

# MANIFOLDS WITH NONNEGATIVE CURVATURE OPERATOR OF THE SECOND KIND

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ABSTRACT. We investigate the curvature operator of the second kind on Riemannian manifolds and prove several classification results. The first one asserts that a closed Riemannian manifold with three-positive curvature operator of the second kind is diffeomorphic to a spherical space form, improving a recent result of Cao-Gursky-Tran assuming two-positivity. The second one states that a closed Riemannian manifold with three-nonnegative curvature operator of the second kind is either diffeomorphic to a spherical space form, or flat, or isometric to a quotient of a compact irreducible symmetric space. This settles the nonnegativity part of Nishikawa's conjecture under a weaker assumption.

## 1. INTRODUCTION

Riemannian manifolds with positive curvature operator have been of great interest in geometry and topology for a long time, and a number of remarkable results have been obtained by various authors. For instance, Meyer [Mey71] proved, using the Bochner technique, that a closed Riemannian manifold with positive curvature operator must be a real homology sphere. A well-known theorem of Tachibana [Tac74] states that an Einstein manifold with positive curvature operator must have constant positive sectional curvature. Indeed, these two results are both consequences of the following celebrated differentiable sphere theorem.

**Theorem 1.1.** *A closed Riemannian manifold with two-positive curvature operator is diffeomorphic to a spherical space form.*

Recall that a Riemannian manifold  $(M^n, g)$  is said to have two-positive curvature operator if the sum of the smallest two eigenvalues of the curvature operator  $\hat{R}_p : \Lambda^2(T_p M) \rightarrow \Lambda^2(T_p M)$  defined via

$$(1.1) \quad \hat{R}(e_i \wedge e_j) = \frac{1}{2} \sum_{k,l} R_{ijkl} e_k \wedge e_l$$

is positive for any  $p \in M$ . Here  $\Lambda^2(T_p M)$  is the space of two-forms and  $\{e_1, \dots, e_n\}$  is an orthonormal basis of  $T_p M$ . In dimension three, two-positive curvature operator is equivalent to positive Ricci curvature, so Theorem 1.1 is due to Hamilton [Ham82], who introduced Ricci flow and used it to prove that three-manifolds

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with positive Ricci curvature are diffeomorphic to spherical space forms. Hamilton [Ham86] also showed the above differentiable sphere theorem for closed four-manifolds with positive curvature operator, which was extended to two-positive curvature operator by Chen [Che91]. In higher dimensions, Theorem 1.1 is due to Böhm and Wilking [BW08]. The proof, in all dimensions, uses the normalized Ricci flow to evolve an initial metric with two-positive curvature operator into a limit metric with constant positive sectional curvature. See also [BS08], [BS09], [Bre08], [AN09], [NW07a], [NW10] and [BS11] for differentiable sphere theorems under other curvature conditions.

Rigidity results hold if the curvature operator is nonnegative or two-nonnegative. Gallot and Meyer [GM75] proved that a closed Riemannian manifold with nonnegative curvature operator is either reducible, or locally symmetric, or its universal cover has the cohomology of a sphere or a complex projective space. Tachibana [Tac74] proved that closed Einstein manifolds with nonnegative curvature operator must be locally symmetric. More generally, the following rigidity result was obtained by Hamilton [Ham86] in dimension three, Hamilton [Ham86] and Chen [Che91] in dimension four, and Ni and Wu [NW07b] in all higher dimensions (see also Wilking's wonderful survey [Wil07, Theorem 1.13]).

**Theorem 1.2.** *A closed simply-connected Riemannian manifold  $(M^n, g)$  with two-nonnegative curvature operator satisfies one of the following statements.*

- (1)  $M$  is diffeomorphic to  $\mathbb{S}^n$ ;
- (2)  $n = 2m$  and  $M$  is a Kähler manifold biholomorphic to  $\mathbb{C}\mathbb{P}^m$ ;
- (3)  $M$  is isometric to a compact irreducible symmetric space;
- (4)  $M$  is isometric to nontrivial Riemannian product.

Next, let's recall the second kind of curvature operator. The Riemann curvature tensor  $R_{ijkl}$  also acts naturally on the space of symmetric two-tensors  $S^2(T_p M)$  (see for example [CV60], [BE69] and [BK78]). This action, denoted by  $\mathring{R} : S^2(T_p M) \rightarrow S^2(T_p M)$  in this paper, is defined by

$$\mathring{R}(e_i \odot e_j) = \sum_{k,l} R_{iklj} e_k \odot e_l,$$

where  $\odot$  denotes the symmetric product. The new feature here is that  $S^2(T_p M)$  is not irreducible under the action of the orthogonal group  $O(T_p M)$ . As in [Nis86], one should consider the induced symmetric bilinear form

$$(1.2) \quad \mathring{R} : S_0^2(T_p M) \times S_0^2(T_p M) \rightarrow \mathbb{R}$$

by restricting  $\mathring{R}$  to the space of traceless symmetric two-tensors  $S_0^2(T_p M)$ . Following Nishikawa's terminology in [Nis86], we call the symmetric bilinear form  $\mathring{R}$  in (1.2) the *curvature operator of the second kind*, to distinguish it from the curvature operator  $\hat{R}$  defined in (1.1), which he called the *curvature operator of the first kind*.

The action of Riemann curvature tensor on symmetric two-tensors indeed has a long history. It appeared for Kähler manifolds in the study of deformation of complex analytic structures by Calabi and Vesentini [CV60]. They introduced the self-adjoint operator  $\xi_{\alpha\beta} \rightarrow R^\rho_{\alpha\beta}{}^\sigma \xi_{\rho\sigma}$  from  $S^2(T_p^{1,0} M)$  to itself, and computed the eigenvalues of this operator on Hermitian symmetric spaces of classical type, with the exceptional ones handled shortly afterward by Borel [Bor60]. In the Riemannian

setting, the operator  $\mathring{R}$  arises naturally in the context of deformations of Einstein structure in Berger and Ebin [BE69] (see also [Koi79a, Koi79b] and [Bes08]). In addition, it appears naturally in the Bochner-Weitzenböck formulas for symmetric two-tensors (see for example [MRS20]), for differential forms in [OT79] and for Riemannian curvature tensors in [Kas93]. In another direction, curvature pinching estimates for  $\mathring{R}$  were studied by Bourguignon and Karcher [BK78], and they also calculated eigenvalues of  $\mathring{R}$  on the complex projective space with the Fubini-Study metric and the quaternionic projective space with its canonical metric.

Given the above-mentioned beautiful theorems for the curvature operator  $\hat{R}$ , it is an intriguing question to classify closed Riemannian manifolds with positive or nonnegative curvature operator of the second kind. Indeed, Nishikawa [Nis86, Conjecture II] proposed the following conjecture in 1986.

**Conjecture 1.3** (Nishikawa [Nis86]). *Let  $(M^n, g)$  be a closed Riemannian manifold.*

- (1) *If  $M$  has positive curvature operator of the second kind, then  $M$  is diffeomorphic to a spherical space form.*
- (2) *If  $M$  has nonnegative curvature operator of the second kind, then  $M$  is diffeomorphic to a Riemannian locally symmetric space.*

The conjecture was made based on two known results. One asserts that  $M$  must be a real homology sphere if it has positive curvature operator of the second kind, which is a result of Ogiue and Tachibana [OT79]. The other one says the conjecture holds if  $M$  is either Einstein or has harmonic curvature tensor, which is a result of Kashiwada [Kas93].

To the best of the author's knowledge, no progress was made on Nishikawa's conjecture until the recent work of Cao, Gursky and Tran [CGT21], in which they proved the positive case of Nishikawa's conjecture under a weaker assumption.

**Theorem 1.4** (Cao-Gursky-Tran [CGT21]). *A closed Riemannian manifold with two-positive curvature operator of the second kind is diffeomorphic to a spherical space form.*

Their key observation is that two-positive curvature operator of the second kind implies the strictly PIC1 condition introduced by Brendle [Bre08] (see also Definition 4.1). The positive case of Nishikawa's conjecture follows immediately from Brendle's result in [Bre08] asserting that the normalized Ricci flow evolves an initial metric satisfying strictly PIC1 into a limit metric with constant positive sectional curvature.

The first purpose of this paper is to prove an improvement of Theorem 1.4 by weakening the assumption to three-positivity. The second purpose is to settle the nonnegativity part of Nishikawa's conjecture.

In dimension two, it is easy to see that both positivity and two-positivity (respectively, nonnegativity and two-nonnegativity) of the curvature operator of the second kind are equivalent to positive (respectively, nonnegative) scalar curvature. Thus, Nishikawa's conjecture is a consequence of the uniformization theorem. So we begin with dimension three.

**Theorem 1.5.** <sup>1</sup> *Let  $(M^3, g)$  be a closed Riemannian manifold of dimension three.*

- (1) *If  $M$  has three-positive curvature operator of the second kind, then  $M$  is diffeomorphic to a spherical space form.*
- (2) *If  $M$  has three-nonnegative curvature operator of the second kind, then  $M$  is either flat or diffeomorphic to a spherical space form.*

To prove this theorem, we show in Proposition 3.1 that in dimension three, three-positive (respectively, three-nonnegative) curvature operator of the second kind implies  $\text{Ric} > \frac{S}{12} > 0$  (respectively,  $\text{Ric} \geq \frac{S}{12} \geq 0$ ), where  $S$  denotes the scalar curvature. Part (1) of Theorem 1.5 then follows from Hamilton's classification of closed three-manifolds with positive Ricci curvature. To prove part (2) of Theorem 1.5, we first observe that the manifold must be locally irreducible if it is not flat, and then apply Hamilton's classification of closed three-manifolds with nonnegative Ricci curvature [Ham86].

We would like to point out that the curvature assumption in Theorem 1.5 is optimal, in the sense that the theorem fails if one only assumes four-positive or four-nonnegative curvature operator of the second kind. In particular, the product manifold  $\mathbb{S}^2 \times \mathbb{S}^1$  has four-positive curvature operator of the second kind, but it does not have three-nonnegative curvature operator of the second kind (see Example 2.6).

In dimensions four and above, we prove that

**Theorem 1.6.** *A closed Riemannian manifold with three-positive curvature operator of the second kind is diffeomorphic to a spherical space form.*

It was shown in [CGT21] that two-positive curvature operator of the second kind implies strictly PIC1. We strengthen their result by showing that three-positive curvature operator of the second kind implies strictly PIC1 (see Proposition 4.1). Theorem 1.6 then follows from this improvement and Brendle's result [Bre08].

The corresponding rigidity result states

**Theorem 1.7.** *Let  $(M^n, g)$  be a closed Riemannian manifold of dimension  $n \geq 4$ . If  $M$  has three-nonnegative curvature operator of the second kind, then one of the following statements holds:*

- (1)  *$(M, g)$  is flat;*
- (2)  *$M$  is diffeomorphic to a spherical space form;*
- (3)  *$n \geq 5$  and the universal cover of  $M$  is isometric to a compact irreducible symmetric space <sup>2</sup>.*

Theorem 1.7 settles the nonnegativity part of Nishikawa's conjecture under the weaker assumption of three-nonnegative curvature operator of the second kind. The key idea is to reduce the problem to the locally irreducible case and make use of the classification of closed locally irreducible Riemannian manifolds with weakly PIC1. For this purpose, we show that

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<sup>1</sup>An improvement has been obtained by the author in a subsequent work [Li22c] assuming  $3\frac{1}{3}$ -positive/nonnegative curvature operator of the second kind.

<sup>2</sup>This possibility has been ruled out by Nienhaus, Petersen and Wink [NPW22], as they proved that a closed  $n$ -dimensional Riemannian manifold with  $\frac{n+2}{2}$ -nonnegative curvature operator of the second kind is either a rational homology sphere or flat.

**Theorem 1.8.** *Let  $(M^n, g)$  be an  $n$ -dimensional non-flat complete Riemannian manifold with  $n$ -nonnegative curvature operator of the second kind. Then  $M$  is locally irreducible<sup>3</sup>.*

Note that the assumption here cannot be weakened to  $(n + 1)$ -nonnegative curvature operator of the second kind, as the product manifold  $\mathbb{S}^{n-1} \times \mathbb{S}^1$  has  $(n + 1)$ -nonnegative curvature operator but it is reducible (see Example 2.6).

It is well-known that nonnegative curvature operator preserves product manifolds, in the sense that if  $M$  is isometric to  $M_1 \times M_2$ , then  $M$  has nonnegative curvature operator if and only if both  $M_1$  and  $M_2$  have nonnegative curvature operator. However, Theorem 1.8 implies that the product manifold  $M_1 \times M_2$  has  $n$ -nonnegative curvature operator of the second kind if and only if both  $M_1$  and  $M_2$  are flat.

It follows from the calculation in [BK78] that the complex projective space  $\mathbb{C}\mathbb{P}^m$  with the Fubini-Study metric does not have four-nonnegative curvature operator of the second kind. We prove a more general result here.

**Theorem 1.9.** *A Kähler manifold with four-nonnegative curvature operator of the second kind is flat<sup>4</sup>.*

Compact Hermitian symmetric spaces are examples of Kähler manifolds with nonnegative curvature operator, but none of them has four-nonnegative curvature operator of the second kind, according to the above theorem. We also point out that the assumption in Theorem 1.9 cannot be weakened to five-nonnegative curvature operator of the second kind in general, as  $\mathbb{C}\mathbb{P}^2$  has five-positive curvature operator of the second kind (see Example 2.5).

The paper is organized as follows. In Section 2, we give an introduction to the curvature operator of the second kind, state some conventions and definitions, and collect some examples on which the eigenvalues of the curvature operator of the second kind can be computed explicitly. In Section 3, we examine the curvature operator of the second kind in dimension three and prove Theorem 1.5. Section 4 is devoted to establishing various algebraic relations between the curvature operator of the second kind and other frequently used curvature conditions such as sectional curvature, Ricci curvature and isotropic curvature. In Section 5, we study the curvature operator of the second kind on product manifolds and prove Theorem 1.8. In Section 6, we prove Theorem 1.9 under a slightly weaker assumption. Finally, the proof of Theorem 1.7 is presented in Section 7.

## 2. THE CURVATURE OPERATOR OF THE SECOND KIND

We give an introduction to the curvature operator of the second kind in this section, and the reader is referred to [CV60, OT79, BK78, Nis86, Kas93, CGT21] for more information and previous results.

Let  $(V, g)$  be a Euclidean vector space of dimension  $n \geq 2$ . We always identify  $V$  with its dual space  $V^*$  via the metric  $g$ . The space of bilinear forms on  $V$  is

<sup>3</sup>Some improvements of this result have been obtained in [NPWW22] and [Li22d].

<sup>4</sup>The curvature operator of the second kind on Kähler manifolds has been further investigated in [Li22b, Li22a] and [NPWW22].

denote by  $T^2(V)$ , and it splits as

$$T^2(V) = S^2(V) \oplus \Lambda^2(V),$$

where  $S^2(V)$  is the space of symmetric two-tensors on  $V$  and  $\Lambda^2(V)$  is the space of two-forms on  $V$ . Our conventions on symmetric products and wedge products are that, for  $u$  and  $v$  in  $V$ ,  $\odot$  denotes the symmetric product defined by

$$u \odot v = u \otimes v + v \otimes u,$$

and  $\wedge$  denotes the wedge product defined by

$$u \wedge v = u \otimes v - v \otimes u.$$

The inner product  $g$  on  $V$  naturally induces inner products on  $S^2(V)$  and  $\Lambda^2(V)$ , respectively. To be consistent with [CGT21], the inner product on  $S^2(V)$  is defined as

$$\langle A, B \rangle = \text{tr}(A^T B),$$

and the inner product on  $\Lambda^2(V)$  is defined as

$$\langle A, B \rangle = \frac{1}{2} \text{tr}(A^T B).$$

In particular, if  $\{e_1, \dots, e_n\}$  is an orthonormal basis for  $V$ , then  $\{e_i \wedge e_j\}_{1 \leq i < j \leq n}$  is an orthonormal basis for  $\Lambda^2(V)$  and  $\{\frac{1}{\sqrt{2}}e_i \odot e_j\}_{1 \leq i < j \leq n} \cup \{\frac{1}{2}e_i \odot e_i\}_{1 \leq i \leq n}$  is an orthonormal basis for  $S^2(V)$ .

The space of symmetric two-tensors on  $\Lambda^2(V)$  has the orthogonal decomposition

$$S^2(\Lambda^2(V)) = S_B^2(\Lambda^2(V)) \oplus \Lambda^4(V),$$

where  $S_B^2(\Lambda^2(V))$  consists of all tensors  $R \in S^2(\Lambda^2(V))$  that also satisfy the first Bianchi identity.  $S_B^2(\Lambda^2(V))$  is called the space of algebraic curvature operators.

By the symmetries of  $R \in S_B^2(\Lambda^2(V))$  (not including the first Bianchi identity), there are (up to sign) two ways that  $R$  can induce a symmetric linear map  $\hat{R} : T^2(V) \rightarrow T^2(V)$ . The first one, denoted by  $\hat{R} : \Lambda^2(V) \rightarrow \Lambda^2(V)$  in this paper, is the so-called curvature operator defined by

$$(2.1) \quad \hat{R}(e_i \wedge e_j) = \frac{1}{2} \sum_{k,l} R_{ijkl} e_k \wedge e_l,$$

where  $\{e_1, \dots, e_n\}$  is an orthonormal basis of  $V$ . Note that if the eigenvalues of  $\hat{R}$  are all greater than or equal to  $\kappa \in \mathbb{R}$ , then all the sectional curvatures of  $R$  are bounded from below by  $\kappa$ . We say  $R \in S_B^2(\Lambda^2(V))$  has positive (respectively, nonnegative) curvature operator if all the eigenvalues of  $\hat{R}$  are positive (respectively, nonnegative). For  $2 \leq k \leq \binom{n}{2}$ , we say  $R \in S_B^2(\Lambda^2(V))$  has  $k$ -positive (respectively,  $k$ -nonnegative) curvature operator if the sum of the smallest  $k$  eigenvalues of  $\hat{R}$  is positive (respectively, nonnegative). We say a Riemannian manifold has positive (respectively, nonnegative,  $k$ -positive,  $k$ -nonnegative) curvature operator if the Riemann curvature tensor at each point has positive (respectively, nonnegative,  $k$ -positive,  $k$ -nonnegative) curvature operator.

The second one, denoted by  $\mathring{R} : S^2(V) \rightarrow S^2(V)$ , is defined by

$$(2.2) \quad \mathring{R}(e_i \odot e_j) = \sum_{k,l} R_{iklj} e_k \odot e_l.$$

However, on contrary to the case of  $\hat{R}$ , all eigenvalues of  $\mathring{R}$  being nonnegative implies all the sectional curvatures of  $R$  are zero, that is,  $R \equiv 0$ <sup>5</sup>. The new feature here is that  $S^2(V)$  is not irreducible under the action of the orthogonal group  $O(V)$  of  $V$ . The space  $S^2(V)$  splits into  $O(V)$ -irreducible subspaces as

$$S^2(V) = S_0^2(V) \oplus \mathbb{R}g,$$

where  $S_0^2(V)$  denotes the space of traceless symmetric two-tensors on  $V$ . The map  $\mathring{R}$  defined in (2.2) then induces a symmetric bilinear form

$$(2.3) \quad \mathring{R} : S_0^2(V) \times S_0^2(V) \rightarrow \mathbb{R}$$

by restriction to  $S_0^2(V)$ . Note that if all the eigenvalues of  $\mathring{R}$  restricted to  $S_0^2(V)$  are bounded from below by  $\kappa \in \mathbb{R}$ , then the sectional curvatures of  $R$  are bounded from below by  $\kappa$ . It should be noted that  $\mathring{R}$  does not preserve the subspace  $S_0^2(V)$  in general, but it does, for instance, when  $R$  is Einstein.

Following [Nis86], we call the symmetric bilinear form  $\mathring{R}$  in (2.3) the *curvature operator of the second kind*, to distinguish it from the map  $\hat{R}$  defined in (2.1), which he called the *curvature operator of the first kind*.

**Remark 2.1.** It was pointed out in [NPW22] that the curvature operator of the second kind can also be interpreted as the self-adjoint operator  $\pi \circ \mathring{R} : S_0^2(V) \rightarrow S_0^2(V)$ , where  $\pi$  is the projection map from  $S^2(V)$  onto  $S_0^2(V)$ . This is equivalent to the interpretation as the symmetric bilinear form in (2.3), as

$$\mathring{R}(\varphi, \psi) = \langle \mathring{R}(\varphi), \psi \rangle = \langle \pi \circ \mathring{R}(\varphi), \psi \rangle = (\pi \circ \mathring{R})(\varphi, \psi)$$

for any  $\varphi, \psi \in S_0^2(V)$ .

Let  $N = \dim(S_0^2(V)) = \frac{(n-1)(n+2)}{2}$  and  $\{\varphi_i\}_{i=1}^N$  be an orthonormal basis of  $S_0^2(V)$ . The  $N \times N$  matrix  $\mathring{R}(\varphi_i, \varphi_j)$  is called the matrix representation of  $\mathring{R}$  with respect to the orthonormal basis  $\{\varphi_i\}_{i=1}^N$ . The eigenvalues of  $\mathring{R}$  refers to the eigenvalues of any of its matrix representation. This is independent of the choices of the orthonormal bases because matrix representations of  $\mathring{R}$  with respect to different orthonormal bases of  $S_0^2(V)$  are similar to each other. We then make the following definitions.

**Definition 2.2.** For a positive integer  $1 \leq k \leq N$ , we say  $R \in S_B^2(\Lambda^2(V))$  has  $k$ -nonnegative (respectively,  $k$ -positive) curvature operator of the second kind if the sum of the smallest  $k$ -eigenvalues of  $\mathring{R}$  is nonnegative (respectively, positive).

**Definition 2.3.** A Riemannian manifold  $(M^n, g)$  is said to have  $k$ -nonnegative (respectively,  $k$ -positive) curvature operator of the second kind if  $R_p \in S_B^2(\Lambda^2 T_p M)$  has  $k$ -nonnegative (respectively,  $k$ -positive) curvature operator of the second kind for each  $p \in M$ .

We conclude this section by collecting some examples on which the eigenvalues of  $\mathring{R}$  are known. These examples are used to demonstrate the sharpness of many results in this paper.

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<sup>5</sup>This follows from the observation that the trace of  $\mathring{R} : S^2(V) \rightarrow S^2(V)$  is equal to  $\frac{S}{2}$  and  $\mathring{R}(g, g) = -S$ .

**Example 2.4.** Let  $(\mathbb{S}^n, g_0)$  be the  $n$ -sphere with constant sectional curvature 1. Its Riemann curvature tensor is given by  $R_{ijkl} = g_{ik}g_{jl} - g_{il}g_{jk}$ . For any traceless symmetric two-tensor  $\varphi$ , we have

$$\begin{aligned} \mathring{R}(\varphi, \varphi) &= \sum_{i,j,k,l=1}^n R_{ijkl} \varphi_{il} \varphi_{jk} \\ &= \sum_{i,j,k,l=1}^n (g_{ik}g_{jl} - g_{il}g_{jk}) \varphi_{il} \varphi_{jk} \\ &= |\varphi|^2 - \text{tr}(\varphi)^2 \\ &= |\varphi|^2. \end{aligned}$$

Thus  $\mathring{R}$  is equal to the identity map on  $S_0^2(T_p\mathbb{S}^n)$  at any point  $p \in \mathbb{S}^n$  and all eigenvalues of  $\mathring{R}$  restricted to  $S_0^2(T_p\mathbb{S}^n)$  are equal to 1.

**Example 2.5.** Let  $(\mathbb{C}\mathbb{P}^m, g_{FS})$  be the complex projective space of complex dimension  $m$  with the Fubini-Study metric  $g_{FS}$ . Then  $\mathring{R}$  restricted to  $S_0^2(T_p\mathbb{C}\mathbb{P}^m)$  has two distinct eigenvalues:  $-2$  with multiplicity  $(m-1)(m+1)$  and  $4$  with multiplicity  $m(m+1)$ . These eigenvalues, together with their corresponding eigenspaces, are determined in [BK78]. In particular,  $\mathbb{C}\mathbb{P}^2$  has five-positive (but not four-positive) curvature operator of the second kind.

**Example 2.6.** Let  $M = \mathbb{S}^{n-1} \times \mathbb{R}$ , with  $\mathbb{S}^{n-1}$  being the  $(n-1)$ -sphere with constant sectional curvature 1. The eigenvalues of  $\mathring{R}$  restricted to  $S_0^2$  are given by

$$\begin{aligned} \lambda_1 &= -\frac{n-2}{n}, \text{ with multiplicity } 1; \\ \lambda_2 &= 0, \text{ with multiplicity } n-1; \\ \lambda_3 &= 1, \text{ with multiplicity } \frac{(n-2)(n+1)}{2}. \end{aligned}$$

The eigenvector associated with  $\lambda_1$  is given by  $\sum_{i=1}^{n-1} e_i \odot e_i - (n-1)e_n \odot e_n$ , where  $\{e_1, \dots, e_{n-1}\}$  is an orthonormal basis of  $T_p\mathbb{S}^{n-1}$  and  $e_n$  is a unit vector in  $T_q\mathbb{R}$ . The eigenspace of  $\lambda_2$  has dimension  $n-1$  and it is spanned by vectors of the form  $u \odot v$  with  $u \in T_p\mathbb{S}^{n-1}$  and  $v \in T_q\mathbb{R}$ . The eigenspace of  $\lambda_3$  is the space of traceless symmetric two-tensors on the  $\mathbb{S}^{n-1}$  factor, whose dimension is  $\frac{(n-2)(n+1)}{2}$ . In particular,  $\mathbb{S}^{n-1} \times \mathbb{R}$  has  $(n+1)$ -positive (but not  $n$ -positive) curvature operator of the second kind.

**Example 2.7.** Let  $M = \mathbb{S}^2 \times \mathbb{S}^2$  with  $\mathbb{S}^2$  being the 2-sphere with constant sectional curvature 1. The eigenvalues of  $\mathring{R}$  are given by  $\lambda_1 = -1$  with multiplicity one,  $\lambda_2 = 0$  with multiplicity 4 and  $\lambda_3 = 1$  with multiplicity 4 (see [CGT21]). If we pick an orthonormal basis  $\{e_1, e_2, e_3, e_4\}$  of  $T_pM$  with  $e_1, e_2$  in the tangent space of the first  $\mathbb{S}^2$  factor and  $e_3, e_4$  in the tangent space of the second  $\mathbb{S}^2$  factor, then the corresponding eigenspaces are given by

$$\begin{aligned} E_1 &= \text{span}\{e_1 \odot e_1 + e_2 \odot e_2 - e_3 \odot e_3 - e_4 \odot e_4\}, \\ E_2 &= \text{span}\{e_1 \odot e_3, e_1 \odot e_4, e_2 \odot e_3, e_2 \odot e_4\}, \\ E_3 &= \text{span}\{e_1 \odot e_2, e_3 \odot e_4, e_1 \odot e_1 - e_2 \odot e_2, e_3 \odot e_3 - e_4 \odot e_4\}. \end{aligned}$$

In particular,  $\mathbb{S}^2 \times \mathbb{S}^2$  has six-nonnegative (but not five-nonnegative) curvature operator of the second kind.

## 3. DIMENSION THREE

In this section, we investigate the curvature operator of the second kind in dimension three.

Let  $(V, g)$  be a three-dimensional Euclidean vector space and let  $\{e_1, e_2, e_3\}$  be an orthonormal basis for  $V$ . The space of traceless symmetric two-tensors  $S_0^2(V)$  has dimension 5, and we can choose an orthonormal basis for it as follows.

$$\begin{aligned}\varphi_1 &= \frac{1}{\sqrt{2}}e_1 \odot e_2, \\ \varphi_2 &= \frac{1}{\sqrt{2}}e_1 \odot e_3, \\ \varphi_3 &= \frac{1}{\sqrt{2}}e_2 \odot e_3, \\ \varphi_4 &= \frac{1}{2\sqrt{2}}(e_1 \odot e_1 - e_2 \odot e_2), \\ \varphi_5 &= \frac{1}{2\sqrt{6}}(e_1 \odot e_1 + e_2 \odot e_2 - 2e_3 \odot e_3).\end{aligned}$$

**Lemma 3.1.** *For the  $\varphi_i$ 's defined as above, we have the following identities.*

$$\begin{aligned}\mathring{R}(\varphi_1, \varphi_1) &= R_{1212}, \\ \mathring{R}(\varphi_2, \varphi_2) &= R_{1313}, \\ \mathring{R}(\varphi_3, \varphi_3) &= R_{2323}, \\ \mathring{R}(\varphi_4, \varphi_4) &= R_{1212}, \\ \mathring{R}(\varphi_5, \varphi_5) &= \frac{2}{3}(R_{1313} + R_{2323}) - \frac{1}{3}R_{1212}.\end{aligned}$$

*Proof.* It follows from direct calculation. For instance, the only nontrivial components of  $\varphi_5$  are  $(\varphi_5)_{11} = (\varphi_5)_{22} = \frac{1}{\sqrt{6}}$  and  $(\varphi_5)_{33} = -\frac{2}{\sqrt{6}}$ , and we calculate that

$$\begin{aligned}\mathring{R}(\varphi_5, \varphi_5) &= \sum_{i,j,k,l=1}^3 R_{ijkl}(\varphi_5)_{il}(\varphi_5)_{jk} \\ &= R_{1221}(\varphi_5)_{11}(\varphi_5)_{22} + R_{1331}(\varphi_5)_{11}(\varphi_5)_{33} \\ &\quad + R_{2112}(\varphi_5)_{22}(\varphi_5)_{11} + R_{2332}(\varphi_5)_{22}(\varphi_5)_{33} \\ &\quad + R_{3113}(\varphi_5)_{33}(\varphi_5)_{11} + R_{3223}(\varphi_5)_{33}(\varphi_5)_{22} \\ &= \frac{2}{3}(R_{1313} + R_{2323}) - \frac{1}{3}R_{1212}.\end{aligned}$$

The other ones are similar.  $\square$

One immediately reads from Lemma 3.1 that two-positive (respectively, two-nonnegative) curvature operator of the second kind implies positive (respectively, nonnegative) sectional curvature, as

$$2R_{1212} = \mathring{R}(\varphi_1, \varphi_1) + \mathring{R}(\varphi_4, \varphi_4).$$

This observation in fact remains valid in all dimensions (see Proposition 4.1).

Another consequence of Lemma 3.1, first pointed out in [CGT21], is that positive (respectively, nonnegative) curvature operator of the second kind implies that  $\text{Ric} > \frac{S}{6} > 0$  (respectively,  $\text{Ric} \geq \frac{S}{6} \geq 0$ ). This follows from

$$\begin{aligned} \mathring{R}(\varphi_5, \varphi_5) &= \frac{2}{3}(R_{1313} + R_{2323}) - \frac{1}{3}R_{1212} \\ &= \frac{2}{3}R_{33} - \frac{1}{3}\left(\frac{S}{2} - R_{33}\right) \\ &= R_{33} - \frac{S}{6}. \end{aligned}$$

Similarly, the following proposition can be easily derived from Lemma 3.1.

**Proposition 3.1.** *Let  $R \in S_B^2(\Lambda^2 V)$  be an algebraic curvature operator on a three-dimensional Euclidean vector space  $V$ . Denote by  $S$  the scalar curvature of  $R$ .*

(1) *If  $R$  has three-positive curvature operator of the second kind, then*

$$\text{Ric} > \frac{S}{12} > 0.$$

(2) *If  $R$  has three-nonnegative curvature operator of the second kind, then*

$$\text{Ric} \geq \frac{S}{12} \geq 0.$$

*Moreover, if the Ricci curvature has an eigenvalue zero, then  $R \equiv 0$ .*

*Proof.* (1). If  $R$  has three-positive curvature operator of the second kind, then

$$\begin{aligned} 0 &< \mathring{R}(\varphi_2, \varphi_2) + \mathring{R}(\varphi_3, \varphi_3) + \mathring{R}(\varphi_5, \varphi_5) \\ &= \frac{5}{3}(R_{1313} + R_{2323}) - \frac{1}{3}R_{1212} \\ &= \frac{5}{3}R_{33} - \frac{1}{3}\left(\frac{S}{2} - R_{33}\right) \\ &= 2\left(R_{33} - \frac{S}{12}\right). \end{aligned}$$

Thus we have  $\text{Ric} > \frac{S}{12}$ . Tracing it yields  $S > \frac{S}{4}$ , which implies  $S > 0$ .

(2). The first part follows similarly as in part (1). If Ricci curvature has an eigenvalue zero, then  $S = 0$ . In view of  $\text{Ric} \geq 0$ , we must have  $\text{Ric} \equiv 0$  and thus  $R \equiv 0$ .  $\square$

We are ready to prove Theorem 1.5.

*Proof of Theorem 1.5.* (1). By part (1) of Proposition 3.1,  $M$  has positive Ricci curvature. The conclusion follows from Hamilton's classification of closed three-manifolds with positive Ricci curvature in [Ham82].

(2). By part (2) of Proposition 3.1,  $M$  has nonnegative Ricci curvature. If  $M$  is not flat, then it must be locally irreducible. Otherwise, the universal cover of  $M$  splits as  $N^2 \times \mathbb{R}$ , whose Ricci curvature has a zero eigenvalue everywhere. Since  $N^2 \times \mathbb{R}$  also has three-nonnegative curvature operator of the second kind, it must be flat by part (2) of Proposition 3.1. The desired conclusion then follows from Hamilton's

classification of closed locally irreducible three-manifolds with nonnegative Ricci curvature [Ham86].  $\square$

#### 4. ALGEBRAIC IMPLICATIONS

In this section, we investigate the curvature operator of the second kind in dimensions four and above, and establish various algebraic relations with other frequently used curvature conditions such as sectional curvature and isotropic curvature. For the reader's convenience, we first recall some definitions regarding isotropic curvature and its variants.

**Definition 4.1.** Let  $R \in S_B^2(\Lambda^2 V)$  be an algebraic curvature operator on a Euclidean vector space  $V$  of dimension  $n \geq 4$ .

- (1) We say  $R$  has nonnegative isotropic curvature if for all orthonormal four-frame  $\{e_1, e_2, e_3, e_4\} \subset V$ , it holds that

$$R_{1313} + R_{1414} + R_{2323} + R_{2424} - 2R_{1234} \geq 0.$$

If the inequality is strict,  $R$  is said to have positive isotropic curvature.

- (2) We say  $R$  has weakly PIC1 if for all orthonormal four-frame  $\{e_1, e_2, e_3, e_4\} \subset V$  and all  $\lambda \in [-1, 1]$ , it holds that

$$R_{1313} + \lambda^2 R_{1414} + R_{2323} + \lambda^2 R_{2424} - 2\lambda R_{1234} \geq 0.$$

If the inequality is strict,  $R$  is said to have strictly PIC1.

- (3) We say  $R$  has nonnegative complex sectional curvature (or weakly PIC2) if for all orthonormal four-frame  $\{e_1, e_2, e_3, e_4\} \subset V$  and all  $\lambda, \mu \in [-1, 1]$ , it holds that

$$R_{1313} + \lambda^2 R_{1414} + \mu^2 R_{2323} + \lambda^2 \mu^2 R_{2424} - 2\lambda \mu R_{1234} \geq 0.$$

If the inequality is strict,  $R$  is said to have positive complex sectional curvature (or strictly PIC2).

The notion of positive isotropic curvature was introduced by Micallef and Moore [MM88] in their study of minimal two-spheres in Riemannian manifolds. They proved that a simply connected closed Riemannian manifold with positive isotropic curvature is homeomorphic to the sphere. The PIC2 curvature condition was introduced by Brendle and Schoen and it played a central role in their proof of the quarter-pinched differentiable sphere theorem [BS09]. Ni and Wolfson [NW07a] discovered that strictly (respectively, weakly) PIC2 is equivalent to positive (respectively, nonnegative) complex sectional curvature. They also gave a simple proof that positive and nonnegative complex sectional curvature are preserved by the Ricci flow. The PIC1 curvature condition was introduced by Brendle [Bre08]. All the above curvature conditions in Definition 4.1 are preserved by the Ricci flow (see for example [Wil13] for a unified simple proof). Moreover, the normalized Ricci flow evolves an initial metric with strictly PIC1 (or the stronger condition strictly PIC2) into a limit metric with constant positive sectional curvature (see [BS09, Bre08, NW07a]).

Next, we summarize the algebraic relations between the curvature operator of the second kind and other curvature conditions.

**Proposition 4.1.** *Let  $R \in S_B^2(\Lambda^2 V)$  be an algebraic curvature operator on a Euclidean vector space  $V$  of dimension  $n \geq 4$ .*

- (1)  *$R$  has  $\frac{(n-1)(n+2)}{2}$ -positive (respectively,  $\frac{(n-1)(n+2)}{2}$ -nonnegative) curvature operator of the second kind if and only if  $R$  has positive (respectively, nonnegative) scalar curvature.*
- (2) *If  $R$  has  $n$ -positive (respectively,  $n$ -nonnegative) curvature operator of the second kind, then  $\text{Ric} > \frac{S}{n(n+1)} > 0$  (respectively,  $\text{Ric} \geq \frac{S}{n(n+1)} \geq 0$ ).*
- (3) *If  $R$  has two-positive (respectively, two-nonnegative) curvature operator of the second kind, then  $R$  has positive (respectively, nonnegative) sectional curvature.*
- (4) *If  $R$  has three-positive (respectively, three-nonnegative) curvature operator of the second kind, then  $R$  has strictly (respectively, weakly) PIC1.*
- (5) *If  $R$  has positive (respectively, nonnegative) curvature operator of the second kind, then  $R$  has positive (respectively, nonnegative) complex sectional curvature.*

*Proof of Proposition 4.1.* (1). Since the dimension of  $S_0^2(V)$  is equal to  $\frac{(n-1)(n+2)}{2}$ ,  $\mathring{R}$  being  $\frac{(n-1)(n+2)}{2}$ -positive (respectively, nonnegative) is equivalent to the trace of  $\mathring{R}$  (restricted on  $S_0^2(V)$ ) being positive (respectively, nonnegative). It can be easily seen that the total trace of  $\mathring{R}$  on  $S^2(V)$  is equal to  $\frac{S}{2}$  and that  $\mathring{R}$  restricted on the one-dimensional subspace  $\mathbb{R}g$  is equal to multiplication by  $-\frac{S}{n}$ . Thus the trace of  $\mathring{R}$  restricted on  $S_0^2(V)$  is equal to  $\frac{n+2}{2n}S$ .

(2). Let  $\{e_1, \dots, e_n\}$  be an orthonormal basis of  $V$  and define

$$\varphi_1 = \frac{1}{2\sqrt{n(n-1)}} \left( (n-1)e_1 \odot e_1 - \sum_{j=2}^n e_j \odot e_j \right),$$

and  $\varphi_i = \frac{1}{\sqrt{2}}e_1 \odot e_i$  for  $2 \leq i \leq n$ . Then  $\{\varphi_1, \dots, \varphi_n\}$  is an orthonormal subset of  $S_0^2(V)$ . Using the observation that the nonzero components of  $\varphi_1$  are  $(\varphi_1)_{11} = \sqrt{\frac{n-1}{n}}$ , and  $(\varphi_1)_{jj} = -\frac{1}{\sqrt{n(n-1)}}$  for  $2 \leq j \leq n$ , we calculate that

$$\begin{aligned} \mathring{R}(\varphi_1, \varphi_1) &= \sum_{i,j=1}^n R_{ijji}(\varphi_1)_{ii}(\varphi_1)_{jj} \\ &= \frac{2}{n} \sum_{j=2}^n R_{1j1j} - \frac{1}{n(n-1)} \sum_{i,j=2}^n R_{ijij} \\ &= \frac{2}{n}R_{11} - \frac{1}{n(n-1)}(S - 2R_{11}) \\ &= \frac{2}{n-1}R_{11} - \frac{1}{n(n-1)}S. \end{aligned}$$

For  $2 \leq i \leq n$ , we have

$$\mathring{R}(\varphi_i, \varphi_i) = R_{1i1i}.$$

Since  $R$  has  $n$ -nonnegative curvature operator of the second kind, we get

$$\begin{aligned} 0 &\leq \mathring{R}(\varphi_1, \varphi_1) + \sum_{i=2}^n \mathring{R}(\varphi_i, \varphi_i) \\ &= \frac{2}{n-1}R_{11} - \frac{1}{n(n-1)}S + \sum_{i=2}^n R_{1i1i} \\ &= \frac{n+1}{n-1}R_{11} - \frac{1}{n(n-1)}S, \end{aligned}$$

which implies that  $\text{Ric} \geq \frac{S}{n(n+1)} \geq 0$ . Similarly,  $n$ -positive curvature operator of the second kind implies  $\text{Ric} > \frac{S}{n(n+1)} > 0$ .

(3). Notice that

$$\begin{aligned} \varphi_1 &= \frac{1}{\sqrt{2}}e_1 \odot e_2, \\ \varphi_2 &= \frac{1}{2\sqrt{2}}(e_1 \odot e_1 - e_2 \odot e_2) \end{aligned}$$

are orthonormal traceless symmetric two-tensors in  $S_0^2(V)$ , where  $\{e_1, e_2\}$  is an orthonormal two-frame in  $V$ . In view of

$$2R_{1212} = \mathring{R}(\varphi_1, \varphi_1) + \mathring{R}(\varphi_2, \varphi_2),$$

we see that two-positive (respectively, two-nonnegative) curvature operator of the second kind implies positive (respectively, nonnegative) sectional curvature.

(4). This is a consequence of Lemma 4.1 below. For any orthonormal four-frame  $\{e_1, \dots, e_4\} \subset V$  and any  $\lambda \in [-1, 1]$ , we have by Lemma 4.1 that

$$\begin{aligned} R_{1313} + \lambda^2 R_{1414} + R_{2323} + \lambda^2 R_{2424} - 4\lambda R_{1234} &\geq 0, \\ R_{1313} + \lambda^2 R_{1414} + R_{2323} + \lambda^2 R_{2424} + 4\lambda R_{1234} &\geq 0. \end{aligned}$$

Therefore, we have

$$R_{1313} + \lambda^2 R_{1414} + R_{2323} + \lambda^2 R_{2424} \geq 4|\lambda R_{1234}| \geq 2|\lambda R_{1234}|,$$

from which we conclude that

$$R_{1313} + \lambda^2 R_{1414} + R_{2323} + \lambda^2 R_{2424} - 2\lambda R_{1234} \geq 0.$$

Hence  $R$  has weakly PIC1.

(5). This follows from Lemma 4.2 below and a similar argument as in (4).  $\square$

**Lemma 4.1.** *Suppose  $n = \dim(V) \geq 4$  and  $R \in S_B^2(\Lambda^2(V))$  has three-positive (respectively, three-nonnegative) curvature operator of the second kind. Then for any orthonormal four-frame  $\{e_1, \dots, e_4\} \subset V$  and any  $\lambda \in \mathbb{R}$ , it holds that*

$$R_{1313} + \lambda^2 R_{1414} + R_{2323} + \lambda^2 R_{2424} - 4\lambda R_{1234} > \text{ (respectively, } \geq) 0.$$

*Proof.* We shall only prove the case that  $R$  has three-nonnegative curvature operator of the second kind, as the three-positive case is similar.

Let  $\{e_1, \dots, e_4\}$  be an orthonormal four-frame in  $V$  and  $\lambda \in \mathbb{R}$ . Define the following traceless symmetric two-tensors on  $V$ .

$$\begin{aligned}\varphi_1 &= \frac{1}{2}(e_1 \odot e_1 + \lambda e_2 \odot e_2 - e_3 \odot e_3 - \lambda e_4 \odot e_4), \\ \varphi_2 &= \lambda e_1 \odot e_4 - e_2 \odot e_3, \\ \varphi_3 &= e_1 \odot e_3 + \lambda e_2 \odot e_4.\end{aligned}$$

One easily verifies that these tensors are mutually orthogonal in  $S_0^2(V)$  and of the same magnitude  $\sqrt{2(1+\lambda^2)}$ .

Next, we compute  $\mathring{R}(\varphi_i, \varphi_i)$  for  $i = 1, 2, 3$ . Notice that the only nonzero components of  $\varphi_1$  are

$$(\varphi_1)_{11} = 1, (\varphi_1)_{22} = \lambda, (\varphi_1)_{33} = -1, (\varphi_1)_{44} = -\lambda.$$

So we calculate

$$\begin{aligned}\mathring{R}(\varphi_1, \varphi_1) &= \sum_{i,j,k,l=1}^n R_{ijkl}(\varphi_1)_{il}(\varphi_1)_{jk} \\ &= \lambda R_{1221} - R_{1331} - \lambda R_{1441} + \lambda R_{2112} - \lambda R_{2332} - \lambda^2 R_{2442} \\ &\quad - R_{3113} - \lambda R_{3223} + \lambda R_{3443} - \lambda R_{4114} - \lambda^2 R_{4224} + \lambda R_{4334} \\ &= 2(-\lambda R_{1212} - \lambda R_{3434} + R_{1313} + \lambda^2 R_{2424} + \lambda R_{1414} + \lambda R_{2323}).\end{aligned}$$

Similarly,

$$\begin{aligned}\mathring{R}(\varphi_2, \varphi_2) &= 2(\lambda^2 R_{1414} + R_{2323} - 2\lambda R_{1234} + 2\lambda R_{1342}), \\ \mathring{R}(\varphi_3, \varphi_3) &= 2(R_{1313} + \lambda^2 R_{2424} - 2\lambda R_{1234} + 2\lambda R_{1423}).\end{aligned}$$

Since  $R$  has three-nonnegative curvature operator of the second kind, we obtain

$$\begin{aligned}0 &\leq \mathring{R}(\varphi_1, \varphi_1) + \mathring{R}(\varphi_2, \varphi_2) + \mathring{R}(\varphi_3, \varphi_3) \\ &= 4(R_{1313} + \lambda^2 R_{2424}) + 2(R_{2323} + \lambda^2 R_{1414}) - 12\lambda R_{1234} \\ &\quad + 2\lambda(-R_{1212} - R_{3434} + R_{1414} + R_{2323}),\end{aligned}$$

where we have used the first Bianchi identity  $R_{1342} + R_{1423} = -R_{1234}$ .

We then replace  $\{e_1, e_2, e_3, e_4\}$  in the above argument by  $\{e_2, e_1, e_3, e_4\}$  and replace  $\lambda$  by  $-\lambda$  to get

$$\begin{aligned}0 &\leq 4(R_{2323} + \lambda^2 R_{1414}) + 2(R_{1313} + \lambda^2 R_{2424}) - 12\lambda R_{1234} \\ &\quad - 2\lambda(-R_{1212} - R_{3434} + R_{2424} + R_{1313}).\end{aligned}$$

Adding the above two inequalities together produces

$$\begin{aligned}0 &\leq 6(R_{1313} + \lambda^2 R_{1414} + R_{2323} + \lambda^2 R_{2424}) - 24\lambda R_{1234} \\ &\quad + 2\lambda(R_{1414} + R_{2323} - R_{2424} - R_{1313}).\end{aligned}$$

Replacing  $\{e_1, e_2, e_3, e_4\}$  by  $\{-e_2, e_1, e_3, e_4\}$  in the above argument yields

$$\begin{aligned}0 &\leq 6(R_{2323} + \lambda^2 R_{2424} + R_{1313} + \lambda^2 R_{1414}) - 24\lambda R_{1234} \\ &\quad + 2\lambda(R_{2424} + R_{1313} - R_{1414} - R_{2323}).\end{aligned}$$

We obtain, by adding the above two inequalities together, that

$$0 \leq 12(R_{2323} + \lambda^2 R_{2424} + R_{1313} + \lambda^2 R_{1414}) - 48\lambda R_{1234}.$$

The proof is complete.  $\square$

**Lemma 4.2.** *Suppose  $n = \dim(V) \geq 4$  and  $R \in S_B^2(\Lambda^2(V))$  has positive (respectively, nonnegative) curvature operator of the second kind. Then for any orthonormal four-frame  $\{e_1, \dots, e_4\} \subset V$  and any  $\lambda, \mu \in \mathbb{R}$ , it holds that*

$$R_{1313} + \lambda^2 R_{1414} + \mu^2 R_{2323} + \lambda^2 \mu^2 R_{2424} - 6\lambda\mu R_{1234} > (\text{respectively, } \geq) 0.$$

*Proof.* Let  $\{e_1, \dots, e_4\}$  be an orthonormal four-frame in  $V$ . Given  $\lambda, \mu \in \mathbb{R}$ , we define  $\varphi$  and  $\psi$  by

$$\varphi = e_1 \odot e_3 + \lambda\mu e_2 \odot e_4$$

and

$$\psi = \mu e_2 \odot e_3 - \lambda e_1 \odot e_4$$

respectively. It's easy to see that both  $\varphi$  and  $\psi$  are traceless symmetric two-tensors on  $V$ .

Noticing that the only non-trivial components of  $\varphi$  are

$$\varphi_{13} = \varphi_{31} = 1 \text{ and } \varphi_{24} = \varphi_{42} = \lambda\mu,$$

we compute that

$$\begin{aligned} \mathring{R}(\varphi, \varphi) &= \sum_{i,j,k,l=1}^n R_{ijkl} \varphi_{il} \varphi_{jk} \\ &= R_{1313} + \lambda\mu R_{1243} + \lambda\mu R_{1423} \\ &\quad + R_{3131} + \lambda\mu R_{3241} + \lambda\mu R_{3421} \\ &\quad + \lambda\mu R_{2134} + \lambda\mu R_{2314} + \lambda^2 \mu^2 R_{2424} \\ &\quad + \lambda\mu R_{4132} + \lambda\mu R_{4312} + \lambda^2 \mu^2 R_{4242} \\ &= 2(R_{1313} + \lambda^2 \mu^2 R_{2424} + 2\lambda\mu R_{1423} + 2\lambda\mu R_{1243}) \end{aligned}$$

Similarly, all components of  $\psi$  are trivial except

$$\psi_{23} = \psi_{32} = \mu \text{ and } \psi_{14} = \psi_{41} = -\lambda,$$

and we calculate

$$\begin{aligned} \mathring{R}(\psi, \psi) &= \sum_{i,j,k,l=1}^n R_{ijkl} \psi_{il} \psi_{jk} \\ &= \mu^2 R_{2323} - \lambda\mu R_{2143} - \lambda\mu R_{2413} \\ &\quad + \mu^2 R_{3232} - \lambda\mu R_{3142} - \lambda\mu R_{3412} \\ &\quad + \lambda^2 R_{1414} - \lambda\mu R_{1234} - \lambda\mu R_{1324} \\ &\quad + \lambda^2 R_{4141} - \lambda\mu R_{4231} - \lambda\mu R_{4321} \\ &= 2(\mu^2 R_{2323} + \lambda^2 R_{1414} - 2\lambda\mu R_{2143} - 2\lambda\mu R_{2413}). \end{aligned}$$

Using the assumption that  $R$  has nonnegative curvature operator of the second kind, we get

$$\begin{aligned}
0 &\leq \frac{1}{2}\mathring{R}(\varphi, \varphi) + \frac{1}{2}\mathring{R}(\psi, \psi) \\
&= R_{1313} + \lambda^2\mu^2R_{2424} + 2\lambda\mu R_{1423} + 2\lambda\mu R_{1243} \\
&\quad + \mu^2R_{2323} + \lambda^2R_{1414} - 2\lambda\mu R_{2143} - 2\lambda\mu R_{2413} \\
&= R_{1313} + \lambda^2R_{1414} + \mu^2R_{2323} + \lambda^2\mu^2R_{2424} - 4\lambda\mu R_{1234} \\
&\quad + 2\lambda\mu(R_{1423} + R_{1342}) \\
&= R_{1313} + \lambda^2R_{1414} + \mu^2R_{2323} + \lambda^2\mu^2R_{2424} - 6\lambda\mu R_{1234},
\end{aligned}$$

where we have used the first Bianchi identity  $R_{1423} + R_{1342} = -R_{1234}$  in the last step. Similarly, if  $R$  has positive curvature operator of the second kind, then the above inequality holds strictly. This finishes the proof.  $\square$

It was shown in [CGT21] that four-positive (respectively, four-nonnegative) curvature operator of the second kind implies positive (respectively, nonnegative) isotropic curvature. Here we prove a slightly stronger result by improving the coefficient in front of  $R_{1234}$  in (4.1) from 2 to 3. The improvement is necessary to prove Theorem 1.9. Moreover, our proof here is a bit cleaner.

**Proposition 4.2.** *Suppose  $n = \dim(V) \geq 4$  and  $R \in S_{\mathbb{B}}^2(\Lambda^2(V))$  has four-positive (respectively, four-nonnegative) curvature operator of the second kind. Then for any orthonormal four-frame  $\{e_1, \dots, e_4\} \subset V$ , we have*

$$(4.1) \quad R_{1313} + R_{1414} + R_{2323} + R_{2424} - 3R_{1234} > (\text{ respectively, } \geq) 0.$$

*Proof.* Let  $\{e_1, \dots, e_4\}$  be an orthonormal four-frame in  $V$  and define

$$\begin{aligned}
\varphi_1 &= \frac{1}{2}(e_1 \odot e_1 + e_2 \odot e_2 - e_3 \odot e_3 - e_4 \odot e_4), \\
\varphi_2 &= \frac{1}{2}(e_1 \odot e_1 - e_2 \odot e_2 + e_3 \odot e_3 - e_4 \odot e_4), \\
\varphi_3 &= e_1 \odot e_4 - e_2 \odot e_3, \\
\varphi_4 &= e_1 \odot e_3 + e_2 \odot e_4.
\end{aligned}$$

Clearly,  $\{\varphi_1, \dots, \varphi_4\}$  are traceless symmetric two-tensors on  $V$  and they are mutually orthogonal with the same magnitude. Direct calculation as before shows that

$$\begin{aligned}
\mathring{R}(\varphi_1, \varphi_1) &= 2(-R_{1212} - R_{3434} + R_{1313} + R_{2424} + R_{1414} + R_{2323}), \\
\mathring{R}(\varphi_2, \varphi_2) &= 2(-R_{1313} - R_{2424} + R_{1212} + R_{3434} + R_{1414} + R_{2323}), \\
\mathring{R}(\varphi_3, \varphi_3) &= 2(R_{1414} + R_{2323} - 2R_{1234} + 2R_{1342}), \\
\mathring{R}(\varphi_4, \varphi_4) &= 2(R_{1313} + R_{2424} - 2R_{1234} + 2R_{1423}).
\end{aligned}$$

If  $R$  has four-nonnegative curvature operator of the second kind, we get

$$\begin{aligned}
0 &\leq \mathring{R}(\varphi_1, \varphi_1) + \mathring{R}(\varphi_2, \varphi_2) + \mathring{R}(\varphi_3, \varphi_3) + \mathring{R}(\varphi_4, \varphi_4) \\
&= 6(R_{1414} + R_{2323}) + 2(R_{1313} + R_{2424}) - 12R_{1234}.
\end{aligned}$$

Replacing  $\{e_1, e_2, e_3, e_4\}$  by  $\{e_2, -e_1, e_3, e_4\}$  in the above argument yields

$$0 \leq 6(R_{2424} + R_{1313}) + 2(R_{2323} + R_{1414}) - 12R_{1234}.$$

We obtain, by adding the above two inequalities together, that

$$0 \leq 8(R_{2424} + R_{1313} + R_{2323} + R_{1414}) - 24R_{1234}.$$

If  $R$  has four-positive curvature operator of the second kind, then all the above inequalities become strict and (4.1) holds strictly. The proof is complete.  $\square$

## 5. FLAT OR LOCALLY IRREDUCIBLE

The goal of this section is to investigate the curvature operator of the second kind on product manifolds and prove Theorem 1.8.

We shall prove a slightly more general statement below, which implies that if the curvature operator of the second kind is  $(k(n-k)+1)$ -nonnegative for some  $1 \leq k \leq \frac{n}{2}$ , then the manifold cannot split off a  $k$ -dimensional factor unless it is flat. In particular,  $n$ -manifolds with  $n$ -nonnegative curvature operator of the second kind must be locally irreducible unless it is flat.

**Proposition 5.1.** *Let  $(M^n, g)$  be an  $n$ -dimensional Riemannian manifold. Suppose that  $M$  is isometric  $M_1 \times M_2$ , where  $M_1$  is a  $k$ -dimensional Riemannian manifold and  $M_2$  is an  $(n-k)$ -dimensional Riemannian manifold. If  $M$  has  $(k(n-k)+1)$ -nonnegative curvature operator of the second kind, then  $M$  is flat.*

**Remark 5.1.** Proposition 5.1 is sharp in the sense that the assumption in general cannot be weakened to  $(k(n-k)+2)$ -nonnegative curvature operator of the second kind. For instance, when  $n = 4$  and  $k = 1$ ,  $\mathbb{S}^3 \times \mathbb{S}^1$  has 5-nonnegative curvature operator of the second kind (see Example 2.6), and when  $n = 4$  and  $k = 2$ ,  $\mathbb{S}^2 \times \mathbb{S}^2$  has 6-nonnegative curvature operator of the second kind (see Example 2.7).

*Proof of Proposition 5.1.* We first show that both  $M_1$  and  $M_2$  have nonnegative sectional curvature. For  $M_1^k$ , it suffices to consider the case  $k \geq 2$ , as one-dimensional Riemannian manifolds are flat. Let  $\sigma \subset T_p M_1$  be a two-plane spanned by two orthonormal vectors  $e_1$  and  $e_2$ . We extend  $\{e_1, e_2\}$  to an orthonormal basis  $\{e_1, \dots, e_k\}$  of  $T_p M_1$ . Let  $\{e_{k+1}, \dots, e_n\}$  be an orthonormal basis of  $T_q M_2$ . Then  $\{e_1, \dots, e_n\}$  is an orthonormal basis of  $T_{(p,q)} M$ .

Define

$$\varphi_{ip} = \frac{1}{\sqrt{2}} e_i \odot e_p \text{ for } 1 \leq i \leq k, k+1 \leq p \leq n,$$

and

$$\xi = \frac{1}{\sqrt{2}} e_1 \odot e_2.$$

It's easy to verify that  $\{\varphi_{ip}\}_{1 \leq i \leq k, k+1 \leq p \leq n} \cup \{\xi\}$  is an orthonormal set of dimension  $k(n-k)+1$  in  $S_0^2(\Lambda^2 T_{(p,q)} M)$ . Since  $M$  has  $(k(n-k)+1)$ -nonnegative curvature operator of the second kind, we have

$$\hat{R}(\xi, \xi) + \sum_{1 \leq i \leq k, k+1 \leq p \leq n} \hat{R}(\varphi_{ip}, \varphi_{ip}) \geq 0.$$

Notice that

$$(5.1) \quad \hat{R}(\varphi_{ip}, \varphi_{ip}) = R_{ipip} = 0 \text{ for } 1 \leq i \leq k, k+1 \leq p \leq n.$$

due to the product structure. We therefore have  $\mathring{R}(\xi, \xi) = R_{1212} \geq 0$ . Since  $\sigma$  is arbitrary, we conclude that  $M_1$  has nonnegative sectional curvature.  $M_2$  is also nonnegatively curved by a similar argument.

The next step is to show that both  $M_1$  and  $M_2$  have vanishing scalar curvature. Let  $\{e_1, \dots, e_n\}$  be an orthonormal basis of  $T_{(p,q)}M$  with  $e_1, \dots, e_k \in T_pM_1$  and  $e_{k+1}, \dots, e_n \in T_qM_2$ . Define  $\varphi_{ip}$  as before and let

$$\psi = \frac{1}{2\sqrt{nk(n-k)}} \left( (n-k) \sum_{i=1}^k e_i \odot e_i - k \sum_{p=k+1}^n e_p \odot e_p \right).$$

One easily verifies that  $\{\varphi_{ip}\}_{1 \leq i \leq k, k+1 \leq p \leq n} \cup \{\psi\}$  is an orthonormal set of dimension  $k(n-k) + 1$  in  $S_0^2(\Lambda^2 T_{(p,q)}M)$ . Since  $M$  has  $(k(n-k) + 1)$ -nonnegative curvature operator of the second kind, we have

$$\mathring{R}(\psi, \psi) = \mathring{R}(\psi, \psi) + \sum_{1 \leq i \leq k, k+1 \leq p \leq n} \mathring{R}(\varphi_{ip}, \varphi_{ip}) \geq 0.$$

We calculate that

$$\begin{aligned} 0 &\leq nk(n-k)\mathring{R}(\psi, \psi) \\ &= (n-k)^2 \sum_{1 \leq i, j \leq k} R_{ijji} - k(n-k) \sum_{1 \leq i \leq k, k+1 \leq p \leq k} R_{ippi} \\ &\quad - k(n-k) \sum_{1 \leq j \leq k, k+1 \leq q \leq n} R_{qqjq} + k^2 \sum_{k+1 \leq p, q \leq n} R_{ppqq} \\ &= -(n-k)^2 S_1(p) - k^2 S_2(q), \end{aligned}$$

where  $S_i$  denotes the scalar curvature of  $M_i$  for  $i = 1, 2$ . The above inequality forces  $S_1(p) \leq 0$  and  $S_2(q) \leq 0$ . On the other hand, we must have  $S_1(p) \geq 0$  and  $S_2(q) \geq 0$ , as we have showed that both  $M_1$  and  $M_2$  are nonnegatively curved. Therefore both  $M_1$  and  $M_2$  must be scalar flat, and thus flat in view of the nonnegativity of their sectional curvatures. Hence  $M$  is flat.  $\square$

*Proof of Theorem 1.8.* Since  $n$ -nonnegative curvature operator of the second kind implies  $(k(n-k) + 1)$ -nonnegative curvature operator of the second kind for all  $1 \leq k \leq n-1$ , the universal cover of  $M$  cannot split as a product of Riemannian manifolds of lower dimensions unless it is flat. Thus  $M$  is either flat or locally irreducible.  $\square$

## 6. KÄHLER MANIFOLDS

In this section, we prove Theorem 1.9. We indeed prove it under a weaker assumption.

**Theorem 6.1.** *Let  $(M^{2m}, g)$  be a Kähler manifold of complex dimension  $m \geq 2$ . Suppose that there exists  $\beta > 1$  such that for any  $p \in M$  and any orthonormal four-frame  $\{e_1, \dots, e_4\} \subset T_pM$ , it holds that*

$$(6.1) \quad R_{1313} + R_{1414} + R_{2323} + R_{2424} - 2\beta R_{1234} \geq 0.$$

Then  $M$  must be flat if  $m \geq 3$ . If  $m = 2$ , then  $M$  is conformally flat, which means locally it is either flat or the product of two complex curves with constant curvature of opposite values.

The curvature assumption in Theorem 6.1 is optimal in the sense that there are many Kähler manifolds, such as compact Hermitian symmetric spaces, satisfying (6.1) with  $\beta = 1$ , namely they have nonnegative isotropic curvature.

*Proof of Theorem 6.1.* The proof here is inspired by Brendle's argument [Bre10, Proposition 9.18] to prove that Kähler-Einstein manifolds with nonnegative isotropic curvature must have positive orthogonal bisectional curvature if it is not flat. The assumption  $\beta > 1$  in some sense compensates the Einstein condition. Most of the calculation below is exactly the same as in Brendle's argument [Bre10, Proposition 9.18].

Recall that on a Kähler manifold there exists a section  $J$  of the endomorphism bundle  $\text{End}(TM)$  with the following properties:

- (1)  $J$  is parallel;
- (2) for each point  $p \in M$ , we have  $J^2 = -\text{id}$  and  $g(X, Y) = g(JX, JY)$  for all  $X, Y \in T_pM$ ;
- (3) the Riemann curvature tensor satisfies

$$R(X, Y, Z, W) = R(X, Y, JZ, JW)$$

for all  $X, Y, Z, W \in T_pM$ .

**Claim A:**  $M$  has vanishing orthogonal bisectional curvature, namely for all unit vectors  $X, Y \in T_pM$  satisfying  $g(X, Y) = g(X, JY) = 0$ , it holds that

$$(6.2) \quad R(X, JX, Y, JY) = 0.$$

*Proof of Claim A:* Consider two unit vectors  $X, Y \in T_pM$  with  $g(X, Y) = g(JX, Y) = 0$ . Notice that by (2) we have

$$g(JX, Y) = g(J^2X, JY) = -g(X, JY) = 0$$

and  $g(JX, JY) = g(X, Y) = 0$ . Thus  $\{X, JX, Y, JY\}$  is an orthonormal four-frame in  $T_pM$ .

Applying (6.1) to the orthonormal four-frame  $\{X, JX, Y, JY\}$  produces

$$\begin{aligned} & R(X, Y, X, Y) + R(JX, Y, JX, Y) \\ & + R(X, JY, X, JY) + R(JX, JY, JX, JY) \\ & \geq 2\beta R(X, JX, Y, JY). \end{aligned}$$

Notice that (6.1) implies

$$(6.3) \quad R_{1313} + R_{1414} + R_{2323} + R_{2424} \geq 0$$

for any orthonormal four-frame  $\{e_1, \dots, e_4\} \subset T_pM$ . Applying (6.3) to the orthonormal four-frame  $\{X, JX, Y, JY\}$  yields

$$\begin{aligned} & R(X, Y, X, Y) + R(JX, Y, JX, Y) \\ & + R(X, JY, X, JY) + R(JX, JY, JX, JY) \geq 0. \end{aligned}$$

On the other hand, it follows from the first Bianchi identity and (3) that

$$\begin{aligned} 2R(X, JX, Y, JY) &= R(X, Y, X, Y) + R(JX, Y, JX, Y) \\ &\quad + R(X, JY, X, JY) + R(JX, JY, JX, JY). \end{aligned}$$

Therefore, we obtain that

$$R(X, JX, Y, JY) \geq 0$$

and

$$2R(X, JX, Y, JY) \geq 2\beta R(X, JX, Y, JY)$$

Since  $\beta > 1$ , we must have

$$R(X, JX, Y, JY) = 0.$$

This finishes the proof of Claim A.

**Claim B:**  $M$  has vanishing scalar curvature.

*Proof of Claim B:* Let  $Z, W \in T_p M$  be two unit vectors satisfying  $g(Z, W) = g(Z, JW) = 0$ . Applying (6.2) to the vectors  $X = \frac{1}{\sqrt{2}}(Z + W)$  and  $Y = \frac{1}{\sqrt{2}}(Z - W)$  yields

$$\begin{aligned} 0 &= R(Z + W, JZ + JW, Z - W, JZ - JW) \\ &= R(Z, JZ, Z, JZ) + R(Z, JZ, W, JW) \\ &\quad - R(Z, JZ, Z, JW) - R(Z, JZ, W, JZ) \\ &\quad + R(W, JW, Z, JZ) + R(W, JW, W, JW) \\ &\quad - R(W, JW, Z, JW) - R(W, JW, W, JZ) \\ &\quad + R(Z, JW, Z, JZ) + R(Z, JW, W, JW) \\ &\quad - R(Z, JW, Z, JW) - R(Z, JW, W, JZ) \\ &\quad + R(W, JZ, Z, JZ) + R(W, JZ, W, JW) \\ &\quad - R(W, JZ, Z, JW) - R(W, JZ, W, JZ) \\ &= R(Z, JZ, Z, JZ) + R(W, JW, W, JW) \\ &\quad + 2R(Z, JZ, W, JW) - 4R(Z, JW, Z, JW) \end{aligned}$$

Similarly, we apply (6.2) to the vectors  $X = \frac{1}{\sqrt{2}}(Z + JW)$  and  $Y = \frac{1}{\sqrt{2}}(Z - JW)$  and obtain

$$\begin{aligned} 0 &= R(Z + W, JZ + JW, Z - W, JZ - JW) \\ &= R(Z, JZ, Z, JZ) + R(W, JW, W, JW) \\ &\quad + 2R(Z, JZ, W, JW) - 4R(Z, W, Z, W). \end{aligned}$$

Averaging the above two equations yields,

$$\begin{aligned} 0 &= R(Z, JZ, Z, JZ) + R(W, JW, W, JW) \\ &\quad + 2R(Z, JZ, W, JW) - 2R(Z, JW, Z, JW) - 2R(Z, W, Z, W). \end{aligned}$$

Noticing that by the first Bianchi identity and (3), we have

$$\begin{aligned} &R(Z, JZ, W, JW) - R(Z, JW, Z, JW) - R(Z, W, Z, W) \\ &= R(Z, JZ, W, JW) + R(Z, JW, JZ, W) + R(Z, W, JW, JZ) \\ &= 0. \end{aligned}$$

Therefore, we have proved that

$$(6.4) \quad R(Z, JZ, Z, JZ) + R(W, JW, W, JW) = 0$$

for all unit vectors satisfying  $g(Z, W) = g(Z, JW) = 0$ . It follows from (6.2) and (6.4) that the scalar curvature of  $M$  vanishes, as the scalar curvature of a Kähler manifold is given by

$$S = 2 \sum_{i,j=1}^m R(e_i, Je_i, e_j, Je_j),$$

where  $\{e_1, \dots, e_m, Je_1, \dots, Je_m\}$  is an orthonormal basis of  $T_p M$ . This proves Claim B.

We continue to prove Theorem 6.1. Clearly (6.1) implies that  $M$  has nonnegative isotropic curvature. Therefore, Theorem 6.1 follows from a result of Micallef and Wang [MW93, Proposition 2.5], which asserts that a Riemannian manifold of real dimension  $n \geq 4$  with nonnegative isotropic curvature and vanishing scalar curvature must be flat if  $n \geq 5$  and must have vanishing Weyl tensor if  $n = 4$ .

Alternatively, one can argue as follows after proving Claim A. Note that (6.2) implies that  $M$  has vanishing orthogonal Ricci curvature  $\text{Ric}^\perp \equiv 0$  ( $\text{Ric}^\perp$  was introduced by Ni and Zheng [NZ18] in the study of comparison theorems for Kähler manifolds). The conclusion then follows from the classification of  $\text{Ric}^\perp$ -flat Kähler manifold by Ni, Wang and Zheng [NWZ21, Theorem 6.1].  $\square$

*Proof of Theorem 1.9.* Since  $M$  has four-nonnegative curvature operator of the second kind, it satisfies (6.1) with  $\beta = \frac{3}{2}$  by Proposition 4.2. By Theorem 6.1,  $M$  must be flat if  $m \geq 3$ . If  $m = 2$ , one also uses Theorem 1.8 to rule out the product of two complex curves with constant curvature of opposite values. Hence  $M$  must be flat.  $\square$

## 7. PROOF OF THEOREM 1.7

*Proof of Theorem 1.7.* Suppose that  $M$  is non-flat. Let  $\widetilde{M}$  be the universal cover of  $M$  with the lifted metric  $\tilde{g}$ . Then  $(\widetilde{M}, \tilde{g})$  also has three-nonnegative curvature operator of the second kind and nonnegative Ricci curvature by part (3) of Proposition 4.1. By the Cheeger-Gromoll splitting theorem [CG72],  $\widetilde{M}$  is isometric to a product of the form  $N^{n-k} \times \mathbb{R}^k$ , where  $N$  is compact. By Theorem 1.8, we know that  $M$  is locally irreducible, which implies that  $k = 0$ . Hence  $\widetilde{M}$  is compact.

On the other hand, it follows from part (4) of Proposition 4.1 that  $\widetilde{M}$  has weakly PIC1. By evoking the classification of closed simply-connected irreducible Riemannian manifold with weakly PIC1 (see for example [Bre10, Theorem 9.33]), we conclude that one of the following statements holds:

- (1)  $\widetilde{M}$  is diffeomorphic to  $\mathbb{S}^n$ ;
- (2)  $n = 2m$  and  $\widetilde{M}$  is a Kähler manifold;
- (3)  $\widetilde{M}$  is isometric to a compact irreducible symmetric space.

Notice that (2) cannot occur in view of Theorem 1.9. Also, (3) cannot occur when  $n \leq 4$  because  $\mathbb{C}\mathbb{P}^2$  does not have three-nonnegative curvature operator of the second kind.

Hence we can conclude that  $M$  is either flat, or diffeomorphic to a spherical space form, or isometric to a quotient of a compact irreducible symmetric space.

In particular,  $M$  is diffeomorphic to a Riemannian locally symmetric space. The proof of Theorem 1.7 is now complete.  $\square$

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