RICCI TENSOR UNDER CONFORMAL CHANGE OF METRIC AS AN ELEMENTARY OBSTRUCTION TO CERTAIN EINSTEIN METRICS

A Thesis by

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The following faculty members have examined the final copy of this thesis for form and content, and recommend that it be accepted in partial fulfillment of the requirement for the degree of Master of Science with a major in Mathematics.

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ABSTRACT

Gregorio Ricci-Curbastro, (1853-1925) Bologna, was an Italian mathematician and a professor at the University of Padua from 1880 -1925. He was the first to introduce the systematic theory of tensor analysis in 1887 with a a major contribution later by his student Tullio Levi-Civita. However, the roots of tensor analysis were laid by the work of German mathematician Bernhard Riemann in Differential Geometry. The beginning of the twentieth century was the emergence of the study of the Ricci Tensor due to its major role in the mathematical formulation of the theory of general relativity of Albert Einstein. Also, as one of the important geometric features that have been used to prove several major theorems in differential geometry and topology. Here I will focus on the idea of the conformal change of metrics and what is the necessary and sufficient condition for a metric to be conformal to another metric on the same manifold, and how geometric objects like Riemannian and Ricci curvature are related under this change.

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CHAPTER 1

Introduction

In this paper, I will start by giving several important geometric definitions that I will use throughout the paper—starting with manifold, inner product, metric, real-valued function, and vector space. I assume that the manifold M is of n- dimension, connected, differentiable, and without boundaries. Let (M,g) be a manifold M with a metric g because we might define several metrics on the same manifold. If $\tilde{g}$ is another metric on M, then g and $\tilde{g}$ are conformal to each other if there exists a function f on M such that $\tilde{g} = e^f g$. The relation g $\rightarrow \tilde{g}$, is called a conformal change of metric. The core work in this paper is to show how the Ricci tensor of the new metric, $\tilde{g}$ is related to that of the original metric g, under the conformal change of metric. The resulting tensor equation is well known and studied, Schoen, Yau [**?**]. However, no detailed proof was given, and here I will present a through and detailed proof. Examining this equation that related the new and the original Ricci tensors. One can ask, can we make the conformal change of metric that the new Ricci tensor, $\tilde{R}$ satisfy a certain property? Namely, can the conformal change be made in such a way that the new metric is Einstein? However, it is not easy to solve such an equation and expect to find the desired metric simply because it is difficult, and there is no guarantee for a solution to exist. Therefore in the hope of solving this equation, I split the equation into a system of PDE's. I derived motivation for the possibility of reduced the system into a single scalar equation which is easier to deal with. I assumed the existence of a special kind of solution; however, I found an obstruction that shows such a solution is trivial and there is no way to find a solution by breaking up the original equation into a system of equations. The method used to reach this conclusion is some basic elementary techniques used in differential geometry and PDE's.

1.1 Preliminary Definitions

Definition :

A manifold M is a topological space that is Hausdorff with a countable basis. Also, $\forall$ p $\in$ M , $\exists$ an open set U $\subset$ M containing p, an open set U' $\subset$ R^n, and a homeomorphism φ : U $\to$ U' and ψ : V $\to$ V'. Also we get
$\psi \circ \varphi^{-1} : \varphi(\mathrm{U} \cap \mathrm{V}) \to \psi(\mathrm{U} \cap \mathrm{V})$, a homeomorphism between two open sets in $\mathbb{R}^n$.
In coordinate $\psi \circ \varphi^{-1} : \varphi(\mathrm{U} \cap \mathrm{V}) \to \psi(\mathrm{U} \cap \mathrm{V})$, is given by continuous functions

$$y^i = h^i(x^1, \ldots, x^n),\ i = 1, \ldots, n$$

given the y-coordinates of each q $\in$ $(U \cap V)$ in term of its x-coordinates. Similarly $\varphi \circ \psi^{-1}$ gives the inverse mapping which expresses the x-coordinates as functions of the y-coordinates

$$x^i = g^i(y^1, \ldots, y^n),\ i = 1, \ldots, n$$

The fact that $\varphi \circ \psi^{-1}$ and $\psi \circ \varphi^{-1}$ are homeomorphism and are inverse to each other is equivalent to the continuity of $h^i(x)$ and $g^j(y)$, i,j= 1. . .,n.

Definition:

We shall say that (U, φ) and (V , ψ) are C^∞ -compatible if U $\cap$ V is nonempty implies $h^i(x)$ and $g^j(y)$ given the change of coordinates are C^∞ ; this is equivalent to requiring that $\varphi \circ \psi^{-1}$ and $\psi \circ \varphi^{-1}$ to be diffeomorphisms of the open subsets $\varphi(U\cap \mathrm{V})$ and $\psi(U\cap \mathrm{V})$

of $\mathbb{R}^n$ [**?**].

Definition :

A differentiable or C^∞ (or smooth structure) on a topological manifold M is the family $\mathcal{U} = \{$ U$_\alpha$, $\varphi_\alpha\}$ of coordinate neighborhood such that :

1) The U_α covers M , ie $\cup_\alpha$ U$_\alpha$ = M.

2) For any α, β the neighborhood U_α, φ_α and U_β, φ_β are C^∞ - compatible.

3) Any coordinate neighborhood (V , ψ) compatible with every (U_α, φ_α) is also in $\mathcal{U}$.

A C^∞ manifold is a topological manifold with a C^∞ differentiable structure(see Fig 1.1).

Theorem

Let M be a Hausdorff space with a a countable basis of open sets. If V =$\{V_\beta, \psi_\beta\}$ is a covering of M by C^∞ -compatible coordinate neighborhoods, then there is a unique C^∞ structure on M containing these coordinate neighborhoods.

Examples :

(1) The Euclidean plane once a unit of length is chosen, the Euclidean plane E^2 becomes a metric space. It is Hausdorff and has a countable basis of open sets; the choice of an origin and mutually perpendicular coordinate axes establishes a homeomorphism (even an isometry).

$\psi : E^2 \to \mathbb{R}^2$, thus we cover E^2 with a single coordinate neighborhood V , ψ with V

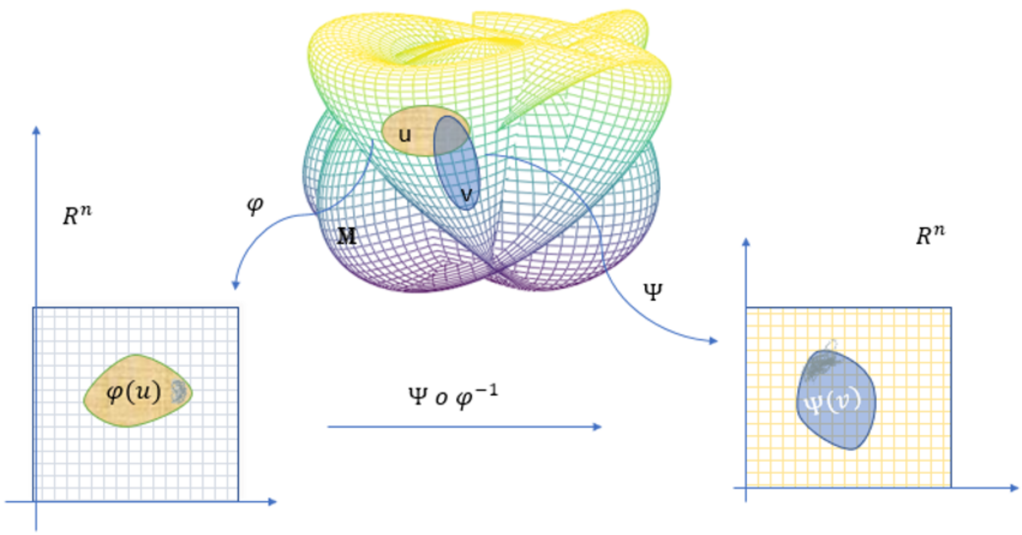

Figure 1.1: M is a differential manifold

$= E^2$ and $\psi(V) = R^2$. It follows that not only that E^2 is a topological manifold, but by above theorem V,ψ determines a differentiable structure, so E^2 is a C^∞ manifold.[**?**]

(2) The sphere $S^2 = \{(x, y, z) \in R^3 | x^2 + y^2 + z^2 = 1\}$ is a C^∞ manifold, following from the fact that the S^2 is a regular surface.

The sphere

$$x^2 + y^2 + z^2 = 1$$

is a regular surface. in fact, it is the set $f^{-1}(0)$ where

$$f(x, y, z) = x^2 + y^2 + z^2 - 1$$

is a differentiable function and 0 is a regular value of f. This follows from the Inverse Function Theorem and the fact that the partial derivatives $f_x = 2x$ $f_y = 2y$, $f_z = 2z$ vanish simultaneously only at the point (0,0,0), which does not belong to $f^{-1}(0)$[**?**].

Let f be a real-valued function defined on an open set W_f, of a C^∞ manifold M. in brief f: $W_f \mapsto R$ If U, ϕ is a coordinate neighborhood such that $W_f \cap U \neq \emptyset$ and if $x^1, ..., x^n$ denotes the local coordinates, then f corresponds to a function $\hat{f}(x^1, ..., x^n)$ on $\phi(W_f \cap U)$ defined by $\hat{f} = f \circ \phi^{-1}$, that is, so that f(p) $= \hat{f}(x^1(p), ..., x^n(p)) = \hat{f}(\phi(p))$ for all p $\in W_f \cap U$(see Fig 1.2).

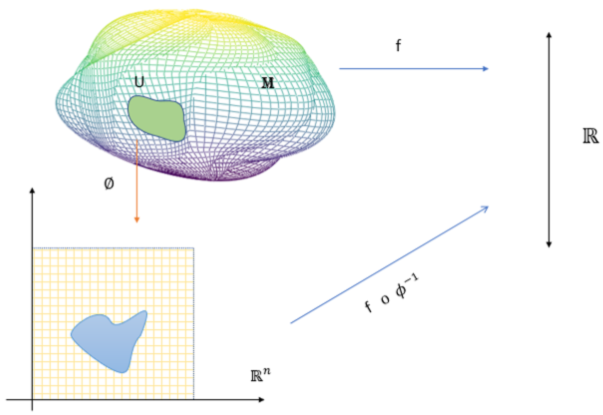

Figure 1.2: A real-valued function f

Using the notation above f: $W_f \mapsto$ R is a C^∞ if each P$\in W_f$, lies in a coordinate neighborhood U, ϕ such that $f \circ \phi^{-1}(x^1 \ldots x^n) = \hat{f}(x^1, ..., x^n)$ is C^∞ on $\phi(W \cap U)$.

One of the most important tools used to study differential manifolds is the tangent space denoted $T_p(M)$. Each element of tangent space X_p can be viewed as an operator on C^∞ functions. Now we are ready to define a tangent space.

Definition :

The tangent space $T_p(M)$ to M at p is the set of all mappings X_p: $C^\infty(M) \mapsto \mathbb{R}$ satisfying for all $\alpha, \beta \in$ R and f, g $\in C^\infty(p)$ the two conditions :

$$(i)X_p(\alpha f+\beta g)=\alpha(X_p f)+\beta(X_p g).$$

$$(ii)X_p(fg)=(X_p f)g(p)+f(p)(X_p g)$$

with the vector space operations in $T_p(M)$ defined by

$$(X_p+Y_p)f=X_p f+Y_p f$$

$$(\alpha X_p f)=\alpha(X_p f)$$

A tangent vector to M at p is any $X_p \in T_p(M)$.

Definition:

A vector field on an open subset U $\subset \mathbb{R}^n$ is a function which assigns to each point p $\in$ U a vector $X_p \in T_p(\mathbb{R}^n)$.

Definition:

An inner product on a real vector space V is a function :

$$\langle\ ,\ \rangle : V\times V \longrightarrow \mathbb{R}$$

In other words it assigns to each pair of vectors v, w in V a number with the following properties:

1) Bilinearity

$$\langle a_1v_1 + a_2v_2, w\rangle = a_1\langle v_1, w\rangle + a_2\langle v_2, w\rangle$$

$$\langle v, b_1w_1 + b_2w_2\rangle = b_1\langle v, w_1\rangle + b_2\langle v, w_2\rangle$$

2) Symmetry:

$$\langle v, w\rangle = \langle w, v\rangle$$

3) Positive Definiteness:

$$\langle v, v\rangle \geq 0 \ ; \ \langle v, v\rangle = 0 \iff v = 0.$$

For example on the vector space R^2 the dot product $\langle v.w\rangle = v_1w_1 + v_2w_2$ is, of course, an inner product, but there are infinitely many others; for instance, $\langle v, w\rangle = 3v_1w_1 + 2v_2w_2$. Basic features of the dot product remain valid for arbitrary inner products. The length of a vector v is $\|v\| = \sqrt{\langle v, v\rangle}$, and vectors are orthogonal if $\langle v, w\rangle = 0$.[**?**]

These inner products are required to vary smoothly in the sense that if v and w are differentiable vector fields on M, then $\langle v, w\rangle$ is a differentiable real-valued function on M. The geometric structure provided by this collection of inner products can be described as a metric tensor g on M, that is, a function on all ordered pairs of tangent vectors v, w at points p of M such that:

$$g_p(v,w) = \langle v,w \rangle_p.$$

On a tangent space we can define the positive definite function $g_{ij} = \langle \frac{\partial}{\partial x_i}, \frac{\partial}{\partial x_j} \rangle$, and so the g_{ij} becomes a smooth function $g_{ij} : U \mapsto R$ defines at each point in the Manifold, which is called a Riemannian Manifold.[**?**]

Definition:

A manifold M on which there is defined a field of symmetric, positive definite, bilinear forms g is called a Riemannian manifold and g the Riemannian metric.

By defining the metric g on a manifold we are able to measure the length of a vector in a single vector space. Also it is possible to find the angle between two vectors.The measure of length of a curve can be define as:

Definition :

Let α : [a,b] $\rightarrow$ M be a piece-wise smooth curve segment in the Riemannian manifold M.

The arc length of α is

$$L(\alpha) = \int_a^b \left| \alpha' \right| ds$$

By definition :

$$\left|\alpha'\right| = \left|\langle\alpha', \alpha'\rangle\right|^{1/2}$$

In coordinate,

$$\left|\alpha'\right| = \left|\sum_{ij} g_{ij} \frac{d(x^i \circ \alpha)}{ds} \frac{d(x^j \circ \alpha)}{ds}\right|^{1/2}$$

From this the distance between two points, p and q, is the infemum of length over all curves from p to q.

By means of natural isomorphism the dot product can be translated to operate on each tangent space. This allows one to perform a natural geometric operations like measuring the length of the tangent vector, or the angle between two tangent vectors.

Let V , W be two vector fields on the Riemannian manifold M, then it is possible to define a new vector field called the covariant derivative of W with respect to V, $\nabla_V W$ as shown below.

Definition :

An affine connection on a C^∞ manifold is a map[?]:

$\nabla : \chi(m) \times \chi(m) \longrightarrow \chi(m)$, such that

$$(X, Y) \longrightarrow \nabla_X Y$$

1) $\nabla_{fX+gY} Z = f\nabla_X Z + g\nabla_Y Z$

2)$\nabla_X (Y + Z) = \nabla_X Y + \nabla_X Z$

3)$\nabla_X fY = f\nabla_X Y + (Xf)Y$

If an inner product is present on M, a connection might also satisfies the following symmetry condition :

4) [X,Y] =$\nabla_X Y - \nabla_Y X$

5) X(Y,Y')= $(\nabla_X Y, Y') + (Y, \nabla_X Y')$

It turns out that if (M,g) is a Riemannian manifold, then a connection ∇ satisfying the last two properties is uniquely determined. This is called the Levi-Civita connection.

CHAPTER 2

Christoffel Symbols

Let $\nabla : \chi(m) \times \chi(m) \longrightarrow \chi(m)$, be an affine connection on a manifold M, where $\chi(m)$ denotes the space of vector fields on M. Let (U,ϕ) be a chart of M. Then on U, ∇ is completely determined by $\nabla_{\frac{\partial}{\partial x^i}} \frac{\partial}{\partial x^j}$, where $x^1, \ldots, x^n$ are coordinates on U. The Christoffel symbols Γ of the connection∇ are now given by:

$$\nabla_{\frac{\partial}{\partial x^i}} \frac{\partial}{\partial x^j} = \sum_k \Gamma^k_{ij} \frac{\partial}{\partial x^k}$$

2.1 Calculation of Christoffel Symbols

The notion of intrinsic geometry of a surface enable us to measure the distance and curvature of the surface without referring to ambient space.

The covariant derivative is the rate of change of the vector field $\nabla_{\frac{\partial}{\partial x_i}} \vec{v}$ in the direction of $\frac{\partial}{\partial x_i}$

$$\nabla_{\frac{\partial}{\partial x_i}} \vec{v} = \frac{\partial \vec{v}}{\partial x_i}$$

Write $\vec{v} = v^j . \frac{\partial}{\partial x_j}$

So the right hand side becomes

$$= \frac{\partial}{\partial x_i}(v^j . \frac{\partial}{\partial x_j})$$

$$= \frac{\partial v^j}{\partial x_i}(\frac{\partial}{\partial x_j}) + v^j \nabla_{\frac{\partial}{\partial x_i}}(\frac{\partial}{\partial x_j})$$

We know that :

$$\nabla_{\frac{\partial}{\partial x^i}} \frac{\partial}{\partial x^j} = \sum_k \Gamma_{ij}^k \frac{\partial}{\partial x^k}$$

Substitute back :

$$\nabla_{\frac{\partial}{\partial x_i}} \vec{v} = \frac{\partial v^j}{\partial x_i}(\frac{\partial}{\partial x_j}) + v^j \sum_k \Gamma_{ij}^k \frac{\partial}{\partial_k}$$

The Levi-Civita connection is determined uniquely by the metric. Because of this, the connection is ultimately determined completely by g, and therefore it is possible to express Γ_{ij}^k in terms of (g_{ij}). Note that the metric tensor g_{ij} is an intrinsic object and it is defined as the dot product of the vector basis $\langle \frac{\partial}{\partial x_i}, \frac{\partial}{\partial x_j} \rangle$ and since it is commutative therefore $\langle \frac{\partial}{\partial x_i}, \frac{\partial}{\partial x_j} \rangle$ is equals to $\langle \frac{\partial}{\partial x_j}, \frac{\partial}{\partial x_i} \rangle$ and from which g_{ij} is equal to g_{ji}. Also the same is true for the

Christoffel symbols, therefore $\sum_k \Gamma^k_{ij} = \sum_k \Gamma^k_{ji}$ for all i,j

For the inner product to hold we have

$$\frac{\partial}{\partial x_k}\langle\frac{\partial}{\partial x_i}\ ,\ \frac{\partial}{\partial x_j}\rangle = \langle\Gamma^l_{ki}\ \frac{\partial}{\partial x_l}\ ,\ \frac{\partial}{\partial x_j}\rangle + \langle\frac{\partial}{\partial x_i}\ ,\ \Gamma^l_{kj}\frac{\partial}{\partial x_l}\rangle$$

It follows that

$$\frac{\partial}{\partial x_k}\langle\frac{\partial}{\partial x_i}\ ,\ \frac{\partial}{\partial x_j}\rangle + \frac{\partial}{\partial x_i}\langle\frac{\partial}{\partial x_j}\ ,\ \frac{\partial}{\partial x_k}\rangle - \frac{\partial}{\partial x_j}\langle\frac{\partial}{\partial x_k}\ ,\ \frac{\partial}{\partial x_i}\rangle = 2\langle\Gamma^l_{ki}\ \frac{\partial}{\partial x_i}\ ,\ \frac{\partial}{\partial x_l}\rangle$$

or,equivalently,

$$\frac{\partial}{\partial x_k}g_{ij} + \frac{\partial}{\partial x_i}g_{jk} - \frac{\partial}{\partial x_j}g_{ki} = 2\sum_l \Gamma^l_{ki}g_{il}$$

$$2\sum_l \Gamma^l_{ki}g_{il}g^{im} = g^{im}(\frac{\partial}{\partial x_k}g_{ij} + \frac{\partial}{\partial x_i}g_{jk} - \frac{\partial}{\partial x_j}g_{ki})$$
$$\sum_l \Gamma^l_{ki}\delta^m_l = (1/2)g^{im}(\frac{\partial}{\partial x_k}g_{ij} + \frac{\partial}{\partial x_i}g_{jk} - \frac{\partial}{\partial x_j}g_{ki})$$

By Kronecker delta rules the above expression becomes:

$$\Gamma^m_{jk} = (1/2)\sum_l g^{im}(\frac{\partial}{\partial x_k}g_{ij} + \frac{\partial}{\partial x_j}g_{ki} - \frac{\partial}{\partial x_i}g_{jk})$$

This is the formula for the Christoffel symbols[**?**].

2.2 Calculating the Christoffel Symbols for a Sphere

For a sphere S^2 with metric g induced by that of $\mathbb{R}^3$ we can calculate the Christoffel symbols. First we find the metric g using a parametrizaion of a sphere :

X(u,v) = (cos(v) sin(u),sin(v) sin(u),cos(u))

$$\vec{X}_u = (cos(v)cos(u), sin(v)cos(u), -sin(u))$$

$$\vec{X}_v = (-sin(v)sin(u), cos(v)sin(u), 0)$$

Since the metric tensor g is the inner product of these first derivatives :

$$\langle \vec{X}_u, \vec{X}_u \rangle = [cos(v)^2 cos(u)^2] + [sin(v)^2 cos(u)^2] + [sin(u)^2]$$

$$= [cos(v)^2 + sin(u)^2]cos(u)^2 + sin(u)^2$$

$$= 1$$

$$\langle \vec{X}_v, \vec{X}_v \rangle = sin(v)^2 sin(u)^2 + cos(v)^2 sin(u)^2$$

$$= [sin(v)^2 + cos(u)^2] sin(u)^2$$

$$= sin(u)^2$$

$$\langle \vec{X}_u, \vec{X}_v \rangle = -cos(v)cos(u)sin(v)sin(u) + cos(v)cos(u)sin(v)sin(u)$$

$$= 0$$

$$g_{ij} = \begin{bmatrix} 1 & 0 \\ 0 & (sin(u))^2 \end{bmatrix}$$

Clearly the derivative of each component is zero except for $\frac{\partial g_{22}}{\partial x_1}$ which equals to $2sin(u)cos(u)$. Since the metric tensor g is invertible, g^{22} is equals to $\frac{1}{sin(u)^2}$:

$$\Gamma^2_{21} = (1/2)g^{22}(\frac{\partial}{\partial x_1} g_{22} + \frac{\partial}{\partial x_2} g_{12} - \frac{\partial}{\partial x_2} g_{21})$$

$$\Gamma^2_{21} = (1/2)\frac{1}{(sin(u))^2}(2sin(u)cos(u))$$

$$= cot(u) = \Gamma^2_{12}$$

$$\Gamma^1_{22} = (1/2)g^{11}(\frac{\partial}{\partial x_2}g_{12} + \frac{\partial}{\partial x_2}g_{21} - \frac{\partial}{\partial x_1}g_{22})$$

$$\Gamma^1_{22} = -(1/2)(2sin(u)cos(u))$$

$$= -(1/2)sin(2u)$$

CHAPTER 3

Conformal Change of Metric

On a Riemannian manifold (M,g) the metric $\tilde{g}$ is confomally related to g if there exists a function f on M such that

$$\tilde{g} = e^f g$$

This means that quantities on the Riemannian manifold remains invariant under this conformal change of metrics.That is, under this change, the length of the vectors get stretched, but the angle between any two vectors is preserved[**?**].

To show that by direct computations, for any vectors v , w

$$\langle v, w \rangle_g = \|v\|_g \|w\|_g \, cos(\theta)_g$$

For the new metric

$$\langle v, w \rangle_{\tilde{g}} = \|v\|_{\tilde{g}} \|w\|_{\tilde{g}} \, cos(\theta)_{\tilde{g}}$$

$$e^f \langle v, w \rangle_g = \sqrt{e^f} \|v\|_g \sqrt{e^f} \|w\|_g \, cos(\theta)_{\tilde{g}}$$

Which simplified to

$$\langle v, w \rangle_g = \|v\|_g \|w\|_g \, cos(\theta)_{\tilde{g}}$$

Thus

$$(\theta)_g = (\theta)_{\tilde{g}}$$

However the change in vector magnitude can be measured

$$\|v\|_{\tilde{g}} = \sqrt{\langle v, w\rangle_{\tilde{g}}} = \sqrt{e^f \langle v, w\rangle_g}$$

$$= \sqrt{e^f}\|v\|_g$$

One can ask the question, if $\tilde{g} = e^f g$ then how is the connection and the curvature and the Ricci curvature of $\tilde{g}$ related to that of g? The answer lies in the following Theorem :

Theorem Let (M,g) be a Riemannian manifold and f a real-valued function on M. Then the Riemannian metric $\tilde{g} = e^f g$ has the following invariants :

1) $\tilde{\Gamma}^m_{jk}, \Gamma^m_{jk}$ are related by

$$\tilde{\Gamma}^m_{jk} = \boxed{(1/2)\ \partial_k f \delta^m_j + (1/2)\partial_j f \delta^m_k - (1/2)\nabla^m f g_{jk} + \Gamma^m_{jk}}$$

2) $\tilde{R}_{ij}, R_{ij}$ are related by

$$\tilde{R}_{ij} = \boxed{\mathrm{R}_{ij} - (1/2)(n-2)(Hf)_{ij} + (1/4)(n-2)(df \otimes df)_{ij} - (1/4)(n-2)\|df\|^2 g_{ij} - (1/2)(\Delta f)g_{ij}}$$

Although the result appears in [SY] [?] and in other sources, I have not found this calculation in the literature. It is possible that it does appear. My proof uses the coordinate point of view, and the results were independently obtained, thereby supplying a proof to the above Theorem.

It is important to introduce the idea of obstruction. In mathematics, when dealing with certain problems and under some constraints, it is possible to face an obstruction. This means that it is impossible to find a way to solve the given problem under such conditions. To be precise, let's consider the following examples :

The Gauss-Bonnet theorem says that the total sectional curvature on a surface is equal to a multiple of the Euler characteristic, which is a topological invariant quantity. For a sphere the Euler characteristic is 2, but that of a 2-handled torus is -2. Now if I bend and deform the two handles torus, its Euler characteristic will remain the same,therefore it is impossible to get a manifold of constant curvature say equals to 1 out of it.

another example :

If a simply connected manifold has constant sectional curvature, then it is $\mathbb{H}^n, \mathbb{R}^n$, or $\mathbb{S}^n$

This imposes extreme topological restrictions on what kind of manifold can admit a metric of constant sectional curvature. Note that $\mathbb{H}^n, \mathbb{R}^n$ are topologically equivalent while

$\mathbb{S}^n$ has a distinct topology. Then any compact manifold with a constant sectional curvature should be isometric to the quotient of $\mathbb{H}^n, \mathbb{R}^n$. So the obstruction here is that it is impossible to find a simply connected manifold that admits a metric of constant sectional curvature, and it is not diffeomorphic to $\mathbb{H}^n, \mathbb{R}^n$ or $\mathbb{S}^n$. These results has been proven by H.Hopf[**?**]

In higher dimension, specifically n=4 the Euler characteristics can be obtained by the following formula :

$$\chi(M) = \frac{1}{8\pi^2} \int (\|U\|^2 - \|Z\|^2 + \|W\|^2)\mu_g.$$

If the manifold M admits Einstein metrics, then Z $\equiv 0$. Therefore $\chi(M)$ is positive and it is zero only if (M,g) is flat.It follow that $T_4\#$ T_4 admits not Einstein metrics. This can proved by direct calculations :
let T_4 be the volume of the regular 4-dimensional ideal geodesic simplex in the real hyperbolic 4-dimensional space. It is commutable and its value is ≈ 0.26889. On the other hand we have:

$$Vol(T_4) = \frac{4.\pi^2}{3}\chi(T_4)$$

From which $\chi(T_4) = 0.01512$. Now use the formula to calculate $\chi(T_4\#T_4) = \chi(T_4) + \chi(T_4) - 2$, which is less than zero[**?**].

Relating to the main problem in this paper, we will find an obstruction, and then we will see that only in a very special case it is possible to find an Einstein metric that solves the system of PDE's.

3.1 The Results of The Conformal Change of metric

As of before we have calculated the Christoffel symbols :

$$\Gamma^m_{jk} = (1/2)g^{im}(\frac{\partial}{\partial x_k}g_{ij} + \frac{\partial}{\partial x_j}g_{ki} - \frac{\partial}{\partial x_i}g_{jk})$$

Now the Christoffel symbols for the conformal metric are calculated as follow :

$$\tilde{g}_{ij} = e^f g_{ij}$$

Knowing that g is invertible then,

$$\tilde{g}^{ij} = e^{-f} g^{ij}$$

$$\tilde{\Gamma}^m_{jk} = (1/2)e^{-f}g^{im}(\frac{\partial}{\partial u^k}(e^f g_{ij}) + \frac{\partial}{\partial u^j}(e^f g_{ki}) - \frac{\partial}{\partial u^i}(e^f g_{jk}))$$

$$= (1/2)e^{-f}g^{im}\left[(\partial_k e^f g_{ij} + e^f \partial_k g_{ij}) + (\partial_j e^f g_{ki} + e^f \partial_j g_{ki}) - (\partial_i e^f g_{jk} + e^f \partial_i g_{jk})\right]$$

Where $\frac{\partial}{\partial x_k} = \partial_k$.

$$= (1/2)e^{-f}g^{im}\left[e^f\partial_k f g_{ij} + e^f\partial_k g_{ij} + e^f\partial_j f g_{ki} + e^f\partial_j g_{ki} - e^f\partial_i f g_{jk} - e^f\partial_i g_{jk}\right]$$

$$= (1/2)e^{-f}g^{im}\left[e^f\partial_k f g_{ij} + e^f\partial_j f g_{ki} - e^f\partial_i f g_{jk} + e^f(\partial_k g_{ij} + \partial_j g_{ki} - \partial_i g_{jk})\right]$$

Distributing $(1/2)e^{-f}g^{im}$ and by putting $e^{-f}e^f = 1$, $g_{ab}g^{bc} = \delta_a^c$ and $g^{ab}\partial_b f = \nabla^a f$ then we will get:

$$\tilde{\Gamma}_{jk}^m = (1/2)\partial_k f\delta_j^m + (1/2)\partial_j f\delta_k^m - (1/2)\nabla^m f g_{jk} + \Gamma_{jk}^m$$

Definition :

∇f is the unique vector field having the property that $\forall$ vector field in V , $\langle\nabla f, V\rangle$ = Vf.

This is a complete characterization of ∇f ; if the projections in all directions are known, then the vector is known (follow from the non-degerenative prperty of $\langle\ ,\ \rangle$)[**?**].

Now we will derive a formula for the grad f. Since the ∇f is a vector field we can express it in local basis ,

$$\nabla f \ = \sum_{i=1}^{n} a_i \frac{\partial}{\partial x_i}$$

Here a_i to solve for, so we have

$$\langle \nabla f, \frac{\partial}{\partial x_j} \rangle = \sum_{i}^{n} a_i \langle \frac{\partial}{\partial x_i}, \frac{\partial}{\partial x_j} \rangle = \sum_{i}^{n} a_i g_{ij}$$

On the other hand ,

$$\langle \nabla f, \frac{\partial}{\partial x_j} \rangle = \frac{\partial f}{\partial x_j}$$

By equating the two expressions we get

$$\sum_{i}^{n} a_i g_{ij} = \frac{\partial f}{\partial x_j}$$

Now we want to solve for a_i , we multiply both sides by g^{jk} we get :

$$\sum_{i}^{n} a_i g_{ij} g^{jk} = \frac{\partial f}{\partial x_j} g^{jk}$$

Summing over j on both sides :

$$\sum_{ij}^{n} a_i g_{ij} g^{jk} = \sum_{j}^{n} \frac{\partial f}{\partial x_j} g^{jk}$$

$$\sum_{i}^{n} a_i \delta_i^k = \sum_j \frac{\partial f}{\partial x_j} g^{jk}$$

$$a_k = \sum_j \frac{\partial f}{\partial x_j} g^{jk}$$

So we get :

$$\nabla f = \sum_{jk} g^{jk} \frac{\partial f}{\partial x_j} \frac{\partial}{\partial x_k}$$

For the j-th componenets

$$\nabla^j f = \sum_k g^{kj} \frac{\partial f}{\partial x_k}$$

Definition :

The gradient grad f of a function f $\in C^{\infty}$ (M) is a vector field metrically equivalent to differential df $\in$ $X(M)$, Thus for all X $\in$ X(M).

$$\langle \nabla f, X \rangle \ = \langle \sum_{ij} g^{ij} \frac{\partial f}{\partial x_j} \frac{\partial}{\partial x_i}, x^k \frac{\partial}{\partial x_k} \rangle$$

$$= g^{ij} \frac{\partial f}{\partial x_i} x^k \langle \frac{\partial}{\partial x_j}, \frac{\partial}{\partial x_k} \rangle$$

$$= g^{ij} g_{jk} \frac{\partial f}{\partial x_i} x^k$$

$$= \frac{\partial f}{\partial x_i} x^i = df(X) = Xf$$

3.1.1 The Hessian

Definition :

The Hessian of a function f is its second covariant differential $\text{H}^f = D(Df)$[**?**] ,Since Df = df

As a consequences we have:

$$H^f(X, Y) = D(df)(X, Y)$$

$$= D_Y(df)(X)$$

$$= Y(df(X)) - df(D_Y X)$$

$$= YXf - df(D_Y X)$$

$$= YXf - (D_Y X)f$$

Because,

$$XY - YX = [X, Y] = D_X Y - D_Y X$$

we can reverse X and Y in the preceding formula showing that the H^f is symmetric. In coordinate the Hessian of a function can be expressed as

$$H^f = \frac{\partial^2 f}{\partial x^i \partial x^j} - \sum_k \Gamma^k_{ij} \frac{\partial f}{\partial x^k}$$

3.2 The Riemannian curvature tensor

Riemann curvature tensor is mainly used to express the curvature in the Riemannian manifold, and its measures the failure of commutativity of the connection.

$$R(X, Y)Z = \nabla_Y \nabla_X Z - \nabla_X \nabla_Y Z + \nabla_{[X,Y]} Z$$

Since the connection ∇ depends on g, R is ultimately determined completly by g. Naturally it is the derivative of g, because it is the change in the inner product as we move from point to point, that gives a manifold it's shape.[**?**]

If X,Y $\in T_pM$ are linearly independent,

$$K(X, Y) = \frac{\langle R(X, Y)X, Y\rangle}{\|X\|^2 \|Y\|^2 - \langle X, Y\rangle^2}$$

is the sectional curvature at p with respect to the 2-plain span(X,Y).

Converting to tensor index notation, the Riemannian curvature tensor cn be expressed in terms of Christoffel symbols as :

$$R^m{}_{imj} = \sum_l \Gamma^l{}_{ij}\ \Gamma^m{}_{ml} - \sum_l \Gamma^l{}_{mj}\ \Gamma^m_{il} + \partial_m \Gamma^m_{ij} - \partial_i \Gamma^m{}_{mj}$$

Now we are interested to measuring the Riemannian tensor of a new conformal metric $\tilde{g}$, which can be written as:

$$\tilde{R}^m{}_{imj} = \sum_l \tilde{\Gamma}^l{}_{ij}\ \tilde{\Gamma}^m{}_{ml} - \sum_l \tilde{\Gamma}^l{}_{mj}\ \tilde{\Gamma}^m_{il} + \partial_m \tilde{\Gamma}^m_{ij} - \partial_i \tilde{\Gamma}^m{}_{mj}$$

So we have:

$$\tilde{R}^m_{imj} = \sum_{ml} \Gamma^l_{ij} \Gamma^m_{ml} + (n/2) \sum_l \Gamma^l_{ij} \partial_l f + (1/2) \sum_m \Gamma^m_{ij} \partial_m f - (1/2) \sum_{lm} \Gamma^l_{ij} g_{lm} \nabla^m f + (1/2) \sum_m \Gamma^m_{mi} \partial_j f$$

$$+(n/4)\partial_j f\partial_i f+(1/4)\partial_j f\partial_i f-(1/4)\sum_m \partial_j f g_{mi}\nabla^m f+(1/2)\partial_i f\sum_m \Gamma^m_{mj}+(n/4)\partial_i f\partial_j f+(1/4)\partial_i f\partial_j f$$

$$-(1/4)\sum_m g_{mj}\partial_i f\nabla^m f-(1/2)g_{ij}\sum\nolimits_{lm}\Gamma^m_{ml}\nabla^l f-(n/4)g_{ij}\sum\nolimits_l \partial_l f\nabla^l f-(1/4)g_{ij}\sum_m \partial_m f\nabla^m f$$

$$+(1/4)g_{ij}\sum_{ml} g_{ml}\nabla^m f\nabla^l f-\sum\nolimits_{ml}\Gamma^l{}_{mj}\Gamma^m_{il}-(1/2)\sum\nolimits_l \Gamma^l_{ij}\partial_l f-(1/2)\partial_i f\sum_m \Gamma^m_{mj}$$

$$+(1/2)\sum_{lm}\sum\nolimits_l g_{il}\Gamma^l_{mj}\nabla^m f-(1/2)\partial_j f\sum_m \Gamma^m_{im}-(1/4)\partial_j f\partial_l f-(n/4)\partial_j f\partial_i f$$

$$+(1/4)\partial_j f\sum_m g_{im}\nabla^m f-(1/2)\sum_m \Gamma^m_{ij}\partial_m f-(1/4)\partial_j f\partial_i f-(1/4)\partial_j f\partial_i f+(1/4)g_{ij}\sum_m \partial_m f\nabla^m f$$

$$+(1/2)\sum_{lm} g_{mj}\Gamma^m_{il}\nabla^l f+(1/4)g_{ij}\sum\nolimits_l \partial_l f\nabla^l f+(1/4)\sum_m g_{mj}\partial_i f\nabla^m f-(1/4)\sum_{lm} g_{mj}g_{il}\nabla^m f\nabla^l f$$

$$+\sum_m \partial_m\Gamma^m_{ij}+(1/2)\frac{\partial^2 f}{\partial_i\partial_j}+(1/2)\frac{\partial^2 f}{\partial_j\partial_i}-(1/2)\sum_m \frac{\partial_{gi}j}{\partial x_m}\nabla^m f-(1/2)g_{ij}\sum_m \frac{\partial\nabla^m f}{\partial x_m}$$

$$-\sum_m \partial_m\Gamma^m_{mj}-(n/2)\frac{\partial^2 f}{\partial_i\partial_j}-(1/2)\frac{\partial^2 f}{\partial_i\partial_j}+(1/2)\frac{\partial g_{mj}}{\partial x_i}\nabla^m f+(1/2)\sum_m g_{mj}\frac{\partial\nabla^m f}{\partial x_i}$$

After cancellations the expression becomes :

$$\tilde{R}^m{}_{imj} = \sum\nolimits_{lm} \Gamma^l_{ij}\Gamma^m_{ml} - \sum\nolimits_{ml} \Gamma^l{}_{mj}\Gamma^m_{il} + \sum_m \partial_m \Gamma^m_{ij} - \sum_m \partial_i \Gamma^m_{mj} + (1/2)(n-1)\sum\nolimits_l \Gamma^l_{ij}\partial_l f$$

$$-(1/2)\sum\nolimits_{lm} \Gamma^l_{ij} g_{lm} \nabla^m f + (1/4)(n-1)\partial_i f \partial_j f - (1/2) g_{ij} \sum\nolimits_{lm} \Gamma^m_{ml} \nabla^l f - (1/4)(n-1) g_{ij} \sum\nolimits_l \partial_l f \nabla^l f$$

$$+(1/4) g_{ij} \sum\nolimits_{lm} g_{ml} \nabla^m f \nabla^l f + (1/2)\sum\nolimits_{lm} g_{il}\Gamma^l{}_{mj} \nabla^m f + (1/2)\sum\nolimits_{lm} g_{mj}\Gamma^m_{il}\nabla^l f - (1/4)\sum\nolimits_{lm} g_{mj} g_{il} \nabla^m f \nabla^l f$$

$$-(1/2)(n-1)\frac{\partial^2 f}{\partial_i \partial_j} - (1/2)\sum_m \frac{\partial g_{ij}}{\partial_m}\nabla^m f - (1/2) g_{ij}\sum_m \frac{\partial \nabla^m f}{\partial_m} + (1/2)\sum_m \frac{\partial g_{mj}}{\partial_i}\nabla^m f + (1/2) g_{mj}\frac{\partial}{\partial x_i}\nabla^m f$$

Lets label this equation as (*)

3.3 Ricci Curvature Tensor

Definition:

Let R be the Riemannian curvature tensor of M. The Ricci curvature tensor of M is the contraction, whose componenets relative to the coordinate system are :

$$R_{ij} = \sum_m R^m_{imj}$$

Ricci tensor can be viewed as a geometric object that depends on the choice of Riemannian metric on the manifold. It can be considered as measure of how much the metric tensor on a given geometry differs locally from that of a standard Euclidean space.

To simplify the Riemannian tensor calculations results, the first four terms adds up to :

$$\sum\nolimits_{lm} \Gamma^{l}{}_{ij}\Gamma^{m}{}_{ml} - \sum\nolimits_{lm} \Gamma^{l}{}_{mj}\Gamma^{m}_{il} + \sum_{m} \partial_m \Gamma^{m}_{ij} - \sum_{m} \partial_i \Gamma^{m}_{mj} = R_{ij}$$

Next we add these two related terms :

$$(1/2)(n-1)\sum\nolimits_{l} \Gamma^{l}_{ij} \frac{\partial f}{\partial x_l} - (1/2)(n-1)\frac{\partial^2 f}{\partial x_i \partial x_j}$$

$$= -(1/2)(n-1)[\frac{\partial^2 f}{\partial x_i \partial x_j} - \sum\nolimits_{l} \Gamma^{l}_{ij} \frac{\partial f}{\partial x_l}]$$

$$= -(1/2)(n-1)(Hf)_{ij}$$

Examine the expression:

$$df \otimes df = \sum_{ij} \frac{\partial f}{\partial x_i} \frac{\partial f}{\partial x_j} dx_i \otimes dx_j$$

Therefore the related terms becomes :

$$(1/4)(n-1)\frac{\partial f}{\partial x_i}\frac{\partial f}{\partial x_j} = (1/4)(n-1)(df \otimes df)_{ij}$$

Now we will simplify each term in order of appearance in th equation (*) : The first coordinate dependent term

$$-(1/2)\sum\nolimits_{lm}\Gamma^l_{ij} g_{lm} \nabla^m f = -(1/2)\sum\nolimits_{klm}\Gamma^l_{ij} g_{lm} g^{mk}\frac{\partial f}{\partial x_k}$$

$$= -(1/2)\sum\nolimits_{kl}\Gamma^l_{ij}\delta^k_l \frac{\partial f}{\partial x_k} = -(1/2)\sum\nolimits_{l}\Gamma^l_{ij}\frac{\partial f}{\partial x_l}$$

The second coordinate dependent term :

$$-(1/2)g_{ij}\sum_{lm}\Gamma^m_{ml}\nabla^l f = -(1/2)g_{ij}\sum_{lm}\Gamma^m_{ml} g^{lk}\frac{\partial f}{\partial x_k}$$

The third coordinate term is :

$$-(1/4)(n-1)g_{ij}\sum_{l}\frac{\partial f}{\partial x_l}\nabla^l f = -(1/4)(n-1)g_{ij}\sum_{lm}\frac{\partial f}{\partial x_l} g^{lm}\frac{\partial f}{\partial x_m}$$

$$= -(1/4)(n-1)g_{ij}\sum_{lm} g^{lm}\frac{\partial f}{\partial x_l}\frac{\partial f}{\partial x_m}$$

Note that :

$$\langle df, df\rangle = \langle \sum_k \frac{\partial f}{\partial x_k} dx_k, \sum_l \frac{\partial f}{\partial x_l} dx_l \rangle$$

$$= \sum_{kl} g^{kl} \frac{\partial f}{\partial x_k} \frac{\partial f}{\partial x_l}$$

Then the third term becomes:

$$-(1/4)(n-1)\|df\|^2 \, g_{ij}$$

The fourth coordinate dependent term :

$$(1/4)g_{ij} \sum_{lm} g_{ml} \nabla^m f \nabla^l f = (1/4)g_{ij} \sum_{lm} g_{ml} (\sum_k g^{mk} \frac{\partial f}{\partial x_k}) \nabla^l f$$

$$= (1/4)g_{ij} \sum_{lk} \delta_l^k \frac{\partial f}{\partial x_k} \nabla^l f = (1/4)g_{ij} \sum_l \frac{\partial f}{\partial x_l} \nabla^l f$$

$$= (1/4)g_{ij} \sum_{lk} \frac{\partial f}{\partial x_l} g^{lk} \frac{\partial f}{\partial x_k} = (1/4)g_{ij}\|df\|^2$$

The fifth coordinate dependent term:

$$(1/2)\sum_{lm} g_{il}\Gamma^{l}_{mj}\nabla^m f = (1/2)\sum_{lmk} g_{il}\Gamma^{l}_{mj}g^{mk}\frac{\partial f}{\partial x_k} = (1/2)\sum_{lmk} g_{il}g^{mk}\Gamma^{l}_{mj}\frac{\partial f}{\partial x_k}$$

The sixth coordinate dependent term :

$$(1/2)\sum_{lm} g_{mj}\Gamma^{m}_{il}\nabla^l f = (1/2)\sum_{lmk} g_{mj}g^{lk}\Gamma^{m}_{il}\frac{\partial f}{\partial x_k}$$

The seventh coordinate dependent term is:

$$-(1/4)\sum_{lm} g_{mj}g_{il}\nabla^m f\nabla^l f = -(1/4)\sum_{lmk} g_{mj}g_{il}g^{mk}\frac{\partial f}{\partial x_k}\nabla^l f$$

$$= -(1/4)\sum_{lk} \delta^k_j g_{il}\frac{\partial f}{\partial x_k}\nabla^l f = -(1/4)\sum_{l} g_{il}\frac{\partial f}{\partial x_j}\nabla^l f$$

$$= -(1/4)\sum_{lk} g_{il}\frac{\partial f}{\partial x_j}g^{lk}\frac{\partial f}{\partial x_k} = -(1/4)\sum_{k} \frac{\partial f}{\partial x_j}\delta^k_i\frac{\partial f}{\partial x_k}$$

$$= -(1/4)\frac{\partial f}{\partial x_i}\frac{\partial f}{\partial x_j} = -(1/4)(df \otimes df)_{ij}$$

The eighth coordinate dependent term is :

$$-(1/2)\sum_{m} \frac{\partial g_{ij}}{\partial x_m}\nabla^m f = -(1/2)\sum_{ml} \frac{\partial g_{ij}}{\partial x_m}g^{ml}\frac{\partial f}{\partial x_l}$$

The ninth coordinate dependent term :

$$-(1/2)g_{ij}\sum_m \frac{\partial}{\partial x_m}\nabla^m f = -(1/2)g_{ij}\sum_{mk}\frac{\partial}{\partial x_m}(g^{mk}\frac{\partial f}{\partial x_k})$$

$$= -(1/2)g_{ij}\sum_{mk}\frac{\partial g^{mk}}{\partial x_m}\frac{\partial f}{\partial x_k} - (1/2)g_{ij}\sum_{mk} g^{mk}\frac{\partial^2 f}{\partial x_m \partial x_k}$$

The tenth coordinate dependent term is :

$$(1/2)\sum_m \frac{\partial g_{mj}}{\partial x_i}\nabla^m f = (1/2)\sum_{mk}\frac{\partial g_{mj}}{\partial x_i} g^{mk}\frac{\partial f}{\partial x_k}$$

The last coordinate dependent term is :

$$(1/2)\sum_m g_{mj}\frac{\partial}{\partial x_i}(\nabla^m f) = (1/2)\sum_{mk} g_{mj}\frac{\partial}{\partial x_i}(g^{mk}\frac{\partial f}{\partial x_k})$$

$$= (1/2)\sum_{mk} g_{mj}\frac{\partial g^{mk}}{\partial x_i}\frac{\partial f}{\partial x_k} + (1/2)\sum_{mk} g_{mj}g^{mk}\frac{\partial^2 f}{\partial x_i \partial x_k}$$

$$= (1/2)\sum_{mk} g_{mj}\frac{\partial g^{mk}}{\partial x_i}\frac{\partial f}{\partial x_k} + (1/2)\sum_{k} \delta_j^k\frac{\partial^2 f}{\partial x_i \partial x_k}$$

$$= (1/2)\sum_{mk} g_{mj}\frac{\partial g^{mk}}{\partial x_i}\frac{\partial f}{\partial x_k} + (1/2)\frac{\partial^2 f}{\partial x_i \partial x_j}$$

Now we can rewire the whole formula as follow:

$$\tilde{R}_{ij} = R_{ij} - (1/2)(n-1)(Hf)_{ij} + (1/4)(n-1)(df \otimes df)_{ij} - (1/4)(n-1)\|df\|^2 g_{ij} + (1/4)g_{ij}\|df\|^2$$

$$-(1/4)(df \otimes df)_{ij} - (1/2)\sum\nolimits_l \Gamma^l_{ij}\frac{\partial f}{\partial x_l} - (1/2)g_{ij}\sum_{lm}\Gamma^m_{ml}g^{lk}\frac{\partial f}{\partial x_k}$$

$$+(1/2)\sum_{lmk} g_{il}g^{mk}\Gamma^l_{mj}\frac{\partial f}{\partial x_k} + (1/2)\sum_{lmk} g_{mj}g^{lk}\Gamma^m_{il}\frac{\partial f}{\partial x_k} - (1/2)\sum_{ml}\frac{\partial g_{ij}}{\partial x_m}g^{ml}\frac{\partial f}{\partial x_l}$$

$$-(1/2)g_{ij}\sum_{mk}\frac{\partial g^{mk}}{\partial x_m}\frac{\partial f}{\partial x_k} - (1/2)g_{ij}\sum_{mk}g^{mk}\frac{\partial^2 f}{\partial x_m \partial x_k} + (1/2)\sum_{mk}\frac{\partial g_{mj}}{\partial x_i}g^{mk}\frac{\partial f}{\partial x_k}$$

$$+(1/2)\sum_{mk} g_{mj}\frac{\partial g^{mk}}{\partial x_i}\frac{\partial f}{\partial x_k} + (1/2)\frac{\partial^2 f}{\partial x_i \partial x_k}$$

The two terms $(1/2)\sum_{mk}\frac{\partial g_{mj}}{\partial x_i}g^{mk}\frac{\partial f}{\partial x_k} + (1/2)\sum_{mk} g_{mj}\frac{\partial g^{mk}}{\partial x_i}\frac{\partial f}{\partial x_k}$ will add up to zero as follow:

$$\sum_m g_{jm}g^{mk} = \delta^k_j$$

Take derivative of both sides we get:

$$\sum_m \frac{\partial g_{mj}}{\partial x_i}g^{mk} + \sum_m g_{mj}\frac{\partial g^{mk}}{\partial x_i} = 0$$

From which we can see that:

$$(1/2)\sum_{mk}\frac{\partial g_{mj}}{\partial x_i}g^{mk}\frac{\partial f}{\partial x_k}+(1/2)\sum_{mk}g_{mj}\frac{\partial g^{mk}}{\partial x_i}\frac{\partial f}{\partial x_k}=0$$

Also notice that the two terms $-(1/2)\sum_l \Gamma^l_{ij}\frac{\partial f}{\partial x_l}$ and $+(1/2)\frac{\partial^2 f}{\partial x_i \partial x_j}$ adds up with the Hessian component so the formula becomes :

$$\tilde{R}_{ij}=R_{ij}-(1/2)(n-2)(Hf)_{ij}+(1/4)(n-2)(df\otimes df)_{ij}-(1/4)(n-2)\|df\|^2 g_{ij}$$

$$-(1/2)g_{ij}\sum_{klm}\Gamma^m_{ml}g^{lk}\frac{\partial f}{\partial x_k}+(1/2)\sum_{lmk}g_{il}g^{mk}\Gamma^l_{mj}\frac{\partial f}{\partial x_k}+(1/2)\sum_{lmk}g_{mj}g^{lk}\Gamma^m_{il}\frac{\partial f}{\partial x_k}-(1/2)\sum_{mk}\frac{\partial g_{ij}}{\partial x_m}g^{mk}\frac{\partial f}{\partial x_k}$$

$$-(1/2)g_{ij}\sum_{mk}\frac{\partial g^{mk}}{\partial x_m}\frac{\partial f}{\partial x_k}-1/2g_{ij}\sum_{mk}g^{mk}\frac{\partial^2 f}{\partial x_m \partial x_k}$$

Now we further simplify by combining $(1/2)\sum_{lmk}g_{il}g^{mk}\Gamma^l_{mj}\frac{\partial f}{\partial x_k}+(1/2)\sum_{lmk}g_{mj}g^{lk}\Gamma^m_{il}\frac{\partial f}{\partial x_k}-(1/2)\sum_{mk}\frac{\partial g_{ij}}{\partial x_m}g^{mk}\frac{\partial f}{\partial x_k}$ we get:

$$(1/2)\sum_{lmk}g_{il}g^{mk}\Gamma^l_{mj}\frac{\partial f}{\partial x_k}+(1/2)\sum_{lmk}g_{mj}g^{lk}\Gamma^m_{il}\frac{\partial f}{\partial x_k}-(1/2)\sum_{mk}\frac{\partial g_{ij}}{\partial x_m}g^{mk}\frac{\partial f}{\partial x_k}$$

$$= (1/2)\sum_{lm} g_{il}\Gamma^l_{mj}\nabla^m f + (1/2)\sum_{lm} g_{mj}\Gamma^m_{il}\nabla^l f - (1/2)\sum_m \frac{\partial g_{ij}}{\partial x_m}\nabla^m f$$

$$= (1/4)\sum_{lm} g_{il}(\sum_k g^{lk}(\frac{\partial g_{mk}}{\partial x_j}+\frac{\partial g_{jk}}{\partial x_m}-\frac{\partial g_{mj}}{\partial x_k}))\nabla^m f+(1/4)\sum_{lm} g_{mj}(\sum_k g^{mk}(\frac{\partial g_{ik}}{\partial x_l}+\frac{\partial g_{lk}}{\partial x_i}-\frac{\partial g_{il}}{\partial x_k}))\nabla^l f$$

$$-(1/2)\sum_m \frac{\partial g_{ij}}{\partial x_m}\nabla^m f$$

$$= (1/4)\sum_{mk} \delta^k_i(\frac{\partial g_{mk}}{\partial x_j}+\frac{\partial g_{jk}}{\partial x_m}-\frac{\partial g_{mj}}{\partial x_k})\nabla^m f+(1/4)\sum_{lk} \delta^k_j(\frac{\partial g_{ik}}{\partial x_l}+\frac{\partial g_{lk}}{\partial x_i}-\frac{\partial g_{il}}{\partial x_k})\nabla^l f-(1/2)\sum_m \frac{\partial g_{ij}}{\partial x_m}\nabla^m f$$

$$= (1/4)\sum_m(\frac{\partial g_{mi}}{\partial x_j}+\frac{\partial g_{ij}}{\partial x_m}-\frac{\partial g_{mj}}{\partial x_i})\nabla^m f+(1/4)\sum_l(\frac{\partial g_{ij}}{\partial x_l}+\frac{\partial g_{lj}}{\partial x_i}-\frac{\partial g_{il}}{\partial x_j})\nabla^l f-(1/2)\sum_m \frac{\partial g_{ij}}{\partial x_m}\nabla^m f$$

$$= (1/2)\sum_m \frac{\partial g_{ij}}{\partial x_m}\nabla^m f - (1/2)\sum_m \frac{\partial g_{ij}}{\partial x_m}\nabla^m f = 0$$

Next we combine the remaining terms $(1/2)g_{ij}\sum_{klm}\Gamma^m_{ml}g^{lk}\frac{\partial f}{\partial x_k}$, $(1/2)g_{ij}\sum_{mk}\frac{\partial g^{mk}}{\partial x_m}\frac{\partial f}{\partial x_k}$ and $(1/2)g_{ij}\sum_{mk}g^{mk}\frac{\partial^2 f}{\partial x_m\partial x_k}$

$$-(1/2)g_{ij}\sum_{klm}\Gamma^m_{ml}g^{lk}\frac{\partial f}{\partial x_k} - (1/2)g_{ij}\sum_{mk}\frac{\partial g^{mk}}{\partial x_m}\frac{\partial f}{\partial x_k} - (1/2)g_{ij}\sum_{mk}g^{mk}\frac{\partial^2 f}{\partial x_m\partial x_k}$$

Note that Laplacian is the trace of the Hessian therefore it can be expressed as :

$$\Delta f = -\sum_{lmk} g^{mk}\Gamma^{l}_{mk}\frac{\partial f}{\partial x_l} + \sum_{mk} g^{mk}\frac{\partial^2 f}{\partial x_m \partial x_k}$$

Now we can write the above expression as :

$$-(1/2)g_{ij}\sum_{klm}\Gamma^{m}_{ml}g^{lk}\frac{\partial f}{\partial x_k} + (1/2)g_{ij}\sum_{mk}\frac{\partial g^{mk}}{\partial x_m}\frac{\partial f}{\partial x_k} - (1/2)(\Delta f)g_{ij} - (1/2)g_{ij}\sum_{lmk} g^{mk}\Gamma^{l}_{mk}\frac{\partial f}{\partial x_l}$$

We know that :

$$\frac{\partial g^{mk}}{\partial x_m} = -\sum_{sl} g^{ks}g^{lm}\frac{\partial g_{sl}}{\partial x_m}$$

Now we only need to find the expression for $\frac{\partial g_{sl}}{\partial x_m}$

$$\frac{\partial g_{sl}}{\partial x_m} = \frac{\partial}{\partial x_m}\langle\frac{\partial}{\partial x_s}, \frac{\partial}{\partial x_l}\rangle = \langle\sum_{t}\Gamma^{t}_{ms}\frac{\partial}{\partial x_t}\ ,\ \frac{\partial}{\partial x_l}\rangle + \langle\frac{\partial}{\partial x_m}\ ,\ \sum_{t}\Gamma^{t}_{ml}\frac{\partial}{\partial x_t}\rangle$$

$$= \sum_{t}(\Gamma^{t}_{ms}g_{tl} + \Gamma^{t}_{ml}g_{st})$$

Write :

$$\frac{\partial g^{mk}}{\partial x_m} = -\sum_{sl} g^{ks}g^{lm}(\sum_{t}\Gamma^{t}_{ms}g_{tl} + \Gamma^{t}_{ml}g_{st})$$

$$= -\sum_{st} g^{ks}\delta_t^m \Gamma_{ms}^t - \sum_{lt} g^{ml}\delta_t^k \Gamma_{ml}^t$$

$$= -\sum_{s} g^{ks}\Gamma_{ms}^m - \sum_{l} g^{ml}\Gamma_{ml}^k$$

So the following expression becomes :

$$(1/2)g_{ij}\sum_{klm}\Gamma_{ml}^m g^{lk}\frac{\partial f}{\partial x_k} + (1/2)g_{ij}\sum_{mk}\frac{\partial g^{mk}}{\partial x_m}\frac{\partial f}{\partial x_k} - (\Delta f)g_{ij} - (1/2)g_{ij}\sum_{lmk} g^{mk}\Gamma_{mk}^l\frac{\partial f}{\partial x_l}$$

$$= -(1/2)g_{ij}\sum_{klm}\Gamma_{ml}^m g^{lk}\frac{\partial f}{\partial x_k} + (1/2)g_{ij}\sum_{mks} g^{ks}\Gamma_{ms}^m\frac{\partial f}{\partial x_k} + (1/2)g_{ij}\sum_{mkl} g^{ml}\Gamma_{ml}^k\frac{\partial f}{\partial x_k} - (\Delta f)g_{ij}$$

$$-(1/2)g_{ij}\sum_{lmk} g^{mk}\Gamma_{mk}^l\frac{\partial f}{\partial x_l}$$

$$= -(1/2)(\Delta f)g_{ij}$$

Then the formula for Ricci tensor will nicely formulated as follow :

$$\tilde{R}_{ij} = \boxed{\mathrm{R}_{ij} - (1/2)(n-2)(Hf)_{ij} + (1/4)(n-2)(df\otimes df)_{ij} - (1/4)(n-2)\|df\|^2 g_{ij} - (1/2)(\Delta f)g_{ij}}$$

We can show that $\|df\|^2$ is equals to $\|\nabla f\|^2$ as follow :

$$\|\nabla f\|^2 = \langle \sum_{ij} g^{ij} \frac{\partial f}{\partial x_I} \frac{\partial}{\partial x_j} \ , \ \sum_{kl} g^{kl} \frac{\partial f}{\partial x_k} \frac{\partial}{\partial x_l} \rangle = \sum_{ij} g^{ij} \sum_{kl} g^{kl} \frac{\partial f}{\partial x_I} \frac{\partial f}{\partial x_k} \langle \frac{\partial}{\partial x_j}, \frac{\partial}{\partial x_l} \rangle$$

$$= \sum_{ij} g^{ij} \sum_{kl} g^{kl} g_{jl} \frac{\partial f}{\partial x_I} \frac{\partial f}{\partial x_k}$$

$$= \sum_{ij} g^{ij} \frac{\partial f}{\partial x_j} \frac{\partial f}{\partial x_j} = \|df\|^2$$

And in an invariant form :

$$\tilde{R} = \boxed{\text{R - (1/2)(n-2)(Hf)+(1/4)(n-2)(df } \otimes df) - (1/4)(n-2)\|\nabla f\|^2 g - (1/2)(\Delta f) g}$$

Hence, the results has been proven and now it is time to state the following Theorem:

Theorem 1 :

If the conformal change of metric was made such that :

$$\tilde{g} = e^f g$$

Then the Ricci tensor of g and $\tilde{g}$ are related by the following formula :

$$\tilde{R} = \boxed{\text{R - (1/2)(n-2)(Hf)+(1/4)(n-2)(df } \otimes df) - (1/4)(n-2)\|\nabla f\|^2 g - (1/2)(\Delta f)g} \quad *$$

By examining the equation above one can ask, can we make a conformal change of metric that makes a Ricci curvature a constant multiple of metric? It is good to emphasize here that both the metric and Ricci curvature are tensors, then the equality means that all components should be equal.

Remark : Here the dimension is n,but we don't have n^2 linearly independent entries, nor n^2 different equations. These tensors are symmetric, so there are $n(n-1)/2$ linearly independent entries.

CHAPTER 4

Einstein metric

4.1 Existence of Einstein metric

The inquiry to investigate whether every compact manifold carry at least one Einstein metric is a huge and wide research area. Even in the two dimensional case it is quite complicated, and becomes more in higher dimensions[**?**].The contribution of this work is unique. Maybe it has been written down somewhere, but I have not seen it. It relies on some strong elementary observations obtained by applying standard material and techniques from PDE or geometric analysis.

A Riemannian manifold (M,g) is Einstein if there exists a local constant c, such that

$$R = cg$$

For the aim to make the equation (*) simpler, let :

$$e^f = u^\alpha$$

for some u > 0, then Lemma 1:

$$df = \frac{\alpha}{u} du$$

Proof

$$f = ln(u^{\alpha}) = (u)$$

Differentiate both sides:

$$df = \frac{\alpha}{u} du$$

Lemma 2:

$$Hf = (\alpha/u)\ Hu - (\alpha/u^2)(du \otimes du)$$

Proof

$$Hf = H(\alpha ln(u)) = \frac{\partial^2}{\partial u^2}(\alpha ln(u))$$
$$\frac{\partial}{\partial u}[\frac{\partial}{\partial u}\alpha ln(u)] = \frac{\partial}{\partial u}[\frac{\alpha}{u}du]$$
$$= (\alpha/u)\frac{\partial^2}{\partial u^2}u + (-\alpha/u^2)(du \otimes du) = (\alpha/u)\ Hu - (\alpha/u^2)(du \otimes du)$$

Lemma 3:

$$\Delta f = (\alpha/u)\Delta u - (\alpha/u^2)\|\nabla u\|^2$$

Proof

$$\Delta f = \Delta \alpha ln(u) = \sum \frac{\partial^2}{\partial u^2}(\alpha ln(u))$$

$$\frac{\partial}{\partial u}[\frac{\partial}{\partial u}(\alpha ln(u))] = \frac{\partial}{\partial u}[\frac{\alpha}{u}du]$$

$$= (\alpha/u)\sum \frac{\partial^2}{\partial u^2}u + (-\alpha/u^2)\langle du, du\rangle = (\alpha/u)\Delta u - (\alpha/u^2)\|\nabla u\|^2$$

By Lemmas 1,2,and 3 the formula of Ricci tensor :

$$\tilde{R} = R - (1/2)(n-2)(\alpha/u)(Hu) + (1/2)(n-2)(\alpha/u^2)(du \otimes du) + (1/4)(n-2)(\alpha/u)^2(du \otimes du)$$

$$-(1/4)(n-2)(\alpha/u)^2 \|\nabla u\|^2 g - (1/2)(\alpha/u)\Delta u g + (1/2)(\alpha/u^2)\|\nabla u\|^2 g$$

Proposition:

The choice of the real number α = -2, eliminates the ($du \otimes du$) term

Proof :

The goal here is to eliminate the ($du \otimes du$)

$$+(1/2)(n-2)(-2/u^2)(du \otimes du) + (1/4)(n-2)(-2/u)^2(du \otimes du)$$

$$-(n-2)/u^2(du \otimes du) + (n-2)/u^2(du \otimes du) = 0$$

Theorem 2

$$\tilde{R} = R + ((n-2)/u)(Hu) - ((n-1)/u^2)\|\nabla u\|^2 g + (1/u)(\Delta u)g$$

Question : if (M,g) is given, so that the Ricci tensor R is determined, and if a symmetric tensor $\tilde{R}$ is chosen in advanced, what equation must u solve in order that the Ricci tensor of $\tilde{g} = u^{-2}g$?

Theorem 3 For a prescribed Ricci curvature $\tilde{R}$, u must solve :

$$\tilde{R} = R + ((n-2)/u)(Hu) - ((n-1)/u^2)\|\nabla u\|^2 g + (1/u)(\Delta u)g$$

This equation is differ than that appears in Theorem two, is that this time, g and $\tilde{R}$ are given. and u is unknown to solve for. This is known as the "Prescribed Ricci curvature problem."

A very important case is the case when $\tilde{R} = c\tilde{g}$ for a real number c. In this case one wants to find an Einstein metric through conformal deformation from a given metric.

Corollary Making the choice $\tilde{R} = c\tilde{g}$ for c $\in \mathbb{R}$, u must satisfy

$$c\frac{1}{u^2}g = R + (n-2)\frac{1}{u}Hu - (n-1)\frac{1}{u^2}\|\nabla u\|^2 g + \frac{1}{u}(\Delta u)g$$

multiplying by u^2,

$$cg = u^2 R + (n-2)uHu - (n-1)\|\nabla u\|^2 g + u(\Delta u)g$$

Questions: M be compact, without boundary if it possible to find u to solve the system

$$u^2 R + (n-2)uHu = 0$$

$$cg = -(n-1)\|\nabla u\|^2 g + u(\Delta u)g$$

Then what conclusion if any can we drawn?

This implies the Einstein condition, but because the converse is not true, requiring that u solve this system is a stronger condition than requiring u to solve the original equation.

Because g is positive-definite as a metric and invertible as a matrix, the second equation is equivalent to the scalar equation

$$c = -(n-1)\|\nabla u\|^2 + u(\Delta u)$$

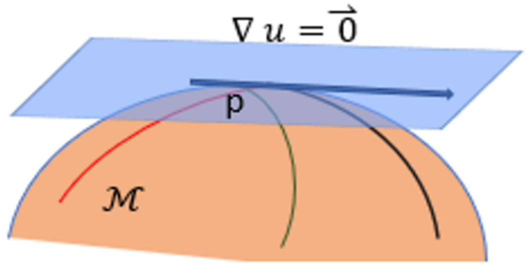

Figure 4.1: The First derivative at the maximum point p is zero

Applying the principle of maximum and minimum, one can observe that at the maximum point p,$\nabla u(p) = \vec{0}$, so $\left\|\nabla u(p)\right\|^2 = 0$, and $\Delta u(p) = \sum_{i=1} \frac{\partial^2 u}{\partial x_i^2}(p) \leq 0$. Using the same argument at the minimum point q, the sum of the second partial derivative is at most zero(see Fig 4.1). This means that the constant c is both positive and negative at the same time therefore, $c = 0$

It interesting to show that there is another way we can prove that c is at most zero, using the integration by part on a compact manifold without boundaries.

$$c \int_M dx = -n \int_M \|\nabla u\|^2 \, dx$$

Now integrating the equation over M and using the Divergence Theorem (ie, integration by parts), we have

$$c.\text{Vol}(M) = -(n-1)\int_M \|\nabla u\|^2\,dx + \int_M u\Delta u\; dx$$

$$c.\text{Vol}(M) = -(n-1)\int_M \|\nabla u\|^2\,dx - \int_M \|\nabla u\|^2\,dx$$

If c is zero, then on the right side of the equation, $\|\nabla u\| = 0$, then u must be constant. Then from the first equation in the system we obtain

$$R = 0$$

So the original metric was already Einstein with zero Ricci curvature.

Theorem 4 if u solves the above system, then $c = 0$. Moreover, u is in this case constant and (M,g) was already Einstein with zero Ricci curvature.

Therefore, hoping to solve for an Einstein metric by solving this system instead of the original tensor equation was a trivial case in there was there was no non-trivial solution. The only time it works is if (M, g) was already, Einstein. Again, the motivation for trying this is what the second equation in the system can be reduced to a single scalar equation. In fact the first equation is equally important but due to lack of time it is to be considered in the near future. Also I show two distinct proofs for $c \leq 0$, one by the Maximum Principle, and an alternate proof by the Divergence Theorem. There's a third proof that is related to Rayleigh quotients,which related to the eigenvalues of the Laplace operator .Back to the original equation, in order to solve for an Einstein metric, one must solve the original equation. And this is beyond the scope of this work, however it is an ambitious project for future work.

CHAPTER 5

Conclusion

The study of conformal change of metric has a comprehensive and extensive application, particularly the Uniformization Theorem—the equation(*) has been widely studied and solved for the case of two and higher dimensions. Many efforts are to made to answer questions about existence, and uniqueness, especially for Einstein metrics due to its importance in Riemannian geometry. For example, is it the case that every compact manifold has at least one Einstein Metric? [1]. These are only some examples in this vast field of differential geometry which has a solid potential to be considered as projects in future researches.

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