THE HOPF CONJECTURE WITH ABELIAN SYMMETRIES

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ABSTRACT

The Hopf Conjecture states that for closed, orietable, even-dimensional manifolds, the Euler characteristic is strictly positive. Results due independently to Püttmann and Searle [13] and Rong [14], and due to Rong and Su [15], showing that the Hopf Conjecture holds under the additional hypothesis of abelian symmetries. In this thesis we detail the proofs of these two results. For the first result, we provide the details of the original proof, whereas for the second, we give a more streamlined proof that relies of the Borel formula.

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CHAPTER 1

INTRODUCTION

The classification of manifolds with positive sectional curvature is a long-standing open problem in Riemannian geometry. To date, the only known topological obstructions are given by two classical theorems: the Bonnet-Myers Theorem and Synge's Theorem. The Bonnet-Myers Theorem tells us that a Riemanian manifold with uniformly positive sectional curvature, M, is compact and has finite fundamental group. Synge's Theorem tells us that for M, a closed Riemannian manifold with positive section curvature, if M is even-dimensional, then its fundamental group, denoted $\pi_1(M)$, is trivial when M is orientable, or $\pi_1(M) = \mathbb{Z}_2$, when M is non-orientable, and if M is odd-dimensional, then M is orientable.

An important topological invariant of a manifold is its Euler characteristic. The Euler characteristic is defined for CW complexes, examples of which are manifolds. Recall that a closed n-cell, e^n, is homeomorphic to an n-dimensional ball, that is, $e^n \cong D^n$. The boundary of an n-cell, ∂e^n, is homeomorphic to S^{n-1}, that is $\partial e^n \cong S^{n-1}$. With this notation established, we can now define a CW complex.

Definition 1.0.1 [CW complex]. *A* CW complex *is a topological space that is defined inductively. In general, an n-*dimensional CW complex *is constructed by taking the disjoint union of a k-dimensional CW complex for some $k < n$ with one or more copies of e^n. For each copy of e^n, there is a map that "glues" ∂e^n to elements of the k-dimensional complex. The topology of the CW complex is the topology of the quotient space defined by these gluing maps.*

Example 1.0.2. *The following are examples of CW complexes.*

1. *e^0 is the CW complex consisting of a point.*
2. *$S^n = e^0 \cup_f e^n$, where $f : \partial e^n \to e^0$.*

3. $S^n = \bigcup_{i=0}^{n} (e_1^i \cup e_2^i)$ *with* $f_i : \partial e_j^i \to S^{i-1}$, *for* $1 \leq j \leq 2$.

We are now ready to define the Euler characteristic.

Definition 1.0.3 [Euler Characteristic]. [8] *For a finite CW complex* X*, the* Euler characteristic *of* X*, which we denote by* $\chi(X)$*, is defined to be the alternating sum* $\sum_n (-1)^n c_n$*, where* c_n *is the number of* $n-$*cells of* X*.*

The Euler characteristic was originally defined for polyhedral surfaces by the classic formula:

$$\chi = v - e + f; \tag{1.1}$$

where

$$\begin{aligned} v &= \text{number of vertices,} \\ e &= \text{number of edges, and} \\ f &= \text{number of faces.} \end{aligned}$$

In dimension two, Poincaré showed that closed, orientable surfaces are completely classified by their Euler characteristic.

In 1931 Heinz Hopf made the following conjecture, relating the geometry of positively curved manifolds to their Euler characteristic.

Hopf Conjecture 1.0.4. *Let* M^{2m} *be a closed, even-dimensional Riemannian manifold admitting a metric of uniformly positive sectional curvature. Then* $\chi(M^{2m}) > 0$*, that is,* M^{2m} *has positive Euler characteristic.*

Heinz Hopf first introduced this idea in a talk he gave in the spring of 1931 in Fribourg, Switzerland and later mentioned it again in the fall of 1931 at Bad Elster. The conjecture was first formally written in the proceedings of the German Mathematical Society in a paper based on talks. To this day, the Hopf conjecture has not been proven or disproven. However, there is evidence to support that it is true:

all known examples of compact even-dimensional manifolds with positive sectional curvature have positive Euler characteristic. At the same time, all known examples of manifolds with positive sectional curvature share a high degree of symmetry. Based on this observation, Karsten Grove proposed the Symmetry Program almost 30 years ago. The Grove Symmetry Program suggests classifying manifolds of positive sectional curvature with the additional hypothesis of symmetries. The initial goal of this program was to consider "large" symmetries, reducing them as needed to prove results for a larger subclass of such manifolds. Applying this approach to other questions related to the study of positively curved manifolds, such as the Hopf conjecture, is then natural.

The isometry group of a compact Riemannian manifold is a compact Lie group by work of Myers and Steenrod [12]. Since every connected, compact Lie group contains a maximal torus, it is natural to first consider the case of abelian symmetries. The results presented below have been obtained in 2001 by Püttman and Searle [13] and Rong [14], independently, in 2005 by Rong and Su [15], in 2007 by Su and Wang [16], and in 2013 by Kennard [9], respectively.

Theorem A. [13],[14] *Let M^{2m} be a closed, orientable Riemannian manifold of positive sectional curvature. Then if T^k acts isometrically on M^{2m} and $k \geq \dfrac{2m-4}{4}$ this implies $\chi(M^{2m}) > 0$*

Theorem B. [15] *Let M^{2m} be a closed $2m-$dimensional manifold of positive sectional curvature on which a torus T^k acts isometrically. For $2m \neq 12$ (respectively, $2m = 12$), if $k \geq \dfrac{2m-4}{8}$ (respectively, $k > 1$), then the Euler characteristic of each T^k-fixed point set component is positive, and thus $\chi(M^{2m}) > 0$.*

Theorem C. [16] *Let M^{2m} be a closed even $2m$-manifold of positive sectional curvature. $\chi(M^{2m}) > 0$, if M admits an isometric $\mathbb{Z}_p^k$-action with prime $p \geq c(2m)$, a constant depending only on the dimension of M, and k satisfies any one of the following conditions:*

(i) $k \geq \dfrac{2m-4}{8}$, *and* $2m \neq 12, 18,$ *or* 20;

(ii) $k \geq \dfrac{2m-2}{10}$, *and* $2m \equiv 0 \mod 4$ *with* $2m \neq 12$ *or* 20; *or*

(iii) $k \geq \dfrac{2m+4}{12}$, *and* $2m \equiv 0, 4,$ *or* $12 \mod 20$ *with* $2m \neq 20$.

Theorem D. [9] *Let M^n be a connected, closed Riemannian manifold with positive sectional curvature. If $n \equiv 0 \mod 4$ and M admits an effective, isometric T^r-action with $r \geq 2\log_2 n - 2$, then $\chi(M) > 0$.*

The graphs in Figures 1.1 and 1.2 below illustrate the lower bounds on the rank of the torus action for each of these four theorems, as well as for the weaker version of the Theorem B, which is stated in Chapter 4 as Theorem 4.0.2. Note that the graph in Figure 1.1 gives the symmetry rank as a continuous function, whereas the graph in Figure 1.2 presents the symmetry rank as the step function it actually is.

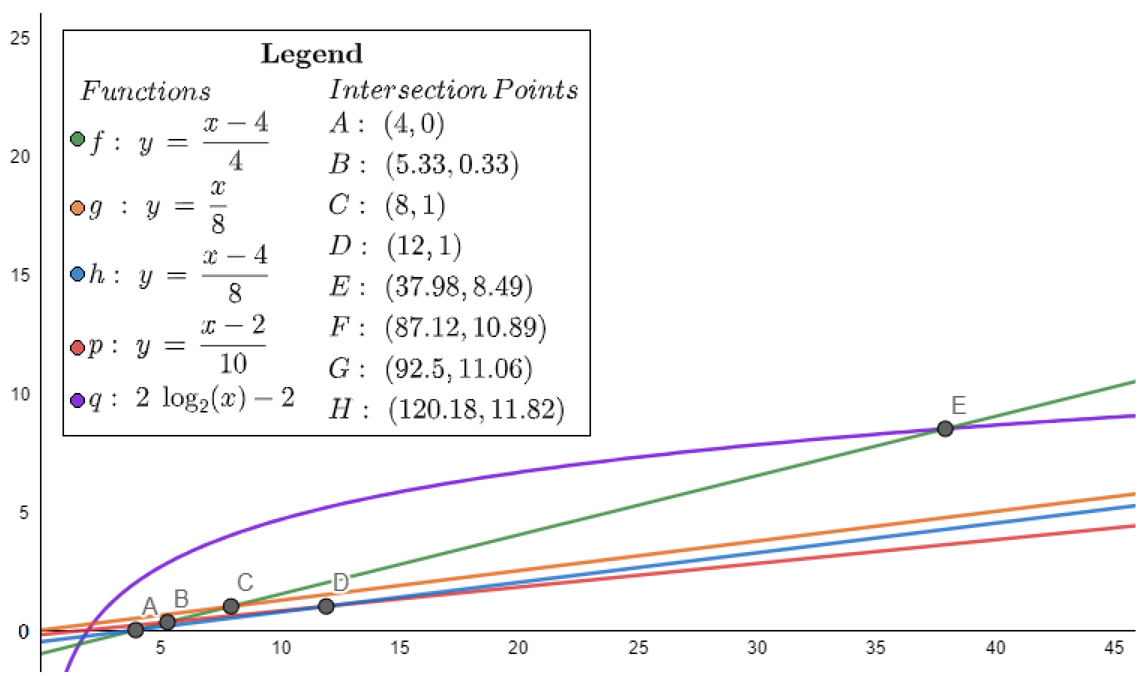

Figure 1.1: A graph of the symmetry rank lower bounds as continuous functions of Theorems A, 4.0.2, B, C, D. Note that points F, G, and H are not depicted in the graph.

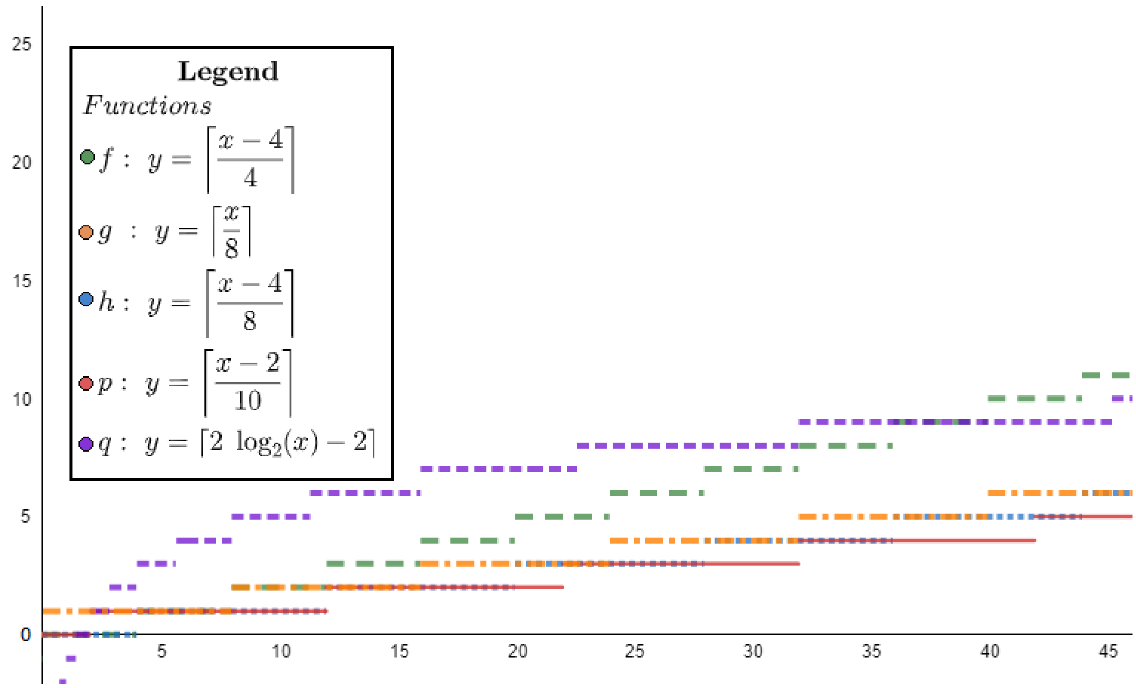

Figure 1.2: A graph of the symmetry rank lower bounds of Theorems A, 4.0.2, B, C, D as step functions.

Our focus in this thesis is to detail the proofs of Theorems A and B. Additionally, we note that while the proof of Theorem A does not differ from the original proof in [13], the proof of Theorem B has been streamlined.

This thesis is organized as follows. In Chapter 2, we establish the building blocks needed to discuss the tools we use to prove our main results. In Chapter 3, we discuss and prove the tools we use to prove our main results. Finally, Chapter 4, we give the proofs of Theorems A, 4.0.2, and B.

CHAPTER 2

PRELIMINARIES

In this chapter, we construct the building blocks needed to talk about the main results of this thesis. We begin by defining the Euler characteristic of a topological space. Then we move on to defining topological manifolds and present some results about the topology of closed, orientable manifolds. We then move on to defining smooth manifolds with the intent of discussing Riemannian manifolds and Riemannian geometry. The next topic we introduce is that of transformation groups. Finally, we describe some important geometric results for manifolds of positive sectional curvature upon which we rely heavily to prove our main results.

2.1 The Euler Characteristic

Recall that for a finite CW complex, X, we define the *Euler characteristic* of X, which we denote by $\chi(X)$, is defined to be the alternating sum $\sum_n (-1)^n c_n$, where c_n is the number of $n-$cells of X.

Since there are many different cellular decompositions possible for a given space X, it is helpful to define $\chi(X)$ in terms of the topology of X. To do so, we introduce the concept of homology. We begin by defining a chain complex.

Definition 2.1.1 [Chain Complex]. [8] *A* chain complex *is an algebraic structure that consists of a sequence of homomorphisms of abelian groups, namely,*

$$\ldots \to C_{n+1} \xrightarrow{\partial_{n+1}} C_n \xrightarrow{\partial_n} C_{n-1} \to \ldots \to C_1 \xrightarrow{\partial_1} C_0 \xrightarrow{\partial_0} 0,$$

with $\partial_n \circ \partial_{n+1} = 0$ *for each* n. *We call the elements of the abelian group* C_k, k*-chains.*

To a topological space X we may then associate a chain complex. One way to do so is by assuming X to have a simplicial structure. In this case, the k-chains are linear combinations of maps from the standard k-simplex to X.

The n^{th} homology group of a space X is defined via its associated chain complex to be a finitely generated abelian group, so before we proceed to its formal definition, we first recall the Fundamental Theorem of Finitely Generated Abelian Groups.

Fundamental Theorem of Finitely Generated Abelian Groups 2.1.2. [5] *Let G be a finitely generated abelian group. Then the following statements hold.*

1. $$G \cong \mathbb{Z}^r \times \mathbb{Z}_{n_1} \times \mathbb{Z}_{n_2} \times \ldots \times \mathbb{Z}_{n_i},$$

   for some integers $r, n_1, n_2, \ldots, n_s$ satisfying the following conditions:

   (a) $r \geq 0$ and $n_j \geq 2$ for all j; and

   (b) $n_{i+1} \big| n_i$ for $1 \leq i \leq s-1$.

2. *The expression in Part 1 is unique: if $G \cong \mathbb{Z}^t \times \mathbb{Z}_{m_1} \times \mathbb{Z}_{m_2} \times \ldots \times \mathbb{Z}_{m_u}$, where t and $m_1, m_2, \ldots, m_u$ satisfy (a) and (b), then $t = r$, $u = s$, and $m_i = n_i$ for all i.*

Definition 2.1.3 [$\mathbf{n^{th}}$ Homology Group]. *Let X be a topological space and $n \in \mathbb{Z}^+$. The n^{th}* homology group *of X is the finitely generated free abelian group defined to be*

$$H_n(X; \mathbb{Z}) = \ker \partial_n / \operatorname{Im} \partial_{n+1}$$

where ∂_n and ∂_{n+1} are the boundary maps of the associated chain complex of X.

We now define the Betti numbers of a topological space X, which provide information about the connectivity of X.

Definition 2.1.4 [$\mathbf{k^{th}}$ Betti Number]. *Let X be a topological space and $k \in \mathbb{Z}^+ \cup \{0\}$. The k^{th}* Betti number *of X, denoted as $b_k(X)$, is defined as follows:*

$$b_k(X) = \operatorname{rank}\left(H_k(X; \mathbb{Z})\right).$$

The following theorem allows us to calculate the Euler characteristic of a space using its Betti numbers.

Theorem 2.1.5. [8] *Let X be a topological space. Then*

$$\chi(X) = \sum_i (-1)^i b_i.$$

We are now ready to compute the Euler characteristic for some important examples. We start with the most basic example: a point.

Example 2.1.6. *Let $X = \{p\}$. The cellular decomposition is given by the 0-cell, e^0. Then $\chi(p) = 1 > 0$.*

Figure 2.1: A triangulation of S^2.

Recall that we define the n-sphere to be $S^n = \{x \in \mathbb{R}^{n+1} : |x| = 1\}$. First, we show how to compute the Euler characteristic of a 2-sphere, then proceed to compute the Euler characteristic of the general n-sphere, for all $n \geq 1$.

Example 2.1.7. *In the tetrahedral triangulation of S^2 given in Figure 2.1, there are four vertices, six edges, and four faces. So the Euler characteristic of S^2 is given by Equation 1.1:*

$$\chi(S^2) = v - e + f = 4 - 6 + 4 = 2.$$

Example 2.1.8. *We compute the Euler characteristic of S^n in two ways: using Definition 1.0.3 and using Theorem 2.1.5.*

Recall from Example 1.0.2 that a cell decomposition of S^n is given by:

$$S^n = e^0 \cup_f e^n,$$

where $f : \partial e^n = S^{n-1} \to e^0$. So $c_0 = 1$, $c_n = 1$, and $c_i = 0$ otherwise. Thus,

$$\chi(S^n) = (-1)^0 c_0 + (-1)^n c_n = 1 \pm 1 = \begin{cases} 2, & \text{if } n \text{ is even,} \\ 0, & \text{if } n \text{ is odd.} \end{cases}$$

The homology of S^n is given as follows:

$$H_i(S^n) = \begin{cases} \mathbb{Z}, & i = 0, n \\ 0, & \text{otherwise.} \end{cases}$$

So, Theorem 2.1.5 gives us that

$$\chi(S^n) = \sum_{i=0}^{n} (-1)^i b_i(S^n) = (-1)^0 b_0 + (-1)^n b_n = 1 \pm 1 = \begin{cases} 2 & \text{if } n \text{ is even,} \\ 0 & \text{if } n \text{ is odd.} \end{cases}$$

The complex projective space, $\mathbb{C}\mathrm{P}^n$, is the $2n$-dimensional quotient space defined to be: $\mathbb{C}\mathrm{P}^n = (\mathbb{C}^{n+1} \setminus \{0\})/z \sim w$, where $z, w \in \mathbb{C}^n$, and $z \sim w$ if $z = \lambda w$ for some $\lambda \in \mathbb{C}$. $\mathbb{C}\mathrm{P}^n$ is also defined as the quotient of the $S^{2n+1} \subseteq \mathbb{C}^{n+1}$ by the free action of the group S^1. We compute its Euler characteristic below.

Example 2.1.9. *The homology of $\mathbb{C}\mathrm{P}^n$ is given by:*

$$H_i(\mathbb{C}\mathrm{P}^n) = \begin{cases} \mathbb{Z}, & \text{if } i = 2k,\ 0 \le k \le n \\ 0, & \text{otherwise.} \end{cases}$$

Then, using Theorem 2.1.5, we have

$$\chi(\mathbb{C}\mathrm{P}^n) = \sum_{i=0}^{n} (-1)^i b_i(\mathbb{C}\mathrm{P}^n) = b_0 + b_2 + \ldots + b_{2n} = n + 1 > 0$$

2.2 Topological Manifolds

We now introduce the concept of a topological manifold, the most general type of manifold.

Definition 2.2.1 [Topological Manifold]. *A topological space M is called a* topological manifold *if it is Hausdorff, second countable, and locally homeomorphic to $\mathbb{R}^n$, for some fixed n. We say that the dimension of M is equal to n, and write* $\dim(M) = n$.

A homeomorphism, which we define below, is the strongest notion of equivalence between topological manifolds.

Definition 2.2.2 [Homeomorphism]. *A* homeomorphism *between two topological spaces M and N is a bijective map $F : M \to N$ such that both F and F^{-1} are continuous.*

In order to introduce a differentiable structure on a topological manifold, M, we need the to define an atlas.

Definition 2.2.3 [Atlas]. *An* atlas *on a topological manifold M, denoted $\{(U_\lambda, \varphi_\lambda)\}$, is an open cover $\{U_\lambda\}$ of M together with a family of maps $\{\varphi_\lambda\}$ such that the following hold.*

1. *For each $(U_\alpha, \varphi_\alpha) \in \{(U_\lambda, \varphi_\lambda)\}$, there exists an open set $V \subset \mathbb{R}^n$ such that*
$$\varphi_\alpha : U_\alpha \to V$$
is a homeomorphism.

2. *For any pair $(U_\alpha, \varphi_\alpha), (U_\beta, \varphi_\beta)$ the map*
$$\varphi_{\alpha,\beta} := \varphi_\beta \circ \varphi_\alpha^{-1} : \varphi_\alpha(U_\alpha \cap U_\beta) \to \varphi_\beta(U_\alpha \cap U_\beta)$$
is a homeomorphism.

We call $\varphi_{\alpha,\beta}$ a transition map and we call each element $(U_\alpha, \varphi_\alpha)$ of an atlas a coordinate chart or simply a chart.

2.2.1 The Topology of Closed, Orientable Manifolds

Before we move on to the definition of a smooth manifold, we first discuss some topological results for closed, orientable manifolds.

Poincaré Duality Theorem 2.2.4. *Let M be orientable and compact. Then*

$$H^k(M;\mathbb{Z}) \cong H_{n-k}(M;\mathbb{Z})$$

That is, the k^{th} cohomology group of M is isomorphic to its $(n-k)^{th}$ homology group.

The following corollary of the Universal Coefficient Theorem adapted for manifolds is a key result that will be used throughout this thesis.

Corollary 2.2.5. [8] *Let M be a manifold. Then*

$$H^k(M;\mathbb{Z}) \cong \mathrm{Fr}(H_k(M;\mathbb{Z})) \oplus T(H_{k-1}(M;\mathbb{Z}))$$

where $\mathrm{Fr}(H_k(M))$ is the free part of $H_k(M;\mathbb{Z})$ and $T(H_{k-1}(M;\mathbb{Z}))$ is the torsion part of $H_{k-1}(M;\mathbb{Z})$. That is, the k^{th} cohomology is isomorphic to the free part of the k^{th} homology plus the torsion action of the $(k-1)^{th}$ homology.

For a compact, orientable manifold M, combining the Poincaré Duality Theorem 2.2.4 and Corollary 2.2.5, we have

$$H_k(M;\mathbb{Z}) \cong H^{n-k}(M;\mathbb{Z}) \cong \mathrm{Fr}(H_{n-k}(M;\mathbb{Z})) \oplus T(H_{n-k-1}(M;\mathbb{Z})).$$

Since $\mathrm{rank}(H_k(M;\mathbb{Z})) = \mathrm{rank}(\mathrm{Fr}(H_k(M;\mathbb{Z})))$ by definition, the following corollary is then immediate.

Corollary 2.2.6. *Let M^n be a compact, orientable manifold. Then,*

$$b_k(M) = b_{n-k}(M),$$

for all $0 \le k \le n$.

Finally, we define the notion of an ℓ-connected inclusion of a subspace $X \subseteq Y$. To do so, we must first define the k^{th} homotopy group of a topological space.

Definition 2.2.7 [$\mathbf{k^{th}}$ Homotopy Group]. *$\pi_k(X)$ is the k^{th}* homotopy group *of a space X. It is defined to be all equivalence classes of maps from (S^k, p) to (X, x_0), where $p \in S^k$ and $x_0 \in X$. Each map sends p to x_0 and two maps are in the same equivalence class if they are homotopic to each other.*

Definition 2.2.8 [ℓ-connected]. *We say the inclusion, $\iota : X \hookrightarrow Y$, is ℓ-*connected *if $\iota_* : \pi_j(X) \hookrightarrow \pi_j(Y)$ is an isomorphism for $0 \leq j < \ell$ and an onto homomorphism for $j = \ell$.*

The following theorem of Whitehead [19] allows us to redefine the notion of $\iota : X \hookrightarrow Y$ being ℓ-connected in terms of the homology groups of X and Y, respectively.

Theorem 2.2.9. [19] *Let $f : X \to Y$ be an ℓ-connected map between connected spaces. Then $f_* : H_q(X) \to H_q(Y)$ is an isomorphism for all $q < \ell$ and an onto homomorphism for $q = \ell$.*

2.3 Smooth Manifolds

We are now ready to define smooth manifolds. Note that smooth n-manifolds may be thought of as spaces that behave like the familiar spaces from Calculus: n-dimensional Euclidean space, $\mathbb{R}^n$.

Recall from Section 2.2, we discussed a way of putting local coordinates on a manifold, so we are ready to define a differentiable manifold.

Definition 2.3.1 [Differentiable Manifold]. *A* differentiable manifold *is a topological manifold M equipped with an atlas $\{(U_\lambda, \varphi_\lambda)\}$ which is maximal relative to the condition that each $\varphi_{\alpha\beta}$ is a differentiable map. We call the associated atlas a differentiable structure on M.*

We define C^k-manifolds as differentiable manifolds whose transitions maps have k continuous derivatives. From this point forward, we assume all manifolds are smooth, that is, infinitely differentiable, or C^∞, unless we state otherwise.

Since a new level of structure to the topological manifold has been introduced, we establish an equivalence criterion to account for this information. A diffeomorphism, which we define below, is the strongest notion of equivalence between differentiable manifolds. Before we define a diffeomorphism, we first define a smooth map.

Definition 2.3.2 [Smooth Map]. *Let M^n and N^m be two smooth manifolds. A map $F : M \to N$ between smooth manifolds is said to be* smooth at $p \in M$ *if given a chart (V, ψ) on N with $F(p) \in V$ there exists a chart (U, φ) in M such that $F(U) \subset V$ and the map $\psi \circ F \circ \varphi^{-1} : \varphi(U) \subset \mathbb{R}^n \to \psi(V) \subset \mathbb{R}^m$ is a* smooth map *at $\varphi(p)$. A map is said to be* smooth *if it is smooth at all points in M.*

In the same way that a homeomorphism is a continuous map preserving the topological information of a manifold, a diffeomorphism is a smooth map that preserves the topological and differential information of a smooth manifold.

Definition 2.3.3 [Diffeomorphism]. *A* diffeomorphism *between two smooth manifolds M and N is a homeomorphism, $F : M \to N$, such that both F and F^{-1} are smooth maps.*

We now give some important examples of smooth manifolds.

Example 2.3.4 [Sphere]. *Recall that we define the n-sphere to be*

$$S^n := \{x \in \mathbb{R}^{n+1} : |x| = 1\}.$$

S^n inherits a manifold topology via the subspace topology in $\mathbb{R}^{n+1}$.

Example 2.3.5 [Complex Projective Space]. *Recall that the $2n$-dimensional quotient space*

$$\mathbb{C}\mathrm{P}^n := (\mathbb{C}^{n+1} \setminus \{0\})/z \sim w$$

where $z, w \in \mathbb{C}^n$, *and* $z \sim w$ *if* $z = \lambda w$ *for some* $\lambda \in \mathbb{C}$. $\mathbb{C}\mathrm{P}^n$ *is also defined as the quotient of the* $S^{2n+1} \subseteq \mathbb{C}^{n+1}$ *by the free action of the group* S^1. *The manifold topology and smooth structure on* $\mathbb{C}P^n$ *are obtained from* S^{2n+1} *via the quotient map.*

Our goal now is to define an affine connection on a manifold so that we can characterize the notion of covariant derivatives for vector fields on smooth manifolds. In order to do so, the notions of tangent bundle and vector fields must first be defined.

Definition 2.3.6 [Tangent Bundle]. *Let* M *be a smooth manifold. We call the set* $TM = \{(p, v) \,|\, p \in M \text{ and } v \in T_pM\}$ *endowed with a smooth structure induced by* M *the* tangent bundle *of* M.

Definition 2.3.7 [Vector Field] [4]. *A* vector field X *on a smooth manifold* M *is a correspondence that associates to each point* $p \in M$ *a vector* $X(p) \in T_pM$. *In terms of mappings,* X *is a mapping of* M *into the tangent bundle* TM. *The field is said to be smooth if the mapping* $X : M \to TM$ *is smooth.*

Definition 2.3.8 [Affine Connection]. [4] *An* affine connection ∇ *on a smooth manifold* M *is a mapping*

$$\begin{aligned} \nabla\colon \mathfrak{X}(M) \times \mathfrak{X}(M) &\to \mathfrak{X}(M), \\ (X, Y) \qquad &\mapsto \nabla_X Y \end{aligned}$$

which satisfies the following conditions for $X, Y, Z \in \mathfrak{X}(M)$ *and* $f, g \in \Omega^\infty(M)$:

1. $\nabla_{fX+gY} Z = f\nabla_X Z + g\nabla_Y Z$,
2. $\nabla_X(Y + Z) = \nabla_X Y + \nabla_X Z$,
3. $\nabla_X(fY) = f\nabla_X Y + X(f)Y$.

These properties guarantee that an affine connection behaves like the usual covariant derivative when $M = \mathbb{R}^n$.

2.4 Riemannian Geometry

Riemannian geometry is the study of smooth manifolds endowed with a Riemannian metric. With the addition of a Riemannian metric to a smooth manifold, we can ask questions similar to those commonly asked in classical Euclidean geometry. In this section we introduce the concept of a Riemannian manifold. We begin with the notion of a Riemannian metric.

Definition 2.4.1 [Riemannian Metric]. [4] *A* Riemannian metric *on a smooth manifold M is an assignment to each $p \in M$ an inner product $\langle \cdot, \cdot \rangle_p$ on T_pM which is smooth in the sense that if $\boldsymbol{x} : U \to M$ is a smooth parametrization at p with $\boldsymbol{x}(q) = \boldsymbol{x}(x_1, x_2, \ldots, x_n) = p \in \boldsymbol{x}(U)$ and $\left.\frac{\partial}{\partial x_i}\right|_p = d\boldsymbol{x}_q(0, \ldots, 1, \ldots, 0)$ with 1 in the i^{th} position then*

$$\left\langle \frac{\partial}{\partial x_i}, \frac{\partial}{\partial x_j} \right\rangle_p = g_{ij}(x_1, x_2, \ldots, x_n)$$

is a smooth real-valued function on U.

We call a smooth manifold M equipped with a Riemannian metric $g = (g_{ij})$ a *Riemannian manifold* and denote it by (M, g).

Example 2.4.2. *Any manifold which can be smoothly embedded in $\mathbb{R}^n$ admits an induced Riemannian metric as follows. Let $f : M \to \mathbb{R}^n$ be a smooth embedding, then M admits a Riemannian metric determined by*

$$\langle u, v \rangle_p = \langle df_p(u), df_p(v) \rangle_{f(p)}$$

where $\langle \cdot, \cdot \rangle_{f(p)}$ is the standard inner product on $T_{f(p)}\mathbb{R}^n$.

Note that any n-dimensional smotoh manifold, M^n, admits a Riemannian metric, via a partition of unity, or with the induced metric as a smoothly embedded submanifold of $\mathbb{R}^{2n}$ (see, for example, Lee [11]).

We now determine a new criterion for equivalence: isometry.

Definition 2.4.3 [Isometry]. *A diffeomorphism,* $F : M \to N$*, between two Riemannian manifolds is called an* isometry *if for all* $p \in M$*, we have*

$$\langle v, w \rangle_p = \langle dF_p(v), dF_p(w) \rangle_{F(p)}.$$

Recall that for smooth manifolds we are able to define affine connections which serve to define a type of covariant derivative for vector fields. In a Riemannian manifold, there is a unique torion-free connection which is compatible with the metric, called the *Levi-Civita Connection.* Throughout this paper, any Riemannian manifold is assumed to be equipped with the Levi-Civita connection.

Definition 2.4.4 [Levi-Civita Connection]. [4] *An affine connection* ∇ *on a Riemannian manifold is called a* Levi-Civita connection *if the following two conditions are satisfied.*

1. *The connection is compatible with the metric, that is,*

$$X\langle Y, Z \rangle = \langle \nabla_X Y, Z \rangle + \langle Y, \nabla_X Z \rangle, \qquad \text{for } X, Y, Z \in \mathfrak{X}(M).$$

2. *The connection is torsion free, that is,*

$$\nabla_X Y - \nabla_Y X = [X, Y].$$

Now that we have a metric and a natural choice of connection to measure the covariant derivatives of vector fields, we are ready to begin the discussion of curvature. We are concerned soley with sectional curvature in this thesis, but note that there are other notions of curvature, such as Ricci and scalar curvature. To begin, we first define the Riemann curvature, which we need to define sectional curvature.

Definition 2.4.5 [Riemann Curvature]. [4] *The* Riemann curvature *R, of a Riemannian manifold* (M, g) *is a correspondence that associates to every pair* $X, Y \in$

$\mathfrak{X}(M)$ a mapping $R(X,Y) : \mathfrak{X}(M) \to \mathfrak{X}(M)$ given by

$$R(X,Y)Z = \nabla_X \nabla_Y Z - \nabla_Y \nabla_X Z - \nabla_{[X,Y]} Z$$

where ∇ is the Levi-Civita connection on (M, g).

The Riemann curvature of a manifold M measures the non-commutativity of the covariant derivative. Equivalently, the Riemann curvature measures how far a space is from being Euclidean, since the covariant derivative in $\mathbb{R}^n$ is commutative. We are now ready to define the sectional curvature of a Riemannian manifold.

Definition 2.4.6 [Sectional Curvature]. [4] *Given a point $p \in M$ and a 2-plane $\sigma_p \subset T_pM$, the* sectional curvature *$K(\sigma_p)$ is defined to be*

$$\frac{\langle R(u,v)v, u\rangle}{\langle u,u\rangle\langle v,v\rangle - \langle u,v\rangle^2}$$

for any pair of linearly independent vectors $u, v \in \sigma_p$.

Now that we have defined the notion of sectional curvature which we use in this paper, we turn to a discussion of curves in Riemannian manifolds. We call such curves geodesics.

Definition 2.4.7 [Geodesic]. [4] *A parametrized curve $\gamma : I \to M$ is a* geodesic *at $t_0 \in I$ if $\frac{D}{dt}\left(\frac{d\gamma}{dt}\right) = 0$ at the point t_0. If γ is a geodesic at t, for all $t \in I$, we say that γ is a geodesic. If $[a,b] \subset I$ and $\gamma : I \to M$ is a geodesic, the restriction of γ to $[a,b]$ is called a geodesic segment joining $\gamma(a)$ to $\gamma(b)$.*

We see that if $\gamma : I \to M$ is a geodesic, then by the metric compatibility of ∇ we have

$$\frac{d}{dt}\Big\|\frac{d\gamma}{dt}\Big\|^2 = \frac{d}{dt}\left\langle \frac{d\gamma}{dt}, \frac{d\gamma}{dt}\right\rangle = 2\left\langle \frac{D}{dt}\frac{d\gamma}{dt}, \frac{d\gamma}{dt}\right\rangle = 2\left\langle 0, \frac{d\gamma}{dt}\right\rangle = 0.$$

So, the magnitude of the velocity vector, $\frac{d\gamma}{dt}$, of a geodesic is constant. In this way, we have identified a type of curve whose length is minimizing.

Given a Riemannian manifold M, we now define a special type of submanifold which depends on the geometry of M. At each point $p \in M$ these submanifolds will be locally equivalent to the image of the exponential map on some small neighborhood of T_pM.

Definition 2.4.8 [Totally Geodesic Submanifold]. *Given a Riemannian manifold, (M, g), a submanifold, $(N, g|_N)$, with the subspace metric is said to be* totally geodesic *if any geodesic γ in N is also geodesic in M.*

The following corollary highlights an important fact about totally geodesic submanifolds

Corollary 2.4.9. *Let $N^k \subseteq M^n$ be a totally geodesic submanifold. Then, for all $p \in N$ and all 2-planes σ_p tangent to N, we have*

$$K_M(\sigma_p) = K_N(\sigma_p).$$

2.5 Transformation Groups

This chapter gives a brief introduction to the theory of group actions on manifolds and presents a few theorems that are important in the proof of our main results. We begin with a general overview of smooth group actions on differentiable manifolds, and then restrict our focus to smooth circle actions. We then proceed to give a general overview of isometric group actions on Riemannian manifolds, and then our focus to the case of isometric circle actions.

There is a rich interplay between the topology of a manifold and the topology of the groups which act on them smoothly. We see in this section that a special type of group, called a Lie group, is a smooth manifold itself and when a Lie group acts on a smooth manifolds we obtain a wealth of information about the orbits. We begin with the definition of a group action.

Definition 2.5.1 [Group Action]. *Let G be a group and let Ω be a set. We say that G* acts *on Ω if there is a map $\varphi : G \times \Omega \to \Omega$ such that the following hold:*

1. *$\varphi(e,x) = x$ for all $x \in \Omega$; and*

2. *$\varphi(h, \varphi(g,x)) = \varphi(hg, x)$ for all $x \in \Omega$ and for all $h, g \in G$.*

We call the map φ a group action*.*

An action is called *effective* if the only element that fixes Ω pointwise is the identity element. An action is called *almost effective* if a finite group fixes Ω pointwise. For a particular element $x \in \Omega$ we set $G_x = \{g \in G : gx = x\}$, the set of all elements of G which fix x. This set forms a subgroup of G, which we call the *isotropy subgroup of x*. In the special case where the isotropy subgroup of any element of Ω is trivial, we say that the action is *free*. If instead the isotropy group of $x \in \Omega$ is G, then we say that x is fixed by G. We set $\text{Fix}(\Omega, G) = \{x \in \Omega : g \cdot x = x \text{ for all } g \in G\}$, and we call this set the *fixed point set of G*.

Definition 2.5.2 [Orbit]. *Let G act on Ω. An* orbit *of the G-action is an equivalence class in Ω given by the relation: $x \sim y$ if and only if $x = g \cdot y$ for some $g \in G$. We denote the orbit of the point x by $G(x)$.*

When a group G acts on a set X, we can use the partial ordering on subgroups of G given by set inclusion to classify orbit types in X. The largest type of orbit in this system is called a *principal orbit*.

Definition 2.5.3 [Principal Orbit]. *For $x \in \Omega$, an orbit is called* principal *if its isotropy subgroup G_x is minimal with respect the the following partial order: $(H) \subseteq (K)$ if and only if H is conjugate to some subgroup of K. The ordering is partial because there may be elements that we cannot compare.*

We distinguish two other types of orbits by their respective isotropy groups. An orbit is called *exceptional* if its isotropy subgroup is not minimal but has the same dimension as the principal isotropy subgroup. An orbit is called *singular* if its isotropy subgroup has dimension strictly larger than that of the principal isotropy subgroup.

An important class of groups are the Lie groups.

Definition 2.5.4 [Lie Group]. *A* Lie group *is a smooth manifold G equipped with a group structure such that the maps*

$$\begin{array}{rlcrl} G\times G \to & G & \textit{and} & G\to & G \\ (g,h) \mapsto & g\cdot h & & g \mapsto & g^{-1} \end{array}$$

are smooth.

The orthogonal group, $O(n)=\{A\in GL(n,\mathbb{R}) : AA^T = Id\}$ and the special orthogonal group $SO(n)=\{A\in O(n) : \det A = 1\}$ are examples of Lie groups. Observe that $\ker SO(2)\cong S^1$ is a circle, which we also denote by T^1. In this thesis, we consider T^k-actions on Riemannian manifolds, where T^k is the k-fold product of circles, that is, $T^k \cong \underbrace{T^1\times\ldots\times T^1}_{k-times}$.

A concept of fundamental importance in the theory of transformation groups is that of the normal slice to an orbit.

Definition 2.5.5 [Slice]. *For an action of a compact Lie group G on a smooth manifold M and a point $p\in G(q)$, the embedded submanifold S_p normal to the orbit $G(q)$ given by $\exp_p v$, $v\in B_\varepsilon^\perp$ is called the* slice *through p. We denote by S the union of the S_p for all $p\in G(q)$.*

The Slice Theorem guarantees the existence of a slice at every point $x\in M$. Moreover, it states that the isotropy subgroup G_x acts linearly on the slice.

Now that we have developed all of the necessary tools for a general discussion of smooth group actions on differentiable manifolds, we are ready to restrict our focus to smooth circle and torus actions.

An important result for smooth torus actions is given by the Borel Formula. Given a T^k-action as a manifold. Let $F=\mathrm{Fix}(M;T^r)$, where $T^r\le T^k$. For $r\ge 2$, the Borel formula gives us information about the codimension of the fixed point sets of T^{r-1} in F, where $T^{r-1} \leq T^r$, based on the codimension of F in M.

The Borel Formula 2.5.6. [2] *Let T^k act smoothly on X^n, where X^n is a Poincaré Duality space. Let*

$$F = X^T = \{x \in X : T^k(x) = x\}.$$

Let $F_1 \in F$ be a connected component of dimension m. For any subtorus $H \leq T^k$, let $F_1(H)$ be the component of X^H containing F_1. Let $n(H) = \dim(F_1(H))$, then

$$n - m = \sum_H (n(H) - m),$$

where the sum ranges over all subtori $H \subset T^k$ of corank one. Namely,

$$H \cong T^{k-1} \leq T^k.$$

2.5.1 Isometric Group Actions

For Riemannian manifolds the strongest possible equivalence between two manifolds M and N is an *isometry* $F : M \to N$. The following lemma characterizes the set of isometries of a Riemannian manifold.

Lemma 2.5.7. *Given a Riemannian manifold (M, g), the set of all isometries of M, $\mathrm{Isom}(M)$, forms a group under composition.*

We say that a group G acts isometrically on (M, g) if $\langle u, v\rangle_p = \langle dh(u), dh(v)\rangle_{g(p)}$ for all $p \in M$, $h \in G$, and $u, v \in T_pM$. The following result is due to Myers and Steenrod [12].

Theorem 2.5.8. [12] *The group $G = \mathrm{Isom}(M)$ of a Riemannian manifold (M, g) is a Lie group with respect to the compact-open topology. For each $x \in M$, the isotropy subgroup G_x is compact. If M is compact, then $\mathrm{Isom}(M)$ is compact.*

The following theorem due to Kobayashi [10] characterizes fixed point sets of isometries.

Theorem 2.5.9. [10] *Let (M, g) be a Riemannian manifold and G be any set of isometries of M. Then each connected component of* $\mathrm{Fix}(M; G)$ *is a totally geodesic submanifold of M. In particular, by Corollary 2.4.9 if* $\sec(M) > 0$, *then for all* $\sec(N) > 0$ *for all* $N \subseteq \mathrm{Fix}(M; G)$.

2.5.2 Isometric $\mathbf{T^k}$-Actions

In this subsection, we show that we are able to extract even more information about the fixed point sets of torus actions when the action is isometric. In particular, the following theorem shows us that for an isometric circle action, the Euler characteristic of its fixed point set is equal to that of the manifold.

Theorem 2.5.10. [10] *Let M be a Riemannian manifold and suppose T^k acts on M by isometries. Then* $\chi(M) = \chi(\mathrm{Fix}(M; T^k))$.

Recall by Theorem 2.5.9 that each connected component of $\mathrm{Fix}(M; G)$ is a totally geodesic submanifold. The next result shows that when $G = S^1$, then each connected component is also closed and orientable if M is.

Theorem 2.5.11. [10] *Let (M, g) be a Riemannian manifold and suppose that $S^1 \times M \to M$ acts isometrically and effectively. Let* $\mathrm{Fix}(M; S^1) = \cup_i N_i$ *be a decomposition of the fixed point set into its connected components, then the following are true.*

1. *Each N_i is a closed totally geodesic submanifold of even codimension.*
2. *If M is orientable then each N_i is orientable.*

2.6 Geometric Results in the Presence of Positive Curvature

In this section, we focus our attention on Riemannian manifolds, M, that have strictly positive sectional curvature, denoted by $\sec(M) > 0$. There is a strong relationship between the sectional curvature of a manifold and its topology. Recall the result due to Synge [18], mentioned in the Introduction.

Synge's Theorem 2.6.1. [18] *Let M be a Riemannian manifold of strictly positive sectional curvature. Then the following are true.*

1. *If M is even-dimensional, then $\pi_1(M) = 0$ if M is orientable, and $\pi_1(M) = \mathbb{Z}_2$, if M is non-orientable.*
2. *If M is odd-dimensional, then M is orientable.*

The following theorem characterizing totally geodesic submanifolds in positively curved spaces will be important in the proof of the main results.

Frankel's Theorem 2.6.2. [6] *For a closed Riemannian manifold M with strictly positive sectional curvature and two totally geodesic submanifolds N_1, N_2 of M, if*

$$\dim(N_1) + \dim(N_2) \geq \dim(M),$$

then N_1 and N_2 have non-trivial intersection.

The next two results due to Grove and Searle [7] are useful for the result of Theorem A.

Lemma 2.6.3. [7] *Suppose* $\operatorname{codim} \operatorname{Fix}(S^1) = 2$*. Then*

1. *Exactly one component, N of* $\operatorname{Fix}(S^1)$ *has codimension 2.*
2. *There is a unique orbit $S^1\ p_0$ at maximal distance from N.*
3. *S^1 acts freely on $M - (N \cup S^1 p_0)$.*

Theorem 2.6.4. [7] *Let S^1 act isometrically and effectively on M^n, an n-dimensional, closed manifold with positive sectional curvature and suppose* $\dim(\operatorname{Fix}(M^n; S^1)) = n - 2$*. Then M^n is diffeomorphic to S^n, $\mathbb{R}\mathrm{P}^n$, $\mathbb{C}\mathrm{P}^{n/2}$, or a lens space.*

These next two results by Wilking [20] are critical to obtaining the result of Theorem B.

Wilking's Connectedness Theorem 2.6.5. [20] *Let M^n be a compact Riemannian manifold with positive sectional curvature.*

1. *Suppose that $N^{n-k} \subset M^n$ is a compact totally geodesic embedded submanifold of codimension k. Then the inclusion map $N^{n-k} \to M^n$ is $(n-2k+1)-$connected. If there is a Lie group G acting isometrically on M^n and fixing N^{n-k} pointwise, then the inclusion map is $(n-2k+1+\delta(G))-$connected, where $\delta(G)$ is the dimension of the principal orbit.*

2. *Suppose that $N_1^{n-k_1}, N_2^{n-k_2} \subset M^n$ are two compact totally geodesic embedded submanifolds, $k_1 \leq k_2$, $k_1 + k_2 \leq n$. Then the intersection $N_1^{n-k_1} \cap N_2^{n-k_2}$ is a totally geodesic embedded submanifold as well, and the inclusion*

$$N_1^{n-k_1} \cap N_2^{n-k_2} \to N_2^{n-k_2}$$

is $(n-k_1-k_2)-$connected.

Figure 2.2 below illustrates Part 2 of the above theorem.

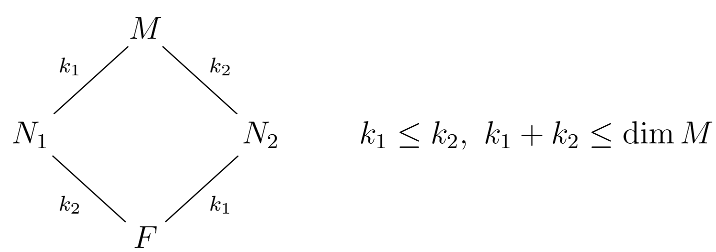

Figure 2.2: An illustration of the inclusion of $F = N_1 \cap N_2$ in N_1, N_2, and M.

Lemma 2.6.6. [20] *Let M^n be a closed orientable manifold, and let N^{n-k} be an embedded compact oriented submanifold without boundary. Suppose the inclusion $\iota : N \to M$ is $n-k-1$ connected and $n-k-2l > 0$. Let $[N] \in H_{n-k}(M;\mathbb{Z})$ be the image of the fundamental class of N in $H_*(M;\mathbb{Z})$, and let $e \in H^k(M;\mathbb{Z})$ be its*

Poincaré dual. Then the homomorphism

$$\cup e : H^i(M;\mathbb{Z}) \to H^{i+k}(M;\mathbb{Z})$$

is surjective for $l \leq i < n-k-l$ *and injective for* $l < i \leq n-k-l$

Two important theorems for manifolds of positive curvature admitting isometric torus actions are Berger's Theorem and Sugahara's Theorem.

Berger's Theorem 2.6.7. [1] *Let* M^{2m} *be a closed, orientable manifold of positive section curvature admitting an isometric* T^k*-action. Then there is a point* $p \in M$ *fixed by* T^k*.*

Sugahara's Theorem 2.6.8. [17] *Let* M^{2m+1} *be a closed, orientable manifold of positive sectional curvature admitting an effective isometric* T^k*-action. Then there is a circle orbit. That is, there exists a circle of points in* M *all fixed by some* T^{k-1} *subgroup of* T^k*.*

CHAPTER 3

TOOLS

Now that our foundation is built, we discuss the tools used to prove our main results, along with their proofs. We begin with the relationship of the Betti numbers between the manifold and a closed, ℓ-connected submanifold.

Lemma 3.0.1. [15] *Let N be a closed submanifold of M. If the inclusion map $\iota : N \hookrightarrow M$ is ℓ-connected then the Betti numbers satisfy*

$$b_j(N) = b_j(M), \quad 0 \leq j \leq \ell - 1, \text{ and} \tag{3.1}$$

$$b_\ell(N) \geq b_\ell(M). \tag{3.2}$$

Proof. By assumption $\iota : N \hookrightarrow M$ is ℓ-connected, and so $\iota_* : H_j(N) \to H_j(M)$ is an isomorphism for all $0 \leq j \leq \ell - 1$ by Theorem 2.2.9. Thus, $\text{rank}(H_j(N)) = \text{rank}(H_j(M))$ for $0 \leq j \leq \ell - 1$. Since $b_j(X) = \text{rank}(H_j(X))$ for a topological space X, we have $b_j(N) = b_j(M)$ for all $0 \leq j \leq \ell - 1$.

Moreover, by definition $\iota_* : H_j(N) \to H_j(M)$ is an onto homomorphism for $j = \ell$. So $\text{rank}(H_\ell(N)) \geq \text{rank}(H_\ell(M))$ by the Fundamental Theorem of Finitely Generated Abelian Groups 2.1.2. Thus, the result holds. □

Observe that if $\phi : H_j(N) \to H_j(M)$ is one-to-one for some j, then again by the Fundamental Theorem of Finitely Generated Abelian Groups 2.1.2, we see that $b_i(N) \leq b_j(M)$. Continuing this observation with Lemma 3.0.1, we obtain the following corollary of Theorem 2.6.6.

Corollary 3.0.2. [15] *Let M^n be a closed orientable manifold, and let N^{n-k} be an embedded compact oriented submanifold without boundary. Suppose the inclusion $\iota : N \to M$ is $n - k - \ell$ connected and $n - k - 2\ell > 0$. Then*

$$\begin{cases} b_i(M) = b_{i+k}(M), & \textit{for } \ell+1 \le i \le n-k-\ell-1, \\ b_\ell(M) \ge b_{\ell+k}(M), & \textit{for } i = \ell, \textit{ and} \\ b_{n-k-\ell}(M) \le b_{n-\ell}(M), & \textit{for } i = n-k-\ell. \end{cases}$$

Proof. Recall that the homology and cohomology groups of a topological space are finitely generated abelian groups and $b_i(M) = \text{rank}(H^i(M))$. Let G and H be finitely generated abelian groups. Set $\text{rank}(G) = r$ and $\text{rank}(H) = s$, and suppose $\phi : G \to H$. Since the rank of a finitely generated abelian groups is determined by its free part and since there are no non-trivial homomorphisms from the torsion part to the free part, then if ϕ is onto, $r \ge s$, and if ϕ is one-to-one, $r \le s$. When ϕ is an isomorphism $r = s$. So the results follow from the fact that $\phi : H^i \to H^{i+k}$ is an isomorphism for $\ell + 1 \le i \le n - k - \ell - 1$, is onto for $i = \ell$, and is one-to-one for $i = n - k - \ell$. □

This next lemma shows that closed, orientable manifolds of dimension 2 or 4 have positive Euler characteristic.

Lemma 3.0.3. *Let N^k be a closed, orientable Riemannian manifold with $k = 2$ or 4, and suppose $H_1(N^k; Z) = 0$. Then $\chi(N^k) > 0$.*

Proof. For $k = 2$, if $H_1(N^2) = 0$, then by Theorem 2.1.5 we calculate

$$\chi(N^2) = b_0(N) - b_1(N) + b_2(N) = b_0(N) + b_2(N) = 2 > 0.$$

Now assume $k = 4$. Since we assume $H_1(N; \mathbb{Z}) = 0$ and N is orientable, we may apply Theorem 2.2.4 to get

$$b_1(N) = b_{4-1}(N) = b_3(N) = 0.$$

Moreover since N is orientable, we have $b_0(N) = b_4(N) = 1$. So,

$$\begin{aligned} \chi(N^4) &= b_0(N) - b_1(N) + b_2(N) - b_3(N) + b_4(N) \\ &= b_0(N) + b_2(N) + b_4(N) \\ &= 2 + b_2(N) > 0. \end{aligned}$$

□

The following corollary is an application that uses the results of Frankel, Kobayashi, and Lemma 3.0.3.

Corollary 3.0.4. *Let M^{2m} be a positively curved manifold admitting an isometric T^1-action. Let $N \in \text{Fix}(M; T^1)$ and suppose the codimension of N is 2, 4, or 6. Then $\chi(M) \geq \chi(N)$.*

Proof. By Theorem 2.5.10, $\chi(M) = \chi(N)$ for all $N \in \text{Fix}(M; T^1)$. Moreover, by Frankel's Theorem 2.6.2, if the codimension of $N \in \text{Fix}(M; T^1)$ is 4 or 6, then the only other components of $\text{Fix}(M; T^1)$ are of dimension less than or equal to 2 or 4, respectively, then $\chi(N) > 0$ by Lemma 3.0.3 and Example 2.1.6. Moreover, if $\dim M \leq 4$, $\chi(M) > 0$ by Lemma 3.0.3. So, $\chi(M) \geq \chi(N)$. □

The following Key Lemma contains three curvature-free results, that we will rely on heavily for the proofs of Theorem 4.0.2 and Theorem B.

Key Lemma 3.0.5. [15] *Let M^{2m} be a simply-connected, closed, even-dimensional manifold. Let N^{2m-k} be a closed, orientable submanifold with $k = 2, 4,$ or 6. Suppose that the inclusion of N in M is i-connected, then the following hold:*

1. *If $k = 2$, $i = 2m - 3$, and $2m \geq 6$, then both $\chi(M)$ and $\chi(N)$ are positive;*

2. *If $k = 4$, $i = 2m - 6$, and $2m \geq 8$, and $\chi(M) \geq \chi(N)$, then $\chi(N) > 0$; and*

3. *If $k = 6$, $i = 2m - 6$, and $2m \geq 8$, and $\chi(M) \geq \chi(N)$, then $\chi(N) > 0$.*

Proof. We prove each part separately.

Proof of Part 1: Suppose $N^{2m-2} \subset M^{2m}$. By Lemma 3.0.1, since we assume N to be $(2m-3)$-connected, we have $b_i(N) = b_i(M)$ for all $i \leq 2m-4$ and $b_{2m-3}(N) \geq b_{2m-3}(M)$. Moreover, by Lemma 2.2.6, we have $b_i(M) = b_{2m-i}(M)$, as M, being simply-connected, is orientable. Therefore,

$$b_{2m-3}(N) \geq b_{2m-3}(M) = b_3(M).$$

By Lemma 2.6.6, since N is $(2m-3)$-connected with $k = \text{codim}(N) = 2$ and so $\ell = 1$, we have

$$H^i(M) \cong H^{i+2}(M),$$

for $2 \leq i \leq 2m-4$. We also have $\cup_e : H^i(M) \to H^{i+2}(M)$ is onto for $i = 1$ and is one-to-one for $i = 2m-3$. Then by Corollary 3.0.2, we see that

$$\begin{cases} b_i(M) = b_{i+2}(M), & 2 \leq i \leq 2m-4 \\ b_1(M) \geq b_3(M). \end{cases} \tag{3.3}$$

Thus,

$$b_1(M) \geq b_3(M) = \ldots = b_{2m-3}(M).$$

But, by hypothesis, $\pi_1(M) = 0 = H_1(M)$, and so $b_1(M) = 0$. Since $b_i \geq 0$ for all i, we have

$$b_1(M) = b_3(M) = \ldots = b_{2m-3}(M) = 0,$$

and by Theorem 2.2.4, $b_1(M) = b_{2m-1}(M) = 0$. Thus, all odd Betti numbers of M vanish. Likewise, by Lemma 2.6.6, we have

$$b = b_2(M) = b_4(M) = \ldots = b_{2m-4}(M) = b_{2m-2}(M) \geq 0.$$

Therefore,

$$\begin{aligned} \chi(M) &= \sum_{i=0}^{2m}(-1)^i b_i(M) \\ &= b_0 + (m-1)(b) + b_{2m} \\ &= 2 + (m-1)b \geq 2 > 0. \end{aligned}$$

Note that $b_0(M) = b_{2m}(M) = 1$, since $H_0(M;\mathbb{Z}) \cong H_{2m}(M;\mathbb{Z}) \cong \mathbb{Z}$ because M is connected and orientable. Then, by the above and Lemma 3.0.1, we have

$$\begin{aligned} \chi(N) &= \sum_{i=0}^{2m-2}(-1)^i b_i(N) \\ &= \sum_{i=0}^{2m-2}(-1)^i b_i(M) \\ &= b_0(M) + (m-2)(b) + b_{2m-2}(M) \\ &= 2 + (m-2)b \geq 2 > 0. \end{aligned}$$

This completes the proof of Part 1.

Proof of Part 2: If $\dim N = 4$ and N is 2-connected, then

$$H_1(N) \cong H_1(M) = 0.$$

This implies N is a closed, simply-connected 4-manifold. Lemma 3.0.3 then gives us that $\chi(N) > 0$, as desired.

So, we may assume $\dim N = 2m - 4 > 4$, that is, $2m \geq 10$. By Display 3.1 of Lemma 3.0.1, since $\iota : N \hookrightarrow M$ is $(2m-6)$-connected, we have

$$b_i(N) = b_i(M), \quad i \leq 2m - 7.$$

Also by Display 3.1 and Display 3.2 of Lemma 3.0.1, combined with Corollary 2.2.6, we have

$$b_2(M) = b_2(N) = b_{(2m-4)-2}(N) = b_{2m-6}(N) \geq b_{2m-6}(M) = b_6(M).$$

Lastly, by Display 3.1 of Lemma 3.0.1, Corollary 2.2.6, and the hypothesis that $\pi_1(M) = 0$, we have

$$b_1(N) = b_1(M) = b_{2m-1}(M) = b_{2m-5}(N) = 0.$$

We summarize the three results above here:

$$\begin{cases} b_i(N) = b_i(M), \quad i \leq 2m-7, \\ b_2(M) \geq b_6(M), \text{ and} \\ b_1(N) = b_1(M) = b_{2m-1}(M) = b_{2m-5}(N) = 0. \end{cases} \tag{3.4}$$

In addition, by Lemma 2.6.6, as $k = 4$ and so $\ell = 2$, we have

$$\cup_e : H^i(M) \to H^{i+4}(M) \tag{3.5}$$

is an isomorphism for $3 \leq i \leq 2m - 7$.

We consider two cases: Case 1, when $2m = 4n$, and Case 2, when $2m = 4n+2$.

Case 1 of Part 2: Let $2m = 4n$ where $2m \geq 10$, so that $n \geq 3$.

Now we evaluate the i^{th} Betti numbers in groups of 4 using the isomorphism in Display 3.5 for $i = 3, 4, 5, 6$ and $i \leq 4n - 7$. We obtain

$$\begin{cases} b_3(M) = b_7(M) = \ldots = b_{4n-9}(M) = b_{4n-5}(M), \\ b_4(M) = b_8(M) = \ldots = b_{4n-8}(M) = b_{4n-4}(M), \\ b_5(M) = b_9(M) = \ldots = b_{4n-7}(M) = b_{4n-3}(M), \text{ and} \\ b_6(M) = b_{10}(M) = \ldots = b_{4n-10}(M) = b_{4n-6}(M). \end{cases} \tag{3.6}$$

Note that the set $\{[3, 4n-3] \cap \mathbb{Z}\}$ has $4n-5$ terms. Each row of equalities in Display 3.6 has $n-1$ terms, except for the row of equalities for $b_6(M)$, which has $n-2$ terms, since $\frac{4n-8}{4} + 1 = n - 1$ and $\frac{4n-12}{4} + 1 = n - 2$, respectively.

Moreover, combining the first and third equations in Display 3.6 with Corollary 2.2.6, and we get

$$b_5(M) = b_{4n-5}(M) = b_3(M). \tag{3.7}$$

In particular, we can then rewrite all the Betti numbers solely in terms of $b_0(M)$, $b_2(M)$, $b_3(M)$, $b_4(M)$, $b_6(M)$, and $b_{4n}(M)$. Also by Corollary 2.2.6, $b_1(M) = b_{4n-1}(M) = 0$ and $b_2(M) = b_{4n-2}(M)$. So, we may rewrite $\chi(M)$ as follows:

$$\begin{aligned} \chi(M) &= \sum_{i=0}^{4n} (-1)^i b_i(M) \\ &= b_0(M) + b_{4n}(M) + \sum_{i=1}^{4n-1} (-1)^i b_i(M) \\ &= 2 - 2b_1(M) + \sum_{i=2}^{4n-2} (-1)^i b_i(M) \\ &= 2 - 0 + 2b_2(M) + \sum_{i=3}^{4n-3} (-1)^i b_i(M) \\ &= 2 + 2b_2(M) + (n-1)b_4(M) + (n-2)b_6(M) - (2(n-1))b_3(M) \\ &= 2 + 2b_2(M) - b_6(M) + (n-1)[b_4(M) + b_6(M) - 2b_3(M)]. \end{aligned} \tag{3.8}$$

So,

$$\chi(M) = 2 + 2b_2(M) - b_6(M) + (n-1)[b_4(M) + b_6(M) - 2b_3(M)].$$

Since N has codimension 4, when we compute a similar expansion for $\chi(N)$, we see that the corresponding summand in the fourth line of Equation 3.8 will be taken from

$i = 3$ to $4n-7$. Now note that the set $\{[3, 4n-7] \cap \mathbb{Z}\}$ has $4n-9$ terms. Restricting to this set, we see that each equality in Display 3.6 loses the last terms because they are no longer in our domain. So, the total number of elements in each equality in Display 3.6 decreases by one. That is, the last equality has $(n-3)$ terms and the rest have $(n-2)$ terms. Also recall from Equation 3.4, $b_1(N) = b_{4n-5}(N) = 0$, and from Corollary 2.2.6, $b_2(N) = b_{(4n-4)-2}(N) = b_{4n-6}(N)$. Using the same method as above, we obtain:

$$
\begin{aligned}
\chi(N) &= \sum_{i=0}^{4n-4} (-1)^i b_i(N) \\
&= b_0(N) + b_{4n-4}(N) + \sum_{i=1}^{4n-5} (-1)^i b_i(N) \\
&= 2 - 2b_1(N) + \sum_{i=2}^{4n-6} (-1)^i b_i(N) \\
&= 2 - 0 + 2b_2(N) + \sum_{i=3}^{4n-7} (-1)^i b_i(N) \\
&= 2 + 2b_2(N) + (n-2)b_4(N) + (n-3)b_6(N) - (2(n-2))b_3(N) \\
&= 2 + 2b_2(M) - b_6(M) + (n-2)[b_4(M) + b_6(M) - 2b_3(M)].
\end{aligned}
$$

Therefore,

$$\chi(N) = 2 + 2b_2(M) - b_6(M) + (n-2)(b_4(M) + b_6(M) - 2b_3(M)). \tag{3.9}$$

Now we want to consider the difference of $\chi(M)$ and $\chi(N)$:

$$
\begin{aligned}
\chi(M) - \chi(N) &= 2 + 2b_2(M) - b_6(M) + (n-1)[b_4(M) + b_6(M) - 2b_3(M)] \\
&\quad - (2 + 2b_2(M) - b_6(M) + (n-2)[b_4(M) + b_6(M) - 2b_3(M)]) \\
&= b_4(M) + b_6(M) - 2b_3(M).
\end{aligned}
$$

So,

$$\chi(M) - \chi(N) = b_4(M) + b_6(M) - 2b_3(M). \tag{3.10}$$

Substituting Equation 3.10 into Equation 3.9, we see that

$$\chi(N) = 2 + [2b_2(M) - b_6(M)] + (n-2)(\chi(M) - \chi(N)). \tag{3.11}$$

Recall from Equation 3.4, that $b_2(M) \geq b_6(M)$, so $b_2(M) - b_6(M) \geq 0$. By hypothesis, we have $\chi(M) \geq \chi(N)$ and $n \geq 3$, so we get $(n-2)(\chi(M) - \chi(N)) \geq 0$. Since the last two terms in Equation 3.11 are non-negative and the first term is positive, we have $\chi(N) > 0$, as desired.

This proves Case 1 of Part 2, where $2m = 4n$.

<u>Case 2 of Part 2:</u> Let $2m = 4n + 2$ where $2m \geq 10$, so $n \geq 2$. In this case, $2m - 7 = (4n+2) - 7 = 4n - 5$.

Again, we will evaluate the i^{th} Betti numbers in groups of 4 using the isomorphism in Display 3.5 for $i = 3, 4, 5, 6$ and $i \leq 4n - 5$. We obtain

$$\begin{cases} b_3(M) = b_7(M) = \ldots = b_{4n-5}(M) = b_{4n-1}(M), \\ b_4(M) = b_8(M) = \ldots = b_{4n-8}(M) = b_{4n-4}(M), \\ b_5(M) = b_9(M) = \ldots = b_{4n-7}(M) = b_{4n-3}(M), \text{ and} \\ b_6(M) = b_{10}(M) = \ldots = b_{4n-6}(M) = b_{4n-2}(M). \end{cases} \tag{3.12}$$

Note that the set $\{[3, 4n-1] \cap \mathbb{Z}\}$ has $4n - 3$ terms. The first equality in Display 3.12 has n terms and all the others have $n - 1$ terms, since $\dfrac{4n-4}{4} + 1 = n$ and $\dfrac{4n-8}{4} + 1 = n - 1$, respectively.

Combining the second and fourth equations in Display 3.12 with Corollary 2.2.6, we get

$$b_6(M) = b_{(4n+2)-6}(M) = b_{4n-4}(M) = b_4(M). \tag{3.13}$$

In particular, we can then rewrite all the Betti numbers solely in terms of $b_0(M)$, $b_2(M)$, $b_3(M)$, $b_4(M)$, $b_5(M)$, and $b_{4n}(M)$. Also by Corollary 2.2.6, $b_1(M) = b_{(4n+2)-1}(M) = b_{4n+1}(M) = 0$ and $b_2(M) = b_{(4n+2)-2}(M) = b_{4n}(M)$. So, we may rewrite $\chi(M)$ as follows:

$$\begin{aligned} \chi(M) &= \sum_{i=0}^{4n+2} (-1)^i b_i(M) \\ &= b_0(M) + b_{4n+2}(M) + \sum_{i=1}^{4n+1} (-1)^i b_i(M) \end{aligned} \tag{3.14}$$

$$= 2 - 2b_1(M) + \sum_{i=2}^{4n} (-1)^i b_i(M)$$
$$= 2 - 0 + 2b_2(M) + \sum_{i=3}^{4n-1} (-1)^i b_i(M)$$
$$= 2 + 2b_2(M) + 2(n-1)b_4(M) - nb_3(M) - (n-1)b_5(M)$$

So,

$$\chi(M) = 2 + 2b_2(M) + 2(n-1)b_4(M) - nb_3(M) - (n-1)b_5(M).$$

When we compute a similar expansion for $\chi(N)$, we see that the corresponding summand in the fourth line of Equation 3.14 will be taken from $i = 3$ to $4n-5$, since N is codimension 4. Now note that the set $\{[3, 4n-5] \cap \mathbb{Z}\}$ has $4n-7$ elements. Restricting to this set, we see that each equality in Display 3.12 loses its last terms because they are no longer in our domain. So, the total number of elements in each equality decreases by one. That is, the first equality has $(n-1)$ elements and the rest have $(n-2)$ elements. Also recall from Equation 3.4, $b_1(N) = b_{4n-3}(N) = 0$, and from Corollary 2.2.6, $b_2(N) = b_{(4n-2)-2}(N) = b_{4n-4}(N)$. We then obtain

$$\chi(N) = \sum_{i=0}^{4n-2} (-1)^i b_i(N)$$
$$= b_0(N) + b_{4n-2}(N) + \sum_{i=1}^{4n-3} (-1)^i b_i(N)$$
$$= 2 - 2b_1(N) + \sum_{i=2}^{4n-4} (-1)^i b_i(N)$$
$$= 2 - 0 + 2b_2(N) + \sum_{i=3}^{4n-5} (-1)^i b_i(N)$$
$$= 2 + 2b_2(N) + 2(n-2)b_4(N) - (n-1)b_3(N) - (n-2)b_5(N)$$
$$= 2 + b_5(M) + 2[b_2(M) - b_4(M)] + (n-1)[2b_4(M) - b_3(M) - b_5(M)].$$

Therefore,

$$\chi(N) = 2 + b_5(M) + 2[b_2(M) - b_4(M)] + (n-1)[2b_4(M) - b_3(M) - b_5(M)]. \quad (3.15)$$

Now we want to consider the difference of $\chi(M)$ and $\chi(N)$:

$$\begin{aligned}\chi(M) - \chi(N) &= 2 + 2b_2(M) + 2(n-1)b_4(M) - nb_3(M) - (n-1)b_5(M) \\ &\quad - (2 + 2b_2(M) + 2(n-2)b_4(M) - (n-1)b_3(M) - (n-2)b_5(M)) \\ &= 2b_4(M) - b_3(M) - b_5(M).\end{aligned}$$

So,

$$\chi(M) - \chi(N) = 2b_4(M) - b_3(M) - b_5(M). \tag{3.16}$$

Substituting Equation 3.16 into Equation 3.15, we see that

$$\chi(N) = 2 + b_5(M) + 2[b_2(M) - b_4(M)] + (n-1)[(\chi(M) - \chi(N)]. \tag{3.17}$$

Using the relations from Equation 3.4 and Equation 3.13, namely, $b_2(M) \geq b_6(M)$ and $b_4(M) = b_6(M)$, respectively, we have $b_2(M) \geq b_4(M)$. Also, by hypothesis we have that $\chi(M) \geq \chi(N)$ and $n \geq 2$, so we get $(n-1)(\chi(M) - \chi(N)) \geq 0$. Since the last three terms of Equation 3.17 are non-negative, we get

$$\chi(N) \geq 2 + 0 + 0 + 0 = 2 > 0.$$

This completes the proof of Case 2 of Part 2 and with it the proof of Part 2.

Proof of Part 3: If $\dim M = 8$ or 10 and N has codimension 6, then $\dim N = 2$ or 4. So by hypothesis, N is 2- or 4-connected, respectively, and so

$$H_1(N) \cong H_1(M) = 0.$$

Thus $\chi(N) > 0$ by Lemma 3.0.3.

So, without loss of generality, we may assume $\dim N = 2m - 6 > 4$. That is, $2m \geq 12$. By Display 3.1 of Lemma 3.0.1, and since $\iota : N \hookrightarrow M$ is assumed to be $(2m-6)$-connected, we have

$$b_i(N) = b_i(M), \quad i \leq 2m - 7.$$

Also by Display 3.1 and Display 3.2 of Lemma 3.0.1, combined with Corollary 2.2.6, we have

$$b_0(M) = b_0(N) = b_{2m-6}(N) \geq b_{2m-6}(M) = b_6(M),$$

and by Display 3.1 of Lemma 3.0.1, Corollary 2.2.6, and our hypothesis, we have

$$b_1(N) = b_1(M) = b_{2m-1}(M) = b_{2m-7}(N) = 0.$$

In summary, we have:

$$\begin{cases} b_i(N) = b_i(M), \quad i \leq 2m-7, \\ b_0(M) \geq b_6(M), \text{ and} \\ b_1(N) = b_1(M) = b_{2m-1}(M) = b_{2m-7}(N) = 0. \end{cases} \tag{3.18}$$

In addition, by Lemma 2.6.6, since N is $(2m-6)$-connected with $k = \operatorname{codim}(N) = 6$ and so $\ell = 0$, we have

$$\cup_e : H^i(M) \to H^{i+6}(M) \tag{3.19}$$

is an isomorphism for $1 \leq i \leq 2m-7$, and a surjection for $i = 0$.

We consider three cases: when $2m = 6n$, when $2m = 6n+2$, and when $2m = 6n+4$.

Case 1 of Part 3: Let $2m = 6n$ where $2m \geq 12$, so that $n \geq 2$.

Now we will evaluate the i^{th} Betti numbers in groups of 6 using the isomorphism in Display 3.19 for $i = 1, 2, \ldots, 6$ and $i \leq 6n-7$. We obtain:

$$\begin{cases} b_1(M) = b_7(M) = \ldots = b_{6n-11}(M) = b_{6n-5}(M), \\ b_2(M) = b_8(M) = \ldots = b_{6n-10}(M) = b_{6n-4}(M), \\ b_3(M) = b_9(M) = \ldots = b_{6n-9}(M) = b_{6n-3}(M), \\ b_4(M) = b_{10}(M) = \ldots = b_{6n-9}(M) = b_{6n-2}(M), \\ b_5(M) = b_{11}(M) = \ldots = b_{6n-7}(M) = b_{6n-1}(M), \text{ and} \\ b_6(M) = b_{12}(M) = \ldots = b_{6n-12}(M) = b_{6n-6}(M). \end{cases} \tag{3.20}$$

Note that the set $\{[1, 6n-1] \cap \mathbb{Z}\}$ has $6n-1$ terms. Each row of equalities in Display 3.20 has n terms, except for the row of equalities for $b_6(M)$, which has $n-1$ terms, since $\dfrac{6n-6}{6} + 1 = n$ and $\dfrac{6n-12}{6} + 1 = n-1$, respectively.

Moreover, combining the first and fifth equations, and the second and fourth equations in Display 3.20 with Corollary 2.2.6, we get

$$\begin{cases} b_5(M) = b_{6n-5}(M) = b_1(M) = 0, \text{ and} \\ b_2(M) = b_{6n-2}(M) = b_4(M). \end{cases} \tag{3.21}$$

In particular, we can then rewrite the Betti numbers solely in terms of $b_0(M)$, $b_2(M)$, $b_3(M)$, $b_6(M)$, and $b_{6n}(M)$. So, we may rewrite $\chi(M)$ as follows:

$$\begin{aligned} \chi(M) &= \sum_{i=0}^{6n} (-1)^i b_i(M) \\ &= b_0(M) + b_{6n}(M) + \sum_{i=1}^{6n-1} (-1)^i b_i(M) \\ &= 2 + 2nb_2(M) + (n-1)b_6(M) - nb_3(M). \end{aligned} \tag{3.22}$$

Therefore,

$$\chi(M) = 2 + 2nb_2(M) + (n-1)b_6(M) - nb_3(M).$$

Since N has codimension 6, when we compute a similar expansion for $\chi(N)$, we see that the corresponding summand in Equation 3.22 will be taken from $i = 1$ to $6n-7$. Now note that the set $\{[1, 6n-7] \cap \mathbb{Z}\}$ has $6n-7$ elements. Restricting to this set, we see that each equality in Display 3.20 loses its last term because it is no longer in our domain. So, the total number of elements in each equality decreases by one. That is, the last equality has $(n-2)$ elements and the rest have $(n-1)$ elements. Also recall from Corollary 2.2.6 that $b_2(N) = b_{6n-7}(N)$. We then obtain

$$\begin{aligned} \chi(N) &= \sum_{i=0}^{6n-6} (-1)^i b_i(N) \\ &= b_0(N) + b_{6n-6}(N) + \sum_{i=1}^{6n-7} (-1)^i b_i(N) \\ &= 2 + 2(n-1)b_2(N) + (n-2)b_6(N) - (n-1)b_3(N) \\ &= 2 + 2(n-1)b_2(M) + (n-2)b_6(M) - (n-1)b_3(M). \end{aligned}$$

Therefore,

$$\chi(N) = 2 + 2(n-1)b_2(M) + (n-2)b_6(M) - (n-1)b_3(M). \tag{3.23}$$

Now we want to consider the difference of $\chi(M)$ and $\chi(N)$:

$$\begin{aligned}\chi(M)-\chi(N) &= 2+2nb_2(M)+(n-1)b_6(M)-nb_3(M)\\ &\quad -(2+2(n-1)b_2(M)+(n-2)b_6(M)-(n-1)b_3(M))\\ &= b_4(M)+b_6(M)-2b_3(M).\end{aligned}$$

So,

$$\chi(M)-\chi(N) = 2b_2(M)+b_6(M)-b_3(M). \tag{3.24}$$

Furthermore, substitute Equation 3.24 into Equation 3.23, and we see

$$\begin{aligned}\chi(N) &= 2+2(n-1)b_2(M)+(n-2)b_6(M)-(n-1)b_3(M)\\ &= 2-b_6(M)+(n-1)[2b_2(M)+b_6(M)-b_3(M)]\\ &= 2-b_6(M)+(n-1)[\chi(M)-\chi(N)].\end{aligned}$$

Recall from Equation 3.18, we have $1 = b_0(M) \geq b_6(M)$, so $0 \leq b_6(M) \leq 1$. By hypothesis we have that $\chi(M) \geq \chi(N)$ and $n \geq 2$, so we get $(n-1)(\chi(M)-\chi(N)) \geq 0$. Therefore,

$$\chi(N) \geq 2-1+0 = 1 > 0.$$

This proves Case 1 of Part 3.

Case 2 of Part 3: Let $2m = 6n+2$ where $2m \geq 12$, so that $n \geq 2$.

Now we will evaluate the i^{th} Betti numbers in groups of 6 using the isomorphism in Display 3.19 for $i = 1, 2, \ldots, 6$ and $i \leq 6n-7$. We obtain

$$\begin{cases} b_1(M) = b_7(M) = \ldots = b_{6n-5}(M) = b_{6n+1}(M),\\ b_2(M) = b_8(M) = \ldots = b_{6n-10}(M) = b_{6n-4}(M),\\ b_3(M) = b_9(M) = \ldots = b_{6n-9}(M) = b_{6n-3}(M),\\ b_4(M) = b_{10}(M) = \ldots = b_{6n-9}(M) = b_{6n-2}(M),\\ b_5(M) = b_{11}(M) = \ldots = b_{6n-7}(M) = b_{6n-1}(M), \text{ and}\\ b_6(M) = b_{12}(M) = \ldots = b_{6n-6}(M) = b_{6n}(M).\end{cases} \tag{3.25}$$

Note that the set $\{[1, 6n+1] \cap \mathbb{Z}\}$ has $6n+1$ terms. The first row of equalities in Display 3.25 has $n+1$ terms because $6n/6+1 = n+1$. Every other has n terms because $(6n-6)/6+1 = n$.

Moreover, combining the second and fourth equations, and the third and fifth equations in Display 3.25 with Corollary 2.2.6, we get

$$\begin{cases} b_2(M) = b_{(6n+2)-2}(M) = b_{6n}(M) = b_6(M), \text{ and} \\ b_3(M) = b_{(6n+2)-3}(M) = b_{6n-1}(M) = b_5(M). \end{cases} \tag{3.26}$$

In particular, we can then rewrite all the Betti numbers solely in terms of $b_0(M)$, $b_2(M)$, $b_3(M)$, $b_4(M)$, and $b_{6n+2}(M)$, since $b_2(M) = b_6(M)$ and $b_3(M) = b_5(M)$. So, we may rewrite $\chi(M)$ as follows:

$$\begin{aligned} \chi(M) &= \sum_{i=0}^{6n+2} (-1)^i b_i(M) \\ &= b_0(M) + b_{6n+2}(M) + \sum_{i=1}^{6n+1} (-1)^i b_i(M) \\ &= 2 + 2nb_2(M) + nb_4(M) - 2nb_3(M). \end{aligned} \tag{3.27}$$

Therefore,

$$\chi(M) = 2 + 2nb_2(M) + nb_4(M) - 2nb_3(M).$$

Since N has codimension 6, when we compute a similar expansion for $\chi(N)$, we see that the corresponding summand in the second line of Equation 3.27 will be taken from $i = 1$ to $6n - 5$. Now note that the set $\{[1, 6n - 5] \cap \mathbb{Z}\}$ has $6n - 5$ elements. Restricting to this set, we see that each equality in Display 3.25 loses its last term because it is no longer in our domain. So, the total number of elements in each equality decreases by one. That is, the first equality has n elements and the rest have $(n - 1)$ elements. We then obtain

$$\begin{aligned} \chi(N) &= \sum_{i=0}^{6n-4} (-1)^i b_i(N) \\ &= b_0(N) + b_{6n-4}(N) + \sum_{i=1}^{6n-5} (-1)^i b_i(N) \\ &= 2 + 2(n-1)b_2(N) + (n-1)b_4(N) - 2(n-1)b_3(N) \\ &= 2 + 2(n-1)b_2(M) + (n-1)b_4(M) - 2(n-1)b_3(M). \end{aligned}$$

Therefore,

$$\chi(N) = 2 + 2(n-1)b_2(M) + (n-1)b_4(M) - 2(n-1)b_3(M). \tag{3.28}$$

Now we want to consider the difference of $\chi(M)$ and $\chi(N)$:

$$\begin{aligned}\chi(M) - \chi(N) &= 2 + 2nb_2(M) + nb_4(M) - 2nb_3(M)\\ &\quad - (2 + 2(n-1)b_2(M) + (n-1)b_4(M) - 2(n-1)b_3(M))\\ &= 2b_2(M) + b_4(M) - 2b_3(M).\end{aligned}$$

So,

$$\chi(M) - \chi(N) = 2b_2(M) + b_4(M) - 2b_3(M). \tag{3.29}$$

Substituting Equation 3.29 into Equation 3.28, we see that

$$\begin{aligned}\chi(N) &= 2 + 2(n-1)b_2(M) + (n-1)b_4(M) - 2(n-1)b_3(M)\\ &= 2 + (n-1)[2b_2(M) + b_4(M) - 2b_3(M)]\\ &= 2 + (n-1)[\chi(M) - \chi(N)].\end{aligned}$$

By hypothesis we have that $\chi(M) \geq \chi(N)$ and $n \geq 2$, so we get $(n-1)(\chi(M) - \chi(N)) \geq 0$. Therefore, $\chi(N) > 0$, since the first term is always positive and the second term is always non-negative.

This proves Case 2 of Part 3.

Case 3 of Part 3: Let $2m = 6n + 4$ where $2m \geq 12$, so that $n \geq 2$.

Now we evaluate the i^{th} Betti numbers in groups of 6 using the isomorphism in Display 3.19 for $i = 1, 2, \ldots, 6$ and $i \leq 6n - 7$. We obtain

$$\begin{cases} b_1(M) = b_7(M) = \ldots = b_{6n-5}(M) = b_{6n+1}(M), \\ b_2(M) = b_8(M) = \ldots = b_{6n-4}(M) = b_{6n+2}(M), \\ b_3(M) = b_9(M) = \ldots = b_{6n-3}(M) = b_{6n+3}(M), \\ b_4(M) = b_{10}(M) = \ldots = b_{6n-9}(M) = b_{6n-2}(M), \\ b_5(M) = b_{11}(M) = \ldots = b_{6n-7}(M) = b_{6n-1}(M), \text{ and} \\ b_6(M) = b_{12}(M) = \ldots = b_{6n-6}(M) = b_{6n}(M). \end{cases} \tag{3.30}$$

Note that the set $\{[1, 6n+3] \cap \mathbb{Z}\}$ has $6n+3$ terms. The first three equalities in Display 3.30 have $n+1$ terms, and the last three equalities in Display 3.30 have n terms, since $\frac{6n}{n} + 1 = n + 1$ and $\frac{6n-6}{n} + 1 = n$, respectively.

Moreover, combining the first and third equalities, and the fourth and sixth equalities in Display 3.30 with Corollary 2.2.6, we get

$$\begin{cases} b_1(M) = b_{(6n+4)-1}(M) = b_{6n+3}(M) = b_3(M) = 0, \text{ and} \\ b_4(M) = b_{(6n+4)-4}(M) = b_{6n}(M) = b_6(M). \end{cases} \tag{3.31}$$

In particular, we can then rewrite the Betti numbers in terms of $b_0(M)$, $b_2(M)$, $b_4(M)$, $b_5(M)$, and $b_{6n+4}(M)$. So, we may rewrite $\chi(M)$ as follows:

$$\begin{aligned} \chi(M) &= \sum_{i=0}^{6n+4} (-1)^i b_i(M) \\ &= b_0(M) + b_{6n+4}(M) + \sum_{i=1}^{6n+3} (-1)^i b_i(M) \\ &= 2 + (n+1)b_2(M) + 2nb_4(M) - nb_5(M). \end{aligned} \tag{3.32}$$

Therefore,

$$\chi(M) = 2 + (n+1)b_2(M) + 2nb_4(M) - nb_5(M).$$

When we compute a similar expansion for $\chi(N)$, we see that the corresponding summand in Equation 3.32 will be taken from $i = 1$ to $6n-3$, since N is of codimension 6. Now note that the set $\{[1, 6n-3] \cap \mathbb{Z}\}$ has $6n-3$ terms. Restricting to this set, we see that each equality in Display 3.30 loses its last term because it is no longer in our domain. So, the total number of elements in each equality decreases by one. That is, the first three equalities have n terms and the last three equalities have $(n-1)$ terms. We then obtain:

$$\begin{aligned} \chi(N) &= \sum_{i=0}^{6n-2} (-1)^i b_i(N) \\ &= b_0(N) + b_{6n-2}(N) + \sum_{i=1}^{6n-3} (-1)^i b_i(N) \\ &= 2 + nb_2(N) + 2(n-1)b_4(N) - (n-1)b_5(N) \\ &= 2 + nb_2(M) + 2(n-1)b_4(M) - (n-1)b_5(M).. \end{aligned}$$

Therefore,

$$\chi(N) = 2 + nb_2(M) + 2(n-1)b_4(M) - (n-1)b_5(M). \tag{3.33}$$

Now we want to consider the difference of $\chi(M)$ and $\chi(N)$:

$$\begin{aligned}\chi(M) - \chi(N) &= 2 + (n+1)b_2(M) + 2nb_4(M) - nb_5(M) \\ &\quad - (2 + nb_2(M) + 2(n-1)b_4(M) - (n-1)b_5(M)) \\ &= 2b_2(M) + b_4(M) - 2b_3(M).\end{aligned}$$

So,

$$\chi(M) - \chi(N) = b_2(M) + 2b_4(M) - b_5(M). \tag{3.34}$$

Furthermore, substitute Equation 3.34 into Equation 3.33, and we see

$$\begin{aligned}\chi(N) &= 2 + nb_2(M) + 2(n-1)b_4(M) - (n-1)b_5(M) \\ &= 2 + nb_2(M) + (b_2(M) - b_2(M)) + 2(n-1)b_4(M) - (n-1)b_5(M) \\ &= 2 + b_2(M) + (n-1)[b_2(M) + 2b_4(M) - b_5(M)] \\ &= 2 + b_2(M) + (n-1)[\chi(M) - \chi(N)].\end{aligned}$$

Therefore,

$$\chi(N) = 2 + b_2(M) + (n-1)[\chi(M) - \chi(N)]. \tag{3.35}$$

Recall that by hypothesis, we have $\chi(M) \geq \chi(N)$ and $n \geq 2$, so we get $(n-1)(\chi(M)-\chi(N)) \geq 0$. Since the last two terms in Equation 3.35 are non-negative and the first term is positive, we have $\chi(N) > 0$.

This proves Case 3 of Part 3, where $2m = 6n + 4$, and completes the proof of Part 3. □

This Key Lemma 3.0.5 gives us the following consequences when M is positively curved, that are important in this thesis.

Lemma 3.0.6. [15] *Let M^{2m} be a closed, orientable, positively curved manifold. Suppose one of the following conditions hold.*

1. *There is a closed totally geodesic, codimension 2 submanifold* $N^{n-2} \subseteq M^n$*;*

2. *There is an isometric* T^1*-action on* M^n *and* $\operatorname{codim}(Fix(M^n;T^1)) = 4$*; or*

3. *There is an isometric* T^1*-action on* M^n*,* $\operatorname{codim}(Fix(M;T^1)) = 6$*, and* N^{n-6} *is* $(n-6)$*-connected in* M^n*;*

then $\chi(N) > 0$*. In particular,* $\chi(M) > 0$ *and if either Condition 2 or 3 holds, then* $\chi(M) \geq \chi(N)$*.*

Proof. We begin with the proof for Condition 1.

Proof for Condition 1: By Part 1 or Wilking's Connectedness Theorem 2.6.5, we have N^{n-2} is $(n-3)$-connected. So, we may apply Part 1 of the Key Lemma 3.0.5 to see that both $\chi(M)$ and $\chi(N)$ are positive.

For the proofs for Conditions 2 and 3, we claim that it suffices to show in each case that $\chi(N) > 0$, as Corollary 3.0.4 guarantees us that $\chi(M) \geq \chi(N)$ and hence $\chi(M) > 0$.

Proof for Condition 2: Let N^{n-4} be the unique codimension 4 component of $\mathrm{Fix}(M;T^1)$. Applying Part 1 of Wilking's Connectedness Theorem 2.6.5 with $k = 4$ and $\delta(T^1) = 1$, we have that N is $(n-6)$-connected. So we may apply Part 2 of the Key Lemma 3.0.5 and the result follows.

Proof for Condition 3: Let N^{n-6} be the unique codimension 6 component of $\mathrm{Fix}(M;T^1)$. Since N is $(n-6)$-connected by assumption, we may apply Part 3 of the Key Lemma 3.0.5, and the result holds. □

The next lemma, which relies on an application of the Borel Formula 2.5.6, is key to proving Theorem B and its weaker version, Theorem 4.0.2.

Lemma 3.0.7. *Let* M^{2m} *be a closed, orientable,* $2m$*-dimensional positively curved manifold admitting an isometric and effective* T^2*-action. Let* $N \in \mathrm{Fix}(M;T^1)$ *and* $F \in \mathrm{Fix}(M;T^2)$*. If* $\operatorname{codim}(N) = 6$*, then* $\chi(M) > 0$*.*

Proof. Our goal is to show that $\chi(N) > 0$, since $\chi(M) \geq \chi(N)$ for the same reasons argued in the proof of Lemma 3.0.6 for Conditions 2 and 3. Let $n = 2m$ and consider $F \in \mathrm{Fix}(M; T^2)$. First note that N admits an induced $T_1^1 = T^2/T^1$-action and that F can also be thought of as lying in $\mathrm{Fix}(N; T^1)$. Moreover, since $N \in \mathrm{Fix}(M; T^1)$ is positively curved. If the codimension of F in N is either 2 or 4, then we see that $\chi(N) \geq \chi(F) > 0$ by Part 1 or 2 of Lemma 3.0.6, respectively. Since $\chi(M) \geq \chi(N)$, the result follows.

So, we may assume $\dim F \geq n - 12$. Since $\chi(N) = \chi(\mathrm{Fix}(N; T_1^1))$ by Theorem 2.5.10, it suffices to show that $\chi(F) > 0$ for all $F \in \mathrm{Fix}(N; T_1^1)$, so, our goal now is to show that $\chi(F) > 0$. Recall that the Borel Formula 2.5.6 gives us

$$\begin{aligned} \sum(\mathrm{codim}(F \subseteq N_i)) &= \mathrm{codim}(F \subseteq M) \\ (\dim N - \dim F) + \alpha &= (\dim M - \dim F), \\ ((n-6) - \dim F) + \alpha &= n - \dim F, \end{aligned}$$

and so,

$$\alpha = 6.$$

Notice that the codimension of F in any fixed point set of a circle transverse to T^1 is less than or equal to 6. We have two cases: when there are two such circles, and when there is one.

If there are two such circles, we may apply Part 1 or 2 of Lemma 3.0.6, as we just did above, respectively, to obtain $\chi(N_i) \geq \chi(F) > 0$. Since the goal at this point is to show $\chi(F) > 0$, we are done with this case. Suppose instead there is only one such circle, and let T_2^1 denote that circle. Let P denote the fixed point set component of T_2^1 that intersects N in F. Since $\dim F \geq n - 12$, then $k_1 = \mathrm{codim}(N \subseteq M) = 6 \leq \mathrm{codim}(P \subseteq M) = k_2$. Then, by Part 2 of Wilking's Connectedness Theorem 2.6.5, F is $(\dim P - 6)$-connected in P. So we may apply

Part 3 of Lemma 3.0.6 to obtain $\chi(P) \geq \chi(F) > 0$. Thus, $\chi(F) > 0$ and so $\chi(N) > 0$. By Corollary 3.0.4, we have $\chi(M) > 0$.

Figures 3.1 and 3.2 below illustrate these two cases.

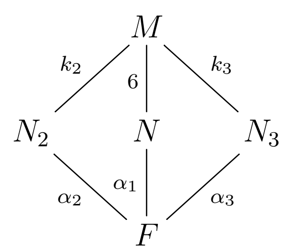

Figure 3.1: Three transverse circles with nontrivial fixed point sets intersecting in F.

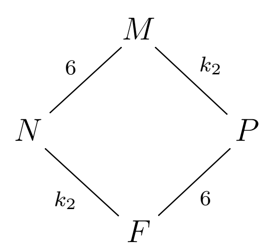

Figure 3.2: Two transverse circles with nontrivial fixed point sets intersecting in F.

□

CHAPTER 4

MAIN RESULTS

In this chapter, we prove Theorems A and B, as well as a weaker version of Theorem B. For the sake of the reader, we recall both theorems in this chapter. We begin with the statement and proof of Theorem A.

Theorem A. [13] *Let M^{2m} be a closed, orientable Riemannian manifold of positive sectional curvature. If T^k acts isometrically on M^{2m} and $k \geq (2m-4)/4$, then $\chi(M^{2m}) > 0$.*

Proof. This proof is by induction on the rank, k. Let $n = 2m$.
Base Case: Consider the case when the rank is 0, that is, $k = 0$. Therefore, $0 \geq (n-4)/4$, so, $n \leq 4$, that is $n \in \{2, 4\}$. By Synge's Theorem 2.6.1, M being orientable implies $\pi_1(M) = 0$, so $H_1(M) = 0$. Thus, by Lemma 3.0.3 our result holds.
Induction Hypothesis: We assume that given a T^k-action on M^n, if $k \geq (n-4)/4$, then $\chi(M) > 0$.

Now we want to show this is true when the rank is equal to $k+1$, namely, for $\dim M \leq 4k+8$. Since M is even-dimensional, by Berger's Theorem 2.6.7, $\mathrm{Fix}(M; T^1) \neq \emptyset$ for any $T^1 \subseteq T^k$. We break the proof into two cases: Case 1, where $\dim(\mathrm{Fix}(M;T^1)) \leq 4k+4$, and Case 2, where $\dim(\mathrm{Fix}(M;T^1)) > 4k+4$.
Case 1: Suppose $\dim(\mathrm{Fix}(M;T^1)) \leq 4k+4$. Let $N \in \mathrm{Fix}(M;T^1)$, then each N admits an induced $T^{k+1}/T^1 \cong T^k$-action, and let $\dim N \leq 4k+4$. So, we haven

$$\frac{\dim N - 4}{4} \leq \frac{(4k+4)-4}{4} = k.$$

Thus, by the Induction Hypothesis, $\chi(N) > 0$. Since N is arbitrary, $\chi(N) > 0$ for all $N \in \dim(\mathrm{Fix}(M;T^1))$, so by Theorem 2.5.10 and Frankel's Theorem 2.6.2, we see that

$$\chi(M) = \chi(\mathrm{Fix}(M;T^1)) = \sum_{N \in \mathrm{Fix}(M;T^1)} \chi(N) > 0,$$

as desired.

<u>**Case 2:**</u> Now suppose $\dim(\mathrm{Fix}(M;T^1)) > 4k+4$. Then there exists an $N \in \mathrm{Fix}(M;T^1)$ such that $\dim N = 4k+6$. Notice that $4k+6 = 4k+8-2$, thus by Theorem 2.6.4, M^{4k+8} is diffeomorphic to S^{4k+8} or $\mathbb{C}\mathrm{P}^{2k+4}$. Recall that these two manifolds have positive Euler characteristic. Thus $\chi(M^{4k+8}) > 0$. This proves Case 2 and completes the proof. □

Remark 4.0.1. *The proof of Case 2 follows as in the original paper. However, we can also prove this result using Part 1 of Lemma 3.0.6.*

As we mentioned above, before we prove the main result of Theorem B, we will first prove a weaker version of the result, as we now have all the tools required to do so. Note that this is still a stronger result than that of Theorem A, provided $\dim(M) > 8$. In fact, for dimensions six and eight, the two results are the same., and for dimensions two and four, Theorem 4.0.2 is weaker. See Figures 4.1 and 4.2.

Theorem 4.0.2. [15] *Let M^{2m} be a closed $2m$-dimensional, orientable, positively curved Riemannian manifold. Suppose T^k acts isometrically and effectively on M^{2m}. If $k \geq 2m/8$, then $\chi(M) > 0$.*

Proof. This proof is again by induction on the rank, k. Let $n = 2m$.

<u>Base Case:</u> When $k = 1$ and $n \in \{2, 4, 6, 8\}$, the result holds by Theorem A.

<u>Induction Hypothesis:</u> Suppose M^n is an even-dimensional, positively curved Riemannian manifold with an isometric and effective T^k-action with $k \geq n/8$. Then $\chi(M^n) > 0$.

We want to show that the statement holds for $k+1$. Assume we have T^{k+1} acting on M^n with $n/8 \leq k+1$. By Berger's Theorem 2.6.7, $\mathrm{Fix}(M;T^1) \neq \emptyset$ for any $T_1 \subseteq M$. Let $N \in \mathrm{Fix}(M;T^1)$, then there is an induced $T^{k+1}/T^1 = T^k$-action on N.

Since $\chi(M) = \chi(\text{Fix}(M; T^1))$ by Theorem 2.5.10, our goal is to show that $\chi(N) > 0$ for all $N \in \text{Fix}(M; T^1)$. If $\dim N \leq n - 8$, then

$$\frac{\dim N}{8} \leq \frac{n-8}{8} = \frac{n}{8} - 1 \leq (k+1) - 1 = k.$$

So, $\chi(N) > 0$ by our Induction Hypothesis, as desired.

Now suppose $\dim N \in \{n-6, n-4, n-2\}$. When $\dim N = n-2$ or $n-4$, then N is of codimension 2 or 4, respectively, so, we may use Part 1 or 2 of Lemma 3.0.6, respectively, to see that $\chi(M) > 0$. When $\dim N = n - 6$, since $k \geq 2$, we consider any $F \in \text{Fix}(M; T^2)$, where the T^1 fixing N is contained in T^2, that is $T^1 \subseteq T^2 \subseteq T^k$, and so $F \subseteq N$. Since $\text{codim}(N) = 6$, we may apply Lemma 3.0.7 to see that $\chi(N) > 0$. This completes the proof. □

We are almost ready to proceed with the proof of Theorem B. The proof is again by induction and in order to prove the base case, we need the following lemma.

Lemma 4.0.3. [15] *Let M^{2m} be a closed $2m-$dimensional manifold of positive sectional curvature on which a torus T^k acts isometrically, with $2m \leq 20$. For $2m \neq 12$ (respectively, $2m = 12$), if $k \geq \dfrac{2m-4}{8}$ (respectively, $k > 1$), then the Euler characteristic of each T^k-fixed point component is positive, and thus the Euler characteristic of M is positive.*

Proof. We set $n = 2m$. First note that $n = 10$ and $n = 20$ are the largest dimensions that require a T^k-action of rank $k = 1$ and $k = 2$, respectively. We claim that if we can prove the lemma for these two cases, the cases for the other dimensions follow in the same manner. Note that the proof depends on showing that all fixed point set components of some subtorus have positive Euler characteristic. This, in turn, relies on considering the components that have dimensions too big to apply the Induction Hypothesis. In particular, for $n < 10$ and for $10 < n < 20$, there are fewer of these cases to consider, justifying the claim. So, it suffices to consider when $n = 10$ and $n = 20$. We prove the two cases separately.

Case 1: Assume $n = 10$ and $k = 1$. By Berger's Theorem 2.6.7, $\text{Fix}(M; T^1) \neq \emptyset$ for any $T^1 \subseteq M$. Let $N \in \text{Fix}(M; T^1)$ and suppose $\dim N \leq 4$. If $\dim N = 0$, then N consists of isolated points, and so $\chi(N) > 0$. If $\dim N$ is 2 or 4, we know $\chi(N) > 0$ by Lemma 3.0.3. So, we may assume $\dim N \geq 6$ for any fixed point set component. If N has dimension $8 = n - 2$, then $\text{codim}(N) = 2$. Thus, by Part 1 of Lemma 3.0.6, we have $\chi(M) \geq \chi(N) > 0$. It remains to consider the case when $\dim N = 6$. Thus, by Part 2 of Lemma 3.0.6, we have $\chi(M) \geq \chi(N) > 0$.

Case 2: Assume $n = 20$ and $k = 2$. Let $T_1^1 \subseteq T^2$ and let $N_1 \in \text{Fix}(M; T_1^1)$. Then N_1 admits an action by $T^2/T^1 \cong T^1$. If $\dim N_1 \leq 10$, then we may apply the results from Case 1 for $n \leq 10$ to see that $\chi(N_1) > 0$. Now assume $\dim N_1 \geq 12$. There are three sub-cases to consider: Case 2.a, when $\dim N_1 = 18$, Case 2.b, when $\dim N_1 = 16$, and Case 2.c, when $12 \leq \dim N_1 \leq 14$. If $\dim N_1 = 18$, then by Part 1 of Lemma 3.0.6, $\chi(M) > 0$. If $\dim N_1 = 16$, then by Part 2 of Lemma 3.0.6, $\chi(M) > 0$. This takes care of Cases 2.a and 2.b.

Case 2.c: Suppose $12 \leq \dim N_1 \leq 14$. Consider any $F \in \text{Fix}(M; T^2)$, then $F \subseteq N_1$. If $\dim N_1 = 14 = n - 6$, then we may apply Lemma 3.0.7 to obtain $\chi(M) > 0$. If $\dim N_1 = 12$, then our goal is to show $\chi(N_1) \geq \chi(F) > 0$. Note that once we show $\chi(N_1) > 0$, then Frankel's Theorem 2.6.2 tells us that the only other components of the fixed point set of T^1 are of dimension $0, 2, 4$, or 6. If the dimension is 0, then the Euler characteristic is positive. If the dimension is 2 or 4, then we have positive Euler characteristic by Lemma 3.0.3. If the dimension is 6, then we can apply Theorem A or the result from Case 1 of this lemma because there is an induced circle action on N_1^6, so $\chi(N_1) > 0$. Then by applying Theorem 2.5.10, we get that $\chi(M) \geq \chi(N_1)$. Thus, showing $\chi(N_1) > 0$ is sufficient because it follows that $\chi(M) > 0$, as desired.

as we can then apply Theorem 2.5.10, Frankel's Theorem 2.6.2, and the Induction Hypothesis to show $\chi(N_1) > 0$. We look at the cases when $\dim F < 12$, in particular, when $\dim F \leq 10$.

If $\dim F \leq 4$, then by Lemma 3.0.3, $\chi(F) > 0$. If $\dim F \in \{8, 10\}$, we see that F is of codimension 4 or 2, respectively. So, $\chi(N_1) \geq \chi(F) > 0$ by Part 1 or 2 of Lemma 3.0.6. Lastly, consider when $\dim F = 6$. Since $T^1_1 \subseteq T^2$ fixes N^{12}_1, the Borel Formula 2.5.6 implies that there is at least one other circle subgroup $T^1_2 \subseteq T^2$ that has trivial intersection with T^1_1 and that fixes some N_2 in M. Explicitly, we have

$$\begin{aligned} \dim N_1 - \dim F + \alpha &= \dim M - \dim F \\ (12 - 6) + \alpha &= 20 - 6 \\ 6 + \alpha &= 14. \end{aligned} \tag{4.1}$$

There are at most three circles in T^2 that have pairwise trivial intersection. If the two not equal to T^1_1 have non-empty fixed point sets N_2 and N_3, then

$$(\dim N_2 - 6) + (\dim N_3 - 6) = \operatorname{codim}(F \subseteq N_2) + \operatorname{codim}(F \subseteq N_3) = 8.$$

Therefore, we have that $(\operatorname{codim}(F \subseteq N_2), \operatorname{codim}(F \subseteq N_3)) \in \{(2, 6), (4, 4), (6, 2)\}$. When $\operatorname{codim}(F \subseteq N_j) = 2$, for some $j \in \{2, 3\}$, then by Part 1 of Lemma 3.0.6, $\chi(N_1) \geq \chi(F) > 0$. When $\operatorname{codim}(F \subseteq N_2) = \operatorname{codim}(F \subseteq N_3) = 4$, then by Part 2 of Lemma 3.0.6, $\chi(N_1) \geq \chi(F) > 0$. So in all cases, we see $\chi(N_1) \geq \chi(F) > 0$. Now, suppose instead that there is just one other circle transverse to T^1_1 fixing N_2, which we denote by T^1_2. Denote its fixed point set by P. Then by Equation 4.1, $\operatorname{codim}(F \subseteq P) = 8$, so $\operatorname{codim}(P \subseteq M) = 6$. Thus, we may apply Lemma 3.0.7 to P and F to obtain $\chi(F) > 0$. In particular, we see that in all cases $\chi(F) > 0$ and hence $\chi(N_1) > 0$, as desired. □

Now we have all of the tools needed to prove Theorem B, which we restate here for the convenience of the reader.

Theorem B. [15] *Let M^{2m} be a closed $2m-$dimensional manifold of positive sectional curvature on which a torus T^k acts isometrically. For $2m \neq 12$ (respectively,*

$2m = 12$), if $k \geq (2m-4)/8$ (respectively, $k > 1$), then the Euler characteristic of each T^k−fixed point component is positive, and thus $\chi(M) > 0$.

Proof. This proof is by induction on the rank. Let $n = 2m$.

Base Case: We consider dimensions less than or equal to 20, and observe that the result holds by Lemma 4.0.3.

Induction Hypothesis: For $j > 2$, we assume that if T^j acts on M^n, an orientable, even-dimensional positively curved manifold, then if $j \geq (n-4)/8$, then $\chi(M^n) > 0$, provided $j \leq k$.

Now we will show the result holds when the rank is equal to $k+1$, that is, when $k + 1 \geq \frac{n-4}{8}$. By Berger's Theorem 2.6.7, $\mathrm{Fix}(M; T^1) \neq \emptyset$ for any $T^1 \subseteq M$. Let $N \in \mathrm{Fix}(M; T^1)$. Then N admits an induced $T^{k+1}/T^1 \cong T^k$-action. To show that our result holds when the rank is $k+1$, we will consider two cases: when $n - 6 \leq \dim N \leq n - 2$, and when $\dim N \leq n - 8$.

Consider when $n - 6 \leq \dim N \leq n - 2$. If $n - 4 \leq \dim N \leq n - 2$, then we may apply Part 2 or 1 of Lemma 3.0.6, respectively, to obtain $\chi(M) > 0$. If $\dim N = n - 6$, then we may apply Lemma 3.0.7 to obtain $\chi(M) > 0$.

Finally, if $\dim N \leq n - 8$, then

$$\frac{\dim N - 4}{8} \leq \frac{(n-8)-4}{8} \leq \frac{n-4}{8} - 1 \leq (k+1) - 1 = k;$$

so, we may apply our Induction Hypothesis to obtain $\chi(M) > 0$. □

Figures 4.1 and 4.2 below are like figures 1.1 and 1.2, but they just display the results we have proven.

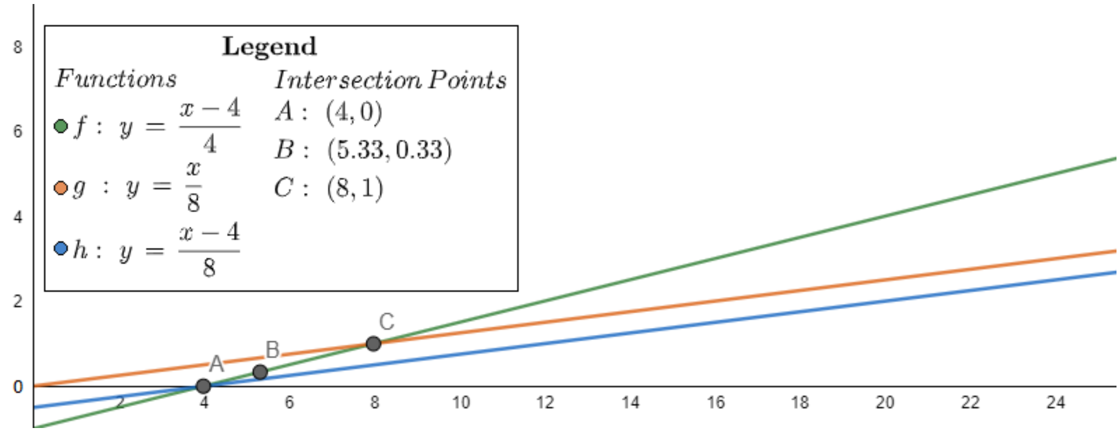

Figure 4.1: A graph of the symmetry rank lower bounds as continuous functions of Theorems A, 4.0.2, B

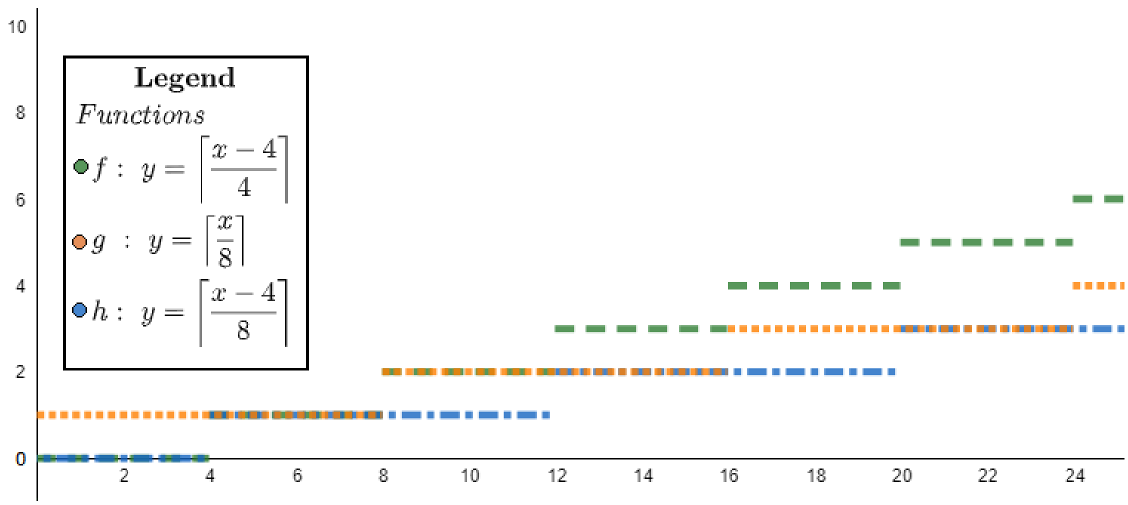

Figure 4.2: A graph of the symmetry rank lower bounds of Theorems A, 4.0.2, B as step functions.

CHAPTER 5

CONCLUSION

In this thesis, we prove results due to Püttmann and Searle [13] and Rong [14], and Rong and Su [15] that address the Hopf conjecture with symmetries. The proofs for both results are by induction on the rank of the torus action and rely heavily on the classical theorems due to Kobayashi [10] and Frankel [6]. The results of Rong and Su [15], which are stronger than those of Püttmann and Searle [13] and Rong [14], rely heavily on Wilking's 2003 Connectedness Theorem [20].

To prove these results, we follow the same general strategy as in Püttmann and Searle [13] and Rong [14], and Rong and Su [15], with one notable exception. Namely, we present a different and simpler proof of the Rong-Su [15] result for the case when the fixed point set of a circle action is of codimension 6. This simplified version relies on a formular due to Borel [2], which was not used in the original proof.

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