) \]

If equation (27) holds, then for each value of \( \omega \),

\[ W_s(j\omega)S(j\omega)e^{j\theta_s} \leq \gamma \quad \forall \omega, \quad (28) \]

must be true for some \( \theta_s \in [0, 2\pi) \), where \( \theta_s = -\angle W_s(j\omega)S(j\omega) \). Consequently, all PID controllers that satisfy equation (26) must lie at the intersection of all controllers that satisfy equation (28) for all \( \theta_s \in [0, 2\pi) \) [38].

To find this region, for each value of \( \theta_s \in [0, 2\pi) \) all PID controllers will be found on the boundary of equation (28). It is easy to show from equation (28) that all PID controllers on the boundary must satisfy

\[ \quad P(\omega, \theta_s, \gamma) = 0, \quad (29) \]
where \( P(\omega, \theta_s, \gamma) = 1 + G_p(j\omega)G_c(j\omega) - \frac{1}{\gamma} W_s(j\omega)e^{j\theta_s} \). By substituting equations (4), (5), (25) and \( e^{j\theta_s} = \cos \theta_s + j \sin \theta_s \) into equation (29), the frequency response of this “modified” characteristic polynomial can be rewritten as

\[
P(\omega, \theta_s, \gamma) = 1 + \left( R_e(\omega) + j I_m(\omega) \right) \left( K_p + \frac{K_i}{j\omega} + K_d j\omega \right) - \left( \frac{1}{\gamma} \left( A_s(\omega) + j B_s(\omega) \right) \left( \cos \theta_s + j \sin \theta_s \right) \right).
\]

Note that equation (30) is reduced to the frequency response of the standard closed-loop characteristic polynomial as \( \gamma \to \infty \). Expanding equation (30) and setting the real and imaginary parts to zero yields

\[
X_{R_p} K_p + X_{R_i} K_i + X_{R_d} K_d = Y_R,
\]

and

\[
X_{I_p} K_p + X_{I_i} K_i + X_{I_d} K_d = Y_I,
\]

where
This is a three-dimensional system in terms of the controller parameters $K_p$, $K_i$, and $K_d$. First, the boundary of equation (29) will be found in the $(K_p, K_i)$ plane for a fixed value of $K_d$. After setting $K_d$ to the fixed value $\tilde{K}_d$, equations (31) and (32) can be rewritten as

$$\begin{vmatrix}
X_{Rp} & X_{Ri} & \left[K_p\right] \\
X_{Ip} & X_{Ii} & \left[K_i\right]
\end{vmatrix}
= \begin{vmatrix}
Y_{R} - X_{Rd}\tilde{K}_d \\
Y_{I} - X_{Id}\tilde{K}_d
\end{vmatrix}.$$  \hspace{1cm} (33)$$

Solving equation (33) for all $\omega \neq 0$ and $\theta_s \in [0,2\pi)$ gives:
\[
K_p(\omega, \theta_S, \gamma) = \frac{1}{\gamma} \left( \frac{A_S(\omega) \cos \theta_S - B_S(\omega) \sin \theta_S}{A_S(\omega) \sin \theta_S + B_S(\omega) \cos \theta_S} \right) \left( R_e(\omega) + \frac{I_m(\omega)}{|G_p(j\omega)|^2} \right),
\]

(34)

and

\[
K_i(\omega, \theta_S, \gamma) = \omega^2 \tilde{K}_d + \left( \frac{-\omega}{\gamma} \left( \frac{A_S(\omega) \sin \theta_S + B_S(\omega) \cos \theta_S}{A_S(\omega) \cos \theta_S - B_S(\omega) \sin \theta_S} \right) R_e(\omega) - \omega I_m(\omega) \right) \left( \frac{1}{|G_p(j\omega)|^2} \right).
\]

(35)

Setting \( \omega = 0 \) in equation (33) yields

\[
\begin{bmatrix}
0 & X_{Ri}(0) \\
0 & X_{Hi}(0)
\end{bmatrix}
\begin{bmatrix}
K_p \\
K_i
\end{bmatrix} = \begin{bmatrix}
0 \\
0
\end{bmatrix},
\]

(36)

and it can be concluded that \( K_p(0, \theta_S, \gamma) \) is arbitrary and \( K_i(0, \theta_S, \gamma) = 0 \), unless

\( I_m(0) = R_e(0) = 0 \), which holds only when \( G_p(s) \) has a zero at the origin.

The procedure can be repeated in the \((K_p, K_i)\) plane. After setting \( K_i \) to a fixed value \( \tilde{K}_i \), equations (31) and (32) can be rewritten as
\[
\begin{bmatrix}
 X_{R_p} & X_{R_d} \\
 X_{I_p} & X_{I_d}
\end{bmatrix}
\begin{bmatrix}
 K_p \\
 K_d
\end{bmatrix}
= \begin{bmatrix}
 Y_R - X_{R_i} \tilde{K}_i \\
 Y_I - X_{I_i} \tilde{K}_i
\end{bmatrix}.
\]  
(37)

Solving equation (37) for all \( \omega \neq 0 \) and \( \theta_s \in [0, 2\pi) \) gives the same expression as equation (34) for \( K_p(\omega, \theta_s, \gamma) \), and the following equation for \( K_d(\omega, \theta_s, \gamma) \):

\[
K_d(\omega, \theta_s, \gamma) = \frac{\tilde{K}_i}{\omega^2} + \left( \frac{1}{\gamma} \left( \frac{A_S(\omega) \sin \theta_s + B_S(\omega) \cos \theta_s}{R_e(\omega)} + \right) + I_m(\omega) \right) \omega |G_p(j\omega)|^2. 
\]  
(38)

At \( \omega = 0 \), \( \tilde{K}_i \) must be equal to zero for a solution to exist. Furthermore, as \( I_m(0) = 0 \) for all real plants, \( \frac{A_S(0) \sin \theta_s + B_S(0) \cos \theta_s}{\gamma} \) must be equal to zero for a solution to exist. This is true, for example, when \( \gamma = \infty \) (stability case) or \( \theta_s = 0 \), since \( B_S(0) \) is generally zero for polynomial weights with real coefficients. In these special cases, \( K_d(0, \theta_s, \gamma) \) is arbitrary and

\[
K_p(0, \theta_s, \gamma) = -\frac{1}{\gamma} \left( \frac{A_S(0) \cos \theta_s - B_S(0) \sin \theta_s}{R_e(0)} \right) - \frac{1}{\gamma}. 
\]  
(39)

Last, the solution is found in the \((K_i, K_d)\) plane. After setting \( K_p \) to a fixed value of \( \tilde{K}_p \), equations (31) and (32) are rewritten as

30
\[
\begin{bmatrix}
    X_{Ri} & X_{Rd} \\
    X_{Li} & X_{Ld}
\end{bmatrix}
\begin{bmatrix}
    K_i \\
    K_d
\end{bmatrix}
= \begin{bmatrix}
    Y_{Ri} - X_{Rp} \hat{K}_p \\
    Y_{Li} - X_{Lp} \hat{K}_p
\end{bmatrix}.
\] (40)

Although the coefficient matrix is singular, a solution will exist in two cases. First, at \( \omega = 0 \), \( K_d(0, \theta_S, \gamma) \) is arbitrary and \( K_i(0, \theta_S, \gamma) = 0 \), unless \( I_m(0) = R_e(0) = 0 \), which holds only when the plant has a zero at the origin. In such a case, a PID compensator should be avoided, as the PID pole cancels the zero at the origin and the system becomes internally unstable. A second set of solutions occurs at any frequency \( \omega_i \), where \( K_p(\omega_i, \theta_S, \gamma) \), from equation (34), is equal to \( \hat{K}_p \). At these frequencies, \( K_d(\omega_i, \theta_S, \gamma) \) and \( K_i(\omega_i, \theta_S, \gamma) \) must satisfy the following straight line equation:

\[
K_d(\omega_i, \theta_S, \gamma) = \frac{K_i(\omega_i, \theta_S, \gamma)}{\omega_i^2} + \frac{1}{\gamma} \frac{\left( A_S(\omega_i) \sin \theta_S + B_S(\omega_i) \cos \theta_S \right) R_e(\omega_i) + I_m(\omega_i)}{\omega_i |G_p(j \omega_i)|^2}.
\] (41)

4.3 Selection of Weighting Functions

There are numerous articles regarding the selection of the weighting function [2], [5], [43], and [44]. The selection of the weighting function for a system provides the designer freedom; consequently, there is no unique solution for all cases. The performance specifications of a system closely correspond to the sensitivity function weight.
The sensitivity function weight should be selected to specify, both bandwidth and the peak sensitivity. A standard form for the sensitivity function weight is given by

\[
W_s = \left( \frac{s + \omega_b k M_s}{s + \omega_b k \sqrt{\varepsilon}} \right)^k ,
\]

where \(M_s\) is the peak sensitivity function, \(\omega_b\) is the closed-loop bandwidth, \(k \geq 1\) and is an integer number, and \(0 \leq \varepsilon \leq 1\) is the upper-bound magnitude on the sensitivity function at low frequency. For a perfect tracking error, \(\varepsilon = 0\), otherwise, it is a real number \(0 < \varepsilon \leq 1\). The steady state error for a unit step input must be less than \(\varepsilon\), i.e., the sensitivity function magnitude must satisfy \(|S(0)| \leq \varepsilon\). More details can be found in Skogestad and Postl ethwaite [5] and Zhou and Doyle [44].

4.4 Numerical Example

In this section, a numerical example demonstrates the application of this method. Consider the second-order plant transfer function [15],

\[
G_p(s) = \frac{-0.5s + 1}{(s + 1)(2s + 1)} e^{-0.6s} .
\]

The goal is to find all PID controllers that stabilize the system and satisfy the weighted sensitivity constraint in equation (26), where \(\gamma = 1\). The closed-loop step response is required
to have an overshoot less than 5 percent, a settling time less than 40 seconds, and zero steady state error.

The sensitivity function weight is chosen to satisfy the performance requirement for the closed-loop system.

\[
W_S(s) = \frac{0.78(s + 0.13)}{s + 0.08}.
\]  

Equations (34) and (35) are used in the \((K_p, K_i)\) plane for a fixed value of \(\tilde{K}_d = 0.4\). As discussed in Section 4.2, the PID stability boundary can be found by setting \(\gamma = \infty\). All PID controllers that satisfy the weighted sensitivity constraint in equation (26) are found by setting \(\gamma = 1\) and finding the intersection of all regions for \(\theta_s \in [0, 2\pi)\) and the given frequency range \(\omega\).

The region that satisfies the weighted sensitivity constraint and the stability boundary is shown in Figure 10. The intersection of all regions inside the stability boundary of the \((K_p, K_i)\) plane is the weighted sensitivity region. To verify the results, an arbitrary controller from this region is chosen, giving the PID controller as

\[
G_c(s) = 0.49 + \frac{0.1}{s} + 0.4 s.
\]

The plot of \(|W_S(j\omega)S(j\omega)|\) for the controller in equation (45) is shown in Figure 11. As can be seen, \(\|W_S(j\omega)S(j\omega)\|_\infty = 0.926\), which is less than \(\gamma = 1\).
Figure 10. Stability boundary and weighted sensitivity region in the \((K_p, K_i)\) plane for a fixed 
\(\tilde{K}_d = 0.4\).
Figure 11. Magnitude of \( W_s S \) for \( G_c(s) = 0.49 + \frac{0.1}{s} + 0.4s \).

The step response of the closed-loop system with the arbitrary PID controller in equation (45) is shown in Figure 12. As can be seen, the closed-loop step response has no overshoot, a setting time less than 40 seconds, and zero steady-state error.
The second method uses equations (34) and (38) in the \((K_p, K_d)\) plane for a fixed value of \(\tilde{K}_i = 0.2\). The PID controller is designed to satisfy the weighted sensitivity constraint with \(\gamma = 1\). The region that satisfies the weighted sensitivity constraint and the stability boundary is shown in Figure 13. The intersection of all regions inside the stability boundary of the \((K_p, K_d)\) plane is the weighted sensitivity region. To verify the results, an arbitrary controller from this region is chosen, giving the PID controller as
\[ G_c(s) = 0.48 + \frac{0.2}{s} + 0.50s . \] (46)

The plot of \( |W_S(j\omega)S(j\omega)| \) for the controller in equation (46) is shown in Figure 14. As can be seen, \( \|W_S(j\omega)S(j\omega)\|_\infty = 0.936 \), which is less than \( \gamma = 1 \).

Figure 13. Stability boundary and weighted sensitivity region in the \((K_p, K_d)\) plane for a fixed \( \tilde{K}_i = 0.2 \).
Figure 14. Magnitude of $W_S S$ for $G_c(s) = 0.48 + \frac{0.2}{s} + 0.50 s$.

The third method is applied in the $(K_i, K_d)$ plane for a fixed value of $\tilde{K}_p = 0.5$. Plots of $K_p(\omega_i, \theta_S, \infty)$ and $K_p(\omega_i, \theta_S, \gamma)$, from equation (34), for various values of $\theta_S \in [0, 2\pi)$ are shown in Figure 15. For each curve, the $\omega_i$ are the frequencies at which $K_p(\omega_i, \theta_S, \gamma) = \tilde{K}_p = 0.5$. Each $\omega_i$ is substituted into (41) to find the required boundaries. In addition, there is the boundary at $K_i(0, \theta_S, \gamma) = 0$. 
Figure 15. Plots of $K_p(\omega, \theta_s, \infty)$ and $K_p(\omega, \theta_s, \gamma)$ for various values of $\theta_s \in [0, 2\pi)$.

The region that satisfies the weighted sensitivity constraint and the stability boundary is shown in Figure 16. The intersection of all regions inside the stability boundary of the $(K_i, K_d)$ plane is the weighted sensitivity region. To verify the results, an arbitrary controller from this region is chosen, giving the PID controller as

$$G_c(s) = 0.5 + \frac{0.11}{s} + 0.39s.$$  (47)
The plot of $\|W_S(j\omega)S(j\omega)\|$ for the controller in equation (47) is shown in Figure 17. As can be seen, $\|W_S(j\omega)S(j\omega)\|_\infty = 0.927$, which is less than $\gamma = 1$. The Matlab program for weighted sensitivity problem can be found in the Appendix.

Figure 16. Stability boundary and weighted sensitivity region in the $(K_i, K_d)$ plane for a fixed $\tilde{K}_p = 0.5$. 

\[ X: 0.1089 \]
\[ Y: 0.386 \]
Figure 17. Magnitude of $W_sS$ for $G_c(s) = 0.5 + \frac{0.11}{s} + 0.39s$.

4.5 Conclusion

In this chapter, a graphical technique was introduced to find all achievable PID controllers that simultaneously stabilized the closed-loop system and satisfied an $H_\infty$ weighted sensitivity constraint. This method was applicable to SISO LTI systems, of arbitrary order with time-delay [38]. This method was simple to understand and required only the frequency response of the plant. A numerical example was presented that demonstrated the application of this method.
CHAPTER 5

COMPLEMENTARY SENSITIVITY PROBLEM

5.1 Introduction

Feedback control of a proper system with integration action results in a complementary sensitivity function with a magnitude of one at low frequencies and a small magnitude at high frequencies. The problem is in the intermediate frequency range, where the feedback control degrades performance. At these frequencies the value of complementary sensitivity peak is larger than one, which is a measure of the worst-case performance degradation. In SISO and LTI systems, $M_T$ is the maximum magnitude of the complementary sensitivity function in the frequency domain. Typically, this magnitude should be less than 1.25 (2 dB) to achieve good performance and robustness. A large value of $M_T$ indicates poor performance as well as poor robustness. A smaller value of $M_T$ indicates a lower percent overshoot in the step response and also introduces a margin of robustness [5].

In this chapter, a technique is developed for finding all achievable PID controllers that simultaneously stabilize the closed-loop system and satisfy an $H_\infty$ complementary sensitivity constraint [37]. This method is applicable for SISO proper transfer functions of any order with time-delay. Since this work builds upon straightforward development in [15], [16], and [36], it does not require the plant transfer function model, only the frequency response of the system. If the plant transfer function is known, the same procedures can be applied by first computing the frequency response.
The remainder of this chapter is organized as follows. In Section 5.2, the design methodology is introduced. In Section 5.3, a numerical example is presented to demonstrate the application of this method. Finally, the conclusion of this chapter is summarized in Section 5.4.

5.2 Design Methodology

Consider the SISO and LTI system shown in Figure 18, where $G_p(s)$ is the plant and $G_c(s)$ is the PID controller. The reference input and the output are $R(s)$ and $Y(s)$, respectively. The output signal $Y(s)$ can be written in terms of the input signal as

$$Y(s) = T(s)R(s),$$  \hspace{1cm} (48)

where $T(s) = \frac{G_p(s)G_c(s)}{1+G_p(s)G_c(s)}$ is the complementary sensitivity function. The plant transfer function can be written as equation (1), and the PID controller is defined as equation (2).

![Figure 18. Block diagram of the system.](image)

All transfer functions in Figure 18 can be expressed in the frequency domain. The plant transfer function can be written as equation (4) and the PID controller is defined as equation (5) in the frequency domain. In this section, the problem is to find all PID controllers that satisfy the complementary sensitivity constraint as
\[ \left\| T(j\omega) \right\|_\infty \leq \gamma, \]  \hspace{1cm} (49)

where \( T(j\omega) = \frac{G_p(j\omega)G_c(j\omega)}{1 + G_p(j\omega)G_c(j\omega)} \) is the complementary sensitivity function in the frequency domain, and \( \gamma \) is a real positive scalar that corresponds to \( M_T \) for a SISO system. The complex function in equation (49) for a SISO system for each value of \( \omega \) can be written in terms of its magnitude and phase angles as

\[ \left\| T(j\omega) e^{\angle T(j\omega)} \right\| \leq \gamma \quad \forall \omega. \]  \hspace{1cm} (50)

If equation (50) holds, then for each value of \( \omega \)

\[ T(j\omega)e^{i\theta_T} \leq \gamma, \]  \hspace{1cm} (51)

must be true for some \( \theta_T \in [0,2\pi) \), where \( \theta_T = -\angle T(j\omega) \). Consequently, all PID controllers that satisfy equation (49) must lie at the intersection of all controllers that satisfy equation (51) for all \( \theta_T \in [0,2\pi) \) [37].

To accomplish this, for each value of \( \theta_T \in [0,2\pi) \), all PID controllers on the boundary of equation (51) will be found. It is easy to show from equation (51), all the PID controllers on the boundary must satisfy
\[ P(\omega, \theta_T, \gamma) = 0, \quad (52) \]

where \( P(\omega, \theta_T, \gamma) = 1 + G_p(j \omega)G_e(j \omega) - \frac{1}{\gamma} G_p(j \omega)G_e(j \omega)e^{j \theta_T} \). By substituting equations (4), (5), and \( e^{j \theta_T} = \cos \theta_T + j \sin \theta_T \) into equation (52), the frequency response of this “modified” characteristic polynomial can be rewritten as

\[
P(\omega, \theta_T, \gamma) = 1 + \left( R_e(\omega) + j I_m(\omega) \right) \left( K_p + \frac{K_i}{j \omega} + K_d j \omega \right) - \frac{1}{\gamma} \left( R_e(\omega) + j I_m(\omega) \right) \left( K_p + \frac{K_i}{j \omega} + K_d j \omega \right) \left( \cos \theta_T + j \sin \theta_T \right). \quad (53) \]

Note that equation (53) reduces to the frequency response of the standard closed-loop characteristic polynomial as \( \gamma \to \infty \). Expanding equation (52) in terms of its real and imaginary parts yields

\[
X_{R_p} K_p + X_{R_i} K_i + X_{R_d} K_d = Y_R, \quad (54) \\
\]

and

\[
X_{I_p} K_p + X_{I_i} K_i + X_{I_d} K_d = Y_I, \quad (55) \]

where

\[
X_{R_p} = \omega R_e(\omega) \left( 1 - \frac{1}{\gamma} \cos \theta_T \right) + I_m(\omega) \left( \frac{1}{\gamma} \sin \theta_T \right), \]

45
\[ X_{Ri} = \left\{ I_m(\omega) \left( 1 - \frac{1}{\gamma} \cos \theta_T \right) - R_e(\omega) \left( \frac{1}{\gamma} \sin \theta_T \right) \right\}, \]

\[ X_{Rd} = -\omega^2 \left\{ I_m(\omega) \left( 1 - \frac{1}{\gamma} \cos \theta_T \right) - R_e(\omega) \left( \frac{1}{\gamma} \sin \theta_T \right) \right\}, \]

\[ Y_R = -\omega, \]

\[ X_{Ip} = \omega \left\{ I_m(\omega) \left( 1 - \frac{1}{\gamma} \cos \theta_T \right) - R_e(\omega) \left( \frac{1}{\gamma} \sin \theta_T \right) \right\}, \]

\[ X_{Ik} = -K_i \left\{ R_e(\omega) \left( 1 - \frac{1}{\gamma} \cos \theta_T \right) + I_m(\omega) \left( \frac{1}{\gamma} \sin \theta_T \right) \right\}, \]

\[ X_{Id} = \omega^2 \left\{ R_e(\omega) \left( 1 - \frac{1}{\gamma} \cos \theta_T \right) + I_m(\omega) \left( \frac{1}{\gamma} \sin \theta_T \right) \right\}, \quad \text{and} \]

\[ Y_i = 0. \]

This is a three-dimensional system in terms of the controller parameters \( K_p, K_i, \) and \( K_d \). This problem is solved in three different planes as a function of three variables \( \omega, \theta_T, \) and \( \gamma \). First, the boundary of equation (51) will be found in the \((K_p, K_i)\) plane for a fixed value of \( K_d \). After setting \( K_d \) to the fixed value \( \tilde{K}_d \), equations (54) and (55) can be rewritten as

\[
\begin{bmatrix}
X_{Rp} & X_{Ri} \\
X_{Ip} & X_{Ik}
\end{bmatrix}
\begin{bmatrix}
K_p \\
K_i
\end{bmatrix}
= 
\begin{bmatrix}
Y_R - X_{Rd} \tilde{K}_d \\
Y_i - X_{Id} \tilde{K}_d
\end{bmatrix}.
\]

Solving equation (56) for all \( \omega \neq 0 \) and \( \theta_T \in [0, 2\pi) \) gives:
\[ K_p(\omega, \theta_T, \gamma) = \frac{1}{\gamma} \left( R_e(\omega) \cos \theta_T - I_m(\omega) \sin \theta_T \right) - R_e(\omega) \left| G_p(j\omega) \right|^2 \left( 1 - \frac{2}{\gamma} \cos \theta_T + \frac{1}{\gamma^2} \right), \]  

(57)

and

\[ K_i(\omega, \theta_T, \gamma) = \omega^2 \tilde{K}_d + \frac{\omega \left( R_e(\omega) \sin \theta_T + I_m(\omega) \cos \theta_T \right) - \omega I_m(\omega)}{\left| G_p(j\omega) \right|^2 \left( 1 - \frac{2}{\gamma} \cos \theta_T + \frac{1}{\gamma^2} \right)} \cdot \]  

(58)

Note that if \( \gamma = 1, \ \theta_T = 0 \) should be avoided, as the denominators of equation (57) will go to zero. Setting \( \omega = 0 \) in equation (56) yields

\[
\begin{bmatrix} 0 & X_{RI}(0) \\ 0 & X_{RI}(0) \end{bmatrix} \begin{bmatrix} K_p \\ K_i \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix},
\]

(59)

and it can be concluded that \( K_p(0, \theta_T, \gamma) \) is arbitrary and \( K_i(0, \theta_T, \gamma) = 0 \), unless \( I_m(0) = R_e(0) = 0 \), which holds only when \( G_p(s) \) has a zero at the origin.

The procedure can be repeated in the \((K_p, K_d)\) plane. After setting \( K_i \) to a fixed value \( \tilde{K}_i \), equations (54) and (55) can be rewritten as

\[
\begin{bmatrix} X_{Rp} & X_{Rd} \\ X_{Ip} & X_{Id} \end{bmatrix} \begin{bmatrix} K_p \\ K_d \end{bmatrix} = \begin{bmatrix} Y_R - X_{RI} \tilde{K}_i \\ Y_I - X_{RI} \tilde{K}_i \end{bmatrix}.
\]

(60)
Solving equation (60) for all \( \omega \neq 0 \) and \( \theta_T \in [0,2\pi) \) gives the same expression as equation (57) for \( K_p(\omega, \theta_T, \gamma) \), and the following equation for \( K_d(\omega, \theta_T, \gamma) \):

\[
K_d(\omega, \theta_T, \gamma) = \frac{\tilde{K}}{\omega^2} + \frac{-1}{\omega \left| G_p(j\omega) \right|^2} \left( \frac{1}{\gamma} \left( R_e(\omega) \sin \theta_T + I_m(\omega) \cos \theta_T \right) + I_m(\omega) \right). \tag{61}
\]

At \( \omega = 0 \), \( \tilde{K} \) must be equal to zero for a solution to exist. Furthermore, as \( I_m(0) = 0 \) for all real plants, \( K_d(0, \theta_T, \gamma) \) is arbitrary and

\[
K_p(0, \theta_T, \gamma) = \frac{-1}{\left( 1 - \frac{1}{\gamma} \cos \theta_T \right) R_e(0)}. \tag{62}
\]

Last, the solution is found in the \((K_i, K_d)\) plane. After setting \( K_p \) to a fixed value of \( \tilde{K}_p \), equations (54) and (55) are rewritten as

\[
\begin{bmatrix}
X_{Ri} & X_{Rd}
\end{bmatrix}
\begin{bmatrix}
K_i
\end{bmatrix}
= \begin{bmatrix}
Y_R - X_{Rp} \tilde{K}_p
\end{bmatrix},
\begin{bmatrix}
X_{Li} & X_{Ld}
\end{bmatrix}
\begin{bmatrix}
K_d
\end{bmatrix}
= \begin{bmatrix}
Y_I - X_{Ip} \tilde{K}_p
\end{bmatrix}. \tag{63}
\]
Although the coefficient matrix is singular, a solution will exist in two cases. First, at \( \omega = 0 \), \( K_d(0, \theta_T, \gamma) \) is arbitrary and \( K_i(0, \theta_T, \gamma) = 0 \), unless \( I_m(0) = R_c(0) = 0 \), which holds only when the plant has a zero at the origin. In such a case, a PID compensator should be avoided, as the PID pole cancels the zero at the origin and the system becomes internally unstable. A second set of solutions occurs at any frequency \( \omega_i \), where \( K_p(\omega_i, \theta_T, \gamma) \), from equation (57), is equal to \( \tilde{K}_p \). At these frequencies, \( K_d(\omega_i, \theta_T, \gamma) \) and \( K_i(\omega_i, \theta_T, \gamma) \) must satisfy the following straight line equation

\[
K_d(\omega_i, \theta_T, \gamma) = \frac{K_i(\omega_i, \theta_T, \gamma)}{\omega_i^2} + \frac{-1}{\gamma} \left( \frac{R_c(\omega_i) \sin \theta_T + I_m(\omega_i) \cos \theta_T}{\omega_i G_p(j\omega_i)} \right) \frac{1}{1 - \frac{2 \gamma}{\cos \theta_T + \frac{1}{\gamma^2}}}.
\]

(64)

5.3 Numerical Example

In this section, a numerical example demonstrates the application of this method. Consider the second-order plant transfer function [15]

\[
G_p(s) = \frac{-0.5s + 1}{(s + 1)(2s + 1)} e^{-0.6s}.
\]

(65)

The goal is to design a PID controller such that the closed-loop system is stable, and the complementary sensitivity constraint in equation (49) is satisfied for \( \gamma = 2 \).
Equations (57) and (58) are used in the \((K_p, K_i)\) plane for a fixed value of \(K_d = 0.1\).

As discussed previously, the PID stability boundary can be found by setting \(\gamma = \infty\). All PID controllers that satisfy the complementary sensitivity constraint in equation (49) are found by setting \(\gamma = 2\) and finding the intersection of all regions for \(\theta_T \in [0, 2\pi)\) and the given frequency range \(\omega\).

The region that satisfies the complementary sensitivity constraint and the stability boundary is shown in Figure 19. The intersection of all regions inside the stability boundary of the \((K_p, K_i)\) plane is the complementary sensitivity region. To verify the results, an arbitrary controller from this region is chosen, giving the PID controller as

\[
G_c(s) = 0.8 + \frac{0.2}{s} + 0.1s.
\]  

(66)

The Bode plot of \(|T(j\omega)|\) for the controller in equation (66) is shown in Figure 20. As can be seen, \(\|T(j\omega)\|_\infty = 1\), which is less than \(\gamma = 2\).
Figure 19. Stability boundary and complementary sensitivity region in the \((K_p, K_i)\) plane for a fixed \(\tilde{K}_d = 0.1\).
Figure 20. Magnitude of $T(j\omega)$ for $G_{c}(s) = 0.8 + \frac{0.2}{s} + 0.1 s$.

The second method uses equation (57) and (61) in the $(K_p, K_d)$ plane for a fixed value of $\tilde{K}_1 = 0.1$. The PID controller is designed to satisfy the complementary sensitivity constraint with $\gamma = 2$. The region that satisfies the complementary sensitivity constraint and the stability boundary is shown in Figure 21. The intersection of all regions inside the stability boundary of the $(K_p, K_d)$ plane is the complementary sensitivity region. To verify the results, an arbitrary controller from this region is chosen, giving the PID controller as
\[ G_c(s) = 0.4 + \frac{0.1}{s} + 0.5s. \]  

(67)

The Bode plot of \( |T(j\omega)| \) for the controller in equation (67) is shown in Figure 22. As can be seen, \( \|T(j\omega)\|_\infty = 1 \), which is less than \( \gamma = 2 \).

Figure 21. Stability boundary and complementary sensitivity region in the \((K_p, K_d)\) plane for a fixed \( \tilde{K}_i = 0.1 \).
The third method is applied in the \((K_i, K_d)\) plane for a fixed value of \(\tilde{K}_p = 0.4\). Plots of \(K_p(\omega, \theta_T, \infty)\) and \(K_p(\omega, \theta_T, \gamma)\) for various values of \(\theta_T \in [0, 2\pi]\) are shown in Figure 23. For each curve, the \(\omega_i\) are the frequencies at which \(K_p(\omega_i, \theta_T, \gamma) = \tilde{K}_p = 0.4\). Each \(\omega_i\) is substituted into equation (64) to find the required boundaries. In addition, there is a boundary at \(K_i(0, \theta_T, \gamma) = 0\).
Figure 23. Plots of $K_p(\omega, \theta_T, \infty)$ and $K_p(\omega, \theta_T, \gamma)$ for various values of $\theta_T \in [0, 2\pi)$.

The region that satisfies the complementary sensitivity constraint and the stability boundary is shown in Figure 24. The intersection of all regions inside the stability boundary of the $(K_s, K_d)$ plane is the $H_\infty$ complementary sensitivity region. To verify the results, an arbitrary controller from this region is chosen, giving the PID controller as

$$G_c(s) = 0.4 + \frac{0.2}{s} + 0.5s.$$  \hspace{1cm} (68)
The Bode plot of $|T(j\omega)|$ for the controller in equation (68) is shown in Figure 25. As can be seen, $\|T(j\omega)\|_{\infty} = 1$, which is less than $\gamma = 2$.

Figure 24. Stability boundary and complementary sensitivity region in the $(K_i, K_d)$ plane for a fixed $\bar{K}_p = 0.4$. 

56
5.4 Conclusion

In this chapter, a technique was developed for finding all achievable PID controllers that simultaneously stabilize the closed-loop system and satisfy an $H_{\infty}$ complementary sensitivity constraint [37]. This method was applicable for SISO proper transfer functions of any order with time-delay. Since this work built upon the straightforward development in [15], [16], and [36] regarding the stability and sensitivity problems, respectively, it did not require the plant transfer function model, only the frequency response of the system. A numerical example was
introduced that demonstrated the application of this method to a given system transfer function with time-delay.
CHAPTER 6

ROBUST STABILITY PROBLEM

6.1 Introduction

In recent years, numerous articles on control systems have focused on designing controllers that make a system robust [2], [4], [5], [1], [9], and [44]. Robust control theory is a bridge between the mathematical model of industrial plant behavior and the actual industrial plant behavior. Robust control methods search for a boundary of uncertainty. This boundary contains a set of mathematical models for the actual industrial plant dynamics. Knowing the uncertainty boundaries allows the designer to find a controller for the worst case plant behavior. The main idea of robust control theory is to ensure that the design specifications are satisfied by the worst case uncertainty [4] and [5]. A control system is robust if the actual systems stay stable for all ranges of the uncertainty set.

In this chapter, weights are added to the complementary sensitivity constraint. This allows for consideration of the robust stability problem for systems with multiplicative uncertainty. The technique is applicable to proper SISO LTI systems of arbitrary order with uncertain time-delay. This work builds upon the straightforward development in [15], [16], [36], [37], [38] regarding the stability, $H_\infty$ sensitivity, $H_\infty$ complementary sensitivity, and weighted sensitivity problems. Consequently, it does not require plant transfer function coefficients, only its frequency response. The ability to include the time-delay in the nominal model of the system will often allow for designs with reduced conservativeness in plant uncertainty and an increase in size of the set of all PID controllers that robustly stabilize the system. If the plant transfer function is known, we can apply the same procedure by first computing the frequency response.
The remainder of this chapter is organized as follows. In Section 6.2, the design methodology is introduced. In Section 6.3, the selection of weighting functions is presented. In Section 6.4, a numerical example is presented to demonstrate the application of this method. Finally, the conclusion of this chapter is summarized in Section 6.5.

6.2 Design Methodology

Consider the SISO and LTI system shown in Figure 26, where $G_{\Delta}(s)$ represents the perturbed plants, $G_p(s)$ is the nominal plant, and $G_c(s)$ is the PID controller. The reference input and the output are $R(s)$ and $Y(s)$, respectively. $W_T(s)$ is the multiplicative weight, and $|\Delta_f(j\omega)| \leq 1$ is the uncertain perturbation. The nominal plant transfer function can be written as equation (1)

The ability to include the time-delay in the nominal model allows the designer to find much tighter uncertainty bounds in systems with known delays than would be possible otherwise [5]. The PID controller is defined as equation (2).

![Figure 26. Block diagram of the system with multiplicative uncertainty.](image)

The transfer functions in Figure 26 can all be expressed in frequency domain. The plant transfer function and the PID controller are defined as equations (4) and (5), respectively. The multiplicative weight $W_T$ is defined in terms of its real and imaginary part as
\[ W_T(j\omega) = A_T(\omega) + jB_T(\omega). \quad (69) \]

In this research, the problem is to find all PID controllers that robustly stabilize the feedback system in Figure 26 for all \( |\Delta(j\omega)| \leq 1 \). The system is robustly stable if the nominal system is stable and if

\[ \left\| W_T(j\omega)T(j\omega) \right\|_{\infty} \leq \gamma, \quad (70) \]

where \( T(j\omega) \) is the complementary sensitivity function and \( \gamma = 1 \). The constraint in equation (70) for a SISO system can be written as

\[ \left\| W_T(j\omega)T(j\omega) \right\| \leq \gamma \quad \forall \quad \omega. \quad (71) \]

Note that the complex function in equation (71) can be written in terms of its magnitude and phase angle as

\[ \left\| W_T(j\omega)T(j\omega) e^{j\theta_T(j\omega)} \right\| \leq \gamma \quad \forall \quad \omega. \quad (72) \]

If equation (72) holds, then for each value of \( \omega \)

\[ W_T(j\omega)T(j\omega)e^{j\theta_T} \leq \gamma \quad \forall \quad \omega, \quad (73) \]
must be true for some \( \theta_T \in [0, 2\pi) \), where \( \theta_T = -\angle W_T(j\omega)T(j\omega) \). Consequently, all PID controllers that satisfy equation (70) must lie at the intersection of all controllers that satisfy equation (73) for all \( \theta_T \in [0, 2\pi) \) [39].

To accomplish this, for each value of \( \theta_T \in [0, 2\pi) \) all PID controllers on the boundary of equation (73) are found. It is easy to show from equation (73) that all PID controllers on the boundary must satisfy

\[
P(\omega, \theta_T, \gamma) = 0, \tag{74}
\]

where

\[
P(\omega, \theta_T, \gamma) = 1 + G_p(j\omega)G_c(j\omega) - \frac{1}{\gamma} W_T(j\omega)G_p(j\omega)G_c(j\omega)e^{j\theta_T}.
\]

By substituting equations (1), (4), (5), (69), and \( e^{j\theta_T} = \cos \theta_T + j\sin \theta_T \) into equation (74), the frequency response of this “modified” characteristic polynomial can be rewritten as

\[
P(\omega, \theta_T, \gamma) = 1 + \left( R_c(\omega) + jI_m(\omega) \right) \left( K_p + \frac{K_i}{j\omega} + K_d j\omega \right) - \frac{1}{\gamma} \left( A_T(\omega) + jB_T(\omega) \right) \left( R_c(\omega) + jI_m(\omega) \right) \left( K_p + \frac{K_i}{j\omega} + K_d j\omega \right) \left( \cos \theta_T + j\sin \theta_T \right).
\]

\[
(75)
\]
Note that equation (75) reduces to the frequency response of the standard closed-loop characteristic polynomial as $\gamma \to \infty$. Expanding equation (75) in terms of its real and imaginary parts yields

$$X_{Rp} K_p + X_{Ri} K_i + X_{Rd} K_d = Y_R,$$  \hspace{1cm} (76)

and

$$X_{Ip} K_p + X_{Ii} K_i + X_{Id} K_d = Y_I,$$  \hspace{1cm} (77)

where

$$X_{Rp} = \omega \left( R_e(\omega) \left( -\alpha A_T(\omega) + \beta B_T(\omega) + 1 \right) + I_m(\omega) \left( \beta A_T(\omega) + \alpha B_T(\omega) \right) \right),$$

$$X_{Ri} = \left( R_e(\omega) \left( -\beta A_T(\omega) - \alpha B_T(\omega) \right) + I_m(\omega) \left( -\alpha A_T(\omega) + \beta B_T(\omega) + 1 \right) \right),$$

$$X_{Rd} = \omega^2 \left( R_e(\omega) \left( \beta A_T(\omega) + \alpha B_T(\omega) \right) + I_m(\omega) \left( \alpha A_T(\omega) - \beta B_T(\omega) - 1 \right) \right),$$

$$X_{Ip} = \omega \left( R_e(\omega) \left( -\beta A_T(\omega) - \alpha B_T(\omega) \right) + I_m(\omega) \left( -\alpha A_T(\omega) + \beta B_T(\omega) + 1 \right) \right),$$

$$X_{Ii} = \left( R_e(\omega) \left( \alpha A_T(\omega) - \beta B_T(\omega) - 1 \right) + I_m(\omega) \left( -\beta A_T(\omega) - \alpha B_T(\omega) \right) \right),$$

$$X_{Id} = \omega^2 \left( R_e(\omega) \left( -\alpha A_T(\omega) + \beta B_T(\omega) + 1 \right) + I_m(\omega) \left( \beta A_T(\omega) + \alpha B_T(\omega) \right) \right),$$

$$Y_R = -\omega, \hspace{0.5cm} Y_I = 0, \hspace{0.5cm} \alpha = \frac{1}{\gamma} \cos \theta_T, \hspace{0.5cm} \text{and} \hspace{0.5cm} \beta = \frac{1}{\gamma} \sin \theta_T.$$

This is a three-dimensional system in terms of controller parameters $K_p$, $K_i$, and $K_d$.

First, the boundary of equation (70) will be found in the $(K_p, K_i)$ plane for a fixed value of $K_d$.

After setting $K_d$ to the fixed value $\tilde{K}_d$, equations (76) and (77) can be rewritten as
\[
\begin{bmatrix}
X_{R_p} & X_{R_i} \\
X_{I_p} & X_{I_i}
\end{bmatrix}
\begin{bmatrix}
K_p \\
K_i
\end{bmatrix}
= \begin{bmatrix}
Y_{R_p} - X_{R_d} \bar{K}_d \\
Y_{I_p} - X_{I_d} \bar{K}_d
\end{bmatrix}.
\] (78)

Solving equation (78) for all \( \omega \neq 0 \) and \( \theta \in [0, 2\pi) \) gives:

\[
K_p(\omega, \theta, \gamma) = \frac{R_e(\omega)\left(\frac{1}{\gamma} + A_T(\omega) \cos \theta_T - \frac{1}{\gamma} B_T(\omega) \sin \theta_T - 1\right) -}{D(\omega)}
\]

\[
I_m(\omega)\left(\frac{1}{\gamma} A_T(\omega) \sin \theta_T + \frac{1}{\gamma} B_T(\omega) \cos \theta_T\right)
\]

\[
K_i(\omega, \theta_T, \gamma) = \omega^2 \bar{K}_d + \frac{\left(R_e(\omega)\left(\frac{1}{\gamma} A_T(\omega) \sin \theta_T + \frac{1}{\gamma} B_T(\omega) \cos \theta_T\right) + \right)}{D(\omega)}
\]

\[
I_m(\omega)\left(\frac{1}{\gamma} A_T(\omega) \cos \theta_T - \frac{1}{\gamma} B_T(\omega) \sin \theta_T - 1\right)
\]

where

\[
D(\omega) = \left|G_p(j\omega)\right|^2 \left(1 - \frac{2}{\gamma^2} (A_T \cos \theta_T - B_T(\omega) \sin \theta_T) + \frac{1}{\gamma^2} \left|W_T(j\omega)\right|^2\right),
\]

\[
\left|G_p(j\omega)\right|^2 = R_e^2(\omega) + I_m^2(\omega), \text{ and } \left|W_T(j\omega)\right|^2 = A_T^2(\omega) + B_T^2(\omega). \] Setting \( \omega = 0 \) in equation (78), yields
\[
\begin{bmatrix}
0 & X_{R_1}(0) \\
0 & X_{R_1}(0)
\end{bmatrix}
\begin{bmatrix}
K_p \\
K_i
\end{bmatrix} =
\begin{bmatrix}
0 \\
0
\end{bmatrix},
\]

(81)

and it can be concluded that \( K_p(0, \theta_T, \gamma) \) is arbitrary and \( K_i(0, \theta_T, \gamma) = 0 \), unless \( I_m(0) = R_e(0) = 0 \), which holds only when \( G_p(s) \) has a zero at the origin.

The procedure can be repeated in the \((K_p, K_d)\) plane. After setting \( K_i \) to a fixed value \( \tilde{K}_i \), equations (76) and (77) can be rewritten as

\[
\begin{bmatrix}
X_{R_p} & X_{R_d} \\
X_{I_p} & X_{I_d}
\end{bmatrix}
\begin{bmatrix}
K_p \\
K_d
\end{bmatrix} =
\begin{bmatrix}
Y_R - X_{R_1} \tilde{K}_i \\
Y_I - X_{I_1} \tilde{K}_i
\end{bmatrix}.
\]

(82)

Solving equation (82) for all \( \omega \neq 0 \) and \( \theta_T \in [0,2\pi) \) gives the same expression as equation (79) for \( K_p(\omega, \theta_T, \gamma) \), and the following equation for \( K_d(\omega, \theta_T, \gamma) \):

\[
K_d(\omega, \theta_T, \gamma) = \frac{\tilde{K}_i}{\omega^2} + \frac{I_m(\omega)\left( -\frac{1}{\gamma} A_T(\omega) \cos \theta_T + \frac{1}{\gamma} B_T(\omega) \sin \theta_T + 1 \right)}{\omega D(\omega)}.
\]

(83)

At \( \omega = 0 \), \( \tilde{K}_i \) must be equal to zero for a solution to exist. Furthermore, as \( I_m(0) = 0 \) for all real plants, \( K_d(0, \theta_T, \gamma) \) is arbitrary and
\[ K_p(0, \theta_T, \gamma) = \frac{-1}{R_e(0) \left( 1 - \frac{1}{\gamma} A_T(0) \cos \theta_T + \frac{1}{\gamma} B_T(0) \sin \theta_T \right)} . \]  

(84)

Last, the solution is found in the \((K_i, K_d)\) plane. After setting \(K_p\) to a fixed value of \(\tilde{K}_p\), equations (76) and (77) are rewritten as

\[
\begin{bmatrix}
X_{Ri} & X_{Rd} & K_i \\
X_{Li} & X_{Ld} & K_d
\end{bmatrix} = \begin{bmatrix}
Y_R - X_{R_{p\tilde{K}_p}} \\
Y_I - X_{I_{p\tilde{K}_p}}
\end{bmatrix} .
\]  

(85)

Although the coefficient matrix is singular, a solution will exist in two cases. First, at \(\omega = 0\), \(K_d(0, \theta_T, \gamma)\) is arbitrary and \(K_i(0, \theta_T, \gamma) = 0\), unless \(I_m(0) = R_e(0) = 0\), which holds only when the plant has a zero at the origin. In such a case, a PID compensator should be avoided as the PID pole cancels the zero at the origin and the system becomes internally unstable. A second set of solutions occurs at any frequency \(\omega_i\), where \(K_p(\omega_i, \theta_T, \gamma)\) from equation (79), is equal to \(\tilde{K}_p\). At these frequencies, \(K_d(\omega_i, \theta_T, \gamma)\) and \(K_i(\omega_i, \theta_T, \gamma)\) must satisfy the following straight line equation
\[ K_i(\omega_i, \theta_i, \gamma) = \frac{K_e(\omega_i, \theta_i, \gamma)}{\omega_i^2} + \begin{pmatrix} -R_e(\omega_i) \left( \frac{1}{\gamma} A_T(\omega_i) \sin \theta_T + \frac{1}{\gamma} B_T(\omega_i) \cos \theta_T \right) + \\ I_m(\omega_i) \left( \frac{1}{\gamma} A_T(\omega_i) \cos \theta_T + \frac{1}{\gamma} B_T(\omega_i) \sin \theta_T + 1 \right) \end{pmatrix} \]

(86)

### 6.3 Selection of Weighting Functions

Uncertainty in a plant model may occur for three main reasons [5]: (1) parametric uncertainty, i.e., the plant model is known but there are some uncertain parameters; (2) neglected and un-modeled dynamic uncertainty, i.e., the model of a real system may not be well known; and (3) lumped uncertainty, i.e., the uncertainty description corresponds to one or several sources of uncertainties that are combined into a single lumped perturbation of a chosen structure. The uncertainty sources can be either case one or case two or both.

Uncertainties in the frequency domain can be lumped as multiplicative uncertainty, which was shown in Figure 26. The perturbed plant can be derived from Figure 26 as

\[ G_\Delta(s) = G_p(s) \left( 1 + W_T(s) \Delta_f(s) \right) \quad \left| \Delta_f(j\omega) \right| \leq 1, \]

\[ = G_p(s) + G_p(s) W_T(s) \Delta_f(s). \]

(87)

Equation (87) can be rearranged as

\[ W_T(s) \Delta_f(s) = \frac{G_\Delta(s) - G_p(s)}{G_p(s)} \quad \left| \Delta_f(j\omega) \right| \leq 1. \]

(88)
It is easy to show from equations (87) and (88) that

\[ |\Delta_f(j\omega)| = \left| \frac{G_\Delta(j\omega) - G_p(j\omega)}{W_T(j\omega)G_p(j\omega)} \right| \leq 1. \quad (89) \]

Substituting the perturbation bound in equation (89) gives

\[ |W_T(j\omega)| \geq \frac{G_\Delta(j\omega) - G_p(j\omega)}{G_p(j\omega)}. \quad (90) \]

The multiplicative weight must be selected to bound all the multiplicative errors of the system [5] and [44]. A typical form for the multiplicative weight is given by

\[ W_T(s) = \frac{s}{\frac{s}{M_h} + \omega_b}, \quad (91) \]

where \( M_h \) is the high frequency bound, and \( \omega_b \) is the bandwidth frequency. Note that the bandwidth frequency is the frequency where the multiplicative weight has a magnitude of 1. These values are chosen for bounding all multiplicative errors of the system. A plot of the magnitude of \( W_T(j\omega) \) is shown in Figure 27.
6.4 Numerical Example

In this section, a PID controller is designed to regulate the shaft position of a DC motor. The feedback loop has an unknown communication delay between 0.05 and 0.15 seconds. The PID controller should stabilize the system and satisfy the robust stability constraint in equation (70) with the uncertain communication delay where $\gamma = 1$. The nominal model of DC motor has been identified as [42]

$$G_p(s) = \frac{65.5}{s(s + 34.6)} e^{-\tau s},$$  \hspace{1cm} (92)

where $\tau$ has been selected to be the mean value of uncertain uniform communication delay, 0.1 seconds. The frequency responses of the multiplicative errors for different communication delays and the multiplicative weight are shown in Figure 28. The multiplicative weight is
\[ W_T(s) = \frac{s}{s + \frac{2.8}{20}} \]  

and is chosen from equation (91) to bound the multiplicative errors. Note, by including the time-delay in the nominal model, it is possible to reduce the conservativeness in the plant uncertainty and increase the size of the set of PID controllers that robustly stabilize the system.

Figure 28. Multiplicative errors for different communication delays and the multiplicative weight.
Equations (79) and (80) are used in the \((K_p, K_i)\) plane for a fixed value of \(K_d = 0.2\). As discussed previously, the PID stability boundary of the nominal system can be found by setting \(\gamma = \infty\). All PID controllers that satisfy the robust stability constraint in equation (70) are found by setting \(\gamma = 1\) and finding the intersection of all regions for \(\theta_f \in [0,2\pi)\).

The regions that satisfy the robust stability constraint and the nominal stability boundary are shown in Figure 29. The intersection of all regions inside the nominal stability boundary of the \((K_p, K_i)\) plane is the robust stability region. To verify the results, an arbitrary controller from this region is chosen, giving the PID controller as

\[
G_c(s) = 2.76 + \frac{7.89}{s} + 0.2s. \tag{94}
\]

The substitution of equations (92), (93), and (94) into equation (70) gives

\[
\|W_T(j\omega)T(j\omega)\|_\infty = 0.558, \text{ which is less than one. This can be seen graphically in Figure 30.}
\]

As the \(H_\infty\) norm is less than one the design goal is met.
Figure 29. Nominal stability boundary and robust stability region in the \((K_p, K_i)\) plane for a fixed \(\tilde{K}_d = 0.2\).
Figure 30. Magnitude of $W_T(j\omega)T(j\omega)$ for $G_c(s) = 2.76 + \frac{7.89}{s} + 0.2s$.

The second method uses equations (79) and (83) in the $(K_p, K_d)$ plane for a fixed value of $\tilde{K}_i = 1$. The PID controller is designed to satisfy the robust stability constraint with $\gamma = 1$. The region that satisfies the robust stability constraint and the nominal stability boundary is shown in Figure 31. The intersection of all regions inside the nominal stability boundary of the $(K_p, K_d)$ plane is the robust stability region. To verify the results, an arbitrary controller from this region is chosen, giving the PID controller as
\[ G_e(s) = 1.63 + \frac{1}{s} + 0.07 \cdot s. \]  

The substitution of equations (92), (93), and (95) into equation (70) gives

\[ \|W_T(j\omega)T(j\omega)\|_{\infty} = 0.227, \] which is less than one. This can be seen graphically in Figure 32.

As the $H_\infty$ norm is less than one the design goal is met.

Figure 31. Nominal stability boundary and robust stability region in the $(K_p, K_d)$ plane for a fixed $K_i = 1$.  

74
Figure 32. Magnitude of $W_T(j\omega)T(j\omega)$ for $G_c(s) = 1.63 + \frac{1}{s} + 0.07s$.

The third method is applied in the $(K_i, K_d)$ plane for a fixed value of $\tilde{K}_p = 0.5$. Plots of $K_p(\omega, \theta_T, \infty)$ and $K_p(\omega, \theta_T, \gamma)$ for various values of $\theta_T \in [0, 2\pi)$ are shown in Figure 33. For each curve, the $\omega_i$ are the frequencies at which $K_p(\omega_i, \theta_T, \gamma) = \tilde{K}_p = 0.5$. Each $\omega_i$ is substituted into equation (86) to find the required boundaries. In addition, there is the boundary at $K_i(0, \theta_T, \gamma) = 0$. 
Figure 33. Plots of $K_p(\omega, \theta_T, \infty)$ and $K_p(\omega, \theta_T, \gamma)$ for various values of $\theta_T \in [0, 2\pi)$.

The region that satisfied the robust stability constraint and the nominal stability boundary is shown in Figure 34. The intersection of all regions inside the nominal stability boundary of the $(K_i, K_d)$ plane is the robust stability region. To verify the results, an arbitrary controller from this region is chosen, giving the PID controller as

$$G_c(s) = 0.5 + \frac{1.30}{s} + 0.07s.$$  \hfill (96)
The substitution of equations (92), (93), and (96) into equation (70) gives
\[ \left\| W_T(j\omega)T(j\omega) \right\|_\infty = 0.197, \] which is less than one. This can be seen graphically in Figure 35.

As the \( H_\infty \) norm is less than one the design goal is met.

Figure 34. Nominal stability boundary and robust stability region in the \((K_i, K_d)\) plane for a fixed \( \tilde{K}_p = 0.5 \).
Figure 35. Magnitude of $W_T(j\omega)T(j\omega)$ for $G_c(s) = 0.5 + \frac{1.30}{s} + 0.07s$.

6.5 Conclusion

In this chapter, a graphical technique was introduced for finding all achievable PID controllers that satisfy the robust stability constraint. This method was applicable to SISO systems with arbitrary order transfer functions and time-delays. Since this work built upon the straight-forward development [15], [16], [36], [37], and [38], it did not require the plant transfer function model. This method was simple to understand and easy to implement, and required only the frequency response of the plant. An example of a DC motor with uncertain communication delay in the feedback path was presented to demonstrate the application of this
method. By including the time-delay in the nominal model, it was possible to reduce the conservativeness in the plant uncertainty and increase the size of the set of PID controllers that robustly stabilized the system. The Matlab program for robust performance problem can be found in the Appendix.
CHAPTER 7

ROBUST PERFORMANCE PROBLEM

7.1 Introduction

An important problem in control theory is the robust performance problem. Controllers that solve this problem must not only meet the performance requirements for the nominal plant, but for the entire set of uncertain plants. In the articles reviewed, only two papers worked on robust performance design of PID controllers [33] and [34]. Consequently, the robust performance design of a PID controller for an arbitrary order transfer function with time-delay is a new problem that has received less attention due to the complexity of the concept.

This chapter focuses on a technique for finding all achievable PID controllers that simultaneously stabilize the closed-loop system and satisfy the robust performance constraint [40] and [41]. To meet these specifications, weights are added to the $H_\infty$ sensitivity and $H_\infty$ complementary sensitivity constraints. The reason for mixing these two problems is to find all achievable PID controllers for all acceptable perturbations that robustly meet the performance requirements of the system. Since this problem has a very straightforward connection to the robust stability problem, time-delay is included inside the nominal model of the system. The ability to include the time-delay in the nominal model allows the designer to find much tighter uncertainty bounds in systems with known delays than would be possible otherwise [5].

In this chapter, the robust performance problem for systems with multiplicative uncertainty is considered. The technique is applicable to proper SISO systems of arbitrary order with time-delay. This work builds upon the straightforward development in [38] and [39] regarding the weighted sensitivity and robust stability problems. As a consequence, this method does not require the rational plant transfer function model but depends on the frequency response
of the system. If the plant transfer function is known, the same procedure can be applied by first computing the frequency response.

The remainder of this chapter is organized as follows. In Section 7.2, the design methodology is introduced. In Section 7.3, a numerical example with a real right-half plane zero is presented to demonstrate the application of this method. Finally, the conclusion of this chapter is presented in Section 7.4.

7.2 Design Methodology

Consider the SISO and LTI system shown in Figure 36, where $G_{\Delta}(s)$ represents the perturbed plants, $G_p(s)$ is the nominal plant, and $G_c(s)$ is the PID controller. The reference input and error signals are $R(s)$ and $Z(s)$, respectively. $W_S$ is the sensitivity function weight, $W_T$ is the multiplicative weight, and $|\Delta_f(j\omega)| \leq 1$ is the uncertain perturbation. The nominal plant transfer function is given by equation (1), and the PID controller is defined in equation (2).

![Figure 36. Block diagram of the system with multiplicative uncertainty.](image)

All transfer functions in Figure 36 can be expressed in the frequency domain. The plant transfer function can be written in terms of its real and imaginary parts as equation (4) and the PID controller is defined in the frequency domain as equation (5). The sensitivity function
weight $W_s$ and the multiplicative weight $W_T$ are defined in equations (25) and (69), respectively. In this research, the problem is to find all PID controllers that satisfy the robust performance constraint of the feedback system in Figure 36 for all $|\Delta_j(j\omega)| \leq 1$.

The robust performance constraint for a SISO system with multiplicative uncertainty can be written as

$$\left|W_s(j\omega)S(j\omega)\right| + \left|W_T(j\omega)T(j\omega)\right| \leq \gamma, \quad \forall \omega. \tag{97}$$

where $S(j\omega)$ is the sensitivity function, $T(j\omega)$ is the complementary sensitivity function, and $\gamma$ is one [5]. The complex functions in equation (97) can be written in terms of their magnitudes and phase angles as

$$\left|W_s(j\omega)S(j\omega)e^{j\theta_s}\right| + \left|W_T(j\omega)T(j\omega)e^{j\theta_T}\right| \leq \gamma, \quad \forall \omega. \tag{98}$$

If equation (98) holds, then for each value of $\omega$

$$W_s(j\omega)S(j\omega)e^{j\theta_s} + W_T(j\omega)T(j\omega)e^{j\theta_T} \leq \gamma, \quad \forall \omega, \tag{99}$$

where $\theta_s = -\angle W_s(j\omega)S(j\omega)$ for some $\theta_s \in [0, 2\pi)$, and $\theta_T = -\angle W_T(j\omega)T(j\omega)$ for some $\theta_T \in [0, 2\pi)$. Consequently, all PID controllers that satisfy equation (97) must lie at the
intersection of all controllers that satisfy equation (99) for some $\theta_S \in [0,2\pi)$ and $\theta_T \in [0,2\pi)$, [40] and [41].

To find this region, for each value of $\theta_S \in [0,2\pi)$ and $\theta_T \in [0,2\pi)$, all PID controllers will be found on the boundary of equation (99). It is easy to show from equation (99) that PID controllers on the boundary must satisfy

$$P(\omega, \theta_S, \theta_T, \gamma) = 0,$$  

(100)

where

$$P(\omega, \theta_S, \theta_T, \gamma) = \left(1 + G_S(j\omega)G_c(j\omega) - \frac{1}{\gamma} W_S(j\omega)e^{j\theta_S} - \frac{1}{\gamma} W_T(j\omega)G_p(j\omega)G_c(j\omega)e^{j\theta_T}\right).$$

By substituting equations (4), (5), (25), (69), $e^{j\theta_S} = \cos \theta_S + j \sin \theta_S$, and $e^{j\theta_T} = \cos \theta_T + j \sin \theta_T$ into equation (100), the frequency response of this “modified” characteristic polynomial can be rewritten as

$$P(\omega, \theta_S, \theta_T, \gamma) = 1 + \left(R_c(\omega) + jI_m(\omega)\left(K_p + \frac{K_i}{j\omega} + K_d j\omega\right)\right) -$$

$$\frac{1}{\gamma}\left((A_S(\omega) + jB_S(\omega))\left(\cos \theta_S + j \sin \theta_S\right)\right) -$$

$$\frac{1}{\gamma}\left((A_T(\omega) + jB_T(\omega))\left(R_c(\omega) + jI_m(\omega)\right)\left(K_p + \frac{K_i}{j\omega} + K_d j\omega\right)\left(\cos \theta_T + j \sin \theta_T\right)\right).$$

(101)
Note that equation (101) reduces to the frequency response of the standard closed-loop characteristic polynomial as $\gamma \to \infty$. Expanding equation (101) in terms of its real and imaginary parts yields

\[ X_{R_p} K_p + X_{R_i} K_i + X_{R_d} K_d = Y_R, \quad (102) \]

and

\[ X_{I_p} K_p + X_{I_i} K_i + X_{I_d} K_d = Y_I, \quad (103) \]

where

\[
X_{R_p}(\omega) = \omega \left( \alpha_T R_e(\omega) + \beta_T I_m(\omega) \right),
\]

\[
X_{R_i}(\omega) = -\beta_T R_e(\omega) + \alpha_T I_m(\omega),
\]

\[
X_{R_d}(\omega) = \omega^2 \left( \beta_T R_e(\omega) - \alpha_T I_m(\omega) \right),
\]

\[
Y_R(\omega) = -\omega \alpha_S,
\]

\[
X_{I_p}(\omega) = \omega \left( -\beta_T R_e(\omega) + \alpha_T I_m(\omega) \right),
\]

\[
X_{I_i}(\omega) = -\alpha_T R_e(\omega) - \beta_T I_m(\omega),
\]

\[
X_{I_d}(\omega) = \omega^2 \left( \alpha_T R_e(\omega) + \beta_T I_m(\omega) \right),
\]

\[
Y_I(\omega) = \omega \beta_S,
\]

\[
\alpha_T = \frac{1}{\gamma} \left( -A_T(\omega) \cos \theta_T + B_T(\omega) \sin \theta_T \right) + 1,
\]

\[
\beta_T = \frac{1}{\gamma} \left( A_T(\omega) \sin \theta_T + B_T(\omega) \cos \theta_T \right),
\]
\[ \alpha_s = \frac{1}{\gamma} \left( -A_s(\omega) \cos \theta_s + B_s(\omega) \sin \theta_s \right) + 1, \]

\[ \beta_s = \frac{1}{\gamma} \left( A_s(\omega) \sin \theta_s + B_s(\omega) \cos \theta_s \right). \]

This is a three-dimensional system in terms of the controller parameters \( K_p, K_i, \) and \( K_d \). First, the boundary of equation (101) will be found in the \((K_p, K_i)\) plane for a fixed value of \( K_d \). After setting \( K_d \) to the fixed value \( \bar{K}_d \), equations (102) and (103) can be rewritten as

\[
\begin{bmatrix}
X_{Rp} & X_{Ri} \\
X_{Ip} & X_{Ii}
\end{bmatrix}
\begin{bmatrix}
K_p \\
K_i
\end{bmatrix}
=
\begin{bmatrix}
Y_{R} - X_{Rd} \bar{K}_d \\
Y_{I} - X_{Id} \bar{K}_d
\end{bmatrix}. \quad (104)
\]

Solving equation (104) for all \( \omega \neq 0, \theta_s \in [0, 2\pi), \) and \( \theta_I \in [0, 2\pi) \) gives the following equations:
$$K_p(\omega, \theta_s, \theta_T, \gamma) =$$

$$-R_e(\omega) \left[ 1 + \frac{1}{\gamma} \left( -A_s(\omega) \cos \theta_s - A_T(\omega) \cos \theta_T + \frac{1}{\gamma} \left( \left(A_s(\omega)A_T(\omega) + B_s(\omega)B_T(\omega) \right) \cos (\theta_s - \theta_T) + \frac{1}{\gamma^2} \left( -B_s(\omega)A_T(\omega) + A_s(\omega)B_T(\omega) \right) \sin (\theta_s - \theta_T) \right) \right] \right]$$

$$- \left[ \frac{1}{\gamma} \left( A_s(\omega) \sin \theta_s + A_T(\omega) \sin \theta_T \right) \right]$$

$$I_m(\omega) \left[ \frac{1}{\gamma} \left( B_s(\omega)A_T(\omega) - A_s(\omega)B_T(\omega) \right) \cos (\theta_s - \theta_T) + \frac{1}{\gamma^2} \left( A_s(\omega)A_T(\omega) + B_s(\omega)B_T(\omega) \right) \sin (\theta_s - \theta_T) \right]$$

$$\left( \frac{1}{\gamma} \left( A_s(\omega) \sin \theta_s + A_T(\omega) \sin \theta_T \right) \right)$$

$$D(\omega)$$

and

(105)
\[ K_i(\omega, \theta_S, \theta_T, \gamma) = \omega^2 \bar{K}_d + \]
\[
\omega R_e(\omega) \left[ \frac{1}{\gamma} \begin{pmatrix} -B_S(\omega) \cos \theta_S + B_T(\omega) \cos \theta_T \\ A_S(\omega) \sin \theta_S + A_T(\omega) \sin \theta_T \end{pmatrix} + \right. + \left. \frac{1}{\gamma^2} \begin{pmatrix} (B_S(\omega)A_T(\omega) - A_S(\omega)B_T(\omega)) \cos (\theta_S - \theta_T) + \\ (A_S(\omega)A_T(\omega) + B_S(\omega)B_T(\omega)) \sin (\theta_S - \theta_T) \end{pmatrix} \right] \]
\[
\omega I_m(\omega) \left[ -1 + \frac{1}{\gamma} \begin{pmatrix} A_S(\omega) \cos \theta_S + A_T(\omega) \cos \theta_T - \\ B_S(\omega) \sin \theta_S - B_T(\omega) \sin \theta_T \end{pmatrix} + \right. + \left. \frac{1}{\gamma^2} \begin{pmatrix} (A_S(\omega)A_T(\omega) - B_S(\omega)B_T(\omega)) \cos (\theta_S - \theta_T) + \\ + (B_S(\omega)A_T(\omega) - A_S(\omega)B_T(\omega)) \sin (\theta_S - \theta_T) \end{pmatrix} \right] \]
\[
D(\omega),
\]

where \[ D(\omega) = |G_p(j\omega)|^2 \left( 1 - \frac{2}{\gamma} \left( A_T(\omega) \cos \theta_T - B_T(\omega) \sin \theta_T \right) + \frac{1}{\gamma^2} |W_T(j\omega)|^2 \right), \]

\[ |G_p(j\omega)|^2 = R_e^2(\omega) + I_m^2(\omega), \text{ and } |W_T(j\omega)|^2 = A_T^2(\omega) + B_T^2(\omega). \]

Setting \( \omega = 0 \) in equation (104), yields

\[
\begin{bmatrix} 0 & X_{Ri}(0) \\ 0 & X_{Ii}(0) \end{bmatrix} \begin{bmatrix} K_p \\ K_i \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix},
\]

(107)
and it can be concluded that $K_p(0, \theta_S, \theta_T, \gamma)$ is arbitrary and $K_i(0, \theta_S, \theta_T, \gamma) = 0$, unless $I_m(0) = R_e(0) = 0$, which holds only when $G_p(s)$ has a zero at the origin.

The procedure can be repeated in the $(K_p, K_d)$ plane. After setting $K_i$ to a fixed value $\tilde{K}_i$, equations (102) and (103) can be rewritten as

$$\begin{bmatrix} X_{Rp} & X_{Rd} \\ X_{Ip} & X_{Id} \end{bmatrix} \begin{bmatrix} K_p \\ K_d \end{bmatrix} = \begin{bmatrix} Y_R - X_{Ri}\tilde{K}_i \\ Y_I - X_{Ii}\tilde{K}_i \end{bmatrix}. \tag{108}$$

Solving equation (108) for all $\omega \neq 0$, $\theta_S \in [0, 2\pi)$ and $\theta_T \in [0, 2\pi)$ gives the same expression as equation (105) for $K_p(\omega, \theta_S, \theta_T, \gamma)$ and the following equation for $K_d(\omega, \theta_S, \theta_T, \gamma)$:
\[ K_d(\omega, \theta_S, \theta_T, \gamma) = \frac{\tilde{K}_i}{\omega^2} + \begin{vmatrix} \frac{1}{\gamma} \left( -B_s(\omega) \cos \theta_S + B_T(\omega) \cos \theta_T - \right) \\ A_s(\omega) \sin \theta_S + A_T(\omega) \sin \theta_T \\ \left. \right|_{-R_e(\omega)} \left( \left( B_s(\omega)A_T(\omega) - A_s(\omega)B_T(\omega) \right) \cos (\theta_S - \theta_T) + \right) \\ + \frac{1}{\gamma^2} \left( A_s(\omega)A_T(\omega) + B_s(\omega)B_T(\omega) \right) \sin (\theta_S - \theta_T) \end{vmatrix} \]

\[ \omega D(\omega) \]

(109)

At \( \omega = 0 \) in equation (108), \( \tilde{K}_i \) must be equal to zero for a solution to exist. Furthermore, as

\[ I_m(0) = 0 \] for all real plants, \( K_d(0, \theta_S, \theta_T, \gamma) \) is arbitrary and

\[ K_p(0, \theta_S, \theta_T, \gamma) = -\frac{1}{\gamma} \left( -A_s(0) \cos \theta_S + B_s(0) \sin \theta_S \right) + 1 \]

\[ \frac{R_e(0)}{1} \left( -A_T(0) \cos \theta_T + B_T(0) \sin \theta_T \right) + 1 \]

(110)

or

89
\[ K_p(0, \theta_S, \theta_T, \gamma) = \frac{-\left( A_S(0) \sin \theta_S + B_S(0) \cos \theta_S \right)}{R_e(0) \left( A_T(0) \sin \theta_T + B_T(0) \cos \theta_T \right)}. \] (111)

Last, the solution is found in the \((K_i, K_d)\) plane. After setting \(K_p\) to a fixed value of \(\tilde{K}_p\), equations (102) and (103) are rewritten as

\[
\begin{bmatrix}
X_{Ri} & X_{Rd} \\
X_{Li} & X_{Ld}
\end{bmatrix}
\begin{bmatrix}
K_i \\
K_d
\end{bmatrix} = \begin{bmatrix}
Y_R - X_{Rp} \tilde{K}_p \\
Y_I - X_{Ip} \tilde{K}_p
\end{bmatrix}. \] (112)

Although the coefficient matrix is singular, a solution exists in two cases. First, at \(\omega = 0\), \(K_d(0, \theta_S, \theta_T, \gamma)\) is arbitrary and \(K_i(0, \theta_S, \theta_T, \gamma) = 0\), unless \(I_m(0) = R_e(0) = 0\), which holds only when the plant has a zero at the origin. In such a case, a PID compensator should be avoided, as the PID pole cancels the zero at the origin and the system becomes internally unstable. A second set of solutions occurs at any frequency \(\omega_i\), where \(K_p(\omega_i, \theta_S, \theta_T, \gamma)\) from equation (105) is equal to \(\tilde{K}_p\). At these frequencies, \(K_d(\omega_i, \theta_S, \theta_T, \gamma)\) and \(K_i(\omega_i, \theta_S, \theta_T, \gamma)\) must satisfy the following straight line equation:
\[
K_d(\omega_i, \theta_s, \theta_T, \gamma) = \frac{K_i(\omega_i, \theta_s, \theta_T, \gamma)}{\omega_i^2} + \left( \begin{array}{c}
\frac{1}{\gamma} \left( -B_S(\omega_i) \cos \theta_s + B_T(\omega_i) \cos \theta_T \right) \\
+ \frac{1}{\gamma^2} \left( B_S(\omega_i) A_T(\omega_i) - A_S(\omega_i) B_T(\omega_i) \right) \cos(\theta_s - \theta_T) \\
\left( A_S(\omega_i) A_T(\omega_i) + B_S(\omega_i) B_T(\omega_i) \right) \sin(\theta_s - \theta_T)
\end{array} \right) \\
- R_e(\omega_i)
\left( \begin{array}{c}
-1 + \frac{1}{\gamma} \left( B_S(\omega_i) \sin \theta_s + B_T(\omega_i) \sin \theta_T \right) \\
\left( -A_S(\omega_i) A_T(\omega_i) - B_S(\omega_i) B_T(\omega_i) \right) \cos(\theta_s - \theta_T) \\
\left( B_S(\omega_i) A_T(\omega_i) - A_S(\omega_i) B_T(\omega_i) \right) \sin(\theta_s - \theta_T)
\end{array} \right) \\
I_m(\omega_i)
\left( \begin{array}{c}
+ \frac{1}{\gamma^2} \left( B_S(\omega_i) A_T(\omega_i) - A_S(\omega_i) B_T(\omega_i) \right) \sin(\theta_s - \theta_T)
\end{array} \right)
\right)
\]

\[\omega_i D(\omega_i)\]

(113)

7.3 Numerical Example

In this section, a numerical example is used to demonstrate the application of this method. Consider the second-order plant transfer function [15] where it is assumed that the feedback loop has an unknown time-delay with a range of 0.4 to 0.6 seconds. The goal is to find all PID controllers that stabilize the system and satisfy the robust performance constraint in equation (97), where \( \gamma = 1 \). The nominal model of the system is given by
where $\tau$ has been selected to be the mean value of the uncertain delay, 0.5 seconds. The frequency responses of the multiplicative errors for different delays and the multiplicative weight are shown in Figure 37. The multiplicative weight

$$W(s) = \frac{s}{1 + 0.3s}$$

is chosen from equation (91) to bound the multiplicative errors. Note, by including the time-delay in the nominal model, the conservativeness in plant uncertainty can be reduced. This will increase the size of the set of PID controllers that robustly meet the performance goals.
Figure 37. Multiplicative errors for different communication delays and the multiplicative weight.

The closed-loop step response is required to have an overshoot less than 5 percent and a settling time less than 40 seconds. The sensitivity weight that is chosen to satisfy the performance requirement for the closed-loop system is

\[ W_s(s) = \frac{0.78(s + 0.13)}{s + 0.08}. \]  \hspace{1cm} (116)

Equations (105) and (106) are used in the \((K_p, K_i)\) plane for a fixed value of \(\tilde{K}_d = 0.2\).

As discussed previously, the PID stability boundary of the nominal system can be found by
setting $\gamma = \infty$ in equations (105) and (106). All PID controllers that satisfy the robust performance constraint in equation (97) are found by setting $\gamma = 1$ in equations (105) and (106) for $\theta_s \in [0, 2\pi)$ and $\theta_T \in [0, 2\pi)$, and then finding the intersection of all regions.

The region that satisfies the robust performance constraint and the nominal stability boundary is shown in Figure 38. The intersection of all regions inside the nominal stability boundary of the $(K_p, K_i)$ plane is the robust performance region.

Figure 38. Nominal stability boundary and robust performance region in the $(K_p, K_i)$ plane for a fixed $\tilde{K}_d = 0.2$.

To verify the results, an arbitrary PID controller from this region is chosen,
The substitution of equations (114), (115), (116), and (117) into equation (97), yields
\[ |W_S(j\omega)S(j\omega)| + |W_T(j\omega)T(j\omega)| \leq 0.98. \] This can be seen graphically in Figure 39. As can be seen, the magnitude of robust performance system is less than one, the design goal is met.

![Bode Diagram](image)

**Figure 39.** Magnitude of \(|W_S(j\omega)S(j\omega)| + |W_T(j\omega)T(j\omega)|\) for \(G_c(s) = 0.49 + \frac{0.17}{s} + 0.2s\).
The second method uses equations (105) and (109) in the \((K_p, K_d)\) plane for a fixed value of \(\tilde{K}_i = 0.1\). As discussed previously, the PID stability boundary of the nominal system can be found by setting \(\gamma = \infty\) in equations (105) and (109). The PID controller is designed to satisfy the robust performance constraint equation (97) by setting \(\gamma = 1\) in equations (105) and (109) for \(\theta_S \in [0, 2\pi)\) and \(\theta_T \in [0, 2\pi)\), and finding the intersection of all regions.

The region that satisfies the robust performance constraint and the nominal stability boundary is shown in Figure 40. The intersection of all regions inside the nominal stability boundary of the \((K_p, K_d)\) plane is the robust performance region.

![Figure 40. Nominal stability boundary and robust performance region in the \((K_p, K_d)\) plane for a fixed \(\tilde{K}_i = 0.1\).](image-url)
To verify the results, an arbitrary controller from this region is chosen, giving the PID controller as

$$G_c(s) = 0.3 + \frac{0.1}{s} + 0.03s.$$  \hfill (118)\]

The substitution of equations (114), (115), (116), and (118) into equation (97) gives

$$|W_S(j\omega)S(j\omega)| + |W_T(j\omega)T(j\omega)| \leq 0.94.$$  

This can be seen graphically in Figure 41. As can be seen, the magnitude of robust performance system is less than one, the design goal is met.

![Bode Diagram](image)

Figure 41. Magnitude of $|W_S(j\omega)S(j\omega)| + |W_T(j\omega)T(j\omega)|$ for $G_c(s) = 0.3 + \frac{0.1}{s} + 0.03s$. 

97
The third method is applied in the \((K_i, K_d)\) plane for a fixed value of \(\bar{K}_p = 0.4\). Plots of \(K_p(\omega, \theta_S, \theta_T, \gamma)\) at \(\gamma = \infty\) and \(K_p(\omega, \theta_S, \theta_T, \gamma)\), from equation (105), for values of \(\theta_S \in [0, 2\pi)\) and \(\theta_T \in [0, 2\pi)\) are shown in Figure 42. For each curve, the \(\omega_i\)s are the frequencies at which the chosen value for \(\bar{K}_p(\omega_i, \theta_S, \theta_T, \gamma) = \bar{K}_p = 0.4\). Each \(\omega_i\) for this chosen constant coefficient of \(\bar{K}_p\) is substituted into equation (113) to find the required boundaries. In addition, we have the boundary at \(K_i(0, \theta_S, \theta_T, \gamma) = 0\).

Figure 42. Plots of \(K_p(\omega, \theta_S, \theta_T, \infty)\) and \(K_p(\omega, \theta_S, \theta_T, \gamma)\) for various values of \(\theta_S \in [0, 2\pi)\) and \(\theta_T \in [0, 2\pi)\).
The region that satisfies the robust performance constraint and the nominal stability boundary is shown in Figure 43. The intersection of all regions inside the nominal stability boundary of the \((K_i, K_d)\) plane is the robust performance region. To verify the results, an arbitrary controller from this region is chosen, giving the PID controller as

\[
G_c(s) = 0.4 + \frac{0.11}{s} + 0.04s.
\]  

(119)

Figure 43. Nominal stability boundary and robust performance region in the \((K_i, K_d)\) plane for a fixed \(\tilde{K}_p = 0.4\).
Substituting equations (114), (115), (116), and (119) into equation (97) gives

\[ |W_S(j\omega)S(j\omega)| + |W_T(j\omega)T(j\omega)| \leq 0.96. \]

This can be seen graphically in Figure 44. As can be seen, the magnitude of robust performance system is less than one, the design goal is met.

![Bode Diagram](image)

**Figure 44.** Magnitude of \(|W_S(j\omega)S(j\omega)| + |W_T(j\omega)T(j\omega)|\) for \(G_c(s) = 0.4 + \frac{0.11}{s} + 0.04s\).

Step responses of the closed-loop system with the PID controller in equation (119) and various time-delays between 0.4 and 0.6 seconds are shown in Figure 45. As can be seen, the closed-loop step responses all have an overshoot less than 5% and a setting time less than 40 seconds. The maximum setting time is 39.9 seconds, and no percent overshoot.
Figure 45. Step response of the closed loop system for various time-delays.

7.4 Conclusion

In this chapter, a graphical technique was introduced for finding all achievable PID controllers that simultaneously stabilize and satisfy the robust performance constraint of an arbitrary order transfer function with uncertainty. This method was simple to understand and required only the frequency response of the system. A numerical example with an unknown time-delay in the feedback path was presented to demonstrate the application of this method. By including the time-delay in the nominal model, it was possible to reduce the conservativeness in plant uncertainty and increase the size of the set of PID controllers that satisfied the robust performance requirements.
8.1 Summary

This dissertation introduced a method that found all achievable PID controllers that simultaneously stabilize and satisfy $H_\infty$ sensitivity, complementary sensitivity, weighted sensitivity, robust stability, or robust performance constraints of any arbitrary order transfer function with an unknown time-delay. Given a particular problem a set of PID controller may or may not exist. If this set exists, this method will find this set of controllers. This method was simple to understand and required only the frequency response of the plant. If the plant transfer function was known, it could be applied to the same procedure by first computing the frequency response.

Chapter 1 provided a brief overview of this dissertation. The literature was reviewed in Chapter 2. Chapter 3 discussed finding the characteristic equations of all achievable PID controllers that simultaneously stabilize and satisfy an $H_\infty$ sensitivity constraint. In this chapter, the technique was applied to the experimental data of a DC motor with a communication delay in the feedback loop. Chapter 4 considered the characteristic equations of all achievable PID controllers that simultaneously stabilize and satisfy a weighted sensitivity constraint. A numerical example with a real right-half plane zero and stable poles demonstrated the application of this method. A generalized characteristic equation for all achievable PID controllers that simultaneously stabilize and satisfy an $H_\infty$ complementary sensitivity constraint was introduced in Chapter 5, using the same example as found in Chapter 4.
The key contribution of this dissertation was to define a general characteristic equation for all achievable PID controllers that simultaneously stabilize and satisfy robust stability or robust performance constraints of an arbitrary order system with time-delay. In Chapter 6 experimental data from a DC motor [42] was used to find all PID controllers that simultaneously stabilized and satisfied a robust stability constraint. This chapter showed that the ability to include the time-delay in the nominal model of the system will often permit designs with reduced conservativeness in plant uncertainty and an increase in size of the set of all PID controllers that robustly stabilize the system. In Chapter 7, a general characteristic equation was introduced for finding robust performance design of all achievable PID controllers in the frequency domain. Robust performance design of PID controllers for an arbitrary order transfer function with uncertainty time delay in the frequency domain was not reported in previous articles that were reviewed in this dissertation. The robust performance design of PID controllers presented in this dissertation was awarded as the best paper at the 8th International Conference on Applications of Electrical Engineering [40].

8.2 Additional Work

Since this work built upon the straight-forward development in [15] and [16], there are many areas in which to extend this research. This method has been extended to a unified approach for sensitivity design of PID controllers for both continuous-time and discrete-time systems [17]. This extension was based on the development in [17] with a slight difference in the definition of the PID controller parameters as reported by Emami and Watkins in [45]. The delta operator was used to describe the controllers, because it not only possesses numerical properties superior to the discrete-time shift operator but also converges to the continuous-time controller as the sampling period approaches zero. A unified approach allows for the use of the
same procedure for discrete-time and continuous-time design of PID controllers [8], [17], [45], [46], [47], [48], [49], [50], [50], and [51].

This research has been extended to a unified approach for weighted sensitivity [51] and complementary sensitivity design [52]. Unified Robust stability, robust performance, and mixed sensitivity design of PID controllers for continuous-time and discrete-time systems are also possible. This research could be extended to other uncertainty models and multiple-input multiple-output (MIMO) systems. It could also be extended to fractional PID controllers. This would give more freedom in the PID controller design [24]. This research could also be applied to other infinite dimensional systems. This method could be applied to the numerous industrial systems, including DC motor shaft position or speed control [42] and [53], bioengineering, bio-robotic devices, and robotic design.
REFERENCES
# LIST OF REFERENCES


[52] T. Emami and J. M. Watkins, “A unified approach for $H_\infty$ complementary sensitivity design of PID controllers applied to a DC motor with communication delay,”

APPENDIX

WEIGHTED SENSITIVITY PROGRAM

This is a script file for solving the weighted sensitivity problem in the continuous time systems.
Author: Tooran Emami & John Watkins
Date: 07/14/2008

s=zpk ('s');

G=(-0.5*s+1)/((s+1)*(2*s+1));
td=0.6;

set(G,'iodelay',td);
w=0.01:0.01:4;
ww=0.001:0.1:200;

po=0.05
phi=atan(-pi/log(po))
zeta=cos(phi)
alpha=sqrt(0.5+0.5*sqrt(1+8*zeta^2))
Ms=alpha*sqrt(alpha^2+4*zeta^2)/sqrt((1-alpha^2)^2+4*zeta^2*alpha^2)
Ts=40
wn=4/(zeta*Ts)
wb=wn/sqrt(2)
AAA=0.8

Wp=(s/Ms+wb)/(s+wb*AAA)

Kdt=0.4;
Kpt=0.5;
Kit=0.2;

teta1=0;
Gamma=inf;
gamat=1;

KIKp=-10:0.1:20;
axd=[-1 3.5 0 1.5];
axi=[-1.2 4 -4 4];
axp=[0 2.5 0 4];
axp2=[-0.5 3 -4 4];

[gama1,gama2,gama3,Gc1,Gc2,Gc3]=piddiss(G,w,ww,Wp,Kdt,Kit,Kpt,KIKp,teta1,Gamma,gamat,axd,axi,axp,axp2);
This is a function file for solving the weighted sensitivity problem in the continuous time
systems.
Author: Tooran Emami & John Watkins
Date: 07/14/2008

function [gama1, gama2, gama3, Gc1, Gc2, Gc3] = piddiss(G, w, ww, Wp, Kdt, Kit, Kpt, KiKp, teta1, Gamma, gamat, axd, axi, axp, axp2);
s = zpk ('s');
Gp = frd(G, w);
om = imag(frd(s, w));
disp('om=');
WP = frd(Wp, w);

Rp = real(Gp);
Ip = imag(Gp);
Gd = abs(Gp);
Gd2 = Gd^2;
A = real(WP);
B = imag(WP);

RAIB = (Rp*A + Ip*B)/Gd2;
RBIA = (Rp*B - Ip*A)/Gd2;

KpGd = Rp/Gd2;
KdGd = (Kdt*om*(Gd2) - Ip)/Gd2;
KiGd = (Kit*(Gd2)/om^2 + Ip/om)/Gd2;

for teta = 0:0.08:2*pi;
    GC = 1/gamat*cos(teta);
    GS = 1/gamat*sin(teta);
    P1 = GC*(RAIB) - GS*(RBIA) - KpGd;
    numKp = P1;
    D1 = -(GC)*(RBIA) - GS*(RAIB) + KdGd;
    numKi = om*D1;
    Kp = numKp;
    Ki = numKi;

    D2 = (GC)*(RBIA)/om + GS*(RAIB)/om + KiGd;
    numKd = D2;
    Kd = numKd;

end
figure(2)
line('Xdata',Kp.respondedata(:, 'Ydata', Ki.respondedata(:, 'color', 'c', 'linewidth', 1))
axis(axd)

figure(3)
line('Xdata', Kp.respondedata(:, 'Ydata', Kd.respondedata(:, 'color', 'c', 'linewidth', 1))
axis(axi)

figure(4)
line('Xdata', w, 'Ydata', Kp.respondedata(:, 'color', 'c', 'linewidth', 1));
axis(axp)
y=[Kp.respondedata(:, Kpt)';
[i,j]=find(abs(diff(sign(y)))>1);
n=sum(i);
wi=zeros(1,n);

hold on
y=[Kp.respondedata(:, Kpt)';
[i,j]=find(abs(diff(sign(y)))>1);
n=sum(i);
iw=zeros(1,n);

for i=1:n;
    wi(i)=interp1(y(j(i):j(i)+1), w(j(i):j(i)+1), 0);
end

figure(4)
plot(wi,Kpt,'*');
gtext on
xlabel('w');
ylabel('K_p');
Kp1=Kpt;
Kp2=Kpt;
Kd1=KiKp*0;

for i=1:n;
    wi(i)=interp1(y(j(i):j(i)+1), w(j(i):j(i)+1), 0);
    Kd1=KiKp*0;
    Gp1=frd(G,wi(i));
    Rp1=real(Gp1);
    Ip1=imag(Gp1);
    Gd1=abs(Gp1);
    Gd21=Gd1^2;
    Wp1=frd(Wp,wi(i));
    A1=real(Wp1);
B1 = \text{imag}(Wp1);
D11 = (GC) *(Rp1*B1 - Ip1*A1)/wi(i) + GS*(Rp1*A1+Ip1*B1)/wi(i) + Ki*Kp*(Gd21)/wi(i)^2 + Ip1/wi(i);
numKd = D11;
Kd2 = numKd/Gd21;
Kd1 = Ki*Kp*0;

figure(5)
line('Xdata',Kd1,'Ydata',KiKp,'color','c');
line('Xdata',KiKp,'Ydata',Kd2,'color','c');
axis(axp2)
end

figure(2)
xlabel('K_p')
ylabel('K_i')
grid on
axis(axd);

figure(3)
xlabel('K_p')
ylabel('K_d')
axis(axi);
grid on

figure(4)
xlabel('w');
ylabel('K_p');
axis(axp);
grid on

figure(5)
xlabel('K_i');
ylabel('K_d');
axis(axp2);
grid on

GC1 = 1/Gamma*cos(teta1);
GS1 = 1/Gamma*sin(teta1);
P1 = GC1*RAIB-GS1*RBIA-KpGd;
numKp = P1;

D1 = -(GC1)*(RBIA)-GS1*(RAIB)+KdGd;
numKi = om*D1;
D2=(GC1)*(RB1A)/om+GS1*(RAIB)/om+KiGd;
numKd=D2;

Kd=numKd;
Kp=numKp;
Ki=numKi;

figure(2)
Ki=Ki.responsedata(:)*0;
line('Xdata',Kp.responsedata(:),'Ydata',Ki,'color','r','linewidth',2);
axis(axd);
hold on

figure(3)
line('Xdata',Kp.responsedata(:),'Ydata',Kd.responsedata(:),'color','r','linewidth',2);
axis(axi);
grid on
xlabel('K_p');
ylabel('K_d');
grid on
hold on

figure(4)
P1=-Rp;
numKp=P1;
D1=Ip/om;
numKd=D1;
Kp=numKp/Gd2;
Kd=numKd/(Gd2);

line('Xdata',w,'Ydata',Kp.responsedata(:),'color','r','linewidth',2);
axis(axp);
hold on
y=[Kp.responsedata(:)-Kpt]';
[i,j]=find(abs(diff(sign(y))))>1);
n=sum(i);
wi=zeros(1,n);
for i=1:n;
    wi(i)=interp1(y(j(i):j(i)+1),w(j(i):j(i)+1), 0);
    figure(4)
    plot(wi,Kpt,'*');
    grid on
    xlabel('w');
ylabel('K_p');
axis(axp);

Kp1=Kp;
Kp2=Kp;
Kd1=Ki*Kp*0;

Gp1=frd(G,wi(i));
Rp1=real(Gp1);
Ip1=imag(Gp1);
Gd1=abs(Gp1);
Gd21=Gd1^2;
Kd2=Ki*Kp/ww(i)^2+Ip1/(ww(i)*Gd21);

figure(5)
line('Xdata',Kd1,'Ydata',Ki*Kp,'color','r','linewidth',2);
line('Xdata',Ki*Kp,'Ydata',Kd2,'color','r','linewidth',2);
axis(axp2);
hold on
end

figure(2)
[Kp1,Ki1]=ginput(1);
line(Kp1,Ki1,'Marker','*')
Gc1=Kp1+Kdt*s+Ki1/s
WP=frd(Wp,ww);
G1=frd(G,ww);
Gcf=frd(Gc1,ww);
L1=G1*Gcf;
S1=1/(1+L1);
gama1=norm(WP*S1,inf)

figure(3)
[Kp2,Kd2]=ginput(1);
line(Kp2,Kd2,'Marker','*')
Gc2=Kp2+Kd2*s+Ki3/s
Gcf=frd(Gc2,ww);
L2=G1*Gcf;
S2=1/(1+L2);
gama2=norm(WP*S2,inf)

figure(5)
[Ki3,Kd3]=ginput(1);
line(Ki3,Kd3,'Marker','*')
Gc3=Kp+Kd3*s+Ki3/s
Gcf=frd(Gc3,ww);
L3=G1*Gcf;
S3=1/(1+L3);
gama3=norm(WP*S3,inf)

figure(6)
WsKd=S1*WP;
bodemag(WsKd);
grid on

figure(7)
WsKi=S2*WP;
bodemag(WsKi);
grid on

figure(8)
WsKp=S3*WP;
bodemag(WsKp);
grid on

figure(9)
T11=feedback(ss(G*Gc1),1);
step(T11);
grid on