ORDER RESTRICTED INFERENCES ABOUT LIFETIMES UNDER CENSORING

A Dissertation by

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DEDICATION

To the All Mighty, the Creator of the earth and the heavens,
ending and endlessness,
mathematics and statistics,
all we can discover and all we cannot

For wisdom is better than rubies;
and all the things that may be desired
are not to be compared to it.

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ABSTRACT

In survival analysis and in the analysis of the life tables an important biometric function of interest is the mean residual life function (MRLF) M whose value at age t is the average future life time given that a subject has survived till time t. Two important classes of the MRLFs are the 'New Better than Used in Expectation (NBUE) class' and 'Decreasing Mean Residual Life (DMRL) class'. These two classes are defined as $\{M(t) : M(t) \leq M(0),\ t\ \geq 0\}$ and $\{M(t) : M(t) \leq M(s),\ \text{if } t \geq s\}$ respectively.

In this dissertation we consider the problem of estimation and testing for the distributions in the NBUE and DMRL classes under random censoring. Our order restricted estimators of M for the NBUE and DMRL classes are respectively $M_n^*(t) = M_n(t) \wedge M_n(0)$ and $M_n^{**}(t) = \inf_{y \leq t} M_n(y)$, where M_n is the Kaplan-Meier estimator of M. We have proven that both these estimators are uniformly strongly consistent, and converge weakly to a Gaussian process. By simulation, we have also shown that our estimators are better than M_n in terms of asymptotic mean sums of squares. Several applications of our estimators have been provided. Tests that identify the NBUE or DMRL behavior have been developed. Both tests are shown to be consistent in their classes. We have derived the asymptotic distributions of our test statistics.

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Chapter 1

INTRODUCTION

Let X be a non-negative random variable denoting the time of occurrence of an event subjects with the distribution function (DF) F, $F(t) = P(X \leq t)$. Let $S(t) = 1 - F(t) = P(X > t)$ be the corresponding survival function (SF). An important biometric function is the mean residual function (MRLF). The mean residual life (MRL) at any time t is the expected remaining life of a unit of age t. It is a conditional concept, if we denote the MRLF by M, then $M(t)$ is the average remaining life among those population members who have survived until time t. Throughout we assume that the mean of X, $E(X) = \int_0^\infty S(u)du < \infty$. Then, MRL at time t, $M(t)$, is defined by

$$M(t) = E[X - t|X > t] = \frac{1}{S(t)} \int_t^\infty S(u)\ du\ I[S(t) > 0], \tag{1.0.1}$$

where $I(A)$ denotes the indicator function of the set A, $I(A)(t) = 1$ if $t \in A$ and 0 if $\notin A$.

The MRLF completely determines the distribution of the random variable, i.e., if we know the MRLF we can recover the corresponding survival function by using the inversion formula (Cox (1962))

$$S(t) = \frac{M(0)}{M(t)}\ exp\{-\int_0^t \frac{1}{M(u)}du\}\ I[M(t) > 0]. \tag{1.0.2}$$

An exponential random variable is characterized by the 'lack of memory property', i.e., if a continuous random variable is such that X has the same distribution as the conditional distribution of $X - t$ given $X > t$ for all $X \geq 0$, then X is distributed as exponential(λ) for some $\lambda > 0$. Using the lack of memory property and the previous inversion formula we can see that the exponential distribution is characterized by a constant MRL. Specifically, X has the density $\lambda\ exp(-\lambda t)$ for $t \geq 0$ for some $\lambda > 0$ if and only if $M(t) = \frac{1}{\lambda}$ for all $t \geq 0$. The importance of the MRLF is due to its wide range of applications. Actuaries apply MRL to setting rates and benefits for life insurance companies. In the social sciences we can use the MRLF for modeling the life-lengths of wars and strikes. The MRLF occurs naturally in areas such as biomedical sciences, optimal disposal of an asset, renewal theory, and reliability. More information on application of the MRL function can be find in Gross and Clark (1975) and Kuo (1984).

Modeling aging can be done in a variety of ways. In many applications, with prior knowledge about the structure of the phenomenon being studied, we may assume that the MRLF has a specific form. Classes of life distributions have been defined according to some monotonic properties of he MRLF, for example, the new better than used in expectation (NBUE) and the decreasing mean residual life IDMRL classes. A life distribution function F is said to be in the NBUE class if $M(0) \geq M(t)$ for all $t \geq 0$; F is in the DMRL class if $M(t)$ is decreasing in t, for $t \geq 0$. Throughout, we use decreasing (increasing) for nonincreasing (nondecreasing). Clearly, DMRL implies NBUE. Each of these classes has a dual class associated with it, i.e., increasing mean residual life (IMRL) and new worse than used in expectation (NWUE). Also, we can have a life distribution with IDMRL, i.e., there exists $t > 0$ such that M is increasing on $[0,\ t]$ and decreasing on $[t,\ \infty)$. Similarly, we may have a distribution with DIMRL.

The DMRL class is a natural one in reliability. The NBUE class is used in renewal theory (see Hall and Wellner (1981)). Life length of humans is described by the IDMRL class. High infant mortality explains the initial IMRL part and aging explains he later DMRL. Note that the exponential distribution belongs to both the DMRL and the NBUE classes.

Some of the parametric distributions frequently used in applications also belong to the DMRL class. The Uniform $(0,\ 1)$ has a strictly decreasing MRL. The Gamma distribution with density

$$f(x) = C(\alpha,\ \beta)\ exp(-\alpha x)\ x^{\beta-1},\ 0 \leq x,\ 0 < \alpha,\ \beta,$$

has a DMRL if $\beta > 1$. The Weibull distribution with the SF

$$S(x) = exp(-x^{\theta}),\ 0 \leq x,\ 0 < \theta,$$

also belongs to the DMRL class if $\theta > 1$.

We have seen how those order restrictions on M occurs naturally in applications. In this dissertation we will be dealing with estimation and testing for distributions in the NBUE and the DMRL classes under censoring. Our estimators are order restricted, i.e., we define estimators that satisfy the order restrictions imposed by the NBUE and DMRL classes.

The idea behind the order restricted estimator is that if we define estimators that use or incorporate the information contained in those order restricted classes we should expect a better estimator than if we do not use that additional information (the unrestricted estimators). Clearly, there is another problem associated with these order restrictions, namely, hypothesis testing under these restriction.

Now, we briefly discuss some of the previously known results for the NBUE and DMRL classes under censoring. Let $X_1^{(0)}, ..., X_n^{(0)}$ be independent random variables with a common continuous distribution function F, and let $U_1, ..., U_n$ be independent positive random variables with possibly discontinuous and defective common distribution function G that are independent of the $X_i^{(0)}$'s. The estimator of the MRLF $M(t)$ considered here is based on the censored date $(X_i,\ \delta_i)$ for $1 \leq i \leq n$, defined by

$$X_i = \min(X_i^{(0)},\ U_i), \quad \text{and} \quad \delta_i = I(X_i^{(0)} \ \leq U_i).$$

Let $X_{(i)}s$ be the order statistics of X_is. Then, for $t \in [X_k,\ X_{k+1}),\ k = 0, 1, ..., n-1$, the Kaplan-Meier estimate of the survival function is given by

$$S_n(t) = \prod_{X_{(i)} \leq t} \left[1 - \frac{1}{n-i+1}\right]^{\delta_{(i)}},\ \text{t} \ \geq 0. \tag{1.0.3}$$

With the usual convention that $X_{(n)}$ is uncensored whether it is censored or not, $M(t)$ is estimated by

$$M_n(t) = \frac{1}{S_n(t)} \int_t^{X_{(n)}} S_n(s)\, ds. \tag{1.0.4}$$

Since $S_n(t)$ is a step-function, for $t \in [X_k,\ X_{k+1}),\ k = 0, 1, ..., n-1,$

$$M_n(t) = \frac{S_n(X_k)(X_{k+1} - t) + S_n(X_{k+1})(X_{k+2} - X_{k+1}) + \quad ... \quad + S_n(X_{n-1})(X_n - X_{n-1})}{S_n(X_k)},$$

$$M_n(t) = 0 \quad \text{for} \quad t \geq X_n.$$

Even in the complete data case, it is well known that M_n is a biased estimator of M,

$$E[M_n(t)] = M(t)\ [1 - F^n(t)] \tag{1.0.5}$$

However, (1.0.5) implies that M_n is asymptotically unbiased. Kumazawa (1987) considered the estimation of the MRL under censoring. He proved that M_n is a strongly uniformly consistent estimator of M on $[0,\ T],$ where $T = \max(X_1, ..., X_n)$. He also proved the following theorems.

Theorem 1. *Suppose the distributions F and G satisfy the conditions*
$(i)\sqrt{n}\ h(T) \to 0$ in probability as $n \to \infty$,
$(ii)\ \int_0^{\tau_H} h^2(t)dC(t) < \infty$, where $H = 1 - S(1-G)$ denotes the distribution of X_1, $\tau_H = \sup\{t : H(t) < 1\}$ and so on, $h(t) = \int_t^{\tau_F} S(s)\ ds$, and $C(t) = \int_0^t [S^2(s)\{1 - G(s^-)\}]^{-1}\ dF(s)$, then the stochastic process $\{\sqrt{n}\ [M_n(t) - M(t)] : 0 \leq t \leq T\}$ converges weakly in $D[0,\ \tau_H]$ as $n \to \infty$ to a Gaussian process $\{B(t) : 0 \leq t \leq T\}$ with zero mean

and covariance function

$$cov\{B(s),\ B(t)\} = \{S(s)S(t)\}^{-1} \int_t^{\tau_H} h^2(u)\ dC(u) \qquad (s \le t).$$

The Gaussian process $\{B(t) : 0 \le t \le T\}$ *is given by*

$$B(t) = Z(t)\ M(t) + \int_t^{\tau_H} Z(s)\, dh(s)/S(t),$$

where $\{Z(t) : 0 \le t \le T\}$ *is a Gaussian process with zero mean and covariance function*

$$cov\{Z(s),\ Z(t)\} = C\{min(s,\ t)\}.$$

Theorem 2. *Under the condition of the Theorem 1, we have, for any* $t < \tau_H,\ \sigma_n^2(t) \to \sigma^2(t)$ *in probability as* $n \to \infty$.

This results can also be used to construct asymptotic confidence bands for $M(t)$ where $\sigma^2(t)$ is increasing in t, which holds, for instance, when F is exponential with arbitrary G. The limiting process of the $\sqrt{n}\ \frac{[M_n(s)-M(s)]}{\sigma_n(t)}$ for $0 \le s \le t$, for fixed $t < \tau_H$ has the same distribution as $W\left(\frac{\sigma(s)}{\sigma(t)}\right)$ for $0 \le s \le t$, where W is a standard Gaussian process on the unit interval $[0,\ 1]$.

Kochar, Mukerjee and Samaniego (2000) introduced an estimator of M under the DMRL assumption for the complete data. Utilizing the fact that M is a DMRL if and only if $M(t) = \inf_{y \le t} M(y)$, they estimated M by the corresponding sample analogue, i.e.,

$$M_n^*(t) = \inf_{y \le t} M_n(y)\ I(t < X_n),$$

where M_n is the empirical MRLE based on the complete data. They showed their estimator to be strongly uniformly consistent on $[0,\ b]$ for all $b < T$. They also showed that $\sqrt{n}\ [M_n^*(t) - M(t)]$ and $\sqrt{n}\ [M_n(t) - M(t)]$ are asymptotically equivalent in probability when M is strictly decreasing, under further analytic properties of M.

For the incomplete data, Koul and Susarla (1980) derived a test for H_0 vs H_1, where

$$\begin{aligned} H_0 &: S(x) = exp\left(-\frac{x}{\lambda}\right),\ x \geq 0,\ \lambda > 0 \text{ (unknown)} \\ H_1 &: F \text{ is NBUE, not an exponential.} \end{aligned}$$

In the case of complete data, tests for the preceding problem have been considered by Hollander and Proschan (1975) and Koul (1978). Koul and Susarla (1980) have considered the parameter

$$J(F) \equiv \mu^{-1} \int_0^\infty \int_0^y S(x)dx\ dF(y) = \frac{\int_0^\infty S^2(x)dx}{\int_0^\infty S(x)dx}.$$

Clearly, $F \in H_0$ implies $J(F) = \frac{1}{2}$. On the other hand, $J(F) = \frac{1}{2}$ and F continuous and NBUE implies that F is in H_0. The more $J(F)$ differs from $\frac{1}{2}$, the more there is evidence in favor of an $F \in H_1$. The test statistic proposed by Koul and Susarla (1980) is a nontrivial analog of the test statistic developed by Hollander and Proschan (1975), which is suitable for the censoring model. With their estimator S_n (modified form of the Kaplan-Meier estimator of the S), their test statistic is

$$T_n \equiv \frac{\int_0^{M_n} S_n^2(x)dx}{\int_0^{M_n} S_n(x)dx}.$$

This test is to reject H_0 if T_n is large. They have proved the folllowing results:

Lemma 1. *Suppose the support of G is* $(0,\ \infty)$ *and* $M_n = O(n^{\alpha})$, *for some* $0 < \alpha < \frac{1}{2}$, $n^{-\frac{1}{2}} \int_0^{M_n} S^{-2}\ (1-G)^{-3} \to 0$. *Then,* $T_n \to J(F)$ *in probability.*

Theorem 3. *Assume that the support of G is* $(0,\ \infty)$ *and that the following conditions are satisfied:*

(i) $n^{-\frac{1}{2}}(log\ logn)^{\frac{1}{2}}\ M_n\{e^{-2\frac{M_n}{\lambda}}\ (1-G)^3(M_n)\}^{-1} \to 0$,

(ii) $\lim\ \inf\ n^{\frac{1}{2}}\ (log\ logn)^{\frac{1}{2}}\ M_n^{-1}\ (1-G)(M_n) > 0$, *and*

(iii) $n^{-\frac{1}{2}}\ \int_0^{M_n} e^{\frac{4t}{\lambda}}\ (1-G)^{-6}\ dt \to 0$.

Then, $\sqrt{n}\ (T_n - \upsilon_n) \Rightarrow_w N(0,\ \sigma^2)$

where, $\upsilon_n \equiv \frac{(1-e^{-2\frac{M_n}{\lambda}})}{2}\ (1-e^{-\frac{M_n}{\lambda}}) = 2^{-1}(1+e^{-\frac{M_n}{\lambda}})$.

The alternative form of the test statistic T_n and its computational formulas can be seen in Koul and Susarla (1980).

Chen, Hollander, and Langberg (1983) derived a test for H_0 vs H_1, where

H_0 : F exponential(λ) for some $\lambda > 0$ (unknown), against

H_1 : F is in DMRL class, not an exponential.

They have considered the parameter

$$\triangle(F) = \int\int_{s<t} S(s)S(t)\{M(s) - M(t)\}dF(s)\ dF(t),$$

which is same as the parameter used by Hollander and Proschan (1975) in the uncensored case. To form the test statistic, they have replaced F with F_n, the Kaplan-Meier estimate of F. Since $\triangle(F_n)$ is not scale-invariant, in order to make it scale-invariant they have used the test statistic

$$V_n^c = \frac{\triangle(F_n)}{\mu_n},$$

where $\mu_n = \int_0^\infty S_n(u)\ du$, which is a consistent estimator of μ, under the assumption that the mean is finite and under suitable regularity on the amount of censoring. The computational form and other details of the test statistic V_n^c can be seen on Chen, Hollander, and Langberg (1983). Some other tests for DMRL using censored data can be found in Lim and Koh (1996) and Lim and Park (1993).

This dissertation is organized as follows. In Chapter 2 we review some standard results on the MRLF and weak convergence to be used later. Chapter 3 is dedicated to estimation in the NBUE class. We define an order restricted estimator under censoring for an NBUE mean residual life,

$$M_n^*(t) = M_n(t) \wedge M_n(0),\ t \geq 0.$$

We show that M_n^* is a strongly uniformly consistent estimator of M, a NBUE MRLF, on $[0,\ b]$ for all b such that $F(b) < 1$. We also obtain the weak convergence, asymptotic bias, and asymptotic mean square error of our estimator. Let the process $\{B_n^* \equiv \sqrt{n}\ [M_n^*(t) - M(t)] : t \in [0,\ b]\}$ and a mean zero Gaussian process $\{B(t) : t \in [0,\ b]\}$ given by

$$B(t) = Z(t)M(t) + \int_t^{\tau_H} Z(s)\, dh(s)/S(t),$$

where $\{Z(t) : t \in [0,\ b]\}$ is a Gaussian process with zero mean and covariance function

$$cov\{Z(s),\ Z(t)\} = C\{\min(s,\ t)\}.$$

Then, we have proven the following result. $If\ M = M(0)$ on $[0,\ b]$, then $B_n^* \Rightarrow_w B \wedge B(0)$ on $[0,\ b]$; if $M < M(0)$ on $(0,\ b]$, then $B_n^* \Rightarrow_w B$ on $[0,\ b]$; and if $M(0) = M(t_0)$ for some $t_0 \in (0,\ b)$ and for some $\gamma > 0,\ M(0) > M(t)$ on $(t_0,\ t_0+\gamma]$ with $t_0+\gamma < b$, then B_n^* does not converge weakly.

If $t \in [0,\ b]$, and $M(t) < M(0)$, $M_n^*(t)$ is asymptotically unbiased since $E(B^*(t)) = E(B(t)) = 0$. If $M(t) = M(0)$, $E(B^*(t)) = \frac{-\sigma_V(t)}{\sqrt{2\pi}}$, where $\sigma_V^2(t) = I(0) + \frac{I(t)}{S^2(t)} - 2\ \frac{I(t)}{S(t)})$, and $I(x) = \int_x^{\tau_H} h^2(u)\ dC(u)$. We have shown that $AMSE[M_n^*] \leq AMSE[M_n]$, where AMSE denotes the asymptotic mean square error. If $M(t) < M(0),\ \ AMSE[M_n^*(t)] = E[B^*(t)]^2 = E[B(t)]^2 = \frac{I(t)}{S^2(t)}$, and if $M(t) = M(0),\ AMSE[M_n(t)] - AMSE[M_n^*(t)] = \frac{1}{2}\,[\frac{I(t)}{S^2(t)} - \frac{I(0)}{S^2(0)}] \geq 0$, when Var(B(t)) is increasing that appears to be the case in most real life data.

Chapter 4 we generalize the order restricted estimator of Kochar, Mukerjee and Samaniego (2000) for the uncensored case to our censored data and define

$$M_n^*(x) = \inf_{y \leq x} M_n(y) I[x < X_n]. \tag{1.0.6}$$

We show that M_n^* is a strongly uniformly consistent estimator of M, on $[0,\ b]$ for all b such that $F(b) < 1$. We also obtain the weak convergence of our estimator. We have derived the limiting distribution of

$$B_n^* = \sqrt{n}\ [M_n^* - M] \text{ on } [0,\ b] \text{ for all } b < T. \tag{1.0.7}$$

For this we define the function l and u on $[0,\ \infty)$ to $[0,\ \infty)$ by

$$l(t) = \inf[s \le t : M(s) = M(t)],\ \text{and } u(t) = \sup[s \ge t : M(s) = M(t)]. \tag{1.0.8}$$

With this, we proved the following.

$$\text{If } B^*(t) = \inf_{l(t)\le s\le t} B(s),\ t \in [0,\ b], \text{then } B_n^* \Rightarrow_w B^* \text{on } [0,\ b]. \tag{1.0.9}$$

We also presented by simulations that $MSE[M_n^*] \le MSE[M_n]$ for finite sample sizes, when population is DMRL.

In Chapter 5, we derived a test in the NBUE case:
$H_0 : F(t) = 1 - e^{-\lambda t}$ for $t \ge 0,\ \lambda > 0$ (unspecified), against
$H_1 : F$ is in NBUE and not exponential.
We consider the following parameter as a measure of the deviation from H_0 to H_1 :

$$\Theta = \sup_{0\le s\le t} \frac{M(0) - M(s)}{\sigma(t)}.$$

Clearly, $\Theta = 0$, under H_0, but positive if H_1 is true. Our test statistic is the sample analogue of Θ,

$$\Theta_n = \sqrt{n} \sup_{0 \le s \le t} \frac{M_n(0) - M_n(s)}{\sigma_n(t)} \quad \text{for some } t < X_n.$$

where M_n and σ_n are the Kaplan- Meier MRLF and SD, respectively. Note that our test statistic is scale invariant and therefore we can assume that H_0 corresponds to $F \sim Exp(\lambda = 1)$. We have also found the limiting distribution of Θ_n as the following theorem.

Theorem 4. *Under H_0,*

$$\Theta_n \to_d \Theta = \sup_{0 \le s \le t} \left\{ W\left[\frac{\sigma(0)}{\sigma(t)}\right] - W\left[\frac{\sigma(s)}{\sigma(t)}\right] \right\} = W\left[\frac{\sigma(0)}{\sigma(t)}\right] - \inf_{0 \le s \le t} W\left[\frac{\sigma(s)}{\sigma(t)}\right] \equiv U - V.$$

In Chapter 6, we developed a test in the DMRL case:

$H_0 : F(t) = 1 - e^{-\lambda t}$ for $t \ge 0$, $\lambda > 0$ unspecified, against

$H_1 : F$ is in DMRL and not exponential.

We considered the following parameter Θ as a weighted measure of the deviation from H_0 to H_1 :

$$\Theta = \sup_{t \ge 0} \sup_{s \le t} S(s)S(t)[M(s) - M(t)].$$

Clearly, $\Theta = 0$ under H_0. If H_1 is true, M is nonincreasing and not constant on $[0, \infty)$. Thus, M has at least one point of decrease, and hence, $\Theta > 0$ if H_1 is true. Our test statistic is the sample analogue of Θ, given by

$$\Theta_n \equiv \sup_{t \ge 0} \sup_{s \le t} S_n(s)S_n(t)[M_n(s) - M_n(t)].$$

In order to make our test statistic scale invariant we set $\Theta_n^* \equiv \frac{\Theta_n}{M_n(0)}$. Therefore, we can assume that H_0 corresponds to $F \sim Exp(\lambda = 1)$. We also showed the consistency of of our test statistic. Although the limiting distribution is intractable, we propose a resampling scheme as given in Lin (1997) assuming that $\tau_H = \infty$, which is reasonable since $\tau_F = \infty$ under H_0.

Chapter 7 contains several ideas for further research and some concluding remarks.

Chapter 2

THE UNRESTRICTED CASE

2.1 Introduction

We now present the MRLF in a more mathematical setting when there are no order restriction. Hall and Wellner (1981) characterized the functions that are MRL functions as follows.

A function M satisfies (i) to (v) if and only if M is the MRLF of a life distribution nongenerate at 0:

(i) $M : [0,\ \infty) \to [0,\ \infty)$,

(ii) $M(0) > 0$,

(iii) M is right continuous,

(iv) $d(t) \equiv M(t) + t$ is increasing in $[0,\ \infty)$,

(v) When there exists t_0 such that $M(t_0^-) = 0$, then $M(t) = 0$ on $[t_0,\ \infty)$. If $M(t^-) \neq 0$ for any t, then $\int_0^\infty \frac{1}{M(u)}\, du = \infty$.

2.2 Censoring

In statistics and engineering, censoring occurs when the value of an observation is only partially known. There are various categories of censoring, which are, briefly, described below.

(i) *Left censoring* - a data point is below a certain value but it is unknown by how much.

(ii) *Interval censoring* - a data point is somewhere on an interval between two values.

(ii) *Right censoring* - a data point is above a certain value but it is unknown by how much.

To deal adequately with censoring in the analysis, one must consider the design which was employed to obtain the data. There are several types of censoring schemes within both left and right censoring.

(i) *Type I censoring* occurs if an experiment has a set number of subjects or items and stops the experiment at a predetermined time, at which point any subjects remaining are right-censored.

(ii) *Type II censoring* occurs if an experiment has a set number of subjects or items and stops the experiment when a predetermined number are observed to have failed; the remaining subjects are then right-censored.

(iii) *Random (or non-informative)* censoring is when each subject has a censoring time that is statistically independent of their failure time. The observed value is the minimum of the censoring and failure times; subjects whose failure time is greater than their censoring time are right-censored.

We now illustrate the computation of a MRLF under random censoring.

2.2.1 An estimator of a MRLF under censoring without ties

Let $X_1^{(0)}, ..., X_n^{(0)}$ be independent random variables with a common continuous distribution function F, and let $U_1, ..., U_n$ be independent positive random variables with possibly discontinuous and defective common distribution function G and independent of the $X_i^{(0)}$'s. The estimator of the MRLF $M(t)$ considered here is based on the censored date $(X_i,\ \delta_i)$ for $1 \le i \le n$, defined by

$$X_i = \min(X_i^{(0)},\ U_i) \text{ and } \delta_i = I(X_i^{(0)} \le U_i).$$

Let $X_{(i)}s$ be the order statistics of $X_i s$.Then for $t \in [X_k,\ X_{k+1})$, $k = 0, 1, ..., n-1$, the Kaplan-Meier survival function is given by

$$S_n(t) = \prod_{X_{(i)} \le t} \left[1 - \frac{1}{n-i+1}\right]^{\delta_{(i)}}. \tag{2.2.1}$$

We follow the usual convention that $X_{(n)}$ is uncensored even if it is censored in order to define S_n on $[0, \infty)$. The empirical estimate of M is given by Yang (1977),

$$M_n(t) = \frac{1}{S_n(t)} \int_t^{X_{(n)}} S_n(s)\ ds\ I(S_n(t) > 0). \tag{2.2.2}$$

Since $S_n(t)$ is a step-function, for $t \in [X_k,\ X_{k+1}),\ k = 0, 1, ..., n-1$,

$$M_n(t) = \frac{S_n(X_k)(X_{k+1} - t) + S_n(X_{k+1})(X_{k+2} - X_{k+1}) + \quad ... \quad + S_n(X_{n-1})(X_n - X_{n-1})}{S_n(X_k)}, \tag{2.2.3}$$

and,

$$M_n(t) = 0 \text{ for } t \geq X_n. \tag{2.2.4}$$

2.2.2 An estimator of a MRLF random censoring with ties

If censored and uncensored observations are tied, consider the uncensored observations to occur just before the censored observations. With this, let $0 \equiv X_{(0)}, X_{(1)}, ..., X_{(l)}$ be the distinct ordered statistics,

$f_k \equiv$ frequency of $X_{(k)}$; $k = 0, 1, ..., l-1$, and

$n_k \equiv$ number alive at time $X_{(k)}^- = n - \sum_{i=0}^{k-1} f_i$.

Then, for $t \in [X_k,\ X_{k+1})$, and $k = 0, 1, ..., l-1$, the Kaplan-Meier survival function is given by

$$S_n(t) = \prod_{i=0}^{k} [1 - \frac{f_i}{n_i}]^{\delta_{(i)}}. \tag{2.2.5}$$

Proceeding as in the previous case, for $t \in [X_k,\ X_{k+1})$, and $k = 0, 1, ..., l-1$,

$$M_n(t) = \frac{S_n(X_k)(X_{k+1} - t) + S_n(X_{k+1})(X_{k+2} - X_{k+1}) + \quad ... \quad + S_n(X_{l-1})(X_l - X_{l-1})}{S_n(X_k)}, \tag{2.2.6}$$

and,

$$M_n(t) = 0 \text{ for } \quad t \geq X_l. \tag{2.2.7}$$

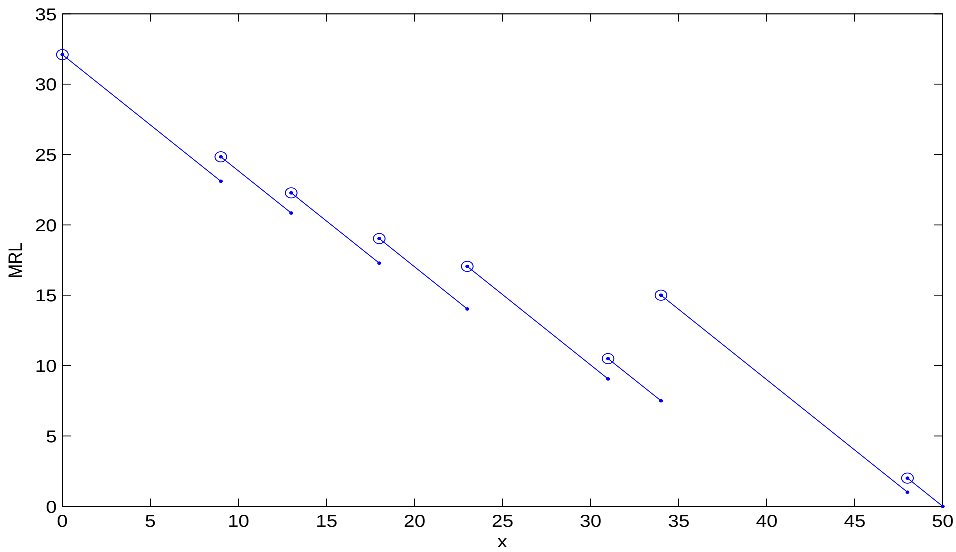
Example: Suppose we took a random sample of n = 15. In Table 2.1, the first four rows form the data set and the remaining rows give the calculated values of S_n and M_n. Notice that M_n consists of line segments with slope = -1, with a jump up at each uncensored order statistic. Note: For $\delta_{(i)}$, 0 = censored, and 1 = uncensored.

Table 2.1: SF and MRLF estimates under random censoring

(i)	0	1	2	3	4	5	6	7	8	9	10
$X_{(i)}$	0	9	13	18	23	28	31	34	45	48	50
Ties, f_i	-	1	2	1	2	2	1	3	1	1	1
$\delta_{(i)}$	-	1	0,1	1	1,1	0,0	1	1,1,1	0	1	1
$S_n(x_{(i)})$	1	.93	.87	.79	.65	.65	.56	.28	.28	.14	0
$M_n(x_{(i)})$	32.10	24.84	22.28	19.03	17.05	12.05	10.50	15	4	2	0
$M_n(x_{(i)}^-)$	0	23.10	20.84	17.28	14.03	12.05	9.05	7.50	4	1	0

Calculation of the Kaplan-Meier SF estimator:

Figure 2.1: Empirical estimate of the MRL with censored data



$S_n(0) = 1,$

$S_n(9) = S_n(0) \times \frac{14}{15} = .93,$

$S_n(13) = S_n(9) \times \frac{13}{14} = .87,$

$S_n(18) = S_n(13) \times \frac{11}{12} = .79,$

$S_n(23) = S_n(18) \times \frac{9}{11} = .65,$

$S_n(28) = S_n(23) \times \frac{9}{9} = .65,$

$S_n(31) = S_n(28) \times \frac{6}{7} = .56,$

$S_n(34) = S_n(31) \times \frac{3}{6} = .28,$

$S_n(45) = S_n(34) \times \frac{3}{3} = .28,$

$S_n(48) = S_n(45) \times \frac{1}{2} = .14,$

$S_n(50) = S_n(48) \times \frac{0}{1} = 0.$

Calculation of a MRL estimator under random censoring:

$M_n(0) = \frac{1(9)+.93(4)+.87(5)+.79(5)+.65(8)+.56(3)+.28(14)+.14(2)}{1} = 32.1,$

$M_n(9) = \frac{.93(4)+.87(5)+.79(5)+.65(8)+.56(3)+.28(14)+.14(2)}{.93} = 24.84,$

$M_n(13) = \frac{.87(5)+.79(5)+.65(8)+.56(3)+.28(14)+.14(2)}{.87} = 22.28,$

$M_n(18) = \frac{.79(5)+.65(8)+.56(3)+.28(14)+.14(2)}{.79} = 19.03,$

$M_n(23) = \frac{.65(8)+.56(3)+.28(14)+.14(2)}{.65} = 17.05,$

$M_n(31) = \frac{.56(3)+.28(14)+.14(2)}{.56} = 10.50,$

$M_n(34) = \frac{.28(14)+.14(2)}{.28} = 15,$

$M_n(48) = \frac{.14(2)}{.14} = 2,$

$M_n(50) = 0.$

2.3 Weak convergence

This section provides an overview of weak convergence, tightness, stochastic convergence, Brownian Motion, Brownian Bridge, and other related concepts that are used throughout.

2.3.1 Stochastic convergence

Let $X \equiv (X_1, X_2, \ldots, X_n) \in R^n$ be a random vector. The distribution function of X is the measurable function F such that $F : x \to P(X \leq x), x \in R^n$.

(i) *Weak convergence.* We say that a sequence of random vectors X_n converges weakly to X, denoted by $X_n \Rightarrow_w X$, if $F_n(x) \to F(x)$ for all x that is a continuity point of F, where F_n is the empirical function associated with X_n.

(ii) *Convergence in probability.* On a metric (S, ρ), we say that X_n converges in probability to X, denoted by $X_n \to_p X$, if $\forall \epsilon > 0$, $\lim_{n\to\infty} P(|X_n - X| < \epsilon) \to 0$. Note that convergence in probability is the usual convergence in measure for the finite measure P.

(iii) *Almost sure convergence.* We say that X_n converges almost surely to X, denoted by $X_n \to_{a.s.} X$, if $\forall \epsilon > 0$, $P(\lim_{n\to\infty} |X_n - X| < \epsilon) \to 0$. Note that a.s. convergence is same as almost everywhere (a.e.) convergence. Thus, in a. s. convergence, $X_n \to X$ except on a possible set of P measure 0. Note that for the convergence in (ii) and (iii), X_n and X need to be defined in the same probability space, but not for the convergence defined in (i). Weak convergence is weaker than convergence in probability, which is weaker than a.s. convergence. The three modes of convergence are related through the following theorem.

Theorem 5. *Let X_n, Y_n, X, Y be random vectors. Then,*

(i) if $X_n \to_{a.s.} X$, $\quad X_n \to_p X$,

(ii) if $X_n \to_p X$, $\quad X_n \Rightarrow_w X$,

(iii) if $X_n \to_p c$, c is constant, if and only if $X_n \Rightarrow_w c$,

(iv) if $X_n \Rightarrow_w X$ and $\rho(X_n,\ Y_n) \to_p 0$, then $Y_n \Rightarrow_w X$,

(v) if $X_n \Rightarrow_w X$ and $X_n \to_p c$, c is constant, then $(X_n,\ Y_n) \Rightarrow_w (X,\ c)$,

(vi) if $X_n \to_p X$ and $Y_n \to_p Y$, then $(X_n, Y_n) \to_p (X,\ Y)$.

2.3.2 Some important results

The following lemma (Billingsley (1999)) gives a series of characterizations of weak convergence.

Lemma 2 (Portmanteau)**.** *For any random vectors X_n and X the following statements are equivalent.*

(i) $X_n \Rightarrow_w X$,

(ii) $E(f(X_n)) \Rightarrow_w E(f(X))$ for all bounded, uniformly continuous function f,

(iii) lim inf $E(f(X_n)) \geq E(f(X))$ *for all nonnegative, continuous function f,*

(iv) lim inf $P(X_n \in G) \geq P(X \in G)$ *for every open set G,*

(v) lim sup $P(X_n \in H) \leq P(X \in H)$ *for every closed set H,*

(vi) $P(X_n \in B) \to P(X \in B)$ for all Borel sets B with $P(X \in \delta B) = 0$, where δB is the boundary of the set B. Note that a Borel set B such that $P(X \in \delta B) = 0$ is called a P-continuity set.

Theorem 6 (Continuous mapping theorem)**.** *If g is continuous on a set C such that $P(X \in C) = 1$, then $\Rightarrow_w$, $\rightarrow_p$, and $\rightarrow_{a.s.}$ are preserved under g, i.e.,*
(i) If $X_n \Rightarrow_w X$, then $g(X_n) \Rightarrow_w g(X)$.
(ii) If $X_n \rightarrow_p X$, then $g(X_n) \rightarrow_p g(X)$.
(iii) If $X_n \rightarrow_{a.s.} X$, then $g(X_n) \rightarrow_{a.s.} g(X)$.

If $X_n \Rightarrow_w X$ and $Y_n \Rightarrow_w c$ and g is a continuous function on $R^k \times \{C\}$, i.e., $\lim_{x \to x_0, y \to c} g(x,\ y) = g(x_0,\ c)$ for all x_0, then $g(X_n,\ Y_n) \Rightarrow_w g(X,\ c)$. In particular, since addition and multiplication are continuous functions, Slutsky's theorem follows.

Theorem 7 (Slutsky)**.** *Let X_n, Y_n, and X, be random vectors. If $X_n \Rightarrow_w X$ and $Y_n \Rightarrow_w c$, c is a constant vector c, then*
(i) $X_n + Y_n \Rightarrow_w X + c$.
(ii) $X_n Y_n \Rightarrow_w cX$.
(iii) $\frac{X_n}{Y_n} \Rightarrow_w \frac{X}{c}$ provided $c \neq 0$.

Tightness. A collection of random vectors $\{X_\alpha : \alpha \in A\}$ ia called uniformly tight if for all $\epsilon > 0$, there exists a constant M such that $\sup_\alpha P(||X_\alpha|| > M) < \epsilon$. The tightness condition and weak convergence are related through Prohorov's theorem.

Theorem 8 (Prohorov)**.** *(i) If $X_n \Rightarrow_w X$, then $\{X_n : n \in N\}$ is tight.*
(ii) If $\{X_n : n \in N\}$ is tight, then there exists a subsequence with $X_{n_k} \Rightarrow_w X$ as $k \to \infty$. The part in proving (ii) lies in the use of Helly's lemma. Helly's lemma states that any sequence of distribution functions $\{F_n\}$ possesses a subsequence $\{F_{n_k}\}$ such that $F_{n_k}(x) \Rightarrow_w F(x)$ at each continuity point x of a possibly defective distribution function

F. A defective distribution function F has all properties of a distribution function with the exception that it has limits less than 1 at ∞ and/or greater than 0 at $-\infty$. But it can be seen that this problem is avoided by requiring $\{X_n : n \in N\}$ to be tight. For a proof of any of the above see Billingley (1999) or A. W. van der Vaart (1998).

2.4 Historical background of the Brownian Motion

The Brownian Motion or Wiener process originated in physics from observing the constant erratic movement of tiny particles suspended in a fluid or gas. In 1827 the English botanist Robert Brown noticed that pollen grains suspended in water jiggled about under the lens of the microscope, following a zigzag path.

The inherent motion of the molecules of the fluid causes the molecules to strike the suspended particles at random. The impact makes the particles move. The first mathematical explanation of the phenomenon was given by Albert Einstein in 1905. Einstein succeeded in stating the mathematical laws governing the movements of particles on the basis of the principles of the kinetic-molecular theory of heat. Brownian Motion was then more generally accepted because it could now be treated as a practical mathematical model. As a result, many scientific theories and applications related to it have been developed. Today, Brownian Motion has applications in areas such as market analysis, investment decisions, and the important multi access queueing networks models for computers, communication and manufacturing systems.

2.4.1 The mathematical setting of the Brownian Motion

Mathematically, the Brownian Motion can be interpreted as the limit of a symmetric random walk process. Consider the symmetric random walk process that at each Δt time units can take a step of size Δx, either to the left or to the right with probability $\frac{1}{2}$. Let $X(t)$ be the position at time t. It follows that

$$X(t) = \Delta x \ (X_1 + X_2 + \ldots + X_{[\frac{t}{\Delta t}]}), \tag{2.4.8}$$

where $X_i = 1$ if the i^{th} step of length Δx is to the right, and $X_i = -1$, if the i^{th} step is to the left, and [ ] is the largest integer function. By the definition of a symmetric random walk, $\{X_i\}$ are independent random variables with $P(X_i = 1) = P(X_i = -1) = \frac{1}{2}$. Hence, $E(X_i) = 0$ and $Var(X_i) = 1$. Therefore, from (2.4.8),

$$E(X(t)) = 0 \text{ and, } Var(X(t)) = (\Delta x)^2 \ \left[\frac{t}{\Delta t}\right].$$

Now, we shall let $\Delta x \to 0$. However, if we let $\Delta x \to 0$, $E(X(t)) \to 0$ and $Var(X(t)) \to 0$. In that case, $X(t) = 0$ with probability 1. If we let $\Delta x = c\sqrt{\Delta t}$ for some positive constant c, and let $\Delta t \to 0$, $E(X(t)) \to 0$ and $Var(X(t)) \to c^2\, t$. In fact, it was shown by Einstein that the distance displaced by a particle suspended in a fluid or gas proportional to $\sqrt{\Delta t}$. By taking $\Delta x = c\sqrt{\Delta t}$ and letting $\Delta t \to 0$, the limiting process has following properties,

(i)$X(t) \sim N(0,\ c^2 t)$.

(ii)$\{X(t) : t \geq 0\}$ has independent increments. Note that the changes of values of the random walk in non overlapping time intervals are independent.

(iii) $\{X(t) : t \geq 0\}$ has stationary increments.

(iv) It can be proved that $X(t)$ is a.s. continuous and nowhere differential.

2.4.2 The Brownian Motion

A stochastic process $\{X(t) : t \geq 0\}$ is said to be a Brownian Motion if,

(i) $X(0) = 0$,

(ii)$\{X(t) : t \geq 0\}$ has stationary and independent increments,

(iii) $X(t) \sim N(0,\ c^2\ t)$ for all $t \geq 0$.

If $c = 1$, we call it a standard Brownian Motion process. It can be shown, using the stationary and independent increments assumptions, that $X(t_1), X(t_2),\ ...,\ X(t_n)$ has a multivariate normal distribution for all $t_1, t_2,\ ...,\ t_n$, i.e., a Brownian Motion is a Gaussian process. Since a Gaussian process is completely determined by its mean $E(X(t)) = 0$, and, covariance function,

$$Cov(X(s),\ X(t)) = Cov(X(s),\ X(s) + X(t) - X(s)) = Var(X(s)) = s, \quad s \leq t,$$

Brownian Motion could also be determined as a Gaussian process having, $E(X(t)) = 0$ and $Cov(X(s),\ X(t)) = s$, where $s \leq t$.

2.4.3 Some properties and /or results about the Brownian Motion

Let B denote a standard Brownian motion.

(i) *Differential Property.*

For any $s > 0$, $\{B_s(t) \equiv B(t+s) - B(s),\ t \geq 0\}$ is a standard Brownian Motion process independent of $\{B(u) : u \leq s\}$.

(ii) *Scaling property.*

For any $c > 0$, $\{\sqrt{c}\ B(\frac{t}{c}) : t \geq 0\} =_d \{B(t) :\ t \geq 0\}$.

(iii) *Symmetry.*

The negative of a Brownian Motion is still a Brownian Motion, that is ,

$-\{B(t)\} =_d \{B(t)\},\ t \geq 0$.

(iv) *Distribution of suprema.*

Let $Q(t) = \sup\limits_{0 \leq s \leq t} B(s)$.

For $a > 0$, $P[Q(t) \geq a] = 2\ P[B(t) \geq a] = P[|B(t)| \geq a\ ]$.

(v) *Time invariance.*

$t\ B(\frac{1}{t}) =_d B(t), \quad t \geq 0$.

2.4.4 The Brownian Bridge

One of the most important stochastic process in the empirical distribution function is the Brownian Bridge. If $\{X(t) : t \geq 0\}$ is a Brownian Motion, a Brownian Bridge is the conditional stochastic process given by

$$\{X(t) : 0 \leq t \leq 1 | X(1) = 0\}.$$

Using this definition it can be shown that,

$$E[X(s)|X(1) = 0] = 0 \text{ for } s < 1, \text{ and } Cov[X(s),\ X(t)] = s(1-t) \text{ for } s \leq t \leq 1.$$

2.4.5 A second characterization of the Brownian Bridge

We can also define a Brownian Bridge in terms of a Brownian Motion as follows. If $\{X(t) : t \geq 0\}$ is a Brownian Motion, set $\{Z(t) : 0 \leq t \leq 1\}$, where $Z(t) = X(t) - t\, X(1)$. From this last definition it follows easily that a Brownian Bridge can be characterized as a Gaussian process having $E[Z(t)] = 0$ and for $s \leq t \leq 1,\ Cov(Z(s),\ Z(t)) = s(1-t)$. Some related results are:

(i) If $X_i,\ i = 1, 2, ..., n$ is a random sample from a continuous distribution function F,

$$\sqrt{n}\ [F_n - F] \Rightarrow_w B^0(F),\ \text{on}\ [0,\ \infty),$$

where B^0 is a Brownian Bridge. (For a detailed proof see Billingsley (1999)).

(ii) For B^0 a Brownian Bridge and $b \geq 0$,

$$P(\sup_{0 \leq t \leq 1} B^0(t) > b) = e^{-2b^2},\ \text{and}\ P(\sup_{0 \leq t \leq 1} |B^0(t)| > b) = 2e^{-2b^2}.$$

Chapter 3

ESTIMATION IN THE NBUE CLASS

3.1 Introduction

In this chapter we define an order restricted estimator of a MRLF, M in the NBUE class. We recall that M is in the NBUE class if $M(0) \geq M(t), \quad t \geq 0$. This is the first time, at least that we know of, that an order restricted estimator based on the Kaplan-Meier estimator under a random censorship model has been considered in estimating a MRLF of a NBUE distribution. According to discussion in Chapter 1, provided that the order restriction in the NBUE class is satisfied, we should expect our estimator to be 'better' than the unrestricted estimator in some sense. In fact, we are able to prove analytically that our estimator produces a smaller asymptotic mean square error.

We show that our restricted estimator, M_n^*, is strongly uniformly consistent on [0, b], for all b such that $F(b) < 1$. The weak convergence, asymptotic bias, and asymptotic mean square error (AMSE) have also been derived. The ASME of our estimator M_n^* is less than or equal to that of the unrestricted estimator. Several applications of the estimators are provided.

3.2 The estimator

Let M be a NBUE MRLF. Then,

$$M(t) = M(T) \wedge M(0),\ t \geq 0. \tag{3.2.1}$$

Let $X_1^{(0)}, ..., X_n^{(0)}$ be independent random variables with a common continuous distribution function F, and let $U_1, ..., U_n$ be independent positive random variables with possibly discontinuous and defective common distribution function G and independent of the $X_i^{(0)}$'s. The estimator of the MRLF $M(t)$ considered here is based on the censored date $(X_i,\ \delta_i)$ for $1 \leq i \leq n$, defined by

$$X_i = \min(X_i^{(0)},\ U_i),\ \text{and}\ \delta_i = I(X_i^{(0)} \leq U_i).$$

Define $H = 1 - S(1 - G)$, which denotes the distribution of X_1. Also define $\tau_F = \sup\{t : F(t) < 1\}$, and $\tau_H = \sup\{t : H(t) < 1\}$. Then we may estimate S by the Kaplan-Meier estimator

$$S_n(t) = \prod_{0 \leq s \leq t} [1 - dN(s)/Y(s)],$$

where

$$N(s) = \sum_{i=1}^{n} I(X_i \leq s,\ \delta_i = 1), \quad Y(s) = \sum_{i=1}^{n} I(X_i \geq s).$$

Then the corresponding empirical estimator of the $M(t)$ is given by

$$M_n(t) = \int_t^T S_n(s)\, ds / S_n(t),$$

where $T = \max(X_1, ..., X_n)$.

We define our restricted estimator to be the sample analogue of (3.2.1);

$$M_n^*(t) = M_n(t) \wedge M_n(0), \quad t \geq 0. \tag{3.2.2}$$

Our estimator preserves the NBUE property. Observe that M_n^* consists of line segments with slope -1 on any interval where $M_n < M_n(0)$ and constant on any interval where $M_n \geq M_n(0)$ (see the figure at the end of this chapter).

3.3 Asymptotic results

3.3.1 Strong Uniform Consistency

Assuming that M is strictly decreasing, the following theorem shows strong uniform consistency of the estimator M_n^*.

Theorem 9. *M_n^* is a strongly uniformly consistent estimator of an NBUE, M on $[0,\ b]$ for all b such that $F(b) < 1$.*

Proof. We need to prove that $\sup\limits_{0 \leq t \leq b} |M_n^*(t) - M(t)| \to_{a.s.} 0$ as $n \to \infty$. Let b be arbitrary but fixed such that $F(b) < 1$. We have

$$\begin{aligned}
\sup_{0\le t\le b} |M_n^*(t) - M(t)| &= \sup_{0\le t\le b} |M_n(t) \wedge M_n(0) - M(t)| \\
&= \sup_{0\le t\le b} |M_n(t) + 0 \wedge [M_n(0) - M_n(t)] - M(t)| \\
&\le \sup_{0\le t\le b} |M_n(t) - M(t)| + \sup_{0\le t\le b} |0 \wedge (M_n(0) + M(t) - M(0) - M_n(t))| \\
&\le \sup_{0\le t\le b} |M_n(t) - M(t)| + |M_n(0) - M(0)| + \sup_{0\le t\le b} |M(t) - M_n(t)| \\
&= 2[\sup_{0\le t\le b} |M_n(t) - M(t)|] + |M_n(0) - M(0)| \to_{a.s.} 0 \text{ as } n \to \infty
\end{aligned}$$

because $|M_n(0) - M(0)| \to_{a.s.} 0$ as $n \to \infty$, by the SLLN, and $\sup_{0\le t\le b} |M_n(t) - M(t)| \to_{a.s.} 0$ as $n \to \infty$, by the strong uniform consistency of M_n on $[0,\ b]$.

3.3.2 Weak convergence

By Stone (1963) and Lindvall (1973), weak convergence on $[0,\ T)$ simply means weak convergence on all finite intervals of the form $[0,\ b]$, where b is a continuity point of the limit process with $b < T \le \infty$. Yang (1977) showed the same result with T replaced by a constant τ with $F(\tau) < 1$, and Kumazawa (1987) has proved the following theorem.

Theorem 10. *Suppose that the distributions F and G satisfy the conditions*
(i)$\sqrt{n}\ h(T) \to 0$ in probability as $n \to \infty$,
(ii) $\int_0^{\tau_H} h^2(t)dC(t) < \infty$, where $H = 1 - S(1 - G)$ denotes the distribution of X_1, $\tau_H = \sup\{t : H(t) < 1\}$ and so on, $h(t) = \int_t^{\tau_F} S(s)\ ds$, and $C(t) = \int_0^t [S^2(s)\{1 - G(s^-)\}]^{-1}\ dF(s)$,. Then the stochastic process $\{\sqrt{n}\ [M_n(t) - M(t)] : 0 \le t \le T\}$, converges weakly in $D[0,\ \tau_H]$ as $n \to \infty$ to a Gaussian process $\{B(t) : 0 \le t \le T\}$ with zero

mean and covariance function

$$cov\{B(s),\ B(t)\} = \{S(s)S(t)\}^{-1} \int_t^{\tau_H} h^2(u)\ dC(u) \quad (s \le t). \tag{3.3.3}$$

The Gaussian process $\{B(t) : 0 \le t \le T\}$ *is given by*

$$B(t) = Z(t)M(t) + \int_t^{\tau_H} Z(s)\, dh(s)/S(t), \tag{3.3.4}$$

where $\{Z(t) : 0 \le t \le T\}$ *is a Gaussian process with zero mean and covariance function*

$$cov\{Z(s),\ Z(t)\} = C\{min(s,\ t)\}. \tag{3.3.5}$$

Define the order restricted MRL process by

$$\begin{aligned}
B_n^*(t) &= \sqrt{n}\ [M_n^*(t) - M(t)] \\
&= \sqrt{n}\ [M_n(t) \wedge M_n(0) - M(t)] \\
&= \sqrt{n}\ [M_n(t) + 0 \wedge (M_n(0) - M_n(t)) - M(t)] \\
&= \sqrt{n}\ [M_n(t) - M(t)] + 0 \wedge \sqrt{n}\ [M_n(0) - M_n(t)] \\
&= \sqrt{n}\ [M_n(t) - M(t)] + 0 \wedge \sqrt{n}\ [M_n(0) - M_n(t) - M(0) \\
&\qquad +M(0) + M(t) - M(t)] \\
&= \sqrt{n}\ [M_n(t) - M(t)] + 0 \wedge \{\sqrt{n}\ [M_n(0) - M(0)] - \sqrt{n}\ [M_n(t) - M(t)] \\
&\qquad +\sqrt{n}\ [M(0) - M(t)].\}
\end{aligned}$$

Thus,

$$B_n^*(t) = B_n(t) + 0 \wedge \{B_n(0) - B_n(t) + \sqrt{n}[M(0) - M(t)]\}. \tag{3.3.6}$$

We now have the following theorem.

Theorem 11. *Consider the process* $\{B_n^*(t) \equiv \sqrt{n}\ [M_n^*(t) - M(t)] : t \in [0,\ b]\}$.
(i) If $M = M(0)$ *on* $[0,\ b]$, *then*

$$B_n^* \Rightarrow_w B \wedge B(0) \ \textit{on}\ [0,\ b].$$

(ii) If $M < M(0)$ *on* $(0,\ b]$, *then*

$$B_n^* \Rightarrow_w B \ \textit{on}\ [0,\ b].$$

(iii) If $M(0) = M(t_0)$ *for some* $t_0 \in (0,\ b)$ *and for some* $\gamma > 0,\ M(0) > M(t)$ *on* $(t_0,\ t_0 + \gamma]$ *with* $t_0 + \gamma < b$, *then* B_n^* *does not converge weakly.*

Proof. From (3.3.6), we have

$$B_n^*(t) = B_n(t) + 0 \wedge [B_n(0) - B_n(t)].$$

By the continuous mapping theorem,

$$B_n^*(t) \Rightarrow_w B(t) + 0 \wedge [B(0) - B(t)] = B(t) \wedge B(0).$$

Hence, (i) follows.

If $M < M(0)$ on $(0,\ b]$, $\sqrt{n}\ [M(0) - M(t)] \to \infty$ on $(0,\ b]$. Thus from (3.3.6),

$$B_n^*(t) \Rightarrow_w B(t) \text{ on } [0,\ b].$$

Part (iii) follows from a lack of tightness argument. Assume that $M(0) = M(t_0)$ for some $t_0 \in (0,\ b)$ and for some $\gamma > 0,\ M(0) > M(t)$ on $(t_0,\ t_0 + \gamma]$ with $t_0 + \gamma < b$. Since $M(0) = M(t_0)$,

$$B_n^*(t_0) = B_n(t_0) + 0 \wedge [B_n(0) - B_n(t_0)].$$

Using (3.3.2) we obtain,

$$[B_n(0) - B_n(t_0)] \to_d N(0,\ \sigma_{t_0}^2)$$

where,

$$\sigma_{t_0}^2 = \sigma^2(0) + \sigma^2(t_0) - 2\ cov\{B(0),\ B(t_0)\}.$$

Also, since F is continuous, B has continuous a.s. paths. Since $B_n \Rightarrow_w B,\ B_n$ is tight on $[0,\ b],\ b < T$. Let $\epsilon,\ \eta > 0$ be arbitrary but fixed. By the tightness of B_n, there exists n_0 in N and $0 < \delta < \gamma$ such that

$$P[\sup_{t_0 \le s \le t_0 + \delta} |B_n(t_0) - B_n(s)| \le \epsilon] \ge 1 - \eta \text{ for all } n \ge n_0. \tag{3.3.7}$$

Now, if $M(0) > M(t)$, for n large enough, with arbitrary high probability, $B_n^*(t) = B_n(t)$.

That is, there exists k_0 in N such that

$$P[B_n^*(t) = B_n(t)] > 1 - \eta \text{ for all } n \geq k_0. \tag{3.3.8}$$

Assume $B_n^*(t)$ is tight on $[0,\ b]$. Then, there exists $0 < \delta' < \delta$ and l_0 in N such that

$$P[\sup_{t_0 \leq s \leq t_0+\delta'} |B_n^*(t_0) - B_n^*(s)| \leq \epsilon] \geq 1 - \eta \text{ for all } n \geq l_0. \tag{3.3.9}$$

Now, fix $t_0 < t < t_0 + \delta'$. Let $n \geq \max\{n_0,\ k_0,\ l_0\}$.

$$\begin{aligned}
P[\sup_{t_0 \leq s \leq t_0+\delta'} |B_n^*(t_0) - B_n^*(s)| \leq \epsilon] &\leq P[|B_n^*(t_0) - B_n^*(t)| \leq \epsilon] \\
&\leq P[|B_n^*(t_0) - B_n(t)| \leq \epsilon] + \eta \text{ (By using (3.3.8))} \\
&= P[|B_n^*(t_0) - B_n(t_0) + B_n(t_0) - B_n(t)| \leq \epsilon] + \eta \\
&= P[-\epsilon - B_n(t_0) + B_n(t) \leq B_n^*(t_0) - B_n(t_0) \\
&\qquad \leq \epsilon - B_n(t_0) + B_n(t)] + \eta \\
&\leq P[|B_n^*(t_0) - B_n(t_0)| \leq 2\epsilon] + 2\eta \\
&= P[|0 \wedge [B_n(0) - B_n(t_0)]| \leq 2\epsilon] + 2\eta \\
&= 1 - P[|0 \wedge [B_n(0) - B_n(t_0)]| > 2\epsilon] + 2\eta \\
&= 1 - P[B_n(0) - B_n(t_0) < -2\epsilon] + 2\eta \\
&\to 1 - \Phi(\frac{-2\epsilon}{\sigma_{t_0}}) + 2\eta,
\end{aligned}$$

where Φ is the standard normal DF. This is a contradiction to the tightness condition of B_n^* in (3.3.9). $\Box$

3.3.3 Asymptotic bias and asymptotic mean square error

We recall that $B_n^* \Rightarrow_w B^*$ on $[0,\ b]$, $\quad b < T$, where

$$B^*(t) = \begin{cases} B(t) \quad \text{if } M(t) < M(0), \text{ on } [0,\ b] \\ B(t) \wedge B(0) \quad \text{if } M(t) = M(0), \text{ on } [0,\ b]. \end{cases}$$

The asymptotic bias of B^*

Let $t \in [0,\ b]$. If $M(t) < M(0)$, $M_n^*(t)$ is asymptotically unbiased since $E(B^*(t)) = E(B(t)) = 0$. If $M(t) = M(0)$,

$$\begin{aligned} E(B^*(t)) &= E(B(t) \wedge B(0)) \\ &= E[B(t) + (B(0) - B(t)) \wedge 0] \\ &= E[(B(0) - B(t)) \wedge 0]. \end{aligned}$$

Let $V = B(0) - B(t) \sim N(0,\ \sigma_V^2(t) = I(0) + \frac{I(t)}{S^2(t)} - 2\frac{I(t)}{S(t)})$, where, $I(x) = \int_x^{\tau_H} h^2(u)\, dC(u)$.Then,

$$\begin{aligned} E(B^*(t)) &= E(V \wedge 0) \\ &= (\int_{-\infty}^{0} \frac{v\, exp(\frac{-v^2}{2\sigma_V^2(t)})}{\sigma_V(t)\sqrt{2\pi}} du) \wedge 0 \\ &= \frac{-\sigma_V(t)}{\sqrt{2\pi}}. \end{aligned}$$

The asymptotic mean square error (AMSE)

We now proceed to show that $E(B^*(t))^2 \leq E(B(t))^2$ for $t \in [0, b]$.

If $M(t) < M(0)$, $AMSE[M_n^*(t)] = E[B^*(t)]^2 = E[B(t)]^2 = \frac{I(t)}{S^2(t)}$.

We recall that $\{B(t) : 0 \le t \le T\}$ is a mean zero Gaussian process with the covariance structure given by $Cov[B(t),\ B(s)] = \{S(s)S(t)\}^{-1}\int_t^{\tau_H} h^2(u)\ dC(u) \quad (s \le t)$, provided that $E(X^2) < \infty$.

If $M(t) = M(0)$, $AMSE[M_n^*(t)] = E[B^*(t)]^2 = E[B(t) \wedge B(0)]^2 = E[B(t) + (B(0) - B(t)) \wedge 0]^2$. Therefore,

$$AMSE[M_n^*(t)] = E[B^2(t) + 2B(t)[(B(0) - B(t)) \wedge 0] + [(B(0) - B(t)) \wedge 0]^2]. \quad (3.3.10)$$

Now, we need $E\{B(t)[(B(0)-B(t))\wedge 0]\}$. To obtain this, we need to use the following well known results. $E[g(X)Y] = E[g(X)E(Y|X)]$, and for $(U,\ V) \sim N_2(\mu_U, \mu_V, \sigma_U^2, \sigma_V^2, \rho)$, $E[U|V] = \mu_U + \rho\frac{\sigma_U}{\sigma_V}(V - \mu_v)$.

Let $U = B(t) \sim N(0, \frac{I(t)}{S^2(t)})$, and $V = B(0) - B(t) \sim N(0,\ \sigma_V^2(t) = I(0) + \frac{I(t)}{S^2(t)} - 2\frac{I(t)}{S(t)})$. Then,

$$\begin{aligned}
E\{B(t)\ [(B(0) - B(t)) \wedge 0]\} &= E[(V \wedge 0)\, U] \\
&= E[(V \wedge 0)\, E(U|V \wedge 0)] \\
&= \rho\, \frac{\sigma_U}{\sigma_V}\, \frac{\sigma_V^2}{2} \\
&= \frac{\rho\, \sigma_U\, \sigma_V}{2} \\
&= \frac{Cov(U,\ V)}{2} \\
&= \frac{1}{2}\, E[B(t)\ (B(0) - B(t))] \\
&= \frac{1}{2}\, [\frac{I(t)}{S(t)} - \frac{I(t)}{S^2(t)}].
\end{aligned}$$

Also,

$$\begin{aligned} E[(B(0) - B(t)) \wedge 0]^2] &= E(V \wedge 0)^2 \\ &= \frac{E(V^2)}{2} \\ &= \frac{\sigma_V^2}{2} \\ &= \frac{1}{2}\,[I(0) + \frac{I(t)}{S^2(t)} - 2\,\frac{I(t)}{S(t)})]. \end{aligned}$$

Putting these together in (3.3.11), and using the linearity of E, we get:

$$AMSE[M_n^*(t)] = \frac{1}{2}\,[I(0) + \frac{I(t)}{S^2(t)})]. \tag{3.3.11}$$

Now, $AMSE[M_n(t)] - AMSE[M_n^*(t)]$
$= \frac{1}{2}\,[\frac{I(t)}{S^2(t)} - \frac{I(0)}{S^2(0)}] \geq 0$, if $Var(B(t))$ is increasing in t that appears to be the case in most real life data.

Table 3.1: Comparison of MSE's of M_n^* and M_n at the Q- quantile ξ_Q for Weibull distribution with $n = 30$, censoring percentage = 15%, based on 50,000 iterations

Q	$M(\xi_Q)$	$Bias(M_n(t))$	$Bias(M_n^*)$	$MSE(M_n)$	$MSE(M_n^*)$	$\frac{MSE(M_n)}{MSE(M_n^*)}$
0.1	0.6363	0.096033	0.096003	0.021121	0.021099	1.001008
0.2	0.5584	0.106719	0.106710	0.023908	0.023901	1.000283
0.5	0.4236	0.100383	0.100327	0.025337	0.025280	1.002188
0.8	0.3225	0.097877	0.096879	0.041034	0.039574	1.036891
0.9	0.2835	0.074081	0.071992	0.058886	0.055595	1.059186

Table 3.2: Comparison of MSE's of M_n^* and M_n at the Q-quantile ξ_Q for Weibull distribution with $n = 50$, censoring percentage = 15%, based on 50,000 iterations

Q	$M(\xi_Q)$	$Bias(M_n(t))$	$Bias(M_n^*)$	$MSE(M_n)$	$MSE(M_n^*)$	$\frac{MSE(M_n)}{MSE(M_n^*)}$
0.1	0.6363	0.097714	0.097713	0.016770	0.016770	1.000017
0.2	0.5584	0.094050	0.094050	0.015955	0.015955	1.000021
0.5	0.4236	0.092374	0.092373	0.017386	0.017384	1.000068
0.8	0.3225	0.094532	0.094403	0.026859	0.026676	1.006839
0.9	0.2835	0.087599	0.086870	0.039376	0.038235	1.029847

3.4 The MSE for finite sample sizes

We have shown in section (3.3.3) that $AMSE[M_n^*(t)] \leq AMSE[M_n(t)]$. We now use simulations to compare the MSE of M_n^* with that of $M_n(t)$ for finite sample size n. We compute the MSE of both estimators at different percentiles for the following NBUE distributions: Weibull with $S(t) = exp(-t^2)$ and $Uniform(0,\ 1)$ with 50,000 replications, using simple exponential censoring with 15% and 25% censoring percentages. The results are shown in Tables $3.1 - 3.8$.

Table 3.3: Comparison of MSE's of M_n^* and M_n at the Q-quantile ξ_Q for Weibull distribution with $n = 30$, censoring percentage = 25%, based on 50,000 iterations

Q	$M(\xi_Q)$	$Bias(M_n(t))$	$Bias(M_n^*)$	$MSE(M_n)$	$MSE(M_n^*)$	$\frac{MSE(M_n)}{MSE(M_n^*)}$
0.1	0.6363	0.137552	0.137512	0.034155	0.034123	1.000942
0.2	0.5584	0.148680	0.148664	0.038679	0.038662	1.000438
0.5	0.4236	0.140277	0.140194	0.040946	0.040838	1.002660
0.8	0.3225	0.107864	0.106944	0.054059	0.052531	1.029093
0.9	0.2835	0.060377	0.058856	0.070014	0.067311	1.040147

Table 3.4: Comparison of MSE's of M_n^* and M_n at the Q-quantile ξ_Q for Weibull distribution with $n = 50$, censoring percentage = 25%, based on 50,000 iterations

Q	$M(\xi_Q)$	$Bias(M_n(t))$	$Bias(M_n^*)$	$MSE(M_n)$	$MSE(M_n^*)$	$\frac{MSE(M_n)}{MSE(M_n^*)}$
0.1	0.6363	0.146371	0.146369	0.031088	0.031086	1.000037
0.2	0.5584	0.144985	0.144985	0.030816	0.030816	1.000000
0.5	0.4236	0.141728	0.141719	0.033065	0.033052	1.000372
0.8	0.3225	0.121664	0.121427	0.040427	0.040033	1.009834
0.9	0.2835	0.087869	0.087130	0.050999	0.049664	1.026888

Table 3.5: Comparison of MSE's of M_n^* and M_n at the Q-quantile ξ_Q for $U(0,1)$ distribution with $n = 30$, censoring percentage = 15%, based on 50,000 iterations

Q	$M(\xi_Q)$	$Bias(M_n(t))$	$Bias(M_n^*)$	$MSE(M_n)$	$MSE(M_n^*)$	$\frac{MSE(M_n)}{MSE(M_n^*)}$
0.1	0.45	0.049257	0.044516	0.005762	0.005058	1.139098
0.2	0.40	0.062471	0.060687	0.007509	0.007099	1.057714
0.5	0.25	0.063032	0.062940	0.007073	0.007035	1.005359
0.8	0.10	0.048911	0.048909	0.005142	0.005140	1.000202
0.9	0.05	0.035714	0.035712	0.004126	0.004125	1.000263

Table 3.6: Comparison of MSE's of M_n^* and M_n at the Q-quantile ξ_Q for $U(0,1)$ distribution with $n = 50$, censoring percentage = 15%, based on 50,000 iterations

Q	$M(\xi_Q)$	$Bias(M_n(t))$	$Bias(M_n^*)$	$MSE(M_n)$	$MSE(M_n^*)$	$\frac{MSE(M_n)}{MSE(M_n^*)}$
0.1	0.45	0.047989	0.046445	0.004351	0.004128	1.054095
0.2	0.40	0.051766	0.051504	0.004655	0.004602	1.011521
0.5	0.25	0.049964	0.049964	0.004017	0.004016	1.000017
0.8	0.10	0.037063	0.037063	0.002501	0.002501	1.000000
0.9	0.05	0.028391	0.028391	0.001885	0.001885	1.000000

Table 3.7: Comparison of MSE's of M_n^* and M_n at the Q-quantile ξ_Q for $U(0,1)$ distribution with $n = 30$, censoring percentage = 25%, based on 50,000 iterations

Q	$M(\xi_Q)$	$Bias(M_n(t))$	$Bias(M_n^*)$	$MSE(M_n)$	$MSE(M_n^*)$	$\frac{MSE(M_n)}{MSE(M_n^*)}$
0.1	0.45	0.071248	0.065789	0.008889	0.007836	1.134326
0.2	0.40	0.086378	0.084158	0.011482	0.010881	1.055150
0.5	0.25	0.084832	0.084714	0.011104	0.011045	1.005247
0.8	0.10	0.058699	0.058688	0.007662	0.007655	1.000966
0.9	0.05	0.038120	0.038116	0.005729	0.005726	1.000552

Table 3.8: Comparison of MSE's of M_n^* and M_n at the Q-quntile ξ_Q for $U(0,1)$ distribution with $n = 50$, censoring percentage = 25%, based on 50,000 iterations

Q	$M(\xi_Q)$	$Bias(M_n(t))$	$Bias(M_n^*)$	$MSE(M_n)$	$MSE(M_n^*)$	$\frac{MSE(M_n)}{MSE(M_n^*)}$
0.1	0.45	0.072479	0.070511	0.007523	0.007152	1.051754
0.2	0.40	0.077584	0.077182	0.008243	0.008143	1.012231
0.5	0.25	0.073149	0.073135	0.007213	0.007210	1.000319
0.8	0.10	0.049291	0.049291	0.004079	0.004079	1.000000
0.9	0.05	0.033521	0.033521	0.002875	0.002875	1.000000

3.5 Recovering an estimator of S from M_n^*

Using the inversion formula,

$$S(x) = \frac{M(0)}{M(x)}\, exp\{-\int_0^x \frac{1}{M(u)} du\}\; I[M(x) > 0],$$

we obtain,

$$\begin{aligned} S^{'}(t) &= -\frac{M(0)}{M^2(t)}\, [exp\{-\int_0^t \frac{1}{M(u)} du\}\; [M^{'}(t)+1] \\ &= -\frac{S(t)}{M(t)}\, [M^{'}(t)+1], \text{where } M^{'}(t) \text{ exist.} \end{aligned} \tag{3.5.12}$$

From (3.5.12) we note that S is flat in any interval where $M^{'} = -1$, and , if $M^{'} = 0$, on an interval, them$S$ corresponds to a shifted exponential distribution with mean equal to the constant value $M(t)$ on the interval. We define S_n^* at the order statistics as follows

$$S_n^*(X_i) = \frac{M_n(0)}{M_n^*(X_{(i)})}\, exp\{-\int_0^{X_{(i)}} \frac{1}{M_n^*(u)} du\}\; I[M_n^*(X_{(i)}) > 0]. \tag{3.5.13}$$

We recall the S_n^* will assign an exponential function where M_n^* is constant and will be constant otherwise. We might obtain S_n^* inductively in the following way. First, on $[0,\ X_{(1)})$, M_n^* is always a line segment with slope = -1, and thus, S_n^* is constant on $[0,\ X_{(1)})$. Hence, $S_n^*(X_{(1)}^-) = 1$. Once we have determined $S_n^*(X_{(i)}^-)$, we can obtain

$S_n^*(X_{(i)})$ for $i = 1, 2, \ldots, n-1$ as follows:

$$\begin{aligned} S_n^*(X_{(i)}) &= \frac{M_n(0)}{M_n^*(X_{(i)})}\, exp\{-\int_0^{X_{(i)}^-} \frac{1}{M_n^*(u)} du\} \\ &= \frac{M_n(0)}{M_n^*(X_{(i)})}\, \frac{M_n^*(X_{(i)}^-)}{M_n^*(X_{(i)}^-)}\, exp\{-\int_0^{X_{(i)}^-} \frac{1}{M_n^*(u)} du\} \\ &= \frac{M_n^*(X_{(i)}^-)}{M_n^*(X_{(i)})}\, S_n^*(X_{(i)}^-). \end{aligned} \tag{3.5.14}$$

Since we already know $S_n^*(X_{(1)}^-) = 1,\ S_n^*(X_{(1)}) = \frac{M_n^*(X_{(1)}^-)}{M_n^*(X_{(1)})}$.

We now can define S_n^* on $[X_{(1)},\ X_{(2)})$. There are three possibilities.

(i) When M_n^* is a line segment with slope = -1 on $[X_{(1)},\ X_{(2)})$

In this case,

$$S_n^*(t) = \frac{M_n^*(X_{(1)}^-)}{M_n^*(X_{(1)})}.$$

Thus, $S_n^*(X_{(2)}^-) = S_n^*(X_{(1)})$.

(ii) When M_n^* is constant on $[X_{(1)},\ X_{(2)})$

Here S_n^* is an exponential in the whole interval $[X_{(1)},\ X_{(2)})$. So,

$$S_n^*(t) = S_n^*(X_{(1)})\, exp\{-\frac{(t - X_{(1)})}{M_n^*(X_{(1)})}\},\ t \in\ [X_{(1)},\ X_{(2)}).$$

By the continuity of the exponential function,

$$S_n^*(X_{(2)}^-) = S_n^*(X_{(1)})\, exp\{-\frac{(X_{(2)} - X_{(1)})}{M_n^*(X_{(1)})}\}.$$

(iii) When there exists $\xi_{(1)}$ such that M_n^* is constant on $[X_{(1)},\ \xi_{(1)})$ and decreasing in $(\xi_{(1)},\ X_{(2)})$

For $t \in\ [X_{(1)},\ \xi_{(1)})$,

$$S_n^*(t) = S_n^*(X_{(1)})\, exp\{-\frac{(t - X_{(1)})}{M_n^*(X_{(1)})}\},$$

and,

$$S_n^*(t) = S_n^*(\xi_{(1)}) \ \text{ on } (\xi_{(1)},\ X_{(2)}).$$

Therefore, $S_n^*(X_{(2)}^-) = S_n^*(\xi_{(1)})$.

In general, suppose S_n^* have been defined on $[X_{(i)},\ X_{(i+1)})$. We define S_n^* on $[X_{(i+1)},\ X_{(i+2)})$ as follows. $S_n^*(X_{(i+1)})$ is obtained from (3.5.14).

(i) When M_n^* is a line segment with slope = -1,

$$S_n^*(t) = \frac{M_n^*(X_{(i+1)}^-)}{M_n^*(X_{(i+1)})} \text{ on } [X_{(i+1)},\ X_{(i+2)})$$

Thus,

$$S_n^*(X_{(i+2)}^-) = \ S_n^*(X_{(i+1)}).$$

(ii) When M_n^* is constant on $[X_{(i+1)},\ X_{(i+2)})$,

$$S_n^*(t) = S_n^*(X_{(i+1)})\, exp\{-\frac{(t - X_{(i+1)})}{M_n^*(X_{(i+1)})}\},\ t \in\ [X_{(i+1)},\ X_{(i+2)})$$

and,

$$S_n^*(X_{(i+2)}^-) = S_n^*(X_{(i+1)})\, exp\{-\frac{(X_{(i+2)} - X_{(i+1)})}{M_n^*(X_{(i+1)})}\}.$$

(iii) When there exists $\xi_{(i+1)}$ such that M_n^* is constant on $[X_{(i+1)},\ \xi_{(i+1)})$ and decreasing in $(\xi_{(i+1)},\ X_{(i+2)})$,

For $t \in\ [X_{(i+1)},\ \xi_{(i+1)})$,

$$S_n^*(t) = S_n^*(X_{(i+1)})\ exp\{-\frac{(t - X_{(i+1)})}{M_n^*(X_{(i+1)})}\},$$

and,

$$S_n^*(t) = S_n^*(\xi_{(i+1)}) \ \text{ on } (\xi_{(i+1)},\ X_{(i+2)}).$$

Therefore,

$$S_n^*(X_{(i+2)}^-) = S_n^*(\xi_{(i+1)}).$$

Table 3.9: Order restricted and empirical estimators of the MRL and S for the death times of kidney transplant patients

$X_{(i)}$	$M_n(X_{(i)})$	$M_n(X_{(i)}^-)$	$M_n^*(X_{(i)})$	$M_n^*(X_{(i)}^-)$	$S_n(X_{(i)})$	$S_n^*(X_{(i)})$
0	2517.78	0	2517.78	0	1.000	1.000
40	2520.50	2477.78	2517.78	2477.78	0.981	0.984
45	2560.42	2515.50	2517.78	2512.78	0.972	0.970
106	2544.87	2499.42	2517.78	2456.78	0.950	0.959
121	2577.60	2529.87	2517.78	2496.78	0.933	0.923
229	2518.99	2469.60	2517.78	2409.78	0.912	0.913
344	2453.05	2403.99	2453.05	2402.78	0.894	0.890
864	1978.01	1933.05	1978.01	1933.05	0.870	0.866
929	1967.66	1913.01	1967.66	1913.01	0.851	0.840
943	2023.44	1953.66	2023.44	1953.66	0.822	0.810
1016	2022.68	1950.44	2022.68	1950.44	0.793	0.774
1196	1919.45	1842.68	1919.45	1842.68	0.762	0.730
2171	987.38	944.45	987.38	944.45	0.731	0.682
2276	962.60	882.38	962.60	882.38	0.676	0.627
2650	654.00	588.60	654.00	588.60	0.601	0.581
3304	-	-	-	-	-	-

3.6 Applications

We illustrate M_n^* and S_n^* on a data set given in the Appendix of the book- Klein and Moeschberger. The observations refer to the survival times (in days) of 61 Kidney transplant patients (Black American women). All patients had their transplant performed at The Ohio State University Transplant Center during the period 1982 - 1992. Patients were censored if they moved from Columbus (lost to follow-up) or if they were alive on June 30, 1992.
Uncensored Observations: 40, 45, 106, 121, 229, 344, 864, 929, 943, 1016, 1196, 2171, 2276, 2650.
Censored Observations :14, 93, 116, 116, 250, 259, 261, 306, 312, 392, 442, 512, 625, 673, 732, 777, 879, 887, 899, 899, 903, 920, 953, 953, 1151, 1291, 1291, 1457, 1508, 1567, 1674, 1736, 1736, 1736, 1739, 1942, 2026, 2268, 2413, 2434, 2463, 2680, 2935, 3072, 3161, 3211, 3304.

Figure 3.1: Empirical estimate M_n of the MRL

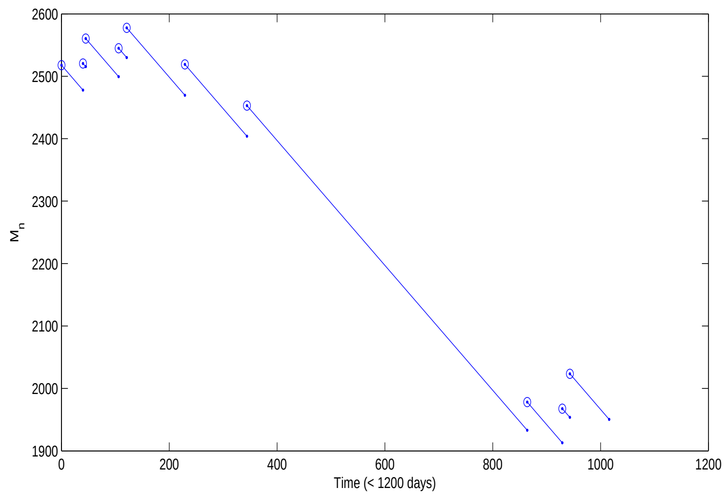


Figure 3.2: Order restricted estimate M_n^* of the MRL

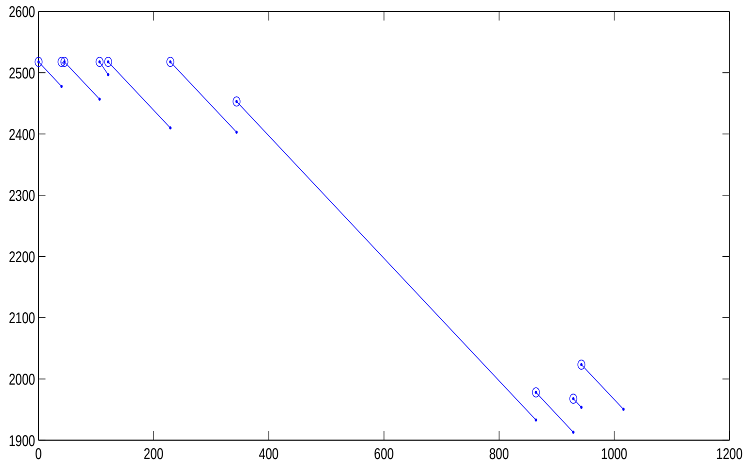


Figure 3.3: Empirical estimate S_n of the SF

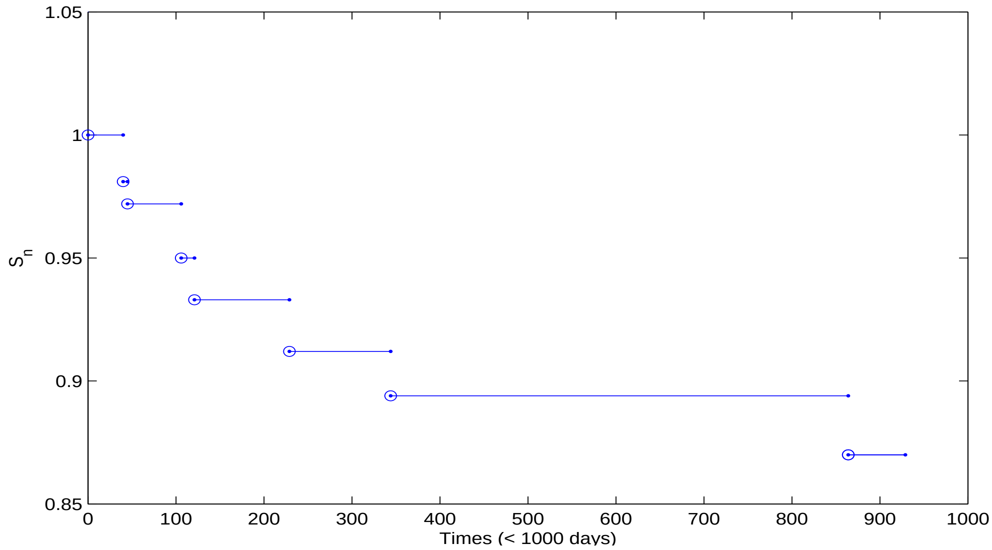
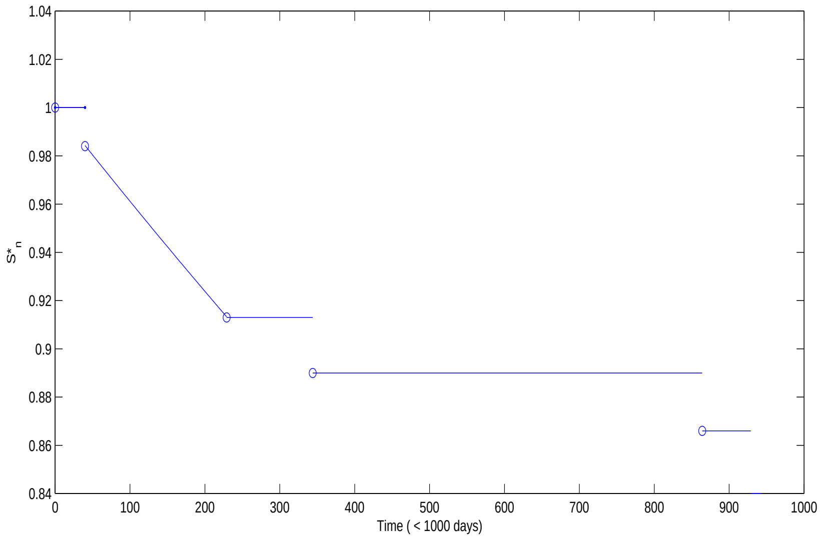


Figure 3.4: Order restricted estimate S_n^* of the SF



Chapter 4

ESTIMATION IN THE DMRL CLASS

4.1 Introduction

We begin this chapter by reviewing some properties of the MRLF that are used later in the proof of the Theorem 16. One property of the MRLF, M, is that $M(t)+t$ is increasing for all t, so that $M' \geq -1$ whenever it exists, where $M'(t) = \frac{dM(t)}{dt}$. Hence, M cannot jump down. Thus, in the DMRL case, M must be continuous, and from (1.0.2), the corresponding SF, S, cannot have a jump expect possibly at T, the right end point of the support of S. The problem of estimating a DMRLF without censoring was considered by Kochar, Mukerjee, and Samaniego (2000). In this chapter we generalize their estimator to the censored case. The estimator introduced by them is,

$$M_n^*(x) = \inf_{y \leq x} M_n(y) I[x < X_n] \text{ for the uncensored case.}$$

4.2 Asymptotic results

4.2.1 Strong uniform consistency

It is well known that M_n is strongly uniformly consistent on [0, b] for any $b < T$. We use this result and the triangle inequality of the sup-norm [see Lemmas 1 and 2, Rojo and Samaniego (1993)] to prove the following theorem.

Theorem 12. *M_n^* is a strongly uniformly consistent estimator of M, on $[0,\ b]$ for all b such that $F(b) < 1$.*

Proof. We need to prove that $\sup\limits_{0\le t\le b} |M_n^*(t) - M(t)| \to_{a.s.} 0$ as $n \to \infty$. Let b be arbitrary but fixed such that $F(b) < 1$. We have

$$\begin{aligned}
\sup_{0\le t\le b} |M_n^*(t) - M(t)| &= \sup_{0\le t\le b} |\inf_{y\le t} M_n(y) - M(t)| \\
&= \sup_{0\le t\le b} |\inf_{y\le t} M_n(y) - \inf_{y\le t} M(y)| \\
&\le \sup_{0\le t\le b} \sup_{y\le t} |M_n(y) - M(y)| \\
&= \sup_{0\le y\le b} |M_n(y) - M(y)| \to_{a.s.} 0 \text{ as } n \to \infty. \quad \square
\end{aligned}$$

4.2.2 Weak convergence

Kumazawa (1987) has proved the following theorem.

Theorem 13. *Suppose that the distributions F and G satisfy the conditions*

(i) $\sqrt{n}\ h(T) \to 0$ *in probability as* $n \to \infty$,

(ii) $\int_0^{\tau_H} h^2(t)dC(t) < \infty$, *where* $H = 1 - S(1-G)$ *denotes the distribution of* X_1,

$\tau_H = \sup\{t : H(t) < 1\}$ *and so on,* $h(t) = \int_t^{\tau_F} S(s)\ ds$, *and* $C(t) = \int_0^t [S^2(s)\{1 - G(s^-)\}]^{-1}\ dF(s)$,*.Then the stochastic process* $\{\sqrt{n}\ [M_n(t) - M(t)] : 0 \leq t \leq T\}$, *converges weakly in* $D[0,\ \tau_H]$ *as* $n \to \infty$ *to a Gaussian process* $\{B(t) : 0 \leq t \leq T\}$ *with zero mean and covariance function*

$$cov\{B(s),\ B(t)\} = \{S(s)S(t)\}^{-1} \int_t^{\tau_H} h^2(u)\ dC(u) \quad (s \leq t). \qquad (4.2.1)$$

The Gaussian process $\{B(t) : 0 \leq t \leq T\}$ *is given by*

$$B(t) = Z(t)M(t) + \int_t^{\tau_H} Z(s)\, dh(s)/S(t),$$

where $\{Z(t) : 0 \leq t \leq T\}$ *is a Gaussian process with zero mean and covariance function*

$$cov\{Z(s),\ Z(t)\} = C\{min(s,\ t)\}.$$

We now consider the weak convergence of $B_n^* = \sqrt{n}\ [M_n^* - M]$ on $[0,\ b]$ for all $b < T$. Define the function l and u on $[0,\ \infty)$ to $[0,\ \infty)$ by $l(t) = \inf[s \leq t : M(s) = M(t)]$, and $u(t) = \sup[s \geq t : M(s) = M(t)]$. By continuity of M, the infimum (supremum) is actually a minimum (maximum, if $u(t) < \infty$).

Theorem 14. *If* $B^*(t) = \inf_{l(t) \leq s \leq t} B(s),\ t \in [0,\ b],$ *then* $B_n^* \Rightarrow_w B^*$ *on* $[0,\ b]$.

Proof. We have

$$\begin{aligned} B_n^*(t) &= \sqrt{n}\min[\inf_{l(t)\le s\le t} M_n(s) - M(t), \quad \inf_{s\le l(t)} M_n(s) - M(t)] \\ &= \sqrt{n}\min[\inf_{l(t)\le s\le t}\{M_n(s) - M(s)\}, \quad \inf_{s\le l(t)} M_n(s) - M(t)]. \qquad (4.2.2) \end{aligned}$$

By the continuity mapping theorem, the process

$$\sqrt{n}\ [\inf_{l(t)\le s\le t}\{M_n(s) - M(s)\} : 0 \le t \le b] \Rightarrow_w B^* \text{ on } [0,\ b].$$

Since B has a. s. continuous paths, a path of B^* is a.s. left-continuous with jump up at $u(t)$ if $l(t) < u(t) < \infty$, and is right-continuous at 0. Let $\tau = \max\ \{t : M(t) = M(0)\}$, i.e., $\tau = u(0)$. Then, for $0 \le t \le \tau$, $\sqrt{n}\ [M_n^*(t) - M(t)] = \sqrt{n}\ \inf_{s\le t}[M_n(s) - M(s)] = \sqrt{n}\ \inf_{l(t)\le s\le t}[M_n(s) - M(s)]$, and thus $B_n^* \Rightarrow_w B^*$ on $[0,\ \tau]$. We will show on $(\tau,\ b]$ the second term in (4.4.2) is convergence equivalent to $B_n \circ l = \sqrt{n}\ [M_n \circ l - M \circ l]$ that converges weakly to $B \circ l$ by the continuous mapping theorem, and thus dominates B^*. The paths of $B \circ l$ are similar to those of B^* except that they are constant on $[l(t),\ u(t)]$ for all t, and may jump up or down at $u(t)$. For the remainder of the proof assume that $\tau < t \leq b$. Now, for an arbitrary $\delta > 0$,

$$\begin{aligned} B_n(l(t)) &= \sqrt{n}\,[M_n(l(t)) - M(l(t))] \\ &\geq \sqrt{n}\,\inf_{s \leq l(t)}[M_n(s) - M(l(t))] \\ &\geq \sqrt{n}\,\inf_{[(l(t)-\delta)\vee 0] \leq s \leq l(t)}[M_n(s) - M(s)] \,\wedge\, \sqrt{n}\,\inf_{s \leq [(l(t)-\delta)\vee 0]}[M_n(s) - M(l(t))] \\ &\geq B_n(l(t)) - w[\sqrt{n}(M_n - M);\ \delta,\ b] \qquad (4.2.3) \\ &\quad \wedge \sqrt{n}\,\inf_{s \leq [(l(t)-\delta)\vee 0]}[M_n(s) - M(l(t))] \end{aligned}$$

where the modulus of oscillation,

$$w[\sqrt{n}\,(M_n - M);\ \delta,\ b] = \sup_{\{s,t \leq b : |M(s) - M(t)| \leq \delta\}} |\sqrt{n}\,[M_n(s) - M(s) - M_n(t) + M(t)]|,$$

satisfies $\lim_{\delta \to 0} \limsup_{n \to \infty} P[w\{\sqrt{n}\ (M_n - M);\ \delta,\ b\} \geq \epsilon]$ for all $\epsilon > 0$, since B_n is tight on $[0,\ b]$. Note that by the Theorem 2.2.1(iv), $\sqrt{n}\,[M_n(l(t)) - M(l(t))]$ has the same asymptotic distribution as $\sqrt{n}\,[M_n(l(t)) - M(l(t)) - w\{\sqrt{n}\ (M_n - M);\ \delta,\ b\}]$.

$$\begin{aligned} \text{Now,} \inf_{s \leq [(l(t)-\delta)\vee 0]} \sqrt{n}\,[M_n(s) - M(l(t))] &\geq \inf_{s \leq [(l(t)-\delta)\vee 0]} \sqrt{n}\,[M_n(s) - M(s) \\ &\qquad + \sqrt{n}\,[M((l(t) - \delta) \vee 0) - M(l(t))] \\ &= O_p(1) + D_n(t, \delta), \end{aligned}$$

where, the $O_p(1)$ term is bounded below by $-\|B_n\|_0^b \to_d -\|B\|_0^b$ uniformly in t, and

$D_n(t,\ \delta) \to \infty$ as $n \to \infty$ uniformly on $[a,\ b]$ for all $\tau < a \leq b$. Thus, the process

$$\{O_p(1) + D_n(t, \delta) : \tau < t \leq b\} \to_p \infty\ I(\tau,\ b]. \tag{4.2.4}$$

Hence, for all n large enough and $t \in (\tau,\ b]$,

$$B_n(l(t)) \geq \sqrt{n} \inf_{s \leq l(t)} [M_n(s) - M(l(t))] \geq B_n(l(t)) - w[\sqrt{n}(M_n - M);\ \delta,\ b], \tag{4.2.5}$$

with arbitrarily high probability. Applying the squeezing theorem in (4.4.5) yields,

$$\sqrt{n} \inf_{s \leq l(t)} [M_n(s) - M(l(t))] \Rightarrow_w B(l(t)) \text{ on}(\tau,\ b],$$

and using (4.4.2), we get $B_n^* \Rightarrow_w B^*$ on $[0,\ b]$. $\square$

Note that $B_n^* \Rightarrow_w B$ on $[0,\ b]$ if $l(t) = t$. Thus, Theorem 14 provides a much stronger result under much weaker assumptions than the weak convergence result of Kochar, Mukerjee and Samaniego (2000) for incomplete data case.

Table 4.1: Comparison of MSE's of M_n^* and M_n at the Q-quantile ξ_Q for Weibull distribution with $n = 30$, censoring percentage = 15%, based on 50,000 iterations

Q	$M(\xi_Q)$	$Bias(M_n(t))$	$Bias(M_n^*)$	$\frac{MSE(M_n)}{MSE(M_n^*)}$
0.1	0.6363	0.0965	0.0791	1.2533
0.2	0.5584	0.1054	0.0969	1.1303
0.5	0.4237	0.1015	0.0990	1.0608
0.8	0.3226	0.0973	0.0915	1.1570
0.9	0.2836	0.0771	0.0677	1.2140

Table 4.2: Comparison of MSE's of M_n^* and M_n at the Q-quantile ξ_Q for Weibull distribution with $n = 30$, censoring percentage = 25%, based on 50,000 iterations

Q	$M(\xi_Q)$	$Bias(M_n(t))$	$Bias(M_n^*)$	$\frac{MSE(M_n)}{MSE(M_n^*)}$
0.1	0.6363	0.0972	0.0923	1.0805
0.2	0.5584	0.0941	0.0934	1.0143
0.5	0.4237	0.0924	0.0923	1.0052
0.8	0.3226	0.0948	0.0939	1.0345
0.9	0.2836	0.0858	0.0829	1.0991

4.3 The MSE for finite sample sizes

We use simulations to compare the MSE of M_n^* with that of $M_n(t)$ for finite sample size $n = 30$. We compute the MSE of both estimators at different percentiles for the Weibull distribution with $S(t) = exp(-t^2)$ with $50,000$ replications, using simple exponential censoring with 15% and 25% censoring percentages. The results are shown in Tables 4.1 − 4.2.

Table 4.3: M_n and M_n^* of the MRL for the survival times of the patients of a melanoma study data

$X_{(i)}$	$M_n(X_{(i)})$	$M_n(X_{(i)}^-)$	$M_n^*(X_{(i)})$	$X_{(i)}$	$M_n(X_{(i)})$	$M_n(X_{(i)}^-)$	$M_n^*(X_{(i)})$
0	143.94	0	143.94	125	48.77	46.56	48.77
16	129.83	127.94	129.83	129	47.13	44.77	47.13
44	103.34	101.83	103.34	132	46.58	44.13	46.58
55	93.79	92.34	93.79	134	47.37	44.58	46.58
67	83.57	81.79	83.57	140	44.13	41.37	44.13
73	79.56	77.57	79.56	147	40.22	37.13	40.22
76	78.62	76.56	78.62	148	43.15	39.22	40.22
80	76.70	74.62	76.70	151	44.61	40.15	40.22
81	77.86	75.70	76.70	152	49.06	43.61	40.22
86	75.00	72.86	75.00	158	49.21	43.06	40.22
93	70.13	68.00	70.13	181	30.58	26.21	30.58
100	65.23	63.13	65.23	190	25.90	21.58	25.90
108	59.28	57.23	59.28	193	28.62	22.90	25.90
114	55.41	53.28	55.41	213	11.50	8.62	11.50
120	51.56	49.41	51.56	215	19.00	9.50	11.50

4.4 Applications

We consider the data used by Susarla and Koul (1980) to explain the implementation and illustration of M_n^*. The data set refers to the survival times (in weeks) of 69 participants of a melonoma study by the central oncology group with headquarters at the University of Wisconsin-Madison.

Uncensored Observations:16, 44, 55, 67, 73, 76, 80, 81, 86, 93, 100, 108, 114, 120, 125, 129, 132, 134, 140, 147, 148, 151, 152, 158, 181, 190, 193, 213, 215.

Censored Observations:13, 14, 19, 20, 21, 23, 25, 26, 27, 31, 32, 34, 37, 38, 40, 46, 50, 53, 54, 57, 58, 59, 60, 65, 66, 70, 85, 90, 98, 102, 103, 110, 118, 124, 130, 136, 138, 141, 194, 234.

Figure 4.1: Empirical estimate of the MRL for a melanoma study data

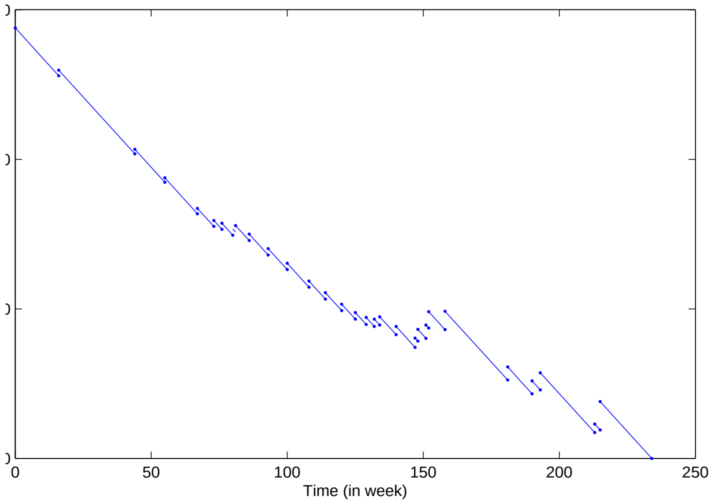
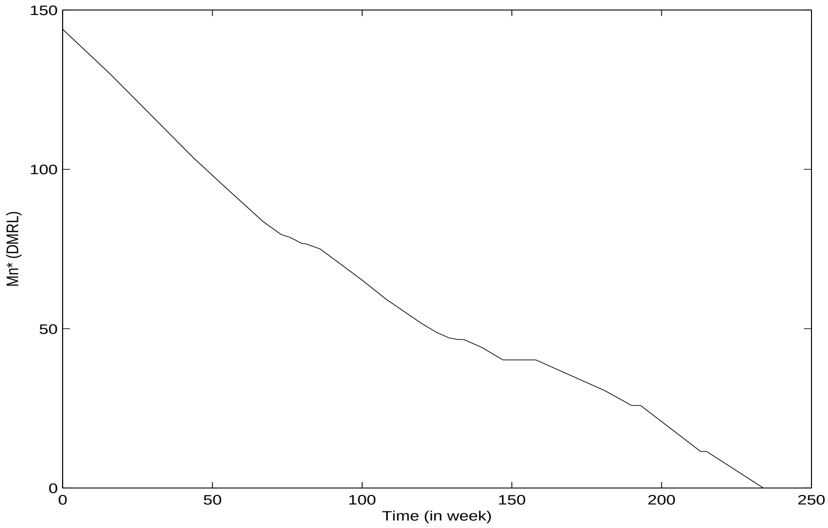


Figure 4.2: Order restricted estimate of the MRL for a melanoma study data



Chapter 5

TESTING THE EXPONENTIAL DISTRIBUTION VS. THE NBUE DISTRIBUTION

5.1 Introduction

In this chapter we consider the problem of testing H_0 against H_1, where

$H_0 : F(t) = 1 - e^{-\lambda t}$ for $t \geq 0$, $\lambda > 0$ unspecified, against.

$H_1 : F$ is in NBUE and not exponential.

This is equivalent to

$H_0 : M(t) = \frac{1}{\lambda}$ for all $t \geq 0$, $\lambda > 0$ unspecified, but fixed, Vs

$H_1 : M(0) \geq M(t)$ and $M(t) \neq \frac{1}{\lambda}$ for all $t \geq 0$.

In Section 5.2 we derived the test statistic Θ_n for the test. We proved the consistency of the test in section 5.3 and derived the limiting distribution of Θ_n in Section 5.4.

5.2 The test statistic and its properties

We consider the following parameter as a measure of the deviation from H_0 to H_1 :

$$\Theta = \sup_{0 \leq s \leq t} \frac{M(0) - M(s)}{\sigma(t)}.$$

We recall Kumazawa's (1987) weak convergence result from section 3.3.2:

$$\sqrt{n}\ [M_n(t) - M_n(t)] \Rightarrow_w B(t) \text{ on } [0,\ \infty) \text{ when } S \text{ is exponential,}$$

where B is a mean-zero Gaussian process with

$$cov\{B(s),\ B(t)\} = \{S(s)S(t)\}^{-1} \int_t^\infty h^2(u)\ dC(u) \quad (s \leq t),$$

with h and C as defined in Theorem 10 in section 3.3.2, and that

$$E[B(t)]^2 \equiv \sigma^2(t) = \{S(t)\}^{-2} \int_t^\infty h^2(u)\ dC(u).$$

Gill (1980) has shown that the empirical estimate, $\sigma_n^2(t)$ of $\sigma^2(t)$ is uniformly consistent on $[0,\ b]$ for any b with $S(b) > 0$. Kumazawa (1987) also noted that, when S is exponential, $\sigma(t)$ is nondecreasing. The assumption that $\sigma(t)$ is nondecreasing is satisfied because for

$S = e^{-t}$,

$$\begin{aligned} \frac{d}{dt}\sigma^2(t) &= e^{2t}\left[2\int_t^\infty \frac{e^{-u}du}{\overline{G}(u)} - \frac{e^{-t}}{\overline{G}(t)}\right] \\ &\geq e^{2t}\left[2\int_t^\infty \frac{e^{-u}du}{\overline{G}(t)} - \frac{e^{-t}}{\overline{G}(t)}\right] \\ &= 2e^{2t}\frac{e^{-t}}{\overline{G}(t)}, \end{aligned}$$

when S is exponential, irrespective of G. Using this and the uniform consistency of $\sigma_n^2(t)$, he showed that, for any fixed $t < \infty$,

$$\left\{\sqrt{n}\ \frac{M_n(s) - M(s)}{\sigma_n(t)} : 0 \leq s \leq t\right\} \Rightarrow_w W\left[\frac{\sigma(s)}{\sigma(t)}\right] \text{ on } [0,\ t],$$

where W is a standard Brownian motion. We use this result to define the test statistic

$$\Theta_n = \sqrt{n}\ \sup_{0\leq s\leq t} \frac{M_n(0) - M_n(s)}{\sigma_n(t)} \quad \text{for some } t < X_n.$$

Note that $\sup_{0\leq s\leq t} [M(0) - M(s)] = 0$ under H_0 while it is positive under the alternative. Thus, we reject H_0 for large values of Θ_n.

5.3 Consistency

We have

$$\begin{aligned}
\Theta_n(t) &= \sqrt{n}\ \sup_{0\le s\le t} \frac{M_n(0)-M_n(s)}{\sigma_n(t)} \text{ for some } t < X_n. \\
&= \sqrt{n}\ \sup_{0\le s\le t} \frac{1}{\sigma_n(t)}\ [(M_n(0)-M(0))-(M_n(s)-M(s))+(M(0)-M(s))] \\
&= \sup_{0\le s\le t} \frac{1}{\sigma_n(t)}\ [B_n(0)-B_n(s)+\sqrt{n}\ (M(0)-M(s))]. \qquad (5.3.1)
\end{aligned}$$

If F is under H_1, the last term in (5.4.1) is zero for all $s \le t$, but if F is in the NBUE class, there exists s_0 in $[0,\ b]$ for some b such that $F(b) < 1$, with $M(0) - M(s_0) > 0$. Hence, $\sup_{0\le s\le t} \frac{1}{\sigma_n(t)} \sqrt{n}\,[M(0)-M(s_0)] \to \infty$, and $\Theta_n(t) \to \infty,$ provided that F is NBUE. Thus, our test is consistent.

5.4 Limiting distribution of Θ_n

Theorem 15. *Under H_0,*

$$\Theta_n \to_d \Theta = \sup_{0\le s\le t}\left\{W\left[\frac{\sigma(0)}{\sigma(t)}\right] - W\left[\frac{\sigma(s)}{\sigma(t)}\right]\right\} = W\left[\frac{\sigma(0)}{\sigma(t)}\right] - \inf_{0\le s\le t} W\left[\frac{\sigma(s)}{\sigma(t)}\right] \equiv U - V.$$

Proof: This is obvious from Kumazawa's result above and the continuous mapping theorem. □

With the change of variable $u = \frac{\sigma(s)}{\sigma(t)}$, we can write

$$V = \inf_{\sigma(0)/\sigma(t) < u \le 1} W(u).$$

Note that U and V are independent from the independent increments of W and

$$U \sim N(0, \sigma(0)/\sigma(t)).$$

Now, for any $c > 0$,

$$\begin{aligned}
P(U - V > c) &= \int P(U - V > c | U = u) d\Phi(u/[\sigma(0)/\sigma(t)]) \\
&= \int P(u - V > c) d\Phi(u/[\sigma(0)/\sigma(t)]) \\
&= \int P(\inf_{\sigma(0)/\sigma(t) < u \leq 1} W(u) \leq u - c) d\Phi(u/[\sigma(0)/\sigma(t)]) \\
&\leq \int P(\inf_{0 < u \leq 1} W(u) \leq u - c) d\Phi(u/[\sigma(0)/\sigma(t)]) \\
&= \int_{u > c} 2\overline{\Phi}(c - u) d\Phi(u/[\sigma(0)/\sigma(t)]),
\end{aligned}$$

where $\Phi = 1 - \overline{\Phi}$ is the standard normal DF and ϕ is its density; the last equality follows from the fact that

$$P(\inf_{0 \leq u \leq 1} W(u) < r) = 2\overline{\Phi}(-r) \text{ for } r \leq 0 \text{ and } 0 \text{ if } r > 0.$$

The integral can be approximated numerically using the estimate of $\sigma(0)/\sigma(t)$ to find the p-value if c is the value of the test statistic.

Chapter 6

TESTING THE EXPONENTIAL DISTRIBUTION VS.THE DMRL DISTRIBUTION

6.1 Introduction

Let X be a lifetime random variable with continuous distribution function (DF) F where $F(0) = 0$. In this chapter we consider the problem of testing H_0 against H_1 where

$H_0 : F(t) = 1 - e^{-\lambda t}$ for $t \geq 0$, $\lambda > 0$ unspecified, against

$H_1 : F$ is in DMRL and not exponential.

This is equivalent to

$H_0 : M(t) = \frac{1}{\lambda}$ for all $t \geq 0$, $\lambda > 0$ unspecified, but fixed, against

$H_1 : M(s) \geq M(t)$ for all $s \leq t$.

In Section 6.2 we derived the test statistic $\sqrt{n}\ \Theta_n$ for the test. We proved the consistency of the test in section 6.3 and derived the limiting distribution of $\sqrt{n}\ \Theta_n^*$ in Section 6.4.

6.2 The test statistic and its properties

Consider the following parameter Θ as a weighted measure of the deviation from H_0 to H_1 :

$$\Theta = \sup_{t\geq 0}\ \sup_{s\leq t} S(s)S(t)\ [M(s) - M(t)]. \tag{6.2.1}$$

Clearly, $\Theta = 0$ under H_0. If H_1 is true, M is nonincreasing and not constant on $[0,\ \infty)$. Thus, M has at least one point of decrease, and hence, $\Theta > 0$ if H_1 is true. Employing the same sampling plan as in chapter 5, our test statistic is the sample analogue of Θ, given by

$$\sqrt{n}\ \Theta_n \equiv \sup_{t\geq 0}\ \sup_{s\leq t} \sqrt{n}\ S_n(s)S_n(t)\ [M_n(s) - M_n(t)]. \tag{6.2.2}$$

In order to make our test statistic scale invariant we set $\Theta_n^* \equiv \frac{\Theta_n}{M_n(0)}$. Therefore, we can assume that H_0 corresponds to $F \sim Exp(\lambda = 1)$.

6.3 Consistency

We note that

$$\begin{aligned}\sqrt{n}\ \Theta_n^*(t) &= \sqrt{n}\ \sup_{t\geq 0}\ \sup_{s\leq t} S_n(s)S_n(t)[M_n(s) - M_n(t)]/M_n(0) \\ &= \sqrt{n}\ \sup_{t\geq 0}\ \sup_{s\leq t}\ S_n(s)S_n(t)[(M_n(s) - M(s)) - (M_n(t) - M(t)) \\ &\quad +(M(s) - M(t))]/M_n(0).\end{aligned}$$

If F is distributed as $Exp(1)$, the last term in the previous expression is zero for all t. However, if F is in the DMRL class, there exists $t_0 \in [0,\ b]$ for some b such that $F(b) < 1$, with $\sup_{s \leq t} S(s)\ S(t_0)[M(s) - M(t_0) > 0$. Hence,

$$\sqrt{n}\ \Theta_n^*(t) \to \infty \text{ as } n \to \infty, \text{ provided that } F \text{ is DMRL.}$$

6.4 Weak convergence

Under H_0,

(i) $M(t)) = 1$,

(ii) $h(t) = M(t)\ S(t) = e^{-t}$,

(iii) $F(s) = 1 - e^{-s}$, and $dF(s) = e^{-s} ds$,

(iv) $C(t) = \int_0^t \frac{e^u}{1-G(u)}\ du$, and

(v) $C'(t) = \frac{e^t}{1-G(t)}$.

Recall that the MRL process, $\{Z_n(t) = \sqrt{n}\ [M_n(t) - M(t)] : t \in [0,\ b]\} \Rightarrow_w \{B(t) : t \in [0,\ b],\ F(b) < 1\}$, where $\{B(t) : t \in [0,\ b],\ F(b) < 1\}$ is a mean zero Gaussian process. Under H_0,

$$B(t) = Z(t) - \frac{\int_t^{\tau_H} Z(s) e^{-s}\, ds}{e^{-t}},$$

where $\{Z(t) : t \in [0,\ b],\ F(b) < 1\}$ is a Gaussian process with zero mean and covariance function:

$$cov\{Z(s),\ Z(t)\} = C\{\min(s,\ t)\}.$$

Now,

$$cov\{B(s),\ B(t)\} = e^{t+s} \int_t^{\tau_H} \frac{e^{-u}}{1 - G(u)}\, du \text{ for } 0 \leq s \leq t \leq \tau_t.$$

Let $0 \le t \le \tau_H$ be fixed and $s \le t$. Note that

$$\begin{aligned} Z_n(t) &= \sqrt{n}\ [M_n(s) - M_n(t)] \\ &= \sqrt{n}\ [(M_n(s) - M(s)) - (M_n(t) - M(t)) + (M(s) - M(t))] \\ &\Rightarrow_w Z(s) - Z(t) - e^s \int_s^{\tau_H} Z(u)e^{-u}\, du + e^t \int_t^{\tau_H} Z(u)e^{-u}\, du. \end{aligned}$$

For fixed t, using Slutsky's theorem, $S_n(s)\ S_n(t)\ \sqrt{n}\ [M_n(s) - M_n(t)] \Rightarrow_w Z_t^*(s)$, where

$$Z_t^*(s) = e^{-(s+t)}\ (Z(s) - Z(t)) - e^{-t} \int_s^{\tau_H} Z(u)e^{-u}\, du + e^{-s} \int_t^{\tau_H} Z(u)e^{-u}\, du. \quad (6.4.3)$$

By the continuous mapping theorem,

$$\sqrt{n}\ \Theta_n^*(t) \Rightarrow_w \sup_{t\ge 0} \sup_{s\le t} Z_t^*(s).$$

Although the limiting distribution is intractable we propose a resampling scheme as given in Lin (1997) assuming that $\tau_H = \infty$, which is reasonable since $\tau_F = \infty$ under H_0.

We first note that, from the integration by parts formula, the limiting distribution in (6.4.3) may be written as

$$\begin{aligned} Z_t^*(s) &= e^{-t}\left[e^{-s}Z(s) - \int_s^{\infty} Z(u)e^{-u}du\right] - e^{-s}\left[e^{-t}Z(t) - \int_t^{\infty} Z(u)e^{-u}du\right] \\ &= -e^{-t}\int_s^{\infty} e^{-u}dZ(u) + e^{-s}\int_t^{\infty} e^{-u}dZ(u). \end{aligned}$$

From Kumazawa (1987),

$$Z_n(t) = \sqrt{n}\ \frac{F_n(t) - F(t)}{S_n(t)} = \sqrt{n}\ \int_0^t \frac{S_n(u^-)}{S_n(u)Y_n(u)} dM_n(u) \Longrightarrow_w Z(t) \text{ on } [0, \infty),$$

where

$$Y_n(u) = \sum_{i=1}^n Y_{in}(u) = \sum_{i=1}^n\ I(X_i \geq u)$$

and

$$M_n(u) = \sum_{i=1}^n M_{in}(u) = N_n(u) - \int_0^u \frac{Y_n(v)}{S_n(v)} dF_n(v) = \sum_{i=1}^n \left[N_{in}(u) - \int_0^u \frac{Y_{in}(v)}{S_n(v)} dF_n(v) \right],$$

where $N_{in}(u) = I(X_i \leq u, \delta_i = 1)$. By replacing $M_n(u)$ by $\sum_{i=1}^n G_{in} N_{in}(u)$, where the G_{in}'s are independent standard normals, the arguments in Lin (1997) show that the distriibution of $Z_t^*(\cdot)$ could be approximated for large n by replacing $dZ(u)$ by

$$d\hat{Z}_n(u) = \sqrt{n}\ \sum_{i=1}^n \frac{S_n(u^-)}{S_n(u)} G_{in} N_{in}(u)$$

from a large number of realizations from $d\hat{Z}_n(\cdot)$ by repeatedly generating $\{G_{in} : 1 \leq i \leq n\}$ while fixing the data $\{S_n(\cdot), Y_{in}, N_{in}(u)\}$. Since we reject H_0 for large values of the test statistic, the p-value could be estimated from the fraction of the generated values under H_0 that exceed the observed value of the test statistic.

Chapter 7

CONCLUSIONS AND RECOMMENDATIONS FOR FUTURE RESEARCH

7.1 Conclusions

We have considered the problem of estimation and testing the mean residual life function (MRLF), M, in two popular classes the New Better than Used in expectation (NBUE) class and the Decreasing Mean Residual Life (DMRL) class of M. In Chapter 3, we considered the problem of estimating and testing in the NBUE class. We have defined an order restricted estimator under censoring for an NBUE MRLF M: $M_n^*(t) = M_n(t) \wedge M_n(0),\ t \geq 0$. We show that M_n^* is a strongly uniformly consistent estimator of M, on $[0,\ b]$ for all b such that $F(b) < 1$. We also obtain the weak convergence, asymptotic bias, and asymptotic mean square error of out estimator. We have shown that $AMSE(M_n^*) \leq AMSE(M_n)$. Using the standard criterion of MSE, we showed by simulation that our order restricted estimator is better than the unrestricted estimator. Letting the process $\{B_n^*(t) \equiv \sqrt{n}\ [M_n^*(t) - M(t)] :\ t \in [0,\ b]\}$ and $\{B(t) :\ t \in [0,\ b]\}$ being a mean zero

Gaussian process given by $B(t) = Z(t)M(t) + \int_t^{\tau_H} Z(s)\,dh(s)/S(t)$, where $\{Z(t) : t \in [0,\ b]\}$ is a Gaussian process with zero mean and covariance function $Cov\{Z(s),\ Z(t)\} = C\{\min(s,\ t)\}$, we have proven the Theorem:

> If $M = M(0)$ on $[0,\ b]$, then $B_n^* \Rightarrow_w B \wedge B(0)$ on $[0,\ b]$, and if $M < M(0)$ on $(0,\ b]$, then $B_n^* \Rightarrow_w B$ on $[0,\ b]$.

In Chapter 4 we considered the problem of estimating and testing in the DMRL class. We have generalized the order restricted estimator of Kochar, Mukerjee and Samaniego (2000) for the uncensored case to our censored data and define $M_n^*(x) = \inf_{y \le x} M_n(y)\ I[x < X_n]$. We showed that M_n^* is a strongly uniformly consistent estimator of M, on $[0,\ b]$ for all b such that $F(b) < 1$. We also obtain the weak convergence of our estimator. We also presented by simulations that $MSE[M_n(t)] \ge MSE[M_n^*(t)]$ for finite sample sizes, when population is DMRL. We have derived the limiting distribution of $\{B_n^*(t) \equiv \sqrt{n}\ [M_n^*(t) - M(t)] :\ t \in [0,\ b]\}$. To do this, we define the function l and u on $[0,\ \infty)$ to $[0,\ \infty)$ by $l(t) = \inf[s \le t : M(s) = M(t)]$, and $u(t) = \sup[s \ge t : M(s) = M(t)]$. Using this result, we have proved the Theorem:

$$\text{If } B^*(t) = \inf_{l(t) \le s \le t} B(s),\ t \in [0,\ b], \text{then } B_n^* \Rightarrow_w B^* \text{on } [0,\ b].$$

This theorem strengthens the results of Kochar, Mukerjee and Samaniego (2000).

We developed tests that identify the NBUE (DMRL) behavior in Chapter 5 (Chapter 6). Both tests are shown to be consistent on their respective classes. We derived the asymptotic distribution of our NBUE test statistic in Section 5.3. However, the limiting distribution of the test statistic considered for the DMRL class is unknown.

Finally, we have shown the importance of the MRLF and how the NBUE and DMRL classes may arise in applications.

7.2 Recommendations for future research

Throughout we have assumed that we have a univariate failure time data. However, bivariate failure time data arise frequently in scientific research because each study subject may experience two types of events or because there exists natural or artificial pairing such that failure times within the same pair are correlated. In such studies, either or both failure times may not be observed due to right-censorship.

A natural extension of our work can be the order restricted estimation and testing of the bivariate Mean Residual Life Function (MRLF) in the New Better than Used in Expectation (NBUE) and the Decreasing Mean Residual Life (DMRL) classes under random censoring. For two lifetime variables X and Y having joint survival function S, $S(x,\ y) = P(X > x,\ Y > y)$, a popular way to define a bivariate MRL is by the vector $(\mu_1(x,\ y),\ \mu_2(x,\ y))$, where

$$\mu_1(x,\ y) = E(X-x|X > x,\ Y > y) \quad \text{and} \quad \mu_2(x,\ y) = E(Y-y|X > x,\ Y > y). \tag{7.2.1}$$

The detail of this definition can be seen in Arnold and Zahedi (1988), and Nair and Nair (1989). Also a complicated form of bivariate MRLF is given by Shaked and Shanthikumar (1993).

Four multivariate generalizations of univariate DMRL are proposed by Buchanan and Singapurwalla (1997) of which two definitions for the bivariate case are given below.

1. Bivariate DMRL-I: If for all $t \geq 0$ for which $S(t,\ t) > 0$, $\int_t^\infty \int_t^\infty S(x,\ y)\, dx\, dy/S(t,\ t)$ is nonincreasing in t, together with a similar condition for the two marginals.

2. Bivariate DMRL-II: If for all $t \geq 0$ for which $S(t,\ t) > 0$, $\int_t^\infty S(x,\ x)\, dx/S(t,\ t)$ is nonincreasing in t, together with a similar condition for the two marginals.

Various definitions of the bivariate NBUE MRLF have been suggested in the literature. One of them is given below. A bivariate MRLF M along with the SF S is in NBUE class if $\int_0^\infty S(x+t,\ y+t)\ dt \le S(x,\ y) \int_0^\infty S(t,\ t)\ dt$ for all $x,\ y \ge 0$. Other definitions can be seen in Basu et al. (1983), Klefsjö (1980), and Basu and Ebrahimi (1986).

Recall that MRLF is completely determined by the SF. Even in the unrestricted case, there is no satisfactory estimator of the bivariate SF if the censoring of the marginals are independent. However, Lin and Ying (1993) have shown that if the same censoring applies to both marginals then there are simple satisfactory estimators. Using this univariate censoring assumption, and some of the various definitions of the bivariate NBUE and DMRL distributions suggested in the literature, our results could be extended to the bivariate, and possibly the general multivariate case.

In this thesis, we have considered the problems of estimation and testing the mean residual life function, M, in two popular classes the New Better than Used in Expectation (NBUE) class and the Decreasing Mean Residual Life (DMRL) class of M. These two classes have their dual classes namely the New Worse than Used in Expectation (NWUE) class and the Increasing Mean Residual Life (IMRL) class. These classes are defined by reversing the direction of monotonicity or inequality. The dual classes are appropriate for describing situations where lifelengths of items improve with age. The boundary members of each of these classes are the exponential distributions which, of course, are appropriate for models where lifelengths neither improve nor deteriorate with age.

The literature does not contain developments of the estimation and testing for the

NWUE and IMRL classes. We believe that the natural replication of our work would be the order restricted inferences about the MRL for the NWUE and IMRL classes under censoring. Taking M_n as an unrestricted estimator of M, a reasonable way to start may be to consider $M_n^*(t) \equiv [M_n(t) \vee M_n(0)]\ I[t < X_n]$, $t \geq 0$, and $M_n^{**}(t) \equiv \sup_{y \leq t} M_n(y)\ I[t < X_n]$, $t \geq 0$ as the estimators of M for the NWUE and IMRL classes respectively and seek their properties in their respective classes . Similarly, tests can be developed for these classes as in their dual classes.

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