# ON THE LÉVY-STEINITZ THEOREM

A Thesis by

Mark Meyer

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The following faculty members have examined the final copy of this thesis for form and content, and recommend that it be accepted in partial fulfillment of the requirement for the degree of Master of Science with a major in Mathematics.

Buma Fridman, Committee Chair

Robert Fraser, Committee Member

Hyuck Kwon, Committee Member

Yueh-Ju Lin, Committee Member

ABSTRACT

This thesis is a further study of Riemann's Theorem of rearrangements of series. The theorem states: (1) If $\sum a_j$ is a conditionally convergent series of real numbers and $a$ is a real number, then there is a rearrangement of the series which converges to $a$. (2) Any rearrangement of an absolutely convergent series in $\mathbb{R}^n$ converges to the same element. The focus of this thesis is to cover the $\mathbb{R}^n$ generalization of Riemann's Theorem and to look at some counter examples in infinite dimensional spaces. Chapter 1 gives the proofs to Riemann's Theorem. Chapter 2 covers the Lévy-Steinitz theorem, which states that the set of sums of convergent rearrangements of a given series is the translate of a subspace of $\mathbb{R}^n$. In chapter 3, a sufficient condition is given for the sum range to be the whole space $\mathbb{R}^n$. The discussion in chapter 4 provides some counter examples proving that in general there is no Lévy-Steinitz theorem in the space $L^2(0,1)$ and for certain $l^p$ spaces. Also given is an example of a series which has sum range equal to $L^2$.

Riemann's Theorem is a well known result in analysis that can be found in many calculus textbooks. Two good references for this result are [4],[5]. The results described in chapters 2 through 4 can be found in [1],[2]. For more recent results related to the Lévy-Steinitz theorem, including results in infinite-dimensional spaces, see [3].

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# CHAPTER 1

## Riemann's Rearrangement Theorem

This section provides the statements and proofs of Riemann's Rearrangement Theorem.

**Theorem 1.1.** Let $\sum_{j=1}^{\infty} a_j$ be an absolutely convergent series in $\mathbb{R}^n$. If $\sum_{j=1}^{\infty} a_j = a$, then $\sum_{j=1}^{\infty} a_{\sigma(j)} = a$ for every permutation $\sigma : \mathbb{N} \to \mathbb{N}$.

*Proof.* By the absolute convergence, the partial sums of the series $\sum_{j=1}^{\infty} \|a_j\|$ are bounded by some $M > 0$. Then $\|a_{\sigma(1)}\| + \cdots + \|a_{\sigma(k)}\| \leq M$ for each natural number $k$. So the series $\sum_{j=1}^{\infty} a_{\sigma(j)}$ is absolutely convergent to $a'$. It remains to show that $a = a'$. Let $\varepsilon > 0$ be given and let $s_k = a_1 + \cdots + a_k$ be the $k$th partial sum of the original series. There exists a natural number $N$ such that if $m > k \geq N$, then $\|a - s_k\| < \varepsilon$ and $\sum_{j=k+1}^{m} \|a_j\| < \varepsilon$. Let $t_q = a_{\sigma(1)} + \cdots + a_{\sigma(q)}$ be a partial sum such that $\|a' - t_q\| < \varepsilon$ and that each of the terms $a_1, ..., a_k$ is in $t_q$. Choose $m$ large enough such that each of the terms $a_{\sigma(1)}, ..., a_{\sigma(q)}$ is in $s_m$. Then

$$\begin{aligned}\|a - a'\| &\leq \|a - s_m\| + \|s_m - t_q\| + \|t_q - a'\| \\ &< \varepsilon + \sum_{n=k+1}^{m} \|a_n\| + \varepsilon < 3\varepsilon\end{aligned}$$

This proves that $a = a'$.

**Theorem 1.2.** Let $\sum_{j=1}^{\infty} a_j$ be a conditionally convergent series in $\mathbb{R}$. If $a \in \mathbb{R}$, then there exists a permutation $\sigma : \mathbb{N} \to \mathbb{N}$ such that $\sum_{j=1}^{\infty} a_{\sigma(j)} = a$.

*Proof.* It will be shown that the series of positive terms and the series of negative terms are each divergent. It may be assumed that $a_j \neq 0$ for each $j$. Let $p_j = \frac{1}{2}(|a_j| + a_j)$ and

$q_j = \frac{1}{2}(|a_j| - a_j)$. Then

$$\sum_{j=1}^{\infty}(p_j + q_j) = \sum_{j=1}^{\infty}|a_j| = +\infty$$

Therefore one or both of the series $\sum_{j=1}^{\infty} p_j$, $\sum_{j=1}^{\infty} q_j$ is divergent. The equality

$$\sum_{j=1}^{k} a_j = \sum_{j=1}^{k} p_j - \sum_{j=1}^{k} q_j$$

implies that both converge or both diverge. It must be that both are divergent.

Let $P_j$ represent the positive terms and let $Q_j$ represent the negative terms. From the work above, each of the series $\sum_{j=1}^{\infty} P_j$, $\sum_{j=1}^{\infty} Q_j$ diverge to positive and negative infinity respectively. The following will be a construction of a rearrangement which converges to $a$.

There exists a smallest $k_1$ such that

$$S_1 = P_1 + \cdots + P_{k_1} > a$$

Then a smallest $l_1$ such that

$$T_1 = S_1 + Q_1 + \cdots + Q_{l_1} < a$$

In general it is possible to pick a smallest $k_m$ such that

$$S_m = T_{m-1} + P_{k_{m-1}+1} + \cdots + P_{k_m} > a$$

and then a smallest $l_m$ such that

$$T_m = S_m + Q_{l_{m-1}+1} + \cdots + Q_{l_m} < a$$

By the choice of $k_m$,

$$0 < S_m - a \leq P_{k_m}$$

By the choice of $l_m$,

$$0 < a - T_m \leq -Q_{l_m}$$

Then $S_m \to a$ and $T_m \to a$ as $m \to \infty$. This completes the proof since any partial sum of this rearrangement must lie between an $S_m$ and a $T_m$.

# CHAPTER 2

## The Lévy-Steinitz Theorem

In this section the Lévy-Steinitz theorem will be proved. That is, it will be shown that given a series in $\mathbb{R}^n$, the set of sums of convergent rearrangements (the sum range) is either the empty set, or the translate of a subspace. To begin, the sum range will be formally defined.

**Definition 2.1 (The sum range of a series)** In $\mathbb{R}^n$ the sum range of the series $\sum_{j=1}^{\infty} v_j$, denoted by $\mathrm{SR}(v_j)$, is the set of all sums of convergent rearrangements of the series. That is, $v \in \mathrm{SR}(v_j)$ if and only if there exists a permutation $\sigma : \mathbb{N} \to \mathbb{N}$ such that $\sum_{j=1}^{\infty} v_{\sigma(j)} = v$.

**Example 2.2** Let $\sum_{j=1}^{\infty} v_j$ be absolutely convergent to $v$ in $\mathbb{R}^n$. By theorem 1.1, $\mathrm{SR}(v_j) = \{v\}$. Let $\sum_{j=1}^{\infty} x_j$ be a conditionally convergent series of real numbers. By theorem 1.2, $\mathrm{SR}(x_j) = \mathbb{R}$.

**Theorem 2.3 (The Polygonal Confinement Theorem)** In $\mathbb{R}^n$ there exists a constant $C_n$, dependent only on $n$, such that if $v_1, \ldots, v_k$ is any finite collection for which $\sum_{j=1}^{k} v_j = 0$ and $\|v_j\| \leq 1$ for each $j$, then there exists a permutation $\sigma$ on $\{2, ..., k\}$ such that $\|v_1 + v_{\sigma(2)} + \cdots + v_{\sigma(r)}\| \leq C_n$ for each $r = 2, 3, \ldots, k$.

*Proof.* This theorem is proved by induction. For the case $n = 1$ assume without loss of generality that $v_1 > 0$. Let $\{p_j\}$ be the set of non-negative $v_j$ terms (excluding $v_1$) and let $\{q_j\}$ be the set of negative $v_j$ terms. There exists a smallest integer $m_1$ such that

$$S_1 = v_1 + q_1 + \cdots + q_{m_1} < 0$$

Then a smallest $n_1$ such that

$$T_1 = S_1 + p_1 + \cdots + p_{n_1} > 0$$

Choose the smallest $m_k$ and $n_k$, k=2,3,... in the same way until there are no more $v_j$ terms remaining. In this arrangement by the choice of $m_k$,

$$0 < -S_k \leq |q_{m_k}|$$

and similarly

$$0 < T_k \leq p_{n_k}$$

Then $|T_k|, |S_k| \leq 1$ for each $k$. Therefore any partial sum in this rearrangement is bounded by 1. Set $C_1 = 1$ and the theorem is proved for $n = 1$.

Suppose that the theorem is true for some $n - 1 \in \mathbb{N}$. Let $L$ be the partial sum with the property

$$\|L\| = \max\{\|S\| : \text{S is a partial sum of } v_1, \ldots, v_k \text{ and begins with } v_1\}$$

Then $L = v_1 + u_1 + \cdots + u_s$ and for the remaining vectors, $L + w_1 + \cdots + w_t = 0$ where the $u_j$ terms and the $w_j$ terms make up the collection $v_2, \ldots, v_k$.

**Claim.** $\langle v_1, L\rangle \geq 0$, $\langle u_j, L\rangle \geq 0$ for each $j$, and $\langle w_j, L\rangle \leq 0$ for each $j$.

*Proof of the claim.* To verify the first inequality assume for condtradiction that $\langle v_1, L\rangle < 0$. If $S_1 = 2v_1 - L$, then

$$\begin{aligned}\langle \frac{-L}{\|L\|}, S_1\rangle &= \langle \frac{-L}{\|L\|}, 2v_1\rangle + \langle \frac{-L}{\|L\|}, -L\rangle \\ &= -2\frac{1}{\|L\|}\langle L, v_1\rangle + \frac{1}{\|L\|}\langle L, L\rangle \\ &> \frac{1}{\|L\|} \cdot \|L\|^2 \\ &= \|L\|\end{aligned}$$

which implies that $\|S_1\| > \|L\|$. This is a contradiction. For the second inequality suppose that $\langle u_j, L\rangle < 0$ for some $j$ and let $S_2 = L - u_j$. Then

$$\begin{aligned}\langle \frac{L}{\|L\|}, S_2\rangle &= \langle \frac{L}{\|L\|}, L\rangle - \langle \frac{L}{\|L\|}, u_j\rangle \\ &= \frac{1}{\|L\|}\langle L, L\rangle - \frac{1}{\|L\|}\langle L, u_j\rangle \\ &> \frac{1}{\|L\|}\cdot \|L\|^2 \\ &= \|L\|\end{aligned}$$

which brings the contradiction $\|S_2\| > \|L\|$. For the third inequality assume that $\langle w_j, L\rangle > 0$ for some $j$ and choose the partial sum $S_3 = L + w_j$. Then

$$\begin{aligned}\langle \frac{L}{\|L\|}, S_3\rangle &= \langle \frac{L}{\|L\|}, L\rangle + \langle \frac{L}{\|L\|}, w_j\rangle \\ &= \frac{1}{\|L\|}\langle L, L\rangle + \frac{1}{\|L\|}\langle L, w_j\rangle \\ &> \frac{1}{\|L\|}\cdot \|L\|^2 \\ &= \|L\|\end{aligned}$$

which gives $\|S_3\| > \|L\|$. This proves the claim.

**Construction of the permutation:** Define the set of vectors orthogonal to $L$ by

$$L^{\perp} := \{u \in \mathbb{R}^n : \langle u, L\rangle = 0\}$$

The set $L^{\perp}$ is a subspace of $n-1$ dimensions. Let $v'$ be the orthogonal projection of a vector $v$ onto the space $L^{\perp}$. Then $v_1' + u_1' + \cdots + u_s' = 0$ and $w_1' + \cdots + w_t' = 0$ (otherwise, $L$ would have a nonzero projection into $L^{\perp}$ which is a contradiction). By the inductive assumption on the $n-1$ dimensional space $L^{\perp}$ there exists a permutation $\sigma_1$ on $\{1, \ldots, s\}$, and a permutation $\sigma_2$ on $\{2, \ldots, t\}$ such that

$$\|v_1' + \sum_{j=1}^{r} u'_{\sigma_1(j)}\| \le C_{n-1}$$

for each $r = 1, \ldots, s$, and that

$$\|w_1' + \sum_{j=2}^{l} w'_{\sigma_2(j)}\| \le C_{n-1}$$

for each $l = 2, \ldots, t$. So as long as the order of the permutations above is not changed, the partial sums of this arrangement projected onto $L^\perp$ must be have norm bounded above by $2C_{n-1}$. Set $\sigma_2(1) = 1$ and then using the claim proved above arrange the vectors as follows. Choose the smallest $m_1$ such that

$$\langle v_1, L \rangle + \sum_{j=1}^{m_1} \langle w_{\sigma_2(j)}, L \rangle < 0$$

Then the smallest $n_1$ such that

$$\langle v_1, L \rangle + \sum_{j=1}^{m_1} \langle w_{\sigma_2(j)}, L \rangle + \sum_{j=1}^{n_1} \langle u_{\sigma_1(j)}, L \rangle > 0$$

Now choose the smallest $m_2$, and so on until there are no more vectors remaining. This ensures that the arrangement of the $\sigma_1$ and $\sigma_2$ permutations has not been changed, but along the direction parallel to $L$, the partial sums are bounded above by 1. If $S$ is a partial sum from this arrangement, then

$$\|S\| = \sqrt{\|S_\perp\|^2 + \|S_\|\|^2} \leq \sqrt{(2C_{n-1})^2 + 1}$$

Where $S_\perp$ is the portion of $S$ orthogonal to $L$ and $S_\|$ is the portion of $S$ parallel to $L$. Therefore, we may set $C_n \leq \sqrt{4C_{n-1}^2 + 1}$ to complete the proof.

**Corollary 2.4** Let $\varepsilon > 0$. If $v_1, \ldots, v_k$ is a finite collection of vectors in $\mathbb{R}^n$ for which $\|\sum_{j=1}^k v_j\| \leq \varepsilon$ and $\|v_j\| \leq \varepsilon$ for each $j$, then there exists a permutation $\sigma$ on the set $\{1, \ldots, k+1\}$ such that $\|v_{\sigma(1)} + \cdots + v_{\sigma(r)}\| \leq \varepsilon(C_n + 1)$ for each $r = 1, 2, \ldots, k+1$.

*Proof.* Let $v_{k+1} = -(\sum_{j=1}^k v_j)$. The set $\{\frac{1}{\varepsilon} v_j\}_{j=1}^{k+1}$ satisfies the hypothesis of the Polygonal Confinement Theorem. Then there exists a permutation $\sigma'$ on $\{1, \ldots, k+1\}$ with $\sigma'(1) = 1$ such that

$$\|v_{\sigma'(1)} + \cdots + v_{\sigma'(r)}\| \leq \varepsilon C_n$$

for $r = 2, \ldots, k+1$. Remove $v_{k+1}$ from this arrangement. Since $\|v_{k+1}\| \leq \varepsilon$ the right side of the inequality changes by $\varepsilon$ which proves the inequality.

**Theorem 2.5 (The Rearrangement Theorem)** Let $\sum_{j=1}^{\infty} v_j$ be a series in $\mathbb{R}^n$ and denote the partial sums by $S_r = \sum_{j=1}^{r} v_j$. If there exists a subsequence $\{S_{r_k}\}$ which converges to $S$ and if the sequence $\{v_j\}$ converges to 0, then there exists a permutation $\sigma$ on $\mathbb{N}$ such that $\sum_{j=1}^{\infty} v_{\sigma(j)}$ converges to $S$.

*Proof.* Let $\delta_k = \|S_{r_k} - S\|$. Then $\delta_k \to 0$. Note that

$$\|\sum_{j=r_k+1}^{r_{k+1}-1} v_j\| = \|S_{r_{k+1}} - S_{r_k} - v_{r_{k+1}}\| \leq \delta_{k+1} + \delta_k + \|v_{r_{k+1}}\|$$

For each $k$ define $\varepsilon_k := \delta_{k+1} + \delta_k + \sup\{\|v_j\| : j \geq r_k\}$. Then $\varepsilon_k \to 0$ and

$$\|\sum_{j=r_k+1}^{r_{k+1}-1} v_j\| \leq \varepsilon_k$$

Also, $\|v_j\| \leq \varepsilon_k$ for $j = r_k + 1, \ldots, r_{k+1} - 1$. Then for each $k$, by corollary 2.4, there exists a permutation $\sigma_k$ on the set $\{r_k + 1, \ldots, r_{k+1} - 1\}$ such that

$$\|\sum_{j=r_k+1}^{r} v_{\sigma_k(j)}\| \leq \varepsilon_k(C_n + 1) \tag{2.1}$$

for each $r$ with $r_k + 1 \leq r \leq r_{k+1} - 1$. Rearrange the vectors in the following way. For each $k$, fix $v_{r_k}$ and then arrange the vectors $\{v_{r_k+1}, \ldots, v_{r_{k+1}-1}\}$ using the permutation $\sigma_k$. If $r \neq r_k$, for any $k \in \mathbb{N}$, there exists a $k$ such that $r_k + 1 \leq r \leq r_{k+1} - 1$. If a partial sum of the new arrangement is $S'_r$, then $S'_r - S_{r_k}$ is a sum of the form (1).

$$\|S'_r - S_{r_k}\| \leq \varepsilon_k(C_n + 1)$$

Since $\{\varepsilon_k\} \to 0$ as $k \to \infty$, if follows from the above inequality that $\{S'_r\} \to S$ as $r \to \infty$.

**Lemma 2.6** Let $v_1, \ldots, v_k$ be a finite collection in $\mathbb{R}^n$, $\varepsilon > 0$, $w = \sum_{j=1}^{k} v_j$, $0 < t < 1$, and $\|v_j\| \leq \varepsilon$ for each $j$. Then there exists a permutation $\sigma$ on $\{2, \ldots, k\}$ and an $r$ with $1 \leq r \leq k$, such that

$$\|v_1 + \sum_{j=2}^{r} v_{\sigma(j)} - tw\| \leq \varepsilon\sqrt{C_{n-1}^2 + 1}$$

When $r = 1$, set $\sum_{j=2}^{1} v_{\sigma(j)} = 0$.

*Proof.* If $w = 0$, then set $r = k$ and let the permutation be the identity function. Suppose that $w \neq 0$. Consider first $n = 1$. Assume without loss of generality that $w > 0$. Then there exists a smallest integer $s$ such that

$$v_1 + \cdots + v_s > tw$$

Then

$$0 < v_1 + \cdots + v_{s-1} + v_s - tw \leq v_s$$

Set $C_0 = 0$, then the lemma is true with $\sigma(j) = j$ for each $j$ and with $r = s$.

Let $n > 1$. Let $v'_j$ represent the projection of $v_j$ onto the space $W^{\perp} := \{v \in \mathbb{R}^n : \langle v, w\rangle = 0\}$. Then

$$v'_1 + \cdots + v'_k = 0$$

Also, $\|v'_j\| \leq \varepsilon$ for each $j$ so that the set $\{\frac{1}{\varepsilon} v'_j\}$ satisfies the hypothesis of the Polygonal Confinement Theorem in the space $\mathbb{R}^{n-1}$. Then there exists a permutation $\sigma$ on $\{2, \ldots, k\}$ such that for $t = 2, \ldots, k$

$$\|v'_1 + v'_{\sigma(2)} + \cdots + v'_{\sigma(t)}\| \leq \varepsilon C_{n-1}$$

Note that

$$\langle v_1, \frac{w}{\|w\|}\rangle + \langle v_{\sigma(2)}, \frac{w}{\|w\|}\rangle + \cdots + \langle v_{\sigma(k)}, \frac{w}{\|w\|}\rangle = \|w\|$$

and that $\|w\|^{-1}|\langle v_j, w\rangle| \leq \varepsilon$ for each $j$. Use the same technique used in the case of $n = 1$. There exists a smallest $r$ such that

$$|\langle v_1, \frac{w}{\|w\|}\rangle + \langle v_{\sigma(2)}, \frac{w}{\|w\|}\rangle + \cdots + \langle v_{\sigma(r)}, \frac{w}{\|w\|}\rangle - t\|w\|| \leq \varepsilon$$

It follows from the inequalties above that

$$\|v_1 + v_{\sigma(2)} + \cdots + v_{\sigma(r)} - tw\|^2 \leq \varepsilon^2 C_{n-1}^2 + \varepsilon^2$$

This proves the lemma for $n > 1$.

**Theorem 2.7 (The Lévy-Steinitz Theorem)** If $\sum_{j=1}^{\infty} v_j$ is a series in $\mathbb{R}^n$, then $\mathrm{SR}(v_j)$ is either the empty set or the translate of a subspace.

*Proof.* A proof is given for the case that $\mathrm{SR}(v_j)$ is nonempty. It may be assumed that $0 \in \mathrm{SR}(v_j)$. If not, then replace $v_1$ by $v_1 - v$ where $v \in \mathrm{SR}(v_j)$ is fixed.

There are two steps to the proof. Step 1 is to show that if $0, s_1, s_2 \in \mathrm{SR}(v_j)$, then $s_1 + s_2 \in \mathrm{SR}(v_j)$. To prove this, define a sequence $\{\varepsilon_l\}$ of positive real numbers which converges to 0. Since $s_1$, 0, $s_2$ are in $\mathrm{SR}(v_j)$, there exist permutations $\sigma_1$, $\sigma_0$, and $\sigma_2$ such that

$$\sum_{j=1}^{\infty} v_{\sigma_1(j)} = s_1, \qquad \sum_{j=1}^{\infty} v_{\sigma_0(j)} = 0, \qquad \sum_{j=1}^{\infty} v_{\sigma_2(j)} = s_2$$

There exists a positive integer $n_1$ such that

$$I(\sigma_1, n_1) := \{\sigma_1(j) : j = 1, 2, \ldots, n_1\}$$

and that $\| \sum_{j=1}^{n_1} v_{\sigma_1(j)} - s_1 \| < \varepsilon_1$. Then an integer $m_1 > n_1$ such that

$$I(\sigma_1, n_1) \subset I(\sigma_0, m_1), \qquad \| \sum_{j=1}^{m_1} v_{\sigma_0(j)} \| < \varepsilon_1$$

Then choose $k_1 > m_1$ such that

$$I(\sigma_0, m_1) \subset I(\sigma_2, k_1), \qquad \| \sum_{j=1}^{k_1} v_{\sigma_2(j)} - s_2 \| < \varepsilon_1$$

For the general case $\varepsilon_l$ with $l \geq 2$, choose $n_l > k_{l-1}$ such that

$$I(\sigma_2, k_{l-1}) \subset I(\sigma_1, n_l), \qquad \| \sum_{j=1}^{n_l} v_{\sigma_1(j)} - s_1 \| < \varepsilon_l$$

Then choose $m_l > n_l$ such that

$$I(\sigma_1, n_l) \subset I(\sigma_0, m_l), \qquad \| \sum_{j=1}^{m_l} v_{\sigma_0(j)} \| < \varepsilon_l$$

Then $k_l > m_l$ such that

$$I(\sigma_0, m_l) \subset I(\sigma_2, k_l), \qquad \|\sum_{j=1}^{k_l} v_{\sigma_2(j)} - s_2\| < \varepsilon_l$$

The following shows how the permutation $\sigma$ is constructed.

For $l = 1$ define

$$\{\sigma(1), \ldots, \sigma(n_1)\} = I(\sigma_1, n_1)$$

$$\{\sigma(n_1 + 1), \ldots, \sigma(m_1)\} = I(\sigma_0, m_1) \setminus I(\sigma_1, n_1)$$

$$\{\sigma(m_1 + 1), \ldots, \sigma(k_1)\} = I(\sigma_2, k_1) \setminus I(\sigma_0, m_1)$$

For $l > 1$ define

$$\{\sigma(k_{l-1} + 1), \ldots, \sigma(n_l)\} = I(\sigma_1, n_l) \setminus I(\sigma_2, k_{l-1})$$

$$\{\sigma(n_l + 1), \ldots, \sigma(m_l)\} = I(\sigma_0, m_l) \setminus I(\sigma_1, n_l)$$

$$\{\sigma(m_l + 1), \ldots, \sigma(k_l)\} = I(\sigma_2, k_l) \setminus I(\sigma_0, m_l)$$

With this construction of $\sigma$, it holds for all $l$ that $n_l < m_l < k_l$ and

$$\|\sum_{j=1}^{n_l} v_{\sigma(j)} - s_1\| < \varepsilon_l \qquad \|\sum_{j=1}^{m_l} v_{\sigma(j)}\| < \varepsilon_l \qquad \|\sum_{j=1}^{k_l} v_{\sigma(j)} - s_2\| < \varepsilon_l$$

Note the inequality $\|\sum_{j=m_l+1}^{k_l} v_{\sigma(j)} - s_2\| < 2\varepsilon_l$. Then

$$\|\sum_{j=1}^{n_l} v_{\sigma(j)} + \sum_{j=m_l+1}^{k_l} v_{\sigma(j)} - (s_1 + s_2)\| < 3\varepsilon_l$$

For each $l$ rearrange the vectors $\{v_{\sigma(n_l)}, \ldots, v_{\sigma(k_l)}\}$ by interchanging the vectors $\{v_{\sigma(n_l+1)}, \ldots, v_{\sigma(m_l)}\}$ with the vectors $\{v_{\sigma(m_l+1)}, \ldots, v_{\sigma(k_l)}\}$. In this new arrangement, call it $\sigma'$, the sequence of partial sums $s_{k_l} = \sum_{j=1}^{k_l} v_{\sigma'(j)}$ converges to $(s_1 + s_2)$. By the Rearrangement Theorem there exists a rearrangement of the series which converges to $(s_1+s_2)$. Therefore $(s_1+s_2) \in \mathrm{SR}(v_j)$.

The second step of the proof is to show that if $t \in \mathbb{R}$ and $0, s \in \mathrm{SR}(v_j)$, then $ts \in \mathrm{SR}(v_j)$. By what was just proved, it is enough to show that this holds for $0 < t < 1$ and then for $t = -1$. Recall that the permutation $\sigma$ gives

$$\|\sum_{j=m_l+1}^{k_l} v_{\sigma(j)} - s_2\| < 2\varepsilon_l$$

Let $\delta_l = \max\{\|v_{\sigma(j)}\| : j = m_l+1, \ldots, k_l\}$ and set $u_l = \sum_{j=m_l+1}^{k_l} v_{\sigma(j)} - s_2$. By lemma 2.6, for each $l$ there exists a permutation $\sigma_l$ on the set $\{\sigma(m_l+1), \ldots, \sigma(k_l)\}$ and a positive integer $r_l$ such that

$$\|\sum_{j=m_l+1}^{r_l} v_{\sigma_l(\sigma(j))} - t(s_2+u_l)\| \le \delta_l \sqrt{C_{n-1}^2+1} = \delta_l M$$

Remove the $tu_l$ term which is bounded by $2\varepsilon_l$ to get

$$\|\sum_{j=m_l+1}^{r_l} v_{\sigma_l(\sigma(j))} - ts_2\| \le 2\varepsilon_l + \delta_l M$$

Then

$$\|\sum_{j=1}^{m_l} v_{\sigma(j)} + \sum_{j=m_l+1}^{r_l} v_{\sigma_l(\sigma(j))} - ts_2\| \le 3\varepsilon_l + \delta_l M$$

This proves that there is a rearrangement for which a subsequence of partial sums converges to $ts_2$. By the rearrangement theorem, there exists a rearrangement which converges to $ts_2$ with $0 < t < 1$. For the case $t = -1$ note that

$$\|\sum_{j=k_l+1}^{m_{l+1}} v_{\sigma(j)} - (-s_2)\| < \varepsilon_{l+1} + \varepsilon_l$$

Then

$$\|\sum_{j=1}^{m_l} v_{\sigma(j)} + \sum_{j=k_l+1}^{m_{l+1}} v_{\sigma(j)} - (-s_2)\| < \varepsilon_{l+1} + 2\varepsilon_l$$

Use the interchange of the vectors technique which was used earlier. Then in the new arrangement there exists subsequence of the sequence of partial sums which converges to $-s_2$. By the Rearrangement Theorem there exists a rearrangement which converges to $-s_2$. This completes the proof.

# CHAPTER 3

## More about the Lévy-Steinitz Theorem

The Lévy-Steinitz Theorem states that the sum range of a series in $\mathbb{R}^n$ is the empty set or the translate of a subspace. The following problems were not answered in Part 1.

**(1)** When is the sum range the empty set?

**(2)** Are there any sufficient conditions so that the sum range is $\mathbb{R}^n$?

**(3)** What form does the subspace have?

To answer problem (1), let $\{v_j\}$ be a sequence in $\mathbb{R}^n$. Define the following set.

$$\mathcal{F} := \{w \in \mathbb{R}^n : \sum_{j=1}^{\infty} \langle w, v_j \rangle^+ < \infty\}$$

For a real number $A$, $A^+$ is equal to $A$ if $A > 0$ and equal to 0 if $A \leq 0$.

**Claim.** If one of the following properties fails, then $\mathrm{SR}(v_j)$ is the empty set.

**(i)** $v_j \to 0$

**(ii)** If $w \in \mathcal{F}$, then $-w \in \mathcal{F}$.

*Proof of the claim.* If property (i) fails, then there is nothing to show. Suppose that property (ii) fails. Let $w$ be a vector in $\mathbb{R}^n$ such that $w \in \mathcal{F}$ but $-w \notin \mathcal{F}$. Then given any permutation $\sigma : \mathbb{N} \to \mathbb{N}$ we have that

$$\begin{aligned}\sum_{j=1}^{k} \langle w, v_{\sigma(j)} \rangle &= \sum_{j=1}^{k} [\langle w, v_{\sigma(j)} \rangle^+ - \langle -w, v_{\sigma(j)} \rangle^+] \\ &= \sum_{j=1}^{k} \langle w, v_{\sigma(j)} \rangle^+ - \sum_{j=1}^{k} \langle -w, v_{\sigma(j)} \rangle^+\end{aligned}$$

Then $\lim_{k\to\infty} |\sum_{j=1}^{k} \langle w, v_{\sigma(j)}\rangle| = \infty$. To see this, first note that the sum on the left is absolutely convergent (since $w \in F$), so by theorem 1.1 any rearrangement converges. The sum on the right is strictly negative and diverges to $\infty$, so any rearrangement must also diverge. Therefore any rearrangement of the series diverges along the direction given by $w$. The claim is proved.

The complete answers to the questions (1), (2), (3) are given as theorems listed below. The proofs are based on the concept of a Lévy vector.

**Theorem 3.1** Let $V$ be a subspace of $\mathbb{R}^n$ with $\dim V \geq 1$ and let $\{v_j\}$ be a sequence of vectors in $V$ for which $v_j \to 0$, and that for each $w \neq 0$ in $V$, $\sum_{j=1}^{\infty} \langle v_j, w\rangle^+ = \infty$. Then $\mathrm{SR}(v_j) = V$.

**Theorem 3.2** If properties (i) and (ii) hold, then $\mathrm{SR}(v_j)$ is non-empty and has the form

$$\mathrm{SR}(v_j) = s + \mathcal{F}^\perp$$

where $s$ is the sum of a convergent rearrangement of the series and the set $\mathcal{F}^\perp$ is a linear subspace of $\mathbb{R}^n$.

The remainder of this chapter will be dedicated to proving theorems 3.1 and 3.2.

**Definition 3.3 (Lévy vector)** A unit vector $u$ is a Lévy vector of a sequence $\{v_j\} \subset \mathbb{R}^n$ if for each $\varepsilon > 0$, $\sum_{j=1}^{\infty} \|v_j^\varepsilon\| = \infty$, where $\{v_j^\varepsilon\} \subset \{v_j\}$ is the set of vectors which are within an angle $\varepsilon$ from $u$.

It is necessary to show that there exist sequences for which the set of Lévy vectors is nonempty. This will be proved in the work below.

**Lemma 3.4** Let $\{v_j\} \subset \mathbb{R}^n$ be a sequence of vectors which satisfies the hypothesis of theorem 3.1. Then the set $L(v_j) = \{\text{Levy vectors of } \{v_j\}\}$ is a non-empty, closed subset of the unit sphere in $\mathbb{R}^n$.

*Proof.* To prove that $L(v_j)$ is non-empty, it is shown that every closed half sphere of the unit sphere must contain a Levy vector. For contradiction, suppose that for some $w \neq 0$ that the closed half sphere

$$H(w) := \{u : \|u\| = 1, \langle u, w \rangle \geq 0\}$$

contains no Levy vectors. Then for each $u \in H(w)$ there exists $\varepsilon > 0$ such that

$$\sum \{\|v_j\| : \frac{v_j}{\|v_j\|} \in B(u, \varepsilon) \cap H(w)\} < \infty$$

By the compactness of $H(w)$ there exists $N \in \mathbb{N}$ and a finite collection $\{u_t\}_{t=1}^N$ in $H(w)$ with corresponding $\varepsilon_t$ such that $\{B(u_t, \varepsilon_t)\}_{t=1}^N$ covers $H(w)$. Then

$$\sum \{\|v_j\| : \frac{v_j}{\|v_j\|} \in H(w)\} \leq \sum \{\|v_j\| : \frac{v_j}{\|v_j\|} \in \bigcup_{t=1}^N (B(u_t, \varepsilon_t) \cap H(w))\} < \infty$$

The above implies that

$$\begin{aligned} \sum_{j=1}^{\infty} \langle v_j, w \rangle^+ &= \sum \{\langle v_j, w \rangle : \frac{v_j}{\|v_j\|} \in H(w)\} \\ &\leq \sum \{\|v_j\| \|w\| : \frac{v_j}{\|v_j\|} \in H(w)\} \\ &< \infty \end{aligned}$$

This contradicts the condition that $\sum_{j=1}^{\infty} \langle v_j, w \rangle^+ = \infty$.

Next it is proved that the set $L(v_j)$ is closed. Let $u \notin L(v_j)$ be a point on the unit sphere. Since $u$ is not a Levy vector, there exists $\varepsilon > 0$ such that $\sum_{j=1}^{\infty} \|v_j^{\varepsilon}\| < \infty$. Now let $u'$ be a unit vector within the angle $\varepsilon$ from $u$. Then $u' \notin L(v_j)$ since an angle $\varepsilon'$ can be chosen so close to 0 that the vectors within the angle $\varepsilon'$ of $u'$ are within the angle $\varepsilon$ of $u$. This gives $\sum_{j=1}^{\infty} \|v_j^{\varepsilon'}\| < \infty$. So $B(u, \varepsilon) \in L(v_j)^c$. Along with the fact that the complement of the unit sphere is open, this proves that $L(v_j)$ is closed.

**Lemma 3.5** If $u \in L(v_j)$, then there exists a subsequence of the form $v'_j = \alpha'_j u + w'_j$ such that $\sum_{j=1}^{\infty} \alpha'_j = \infty$, $\alpha'_j \geq 0$ and $\sum_{j=1}^{\infty} \|w'_j\| < \infty$.

*Proof.* Write $v_j = \alpha_j u + w_j$ where $\alpha_j \in \mathbb{R}$ and $w_j \perp u$. The construction of this sequence is based on the following claims.

**Claim 1.** If $t \in \mathbb{N}$, then there exists $\varepsilon > 0$ such that the vectors $\{v_j^\varepsilon\}$ satisfy

$$\alpha_j^\varepsilon > (1 - \frac{1}{4t^2})^{1/2} \|v_j^\varepsilon\| > \frac{1}{2}\|v_j^\varepsilon\|$$
$$\|w_j^\varepsilon\| < \frac{1}{2t}\|v_j^\varepsilon\|$$

*Proof of claim 1.* Since $\cos\theta$ increases to 1 as $\theta$ decreases to 0, there exists $\varepsilon > 0$ such that if $0 \leq \theta < \varepsilon$, then

$$\cos\theta > \cos\varepsilon > (1 - \frac{1}{4t^2})^{1/2}$$

If $\theta_j$ represents the angle between $v_j^\varepsilon$ and $u$, then $0 \leq \theta_j < \varepsilon$ and

$$\begin{aligned}
\alpha_j^\varepsilon &= \langle \alpha_j^\varepsilon u + w_j^\varepsilon, u \rangle \\
&= \|v_j^\varepsilon\| \|u\| \cos\theta_j \\
&> \|v_j^\varepsilon\| (1 - \frac{1}{4t^2})^{1/2} \\
&> \frac{1}{2}\|v_j^\varepsilon\|
\end{aligned}$$

So $\alpha_j^\varepsilon > (1/2)\|v_j^\varepsilon\|$. This implies that

$$\begin{aligned}
\|w_j^\varepsilon\|^2 &= \|v_j^\varepsilon\|^2 - |\alpha_j^\varepsilon|^2 \\
&< \|v_j^\varepsilon\|^2 - (1 - \frac{1}{4t^2})\|v_j^\varepsilon\|^2 \\
&= \frac{1}{4t^2}\|v_j^\varepsilon\|^2
\end{aligned}$$

This gives $\|w_j^\varepsilon\| < (1/2t)\|v_j^\varepsilon\|$ which proves the claim.

**Claim 2.** If $t \in \mathbb{N}$ and a finite number of $\{v_j\}$ terms have been removed, then there exists a finite set of the remaining vectors

$$A_t := \{v_{1,t}, v_{2,t}, \ldots, v_{m,t}\}$$

such that

$$\sum_{j=1}^{m} \|w_{j,t}\| < \frac{1}{t^2}, \qquad \sum_{j=1}^{m} \alpha_{j,t} > \frac{1}{2t}$$

These vectors can be chosen such that the indices of the terms are larger than the largest index of the removed terms.

*Proof of claim 2.* By claim 1, there exists $\varepsilon > 0$ such that the vectors $\{v_j^\varepsilon\}$ satisfy

$$\alpha_j^\varepsilon > (1/2)\|v_j^\varepsilon\|, \qquad \|w_j^\varepsilon\| < (1/2t)\|v_j^\varepsilon\|$$

Since $u$ is a Levy vector, $\sum_{j=1}^{\infty} \|v_j^\varepsilon\| = \infty$. Then there is a finite subset $\{v_{1,t}, \ldots, v_{m,t}\}$ of the terms $\{v_j^\varepsilon\}$ such that the indices are larger than the largest index of any removed terms and

$$\frac{1}{t} < \sum_{j=1}^{m} \|v_{j,t}\| < \frac{2}{t}$$

This can be done since the $\|v_j^\varepsilon\|$ terms converge to 0 and sum to infinity. By the inequalities above we get

$$\sum_{j=1}^{m} \alpha_{j,t} > \frac{1}{2} \sum_{j=1}^{m} \|v_{j,t}\| > \frac{1}{2t}$$
$$\sum_{j=1}^{m} \|w_{j,t}\| < \frac{1}{2t} \sum_{j=1}^{\infty} \|v_{j,t}\| < \frac{1}{t^2}$$

This proves claim 2.

Now to prove the lemma, use claim 2 to construct the sets $\{A_t\}_{t=1}^{\infty}$ which have increasing indices. Then set $\{v_j'\}_{j=1}^{\infty} = \bigcup_{t=1}^{\infty} A_t$.

**Lemma 3.6** Let $\{v_j\}$ be a sequence which satisfies the hypothesis of Theorem 3.1. Suppose that there exist subsequences of the form $\{\alpha_j u\}$, $\{\beta_j(-u)\}$ such that $\alpha_j \geq 0$, $\beta_j \geq 0$, $\sum_{j=1}^{\infty} \alpha_j = \infty$, $\sum_{j=1}^{\infty} \beta_j = \infty$. Then SR $= V$.

*Proof.* This lemma will be proved by induction on the dimension of $V$. If $\dim V = 1$, then $V$ is a line in $\mathbb{R}^n$ and the exact same technique used in Riemann's theorem can be used here. Suppose that the lemma is true in any subspace satisfying the hypothesis of theorem 3.1 with dimension equal to $n-1$. Suppose that $\dim V = n$ and let $v \in V$. It will be shown that there exists a rearrangment of the vectors which converges to $v$. Write $v = v' + \gamma u$ where $v' \perp u$. The remaining vectors (those not included in the subsequences described above) have the form $v_j = v'_j + \gamma_j u$ where $v'_j \perp u$. For any vector $w \perp u$, $w \neq 0$ it holds that $\sum_{j=1}^{\infty} \langle v_j, w\rangle^+ = \sum_{j=1}^{\infty} \langle v'_j, w\rangle^+ = \infty$. So the hypothesis for theorem 3.1 is satisfied on the $n-1$ dimensional space $U^\perp := \{v \in \mathbb{R}^n : \langle v, u\rangle = 0\}$. By the inductive assumption there exists a permutation $\sigma$ such that $\sum_{j=1}^{\infty} v'_{\sigma(j)} = v'$. Rearrange the vectors as follows. Assume that $\gamma > 0$. Add the $\alpha_j$s until the sum is just larger than $\gamma$. Then add $\gamma_{\sigma(1)}$ and subtract $\beta_1$. Then subtract the $\beta_j$s until the sum is just less than $\gamma$. Then add $\gamma_{\sigma(2)}$ and add the next $\alpha_j$ and so on. In this arrangement the order of $\sigma$ is unchanged and the portion of the $v_j$s parallel to $u$ converge to $\gamma u$. Therefore the rearrangement converges to $v = v' + \gamma u$.

The next result will be proved only for the case $n = 2$. (See figure 3.1)

**Theorem 3.7** If $L$ is a closed subset of the unit sphere of $\mathbb{R}^n$ and if every closed half sphere of the unit sphere contains at least one vector from $L$, then for some $r > 0$ there exist vectors $u_0, u_1, \ldots, u_r$ in $L$ and positive real numbers $p_1, \ldots, p_r$ such that $-u_0 = p_1 u_1 + \cdots + p_r u_r$.

*Proof.* If $u_0, -u_0 \in L$, then the theorem is automatically true. Assume that for any $u \in L$, that $-u \notin L$. Let $u_1 \in L$, and choose $u_2 \in L$ such that $\|u_1 - u_2\| \geq \|u_1 - u\|$ for each $u$

in $L$. Then $-u_1 \notin L$ and in the angle between $-u_1$ and $u_2$ there are no vectors from L. Also, $-u_2 \notin L$. Since $L$ is closed, there exists an open set which contains $-u_2$ and is in the complement of $L$. Choose a vector $u'$ in the angle between $-u_2$ and $u_1$ such that $u' \notin L$ and there are no vectors from $L$ in the angle between $-u_2$ and $u'$. On the side of $-u'$ and $u'$ which does not contain $u_1$ there exists a vector $u_0 \in L$ which is in the angle between $-u_1$ and $-u_2$. Then there exist $p_1, p_2 > 0$ such that $u_0 = p_1(-u_1) + p_2(-u_2)$.

**Remark:** While proving theorem 3.1, any number of absolutely convergent series can be added to prove the theorem and the result will still hold for the original series. Let the series $\sum_{j=1}^{\infty} w_j$ be absolutely convergent to $w$. If theorem 3.1 is true for the series $\sum_{j=1}^{\infty}(v_j + w_j)$, then it is also true for the original series $\sum_{j=1}^{\infty} v_j$. To see this, let $v \in \mathbb{R}^n$. Assuming the theorem is true for the modified series, there exists a permutation $\sigma$ such that $\sum_{j=1}^{\infty}(v_{\sigma(j)} + w_{\sigma(j)}) = v + w$. Then

$$\lim_{k\to\infty} \sum_{j=1}^{k} v_{\sigma(j)} = \lim_{k\to\infty} [\sum_{j=1}^{k}(v_{\sigma(j)} + w_{\sigma(j)}) - \sum_{j=1}^{k} w_{\sigma(j)}] = v$$

That is, $\sum_{j=1}^{\infty} v_{\sigma(j)} = v$. So while proving theorem 3.1, if a certain property can be obtained by adding an absolutely convergent series, it will be assumed for simplicity that the original series has that property.

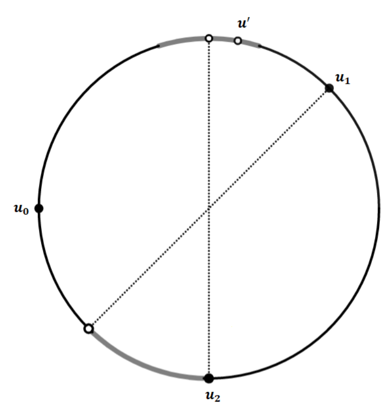


Figure 3.1: The proof of theorem 3.7 for n=2

*Proof of theorem 3.1.* Let $L(v_j) = \{\text{Levy vectors of } \{v_j\}\}$. By theorem 3.7, for some $r \geq 1$ there exist $u_0, u_1, \ldots, u_r \in L(v_j)$ and $p_1, \ldots, p_r > 0$ such that $-u_0 = p_1 u_1 + \cdots + p_r u_r$. By lemma 3.5, for $0 \leq i \leq r$, there exist subsequences of the $\{v_j\}$ of the form $\alpha_{j,i} u_i + w_{j,i}$ with the properties $\alpha_{j,i} \geq 0$, $\sum_{j=1}^{\infty} \alpha_{j,i} = \infty$, $\sum_{j=1}^{\infty} \|w_{j,i}\| < +\infty$. It may be assumed that the subsequences are of the form $\{\alpha_{j,i} u_i\}_{j=1}^{\infty}$. It will be shown that for each fixed $i = 1, \ldots, r$ there exist finite sets $A_{k,i} \subset \{\alpha_{j,i} u_i\}$ for $k \in \mathbb{N}$ with increasing indices such that

$$\| \sum_{v \in A_{k,i}} v - \frac{p_i u_i}{k} \| < \frac{1}{k^2}$$

To construct such sets, first fix $i$, then set $k = 1$. Since the vectors converge to 0, there exists $N_1 > 0$ such that if $j > N_1$, then $\alpha_{j,i} < (1/10k^2)$. Then choose the smallest $m_1$ such that

$$\sum_{j=N_1+1}^{N_1+m_1} \alpha_{j,i} > \frac{p_i}{k}$$

By choice of $m_1$, we get

$$\frac{p_i}{k} < \sum_{j=N_1+1}^{N_1+m_1} \alpha_{j,i} \leq \frac{p_i}{k} + \frac{1}{10k^2}$$

Then

$$| \sum_{j=N_1+1}^{N_1+m_1} \alpha_{j,i} - \frac{p_i}{k} | < \frac{1}{k^2}$$

Set $A_{1,i} = \{\alpha_{N_1+1,i} u_i, \ldots, \alpha_{N_1+m_1,i} u_i\}$. This proves the inequality for fixed $i$ and $k = 1$. Now set $k = 2$ and repeat. There exists $N_2 > N_1 + m_1$ such that if $j > N_2$, then $\alpha_{j,i} < (1/10k^2)$. Then choose the smallest $m_2$ in the same way as for $k = 1$, and set $A_{2,i} = \{\alpha_{N_2+1,i} u_i, \ldots, \alpha_{N_2+m_2,i} u_i\}$. The sets $A_{k,i}$ for $k \geq 3$ are constructed in exactly the same way. This proves the existance of the sets $A_{k,i}$. Now define the remainder term by

$$r_{k,i} = \sum_{v \in A_{k,i}} v - \frac{p_i u_i}{k}$$

Now, subtract each $r_{k,i}$ from the first term of the corresponding set $A_{k,i}$ to arrive at a new set $A'_{k,i}$. Then

$$\| \sum_{v \in A'_{k,i}} v - \frac{p_i u_i}{k} \| = 0$$

Since $\|r_{k,i}\| < (1/k^2)$, the series $\sum_{k=1}^{\infty} r_{k,i}$ is absolutely convergent for each $i = 1, \ldots, r$. So by subtracting these remainder terms, an absolutely convergent series was added to the original series. Therefore, we may assume that the equality above holds for the original series. That is,

$$\sum_{v \in A_{k,i}} v = \frac{p_i u_i}{k}$$

Collect the vectors for $i = 1, \ldots, r$ into the blocks

$$B_k := \{A_{k,1}, A_{k,2}, \ldots, A_{k,r}\}$$

for each integer $k \geq 1$. Each block sums to $\frac{-u_0}{k}$. In the following, we will equate $B_k$ with the sum of its vectors. Apply lemma 3.6 on the sets of vectors $\{\alpha_{j,0}u_0\}_{j=1}^{\infty}$, $\{B_{2k}\}_{k=1}^{\infty}$, and the set which contains the odd $B_k$s with any remaining vectors. The odd $B_k$ sets are included in the third set to insure that the hypothesis of theorem 3.1 holds in the space $U_0^{\perp}$ (which allows for the induction step in lemma 3.6). The rearrangement still converges when the brackets are removed from each block since for any partial sum $P$ of $B_k$: $\|P\| \leq (1/k)\sum_{i=1}^{r} p_i \to 0$ as $k \to \infty$.

*Proof of theorem 3.2.* For each $v_n$ write $v_n = \alpha_n + \beta_n$ where the $\alpha_n$ is in $\mathcal{F}$ and the $\beta_n$ is in $\mathcal{F}^{\perp}$. The series $\sum_{n=1}^{\infty} \alpha_n$ is absolutely convergent to $\alpha$. By theorem 1.1, every rearrangement converges to the same element. That is, $\sum_{n=1}^{\infty} \alpha_{\sigma(n)} = \alpha$ for each $\sigma$ on $\mathbb{N}$. If $u \neq 0$ is in $\mathcal{F}^{\perp}$, then $\sum_{n=1}^{\infty} \langle u, v_n \rangle^+ = \sum_{n=1}^{\infty} \langle u, \beta_n \rangle^+ = \infty$. By theorem 3.1, $\mathrm{SR}(\beta_n) = \mathcal{F}^{\perp}$. It follows that the sum range takes the form

$$\mathrm{SR}(v_n) = \alpha + \mathcal{F}^{\perp}$$

so that the sum $s$ of a convergent rearrangement has the form $s = \alpha + \beta$ where $\beta \in \mathcal{F}^{\perp}$. This proves the theorem.

CHAPTER 4

# Examples in infinite-dimensional spaces

In this section it will be shown that the sum range of a series in $L^p$ or $l^p$ may not be linear, so the Lévy-Steinitz theorem is not true for these spaces. The sum range may not even be closed (See [3]). If an extra condition is added, then there is a Lévy-Steinitz type theorem for the normed spaces (See [3]). The space $\mathbb{R}^\infty$ of all real sequences with pointwise convergence has the Lévy-Steinitz theorem with no additional conditions needed (See [2]).

**Example 4.1**. There exists a series in $L^2(0,1)$ such that SR is not linear.

Let $\phi$ be the characteristic function of $(0,1)$ and for each $n \in \mathbb{N}$ let $\phi_n^{(j)}$ be the characteristic function of $(\frac{j-1}{2^n}, \frac{j}{2^n})$, $j = 1, 2, \ldots, 2^n$. Define

$$A := \{\pm\phi, \pm\phi_1^{(1)}, \pm\phi_1^{(2)}, \ldots\}$$

Both $\phi$ and 0 belong to SR$(A)$ because

$$\phi = \phi + (\phi_1^{(1)} + \phi_1^{(2)} - \phi) + (\phi_2^{(1)} + \phi_2^{(2)} - \phi_1^{(1)}) + \ldots$$

and

$$0 = (\phi - \phi) + (\phi_1^{(1)} - \phi_1^{(1)}) + (\phi_2^{(1)} - \phi_2^{(1)}) + \ldots$$

For the convergence to hold it is necessary that $\phi_n^{(j)} \to 0$ as $n \to \infty$.

Any partial sum $s$ of the functions in $A$ will be a function which takes integer values $\alpha_1, \alpha_2, \ldots, \alpha_r$ on the intervals $(a_0, a_1), (a_1, a_2), \ldots, (a_{r-1}, a_r)$ respectively, where $a_0 = 0$, $a_r = 1$, $a_0 < a_1 < \cdots < a_r$. Choose $k$ in the interval $(0, 1)$. Then

$$(\|s - k\phi\|_2)^2 = |\alpha_1 - k|^2(a_1 - a_0) + \cdots + |\alpha_r - k|^2(a_r - a_{r-1})$$

For each $n$, $|\alpha_n - k| \geq m := \min(k, 1-k)$ it follows that

$$(\|s - k\phi\|_2)^2 \geq m^2 \sum_{n=1}^{r} (a_n - a_{n-1}) = m^2$$

So for any partial sum $s$ of elements in $A$, $\|s - k\phi\|_2 \geq m$. Therefore there is no rearrangement of elements in $A$ which converges to $k\phi$. That is, $k\phi \notin \text{SR}(A)$ so SR is not linear.

**Example 4.2** There exists a series in $l^p$ with $0 < p \leq 2$ such that SR is not linear.

The functions from the set $A$ can be used to make an orthonormal basis for $L^2(0,1)$. The orthonormal basis $\{v_k\}$ consists of the Haar functions

$$v_1 = \phi, \qquad v_2 = \phi_1^{(1)} - \phi_1^{(2)}, \qquad v_3 = \sqrt{2}(\phi_2^{(1)} - \phi_2^{(2)}), \ldots \qquad v_k = \sqrt{2^{n-1}}(\phi_n^{(1)} - \phi_n^{(2)}), \ldots$$

Define a function $T$ which maps $v_k$ into a sequence which has a one in the $k$th position and zero everywhere else. Then $T$ as a linear extension maps the function $\phi_n^{(j)}$ into a sequence $a = \{a_i\}_{i=1}^{\infty}$ which has exactly $n+1$ non-zero terms:

$$\pm\frac{1}{2^n}, \pm\frac{1}{2^n}, \pm\frac{\sqrt{2}}{2^n}, \ldots, \pm\frac{\sqrt{2^{n-1}}}{2^n}$$

(Where we can take plus or minus but not both in each number above). This gives the sum

$$\sum_{i=1}^{\infty} a_i^2 = \frac{1}{(2^n)^2}(1 + 2^n - 1) = \frac{1}{2^n}$$

So for each $i$, $|a_i| \leq (\frac{1}{2^n})^{1/2}$ and non-zero for only $n+1$ values of $i$. For $0 < p < \infty$, $a$ is in $l^p$ with $(\|a\|_p)^p \leq \frac{n+1}{(2^{n/2})^p}$. Then $\|a\|_p \to 0$ as $n \to \infty$. That is, $T(\phi_n^{(j)}) \to 0$ as $n \to \infty$.

Define $B := \{T(f) : f \in A\}$, and let SR$(B)$ denote the sum range of the series with the elements in $B$. Both $T(0)$ and $T(\phi)$ are in SR$(B)$ (see the construction for counter example 4.1 and use the linearity of $T$). If $f$ is any function in $L^2(0,1)$, then $T(f) = T(\sum_{i=1}^{\infty} \alpha_i v_i) = \{\alpha_i\}_{i=1}^{\infty}$. When $0 < p \leq 2$,

$$(\|T(f)\|_p)^p = \sum_{i=1}^{\infty} |\alpha_i|^p \geq \min(1, \sum_{i=1}^{\infty} |\alpha_i|^2) = \min(1, (\|f\|_2)^2)$$

The norm on the right is the $L^2(0,1)$ norm. Then if $0 < k < 1$, and $s$ is a partial sum of the elements in $B$,

$$(\|s - kT(\phi)\|_p)^p \geq \min(1, m^2)$$

where $m = \min(k, 1-k)$. Therefore the set $\mathrm{SR}(B)$ does not contain $kT(\phi)$, so SR is not linear.

It will be shown how to construct a sequence $\{f_k\}$ in $L^2$ such that $\mathrm{SR}(f_k) = L^2$. The proof requires a lemma which can be proved in a way similar to theorem 1.2.

**Lemma.** Given the conditionally convergent series $\sum_{n=t}^{\infty} \frac{(-1)^{n+1}}{n} = a_t + a_{t+1} + \ldots$ where $t \in \mathbb{N}$, and given $a \in \mathbb{R}$. There exists a permutation $P$ on $\{t, t+1, \ldots\}$ such that $\sum_{n=t}^{\infty} a_{P(n)} = a$ and such that for any partial sum $r_k = \sum_{n=t}^{k} a_{P(n)}$, $|r_k| \leq \max(|a|, \frac{3}{t})$.

**Construction of the sequence.** Let $\phi_k$ be a complete orthonormal system in $L^2$. For each $n \in \mathbb{N}$ set $A_n = \{\frac{(-1)^{k+1}}{k}\phi_n\}_{k=n}^{\infty}$ and set $A = \bigcup_{n=1}^{\infty} A_n$. Now we may write $A = \{a_j\}_{j=1}^{\infty}$. For example, let each $A_n$ represent a row of functions:

$$A_1 : a_{1,1}, a_{1,2}, a_{1,3}, \ldots$$

$$A_2 : a_{2,1}, a_{2,2}, a_{2,3}, \ldots$$

$$\vdots$$

$$A_n : a_{n,1}, a_{n,2}, a_{n,3}, \ldots$$

$$\vdots$$

Then enumerate these functions using the same idea as in the proof that the rational numbers are countable. Let $f \in L^2$. Then $f = \sum_{k=1}^{\infty} c_k\phi_k$ and $\sum_{k=1}^{\infty} |c_k|^2 < \infty$. For each $n \in \mathbb{N}$ permute the terms in $A_n$ according to the lemma so that the rearrangment converges to $c_n$. Then define the permutation $P$ to be the composition of all these permutations each restricted to only the elements in its respective set $A_n$. It will be shown that this permutation $P$ defines a rearrangement of the functions in $A$ which converges to $f$.

Let $\varepsilon > 0$ be given, then fix $N \in \mathbb{N}$ such that $\|f - \sum_{k=1}^{N} c_k\phi_k\| < \varepsilon$ and

$$\sum_{N+1}^{\infty} \max(|c_k|^2, \frac{9}{k^2}) < \varepsilon^2$$

Now let $s_t$ denote a partial sum of the rearrangement, and let $s_{t,k}$ represent the portion of $s_t$ which has the $\phi_k$ terms. There exists $N' > 0$ such that if $t > N'$, then $\|c_k\phi_k - s_{t,k}\| < \frac{\varepsilon}{N}$ for $k = 1, \ldots, N$. Then if $t > N'$,

$$\begin{aligned}
\|f - s_t\| &\leq \|f - \sum_{k=1}^{N} c_k\phi_k\| + \|\sum_{k=1}^{N} c_k\phi_k - s_t\| \\
&\leq \varepsilon + \sum_{k=1}^{N} \|c_k\phi_k - s_{t,k}\| + \|\sum_{k=N+1}^{\infty} s_{t,k}\| \\
&\leq 2\varepsilon + (\sum_{k=N+1}^{\infty} \max(|c_k|^2, \frac{9}{k^2}))^{\frac{1}{2}} \\
&< 3\varepsilon
\end{aligned}$$

This proves that $s_t \to f$ which proves the claim.

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LIST OF REFERENCES

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