FREQUENCY ESTIMATION USING SUBSPACE METHODS

A Thesis by

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The following faculties have examined the final copy of this thesis for form and content, and recommend that it be accepted in partial fulfillment of the requirement for the degree of Master of Science with a major in Electrical Engineering.

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Edwin Sawan, Committee Chair

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Rajiv Bagai, Committee Member

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Gamal Weheba, Committee Member

DEDICATION

To my mother and father

To my lovely husband

With love

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My husband, Mahmoud Qasaymeh, has always been with me and suffered through all the struggle of getting this thesis completed. For his patience and the strength that he gave me by never stopping to believe in me, this thesis is as much his as it is mine. Also, I am very thankful for his staying power and serenity as my work could have never been done without his help and support and his huge technical direction in my research.

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ABSTRACT

Complex frequency estimation problem plays a significant role in many engineering applications. The estimation process was traditionally achieved by the Eigenvalue Decomposition (EVD) of the spatial correlation matrix of observations. Frequency estimation has fundamental significant and wide relevance for many reasons. First, any arbitrary signal may be modeled as a sum of frequencies. Hence, any signal estimation problem may be expressed in terms of frequency estimation problems. Second, many parameter estimation applications may be mathematically expressed as a frequency estimation problem.

In this thesis an improved frequency estimation technique is presented based on the unitary transformation, which was basically applied in the direction of arrival problem. The key idea of the proposed technique is to convert the complex valued autocorrelation, cumulant, or the direct data matrix in Hankel like shape into a real valued data matrix with the same dimension. The resultant real valued matrix will be used to extract the noise and/or the signal subspace instead of the original complex one. It is well known that real manipulations are easier and faster than the complex ones.

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LIST OF ABBREVIATIONS

AWGN	Additive White Gaussian Noise
BER	Bit Error Rate
BEWE	Bearing Estimation Without Eigen decomposition
CFO	Carrier Frequency Offset
CIR	Channel Impulse Response
D/A	Digital-to-Analog
DOA	Direction of Arrival
ESPRIT	Rotational Invariance Technique
EVD	Eigen Value Decomposition
ML	Maximum Likelihood
MSE	Mean Square Error
MUSIC	MUltiple SIgnal Classification
OFDM	Orthogonal Frequency Division Multiplexing
OPM	Orthonormal PM
PM	Propagator Method
QAM	Quadrature Amplitude Modulation
RRQR	Rank Revealing QR
SNR	Signal-to-noise ratio
SVD	Singular Value Decomposition
SWEDE	Subspace Methods Without Eigen Decomposition

CHAPTER 1

OVERVIEW

1.1 Introduction and Motivation

Identification of sinusoidal components plays a significant role in many engineering applications [1]. This identification process was traditionally achieved by the Eigenvalue Decomposition (EVD) of the spatial correlation matrix of observations. Frequency estimation has fundamental significant and wide relevance for many reasons. First, any arbitrary signal may be modeled as a sum of frequencies. Hence, any signal estimation problem may be expressed in terms of frequency estimation problems [2]. Second, many parameter estimation applications may be mathematically expressed as a frequency estimation problem [3]. An example is active noise and vibration control [4]. For any machine with rotating components, the resulting noises or vibrations are often modeled as periodic signal. Examples of this class of applications include noises in Turboprop aircraft [5], in helicopters [6], in Heating, Ventilation, and Air Conditioning (HVAC) systems [7], disk drives, and magnetic bearings [8]. Health monitoring is another application of sinusoidal identification in which the interest involves online monitoring such as sonar signal processing [9] and speech processing. Another field in which this problem arises is in digital telephony [10] when we wish to estimate one of a number of possible Caller ID tones. Furthermore, sinusoidal identification is also essential in radar, direction of arrival, [11]-[13] time delay in Frequency Hopping (FH) system [14], [15] Carrier Frequency Offset (CFO) in Orthogonal Frequency Division Multiplexing (OFDM) [16], [17] and joint time delay and frequency estimation problems [18], [19].

The model for all frequency estimators of parameters in white Gaussian noise is fairly standard. The objective is to extract the frequency components from a sum of complex sinusoids embedded in white Gaussian noise.

1.2 Thesis Contribution

In this thesis an improved frequency estimation technique is presented. The key idea of the proposed technique is to convert the complex valued autocorrelation, cumulant, or the direct data matrix in Hankel like shape into a real valued data matrix with the same dimension [12]. The resultant real valued matrix will be used to extract the noise and/or the signal subspace instead of the original complex one [2]. It is well known that real manipulations are easier and faster than the complex ones. In this thesis, a simple frequency estimator is presented based on the unitary ESPRIT transformation [24], which was basically applied in the direction of arrival problem.

1.3 Thesis Outline

This thesis consists of five chapters. Chapter one gives an overview for the problem of frequency estimation, thesis contribution and the organization of this thesis. Chapter two presents a brief summary of the work done in this area and formulates the problem. Chapter three discusses the subspace frequency estimator via rank revealing QR factorization [26]-[28], and then introduces the proposed method. Both methods are compared to each other; they are almost showing the same performance with apparent simplicity in the proposed one as it is dealing with real numbers. MATLAB simulations are used to show the performance of both estimators.

Chapter four gives some concluding remarks and possible future extensions. Finally, the appendix presents some proofs and the accepted papers related to this research.

CHAPTER 2

FREQUENCY ESTIMATION

2.1 Introduction

In 1973, Pisarenko considered the problem of estimating the frequencies of a sum of complex exponentials in white noise [20]. He demonstrated that the frequencies could be derived from the eigenvector corresponding to the minimum eigenvalue of the autocorrelation matrix. In 1979, an improvement to the Pisarenko method known as the Multiple Signal Classification method (MUSIC) was introduced by Schmidt [21]. Like Pisarenko's method, the MUSIC algorithm is a frequency estimation technique based on the autocorrelation matrix. In 1982, EigenVector (EV) method was introduced by Johnson and DeGraaf [22]. Another Eigen Value Decomposition- based (EVD) method of interest is the minimum norm method [23]. All these methods are classified as noise subspace methods. Signal subspace methods are closely related to the noise subspace methods, as the two subspaces are orthogonal to each other [33]. Minimum Variance (MV) [2] and autoregressive spectrum estimation using the autocorrelation matrix are well known signal subspace methods. In spectral MUSIC the frequencies of the components can be obtained from the estimate signal pseudo spectrum by finding the position of the maxima. Alternative approach is possible by constructing the polynomials using the eigenvectors spanning the noise subspace. The roots of each of such polynomials correspond to signal zeros, which is known a root MUSIC [25]. In 1989, Paulraj, Roy and Kailath presented the original ESPRIT (Estimation of Signal Parameter via Rotational Invariance Technique) [24]. It is based on a naturally existing shift invariance between the discrete time series which leads to rotational invariance between the corresponding signal subspaces. The Total Least Squares (TLS) solution minimizes the Frobenius norm of the error matrix, and the solution is given by [27].

2.2 Problem Formulation

Frequency estimation is the process of estimating the complex frequency components of a signal in the existence of noise. The most common frequency estimation methods involve identifying the noise subspace to extract these components. For example, consider a signal, $x(n)$, consisting of a sum of P complex exponentials in the presence of white noise, $w(n)$. This may be represented as:

$$x(n) = s(n) + w(n)$$

$$s(n) = \sum_{i=1}^{P} a_i e^{jn\omega_i}, \qquad n = 0,1,2,\dots,N-1 \tag{2.1}$$

The source signal $x(n)$ is modeled by a sum of P complex sinusoids where the amplitudes a_i are unknown, complex-valued constants, and the normalized radian frequencies $\omega_i$are different. Without a loss of generality, we considered $\omega_1 < \omega_2 \dots < \omega_P$. Parametric methods are those which take advantage of known parameters of the signal, such as the number of tones it contains. Non–parametric methods do not make such assumptions a priori. To simplify the problem we have assumed the number of sources P either known or pre estimated [31]. The term $w(n)$ is representing zero mean, additive white complex gaussian noise process. Also, parameters N represent the number of samples collected at the receiver. Let us define the signal vector (noiseless data), the noise vector, and the measured or received noisy data vector as **s, w, x** respectively.

$$\mathbf{s} = \begin{bmatrix} s(0) \\ s(1) \\ s(2) \\ \vdots \\ s(N-1) \end{bmatrix} \qquad \mathbf{w} = \begin{bmatrix} w(0) \\ w(1) \\ w(2) \\ \vdots \\ w(N-1) \end{bmatrix} \qquad \mathbf{x} = \begin{bmatrix} x(0) \\ x(1) \\ x(2) \\ \vdots \\ x(N-1) \end{bmatrix} \tag{2.2}$$

Rewriting (2.1) in vector form

$$\mathbf{x} = \mathbf{s} + \mathbf{w} \tag{2.3}$$

To summarize, the problem in hand now is to estimate the frequencies from measured data recorded **x**. The power spectrum of $x(n)$ consists of P impulses in addition to the power due to noise. The generic steps of the subspace methods of frequency estimation are summarized by three steps:

1. Construct a matrix from the vector x given by (2.2). This matrix could appear in different forms like:
    1. Hankel matrix
    2. Autocorrelation matrix
    3. Cumulant matrix
2. Derive the noise subspace and /or the signal subspace. This step could be achieved through decomposition techniques or non decomposition techniques. The popular matrix decomposition techniques are listed below:
    1. Eigenvalue decomposition (EVD)
    2. Singular value decomposition (SVD)
    3. Orthogonal projection QR factorizations
    4. Orthogonal projection LU factorizations.

   The most used non matrix decomposition techniques are:
    1. Propagator Method (PM) [18].
    2. Orthogonal Propagator Method (OPM).

3. Frequency estimation function is used to find the component frequencies from the noise subspace or signal subspace. The most popular methods of noise subspace based frequency estimation are:

    1. Pisarenko's Method
    2. MUSIC
    3. The eigenvector solution (EV)
    4. The minimum norm solution

The most popular methods of signal subspace based frequency estimation are:

1. Blackman-Tukey
2. Minimum variance
3. Autoregressive

In Figure 1 a block diagram is showing the general steps of the frequency estimation process. In general the constructed matrix in the first step is a complex matrix, and then all the manipulations in the second and third steps are dealing with this complex matrix. In order to reduce the computational complexity the unitary transformation is used to convert the complex-valued matrix **X** to real-valued matrix of the same size. Figure 2 is showing a block diagram of the steps in the proposed frequency estimation process.

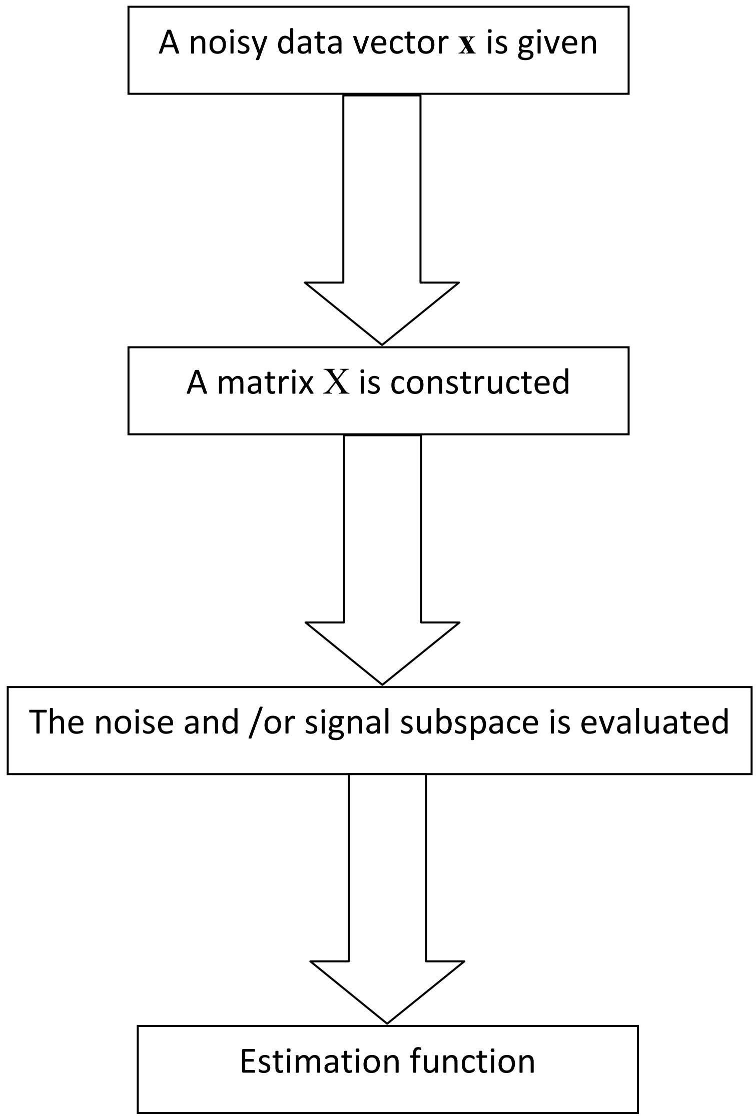

Figure 2.1 Block diagram showing the general steps of the frequency estimation process.

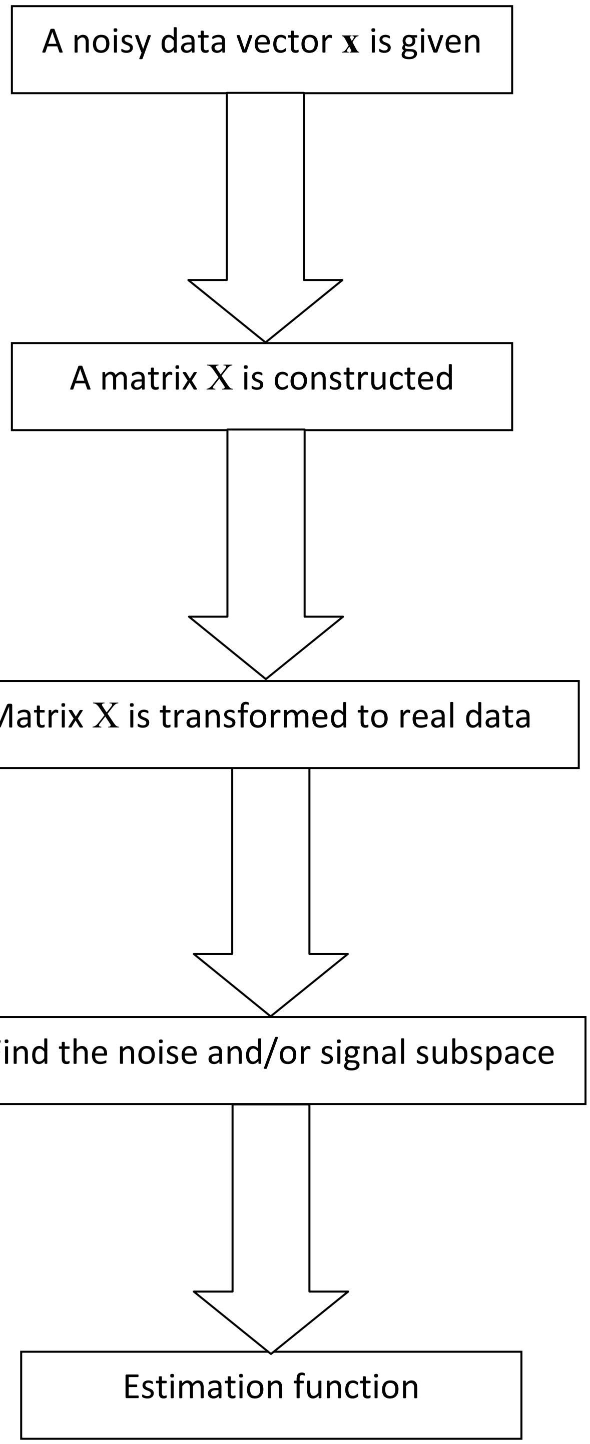

Figure 2.2 Block diagram showing the proposed steps of the frequency estimation process.

CHAPTER 3

FREQUENCY ESTIMATION VIA RANK REVEALING QR FACTORIZATION

3.1 Introduction

The QR decomposition (also called the QR factorization) of a complex matrix is a decomposition of the matrix into a unitary and aright triangular matrix. The QR factorization is used to find orthonormal vectors (bases) that span the successive spaces spanned by the columns of matrix A.

- Let $\boldsymbol{A}$ be a complex $m \times n$ matrix, where $m \geq n$, and *rank* $(A) = n$ then matrix $\boldsymbol{A}$ can be factorized using QR factorization as:

  $$\boldsymbol{A} = \boldsymbol{QR} = [\boldsymbol{Q}_1 \quad \boldsymbol{Q}_2] . \begin{bmatrix} \boldsymbol{R}_1 \\ \boldsymbol{0} \end{bmatrix}$$

  where $\boldsymbol{Q}$ is an m by m unitary matrix, $\boldsymbol{R}$ is an m by n upper triangular matrix, $\boldsymbol{Q_1}$ is an m by n unitary matrix, $\boldsymbol{Q_2}$ is an m by $(m\text{-}n)$ unitary matrix, $\boldsymbol{R_1}$ is an n by n , non singular, upper triangular matrix, and $\boldsymbol{0}$ matrix of size $(m\text{-}n)$ by n.

- Let $\boldsymbol{A}$ be an $m \times n$ matrix, where $m \geq n$, and *rank* $(A) = r$ then matrix $\boldsymbol{A}$ can be factorized using QR factorization with column interchanges as:

  $$\boldsymbol{A} = \boldsymbol{QR} = [\boldsymbol{Q}_1 \quad \boldsymbol{Q}_2] . \begin{bmatrix} \boldsymbol{R}_1 \\ \boldsymbol{0} \end{bmatrix} . \boldsymbol{P}^H$$

  where $\boldsymbol{Q}$ is an m by m unitary matrix, $\boldsymbol{R}$ is an m by n upper triangular matrix, $\boldsymbol{Q_1}$ is an m by r unitary matrix, $\boldsymbol{Q_2}$ is an m by $(m\text{-}r)$ unitary matrix, $\boldsymbol{R_1}$ is an r by n , non singular, upper trapezoidal matrix, and $\boldsymbol{0}$ matrix of size $(m\text{-}r)$ by n.

In this chapter we will follow the notation in [19]. In [19] a complex Hankel matrix is formulated from the noisy data rescored, rank revealing QR factorization (RRQR) was applied, and then the null space or the noise space is found in order to apply one of the frequency estimation functions (MUSIC). The contribution of this thesis is to transform the complex data matrix in step one in the generic algorithms to real data matrix. This conversion would be achieved via the unitary transformation in [11] and [12]. This reduces the processing time by almost a factor of four, since the cost of complex multiplication is four times that of real multiplication.

3.2 Frequency Estimation using Rank-Revealing QR factorization

Using the *N* points of the received noisy data vector given by (2.2), to form the Hankel Matrix $\boldsymbol{X}$ of size $L \times L$ assuming that *N* is an odd number (*N=2L-1*)

$$\boldsymbol{X} = \begin{bmatrix} x(0) & x(1) & \dots & x(L-1) \\ x(1) & x(2) & \dots & x(L) \\ \vdots & \vdots & \ddots & \vdots \\ x(L-1) & x(L) & \dots & x(N-1) \end{bmatrix} \tag{3.1}$$

Using (2.3) we may write (3.1) as

$$\boldsymbol{X} = \boldsymbol{S} + \boldsymbol{W}$$

where

$$\boldsymbol{S} = \begin{bmatrix} s(0) & s(1) & \dots & s(L-1) \\ s(1) & s(2) & \dots & s(L) \\ \vdots & \vdots & \ddots & \vdots \\ s(L-1) & s(L) & \dots & s(N-1) \end{bmatrix}$$

$$\boldsymbol{W} = \begin{bmatrix} w(0) & w(1) & \dots & w(L-1) \\ w(1) & w(2) & \dots & w(L) \\ \vdots & \vdots & \ddots & \vdots \\ w(L-1) & w(L) & \dots & w(N-1) \end{bmatrix}$$

Equation (3.1) can be rewritten as:

$$\boldsymbol{X} = [\boldsymbol{r}(0)\ \boldsymbol{r}(1) \dots \dots \boldsymbol{r}(L-1)]$$

where the i^{th} column of $\boldsymbol{X}$ is given by:

$$\boldsymbol{r}(i) = \boldsymbol{A}(\omega)(\boldsymbol{\varphi}(\omega))^i\boldsymbol{a} + \boldsymbol{w}(i)$$

$$i = 0,1,\dots,L-1$$

$$\boldsymbol{A}(\omega) = \begin{bmatrix} 1 & 1 & \cdots & 1 \\ e^{j\omega_1} & e^{j\omega_2} & \cdots & e^{j\omega_P} \\ \vdots & \vdots & \ddots & \vdots \\ e^{j(L-1)\omega_1} & e^{j(L-1)\omega_2} & \cdots & e^{j(L-1)\omega_P} \end{bmatrix}$$

$$\boldsymbol{\varphi}(\omega) = diag\left(e^{j\omega_1} \;\; e^{j\omega_2} \dots \;\; e^{j\omega_P} \right)$$

$$\boldsymbol{a} = [a_1, a_2, a_3, \dots, a_P]^{\mathrm{T}} \tag{3.2}$$

where $\boldsymbol{w}(i)$ is the i^{th} column of $\boldsymbol{W}$. We can formulate the received data matrix as:

$$\boldsymbol{X} = \left[\boldsymbol{A}(\omega)\boldsymbol{a} \;\; \boldsymbol{A}(\boldsymbol{\omega})\boldsymbol{\varphi}(\omega)\boldsymbol{a} \dots \dots \boldsymbol{A}(\omega)\left(\boldsymbol{\varphi}(\omega)\right)^{L-1}\boldsymbol{a}\right] + \boldsymbol{W}$$

$$= \boldsymbol{A}(\omega)\left[\boldsymbol{a} \;\; \boldsymbol{\varphi}(\omega)\boldsymbol{a} \dots \left(\boldsymbol{\varphi}(\omega)\right)^{L-1}\boldsymbol{a}\right] + \boldsymbol{W} \tag{3.3}$$

The signal space of the Hankel data matrix in (3.3) has full rank which implies that all the incident sources frequencies can be detected. Define matrix

$$\boldsymbol{\Psi} = \left[\boldsymbol{a} \;\; \boldsymbol{\varphi}(\omega)\boldsymbol{a} \dots \dots \left(\boldsymbol{\varphi}(\omega)\right)^{L-1}\boldsymbol{a}\right]$$

$$= \begin{bmatrix} a_1 & & 0 \\ & a_2 & \\ & & \ddots \\ 0 & & & a_p \end{bmatrix} \cdot \begin{bmatrix} 1 & e^{j\omega_1} & e^{j2\omega_1} & \cdots & e^{j(L-1)\omega_1} \\ 1 & e^{j\omega_2} & e^{j2\omega_2} & \cdots & e^{j(L-1)\omega_2} \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ 1 & e^{j\omega_p} & e^{j2\omega_p} & \cdots & e^{j(L-1)\omega_p} \end{bmatrix}$$

$$= \boldsymbol{\alpha}\boldsymbol{A}^T(\omega) \tag{3.4}$$

where $(.)^T$ denotes matrix transpose, and

$$\boldsymbol{\alpha} = \begin{bmatrix} a_1 & & 0 \\ & a_2 & \\ & & \ddots \\ 0 & & & a_p \end{bmatrix}$$

The signal part can be written as

$$\boldsymbol{S} = \boldsymbol{A}(\omega).\boldsymbol{\Psi}$$

$$\boldsymbol{S} = \boldsymbol{A}(\omega).\boldsymbol{\alpha}.\boldsymbol{A}^{\boldsymbol{T}}(\omega) \tag{3.5}$$

Using the fact,

$$\textit{if } \boldsymbol{A} \in R^{m\times n}, \boldsymbol{B} \in R^{n\times n} \textit{ and } \boldsymbol{C} = \boldsymbol{A}.\boldsymbol{B}.\boldsymbol{A}^{T} \textit{ then}$$

$$rank(\boldsymbol{C}) = min\{rank(\boldsymbol{A}), rank(\boldsymbol{B})\} = min\{m, n\}$$

It is obvious that the signal part (3.3) is equal to rank of $\boldsymbol{\Psi}$ in (3.4). The matrix $\boldsymbol{A}^{T}(\omega)$ has the structure of the Vandermonde matrix with rank $P \leq L$. This rank represents the number of sources in the received signal. When applying QR factorization to the Hankel data matrix $\boldsymbol{X}$, it can be expressed as a product of a unitary matrix and rank-revealing upper triangular matrix as

$$\boldsymbol{X} = \boldsymbol{QR} = [\boldsymbol{Q_1}\ \ \boldsymbol{Q_2}]\begin{bmatrix} \boldsymbol{R_{11}} & \boldsymbol{R_{12}} \\ \boldsymbol{0} & \boldsymbol{R_{22}} \end{bmatrix} \tag{3.6}$$

where

$\boldsymbol{Q}_1$ is $L \times P$ Unitary matrix

$\boldsymbol{Q}_2$ is $L \times (L-P)$ Unitary matrix

$\boldsymbol{R}_{11}$ is $P \times P$ upper triangular matrix

$\boldsymbol{R}_{12}$ is $P \times (L-P)$ matrix

$\boldsymbol{0}$ is $(L-P) \times P$ matrix

$\boldsymbol{R}_{22}$ is $(L-P) \times (L-P)$ upper triangular matrix

Since $\boldsymbol{R}_{22}$ has small norm we can easily extract the basis of the noise space form matrix

$$\widetilde{\boldsymbol{R}} = [\boldsymbol{R}_{11} \quad \boldsymbol{R}_{12}] \tag{3.7}$$

The Hankel data matrix $\boldsymbol{X}$ (3.6) is approximated by

$$\boldsymbol{X} \approx \boldsymbol{Q_1}\widetilde{\boldsymbol{R}} = \boldsymbol{Q_1}.[\boldsymbol{R}_{11} \quad \boldsymbol{R}_{12}] \tag{3.8}$$

Here the matrix $\boldsymbol{G}$ is defined as the null space of $\boldsymbol{X}$

$$\boldsymbol{G} = N(\boldsymbol{X}) \tag{3.9}$$

Clearly, any vector that belongs to null space should satisfy

$$\boldsymbol{XG} = \boldsymbol{0} \tag{3.10}$$

Using (3.8), and assuming that $= \begin{bmatrix} \boldsymbol{g}_1 \\ \boldsymbol{g}_2 \end{bmatrix}$, with none zero $\boldsymbol{g}_1, \boldsymbol{g}_2$ we may rewrite (3.10)

$$\boldsymbol{XG} = \boldsymbol{Q}_1.\begin{bmatrix} \boldsymbol{R}_{11} & \boldsymbol{R}_{12} \end{bmatrix} \begin{bmatrix} \boldsymbol{g}_1 \\ \boldsymbol{g}_2 \end{bmatrix} = \boldsymbol{0}$$

Or simply

$$\begin{bmatrix} \boldsymbol{R}_{11} & \boldsymbol{R}_{12} \end{bmatrix} \begin{bmatrix} \boldsymbol{g}_1 \\ \boldsymbol{g}_2 \end{bmatrix} = \boldsymbol{0}$$

$$\boldsymbol{R}_{11}\boldsymbol{g}_1 + \boldsymbol{R}_{12}\boldsymbol{g}_2 = \boldsymbol{0} \qquad (3.11)$$

so that $\boldsymbol{R}_{11}\boldsymbol{g}_1 = -\boldsymbol{R}_{12}\boldsymbol{g}_2$ Since $\boldsymbol{R}_{11}$ is an invertible matrix, $\boldsymbol{g}_1$ can be written in terms of $\boldsymbol{g}_2$ as

$$\boldsymbol{g}_1 = -\boldsymbol{R}_{11}^{-1}\boldsymbol{R}_{12}\boldsymbol{g}_2$$

Then $\boldsymbol{G}$ can be written as

$$\boldsymbol{G} = \begin{bmatrix} \boldsymbol{g}_1 \\ \boldsymbol{g}_2 \end{bmatrix} = \begin{bmatrix} -\boldsymbol{R}_{11}^{-1}\boldsymbol{R}_{12} \\ \boldsymbol{I}_{(L-P)} \end{bmatrix} \boldsymbol{g}_2 = \boldsymbol{H}\boldsymbol{g}_2 \qquad (3.12)$$

Substitute (3.12) in (3.10)

$$\boldsymbol{XH}\boldsymbol{g}_2 = \boldsymbol{0}$$

Or simply

$$\boldsymbol{X}.\boldsymbol{H} = \boldsymbol{0} \qquad (3.13)$$

Here the matrix $\boldsymbol{H}$ is defined as the null space of $\boldsymbol{X}$

$$\boldsymbol{H} = N(\boldsymbol{X})$$

Using (3.5) in (3.13)

$$\boldsymbol{A}(\omega).\boldsymbol{\alpha}.\boldsymbol{A}^{\boldsymbol{T}}(\omega).\boldsymbol{H} = \boldsymbol{0}$$

Or simply

$$\boldsymbol{A}^{T}(\omega).\boldsymbol{H} = \boldsymbol{0} \qquad (3.14)$$

It can be observed here that the columns of the basis of the null space $\boldsymbol{H}$ are not orthonormal. To satisfy orthonormality we use orthogonal projection onto this subspace in order to improve the performance by making the basis of null space of $\boldsymbol{H}$ orthonormal.

$$\boldsymbol{H}_o = \boldsymbol{H}(\boldsymbol{H}^H\boldsymbol{H})^{-1}\boldsymbol{H}^H \tag{3.15}$$

The columns of the $L \times L$ square matrix $\boldsymbol{H}_o$ are forming the bases of the null subspace, and it may be written as:

$$\boldsymbol{H}_0 = [\boldsymbol{h}_o^1 \quad \boldsymbol{h}_o^2 \quad \dots \quad \boldsymbol{h}_o^L]$$

Apply MUSIC-like search algorithm [2] to estimate the frequencies using the following function

$$\hat{P}_{MU}(e^{j\omega}) = \frac{1}{\sum_{i=1}^{L} \left|\boldsymbol{e}^H.\boldsymbol{h}_o^i\right|^2} \tag{3.16}$$

where $\boldsymbol{e}$ is the search vector given by:

$$\boldsymbol{e} = [1 \quad e^{j\omega} \quad e^{j2\omega} \quad \dots \quad e^{jP\omega}]^T$$

Instead of searching for the peaks in (3.16), an alternative is to use a root-MUSIC [13]. The frequency estimates may be taken to be the angles of the p roots of the polynomial D(z) that are closest to the unit circle

$$D(z) = \sum_{i=0}^{L-1} \boldsymbol{V}_i(z)\boldsymbol{V}_i^*(1/z^*) \tag{3.17}$$

where $\boldsymbol{V}_i(z)$ is the Z-transform of $\boldsymbol{h}_o^i$.

$$\boldsymbol{h}_o^i = \begin{bmatrix} h_o^{1i} \\ h_o^{2i} \\ \vdots \\ h_o^{Li} \end{bmatrix}$$

3.3 Frequency Estimation using Real data with Rank-Revealing QR Factorization

Convert the Hankel matrix of size $L \times L$ in (3.1) to the Toeplitz matrix $\boldsymbol{Z}$ of the same size using the $L \times L$ exchange matrix:

$$\overline{\boldsymbol{X}} = \boldsymbol{J}.\boldsymbol{X}$$

$$= \boldsymbol{J}.(\boldsymbol{S} + \boldsymbol{W}) = \overline{\boldsymbol{S}} + \overline{\boldsymbol{W}} \qquad (3.18)$$

where

$$\overline{\boldsymbol{X}} = \begin{bmatrix} x(L-1) & x(L) & \dots & x(N-1) \\ x(L-2) & x(L-1) & \dots & x(N-2) \\ \vdots & \vdots & \ddots & \vdots \\ x(0) & x(1) & \dots & x(L-1) \end{bmatrix}$$

Equation (3.18) can be rewritten as

$$\overline{\boldsymbol{X}} = [\boldsymbol{c}(0)\ \boldsymbol{c}(1) \dots \dots \dots \boldsymbol{c}(L-1)]$$

where the i^{th} column of $\overline{\boldsymbol{X}}$ is given by

$$\boldsymbol{c}(i) = \boldsymbol{J}\boldsymbol{A}_{\boldsymbol{L}}(\omega)(\boldsymbol{\varphi}\,(\omega))^i\boldsymbol{a} + \overline{\boldsymbol{w}}_i, \qquad i = 0,1, \dots, L-1$$

where

$$\boldsymbol{A}_{\boldsymbol{L}}(\omega) = \begin{bmatrix} 1 & 1 & \dots & 1 \\ e^{j\omega_1} & e^{j\omega_2} & \dots & e^{j\omega_P} \\ \vdots & \vdots & \ddots & \vdots \\ e^{j(L-1)\omega_1} & e^{j(L-1)\omega_2} & \dots & e^{j(L-1)\omega_P} \end{bmatrix}$$

$$\boldsymbol{\varphi}(\omega) = diag\left(e^{j\omega_1}\ \ e^{j\omega_2} \dots \dots\ \ e^{j\omega_P} \right)$$

$$\boldsymbol{a} = [a_1, a_2, \dots \dots a_P]^T$$

$$\boldsymbol{w}_i = [w(i+L-1)\ w(i+1) \dots \dots w(i)]^T \qquad (3.19)$$

The received data matrix can be reformulated as:

$$\overline{\boldsymbol{X}} = \left[\boldsymbol{A}_{\boldsymbol{L}}(\omega)\boldsymbol{a}\ \ \boldsymbol{A}_{\boldsymbol{L}}(\omega)\boldsymbol{\varphi}\,(\omega)\boldsymbol{a} \dots \dots \boldsymbol{A}_{\boldsymbol{L}}(\omega)\left(\boldsymbol{\varphi}\,(\omega)\right)^{N-L}\boldsymbol{a}\right] + \overline{\boldsymbol{U}} \qquad (3.20)$$

Let $\boldsymbol{J}_k$ represents the exchange matrix (given in the appendix) and $\boldsymbol{I}_k$ represents the identity matrix of size $k \times k$. Let L defined as

$$L = \begin{cases} 2M & if\ L\ is\ even \\ \\ 2M+1 & if\ L\ is\ odd \end{cases} \tag{3.21}$$

Let **U** be an $L\times L$ unitary matrix defined as

$$\boldsymbol{U} = \begin{cases} \frac{1}{\sqrt{2}}\begin{bmatrix} \boldsymbol{I}_M & \boldsymbol{J}_M \\ i\boldsymbol{J}_M & -i\boldsymbol{I}_M \end{bmatrix} & if\ L\ is\ even(L=2M) \\ \\ \frac{1}{\sqrt{2}}\begin{bmatrix} \boldsymbol{I}_M & \boldsymbol{0}_{M\times 1} & \boldsymbol{J}_M \\ \boldsymbol{0}_{1\times M} & \sqrt{2} & \boldsymbol{0}_{1\times M} \\ i\boldsymbol{J}_M & \boldsymbol{0}_{M\times 1} & -i\boldsymbol{I}_M \end{bmatrix} & if\ L\ is\ odd(L=2M+1) \end{cases} \tag{3.22}$$

Matrix **U** satisfied (The proof is shown in the appendix A.3, A.4)

$$\boldsymbol{U}\boldsymbol{U}^H = \boldsymbol{I} \tag{3.23}$$

$$\boldsymbol{U}^*\boldsymbol{J} = \boldsymbol{U}\ \boldsymbol{I} \tag{3.24}$$

Let

$$\boldsymbol{\Psi} = \overline{\boldsymbol{X}} + \boldsymbol{J}\overline{\boldsymbol{X}}^*\boldsymbol{J}$$

Construct the output real data matrix $\boldsymbol{X}^{real}$ of size $L\times L$ by pre multiplying $\boldsymbol{\Psi}$ by $\boldsymbol{U}$ and post multiply by $\boldsymbol{U}^H$

$$\boldsymbol{X}^{real} = \boldsymbol{U}\boldsymbol{\Psi}\boldsymbol{U}^H \tag{3.25}$$

To show that $\boldsymbol{X}^{real}$ is indeed a real matrix, the properties of the unitary, exchange and Toeplitz matrices would be used to show that

$$\boldsymbol{X}^{real} = (\boldsymbol{X}^{real})^*$$

The proof is shown in appendix (A.5). The decomposition of $\boldsymbol{U\Psi U^H}$ requires only a real computation which means the computational load and cost will reduce significantly without effecting the accuracy of the frequencies. The RRQR factorization is applied for $\boldsymbol{X}^{real}$ given by (3.25), and the bases of the null space are extracted similar to (3.15). The columns of the $L\times L$ square matrix $\boldsymbol{H}_o$ are forming the bases of the null subspace before transformation. Apply unitary transformation on the basis of the original noise space by pre-multiplying $\boldsymbol{H}_0$ by $\boldsymbol{U}$ and post-multiply by $\boldsymbol{U}^H$

$$\boldsymbol{\Lambda} = \boldsymbol{U}\, \boldsymbol{H}_0 \boldsymbol{U}^H$$

where the columns of $\boldsymbol{\Lambda}$ are given by $\boldsymbol{\Lambda} = [\Lambda^1 \quad \Lambda^2 \quad \ldots \quad \Lambda^L]$. Apply MUSIC like search algorithm [2] to estimate the frequencies using the following function:

$$\hat{P}_{MU}(e^{j\omega}) = \left[\sum_{i=1}^{L} \left|\boldsymbol{e}^H . \Lambda^i\right|^2\right]^{-1} \tag{3.26}$$

where $\boldsymbol{e} = [1 \quad e^{j\omega} \quad e^{j2\omega} \quad \ldots \quad e^{jP\omega}]^T$ is the search vector. Instead of searching for the peaks in (26) an alternative is to use the root-MUSIC [30]; where the power spectrum of the MUSIC algorithm is converted into a polynomial whose roots contain information about the frequencies. The p roots of the polynomial $D(z)$ which are closest to the unit circle

$$D(z) = \sum_{i=1}^{L} \boldsymbol{V}_i(z)\boldsymbol{V}_i^*(1/z^*) \tag{3.27}$$

where $\boldsymbol{V}_i(z)$ is the z-transform of Λ^i.

3.4 SIMULATION RESULTS

Extensive computer simulations were done to validate our proposed method. In the First experiment we considered one complex sinusoidal signal with amplitudes $a_1 = 1$, $\omega_1 = 1.5$ rad/s. We simulated the performance under AWGN environment. The signal model is given by

$$x(n) = A_1 e^{jn\omega_1} + w(n) = e^{j(1.5)n} + w(n) \qquad n = 0,1,2,\ldots,N-1$$

The number of data points is assumed to be 199. In Figure 1 the spectrum of the reference method [19] (using the Hankel complex matrix) has an overlay plot of 500 spectrum estimates at

SNR =0dB. In Figure 2 the spectrum of the proposed method (real data) has an overlay plot of 500 spectrum estimates. The two figures are showing similar results to each other. Figure 3 plots the average spectrum of the reference and the proposed method with 500 realizations at 0 dB signal to noise ratios (SNR). Figure 4 plots the average spectrum estimate of 500 realizations for the two methods at 10 dB signal to noise ratios. Figure 5 plots the performance curve of the reference and the proposed methods in terms of MSE as a function of SNR with *100* realizations and *N=499*. From this figure, it is obvious that we almost obtained the same performance. In order to see the effect of the data record length, Figures *6, 7* and *8* plot the performance curve of the reference and the proposed methods with *50* realizations at *10, 20*, and *5* dB, respetivly as a function of the length of data record. Again, almost similar results are obtained for the two methods.

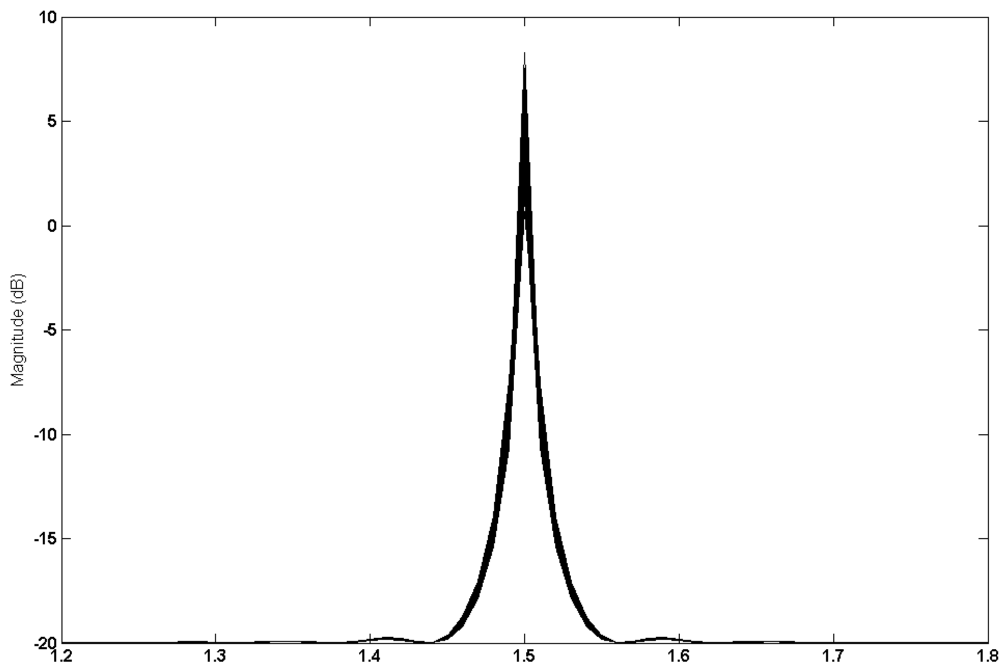

Figure 3.1 The spectrum of the reference method with overlay plot of 500 spectrum estimates at SNR =0dB

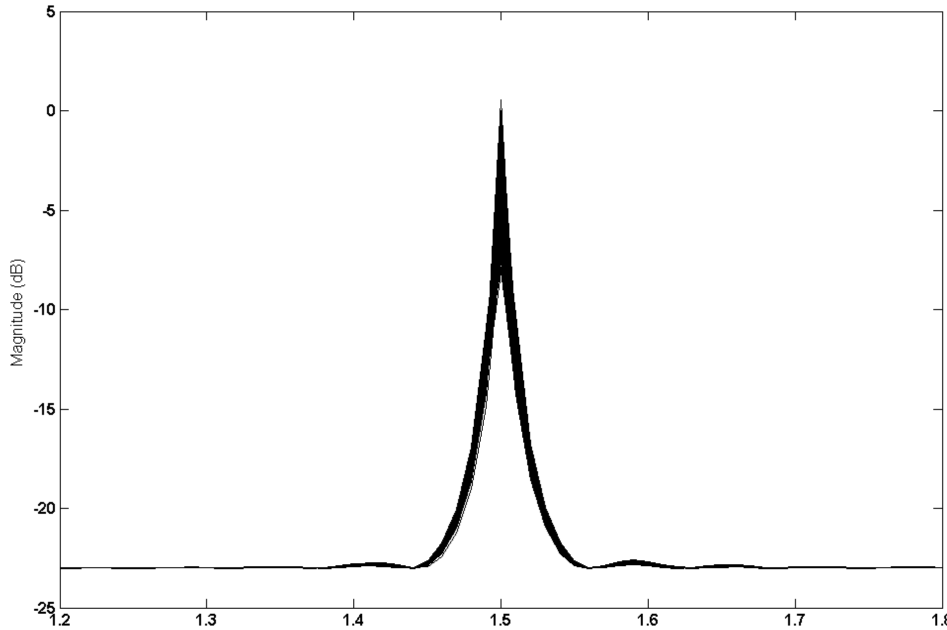

Figure 3.2 The spectrum of the proposed method with overlay plot of 500 spectrum estimates at SNR =0dB

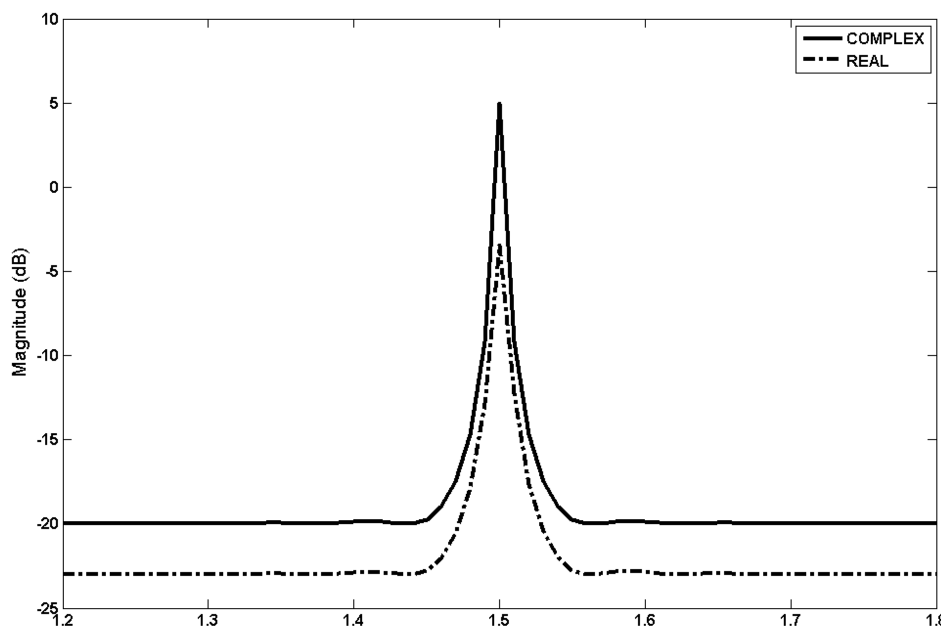

Figure 3.3 The average spectrum of the reference and the proposed methods with 500 realizations at 0 dB.

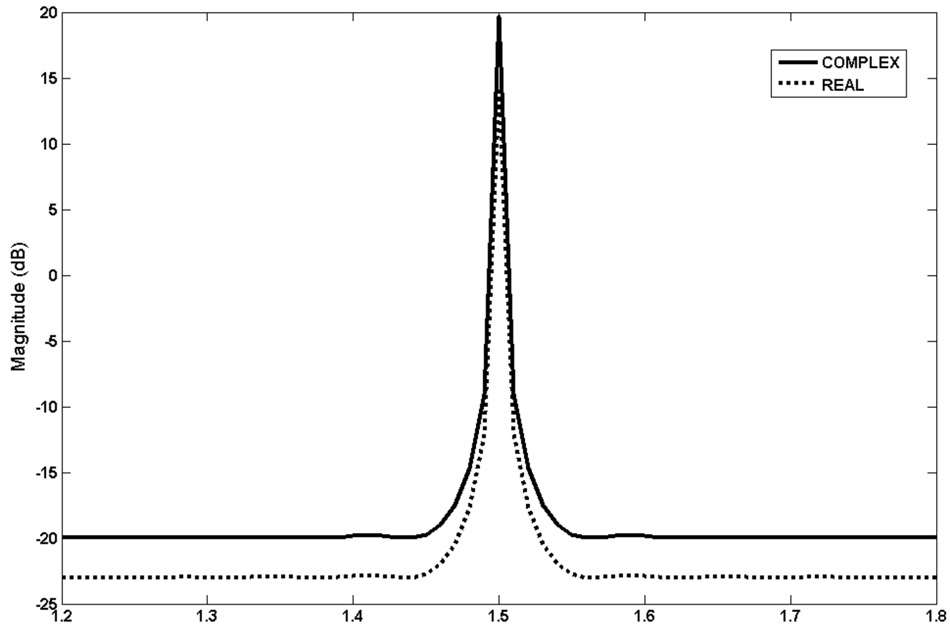

Figure 3.4 The average spectrum of the reference and the proposed methods with 500 realizations at 10 dB.

The performance of the estimators is measured by the mean square error (MSE) which is defined as:

$$MSE_{dB} = 10log_{10}\left(\frac{1}{N_t P}\sum_{i=1}^{N_t}\sum_{j=1}^{P}\left(\omega_j - \widehat{\omega}_j\right)^2\right) \tag{3.27}$$

where $\widehat{\omega}_j$ is the estimate of ω_j, and N_t is the number of Monte Carlo trials. The signal to noise ratio (SNR) is defined as:

$$\text{SNR} = 20\log\frac{|s(n)|}{|w(n)|} \tag{3.28}$$

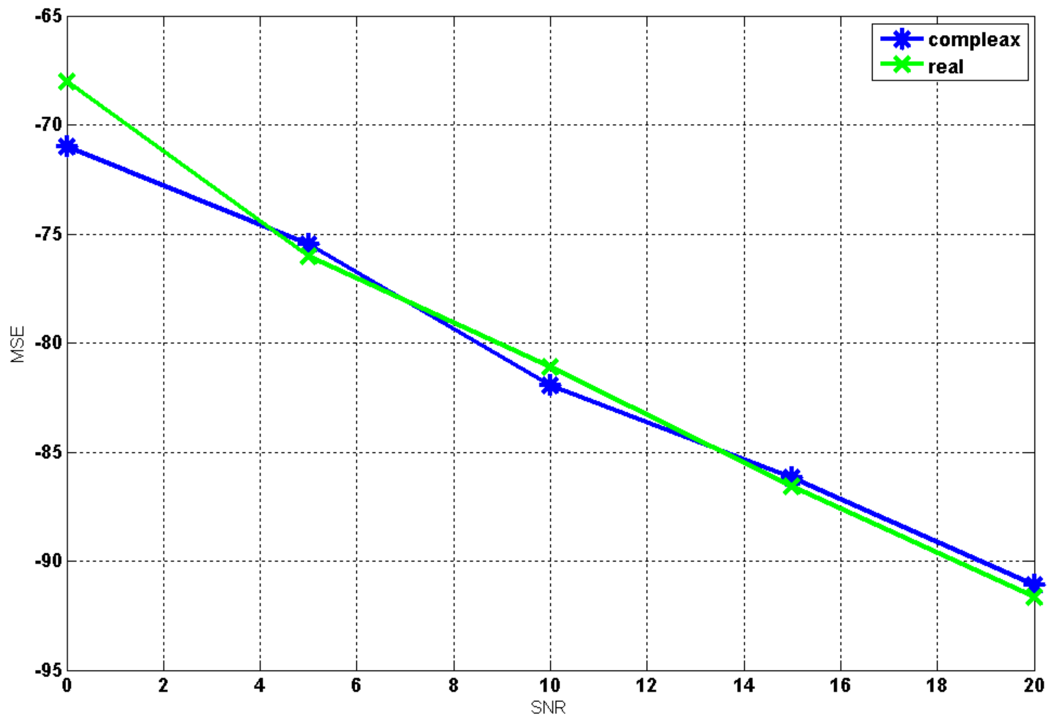

Figure 3.5 The performance curve of the reference and the proposed methods with 100 realizations with N=499.

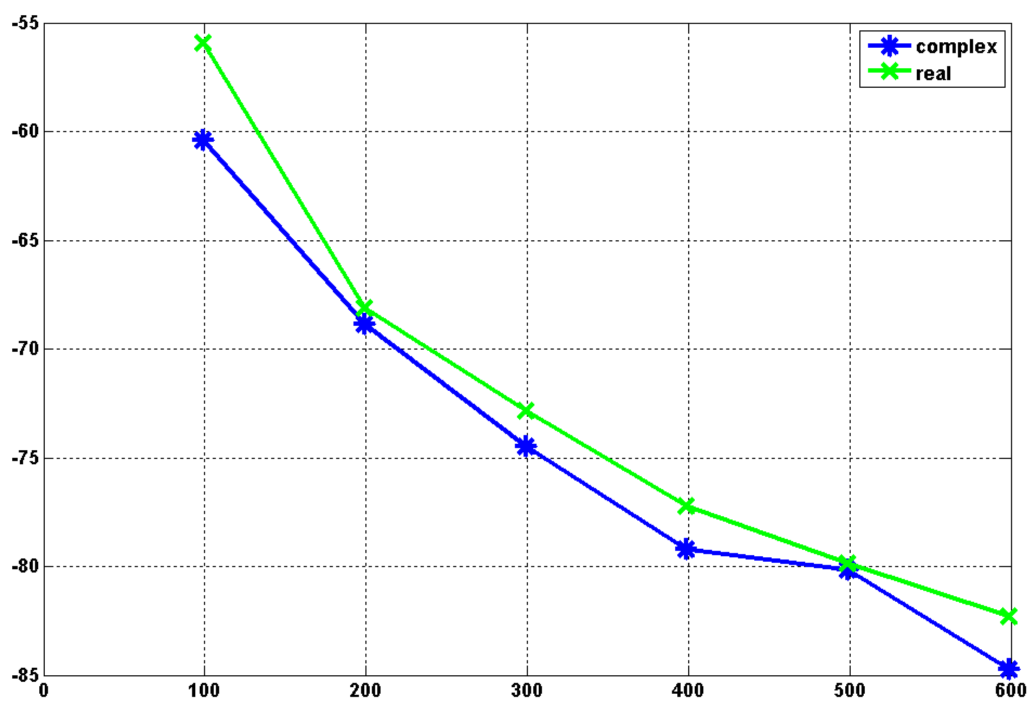

Figure 3.6 The performance curve of the reference and the proposed methods with 50 realizations at 10 dB.(P=1, *a=1, w1=1.2*)

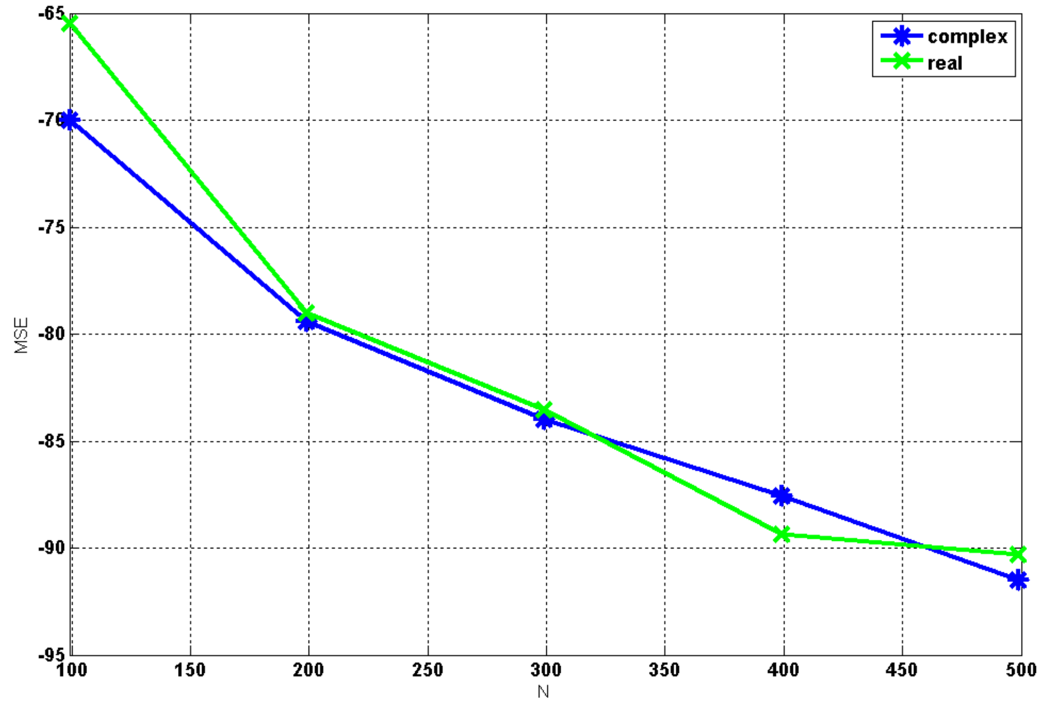

Figure 3.7 The performance curve of the reference and the proposed methods with 100 realizations at 20 dB.(P=1, *a=1, w1=1.2*)

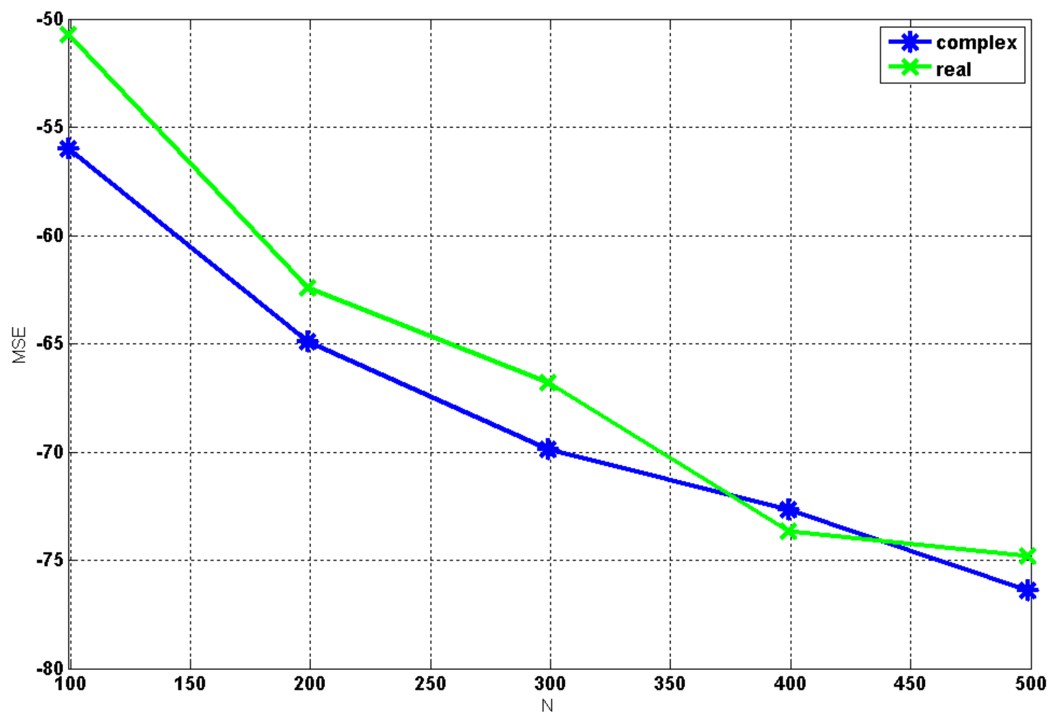

Figure 3.8 The performance curve of the reference and the proposed methods with 100 realizations at 5 dB.(P=1, *a=1, w1=1.2*)

In the second experiment we considered two complex sinusoidal signals with amplitudes $a_1 = \frac{1}{\sqrt{2}}$,$\omega_1 = 1.2$ rad/s, $a_2 = \frac{1}{\sqrt{2}}$, $\omega_2 = 1.5$ rad/s. We simulated the performance under AWGN environment. The signal model is given by

$$x(n) = a_1 e^{jn\omega_1} + a_2 e^{jn\omega_2} + w(n)$$

$$= \frac{1}{\sqrt{2}} . e^{j(1.2)n} + \frac{1}{\sqrt{2}} . e^{j(1.5)n} + w(n) \qquad n = 0,1,2, \dots , N-1$$

The number of data points is assumed to be 199. In Figure 9 the spectrum of the reference method has an overlay plot of 500 spectrum estimates at SNR =0dB. In Figure 10 the spectrum of the proposed method has an overlay plot of 500 spectrum estimates at SNR =0dB. Figures 11, 12, and 13 plot the average spectrum of the reference and the proposed method, with 500 realizations at 10, 5, and 0 dB.

Figures 14 and 15 plot the performance curve of the reference and the proposed methods using 100 realizations with N=299 and 499. Figures 16, 17 and 18 plot the performance curve of the reference and the proposed methods with 100 realizations at 20, 10, and dB.

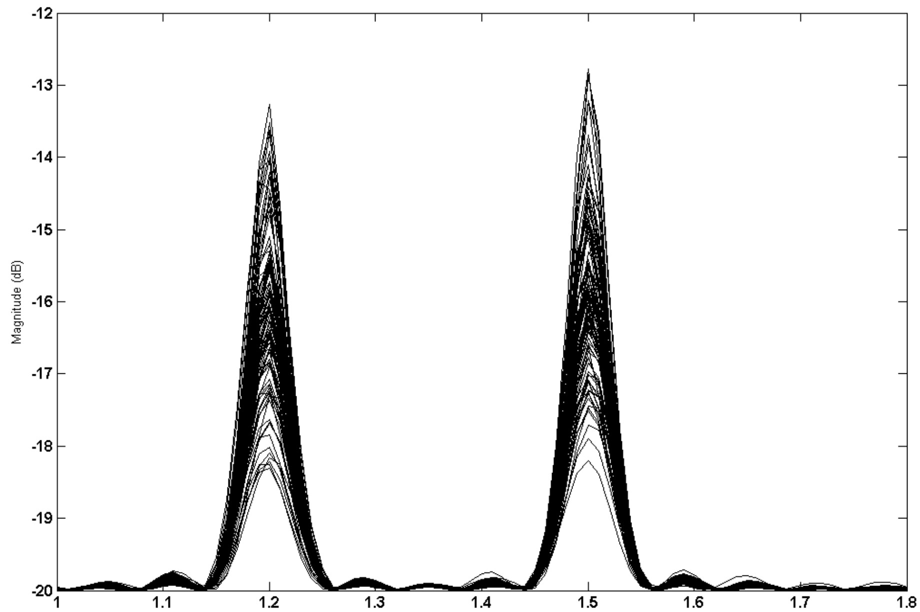

Figure 3.9 The spectrum of the reference method of two complex frequencies in white noise with overlay plot of 500 spectrum estimates at SNR =0dB

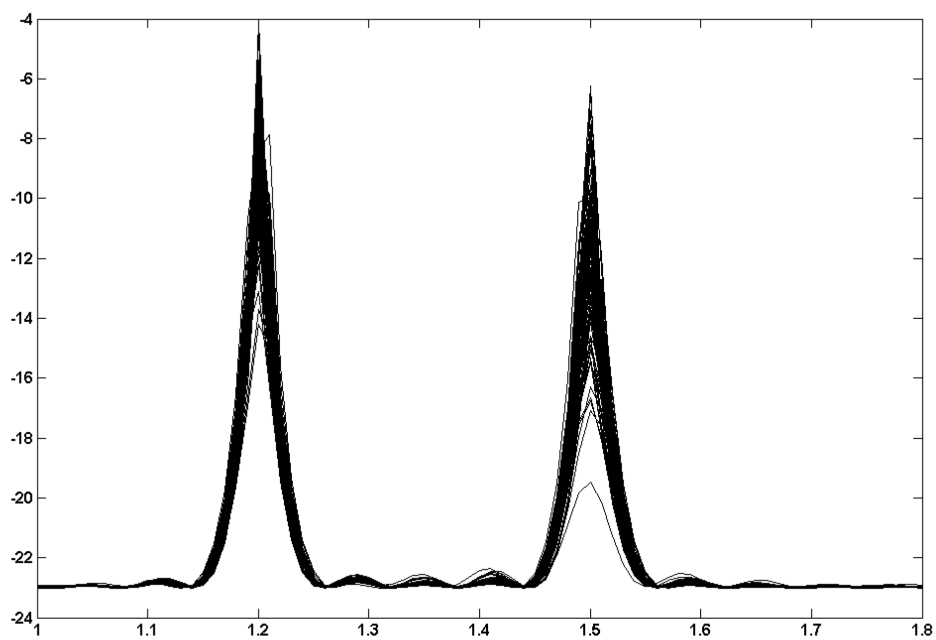

Figure 3.10 The spectrum of the proposed method of two complex frequencies in white noise with overlay plot of 500 spectrum estimates at SNR =0dB

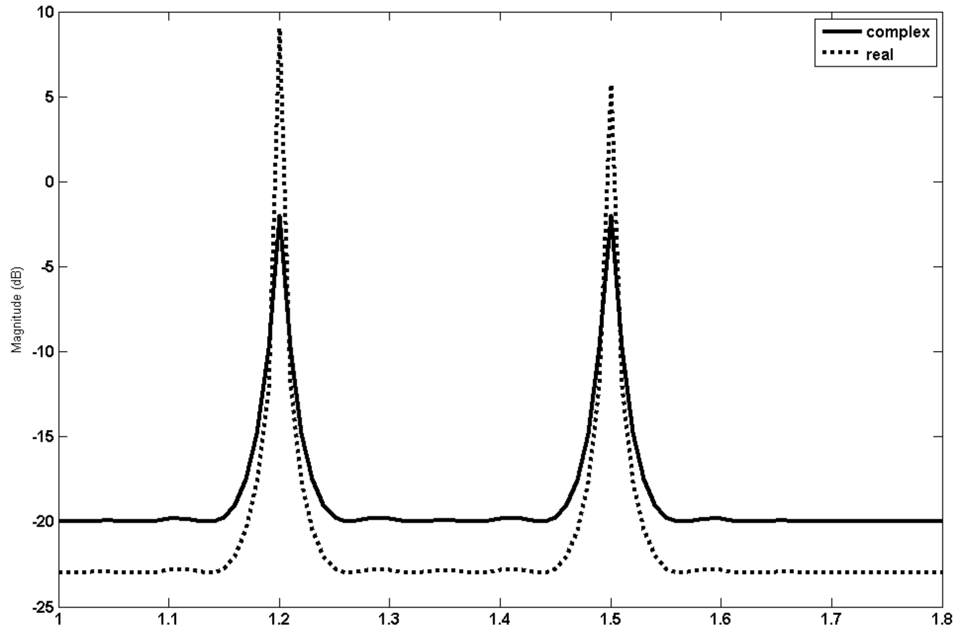

Figure 3.11 The average spectrum of the reference and the proposed methods with 500 realizations at 10 dB.

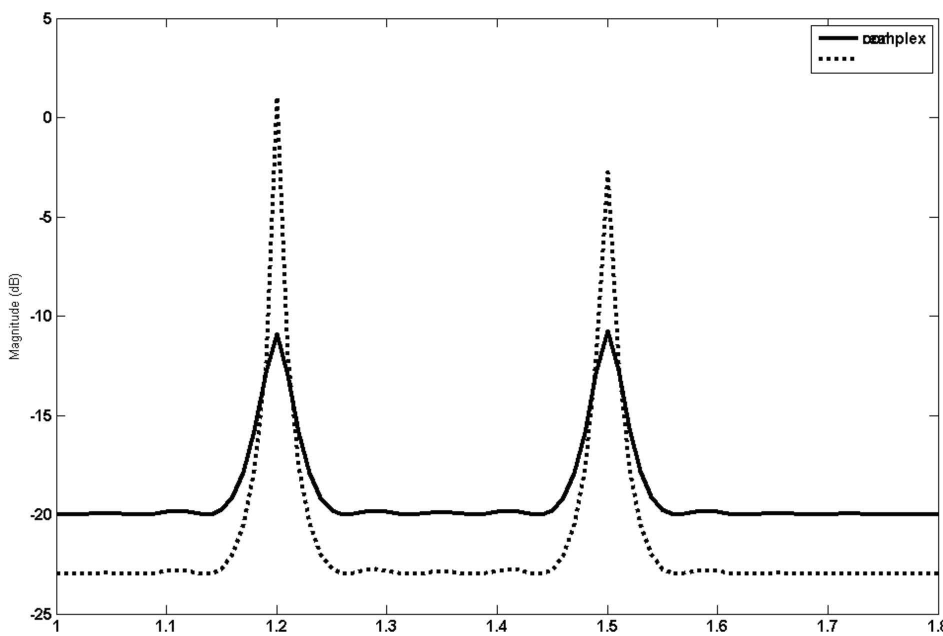

Figure 3.12 The average spectrum of the reference and the proposed Methods using 500 realizations at 5 dB.

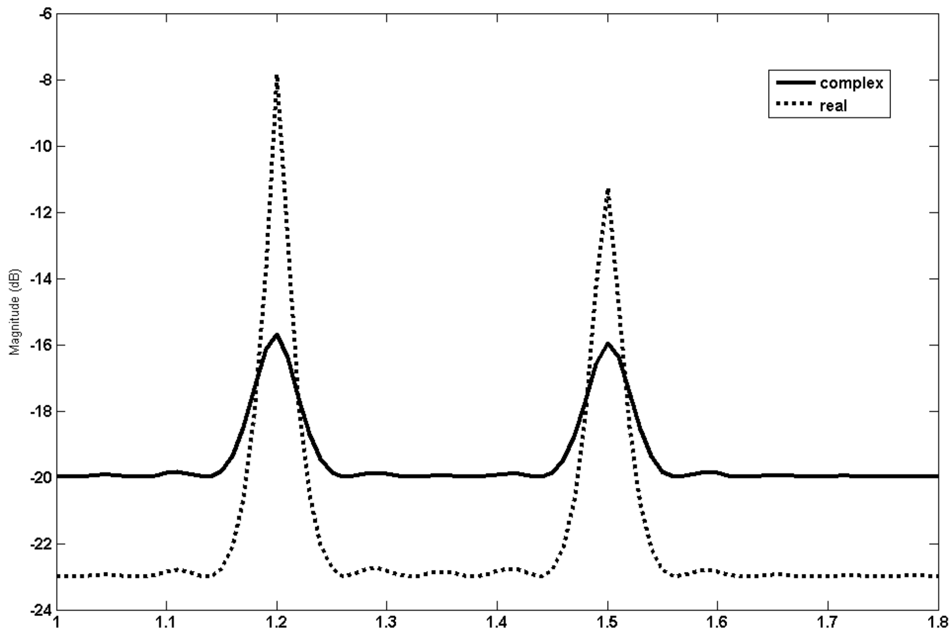

Figure 3.13 The average spectrum of the reference and the proposed methods with 500 realizations at 0 dB.

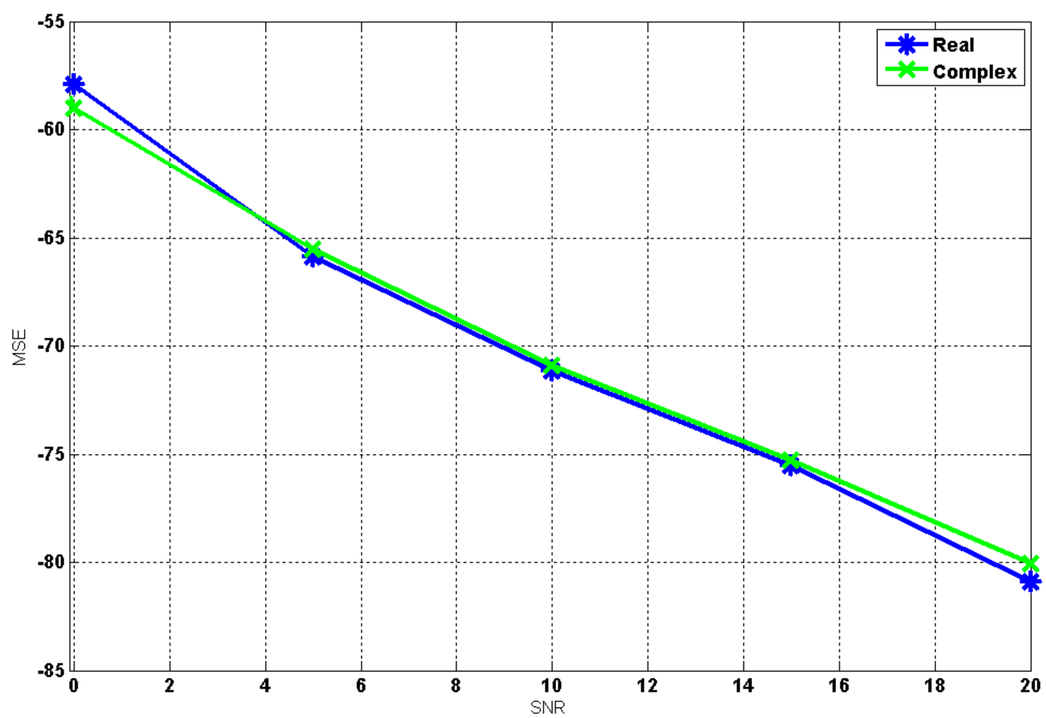

Figure 3.14 The performance curve of the reference and the proposed methods using 100 realizations with N=299.

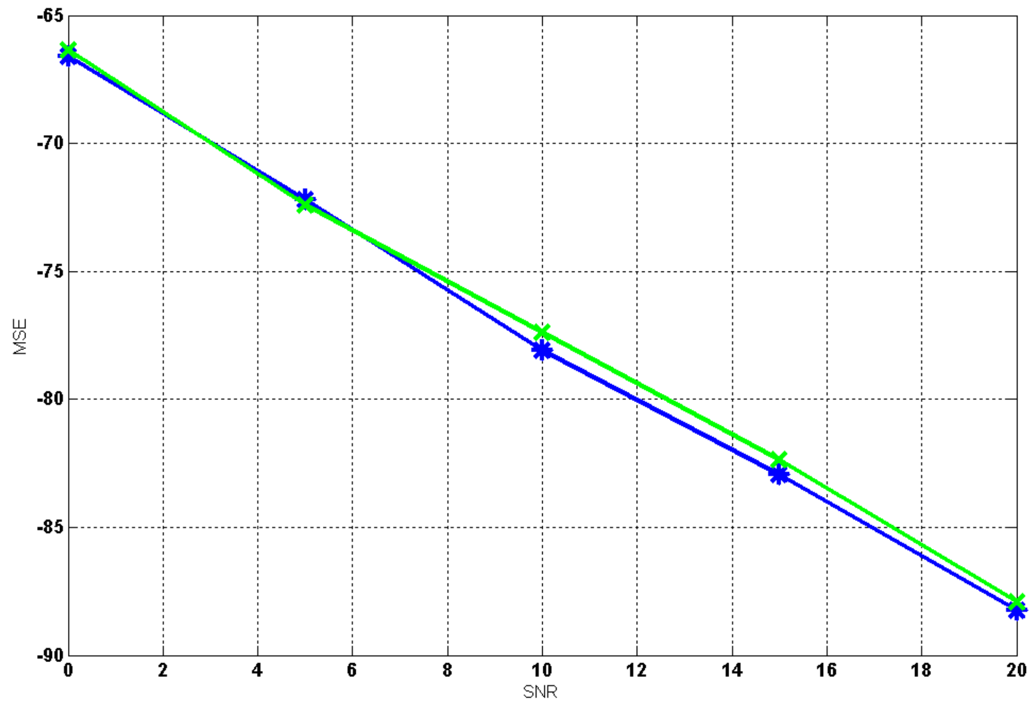

Figure 3.15 The performance curve of the reference and the proposed methods with 100 realizations with N=499.

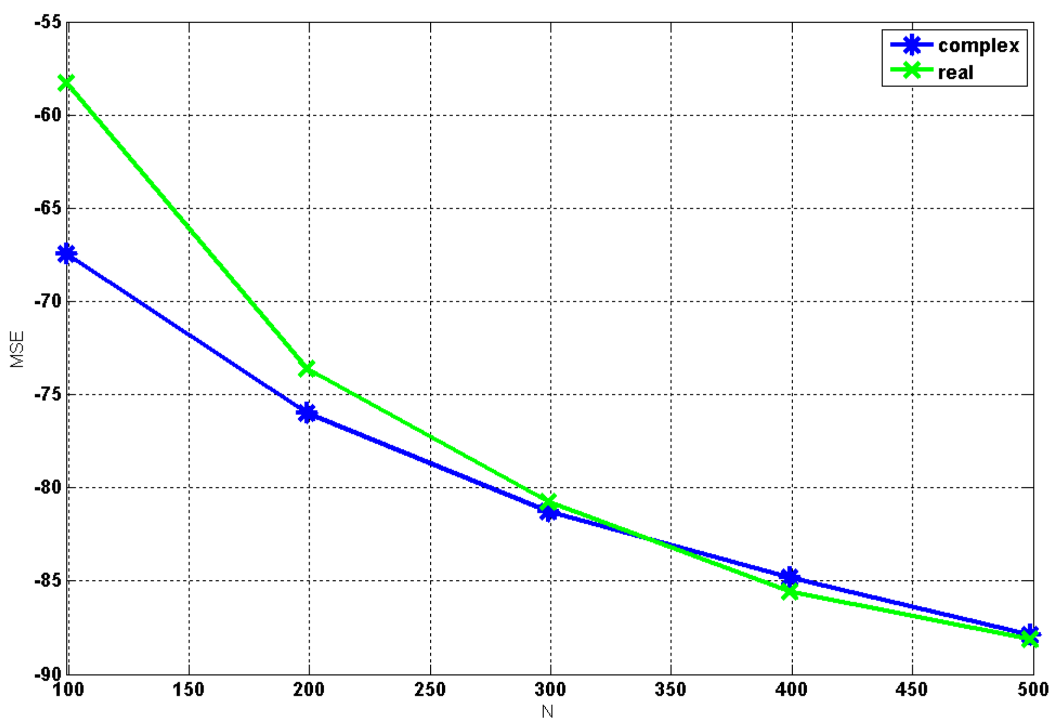

Figure 3.16 The performance curve of the reference and the proposed methods with 100 realizations at 20 dB.

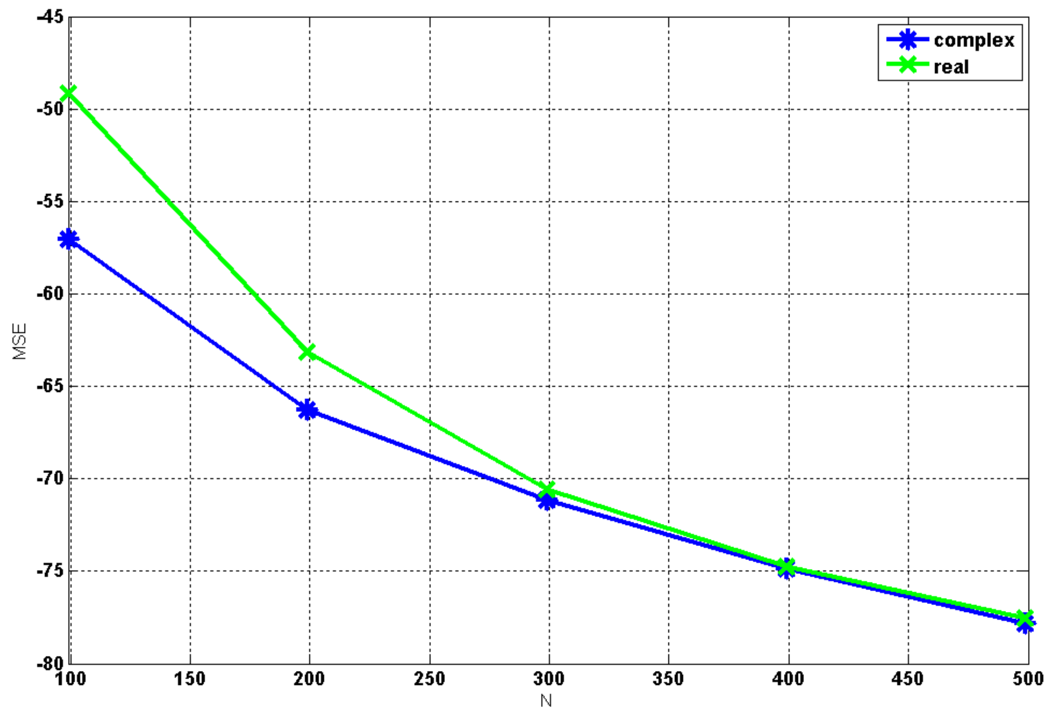

Figure 3.17 The performance curve of the reference and the proposed methods with 100 realizations at 10 dB.

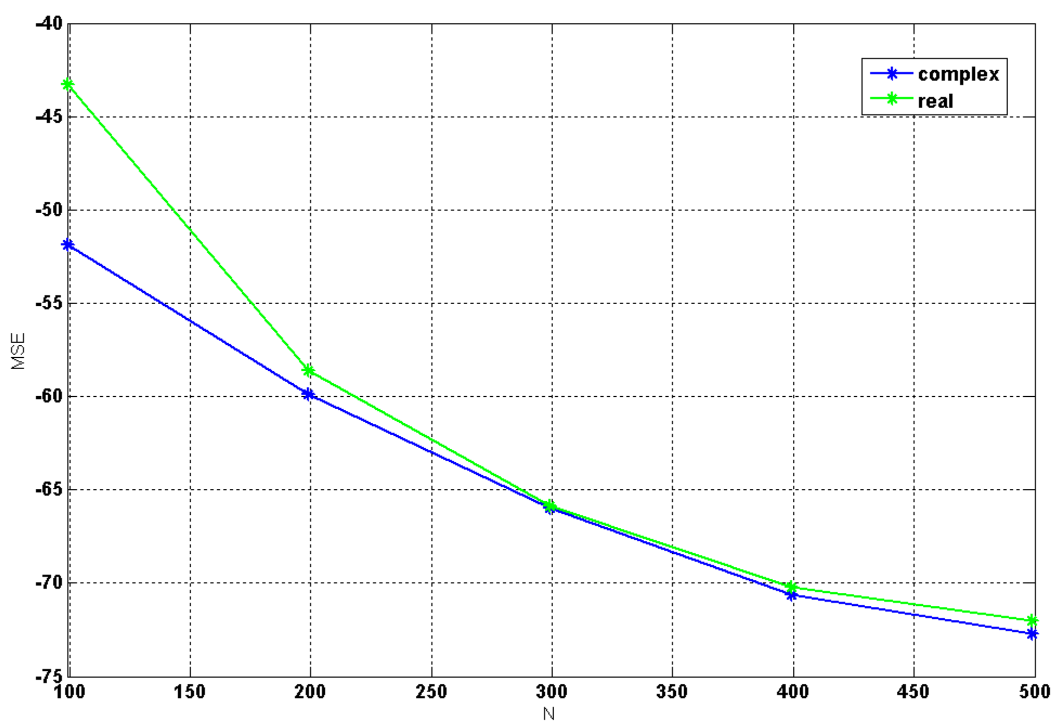

Figure 3.18 The performance curve of the reference and the proposed methods with 100 realizations at 5 dB.

CHAPTER FOUR

CONCLUSIONS AND FUTURE WORK

Frequency estimation is the process of estimating the complex frequency components of a signal in the existence of noise. The most common frequency estimation methods involve identifying the noise subspace to extract these components. Frequency estimation has fundamental significance and wide relevance for many reasons. First, any arbitrary signal may be modeled as a sum of frequencies. Hence, any signal estimation problem may be expressed in terms of frequency estimation problems [2]. Second, Many parameter estimation applications may be mathematically expressed as a frequency estimation problem [3].

In this thesis a new frequency estimation technique is presented. The key idea of the proposed technique is to convert the Hankel complex valued data matrix into a real valued data matrix with the same dimension. This conversion would be achieved via unitary ESPRIT transformation, which was developed basically for the direction of arrival problems. The resultant real valued matrix will be used to extract the noise and/or the signal subspace instead of the original complex one [2]. It is well known that real manipulations are easier and faster than the complex ones. The two methods were compared to each other in terms of SNR and length of the data record. We conclude that the proposed method converge to the reference at data record of length of 300 points and more. In terms of future extension, it is worth to analyze the computational load precisely and to extend this idea to be applied on the higher order frequency estimation problems.

REFERENCES

LIST OF REFERENCES

[1] P. Stoica, “List of references on spectral line analysis,” Signal Process., vol. 31, no. 3, pp. 329–340, Apr. 1993.

[2] Hayes, “Statistical Digital Signal Processing”, Wiley, 1996.

[3] Stoica, P., and Moses, R., Spectral Analysis of Signals, Prentice Hall, Upper Saddle River, New Jersey, 2003.

[4] C. Fuller and A. von Flotow, “Active control of sound and vibration,” IEEE Contr. Syst. Mag., vol. 15, no. 6, pp. 9–19, Dec. 1996.

[5] U. Emborg and C. F. Ross, "Active Control in the Saab 340," Proc. 2nd Conf. Recent Adv. Active Control Sound Vibrations (Suppl.), Blackburg, VA, pp. 67-73.

[6] S. Hall and N. Wereley, “Performance of higher harmonic control algorithms for helicopter vibration reduction,” J. Guidance Contr. Dynam., vol. 116, no. 4, pp. 793–797, 1993.

[7] L.J. Eriksson, "A Practical System for Active Attenuation in Ducts," Sound and Vibration, 22(2), 30-34, 1988.

[8] R. Herzog, P. Buhler, C. Gahler, and R. Larsonneur, “Unbalance compensation using generalized notch filters in the multivariable feedback of magnetic bearings,” IEEE Trans. Contr. Syst. Technol., vol. 4, pp. 580–586, Sept. 1996.

[9] P. Lozano, N. Ho, M. Martinez-Sanchez, R. Mangoubi, “The Use of Vibration Data for Liquid Rocket Health Monitoring," 34th A1AA/ASME/SAE/ASEE Joint Propulsion Conference, Cleveland OH, July 1998.

[10] A. Samant and S. Shearman. High-resolution frequency analysis with a small data record. IEEE Spectrum, pages 82–86, Sep. 1999.

[11] Keh-Chiarng Huarng, Chien-Chung Yeh **“**A unitary transformation method for angle-of-arrival estimation,**”** IEEE Transactionson Signal Processing, Volume 39, Issue 4, Apr 1991 Page(s): 975 - 977

[12] K. C. Huarng and C. C. Yeh, “A unitary transformation method for angle of arrival estimation," IEEE Trans. Acoust., Speech, Signal Processing, Vol. 39, pp. 975-977, Apr. 1991

[13] H. Krim and M. Viberg, “Two decades of array signal processing research: The parametric approach,” IEEE Signal Processing Magazine, vol. 13, pp. 67-94, July 1996.

[14] Gami Hiren, Qasaymeh M. M, Tayem Nizar, Ravi Pendse, M. E. Sawan, “Time Delay Estimator for Frequency Hopping System without Eigen Decomposition,” Barcelona, Spain, VTC 2009.

[15] Qasaymeh M. M., Gami Hiren, Nizar Tayem, M. E. Sawan, Ravi Pendse, “Triangular Factorization for Joint Time Delay and Frequency Estimation,” IEEE Signal Processing Letters.

[16] Qasaymeh M. M., Gami Hiren, Nizar Tayem, M. E. Sawan, Ravi Pendse, “Subspace-Based Blind CFO Estimation for OFDM by Exploiting Used Carriers,” IEEE Sarnoff Symposium, Princeton, NJ, USA, 2009.

[17] Gami Hiren, Qasaymeh M. M, Tayem Nizar, Ravi Pendse, M. E. Sawan, “Efficient Structure-Based Carrier Offset Estimator for OFDM System,” Barcelona, Spain, VTC 2009.

[18] Qasaymeh M. M, Gami Hiren, Nizar Tayem, Ravi Pendse, M. E. Sawan “Propagator Method for Joint Time Delay and Frequency Estimation,” IEEE Conference on Signal, System & Computers, Pacific Grove, CA, Oct. 2008

[19] Qasaymeh M. M, Gami Hiren, Tayem Nizar, M. E. Sawan, Ravi Pendse, “Rank Revealing QR Factorization for Jointly Time Delay and Frequency Estimation,” Barcelona, Spain, VTC 2009.

[20] V. F. Pisarenko, “The retrieval of harmonics from a covariance functions,” Geophys. J. Royal Astron. Sci., vol. 33, pp. 347–366, 1973.

[21] Schmidt, R. O. “Multiple emitter location and signal parameter estimation,” Proc. RADC, Spectral Estimation Workshop, Rome, NY, pp. 243-258, 1979.

[22] D. H. Johnson and S. R. DeGraff, "Improving the resolution of bearing in passive sonar arrays by eigenvalue analysis," IEEE Trans. Acoust., Speech, Signal Processing, vol. ASSP-29, pp. 401--413, 1982.

[23] R. Kumaresan and D. W. Tufts” Estimating the angles of arrival of multiple plane Waves,” IEEE Trans. on Aerospace and Elec. Syst. Vol AES-19, pp.134-139, Jan. 1983.

[24] Roy R., Kailath T., ESPRIT – Estimation of Signal Parameters via Rotational Invariance Techniques, IEEE Transactions on Acoustics, Speech, and Signal Processing, Vol. ASSP– 37, 1989, pp. 984–995.

[25] Rao B. D., Hari K. V. S., Performance Analysis of Root–Music, IEEE Trans. on Acoustics, Speech and Signal Processing, Vo. 37, No. 12, 1989, pp. 1939–1949.

[26] M. Bouri and S. Bourennane," High resolution methods based on rank revealing triangular factorization," Transactions on Engineering, Computing and Technology,

vol. 2, pp. 35-38, Dec 2004.

[27] M. P. Fargues and M. P. Ferreira," Investigations in the numerical behavior of the adaptive rank-revealing QR factorization," IEEE Transactions on Acoustics, Speech, and Signal Processing, vol. 43, pp. 2787-2791, Nov. 1995.

[28] C. H. Bischof and G. M. Shroff," On updating signal subspaces," IEEE Transactions on Acoustics, Speech, and Signal Processing, vol. 40, pp. 96-105, Jan 1992.

[29] A. B¨ottcher and S.M. Grudsky, Toeplitz Matrices, Asymptotic Linear Algebra, and Functional Analysis, Birkh¨auser, 2000.

[30] Naidu P. S., Modern Spectrum Analysis of Time Series, Boca Raton, Florida, CRC Press. 1996.

[31] H. T. Wu, J. F. Yang, and F. K. Chen, "Source number estimators using transformed gerschgorin radii," IEEE Trans. Signal Processing, vol. 43, pp. 1325–1333, June 1995.

APPENDIX

APPENDIX

A.1 Exchange Matrix

The exchange matrix is a special case of a permutation matrix, where the 1 element locates on the counter diagonal and all other elements are zero. In other words, it is a 'row-reversed' or 'column-reversed' version of the identity matrix. An exchange matrix of size $N\times N$ is denoted by $\boldsymbol{J}_N$, for example $\boldsymbol{J}_5$ is given by:

$$\boldsymbol{J}_5 = \begin{bmatrix} 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 \end{bmatrix}$$

It is worth to mention that $\boldsymbol{J}_N$ is a real symmetric matrix ($\boldsymbol{J}_N = \boldsymbol{J}_N^T = \boldsymbol{J}_N^H$) and $\boldsymbol{J}_N$ is its own inverse.

$$\boldsymbol{J}_N.\boldsymbol{J}_N = \boldsymbol{J}_N^2 = \boldsymbol{I}_N \tag{a.1}$$

If a matrix **A** of size $N\times N$ is multiplied on the left by $\boldsymbol{J}_N$, the effect is to reverse the order of each column. For example if

$$\boldsymbol{A} = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}$$

then

$$\boldsymbol{J}_3^T.\boldsymbol{A} = \begin{bmatrix} a_{31} & a_{32} & a_{33} \\ a_{21} & a_{22} & a_{23} \\ a_{11} & a_{12} & a_{13} \end{bmatrix}$$

Similarly, if **A** is multiplied on the right by $\boldsymbol{J}_N$, the effect is to reverse the order of each row.

$$\boldsymbol{A}.\boldsymbol{J}_3 = \begin{bmatrix} a_{13} & a_{12} & a_{11} \\ a_{23} & a_{22} & a_{21} \\ a_{33} & a_{32} & a_{31} \end{bmatrix}$$

Finally, if **A** is multiplied on the right and on the left by $\boldsymbol{J}_N$, the effect is to reflect each element of **A** about the central element.

$$J_3^T.A.J_3 = \begin{bmatrix} a_{33} & a_{32} & a_{31} \\ a_{23} & a_{22} & a_{21} \\ a_{13} & a_{12} & a_{11} \end{bmatrix}$$

Any matrix $\boldsymbol{A}$ satisfying the condition $\boldsymbol{AJ} = \boldsymbol{JA}$ is said to be centrosymmetric. Also any matrix A satisfying the condition $\boldsymbol{AJ} = \boldsymbol{JA}^{\mathrm{T}}$ is said to be persymmetric.

A.2 Toeplitz Matrix

A Toeplitz matrix is a matrix which has along each of the diagonals the same elements. It is n × n matrix $T_N = [t_{kj}: k, j = 0,1,2, \dots, N-1]$ where $t_{kj} = t_{k-j}$ i.e., a matrix of the form

$$\boldsymbol{X}_T = \begin{bmatrix} x(L-1) & x(L) & \dots & x(N-1) \\ x(L-2) & x(L-1) & \dots & x(N-2) \\ \vdots & \vdots & \ddots & \vdots \\ x(0) & x(1) & \dots & x(L-1) \end{bmatrix} \quad \text{(a.2)}$$

Clearly, the matrix (a.2) is completely determined by its entries in the first row and first column. For any Toeplitz matrix $\boldsymbol{X}_T$

$$\boldsymbol{J}.\boldsymbol{X}_T{}^*.\boldsymbol{J} = \boldsymbol{X}_T{}^H \quad \text{(a.3)}$$

Proof: Let

$$\boldsymbol{X}_T = \begin{bmatrix} x(L-1) & x(L) & \dots & x(N-1) \\ x(L-2) & x(L-1) & \dots & x(N-2) \\ \vdots & \vdots & \ddots & \vdots \\ x(0) & x(1) & \dots & x(L-1) \end{bmatrix}$$

$$\boldsymbol{X}_T{}^H = \begin{bmatrix} x^*(L-1) & x^*(L-2) & \dots & x^*(0) \\ x^*(L) & x^*(L-1) & \dots & x^*(1) \\ \vdots & \vdots & & \vdots \\ x^*(N-1) & x^*(N-2) & \dots & x^*(L-1) \end{bmatrix}$$

$$\boldsymbol{X}_T{}^* = \begin{bmatrix} x^*(L-1) & x^*(L) & \dots & x^*(N-1) \\ x^*(L-2) & x^*(L-1) & \dots & x^*(N-2) \\ \vdots & \vdots & \ddots & \vdots \\ x^*(0) & x^*(1) & \dots & x^*(L-1) \end{bmatrix}$$

$$\boldsymbol{J}.\boldsymbol{X_T}^* = \begin{bmatrix} x^*(0) & x^*(1) & \dots & x^*(L-1) \\ x^*(1) & x^*(2) & \dots & x^*(L) \\ \vdots & \vdots & & \vdots \\ x^*(L-1) & x^*(L) & \dots & x^*(N-1) \end{bmatrix}$$

$$\boldsymbol{J}.\boldsymbol{X_T}^*.\boldsymbol{J} = \begin{bmatrix} x^*(L-1) & x^*(L-2) & \dots & x^*(0) \\ x^*(L) & x^*(L-1) & \dots & x^*(1) \\ \vdots & \vdots & & \vdots \\ x^*(N-1) & x^*(N-2) & \dots & x^*(L-1) \end{bmatrix}$$

A.3 Hankel Matrix

A Hankel matrix is a square matrix with constant (positive sloping) skew-diagonals. For a given vector $\mathbf{x} = [x(0),\ x(1),\ x(2), \dots, x(N-1)]$ of length $N=2L+1$, a Hankel matrix **X** of size $L \times L$ can be constructed as:

$$\boldsymbol{X}_H = \begin{bmatrix} x(0) & x(1) & \dots & x(L-1) \\ x(1) & x(2) & \dots & x(L) \\ \vdots & \vdots & \ddots & \vdots \\ x(L-1) & x(L) & \dots & x(N-1) \end{bmatrix} \tag{a.4}$$

Hankel matrices are symmetric. The Hankel matrix is closely related to the Toeplitz matrix (a Hankel matrix is an upside-down Toeplitz matrix). The Hankel matrix in (a.4) is related to the Toeplitz matrix in (a.2) by:

$$\boldsymbol{X}_H = \boldsymbol{J}.\boldsymbol{X}_T \quad or \quad \boldsymbol{X}_T = \boldsymbol{J}.\boldsymbol{X}_H \tag{a.5}$$

A.4 Equation (3.23) Proof

Matrix **U** is a unitary matrix satisfying

$$\boldsymbol{U}\boldsymbol{U}^{\boldsymbol{H}} = \boldsymbol{I} \tag{3.23}$$

If L is even $(L = 2M)$

$$\boldsymbol{U} = \frac{1}{\sqrt{2}}\begin{bmatrix} \boldsymbol{I}_M & \boldsymbol{J}_M \\ i\boldsymbol{J}_M & -i\boldsymbol{I}_M \end{bmatrix} \Rightarrow \boldsymbol{U}^H = \frac{1}{\sqrt{2}}\begin{bmatrix} \boldsymbol{I}_M & -i\boldsymbol{J}_M \\ \boldsymbol{J}_M & +i\boldsymbol{I}_M \end{bmatrix}$$

$$\boldsymbol{U}\boldsymbol{U}^H = \frac{1}{2}\begin{bmatrix} \boldsymbol{I}_M.\boldsymbol{I}_M + \boldsymbol{J}_M.\boldsymbol{J}_M & \boldsymbol{I}_M.-i\boldsymbol{J}_M + \boldsymbol{J}_M.i\boldsymbol{I}_M \\ i\boldsymbol{J}_M.\boldsymbol{I}_M - i\boldsymbol{I}_M.\boldsymbol{J}_M & -i\boldsymbol{J}_M.i\boldsymbol{J}_M - i\boldsymbol{I}_M.i\boldsymbol{I}_M \end{bmatrix}$$

$$= \frac{1}{2}\begin{bmatrix} \boldsymbol{I}_M + \boldsymbol{I}_M & \boldsymbol{0} \\ \boldsymbol{0} & \boldsymbol{I}_M + \boldsymbol{I}_M \end{bmatrix} = \boldsymbol{I}_{2M} = \boldsymbol{I}_L$$

If L is odd ($L = 2M + 1$)

$$\boldsymbol{U} = \frac{1}{\sqrt{2}}\begin{bmatrix} \boldsymbol{I}_M & \boldsymbol{0}_{M\times1} & \boldsymbol{J}_M \\ \boldsymbol{0}_{1\times M} & \sqrt{2} & \boldsymbol{0}_{1\times M} \\ i\boldsymbol{J}_M & \boldsymbol{0}_{M\times1} & -i\boldsymbol{I}_M \end{bmatrix} \Rightarrow \boldsymbol{U}^H = \frac{1}{\sqrt{2}}\begin{bmatrix} \boldsymbol{I}_M & \boldsymbol{0}_{M\times1} & -i\boldsymbol{J}_M \\ \boldsymbol{0}_{1\times M} & \sqrt{2} & \boldsymbol{0}_{1\times M} \\ \boldsymbol{J}_M & \boldsymbol{0}_{M\times1} & i\boldsymbol{I}_M \end{bmatrix}$$

$$\boldsymbol{U}\boldsymbol{U}^H = \frac{1}{2}\begin{bmatrix} \boldsymbol{I}_M.\boldsymbol{I}_M + \boldsymbol{0}_{M\times1}.\boldsymbol{0}_{1\times M} + \boldsymbol{J}_M.\boldsymbol{J}_M & \boldsymbol{0}_{M\times1} & \boldsymbol{I}_M.-i\boldsymbol{J}_M + \boldsymbol{0}_{M\times1}.\boldsymbol{0}_{1\times M} + \boldsymbol{J}_M.i\boldsymbol{I}_M \\ \boldsymbol{0}_{1\times M} & 2 & \boldsymbol{0}_{1\times M} \\ i\boldsymbol{J}_M.\boldsymbol{I}_M + \boldsymbol{0}_{M\times1}.\boldsymbol{0}_{1\times M} - i\boldsymbol{I}_M.\boldsymbol{J}_M & \boldsymbol{0}_{M\times1} & -i\boldsymbol{J}_M.i\boldsymbol{J}_M + \boldsymbol{0}_{M\times1}.\boldsymbol{0}_{1\times M} - i\boldsymbol{I}_M.i\boldsymbol{I}_M \end{bmatrix}$$

$$= \frac{1}{2}\begin{bmatrix} \boldsymbol{I}_M + \boldsymbol{I}_M & \boldsymbol{0}_{M\times1} & \boldsymbol{0}_{M\times M} \\ \boldsymbol{0}_{1\times M} & 2 & \boldsymbol{0}_{1\times M} \\ \boldsymbol{0}_{M\times M} & \boldsymbol{0}_{M\times1} & \boldsymbol{I}_M + \boldsymbol{I}_M \end{bmatrix} = \boldsymbol{I}_{2M+1} = \boldsymbol{I}_L$$

A.5 Equation (3.24) Proof

$$\boldsymbol{U}^*\boldsymbol{J} = \boldsymbol{U} \tag{3.24}$$

If L is even ($L = 2M$)

$$\boldsymbol{U} = \frac{1}{\sqrt{2}}\begin{bmatrix} \boldsymbol{I}_M & \boldsymbol{J}_M \\ i\boldsymbol{J}_M & -i\boldsymbol{I}_M \end{bmatrix} \Rightarrow \boldsymbol{U}^* = \frac{1}{\sqrt{2}}\begin{bmatrix} \boldsymbol{I}_M & \boldsymbol{J}_M \\ -i\boldsymbol{J}_M & +i\boldsymbol{I}_M \end{bmatrix}$$

$$\boldsymbol{U}^*\boldsymbol{J} = \frac{1}{\sqrt{2}}\begin{bmatrix} \boldsymbol{I}_M & \boldsymbol{J}_M \\ -i\boldsymbol{J}_M & +i\boldsymbol{I}_M \end{bmatrix}.\begin{bmatrix} \boldsymbol{0}_{M\times M} & \boldsymbol{J}_M \\ \boldsymbol{J}_M & \boldsymbol{0}_{M\times M} \end{bmatrix} = \frac{1}{\sqrt{2}}\begin{bmatrix} \boldsymbol{I}_M & \boldsymbol{J}_M \\ -i\boldsymbol{J}_M & +i\boldsymbol{I}_M \end{bmatrix} = \boldsymbol{U}$$

If L is odd ($L = 2M + 1$)

$$\boldsymbol{U} = \frac{1}{\sqrt{2}}\begin{bmatrix} \boldsymbol{I}_M & \boldsymbol{0}_{M\times1} & \boldsymbol{J}_M \\ \boldsymbol{0}_{1\times M} & \sqrt{2} & \boldsymbol{0}_{1\times M} \\ i\boldsymbol{J}_M & \boldsymbol{0}_{M\times1} & -i\boldsymbol{I}_M \end{bmatrix} \Rightarrow \boldsymbol{U}^* = \frac{1}{\sqrt{2}}\begin{bmatrix} \boldsymbol{I}_M & \boldsymbol{0}_{M\times1} & \boldsymbol{J}_M \\ \boldsymbol{0}_{1\times M} & \sqrt{2} & \boldsymbol{0}_{1\times M} \\ -i\boldsymbol{J}_M & \boldsymbol{0}_{M\times1} & +i\boldsymbol{I}_M \end{bmatrix}$$

$$U^*J = \frac{1}{\sqrt{2}}\begin{bmatrix} \boldsymbol{I}_M & \boldsymbol{0}_{M\times1} & \boldsymbol{J}_M \\ \boldsymbol{0}_{1\times M} & \sqrt{2} & \boldsymbol{0}_{1\times M} \\ -i\boldsymbol{J}_M & \boldsymbol{0}_{M\times1} & +i\boldsymbol{I}_M \end{bmatrix}.\begin{bmatrix} \boldsymbol{0}_{M\times M} & \boldsymbol{0}_{M\times1} & \boldsymbol{J}_M \\ \boldsymbol{0}_{1\times M} & 1 & \boldsymbol{0}_{1\times M} \\ \boldsymbol{J}_M & \boldsymbol{0}_{M\times1} & \boldsymbol{0}_{M\times M} \end{bmatrix}$$

$$= \frac{1}{\sqrt{2}}\begin{bmatrix} \boldsymbol{I}_M & \boldsymbol{0}_{M\times1} & \boldsymbol{J}_M \\ \boldsymbol{0}_{1\times M} & \sqrt{2} & \boldsymbol{0}_{1\times M} \\ i\boldsymbol{J}_M & \boldsymbol{0}_{M\times1} & -i\boldsymbol{I}_M \end{bmatrix} = \boldsymbol{U}$$

A.6 Equation (3.26) Proof

To show that $\boldsymbol{X}^{real}$ is indeed a real valued matrix we have to prove the following:

$$\boldsymbol{X}^{real} = (\boldsymbol{X}^{real})^* = (\boldsymbol{U\Psi U}^H)^*$$

Use

$$= (\boldsymbol{U}(\overline{\boldsymbol{X}} + \boldsymbol{J}\overline{\boldsymbol{X}}^*\boldsymbol{J})\boldsymbol{U}^H)^*$$

$$= (\boldsymbol{U}\overline{\boldsymbol{X}}\boldsymbol{U}^H + \boldsymbol{U}\boldsymbol{J}\overline{\boldsymbol{X}}^*\boldsymbol{J}\boldsymbol{U}^H)^*$$

$$= (\boldsymbol{U}\overline{\boldsymbol{X}}\boldsymbol{U}^H)^* + (\boldsymbol{U}\overline{\boldsymbol{X}}^H\boldsymbol{U}^H)^*$$

$$= \boldsymbol{U}^*\overline{\boldsymbol{X}}^*\boldsymbol{U}^{*H} + \boldsymbol{U}^*\overline{\boldsymbol{X}}^{*H}\boldsymbol{U}^{*H}$$

$$= \boldsymbol{U}^*\overline{\boldsymbol{X}}^*\boldsymbol{U}^T + \boldsymbol{U}^*\overline{\boldsymbol{X}}^T\boldsymbol{U}^T$$

Using

$$(\boldsymbol{U\Psi U}^H)^* = \boldsymbol{U}^*\underbrace{\boldsymbol{I}}_{\boldsymbol{J}\cdot\boldsymbol{J}}\overline{\boldsymbol{X}}^*\underbrace{\boldsymbol{I}}_{\boldsymbol{J}\cdot\boldsymbol{J}}\boldsymbol{U}^T + \boldsymbol{U}^*\underbrace{\boldsymbol{I}}_{\boldsymbol{J}\cdot\boldsymbol{J}}\overline{\boldsymbol{X}}^T\underbrace{\boldsymbol{I}}_{\boldsymbol{J}\cdot\boldsymbol{J}}\boldsymbol{U}^T$$

$$= \underbrace{\boldsymbol{U}^*\boldsymbol{J}}_{\boldsymbol{U}}.\underbrace{\boldsymbol{J}\overline{\boldsymbol{X}}^*\boldsymbol{J}}_{\overline{\boldsymbol{X}}^H}.\overbrace{\boldsymbol{J}\boldsymbol{U}^T}^{\boldsymbol{U}^H} + \overbrace{\boldsymbol{U}^*\boldsymbol{J}}^{\boldsymbol{U}}\underbrace{\boldsymbol{J}\overline{\boldsymbol{X}}^T\boldsymbol{J}}_{\overline{\boldsymbol{X}}}\underbrace{\boldsymbol{J}\boldsymbol{U}^T}_{\boldsymbol{U}^H}$$

$$= \boldsymbol{U}.(\overline{\boldsymbol{X}}^H + \overline{\boldsymbol{X}}).\boldsymbol{U}^H$$

$$(\boldsymbol{U}.\boldsymbol{\Psi}.\boldsymbol{U}^H)^* = \boldsymbol{U}.\boldsymbol{\Psi}.\boldsymbol{U}^H$$