CONDITION BASED MAINTENANCE OF A SINGLE SYSTEM UNDER SPARE PART INVENTORY CONSTRAINTS

A Thesis by

Mitchell T. Rausch

Bachelor of Science, Wichita State University, 2006

Submitted to the Department of Industrial Engineering
and the faculty of the Graduate School of
Wichita State University
in partial fulfillment of
Master of Science

August 2008

CONDITION BASED MAINTENANCE OF A SINGLE SYSTEM UNDER SPARE PART INVENTORY CONSTRAINTS

I have examined the final copy of this thesis for form and content, and recommend that it be accepted in partial fulfillment of the requirement for the degree of Master of Science with a major in Industrial Engineering.

____

Haitao Liao, Committee Chair

We have read this thesis and recommend its acceptance:

____

Chunsheng Ma, Committee Member

____

Gamal Weheba, Committee Member

ACKNOWLEDGEMENTS

I would like to thank my advisor, Dr. Haitao Liao, for his years of patience, guidance, and support. His influence has contributed significantly to my academic and professional career development. By continually challenging me with thought provoking questions and support during my research endeavors, I owe him much gratitude and respect. I would also like to thank my committee members, Dr. Chunsheng Ma and Dr. Gamal Weheba for their thoughtful review and comments during the development of this work.

The underpinning of my educational pursuits and success can be attributed to my devoted parents, Thomas and Jada Rausch, who shaped me into the person I am today. Their love, unwavering support, and guidance have provided me the assurance and ability to continue my education with success. To my parents, I am eternally grateful.

The faculty and staff of the Industrial and Manufacturing Engineering department have made my experience at Wichita State University a positive and rewarding endeavor. I would especially like to thank Dr. Don Malzahn for promoting the industrial engineering profession to me nearly eight years ago. His encouragement, advice, and sympathetic ear have been invaluable over the years.

ABSTRACT

The problem of effectively integrating condition based maintenance and spare part inventory control is studied and a solution methodology demonstrated. Degradation modeling, Bayesian analysis, and optimization techniques are utilized to define a condition based maintenance model for a single production system under spare part inventory constraints. Specifically, the gamma process is used to model the degradation process for a system that has a monotonically increasing degradation behavior. The initial gamma process parameters are inferred during product testing and utilized to define a spare part optimization model. This optimization model is used to ascertain the stockout probability to support the system. To address the uncertainty in parameter estimates, the gamma process parameters are updated through a Bayesian updating technique as more degradation data is collected over time for a real time remaining useful life prediction of the component. Finally, condition based maintenance and spare part inventory control are tied together into a overall production decision model. The production decision model generates an optimal degradation limit maintenance policy which provides a means to make component replacement decisions while addressing the relationship among outstanding orders, the number of spares, and the degradation state.

One can see that the methodology developed in this thesis effectively ties together condition based maintenance, production, and spare parts inventory control. This body of work is important in the area of reliability and maintenance engineering since it provides a way of controlling spare parts in conjunction with condition based maintenance and production. This concept addresses a relationship which is not well developed in the literature, yet has a significant practical value.

TABLE OF CONTENTS

TABLE OF CONTENTS (continued)

Chapter Page

LIST OF FIGURES

LIST OF FIGURES (continued)

CHAPTER 1

INTRODUCTION

1.1 Background

Companies are faced with extensive competition from global markets that must be mitigated with keen engineering designs and robust manufacturing processes. Customers expect products to arrive on time and meet the product design specifications at the lowest cost possible. In order to necessitate effective manufacturing processes, maintenance activities are planned and executed to reduce the instances of unexpected system failures that can affect product quality, attainment of customer schedules, worker safety, and the ability to effectively produce revenue. The costs associated with system failures can be exceedingly detrimental to the ability of a company to compete in the market. According to Wireman (1990), maintenance costs for industrial organizations have increased by 10-15% per year since 1979. The continual increases in maintenance costs reflect a significant opportunity to optimize maintenance practices, reduce manufacturing costs, and promote a competitive business structure in the ever increasing global market.

The day to day operation of a complex system requires an effective maintenance program that optimizes performance. In order to provide a basic understanding of the general maintenance practices available, a review of some of the popular maintenance strategies are provided for clarity within Section 1.1. It is assumed that the reader has a general understanding of reliability engineering concepts, maintenance plans, and basic statistical analysis. This research primarily investigates a condition based maintenance framework that monitors a single system coupled with spare part inventory control. A full treatment and discussion of condition based maintenance, spare part inventory control, and implementation approaches in the literature will be provided in Chapter 2.

With the continual focus on maintenance optimization, there has been a significant amount of research pertaining to the development and execution of maintenance plans. Blischke & Murthy (2003), provide a high level review of many of the maintenance policies available in practice. Also, Wang (2002) provides an extensive review of the literature pertaining to maintenance policies. Maintenance encompasses the actions taken to mitigate a failure from occurring or repairing a failure that has already occurred in order to bring a system back into operation. The two types of maintenance commonly utilized in practice include preventive maintenance and corrective maintenance. Preventive maintenance involves shutting an operational system down and performing actions to improve the system's health status. Corrective maintenance encompasses the actions taken to return a failed system back to operational capability through repair.

Preventive maintenance is a broad category that includes several different types of maintenance plans. The common preventive maintenance practices include time based maintenance and condition based maintenance. Typical time based maintenance policies include clock-based maintenance, age-based maintenance, and usage-based maintenance. Figure 1.1 provides a hierarchy structure of the aforementioned common maintenance policies in practice (Blischke & Murthy, 2003).

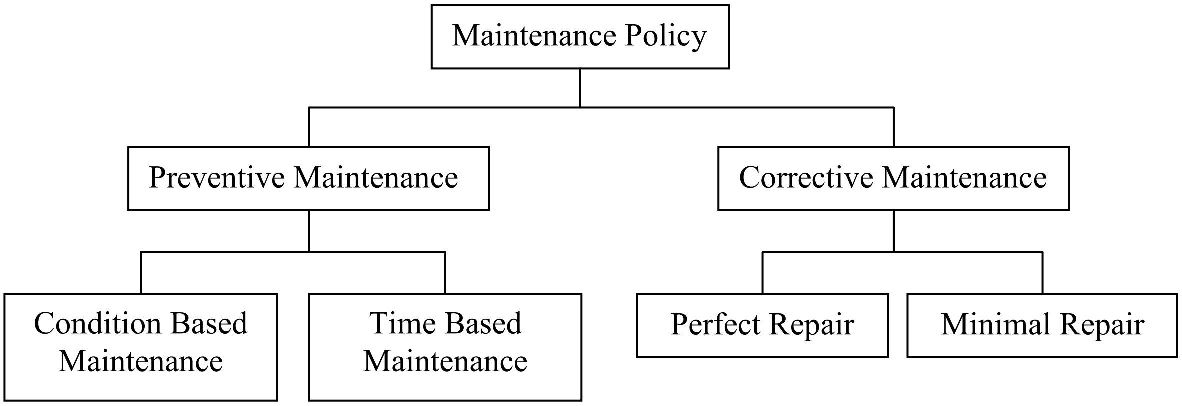

Figure 1.1: Maintenance policy framework

Condition based maintenance can be initiated according to the state of a degrading system that is monitored directly or indirectly through vibration, temperature, fluid particulate, or any other characteristic measure that describes the state of the system. Once the system degradation characteristic crosses a specified failure threshold, the maintenance activity is initiated. Condition based maintenance updates the knowledge of the failure time of the system and provides a means to determine inspection and maintenance activities as needed. Clock-based maintenance is carried out after a specified period of clock time. Age-based maintenance, on the other hand, is carried out after a component has been in the system for a specified age. Finally, usage-based maintenance is carried out after a predetermined amount of system usage.

Corrective maintenance encompasses repairs that are required to return a failed system back to operation. The two common types of repair include perfect repair and minimal repair (Blischke & Murthy, 2003). With perfect repair, the failed component is replaced or repaired leaving a system with a failure time distribution equivalent to a new component. On the other hand, minimal repair returns a failed component to operational state but the system is considered to have the same failure rate as the time when it failed.

Each maintenance approach differs with respect to implementation and management. Corrective maintenance usually does not provide a direct business advantage since the system is allowed to fail before the maintenance activity occurs. Therefore, unexpected system downtime is experienced with corrective maintenance and is typically utilized when the failure rate of the system is constant or decreasing. Running a machine or system to failure with no precursor to the failure event may be undesirable in practice. Preventive maintenance approaches are proactive in nature since the system is scheduled for maintenance downtime while corrective maintenance policies are reactive in nature since the failure event occurs before maintenance actions are commenced.

Clock-based, age-based, and usage-based maintenance practices do not address the actual physical state of the component being replaced and many times utilize failure time distributions for maintenance predictions. Components being removed and replaced can have additional operational life available that is not captured or addressed by clock-based, age-based, and usage-based maintenance policies. The practice of removing components that still have useful life remaining can incur unnecessary operational costs to an organization. Condition based maintenance has emerged as an effective maintenance policy that provides actual system state indications that aid in making predictions about the remaining useful life of a component. The remaining useful life predictions allow for the effective replacement of components that are truly approaching the end of operational life.

Condition based maintenance requires the installation of sensors and instrumentation that monitor the state of a degrading component or system. Nowadays, systems have continually become more complex while sensors have persistently become cheaper and more reliable. The ability to integrate sensors into maintenance programs continues to become more economically feasible and justified in practice. Critical components should be pinpointed for the application of condition based maintenance. Some of the typical components monitored using condition based maintenance include, but are not limited to, bearings, valves, gears, engines, motors, and fluids.

Once sensors are installed along with monitoring equipment, the next challenge faced in practice is the development of an effective data analysis approach. Figure 1.2 presents a hypothetical degradation process, *D(t),* which is a stochastic process that would be tracked and plotted over time from sensor readings. *D(t)* starts at approximately zero and tends toward the failure threshold, D_f, over time as additional degradation occurs. The system is considered failed at time t_f when *D(t)* crosses the failure threshold.

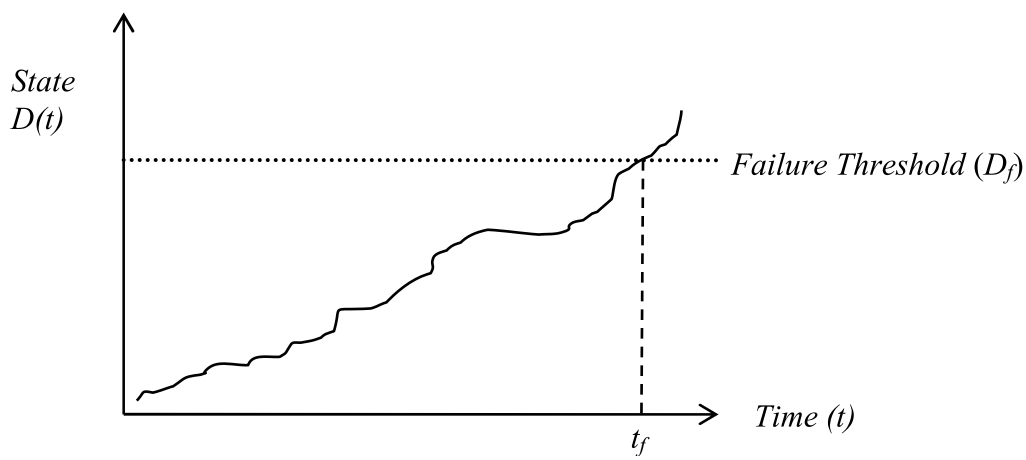

Figure 1.2: Example degradation process for condition based maintenance

Let *T* be the time to failure of a degrading component. The relationship between degradation and the time to failure of the component is defined according to the following relationship:

$$R(t) = \boldsymbol{Pr}(T \geq t) = \boldsymbol{Pr}\left(D(t) < D_f\right) \tag{1.1}$$

where $R(t)$ represents the reliability of the component at time *t*.

As data is collected, effective estimation techniques must be implemented to determine the remaining useful life (*RUL*) of a component and the maintenance actions that must be commenced to mitigate a failure. Stochastic processes, Kalman filters, Markov processes, neural networks, polynomials, and cumulative damage are some of the techniques utilized to define the degradation process and derive the *RUL* of a component. Chapter 2 will provide a review of the popular degradation tracking models in the literature. The goal of monitoring the degradation process is to develop a model that mimics the degradation process and is utilized to mitigate the occurrence of a failure.

1.2 Motivation and Significance of this Research

The ability to predict an eminent failure using condition based maintenance allows an operations manager, in most cases, to schedule maintenance, acquire the necessary spare parts, and perform the maintenance activities before the failure occurrence. In order to effectively manage systems and perform maintenance, the operations manager of a company must have spare parts available to replace components on a preventive based schedule or upon failure, per the maintenance strategies described in Section 1.1. Companies must determine the number of spare parts to hold in inventory to perform maintenance that ensures an effective operation with minimal downtime. From an inventory control standpoint, the time required to procure spare parts must be quantified and addressed given a system failure (Huiskonen, 2001). For spare parts that have a significant lead time, more spare part inventory should be held on hand to buffer the time required to order and receive additional spare parts. Spare parts that can be received on demand do not require a local stock to mitigate failures. The ability of an operation to predict system failures and effectively manage spare part inventory provides a distinct competitive business advantage.

Condition based maintenance alone is a useful approach to monitor and predict failures of a system but does not directly address the spare part inventory control. Once the failure occurrence is predicted, the operations manager is still left with managing the acquisition of the spare parts required for maintenance. Current literature discusses spare part inventory control and condition based maintenance but does not effectively link the strategies together into a comprehensive model. Developing a condition based maintenance model tied to a spare part inventory strategy reveals an important research problem that is not effectively addressed in the literature.

1.3 Objectives

The overall goal of this thesis is to investigate a condition based maintenance model that can be utilized to monitor a single system under spare part inventory constraints. The integrated condition based maintenance and spare part inventory control model will track the degradation process for a single system with a maintenance threshold while optimizing the number of spare parts required for the system in order to meet production demand. Finally, the number of spare parts required for system operation is incorporated into a production and spare part inventory decision model.

The objective of the thesis is to develop a comprehensive spare part and production decision model that is achieved by:

1.) Developing a model to optimize the number of spare parts to satisfy a predetermined stockout probability requirement. The gamma process is utilized to model the degradation process over time.

2.) Incorporating the spare parts model and *RUL* model into a comprehensive production decision model. The production decision model will incorporate outstanding orders, number of spare parts available, and the degradation process into an integrated model.

Ultimately, the production decision model will provide an optimal static threshold preventive maintenance policy.

3.) Establishing a *RUL* model that utilizes Bayesian analysis to update the degradation process as additional degradation data is obtained. Bayesian analysis techniques similar to Gebraeel et al. (2005) are implemented to update the estimate of the system state as degradation data is collected over time. The analysis presented and reviewed is centered on the degradation tracking and inventory control of a single unit (i.e. motor, pump, etc.).

1.4 Thesis Overview

The remainder of this thesis will be organized according to the following structure. Chapter 2 covers literature discussing condition based maintenance, degradation models, especially the gamma process, Bayesian analysis, and spare part inventory control techniques. Chapter 3 provides the theoretical development of a comprehensive condition based maintenance strategy combined with spare part inventory control. Chapter 4 provides a fully developed example that implements the theoretical developments in Chapter 3. Finally, Chapter 5 provides conclusions from the thesis along with recommendations for further research extensions.

CHAPTER 2

LITERATURE REVIEW

2.1 Introduction

Chapter 2 provides an overview of the literature pertaining to condition based maintenance, degradation tracking models, Bayesian analysis, and spare part inventory models coupled with maintenance policies. Within this review, some of the typical time based preventive maintenance policies are briefly presented to provide a comparison between time based and condition based maintenance policies. Typical condition monitoring approaches are discussed which include vibration analysis, process parameter monitoring, thermography, tribology, and visual inspection. The majority of the review focuses on the typical degradation modeling techniques in the literature which include polynomial models, gamma process, Brownian motion, Markov chains, Markov decision processes, neural networks, and Kalman filters. In addition, literature pertaining to Bayesian techniques coupled with condition based maintenance will be provided. The literature review is concluded with a review of research pertaining to maintenance policies that incorporate spare part inventory control.

Chapter 2 is organized as follows: Section 2.2 provides a review of time based preventive maintenance versus condition based maintenance along with applications of condition based maintenance in practice. Section 2.3 covers degradation tracking methods and models that can be applied to condition based maintenance. Section 2.4 addresses a number of degradation tracking and condition based maintenance examples available in the literature. Section 2.5 addresses maintenance and spare part inventory control models. Finally, Section 2.6 presents the shortcomings of the existing models and techniques along with the research extensions addressed in Chapter 3.

2.2 Applications of Condition Based Maintenance

Preventive maintenance allows for better downtime scheduling, operational agility, efficient resource utilization, and more effective spare part inventory control (Endrenyi et al., 2001). In practice, there are many different methodologies utilized for preventive maintenance and there is a dilemma in practice between the utilization of condition based maintenance versus time based preventive maintenance. Many of the methods utilized are based on failure time distributions that do not directly address the current state of the component. Unlike time based preventive maintenance, the application of condition based maintenance requires the use of monitoring technology that provides a status of condition parameters for the system. Maintenance engineers must select degradation characteristics for a system that provide an indication that can be utilized to determine the *RUL* of the system. There are a number of different monitoring methods and measures utilized in practice that must be selected appropriately to produce an effective condition based maintenance program.

2.2.1 Time Based Preventive Maintenance vs. Condition Based Maintenance

Wang (2002) provides a thorough review of time based preventive maintenance approaches in the literature. The author addresses age dependent preventive maintenance policies, periodic preventive maintenance policies, failure rate limit policies, sequential preventive maintenance policies, repair limit policies, repair number counting and reference time policies, opportunistic maintenance policies, and optimization approaches for maintenance policies.

Section 1.1 addresses the basics behind age based and usage based maintenance approaches. The additional preventive maintenance polices not covered in Section 1.1 are addressed by Wang (2002).

1.) *Failure rate limit policies* initiate maintenance when the system reaches a predetermined failure rate. State variables such as wear, stress, or damage are monitored to update the failure rate function. When the failure rate reaches the predetermined maintenance failure rate, the preventive maintenance activities are commenced.

2.) *Sequential maintenance policies* initiate maintenance according to unequal preventive maintenance time intervals. As the age of the component increases, the time between maintenance activities is reduced.

3.) *Repair limit polices* utilize a cost basis to determine the action taken when a component fails. When the component fails, the cost of repair is compared to the cost of replacement. The component will be repaired if the cost of repair is less than the cost to replace, otherwise the component is replaced.

4.) *Repair number counting polices* allow for a component to fail n times before the component is replaced. The failures up to and including n-1 failure are mitigated with minimal repair.

5.) *Repair number counting and reference polices* are an enhancement to repair number counting policies by adding an additional variable T that represents a positive operating time. Under the reference policy, the component is allowed to fail n times but is not replaced at the n^{th} failure if the operational time has not reached the predetermined T value. If T has not been reached, the component is minimally repaired and replaced on the n+1 failure.

6.) *Opportunistic maintenance policies* address dependencies that occur in large systems. Failure of a component within a large system of components may require the removal of non-failed components to access the failed component. Given this situation, there is opportunity to replace or repair non-failed components according to criterion such a hazard rate or cost.

7.) *Optimization of preventive maintenance policies* is conducted by analyzing cost and system reliability measurements. The optimization approach generates preventive maintenance intervals by minimizing costs or ensuring that a desired system reliability is achieved.

Mann et al. (1995) provides a review of time based preventive maintenance models and condition based maintenance. Within the review, a time based preventive maintenance cost model is established along with the shortcomings of utilizing time based preventive maintenance. Once the time based methods are established, condition based maintenance equipment, techniques, degradation measures, and failure diagnosis systems are addressed.

Mann et al. (1995) defines Eq. (2.1) which represents a time based cost model for preventive maintenance that minimizes the average cost per unit time. Tsang (1995) also provides a similar cost model approach for the optimization of the preventive maintenance intervals. The cost model utilizes the following notation: C_p is defined as the cost of planned preventive maintenance, C_f is the cost of corrective maintenance, *TC* is the total cost of maintenance per unit time, *f(x)* is the probability density function (*pdf*) for the component failure distribution, and *T* is the repair/replace time.

$$TC\left(T\right)=\frac{\text{Expected cost per cycle}}{\text{Expected cycle length}}=\frac{C_f\int_0^T f\left(x\right)dx+C_p\int_T^\infty f\left(x\right)dx}{\int_0^T xf\left(x\right)dx+T\int_T^\infty f\left(x\right)dx} \tag{2.1}$$

The total cost per unit time is minimized to determine the appropriate preventive maintenance interval *T*. The disadvantage of utilizing time based methods, like Eq. (2.1), become apparent when the standard deviation of the failure time distribution is large. When a large standard deviation is present, the ability to acquire the correct preventive maintenance interval is jeopardized. Excessive maintenance, emergency maintenance, and unnecessary maintenance costs are incurred when there is a high degree of variability within the system. Time based preventive maintenance also ignores the operating conditions of the system that contribute to the appropriate maintenance intervals.

Condition based maintenance allows for maintenance activities to be performed when the state of the system warrants the need for maintenance which reduces the number of unnecessary maintenance activities and their associated costs. Condition based maintenance techniques are justified for highly critical systems that require effective maintenance planning and execution. On the other hand, time based maintenance is an effective strategy for less critical systems that have a small degree of variability in the failure time distribution (Mann et al., 1995).

The question of which maintenance policy should be selected can be addressed by the use of reliability-centered maintenance (RCM). RCM follows a structured approach to classify and determine the appropriate preventive maintenance activities for large complex systems (Tsang, 1995). Maintenance policies are compared using RCM to determine the most cost effective maintenance policy for a given system (Endrenyi et al., 2001). Criticality of systems and components are determined through the use of RCM which provides a quantitative method to justify the adoption of a maintenance approach. Saranga (2002) proposes a structured method called relevant condition parameter (RCP) maintenance approach. The RCP approach selects maintenance significant items according to a risk priority number. Once the maintenance items are identified by the risk priority number, the relevant condition parameters are identified along with monitoring techniques. Eisinger & Rakowsky (2001) develop an RCM approach that integrates decision uncertainty into the maintenance policy selection by utilizing a belief probability in the RCM framework. For additional information concerning RCM, consult Zhou et al. (2007) and Rausand (1998).

2.2.2 Condition Monitoring Methods and Measures

Degradation measures must be identified that effectively relate the state of the component to the *RUL*. The degradation measure is identified according to the type of system being monitored and the feasibility of implementing conditioning monitoring technology. The next step after selecting a degradation measure is to identify the failure threshold. The threshold represents the limit set by the maintenance engineer that signifies a point at which unacceptable performance occurs with respect to the monitored degradation measure. For instance, vibration may exceed the acceptable limits on a machine cutting spindle that affects a product quality characteristic in production. The vibration reading that exceeds the acceptable limit would be defined as the failure threshold for the spindle. Once the vibration reading for the spindle exceeds the failure threshold, the system is considered failed and maintenance must be scheduled. See Dimla (2002) and Mannan & Stone (1998) for examples of cutting tool vibration monitoring methods.

Tsang (1995) states that many degrading mechanical systems utilize vibration monitoring, process parameter monitoring, thermography, tribology, and visual inspection for implementing condition based maintenance. Endrenyi et al. (2001) states that some of the common condition monitoring techniques in the power industry include visual inspection, optical inspection, neutron analysis, radiography, eddy current analysis, ultrasonic testing, vibration analysis, lubricant analysis, temperature analysis, magnetic flux leakage analysis, and acoustic emission monitoring.

The common monitoring methods in general practice include (Tsang, 1995):

1) *Vibration monitoring* can detect wear, fatigue, imbalance, misalignment, and loose assemblies for rotating equipment such a bearings, gear boxes, shafts, pumps, motors, and engines. Vibration readings are collected from the component or system over time and compared to the baseline and alarm limits. Maintenance personal are alarmed of an imminent failure when the vibration readings tend toward the alarm limits.
2) *Process parameter monitoring* involves tracking a wide variety of operational characteristics such as process efficiency, system temperature, electrical current, and pressure that can be linked to the health status of the system.
3) *Thermography* is a method of capturing the infrared emissions of a component to determine if the operating temperature conditions are fluctuating outside of normal operation. Abnormal temperature changes can be a precursor to failure.
4) *Tribology* is the study of the effects of friction between two mating surfaces. Friction causes the generation of particulate that can be monitored through wear particulate analysis. Lubrication analysis can also be conducted to determine the appropriate time to change lubricating fluids in a system.
5) *Visual inspection* is the easiest method to implement and involves identifying loose components, structural cracks, and abnormal operating characteristics.

Endrenyi et al. (2001) surveyed numerous electrical power companies to determine the common preventive maintenance practices utilized on power generation equipment. The authors found that many electrical providers depend highly on time based maintenance practices but have started to move toward the utilization of condition based maintenance programs along with probabilistic methods for maintenance prediction.

2.3 Degradation Methods and Models Applied to Condition Based Maintenance

To model degradation, there are continuous time, discrete time, continuous state, and discrete state degradation representations. Many of the discrete state/time methods involve Markov methods that require the definition of discrete degradation states and state transition probabilities. The discrete state/time degradation models are less realistic in application compared to a continuous state/time representation. Continuous state/time degradation models describe the true operational conditions over a continuous state/time domain. Therefore, continuous state/time models are more effective at describing degradation. Some of the discrete models include Markov chains and Markov decision processes. On the other hand, some of the continuous degradation models include polynomials, cumulative damage, Brownian motion, and gamma process.

2.3.1 Markov Chain Method

Markov chains represent a discrete time and discrete state stochastic process and are utilized to describe state transitions mathematically. Markov chain methods are utilized in physics, statistics, queuing theory, and many other applications. The Markov chain methods have been applied to condition based maintenance policies to establish the relationship between state transitions for a degradation process. The degradation phenomenon is defined according to discrete states that are modeled with a Markov chain. To utilize Markov methods, multiple states must be identified which can be challenging to define in practice and arbitrary in many instances. Along with the state definitions, the Markov methods require transition probabilities between states that can be difficult to define in practice. Incorrectly or arbitrarily defining states and transition probabilities can negate the value of the maintenance policy. Kharoufeh & Cox (2005), Chen & Trivedi (2002), and Saranga & Knezevic (2001) develop condition based maintenance models that rely on Markov chains.

Kharoufeh & Cox (2005) integrates sensor data and a continuous time Markov chain to estimate a full and residual lifetime distribution for preventive maintenance. The degradation is described by a cumulative stochastic process $X(t)=\int_0^t r\left(Z(u)\right)du$ where *Z(t)* represents the state of the random environment at time *t*. The system is considered failed when the accumulated degradation exceeds the failure threshold. The model is applied to propagation of a fatigue crack in 2024-T3 aluminum where the environmental transitions are modeled according to a Markov process.

Chen & Trivedi (2002) address condition based maintenance through the development of a closed form continuous time Markov chain for minimal and major maintenance. The states defined in the model are: (i,0) represents the state where the unit is operational and in the i^{th} deterioration state, (i,1) represents the state when the unit is in the i^{th} deterioration and under inspection, (i,2) represents the state when the unit is in the i^{th} deterioration and under maintenance, and state *F* represents a failed unit. The states are integrated into a Markov chain with a minimal and major maintenance threshold. The closed form steady state probabilities, availability, and MTTF formulas are derived and the optimal inspection intervals are generated and applied to a theoretical example.

Saranga & Knezevic (2001) utilize relevant condition predictors (RCP) that are applied to components in a system that are affected by degradation. Condition predictors are established for each failure mechanism within the system. The authors state that each RCP can transition between multiple states within the failure threshold with a probability of detection equal to p_i and a probability of not detecting the deterioration equal to q_i. The probability of detection increases in the $(k+j)^{th}$ state as compared to the $(k+i)^{th}$ state where $i<j$. To model the random fluctuation between states, a Markov Chain is utilized along with five states where zero is the initial state and five is the failure state. Reliability equations are developed by the author and implemented for several data sets.

2.3.2 Markov Decision Process Method

Markov processes represent a discrete state continuous time stochastic process that is defined by a set of states *s* in which multiple actions *a* are available to the decision maker at each state. State transition probabilities are defined for each state *s* and action *a* that establish the probability of transition to the next state. For each state traversed, the decision maker receives an award which establishes the decision made for that time period. Markov decision processes utilize actions and rewards unlike Markov chains. Kaufman & Lewis (2007), Iravani & Duenyas (2002), Maillart (2006), Hontelez et al. (1996), Sloan & Shanthikumar (2002) utilize Markov decision processes to optimize maintenance decisions for a discrete multi state system.

Kaufman & Lewis (2007) develop a repair and replacement model that considers the relationship between optimal maintenance decisions and optimal production decisions. The model addresses repair and replacement costs that are related to operational state of the queue. The deterioration process is described according to a discrete number of states that deteriorate independently of the work occurring at the machine. The demand on the system is modeled by a Poisson process and the service rate is modeled by an exponential distribution. The overall problem is modeled with a semi-Markov decision process with the actions defined as repair (replace) and do not repair (replace). Given the number of components in the queue and a discrete state, the decision to repair (replace) or do not repair (replace) is established by minimizing the infinite horizon expected discounted cost. For each combination of state and number of parts in queue, the repair (replace) or do not repair (replace) decision is established that generates a policy called a switching curve. The switching curve is generated by the decision boundary between the repair (replace) and no not repair (replace) decision for each number in the queue. The switching curves provide a policy for production to determine the appropriate repair (replace) decision considering the impact on the overall production facility. Iravani & Duenyas (2002) consider a single machine and address the production and maintenance decisions according to a Markov Decision process. The objective of Iravani & Duenyas (2002) is to determine the best joint production/inventory and repair/maintenance policy that minimizes the total average cost per unit time. The decisions for each state and inventory combination include continue to produce, repair, or remain idle. The authors present a double threshold policy that generates a policy decision for each combination of inventory and state which is similar to the switching curve concept developed by Kaufman & Lewis (2007).

Maillart (2006) develops a condition monitoring optimization problem that allows the decision maker to observe a wear related variable to establish the system degradation. An adaptive model is developed to determine the appropriate observation schedule and preventive maintenance actions for a multistate Markov system. The long run average cost per unit is minimized to generate a decision policy that allows for preventive maintenance, do nothing, or collect an observation.

Hontelez et al. (1996), utilize a Markov decision process to attain an optimum maintenance policy with a finite number of conditions that allow for an inspection, replacement, or repair. Knowledge of the system is considered partial since inspections are required to gain the precise condition of the system (i.e. the system is not continuously monitored). The degradation is defined by $Y(t) = g(t) + b\sqrt{t}U$ where *g(t)* represents a drift function and *U* represents a Normal random *N*(0,1) variable. The degradation function is divided into *N*+1 intervals where condition level 0 represents the new condition and condition level *N* is the failure condition. The progression through the states is modeled according to a Markov decision process with transition probabilities and detection probabilities. The overall objective of the model is to determine the cost-optimal policy that generates the actions that should be taken for every possible state. The results are validated by applying the model to monitor carbonation in concrete.

Sloan & Shanthikumar (2002) analyze the production and maintenance schedules of semiconductor wafer fabrication plant where the condition of the equipment affects the process yield. The authors model the system with a Markov decision process that allows for semiconductors to make multiple visits to the same workstation. The machine condition is described with a particle monitor that tracks particulate in the machine over time. The continuous degradation measurement is divided into discrete states and the transition probabilities are established for each state transition. The maintenance policy is established by maximizing the total reward for the system. The rewards are established to reflect the implications of yield on the maintenance decisions for stop production and clean or continue production decisions. The authors develop a simulation of a semiconductor operation and provide results that link the yield information to the condition state of the equipment.

2.3.3 Neural Network and Artificial Intelligence Method

Neural networks represent a set of nodes that perform computations and are arranged in patterns similar to neural nets (Chinnam, 1999). Each of the processing elements are connected through synapses with associated weights that modify a signal as it propagates through connections. Neural networks have the ability to learn from the environment and adjust the synaptic weights. In practice, neural networks have been applied to monitor and forecast degradation trends that are linked to maintenance decisions. Mann et al. (1995) provides a cohesive review of artificial intelligence techniques that include neural networks, expert systems, fuzzy logic, and model-based systems. Yam et al. (2001) and Chinnam (1999) utilize a neural network approach to reliability estimation and maintenance decision making.

Yam et al. (2001) develop an intelligent decision support system that monitors equipment, provides condition-based fault diagnosis, and trend prediction for the equipment deterioration. The fault diagnosis is achieved through one of three methods that include rule-based diagnostics, case-based diagnostics, or model-based diagnostics. Rule-based diagnostics identify faults according to rules that relate the fault to the monitored equipment condition. Case-based diagnostics use historic data to determine the current state of the system. Model-based diagnostics utilize mathematical models or neural networks to compare the model to the current condition to identify system faults. Neural networks have shown to effectively model, predict, and forecast the fault trend in order to dispatch maintenance before the system failure occurrence. The authors apply a neural network to track a planetary gear in a power generation facility. Vibration is monitored and data is collected to train the neural network to verify the ability of the model to predict wear characteristics of the gear over time. The neural network is shown to effectively model the gear and provide a condition based maintenance model.

Chinnam (1999) uses a neural network to model a degradation measure and to predict the reliability of twist drill bits as they degrade with use. According to Chinnam (1999), neural networks have proven to be extremely useful at time series forecasting and function approximation. The degradation of the twist drill bits are described by a non-linear regression model of order p equal to $y(s) = f\left[y(s-1), y(s-2), ..., y(s-p)\right] + \varepsilon(s)$. The neural network is trained and is capable of working with multiple degradation signals and can make different predictions for different signal orders. Reliability estimates are generated from the neural network state estimate and compared to the failure threshold for each bit.

2.3.4 Kalman Filter Method

Kalman filters provide a recursive method for the inference of the state of nature θ_t. The filter includes two main equations that include the observation equation and the system equation. Each of the two equations complement each other since the observation equation is a function of the system equation. The observation equation is defined by $Y_t = F_t\theta_t + v_t$ where Y_t denotes the observed parameter of interest at time t, F_t is a known quantity, and v_t is the observation error that is assumed to be a Normal $N(0,V_t)$ random variable. The system equation is defined by $\theta_t = G_t\theta_{t-1} + w_t$ where G_t is a known quantity, and w_t is a $N(0,W_t)$ variable. Applications of the Kalman filter include but are not limited to satellite tracking, underwater sonar, statistical quality control, and state estimation for maintenance. The Kalman filter provides a recursive method to collect indirect measurements and describe a system state parameter through the measurements. Refer to Meinhold & Singpurwalla (1983) and Zarchan & Musoff (2005) for a review of the theory of Kalman filters. Barone et al. (2007), Yang (2003), Pedregal & Carnero (2006), Yang (1999), and Yang (2002) utilize Kalman filters to develop preventive maintenance strategies.

Barone et al. (2007) developed an on board diagnostic system utilized on a vehicle oxygen sensor during product development that collects operational data during testing. The authors utilize a continuous time autoregressive (CAR) model for the data time series. Kalman recursion is utilized to estimate the unknown model parameters. The data is collected during testing and utilized to detect degradation trends and predict failure times of the sensors.

Shewhart control charts for individual measurements were plotted for the residuals to detect out of control diagnostic readings. The out of control readings were investigated to determine system faults.

Yang (2003) presents a condition based maintenance model that uses a Petri net and Kalman filter. The Petri net is a graphical representation of a system that relates conditions with events and is an effective model for static and dynamic systems. Each of the events within the Petri net is monitored and processed using Kalman filters for state estimation. The Kalman filter allows for making a state estimate from measured data and utilizes the prior state estimate in generating *N*-step state predictions. The Petri net transitions occur when the state estimate from the Kalman filter exceeds the failure threshold for the system. The Petri net and Kalman filter model is applied to a thermal power plant for condition monitoring.

Pedregal & Carnero (2006) utilize a state space model for condition monitoring of vibration of a centrifugal compressor. The forecasting model is defined by $z_k = signal + v_k$ where z_k is a bivariate output series and v_k is a bivariate random Gaussian variable. The signal is modeled according to a bivariate random walk. The forecast model is cast into a state space model that is defined according to a state equation and observation equation. To determine whether to replace a component, the authors develop an expected cost per unit time model that integrates the Kalman filter estimate of the system state to provide a replacement determination.

Yang (1999) and Yang (2002) develop a Kalman filter predictive maintenance application for an armature controlled DC motor. The physical state model for the Kalman filter utilizes several mathematical properties for a DC motor. The rotating speed of the motor output is monitored and considered failed if the output is at or below 95% of the normal value. The

Kalman filter is utilized to make *n* step predictions of the motor performance which are compared to the failure threshold for maintenance decisions.

2.3.5 Miscellaneous Degradation Models

The most basic models for degradation are used to describe simple linear, convex, and concave degradation paths. Polynomials and differential equations are utilized to describe linear, convex, and concave degradation behavior (Meeker & Escobar, 1998). Many of the basic methods utilize regression techniques to define the degradation path and can describe linear and non-linear behavior effectively. Meeker et al. (2001), Lu & Meeker (1993), Lu et al. (1997), Liao et al. (2005), Kim & Kolarik (1992), and Chinnam (2002) provide several examples of linear and nonlinear models applied to degradation modeling.

Meeker et al. (2001) model the degradation of a gallium arsenide laser utilizing linear regression analysis for the degradation data. Pseudo failure times are generated by extrapolating the degradation model to the failure threshold and determining the failure time for the components. Linear degradation paths represent the most basic and well behaved degradation phenomenon that can be modeled in practice. Meeker et al. (2001) state that more advanced analysis techniques are required to model degradation data if the path is not linear, there is significant measurement error, and/or the failure of the component can occur instantly and does not correlate to the degradation path.

Lu & Meeker (1993) utilize a general path model to describe fatigue-crack growth. The degradation path model utilized by Lu & Meeker (1993) is defined by y_{ij} where $y_{ij}=\eta_{ij}+\varepsilon_{ij}=\boldsymbol{\eta}\left(t_j;\phi,\Theta_i\right)+\varepsilon_{ij}$ and $\varepsilon_{ij}=N(0,\sigma^2_\varepsilon)$. t_j= time of the j^{th} measurement, ε_{ij} =measurement error with constant variance, η_{ij}=degradation path for the i^{th} unit at time t_j, ϕ=unknown fixed-effect parameter vector, and Θ=unknown random-effect parameter vector. Θ is defined according to a multivariate distribution function. Several path model examples are provided which utilize the Weibull distribution, Normal distribution, and multivariate Normal distribution to describe the degradation rate parameter Θ. For the highly complex cumulative distribution function *F(t)*, the authors develop a Monte Carlo estimation technique to determine the estimate of *F(t)*. Bootstrap techniques are utilized to provide the confidence intervals for the estimate of *F(t)*.

Lu et al. (1997) model a linear degradation phenomenon for MOS field transistors (MOSFET). The general linear model is defined by $y_{ij} = \theta_{0i} + \theta_{1i}\,\boldsymbol{log}\left(t_{ij}\right) + \varepsilon_{ij}$ where y_{ij} represents the j^{th} successive measurement of the degradation of the i^{th} device and ε_{ij} corresponds to the error term. The θ_{0i} and θ_{1i} variables are modeled according to a multivariate Normal distribution and ε_{ij} is modeled by an independent identically distributed Normal $N\left(0, \sigma_{ij}^{2}\right)$ variable. Maximum likelihood estimation is utilized to establish the model parameters along with bootstrap techniques to acquire the confidence intervals for the *CDF*.

Liao et al. (2005) applies the proportional hazard model to determine the remaining useful life of a bearing. The proportional hazard model is effective at modeling multiple degradation processes simultaneously. Refer to Kumar & Klefsjo (1994) for a review of the proportional hazard model. The vibration characteristics of the bearings were collected and a time window nonlinear regression model was utilized to describe the RMS and Kurtosis of the data. The RMS and Kurtosis values are integrated into the proportional hazard model to determine the remaining useful life of the bearings.

Kim & Kolarik (1992) provide a real time conditional reliability method for cutting tools that uses thrust data for each independent tool and thrust data from the population for modeling. The authors use a first, second, and third order regression model along with an exponential model to describe the thrust force as a function of each hole drilled. The quadratic model was chosen as the best model according to the R^2 value from the regression procedure. Reliability estimates are generated from the thrust data by defining a thrust failure threshold along with a time to failure distribution.

Chinnam (2002) utilize a general polynomial model to describe degradation and allow for first order residual autocorrelation. The polynomial model is defined by $y_j = \eta_j + \varepsilon_j$ where η_j is the degradation path and ε_j is error disturbance. To address autocorrelation in the degradation data, the authors add an autocorrelation parameter ρ to ε_j. The authors discuss several methods to determine ρ along with a brief discussion of autoregressive models, moving average models, and autoregressive moving average models. Reliability estimation techniques are provided for polynomial models along with an application to drilling operation and fatigue crack growth.

2.3.6 Cumulative Damage Model

Cumulative damage represents a process where successive damage accumulates and failure occurs when the maximum cumulative damage crosses a specified threshold. Damage can occur at successive time intervals and the stresses may vary over time. Giorgio et al. (2007) apply the cumulative damage model in defining a degradation process.

Giorgio et al. (2007) develop a condition based maintenance model for heavy duty diesel engine cylinder liners for marine applications. The internal surface wear of the liner is identified as the degradation characteristic for the engines. The degradation process is defined according to the cumulative damage model since it is effective at capturing successive wear phenomenon. The wear events are modeled according to the power-law process with the wear accumulated up to time t defined by $W(t)=cN(t)$. The parameters of the power law process are established using a maximum likelihood procedure along with a moment estimation approach. Bootstrap techniques are utilized to determine the confidence intervals for the estimate parameters and reliability function. The degradation model is utilized to define a condition based maintenance methodology. The probability of the wear for each cylinder exceeding the failure threshold is established for each inspection time. If the probability of exceeding the failure threshold is insignificant (determined by the model user), the inspection will not be commenced. If the probability of exceeding the failure threshold is significant, the inspection will be commenced and liners that have a high probability of failure will be replaced.

2.3.7 Gamma Process & Brownian Motion Models

The gamma process and Brownian motion represent continuous time and continuous state stochastic processes that define an increment between two time periods. For degradation modeling, the degradation increment between two time intervals can be defined by the gamma process or Brownian motion. The degradation increment, for the gamma process, is defined by the gamma distribution where the Brownian motion increment is defined by the Normal distribution. Both of the processes have independent increment properties. Chapter 3 provides a full mathematical treatment of the gamma process. Liao et al. (2006), Grall et al. (2002), Lawless and Crowder (2004), Park & Padgett (2005), and Joseph & Yu (2006) apply the gamma process and/or Brownian motion to model degradation phenomenon.

Liao et al. (2006) utilize the gamma process to develop a condition-based availability limit policy. The objective of the model is to maximize the availability of the system by establishing the preventive maintenance threshold. An algorithm is established to solve the optimization model. Within this work, Liao et al. (2006) prove that a degradation path defined according to the gamma process has an increasing failure rate over the whole degradation path. To justify preventive maintenance activities, the system must exhibit an increasing failure rate over time. Performing maintenance on a system that does not exhibit an increasing failure rate is not economical or practical. Therefore, performing preventive maintenance to a system described by the gamma process is justified.

Grall et al. (2002) focus on modeling a condition based maintenance and inspection policy for a stochastically continuously deteriorating system where the degradation process is defined according to a gamma process. When the system condition exceeds the failure threshold, corrective maintenance is dispatched with as-good-as-new repair policy. Preventive maintenance is enacted when the degradation process crosses a critical maintenance threshold λ. The main optimization parameters for the condition based maintenance model include inspection dates and critical threshold values. The authors develop a model to determine λ and the inspection dates that minimize the long run expected maintenance cost per unit time. To determine the long run cost the authors define three cost parameters that include the cost of inspection, cost of preventive maintenance, and cost of corrective replacement. The second development within this work is an inspection scheme that allows for irregular inspection intervals that are updated according to the system state. Grall et al. (2002) states that the gamma process is effective at modeling the degradation of hydraulic structures subject to erosion, cylinders subject to mechanical damage, induction furnaces, and concrete bridge structures.

Lawless and Crowder (2004) provide the likelihood functions for the estimation of the gamma process parameters, the failure distributions, and applications of the gamma process to modeling crack growth in materials. The authors state that the gamma process is an effective model of degradation that is monotonic and has conditionally independent increments and is strictly increasing with time. Some of the common applications include degradation modeling of materials, accumulation of flow in systems, and other natural degradation phenomenon (Lawless & Crowder, 2004).

Park & Padgett (2005) present an accelerated degradation model that utilizes the geometric Brownian motion and gamma process to describe the degradation path. The accelerated degradation test exposes products to stresses outside the use environment to obtain degradation information for the product in a timely manner. The geometric Brownian motion is a strictly positive degradation process where the Brownian motion process can be negative. For systems that accumulate damage, the degradation process should be strictly positive which lends to the utilization of geometric Brownian motion or gamma process which are strictly positive. The authors present derivations for the *pdf* and *CDF* of the Brownian motion and gamma process along with the *pdf* and *CDF* for the first passage time of the failure threshold by the degradation process. The maximum likelihood estimation for the unknown models parameters are presented for a multiple stress accelerated degradation model. The results are applied to a carbon film resistor and to fatigue crack growth experiment.

Joseph & Yu (2006) utilize a Brownian motion model to describe the degradation of a window wiper switch. The Brownian motion model is integrated with a design of experiment to optimize the control factors for the product. The authors develop an optimization approach to minimize the quality loss of the product for a specified degradation characteristic. The degradation characteristic of the product is defined as a function of the product noise, environmental noise, control factors, and measurement error. Control factors are established by the design of experiment to minimize the expected total loss function.

2.3.8 Applied Bayesian Analysis

Practitioners utilize Bayesian techniques in many applications to effectively utilize prior parameter knowledge along with additional data to consistently update the parameters of interest. Bayesian techniques are effective at dealing with uncertainty in the system state and refining the estimate as additional information becomes available. Bayes theorem was initially developed by Thomas Bayes (Barnard & Bayes, 1958) and provides a method to utilize prior information along with new evidence to update or revise a parameter of interest. Bayesian analysis is widely utilized in practice to address uncertainly in parameter estimates and to update the current parameter estimate as additional information becomes available. The mathematical equations for Bayes theorem are provided in Chapter 3 for reference. Gabraeel et al. (2005), Hamada (2005), and Merrick et al. (2003) integrate Bayesian techniques and preventive maintenance into comprehensive models.

Gebraeel et al. (2005) utilize Bayesian updating to combine information from the distribution of the population parameters and the real time sensor information. The degradation model and Bayesian update is utilized to make predictions of the residual life distribution. Two exponential signal models are developed by Gabraeel et al. (2005) in which one model assumes independent identically distributed random error and the second model assumes the error is modeled by Brownian motion. The Bayesian analysis is utilized to improve the estimation of the parameters of the exponential degradation model. The degradation model is defined by a continuous stochastic process $S(t_i)$ where $S(t_i)=\phi+\theta\, \boldsymbol{exp}(\beta t_i)\boldsymbol{exp}\left(\varepsilon(t_i)-\sigma^2/2\right)$. ϕ is defined as a known constant, θ is a lognormal random variable, β is a Normal random variable, and $\varepsilon(t_i)$ is a random error term. The log of the signal is utilized to develop the posterior distribution for the model parameters using Bayesian analysis. The posterior distribution is shown to be a bivariate

Normal distribution and is utilized to define the residual life distribution for the component. Each time that new sensor information is available, the residual life distribution is updated. The theoretical developments are applied to degradation data obtained from testing bearings and is shown to be an effective modeling approach to incorporate sensor data into the residual life estimation.

Hamada (2005) provides a degradation modeling example for a laser along with a Bayesian approach to update the model parameters. The degradation at time t_i is defined by $D_i(t) = (1/\theta_i)t$ where $1/\theta_i$ is the slope. θ_i is defined as a random variable represented by Weibull(β, λ) where β is the shape parameter and λ is the scale parameter. Measurement error is defined by ε which is assumed to be independently distributed as $N(0, \sigma_\varepsilon^2)$. Bayesian analysis is utilized to estimate the model parameter vector $\boldsymbol{\eta}$ = $(\lambda, \beta, \sigma_\varepsilon^2)$. Prior information for $\boldsymbol{\eta}$ is incorporated by using Bayes theorem to calculate the posterior distribution. The reliability estimate for the laser degradation is provided with the incorporation of Bayesian analysis.

Merrick et al. (2003) uses a proportional hazard model to describe the failure rate of cutting tools with p covariates that describe the environment of the machine tools. Bayesian analysis is applied to the unknown parameters for each of the machine tools. The method is applied to a number of machine tools and each covariate is studied to determine the posterior densities for each parameter. Cost models are developed to determine the appropriate replacement time for each of the cutting tools.

2.3.9 Summary of Degradation Methods and Models

From Sections 2.3.1 through 2.3.7, one can see that there are numerous methods and models that can be used to define a degradation process. The question of which model is better then another model becomes challenging in practice. Selection of a model should be based on the ability of the model to accurately describe the degradation process and make effective state extrapolations into the future for maintenance decisions. Ineffective models will generate erroneous maintenance decisions that can jeopardize the effectiveness of a condition based maintenance system. The degradation model must ensure that that physical degradation phenomenon is captured in the most realistic and tractable method available for implementation.

2.4 Examples of Condition Monitoring

Degradation modeling is a critical and challenging aspect of the implementation of a condition based maintenance program. Typically, degradation measurements traverse upward or downward toward a failure threshold and the system is considered failed at the time when the measurement crosses a predetermined failure threshold. The failure mechanisms for the system must be understood so that an appropriate degradation model is developed and employed in practice. Typically, degradation phenomena are characterized by a linear, convex, or concave degradation path. There are a number of models that characterize degradation models and predominately utilize statistical modeling and parameter estimation.

The advantage of monitoring degradation behavior is the ability of the operation to detect a preeminent failure before the actual occurrence. Time based maintenance policies and corrective maintenance policies do not have the ability to determine the current system state and predict the onset of a failure. Complex systems require a degradation measure that indirectly monitors a failure mechanism. For instance, the breakdown of a bearing is monitored with vibration which is an indirect measure of the actual degradation. Indirect measurements are required to avoid extensive inspection of equipment that can lead to exorbitant downtime and the possibility of introducing undue damage to the system during inspection. There are many applications of indirect measurements that are used for degradation monitoring in the literature that are applied to motors, bearings, gearboxes, cutting tools, structures, rotating equipment, and a myriad of other systems.

Moseler & Isermann (2000) model a fault detection method for a brushless DC motor. Signal monitoring along with a mathematical model are utilized to track the supply voltage, DC current, and motor angular velocity. Differential state equations are utilized to describe the brushless DC motor and to describe the performance of the motor from the input and output signals. Faults are identified by comparing the mathematical model to the nominal performance values. Habetler et al. (2002) propose a condition monitoring method for a low voltage, line connected induction motor. The method developed by Habetler et al. (2002) monitors the stator winding temperature, stator turn faults, rotating faults, and bearing faults. Several current monitoring techniques are presented that utilize neural networks for fault determination. Onel et al. (2005) and Schoen et al. (1995) provide a review of motor stator current analysis to detect rolling element bearing damage. The frequency of the stator current is affected by changing air gap flux density that is caused by bearing defect vibration. Both Onel et al. (2005) and Schoen et al. (1995) provide a case study that verifies the relationship between bearing vibration and stator current harmonics.

Isermann (1984) provides a review of fault detection methodologies and two applications in practice to a centrifugal pump and leak detection in pipelines. Methods to address measurable signals, non-measurable state variables, non-measurable process parameters, and non-measurable characteristic quantities are presented.

1.) *Measurable signals* represent input or output signals that directly monitor changes in the process. Measurable signals are compared to maximum and minimum values for the system in the determination of the health status of the system. Limits for the measured signal are established to allow appropriate time for interaction yet minimize the false alarm occurrence.

2.) *Non-measurable state variables* are process state variables that can not be directly measured. Process models are required to estimate the non-measurable variable. Kalman filter techniques are surveyed to estimate the non-measurable state with other system variable measurements.

3.) *Non-measurable process parameters* are time dependent coefficients that appear in the input and output process models.

4.) *Non-measurable characteristic quantities* provide state information for large systems. Examples included system efficiency, resource consumption, and tool usage per unit time.

Parameter estimation for continuous time models can be established with least square parameter estimation, determination of time derivatives, instrumentation parameter estimation, and discrete time models.

Fault diagnosis can be conducted through the use of signal based or model based approaches. Harihara et al. (2003) compare the complexity and differences between signal based and model based diagnostic systems. Signal based detection utilizes raw system measurements, such as vibration, and determines a diagnosis with no model processing. On the other hand, model based diagnostics process signals and then utilize a model to make state estimates. Harihara et al. (2003) investigate the performance of signal based versus model based diagnostics and find that model based approaches are less prone to false alarms compared to signal based approaches. Experimental results are provided that prove that a model based diagnostic system reduces the likelihood of false alarms for an induction motor.

Li et al. (2000) and Mathew & Alfredson (1984) apply condition monitoring techniques to rolling element bearings. Mathew & Alfredson (1984) apply time domain and frequency domain analysis to the vibration signals. Time domain analysis monitors the vibration signal and can be used to determine the root mean square (RMS) and peak acceleration over time which are compared to recommended values for fault detection. Frequency domain analysis involves the use of spectral analysis to analyze vibration signals which are tracked and compared to predetermined failure limits for diagnosis. Monk (1972) provides a review of some basic vibration levels with acceptable vibration limits for different types of systems. The frequency and time domain approach do not utilize a model for diagnostics and rely solely on signal processing for fault determination. Li et al. (2000) develop a stochastic model to describe the defect propagation of rolling element bearings. Recursive least squares estimation is utilized to update the model parameters as online data is collected and material uncertainty is modeled by a random variable.

There are numerous applications of condition based maintenance in practice. Within the literature, many of sources study mechanical systems since they exhibit an increasing failure rate in most cases. The increasing failure rate justifies the application of preventive maintenance. Polynomial degradation models, gamma process, Brownian motion, state space equations, and Bayesian analysis are widely applied in the literature to model degradation phenomenon. Condition based maintenance can be applied to cutting tools, hydraulic structures, brake linings, airplane compressor blades, rotating equipment, bearings, crack growth, erosion, corrosion, and fatigue (Grall et al., 2002), (Ray & Tangirala, 1996), (Ma & Li, 1996), and (Zhan et al., 2006).

2.5 Maintenance and Spare Part Inventory Control Models

In order to conduct maintenance activities, an operation requires spare part resources to replace failed components and assist maintenance crews in keeping equipment operational. Spare parts are carried as insurance against unnecessary downtime that can occur when a part or system fails. Carrying spare parts provides risk mitigation against downtime but requires a significant capital investment with minimal direct financial return. Companies strive to reduce the amount of spare part inventory held due to holding costs and spare part obsolescence. Therefore, the goal of many organizations is to optimize the number of spare parts held to ensure a rapid response to failure. Much of the literature in spare part inventory control develops models to determine when to place a spare part order and how many to order with the objective of minimizing cost and ensuring a high service rate. Kennedy et al. (2002) provide a thorough review of spare part management issues and the current trends in the literature. Most of the literature addressing spare part inventory control utilizes cost optimization approaches to establish the ordering and spare part management policy.

Aronis et al. (2004) present a spare part model that is based on a case study of a circuit pack utilized by a communications company. The electronic component is modeled with a constant failure rate λ. The communications company manages spare part inventory according to an (S-1, S) policy. The lowest S value is calculated by the company to ensure a specified service level p is achieved. The service level p represents the probability of immediately satisfying demand for a spare from the spare parts stock. The number of spares required during a replenishment lead time L is defined by the Poisson distribution and minimized to ensure that the probability of demand for a spare is immediately satisfied by the S-1 spares available. Therefore, the relationship defined by Eq. (2.2) is utilized to establish the minimum number of spare parts required to satisfy the service level p after one part is retrieved from stock for n installed circuit packs.

$$\sum_{k=0}^{S-1} \frac{(\lambda nL)^{k} \boldsymbol{exp}(-\lambda nL)}{k\,!} \geq p \tag{2.2}$$

The initial failure rate estimate λ_o is established from MIL-HDBK-217F part count prediction methods. As field data is collected, the estimate of λ is updated with Bayesian analysis. The failure rate, λ, is defined as the number of failures that occurred in a 1,000 circuit packs per year. The gamma distribution is selected as the prior distribution and applied with Bayes theorem to determine the posterior distribution of λ. The compound Gamma-Poisson probability function is utilized to define the number of failures k during the replenishment lead time L where the minimum value of S is established to satisfy:

$$\sum_{k=0}^{S-1} \int_{0}^{\infty} \frac{(\lambda nL/1000)^{k} \boldsymbol{exp}(-\lambda nL/1000)}{k\,!} \frac{\beta^{\alpha} \lambda^{\alpha-1} \boldsymbol{exp}(-\lambda\beta)}{\Gamma(\alpha)} d\lambda \geq p \tag{2.3}$$

As failure data is collected, the posterior estimate of λ is updated. The prior distribution parameters of α and β are initially established by solving the following two equations:

$$\frac{\alpha}{\beta} = \omega\lambda_0 \tag{2.4}$$

$$P(\lambda \leq \delta\lambda_0) = 0.95 \Rightarrow \int_{0}^{\delta\lambda_0} \frac{\beta^{\alpha} \lambda^{\alpha-1} \boldsymbol{exp}(-\lambda\beta)}{\Gamma(\alpha)} d\lambda = 0.95 \tag{2.5}$$

where δ and ω are variables established by the decision maker reflecting the certainty of the initial estimate λ_o. Once the initial prior parameters are established, the Bayesian analysis updates the failure rate λ and subsequently the number of spares required to ensure the service level is satisfied.

Rustenburg et al. (2000) consider a single system with multiple different components that can fail at different times and are replaced instantaneously given a spare part is available. If the spare part is not available, the system experiences downtime until the part is received. The time between failures is assumed to be exponentially distributed for each component. The operation has a limited capital budget of *C* that can be spent on purchasing spare items each year. The goal of the model is to determine the number of spares required for each component that meets the overall budget constraint while maximizing the system availability. To optimize the initial selection of spare parts, the authors minimize the probability of a backorder while satisfying the capital expenditure constraint. The probability of backorder (PBO) is defined according to the Poisson distribution and the availability given spare S_i for product *i* is defined by:

$$A(S_1, ..., S_I) = \prod_{i=1}^{I} \left(1 - PBO_i(S_i)\right) \tag{2.6}$$

The initial optimization model establishes the spare part quantities under the capital expenditure constraint. Once the initial spares are determined, the authors develop a resupply problem that determines the ordering strategy when components fail given a yearly capital budget constraint. The ordering strategy is developed to maximize the availability probability over the entire planning horizon. Overall, the authors provide a model to determine the spare part combinations required to meet an initial budget and define system availability for each spare part position in order to establish the reorder strategy.

Armstrong & Atkins (1996) develop a joint optimization order and replace policy for a system with one component that is subject to random failure with only space to carry one spare. Two initial cost functions are developed which include the replacement cost function and the ordering cost function. The replacement cost function is comprised of the part cost and a breakage cost and is minimized to determine the replacement time t_r. The ordering cost function is comprised of the shortage cost and holding cost which is optimized to balance the cost of holding spares versus the cost of a shortage occurrence. The time to order t_o is established to minimize the ordering cost function. The replacement cost function and ordering cost functions are combined into a joint cost function that represents the expected operating cost per unit time. The optimal solution to the joint cost function can be obtained through non-linear programming to establish the time to replace and the time to order. The authors show that the joint optimization approach to the determination of t_r and t_o is more effective from a system cost perspective as compared to sequential optimization of t_r and t_o.

Shibuya et al. (1998) develop an inventory model with stochastic lead times and the option of initiating expedited orders. Two models are developed with the first model considering continuous inventory monitoring and the second model considering periodic inventory monitoring. The total expected cost for one cycle is developed for the continuous and periodic inventory monitoring cases. The objective of the model is to determine the order time t_o and the order quantity Q to minimize the total expected cost.

Almeida (2001) generates a multicriteria decision model for two maintenance problems that include repair contract selection and spares provisioning. The first decision model is defined by a set of actions $\{a_1, a_2, ..., a_m\}$ that represent the set of all possible maintenance contracts available to the decision maker. The actions each have an associated cost and availability performance that is affected by the system interruption time. The author provides an expected utility function that is maximized to determine the most appropriate maintenance contract a_i. The second model involves provisioning spare parts by maximizing a multicriteria decision model that is a function of the total spare cost and probability of stockout. The author maximizes the expected utility function to determine the optimum number of spare parts required for the system.

Srinivasan & Lee (1996) model a system with a single production part type with demand occurring according to a Poisson process that is managed with an (*S*, *s*) policy. The authors analyze the effects of preventive maintenance coupled with production inventory decisions when the facility has an increasing failure rate. The best time to schedule preventive maintenance for this system is when the inventory level for the production parts reaches *S* and the facility is waiting to return to *s*. When the production part inventory reaches *S*, the system is reserved for preventive maintenance. The expected cost per unit time over the long run is established for each renewal epoch that occurs when the inventory level reaches *S*. Each of the components of the expected cost are derived and a solution approach is established to minimize the expected cost by determining $r = S\text{-}s$ and *S*.

Vaughan (2005) develops an $\left(s(k), S(k)\right)$ optimal spare part inventory policy where k represents the number of periods until the next scheduled preventive maintenance. Each component is modeled with a constant failure rate and the total number of part failures during any period is defined by the Poisson distribution. Scheduled replacement of components occurs every *T* periods and demand must be met for failures with existing inventory or expedited to satisfy the demand. Dynamic programming is utilized to determine the optimal policy for ordering in each period to perform preventive maintenance. The solution determines the appropriate $\left(s(k), S(k)\right)$ values for each k by minimizing the total cost function. The cost equation is a function of holding cost, ordering cost, inventory shortage cost at preventive maintenance intervals, and inventory shortage cost given a random failure between preventive maintenance intervals.

Chelbi & Ait-Kadi (2001) provides a block replacement preventive maintenance and spare part inventory policy. The authors seeks to minimize the average cost per unit of time over an infinite horizon by establishing T, R, and s which represent the replacement period, replacement cycle, and stock level respectively. The cost model is comprised of preventive maintenance cost, corrective maintenance cost, holding cost, ordering cost, and shortage cost. The authors provide a solution algorithm to solve the optimization problem and establish the optimal block replacement spare part policy.

Ghodrati & Kumar (2005) provide a model to determine the number of spare parts required considering environmental factors such as dust, temperature, humidity, pollution, and vibration that affect the reliability of the system and the number of spare parts required. The proportional hazard model is utilized to model the effects of multiple environmental conditions that correlate to the hazard rate of the component. The renewal function is utilized to define the number of spares required during the period with a probability of shortage equal to p. The Weibull reliability model is employed to describe the reliability of the component and is integrated with the proportional hazard model to reflect multiple environmental conditions. The Weibull-proportional hazard model is combined with the renewal function to determine the number of spare parts required.

Sarker & Hague (2000) utilize a simulation to model a block replacement and continuous review inventory policy. The author states that block replacement is an effective approach for inventory items such as screws, springs, coils, disks, valves, bearings, and bushings. Multiple production lines are modeled with the following characteristics: the failure rate of each item increases, the cost of items are constant, emergency orders can be placed, maintenance occurs according to block replacement, and the production lines are balanced. Cost formulations that address the cost per unit, individual failure replacement cost, inventory costs, and emergency ordering costs are utilized in the simulation model. The authors show that a jointly optimized (*S, s, T*) policy yields more cost effective solutions compared to optimizing the inventory policy (*S, s*) and block replacement policy (*T*) separately. Therefore, jointly optimizing the spare part policy and the maintenance policy provides a more robust solution compared to individual optimization problems that address the spare part and maintenance policy separately.

Marseguerra et al. (2005) generate a multiobjective spare part model that maximizes the total net profit while minimizing the total spares volume. The optimization problem is solved with a genetic algorithm and the failure, repair, and replacement distributions are modeled using Monte Carlo simulation. The total net profit objective function is defined as the plant profit minus the acquisition of spare parts, handling costs of spares, and system downtime costs. Monte Carlo simulation is utilized to model the failure, repair, and replace system dynamics. The Monte Carlo simulation model is integrated with the genetic algorithm to maximize the total net profit and the inverse number of spares. For each chromosome generated by the genetic algorithm, the Monte Carlo simulation is executed for a limited number of iterations. The estimates are updated each time that a chromosome is generated to determine a statistically significant result. Pareto optimality and dominance are utilized to compare two solutions with respect to several objectives. Chromosomes are ranked according to Pareto dominance and non dominated chromosomes are considered for the selection of parents within the genetic algorithm. The minimum number of spares required to maximize the system profit is established with a Monte Carlo and genetic algorithm combination.

Smidt-Detombes et al. (2006) analyze a *k*-out-of-*N* system with identical repairable components with hot standby redundancy that is managed with a condition based maintenance policy. The system availability is controlled by the maintenance policy, spare part inventory level, the repair capacity, and the repair job priority setting. Maintenance is initiated if the number of failed components passes a critical level m. The components are modeled using three states that include a good-as new state, degraded state, and failed state. The time in each state and the repair time are defined by the exponential distribution. Maintenance initiation and setup are modeled with a constant lead time L. During maintenance, all failed and possibly all degraded components are replaced by the spare pool S. Repair capacity is defined by c parallel servers that process failed components. The authors focus on the limiting and steady state system availability considering the impact of the maintenance threshold m, spare parts pool S, repair capacity c, total number of components in the system N, and the minimum number of components k that are required for system functionality. The exact availability and approximate availability are generated for multiple system configurations.

2.5.1 Summary of Maintenance and Spare Part Inventory Control Models

Within Section 2.5 there are a number of spare part models that typically minimize the cost of holding spares compared to the cost of stockout. Along with cost optimization, the spare part models are defined by a failure time distribution that does not consider the current system condition. Condition based maintenance models applied to spare part inventory control are not widely studied or available in the literature.

2.6 Thesis Contribution

The current literature does not effectively integrate condition based maintenance models with spare part inventory control. Therefore, an opportunity to expand the existing literature becomes apparent when analyzing the work done on incorporating spare part inventory control with condition based maintenance.

The available literature is effective at defining numerous approaches for degradation modeling and estimation which are discussed in 2.3.1 through 2.3.7. Likewise, the spare part inventory control literature defines a number of approaches to optimize spare part policies which is discussed in Section 2.5. These two research bodies are widely studied but there is minimal work available that addresses the integration of spare part inventory control and condition based maintenance. The integration of production and maintenance has been addressed by the semi-Markov decision processes in Section 2.3.2. The semi-Markov decision process provides a method of making production and maintenance decisions by minimizing the operational costs for each state but does not address spare part inventory control and condition based maintenance.

This thesis expands on the current literature by developing a comprehensive condition based maintenance and spare part inventory control model. Within the current literature, there are numerous modeling methods to define degradation phenomenon that include continuous and discrete state/time models. Continuous state/time models are more effective at describing a degradation path as compared to discrete state/time models that require state identification and transition probabilities. With this realization, the degradation model utilized for this thesis is defined by a continuous state/time model to avoid discrete state identification and transition probability estimation. With regards to the spare part inventory control policy, much of the literature defines the optimization models as a function of holding cost and stockout cost. The spare part determination model developed in this thesis does not directly address costs but ensures that a minimum number of spare are held to maintain a specified stockout probability. Therefore, the spare part model in this thesis indirectly minimizes cost since the minimum number of spares are held to guarantee a specified performance. Finally, the condition based maintenance model and spare part inventory control model are integrated to provide a comprehensive production decision model that is not provided in the literature for continuous time and state degradation models.

CHAPTER 3

METHODOLOGY

3.1 Objectives and Model Framework

The objective of this research is to utilize condition based maintenance, optimization techniques, and Bayesian analysis to develop a cohesive model to manage the maintenance and spare part inventory for a single production system. The development of this research is achieved through a three-step process that supports the overall objective. The overall development process in this thesis is given in Figure 3.1. The first step develops an optimization model for the determination of the number of spare parts for the system. The second step addresses the integration of the degradation model, spare parts optimization model, and the production order constraints into one integrated production decision model. Finally, the third step involves developing a Bayesian update method.

More specifically, the first step involves determining the number of spare parts that are required to support a predetermined stockout probability. A stochastic optimization model is formulated which minimizes the number of spare parts required in stock while ensuring that the probability of stockout is adequate to support production requirements. The derivation utilizes a Laplace transform approach and constrained regression approximation to address the convolution of random variables within the optimization model.

The second step in the model development derives a degradation model utilizing the gamma process. This step ties together production with the degradation process and spare part optimization models from step one. The final model provides a comprehensive decision model that addresses outstanding orders, number of spare parts available, and the degradation process.

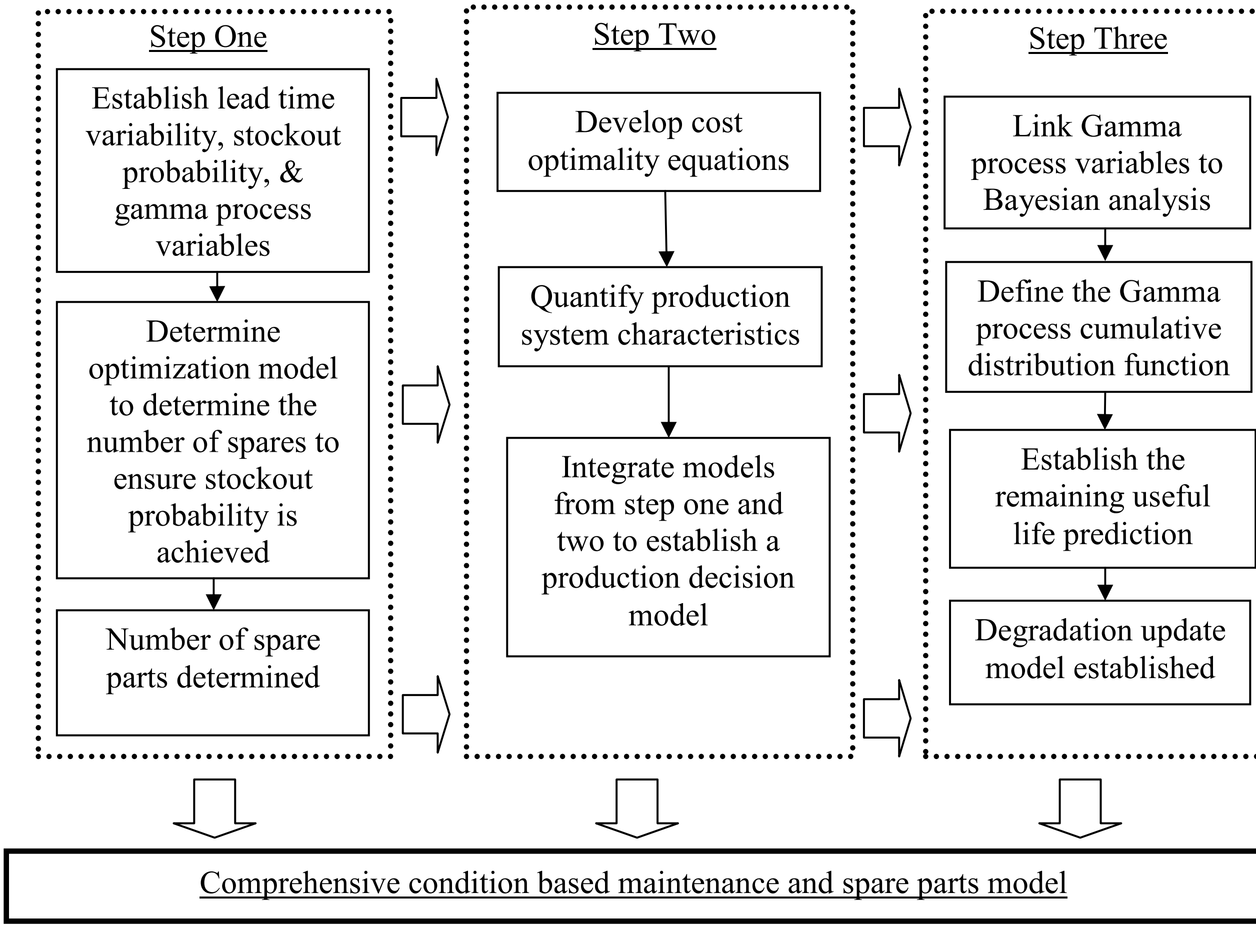

Figure 3.1: Thesis development process

Finally, the gamma process model is developed in conjunction with a Bayesian updating technique that updates the estimates of the model parameters as additional degradation data is collected. The Bayesian analysis allows for improvement in the state estimate as more degradation data becomes available through condition monitoring.

This thesis addresses components and systems as synonymous entities like a motor, pump, or compressor in the development of a condition based maintenance model. Each complex system may include thousands of individual components that will be treated as one component in the application of condition based maintenance. This simplification allows for the effective application of condition based maintenance techniques since sensors can be incorporated into the unit to provide the necessary means to collect degradation data. In most cases, maintenance engineers are concerned with the aggregate performance of all components comprised of a system and modeled as one entity.

3.2 Gamma Process

Definition 3.1*—Gamma Process* (Lawless & Crowder, 2004)

The gamma process defines a non-negative monotonically increasing degradation process. Let $D(t_i)$ be the degradation measurement at time t_i, and $D_i = D(t_i) - D(t_{i-1})$ be the increment of degradation measures between time t_i and t_{i-1}. The gamma process has the following properties:

1.) The random increments D_i are independent and nonnegative;

2.) D_i has a gamma *pdf* $D_i \sim gamma\left(\alpha \cdot (t_i - t_{i-1}), \beta\right)$ defined according to the shape parameter α and inverse scale parameter β in Eq. (3.1).

$$D_i \sim \frac{d_i^{\alpha \cdot (t_i - t_{i-1}) - 1} \boldsymbol{exp}(-\beta d_i)\, \beta^{\alpha \cdot (t_i - t_{i-1})}}{\Gamma\left(\alpha \cdot (t_i - t_{i-1})\right)} \tag{3.1}$$

Figure 3.2 provides three example gamma process degradation paths. Note that the paths represent three degradation processes that are strictly increasing with time.

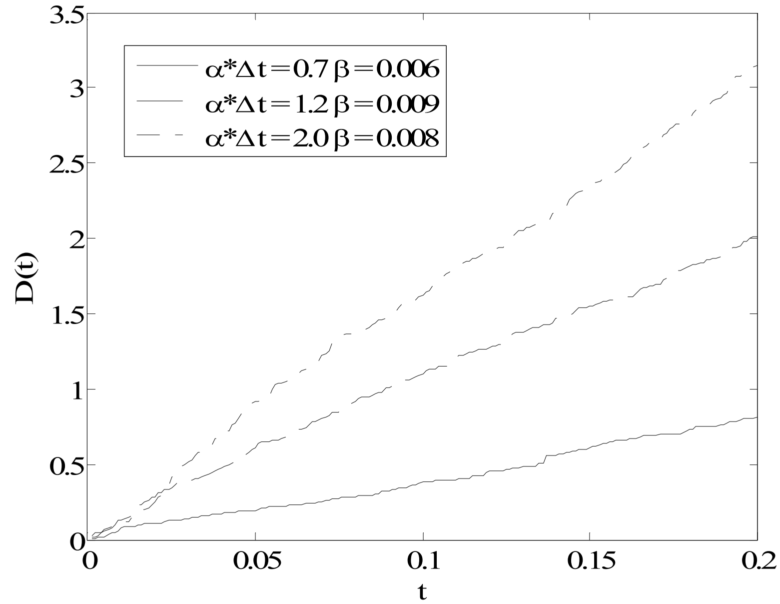

Figure 3.2: Three gamma process degradation paths

The cumulative distribution function (*CDF*) for the failure time is defined by:

$$F_{\alpha,\beta}(t) = \boldsymbol{Pr}(T < t) = \boldsymbol{Pr}\left(D(t) > D_f\right) \tag{3.2}$$

where T is the failure time and D_f is the failure threshold. The *CDF* of the failure time associated with the gamma process is given by Park and Padgett (2005) in Eq. (3.3), where x_0 represents the initial starting point for the degradation process.

$$F_{\alpha,\beta}(t) = \frac{\Gamma\left(\alpha t, \beta\left(D_f - x_0\right)\right)}{\Gamma(\alpha t)} \tag{3.3}$$

$\Gamma\left(\alpha t, \beta\left(D_f - x_0\right)\right)$ represents the incomplete gamma function given by:

$$\Gamma\left(\alpha t, \beta\left(D_f - x_0\right)\right) = \int_{\beta\left(D_f - x_0\right)}^{\infty} \xi^{\alpha t - 1}\, \boldsymbol{exp}(-\xi)\; d\xi \tag{3.4}$$

Figure 3.3 provides a graphical representation of the *CDF* of the failure time for the gamma process.

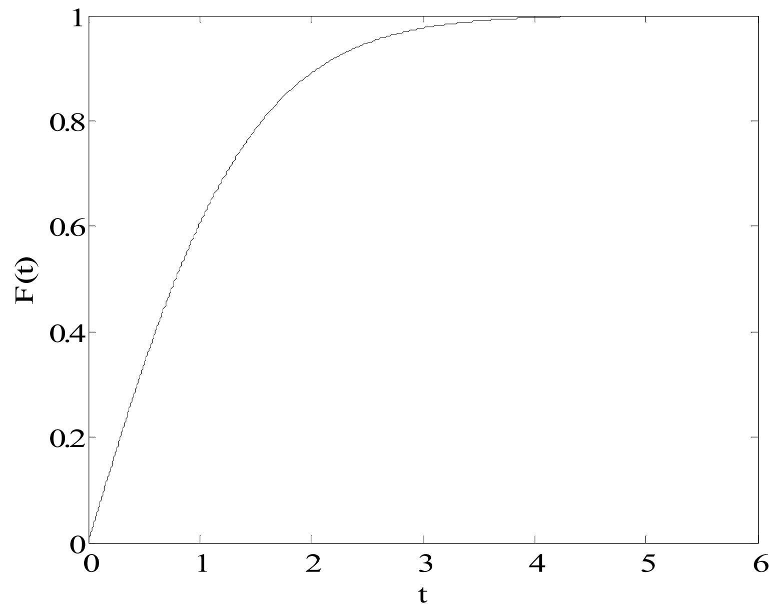

Figure 3.3: Gamma process failure time *CDF* ($\alpha = 0.7$, $\beta = 0.006$, $D_f = 45$)

The *pdf* of the failure time of the gamma process is defined by Park and Padgett (2005):

$$f_{\alpha,\beta}(t) = \alpha\left[\Psi(\alpha t) - \boldsymbol{log}\left(\beta\left(D_f - x_0\right)\right)\right]\left[1 - \frac{\Gamma\left(\alpha t, \beta\left(D_f - x_0\right)\right)}{\Gamma(\alpha t)}\right] + \frac{\alpha}{\Gamma(\alpha t)} \frac{\left(\beta\left(D_f - x_0\right)\right)^{\alpha t}}{(\alpha t)^2} {}_2F_2 \qquad (3.5)$$

where $\Psi(\alpha t) = \frac{\Gamma'(\alpha t)}{\Gamma(\alpha t)}$ represents the digamma function and ${}_pF_q$ represents the generalized hypergeometric function defined by:

$${}_pF_q\left(\{a_1, ..., a_p\}, \{b_1, ..., b_q\}, z\right) = \sum_{k=0}^{\infty} \frac{(a_1)_k, ..., (a_p)_k}{(b_1)_k, ..., (b_q)_k} \frac{z^k}{k!} \qquad (3.6)$$

where $(a_i)_k = a_i \cdot (a_i + 1) .... (a_i + k - 1)$ and ${}_2F_2\left(\{\alpha t, \alpha t\}, \{\alpha t + 1, \alpha t + 1\}, -\beta\left(D_f - x_0\right)\right)$ is defined by:

$${}_2F_2 = 1 + \sum_{k=1}^{\infty} \left(\frac{\alpha t}{\alpha t + k}\right)^2 \frac{\left(-\beta\left(D_f - x_0\right)\right)^k}{k!} \qquad (3.7)$$

Figure 3.4 shows an example *pdf* curve for the gamma process.

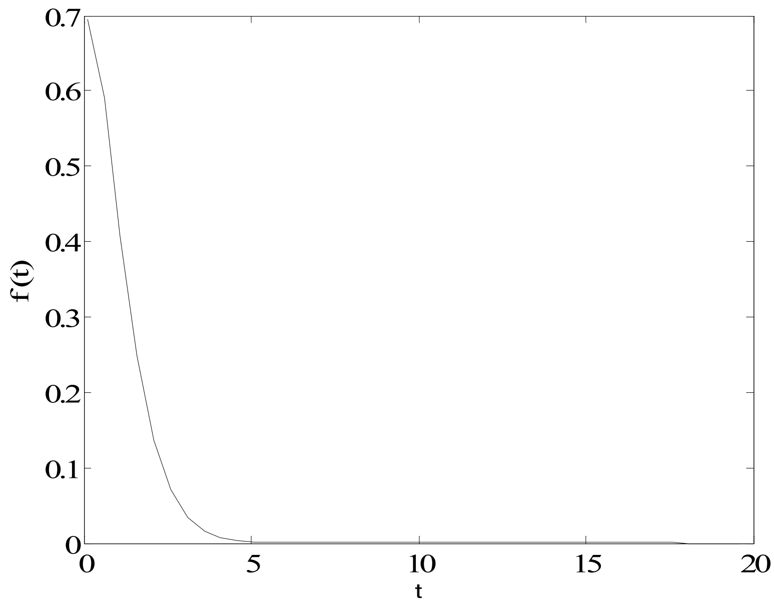

Figure 3.4: Gamma process failure time *pdf* ($\alpha = 0.7$, $\beta = 0.006$, $D_f = 45$)

The MATLAB code utilized to generate the figures within the thesis are provided in the Appendix for reference. Please note that the MATLAB statistics toolbox and optimization toolbox are required to execute the code generated for this thesis.

3.3 Step One: Spare Part Determination Under Stockout Probability Constraint

The first step in the development of the overall production decision model is to determine the number of spare parts required to support the operation. There are numerous models in the literature that optimize the number of spare parts. Many of the models utilize cost criteria in the optimization models to determine the number of spares required. The spare part model developed in this thesis minimizes the number of spare parts required under the stockout probability constraint for the gamma process described by Eq. (3.1). For instance, the maintenance engineer would determine that the probability of stockout must be less than or equal to 0.1. Therefore, the number of spares is to be established to ensure that the probability of stockout is less than or equal to 10% in this case.

3.3.1 Spare Part Determination Model

Figure 3.5 is a representation of the spare part determination model. The number of spares is minimized to ensure that the sum of T_i values is less than or equal to the lead time for the component with a specified stockout probability.

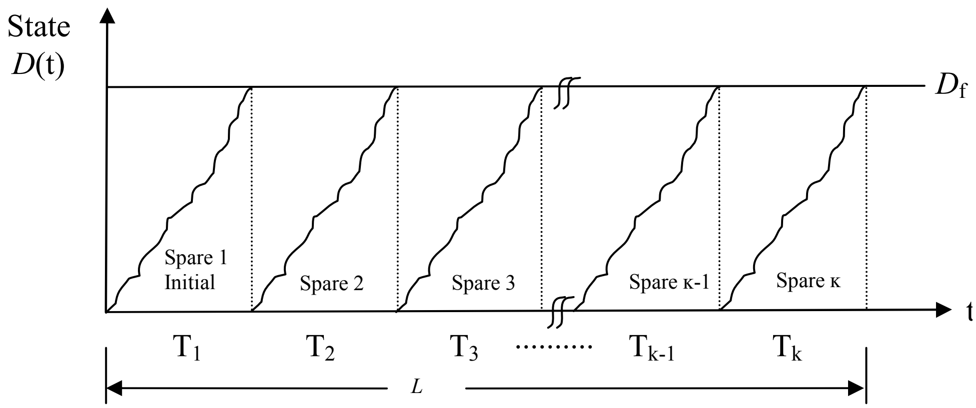

Figure 3.5: Degradation and lead time model for multiple spares

The minimum number of spare parts that satisfies the specified stockout probability constraint is obtained by solving the following stochastic optimization problem.

Problem L:

$$\min \kappa$$

subject to

$$\boldsymbol{Pr}\left(\sum_{i=1}^{\kappa} T_i < L\right) = \int_0^{\infty} \boldsymbol{Pr}\left(\varpi_\kappa = \sum_{i=1}^{\kappa} T_i < L \mid L = l\right) g(l)\,dl = \int_0^{\infty} g(l) \int_0^{l} f_{\varpi_\kappa}(t)\,dt\,dl = \int_0^{\infty} g(l) F_{\varpi_\kappa}(l)\,dl = \alpha \le \upsilon \quad (3.8)$$

$$\kappa \ge 0$$

$$\kappa \text{ int}$$

where:

κ = number of spares parts required,

L = random variable representing the lead time,

g(l) = *pdf* of the lead time distribution,

υ = minimum stockout probability required,

α = calculated stockout probability,

T_i = random time for the degradation process of component i to reach the failure threshold D_f;

$f_{\varpi_\kappa}(t)$ = *pdf* of $\sum_{i=1}^{\kappa} T_i$, and

$F_{\varpi_\kappa}(l)$ = *CDF* of $\sum_{i=1}^{\kappa} T_i$ evaluated at lead time l.

Problem L is based on the assumptions that the preventive maintenance threshold D_{pm} equals the failure threshold D_f and the replacement of components occur instantaneously when the degradation measure reaches D_f. The solution to *Problem L* provides the optimal number of spare parts, κ^*, required to ensure a specified stockout probability given the degradation of a component is defined according to the gamma process.

From Chapter 2, Aronis (2004) presents a methodology to determine the number of spares required during a replenishment lead time to ensure a specified service level. The author utilizes Bayesian analysis and parametric distributions in the solution procedure. *Problem L* has a similar objective to Aronis (2004) but addresses condition based maintenance and degradation monitoring while determining the number of spare parts required during the replenishment lead time. The incorporation of degradation modeling extends and enhances the concepts presented by Aronis (2004).

3.3.2 Convolution Properties

To solve the above optimization problem, Eq. (3.8) needs to be evaluated. Consider the sum of two failure times where $Z = T_1+T_2$. T_1 has a *pdf* defined by $f_{T_1}(t)$ and T_2 has a *pdf* defined by $f_{T_2}(t)$. The *pdf* of Z is acquired by taking the convolution of $f_{T_1}(t)$ and $f_{T_2}(t)$ and is defined by the convolution operator * as:

$$f_Z(t) = f_{T_1}(t) * f_{T_2}(t) \tag{3.9}$$

where:

$$f_Z(t) = \int_{-\infty}^{\infty} f_{T_1}(t-\tau) f_{T_2}(\tau) d\tau \tag{3.10}$$

The results of Eq. (3.9) can be generalized for ϖ_κ as follows due to the commutative property of convolutions:

$$f_{\varpi_\kappa}(t) = f_\kappa(t) * f_{\kappa-1}(t) * \ldots * f_2(t) * f_1(t) \tag{3.11}$$

3.3.3 Laplace Transform

When evaluating the sum of random variables, it is more tractable to use the Laplace transform. The Laplace transform can be applied to the convolution of random variables to avoid the evaluation of the integral expression in Eq. (3.10). The Laplace transform for a positive domain density function $f(t)$ is defined by:

$$L\left(f(t)\right) = f^*(s) = \int_0^{\infty} \boldsymbol{exp}(-st) f(t)\, dt, \text{ for } t > 0 \tag{3.12}$$

The follow relationship holds for the inverse Laplace transform:

$$L^{-1}\left[L\left(f(t)\right)\right] = L^{-1}\left[f^*(s)\right] = f(t) \tag{3.13}$$

The sum of random variables defined by Eq. (3.10) can be evaluated by the inverse of the product of the Laplace transform density functions with the following relationship:

$$f_Z(t) = \int_{-\infty}^{\infty} f_{T_1}(t-\tau) f_{T_2}(\tau) d\tau = L^{-1}\left[f^*{}_{T_1}(s) f^*{}_{T_2}(s)\right] \tag{3.14}$$

Therefore, the general case of Eq. (3.14) is defined by:

$$f_{\varpi_\kappa}(t) = f_\kappa(t) * f_{\kappa-1}(t) * \ldots * f_2(t) * f_1(t) = L^{-1}\left[f^*{}_\kappa(s) f^*{}_{\kappa-1}(s) \mathbf{....} f^*{}_2(s) f^*{}_1(s)\right] \tag{3.15}$$

Assuming that the *pdf* of the time to failures for each of the κ units are identical, Eq. (3.15) simplifies to:

$$f_{\varpi_\kappa}(t) = f_\kappa(t) * f_{\kappa-1}(t) * \ldots * f_2(t) * f_1(t) = L^{-1}\left[f^*(s)^\kappa \right] \tag{3.16}$$

There are several important properties for Laplace transforms that will be utilized in the solution procedure of Eq. (3.8) that are provided in Eq.s (3.17) through (3.19).

$$L(F(t)) = F^*(s) = \frac{1}{s} f^*(s) \tag{3.17}$$

$$\lim_{t\to 0} F(t) = \lim_{s\to\infty} sF^*(s) = 0 \tag{3.18}$$

$$\lim_{t\to\infty} F(t) = \lim_{s\to 0} sF^*(s) = 1 \tag{3.19}$$

Expanding Eq. (3.17) (Rehmert, 2000) provides the *CDF* of ϖ_κ:

$$\begin{aligned} F_{\varpi_\kappa} &= \boldsymbol{Pr}\left[\varpi_\kappa \le t\right] = \boldsymbol{Pr}\left[T_1 + T_2 + \ldots + T_{\kappa-1} + T_\kappa \le t\right] \\ &= L^{-1}\left[F_{\varpi_\kappa}(s)\right] = L^{-1}\left[\frac{1}{s} f^*{}_{\varpi_\kappa}(s)\right] \\ &= L^{-1}\left[\frac{1}{s} f^*{}_{T_1}(s) f^*{}_{T_2}(s) \ldots f^*{}_{T_{\kappa-1}}(s) f^*{}_{T_\kappa}(s)\right] \end{aligned} \tag{3.20}$$

3.3.4 Numerical Method to Calculate the Laplace Transform

A numerical approximation technique that utilizes the Widder Laplace transform inversion formula is provided by Jagerman (1978) and Jagerman (1982) to approximate the inverse Laplace transform. The method allows for a tractable method to invert complex Laplace transform functions. Jagerman (1978) & (1982) shows that the approximation sequence converges uniformly and maintains the structural characteristics of the original function.

For the Laplace transform defined in Eq. (3.12) there exists a series of functionals, σ_n that converge to the inverse Laplace transform estimate:

$$L^{-1}\left[f^*(s)\right] \approx \sigma_n f = \sigma_n(t) = \left(2+\frac{1}{n}\right) f_{2n}(t) - \left(1+\frac{1}{n}\right) f_n(t) \quad (3.21)$$

where:

$$f_n(t) = \frac{(-1)^n}{n!} s^{n+1} \frac{d^n f}{ds^n}\bigg|_{s=\frac{n+1}{t}} \quad (3.22)$$

Example:

Consider the *CDF* of the exponential distribution defined by:

$$F(t) = 1 - \boldsymbol{exp}(-\lambda t) \quad (3.23)$$

The Laplace transform is defined by:

$$F(s) = \frac{\lambda}{s(s+\lambda)} \quad (3.24)$$

Eq. (3.21) and (3.22) can be utilized to determine the inverse Laplace transform of Eq. (3.24) to verify the functionality of the numerical approximation. From Eq. (3.13), the results of the inverse Laplace transform of Eq. (3.24) should converge to the results of Eq. (3.23). MATLAB code was generated to verify the results of the approximation method and is shown in the Appendix. Figure 3.6 plots Eq. (3.23) compared to the results of the inverse Laplace transform of Eq. (3.24). Since the actual and estimated *CDF* show near equivalency in Figure 3.6, the difference between the two curves was calculated and plotted in Figure 3.7. The difference between the estimated inverse Laplace transform and the actual *CDF* shown in Figure 3.7 is of the magnitude of 10^{-4} and thus negligible for this application. Therefore, the estimation approach provided by Eq. (3.21) is an effective method to estimate the inverse Laplace transform. Obi (1990) provides the error function for the Jagerman estimation equations.

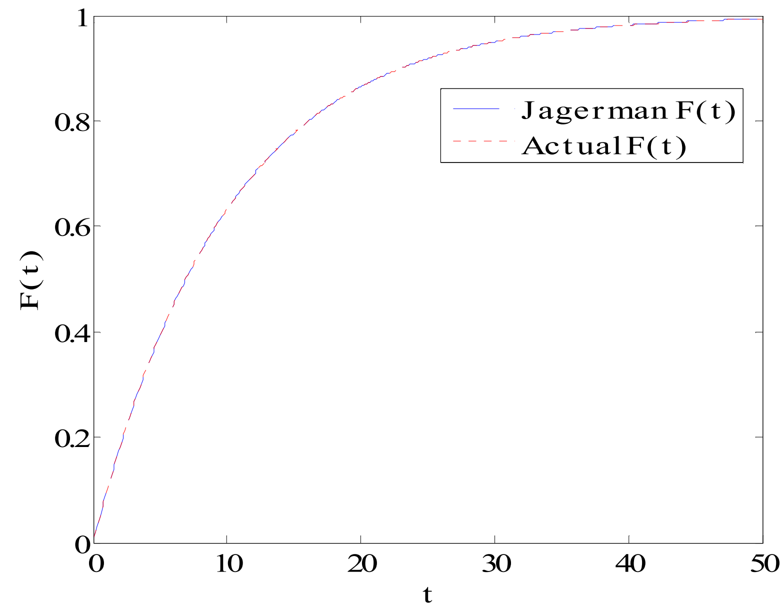

Figure 3.6: Jagerman *CDF* estimate $L^{-1}[F(s)]$ versus the actual *CDF* (λ=0.1, n=25)

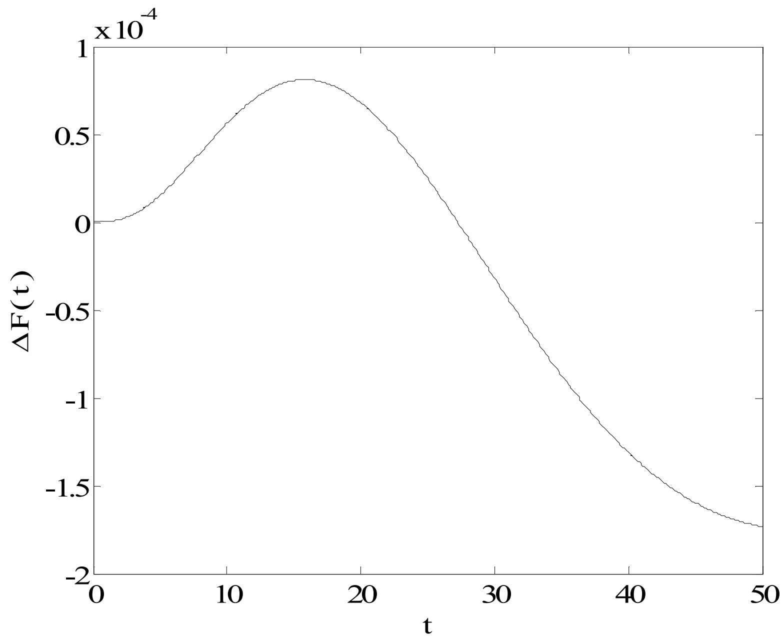

Figure 3.7: Difference between Jagerman *CDF* estimate $L^{-1}[F(s)]$ and the actual *CDF* (λ=0.1, n=25)

3.3.5 Constrained Regression Model

For complex functions, obtaining closed form solutions to Eq. (3.12) can be difficult. To address this issue, the integral can be solved by simulation according to the following relationship:

$$L\left[f(t)\right]=\int_0^{\infty} e^{-st} f(t)\,dt = E\left(e^{-st}\right) \approx \lim_{n\to\infty} \frac{1}{n}\sum_{i=1}^{n} e^{-s\vartheta_i} \qquad \forall s,\ \vartheta_i = \left\{t \mid H(t) = U_i(0,1)\right\} \tag{3.25}$$

where $H(t)$ is the *CDF* of the density function $f(t)$ and $U_i(0,1)$ is a uniform random variable with the range [0, 1]. The simulation calculates the expectation of e^{-st} for each value of *s* by taking a large number of random samples ϑ_i from the failure time distribution with *pdf* $f(t)$. Figure 3.8 represents a basic pictorial example of a Laplace transform for the exponential *pdf* using Eq. (3.25). For each value of *s*, the simulation takes an average of a large number of random samples ϑ_i to determine the value of *L*[*f(t)*] at this point.

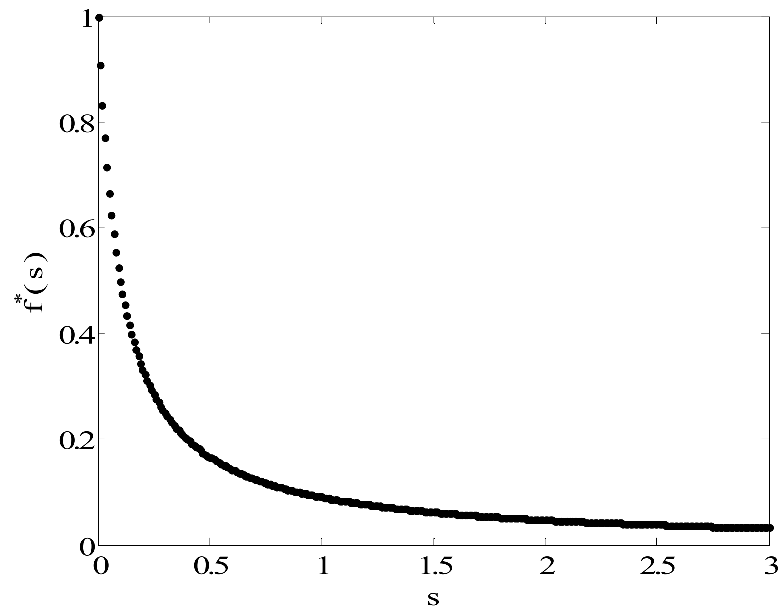

Figure 3.8: Laplace transform integral by simulation for the exponential *pdf* (λ=0.1)

The inverse Laplace transform defined by Eq. (3.21) requires a continuous function that is established from the Laplace transform curve generated by Eq. (3.25). A constrained least squares approach is employed in fitting a continuous function to $f^*(s)$ by:

$$MSE = \frac{1}{n}\sum_{j=1}^{n}\left(f^*\left(s_j\right) - \hat{f}\left(s_j;\eta\right)\right)^2 \tag{3.26}$$

where $f^*(s_j)$ represents the actual Laplace transform, $\hat{f}(s_j;\eta)$ is the estimated function value defined by the parameter vector η. Therefore, the following constrained least squares estimation method is utilized to determine the unknown parameter vector η:

$$\begin{aligned}&\boldsymbol{min}\ \frac{1}{n}\sum_{j=1}^{n}\left(f^*(s_j)-\hat{f}(s_j;\eta)\right)^2\\&\text{subject to}\\&\qquad \lim_{s\to\infty} s\hat{f}(s;\eta)=0 \qquad \text{(initial value theorem)}\\&\qquad \lim_{s\to 0} s\hat{f}(s;\eta)=1 \qquad \text{(final value theorem)}\end{aligned} \tag{3.27}$$

The two constraints in Eq. (3.27) represent the initial value theorem and final value theorem of Laplace transforms defined by Eq. (3.18) and (3.19), respectively. These two constraints ensure that model utilized to define the Laplace transform has the appropriate limit behavior in the time domain.

3.3.6 Solution Procedure for the Spare Part Model

The solution of Eq. (3.8) requires a multi-step procedure that utilizes methods from Sections 3.3.2 through 3.3.5. A MATLAB program is developed to determine the minimum number of spare parts required to ensure that the stockout probability is satisfied. The general solution framework is:

1.) Initialize the gamma process by defining the shape parameter α, inverse scale parameter β, failure threshold D_f, stockout probability υ, lead time distribution with associated parameters, and a count variable i=1.

2.) For the base case, set the number of spares equal to one (i = 1) where $\kappa_i = i$ and calculate the stockout probability by:

$$\int_0^\infty g(l)F_{\varpi_1}(l)dl = E\left(F_{\varpi_1}(l)\right) \approx \lim_{n\to\infty}\frac{1}{n}\sum_{i=1}^{n}F_{\alpha,\beta}(l_i) \quad l_i=\{t\,|\,G(t)=U_i(0,1)\} \tag{3.28}$$

where $G(\cdot)$ represents the *CDF* of the lead time distribution and $F_{\alpha,\beta}(\cdot)$ is the failure time *CDF* of the gamma process defined in Eq. (3.3). If the calculated stockout probability is less than υ from step 1, the operation should carry one spare and the solution procedure is complete. If the calculated stockout probability is greater than υ from step 1, proceed to step 3.

3.) Calculate the Laplace transform of the failure time *pdf* of the gamma process by:

$$f^*(s)=\int_0^\infty e^{-st}f_{\alpha,\beta}(t)\,dt=E\left(e^{-st}\right)\approx \lim_{n\to\infty}\frac{1}{n}\sum_{j=1}^{n}e^{-s\tau_j}\quad \forall s,\ \tau_j=\left\{t\mid F_{\alpha,\beta}(t)=U_j(0,1)\right\} \tag{3.29}$$

4.) Increment $i=i+1$ and $\kappa_i=\kappa_{i\text{-}1}+1$. Calculate the Laplace transform of the failure time *CDF* for $\kappa_i\geq 1$ by:

$$L\left(F_{\varpi_{\kappa_i}}(t)\right)=F_{\varpi_{\kappa_i}}{}^*(s)=\frac{1}{s}\left[f^*(s)\right]^{\kappa_i} \tag{3.30}$$

5.) Utilize the constrained least squares model defined by Eq. (3.27) to fit a model to $F_{\varpi_{\kappa_i}}{}^*(s)$ from step 4.

6.) Utilize Jagerman's method, Eq. (3.21), to acquire the inverse Laplace transform $F_{\varpi_{\kappa_i}}(t)=L^{-1}\left[F_{\varpi_{\kappa_i}}{}^*(s)\right]$ from the model established in step 5.

7.) Calculate the stockout probability given $\kappa_i=i$ spares by:

$$\int_0^\infty g(l)F_{\varpi_{\kappa_i}}(l)\,dl=E\left(F_{\varpi_{\kappa_i}}(l)\right)=\lim_{n\to\infty}\frac{1}{n}\sum_{j=1}^{n}F_{\varpi_{\kappa_i}}(l_j)\quad l_j=\left\{t\mid G(t)=U_j(0,1)\right\} \tag{3.31}$$

8.) If the calculated stockout probability from step 7 is less than υ from step 1, the operation should carry $\kappa_i=i$ spares and the solution procedure is complete. If the calculated stockout probability is greater than υ from step 1, return to step 4.

The solution methodology defined above is summarized in Figure 3.9.

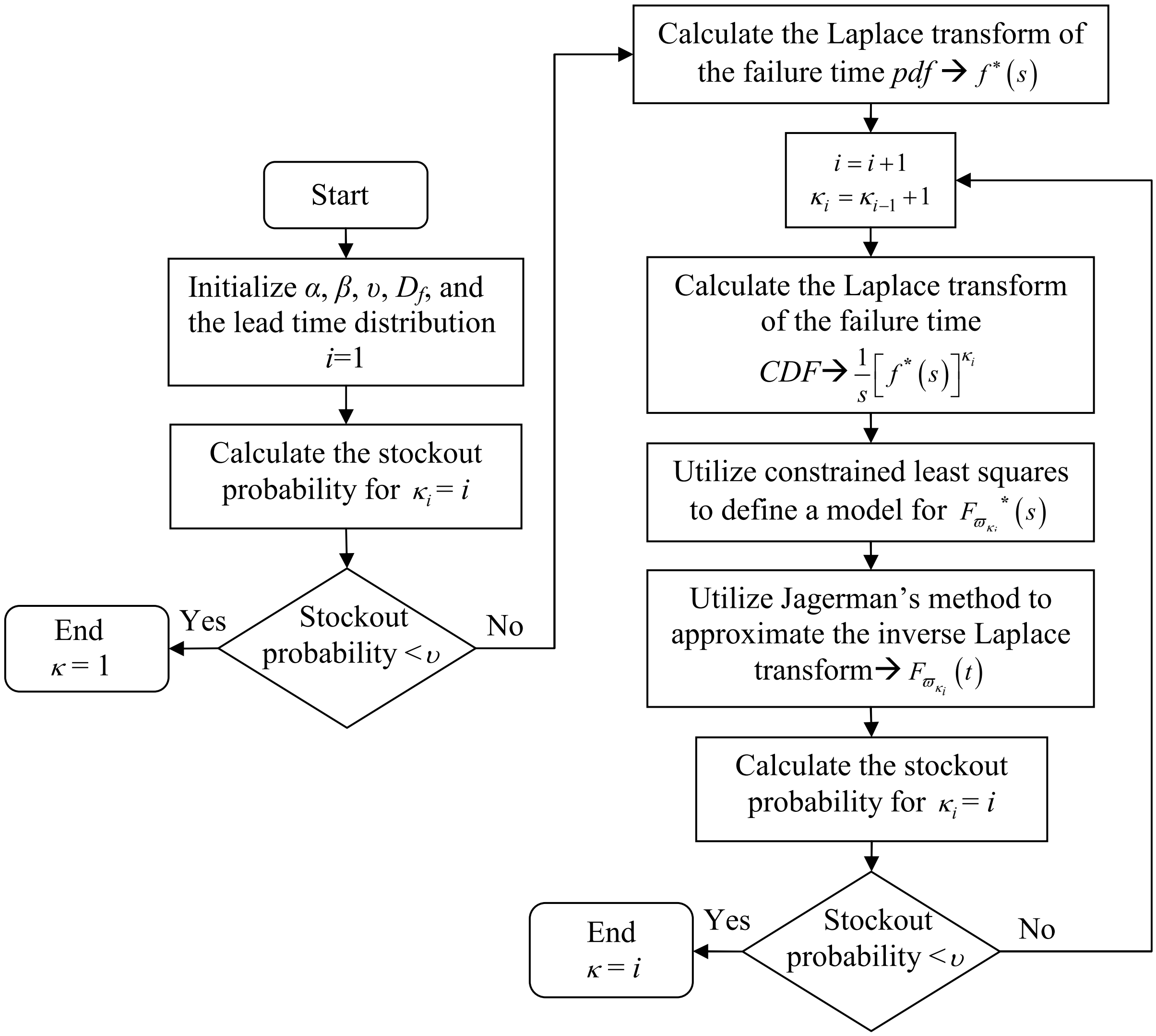

Figure 3.9: Flow chart for solving *Problem L*

3.4 Step Two: Integrated Production Decision Model

A comprehensive production decision model is established that integrates the spare part model with production constraints. The production decision model is based on a cost optimality derivation. Optimization techniques are implemented to determine the appropriate static preventive maintenance degradation threshold for each spare part utilized to process an order.

3.4.1 Assumptions for the Integrated Production Decision Model

The production decision model minimizes the cost of producing an order by establishing the degradation threshold while guaranteeing that the stockout probability constraint is satisfied. The production decision model is established according to the following assumptions:

1.) The objective function is based on a cost derivation and is minimized to generate the preventive maintenance threshold for each spare part. The cost of a unit replacement, the cost of failure, and the cost per unit of backlog is included in the objective function.

2.) Components that run to failure take longer to replace than components that are replaced before failure. The time to replace a component increases exponentially as the degradation state moves toward the failure threshold D_f.

3.) The production rate of the system is fixed. If a distribution for the production rate is provided, the expected value is utilized to define the production rate.

4.) The time available to process an order is known and fixed.

5.) Outstanding orders are processed with the goal of satisfying all orders during the fixed order processing time. Each unit in the order that is not processed during that time period incurs a backorder cost.

6.) The cost for replacement increases exponentially as the degradation state moves toward the failure threshold D_f.

7.) The preventive maintenance threshold is selected to support the stockout constraint in *Problem L*.

8.) The gamma process is the link between the degradation domain and the time domain.

9.) The fixed ordering cost includes the cost of the component along with the administrative costs to generate a spare part order.

10.) Spare parts are replenished using a base stock policy.

The stockout probability constraint in *Problem L* is guaranteed by not satisfying the instantaneous replacement assumption. For the production decision model, there is a replacement time for each maintenance event that extends the total time to stockout during the lead time *L* compared to *Problem L* that has instantaneous replacement. The time extension that occurs in the production decision model, from not assuming instantaneous replacement, maintains or reduces the stockout probability and therefore guarantees *Problem L* is satisfied. For the production decision model, the degradation thresholds are $D_{pm} \leq D_f$. When $D_{pm}=D_f$ the assumptions of *Problem L* are satisfied. The stockout probability α increases when $D_{pm} < D_f$ given an upper failure threshold and decreases when the replacement time is greater than zero. As the maintenance time increases, one can perform more preventive maintenance instead of corrective maintenance. The following theorem states the model relationships between *Problem L* and the production decision model.

Theorem 1 Given that α and β are known, the spare part model guarantees that a feasible solution can be generated for the production decision model which satisfies the stockout probability requirement υ.

Proof. Consider the following conditions for the production decision model:

Condition 1: $D_{pm} \leq D_f$ where D_{pm} is a decision variable, and

Condition 2: The total replacement time over the lead time *L* is defined by $T_M > 0$.

Define the stockout probability constraint as:

$$\boldsymbol{Pr}\left(\sum_{i=1}^{\kappa} T_i + T_M < L\right) = \int_0^{\infty} \boldsymbol{Pr}\left(\varpi'_{\kappa} = \sum_{i=1}^{\kappa} T_i + T_M < L \mid L = l\right) = \int_0^{\infty} g(l) \int_0^{l} f_{\varpi'_{\kappa}}(t)\, dt\, dl = \int_0^{\infty} g(l) F_{\varpi'_{\kappa}}(l) dl = \alpha' \quad (3.32)$$

where the stockout probability is equal to α' and is decreasing function with the addition of the total maintenance time $T_M > 0$. Therefore, $\alpha' \le \alpha \le \upsilon$ indicates that *Problem L* guarantees the stockout probability in the production decision model. □

3.4.2 Integrated Production Decision Model Framework

The conceptual framework of the production decision model is show in Figure 3.10. Each layer in Figure 3.10 is a surface that is defined according the degradation state *D(t)* and the outstanding units *X(t)*.

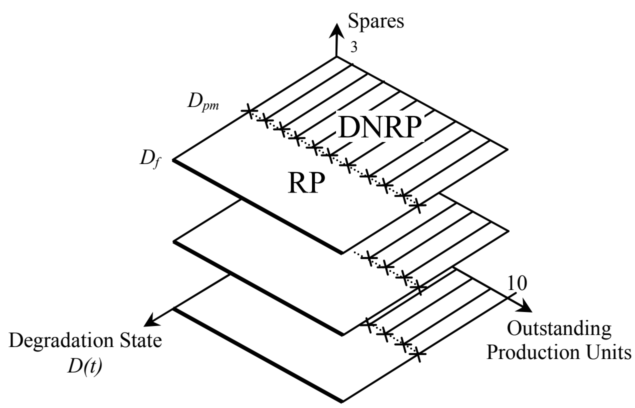

Figure 3.10: Conceptual framework of the production decision model

Figure 3.11 provides a breakout of one of the surfaces in Figure 3.10. The D_{pm} is established to minimize the cost of production while ensuring the spare part model is satisfied.

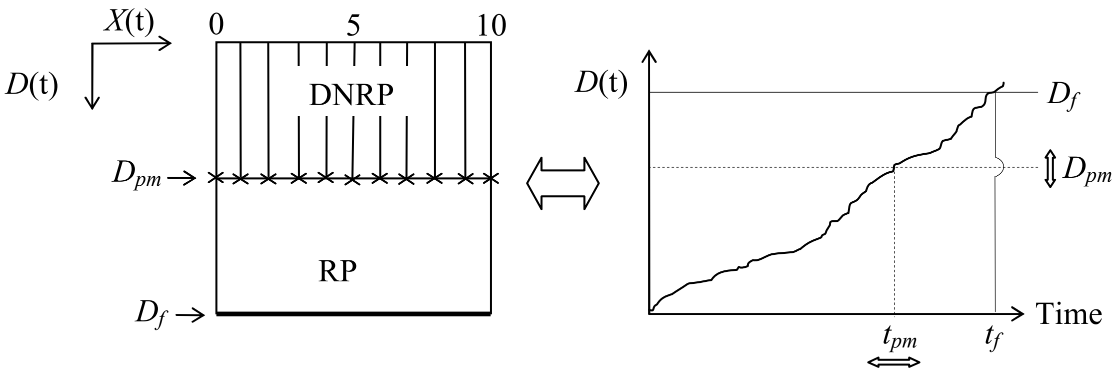

Figure 3.11: Relationship of one surface in Figure 3.10 to the preventive maintenance schedule

Given the number of outstanding units, there is a D_{pm} that generates a boundary between the do not replace decision (DNRP) and the replace decision (RP). For each spare part layer, there is static D_{pm} established that is selected in order to optimize the system by ensuring the spare part inventory constraints are upheld and the cost associated with performing maintenance is minimized.

3.4.3 Spare Part Ordering Policy

The stockout probability is considered a critical performance measure for managing spare parts. *Problem L* generates the minimum number of spare parts required to ensure the stockout probability requirement is satisfied. Given the results of *Problem L*, the ordering policy must be established to replenish spare parts while ensuring that the stockout constraint is satisfied during production. The replenishment of spare parts is managed by utilizing a base stock model. An order for one spare part is generated each time that a spare part is utilized to perform maintenance. Hopp & Spearman (2001) define the base stock model according to the following assumptions:

1.) The product can be analyzed individually with no shared resources,

2.) The demand for the product occurs one at a time and there are no batches,

3.) Demand that is not satisfied incurs a backorder cost and is not lost, 4.) Replenishment of the product occurs one at a time and there is no penalty or restriction on the number of orders that can be placed over time, and

5.) Replenishment lead times are known.

An order is placed when there are r units of spare part inventory on hand and the optimal base stock level is equal to r+1. Therefore, the reorder point for a base stock model is equal to r. *Problem L* establishes the minimum number of spare parts required to satisfy the demand over the lead time for a spare part order while ensuring a specified service level (i.e. stockout probability).

The inventory position is equal to the on-hand inventory minus backorders plus orders. For a base stock model, the inventory position is a constant value of $R=r+1$. For the production decision model, the replenishment of spare parts will occur according to a base stock model. Each time that a demand for a spare part occurs, an order for a spare part will be placed. Therefore, the inventory position will be equal to κ^*+1. An order for a spare part will occur when the number of spare parts on hand equals κ^*.

3.4.4 Optimization Criteria and Model

The preventive maintenance degradation threshold is established by solving the following optimization problem for a production order.

Problem M:

$$\min_{D_{pm}} E\left(c_f\, \boldsymbol{exp}\left(1-\frac{D_f}{D_{pm}}\right) N_{D_{pm}} + \left[c_B \left(Q - \lambda \sum_{p=1}^{N_{D_{pm}}} \Im_{D_{pm}}^{(p)} \right)\right]^{\dagger} + c_{reg} N_{D_{pm}} \right) \tag{3.33}$$

subject to

$$N_{D_{pm}} = \boldsymbol{inf}\left\{ \boldsymbol{sup}\left\{ i \;\middle|\; \left(\sum_{p=1}^{i} \Im_{D_{pm}}^{(p)} + \sum_{p=1}^{i} M_f\, \boldsymbol{exp}\left(1-\frac{D_f}{D_{pm}}\right) + \sum_{p=\kappa^*+2}^{i} \phi_p \right) \le T_Q \right\},\ \boldsymbol{sup}\left\{ j \;\middle|\; \left\lfloor Q - \lambda \sum_{p=1}^{j} \Im_{D_{pm}}^{(p)} \right\rfloor \ge 0 \right\} \right\}$$

$$E\left[\frac{\sum_{p=\kappa^*+2}^{\xi_{D_{pm}}} \partial_p}{\xi_{D_{pm}}}\right] \le \upsilon$$

where:

$$\phi_p = \begin{cases} \boldsymbol{inf}\left\{L_p \mid L_p \ge T_{cum}^{(p)}\right\} - T_{cum}^{(p)} & \text{if } \mathrm{I_p} = 0 \\ 0 \text{ otherwise} \end{cases} \quad \forall p \in \left[\kappa^* + 2, \kappa^* + 3, \ldots, N_{D_{pm}}\right]$$

$$\partial_p = \begin{cases} 1 \text{ if } \mathrm{L_p} > \kappa^* E\left(\Im_{D_{pm}}\right) \\ 0 \text{ otherwise} \end{cases}$$

λ =fixed production processing rate,

Q =production quantity that is to be completed during the available production time T_Q,

c_f =worst case cost of performing maintenance,

c_B =cost per unit of production not processed during T_Q,

c_{reg} =cost to procure one spare part,

υ =minimum stockout probability requirement,

$\xi_{D_{pm}}$ =total number of part replenishment orders,

$N_{D_{pm}}$ =total number of parts utilized during T_Q,

L_p =clock time to receive part p, $p \in \left[\kappa^* + 2, \kappa^* + 3, \ldots, \xi_{D_{pm}}\right]$

κ^* =number of spare parts required to satisfy the stockout probability requirement,

I_p =on hand inventory after part p is removed from the machine, $p \in \left[1, 2, \ldots, N_{D_{pm}}\right]$

D_{pm} =preventive maintenance threshold,

D_f =failure threshold,

T_Q=available production time to complete an order,

$\Im_{D_{pm}}^{(p)}$=total operational time for part p, $p \in \left[1, 2, \ldots, N_{D_{pm}}\right]$,

$T_{cum}^{(p)}$=total accumulated clock time when part p is removed from the machine, $p \in \left[1, 2, \ldots, N_{D_{pm}}\right]$,

M_f=maximum time required to perform corrective maintenance

3.4.5 Solution Procedure for the Production Decision Model

Problem M is solved using simulation based optimization due to the complexity of the problem and the interdependencies that occur between the model variables and D_{pm}. A MATLAB program was developed to determine the optimal degradation threshold. The solution methodology is summarized in Figure 3.12 and the MATLAB code is provide in the Appendix.

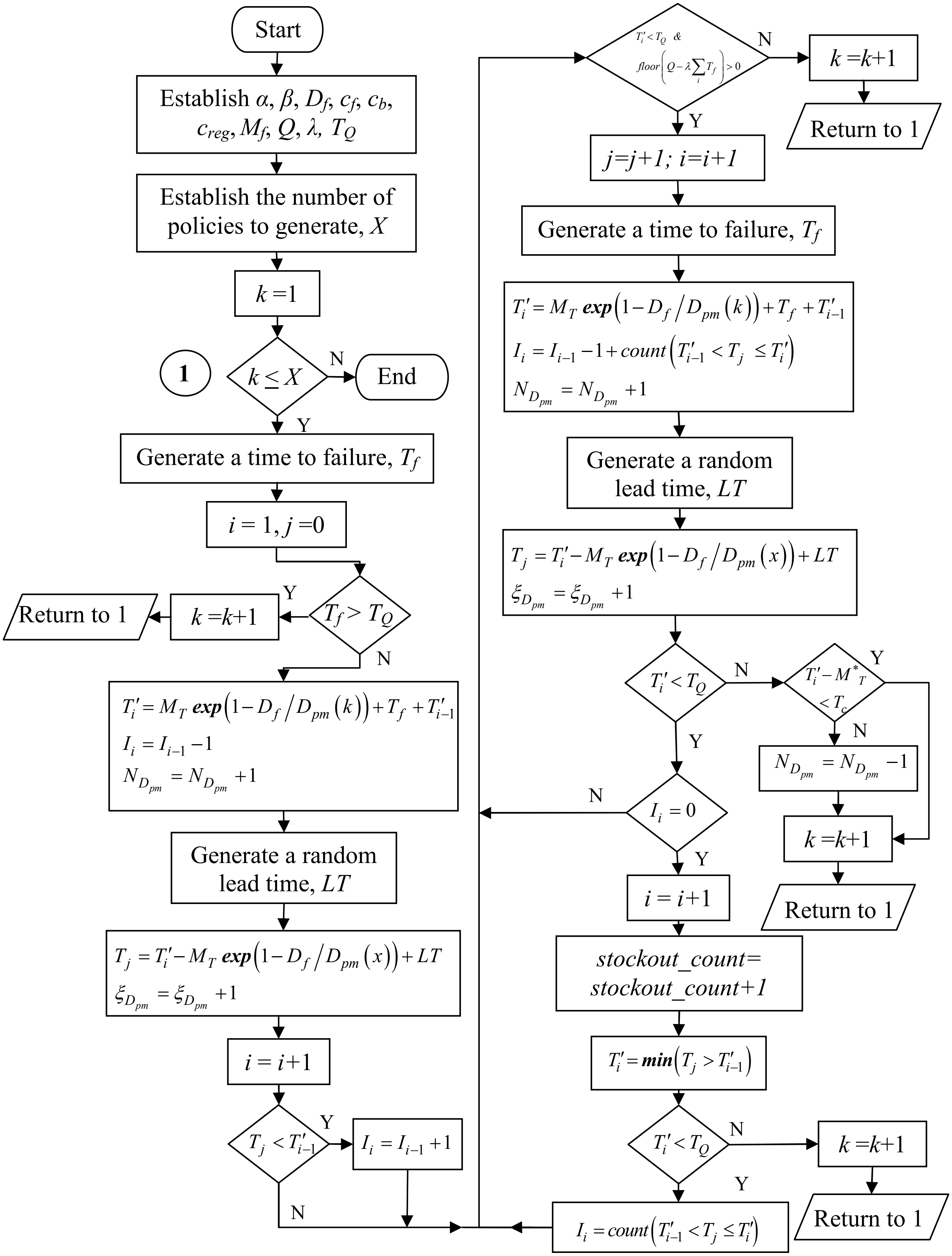

Figure 3.12: Flow chart for solving *Problem M*

The MATLAB code generates a large number of iterations for each maintenance policy (D_{pm}) that are averaged to obtain one value for the objective function. For each D_{pm}, there is an objective function value and an associated stockout probability value. The stockout probability value is generated to ensure that the maintenance policy supports the stockout probability constraint from *Problem L*. The simulation generates many degradation thresholds starting at the failure threshold D_f and decreasing toward zero. The minimum cost maintenance policy (D_{pm}) that supports the stockout probability is selected as the optimal condition based maintenance policy.

3.5 Step Three: Bayesian Analysis and Remaining Useful Life

The third step in the development of the overall model is to develop a Bayesian approach to update the degradation model. Bayesian analysis provides an effective approach to address uncertainty in model development since parameter estimates can be updated as additional degradation data concerning the process is obtained. In the development of the degradation model, the parameters of the gamma process are defined with previous knowledge obtained during product development testing. The parameters established during testing for the gamma process can have a degree of uncertainty that can be reduced with Bayesian analysis. Degradation model parameters are updated as additional degradation data is obtained over time.

Degradation measurements are collected during processing for each spare part to update the degradation process for each individual part. The inspection interval for obtaining a degradation measure for the Bayesian update is assumed to be known and fixed. Given the continual update of the gamma process parameters, the remaining useful life estimate is also updated to represent the additional knowledge gained by the collection of degradation data.

3.5.1 Bayes Theorem

Definition 3.2*—Bayes Theorem*

Let θ be a variable of interest such as a statistical distribution parameter, failure rate, mean time to failure, etc. Given θ is a continuous variable, let $h(\theta)$ be a prior distribution of the model parameter. Let $l(data \mid \theta)$ be the likelihood function of the parameter with respect to the newly observed data. According to Bayes Theorem, the posterior distribution of θ when new observations are obtained can be expressed as:

$$f(\theta \mid data) = \frac{h(\theta)\ l(data \mid \theta)}{\int_{-\infty}^{\infty} h(\theta)\ l(data \mid \theta)\ d\theta} \tag{3.34}$$

Ignoring constant terms in Eq. (3.34) provides the proportional posterior *pdf* expressed as:

$$f(\theta \mid data) \propto h(\theta)\ l(data \mid \theta) \tag{3.35}$$

The Bayesian approach updates the prior distribution with additional information gained from data to generate the posterior distribution. Performing Bayesian analysis involves constructing a likelihood function for the distribution of interest, developing estimates of the prior distribution parameters, and estimating the posterior distribution of the parameters.

3.5.2 Conjugate Prior for the Gamma Process

Definition 3.3*—Conjugate Prior Distribution* (Degroot, 1970) & (Martz & Waller, 1982)

For the parameter θ, there is a standard family of distributions that obey the following property: Suppose the prior distribution of θ belongs to this family, then the posterior distribution must belong to the same family given any arbitrary sample size *n* of observations. The family is called a conjugate family of distributions. In other words, given the conjugate prior distribution $g(\theta)$ and the posterior distribution $g(\theta \mid data)$, the prior and posterior are members of the same family of distributions.

The utilization of the conjugate prior provides analytical tractability in the derivation and solution of Eq. (3.34). An arbitrary selection of the prior distribution can generate an integral in the denominator of Eq. (3.34) that may not be solved analytically. By selecting a conjugate prior distribution, the posterior distribution is defined according to the same distribution class. The parameters of the prior distribution are called hyperparameters if they are not fixed at a particular value (Gelman, Carlin, Stern, & Rubin, 1995). Eq. (3.35) is utilized to develop the posterior distributions that are updated to refine the estimate of the parameter θ as additional data is collected. Consult Arnold, Castillo, & Sarabia (1998), Miller (1980), Lwin & Singh (1974), and Demsleth (1975) for additional discussion on applications of conjugate priors for the gamma distribution.

3.5.3 Bayesian Analysis for the Gamma Process

Assume the performance or physical damage (degradation) of a system follows Eq. (3.1) where α is a known constant representing the shape parameter and β is the inverse scale parameter. Assume β is a random variable and consider the conjugate prior distribution for β:

$$h\left(\beta;\alpha_0,\beta_0\right)=\frac{\beta^{\alpha_0-1}\,\boldsymbol{exp}\left(-\beta_0\beta\right)\beta_0^{\alpha_0}}{\Gamma\left(\alpha_0\right)} \tag{3.36}$$

The prior hyperparameters for Eq. (3.36) are defined by α_0 and β_0 which represent the shape parameter and inverse scale parameter of β, respectively.

To define Eq. (3.35) with respect to the gamma process, the likelihood function is expressed as:

$$L\left(all\ data\ d_i \mid \alpha,\beta\right)=\prod_{i=1}^{n} f\left(d_i\right)=\prod_{i=1}^{n}\frac{d_i^{\alpha\cdot\left(t_i-t_{i-1}\right)-1}\,\boldsymbol{exp}\left(-\beta d_i\right)\beta^{\alpha\cdot\left(t_i-t_{i-1}\right)}}{\Gamma\left(\alpha\cdot\left(t_i-t_{i-1}\right)\right)} \tag{3.37}$$

Simplifying Eq. (3.37) yields:

$$L\left(all\ data\ d_i \mid \alpha,\beta\right)=\left[\frac{d_1^{\alpha\cdot(t_1-t_0)-1}\,\boldsymbol{exp}\left(-\beta d_1\right)\beta^{\alpha\cdot(t_1-t_0)}}{\Gamma\left(\alpha\cdot(t_1-t_0)\right)}\right]\times\ldots\times$$

$$\left[\frac{d_2^{\alpha\cdot(t_2-t_1)-1}\,\boldsymbol{exp}\left(-\beta d_2\right)\beta^{\alpha\cdot(t_2-t_1)}}{\Gamma\left(\alpha\cdot(t_2-t_1)\right)}\right]\times\cdots\times\left[\frac{d_n^{\alpha\cdot(t_n-t_{n-1})-1}\,\boldsymbol{exp}\left(-\beta d_n\right)\beta^{\alpha\cdot(t_n-t_{n-1})}}{\Gamma\left(\alpha\cdot(t_n-t_{n-1})\right)}\right] \quad (3.38)$$

$$=\left[\frac{\prod_{i=1}^{n}\left(d_i^{\alpha\cdot(t_i-t_{i-1})-1}\right)\times\boldsymbol{exp}\left(-\beta\sum_{i=1}^{n}d_i\right)\times\beta^{\alpha\cdot(t_n-t_0)}}{\prod_{i=1}^{n}\Gamma\left(\alpha\cdot(t_i-t_{i-1})\right)}\right]$$

Then the posterior distributions of α and β can be obtained from the following equation:

$$f\left(\beta \mid all\ data\ d_i\right)\propto\left[\frac{\prod_{i=1}^{n}\left(d_i^{\alpha\cdot(t_i-t_{i-1})-1}\right)\times\boldsymbol{exp}\left(-\beta\sum_{i=1}^{n}d_i\right)\times\beta^{\alpha\cdot(t_n-t_0)}}{\prod_{i=1}^{n}\Gamma\left(\alpha\cdot(t_i-t_{i-1})\right)}\right]\left[\frac{\beta^{\alpha_0-1}\,\boldsymbol{exp}\left(-\beta_0\beta\right)\beta_0^{\alpha_0}}{\Gamma\left(\alpha_0\right)}\right]$$

$$\propto\left[\frac{\beta_0^{\alpha_0}\prod_{i=1}^{n}\left(d_i^{\alpha\cdot(t_i-t_{i-1})-1}\right)\times\boldsymbol{exp}\left(-\beta\sum_{i=1}^{n}d_i\right)\times\beta^{\alpha\cdot(t_n-t_0)}\beta^{\alpha_0-1}\,\boldsymbol{exp}\left(-\beta_0\beta\right)}{\Gamma\left(\alpha_0\right)\prod_{i=1}^{n}\Gamma\left(\alpha\cdot(t_i-t_{i-1})\right)}\right] \quad (3.39)$$

From Eq. (3.39), rearranging the coefficients provides the functional form of the posterior distribution shown in Eq. (3.40).

$$f\left(\beta \mid all\ data\ d_i\right)\propto\boldsymbol{exp}\left(-\beta\sum_{i=1}^{n}d_i-\beta_0\beta\right)\times\beta^{\alpha\cdot(t_n-t_0)+\alpha_0-1}$$

$$\propto\boldsymbol{exp}\left(-\beta\left(\sum_{i=1}^{n}d_i+\beta_0\right)\right)\times\beta^{\alpha\cdot(t_n-t_0)+\alpha_0-1} \quad (3.40)$$

The posterior distribution follows a gamma distribution which is a result of the conjugate prior distribution properties. The posterior hyperparameters for the gamma process with known α are revealed from Eq. (3.40) as:

$$\alpha_0'=\alpha\cdot\left(t_n-t_0\right)+\alpha_0 \quad (3.41)$$

$$\beta_0' = \beta_0 + \sum_{i=1}^{n} d_i \tag{3.42}$$

The final posterior *pdf* for β is given by:

$$f\left(\beta \mid all\ data\ d_i\right) = \frac{\beta^{\alpha_0'-1}\, \boldsymbol{exp}\left(-\beta_0'\beta\right) \beta_0'^{\alpha_0'}}{\Gamma\left(\alpha_0'\right)} \tag{3.43}$$

During condition monitoring, as each new degradation observation is collected, the *pdf* expressed in Eq. (3.43) is updated through the posterior hyperparameters. Each new degradation measurement represents additional knowledge regarding the true β value. The continual update of β provides a refined degradation model that is utilized to make prediction of the remaining useful life of the system. Figure 3.13 depicts the Bayesian update process for the *pdf* of β utilizing Eq.s (3.41) through (3.43). Each *pdf* curve in Figure 3.13 represents the additional knowledge acquired from one degradation measurement. As additional degradation data is collected, the Bayesian update of the *pdf* for β converges to the actual values. The Bayesian approach reduces the uncertainty in the initial model parameters since additional data is utilized to update the initial parameter estimates.

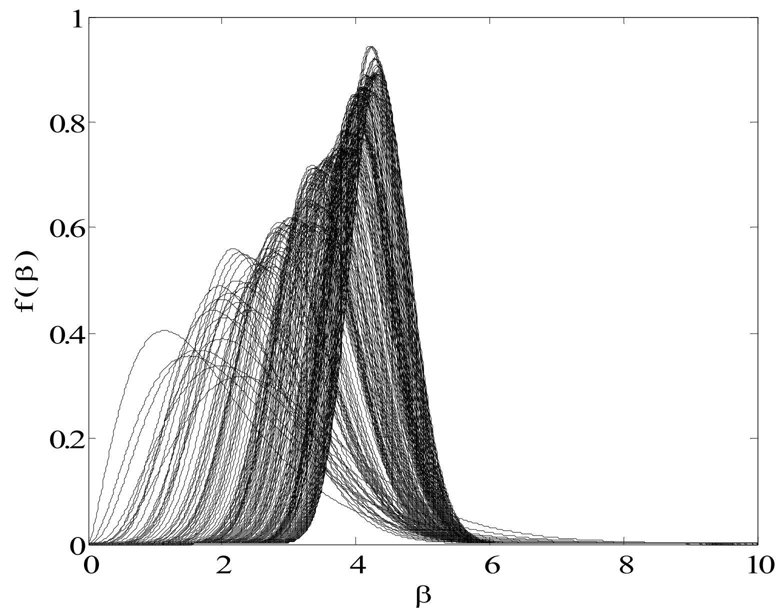

Figure 3.13: β posterior *pdf* update using Bayesian analysis (α=0.5, α_o=2, β_o=1.3)

3.5.4 Remaining Useful Life for the Gamma Process

The posterior hyperparameters defined by Eq. (3.41) and (3.42) update the *pdf* for the inverse scale parameter β as each new degradation measurement is obtained from the degradation process. Each time that Eq. (3.43) is updated, the *RUL* estimate for the system is also updated to reflect the additional knowledge about parameters gained from the degradation data. The *CDF* of the *RUL* is described by:

$$F\left(t \mid all\ data\ d_i\right) = \int_0^\infty F\left(t \mid \beta = \beta''\right) f\left(\beta''\right) d\beta'' \qquad (3.44)$$

Substituting Eq. (3.3) and (3.43) provides the *RUL* for the failure time *CDF* of the gamma process:

$$F\left(t \mid all\ data\ d_i\right) = \int_0^\infty \frac{\Gamma\left(\alpha t, \beta''\left[D_F - \left(x_0 + \sum d_i\right)\right]\right)}{\Gamma\left(\alpha t\right)} \frac{\beta''^{\alpha_0'-1} \boldsymbol{exp}\left(-\beta_0 \beta''\right) \beta_0'^{\alpha_0'}}{\Gamma\left(\alpha_0'\right)} d\beta'' \quad t > 0 \qquad (3.45)$$

Eq. (3.45) updates the *RUL* of the component as additional degradation data is collected. The *RUL* update is caused by the Bayesian update of the parameter β that is represented in Figure

3.13. As the posterior *pdf* of β converges, the *RUL* curves shift to represent the parameter update provided by Bayes theorem. Figure 3.14 illustrates the shift in the CDF as degradation data is collected that subsequently updates the *RUL*.

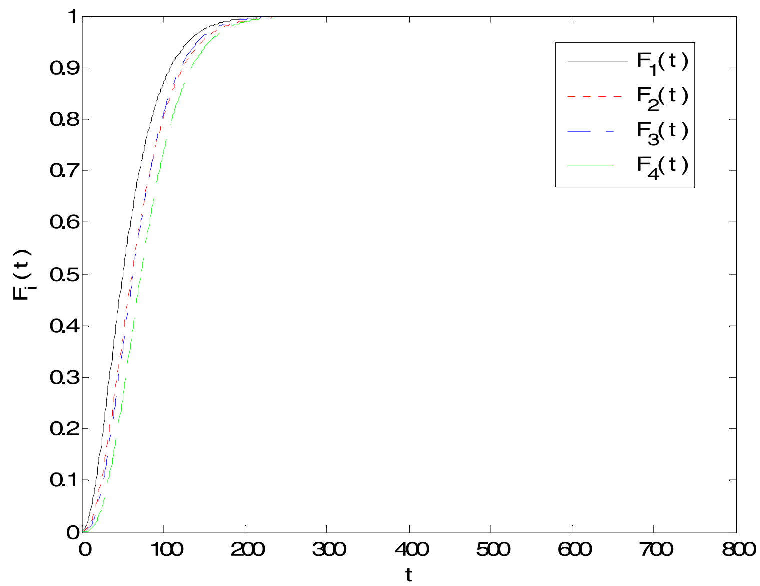

Figure 3.14: *RUL* distribution update (α=0.5, α_o=2, β_o=1.3)

3.6 Summary of the Methodology

Chapter 3 has presented a three step methodology to conduct condition based maintenance, spare parts inventory control, and production management. The methodology requires establishing the number of spare parts to ensure a specified stockout probability which is followed by a comprehensive production decision model that generates a static preventive maintenance threshold policy. The methodology culminates into the integration of the gamma process and Bayesian analysis to update the *RUL*. Figure 3.15 provides a review of the methodology framework.

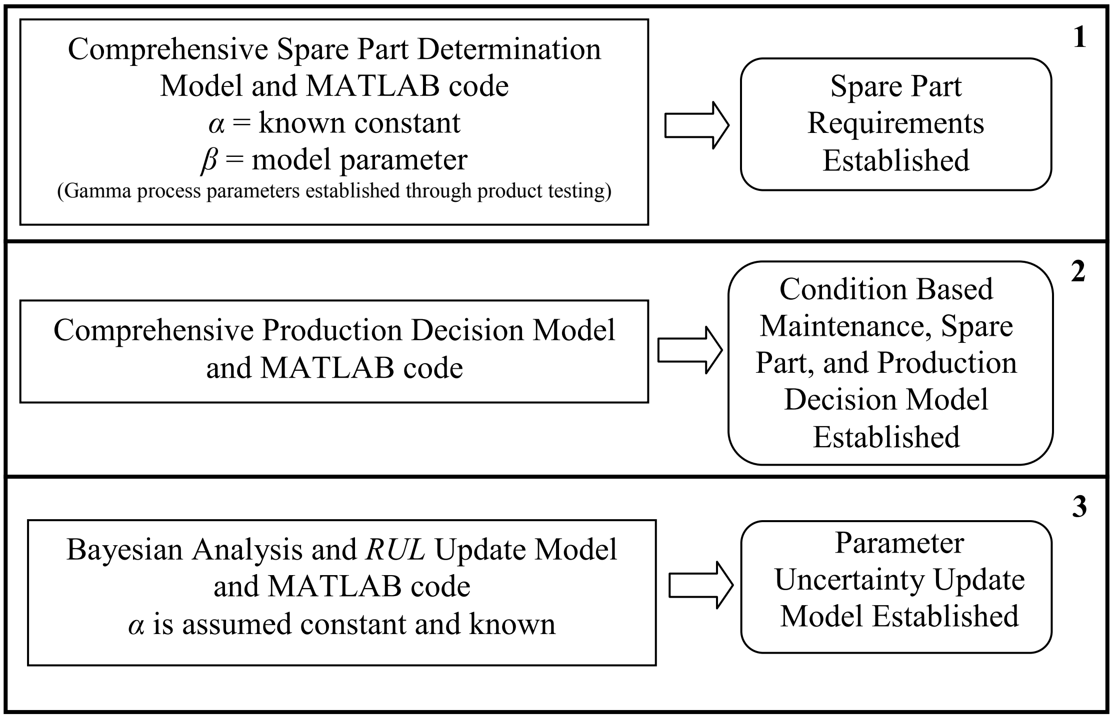

Figure 3.15: Methodology overview

CHAPTER 4

ANALYSIS & DISCUSSION

4.1 Introduction

The methodology developed in Chapter 3 is based on a three-step development process that includes the spare part model, a production decision model, and a Bayesian analysis/*RUL* update procedure. Within Chapter 4, a comprehensive example will be presented that applies the methodology generated in Chapter 3. The structure of Chapter 4 will be as follows:

1.) The spare part model will be utilized to generate the optimal number of spare parts required to maintain an established stockout probability.

2.) Upon the completion of Step 1, the production decision model will be utilized to establish the optimal degradation threshold which minimizes the cost to process an order.

3.) Finally, the degradation parameters for a component will be updated using the Bayesian model as real time degradation data is acquired.

4.2 Problem Statement

Suppose a critical component in a manufacturing system has been identified for the implementation of condition based maintenance. Chapter 2 provided several methods for identifying critical components in a system that require preventive maintenance strategies such as condition based maintenance. Upon the identification of the component, a reliability test is developed to ascertain the degradation behavior of the component to be utilized for condition based maintenance. Consider that the component in the test exhibits a degradation behavior which is strictly increasing, positive, and defined according to the gamma process. There are a number of components that exhibit a degradation behavior that is defined according to the gamma process and are provided in Chapter 2. Once the test is conducted, the shape parameter α and the inverse scale parameter β are defined from the degradation process. Table 4.1 provides values that will be utilized to illustrate the model developments provided in Chapter 3.

Table 4.1: Model Parameter Values

Model Variable	Value	Description
α	0.7	Shape parameter defined from product testing for the gamma process
β	0.006	Inverse scale parameter defined from product testing for the gamma process
D_f	45	Failure threshold defined by the maintenance engineer where unacceptable system performance occurs
υ	0.1	Minimum stockout probability requirement
$g(l)$	Logn(0.02, 0.05)	Lead time distribution to receive a spare part
α_0	0.05	Prior hyperparameter representing the shape parameter
β_0	25	Prior hyperparameter representing the inverse scale parameter
c_f	\$750	Worst case cost of performing maintenance
c_B	\$450	Cost per unit of production not processed in the available production period
c_{reg}	\$550	Cost to procure one spare part
Q	35	Production quantity that is to be completed during the available production time
λ	3.25	Production processing rate
M_f	0.4	Maximum time required to perform maintenance
T_Q	10	Available production time to complete an order

4.3 Spare Part Model

Problem L, defined in Chapter 3, is solved to determine the optimal number of spare parts κ^* required to guarantee the stockout probability requirement in Table 4.1. MATLAB code, provided in the Appendix, is executed using the gamma process variables α and β, the failure threshold D_f, the *lognormal(0.02, 0.05)* lead time distribution, and the stockout probability requirement υ. The spare part model provides the optimal number of spare parts required to ensure that the minimum stockout probability is satisfied. Once the model parameters are initialized, the MATLAB code executes according to the solution procedure defined in Figure 3.9. The MATLAB code first determines whether carrying the part on the machine is sufficient to satisfy the stockout probability requirement utilizing Eq. (3.28). The code will proceed to calculate the Laplace transform of the failure time probability density function using Eq. (3.29) if the results of Eq. (3.28) do not satisfy the minimum stockout probability requirement. The Laplace transform is then utilized to determine the failure time *CDF* using Eq. (3.30) and constrained regression. Jagerman numerical inversion is then used to generate the inverse Laplace transform which is utilized in Eq. (3.31) to calculate the stockout probability. The above iterative process continues until the stockout probability is satisfied and the minimum number of spare parts is obtained. Table 4.2 provides the stockout probability values generated from *Problem L*.

Table 4.2: Stockout probabilities from the spare part model

κ	Stockout Probability
1	0.613
2	0.215
3*	0.060

The optimum number of spare parts required to satisfy the stockout probability requirement for υ=0.1 is found to be κ^*=3. The first iteration, using Eq. (3.28), determined that the stockout probability could not be satisfied when κ=1. Since the stockout probability could not be satisfied when κ=1, the MATLAB code proceeds to calculate the Laplace transform of the failure time *CDF* using Eq. (3.30). Eq. (3.27) is utilized to generate a continuous function from the discrete Laplace transform estimates of Eq. (3.30). The parameter vector in Eq. (3.27) is defined by $\eta = (b_1, b_2, b_3, b_4)$ and solved to minimize the mean squared error between Eq. (4.1) and the results of Eq. (3.30).

$$\hat{f}\left(s_j;\eta\right)=\frac{b_1 s_j^2+b_2 s_j}{s_j^4+b_3 s_j^3+b_4 s_j^2} \tag{4.1}$$

For $\kappa > 1$, the MATLAB code generates the Laplace transform of the failure time *CDF* and utilizes constrained regression to acquire values for the parameter vector η. Figure 4.1 and Figure 4.3 plot the calculated Laplace transform, $F_{\varpi_{\kappa_i}}{}^*(s)$, and the constrained regression results for $\kappa = 2$ and $\kappa = 3$, respectively. Figure 4.2 and Figure 4.4 represent the difference between the calculated Laplace transform $F_{\varpi_{\kappa_i}}{}^*(s)$ and the constrained regression results for $\kappa = 2$ and $\kappa = 3$, respectively.

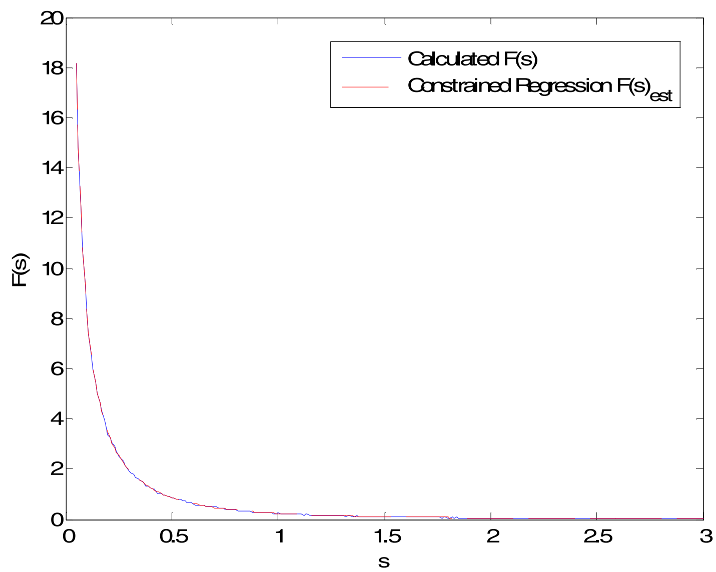

Figure 4.1: Calculated Laplace transform and constrained regression—κ=2

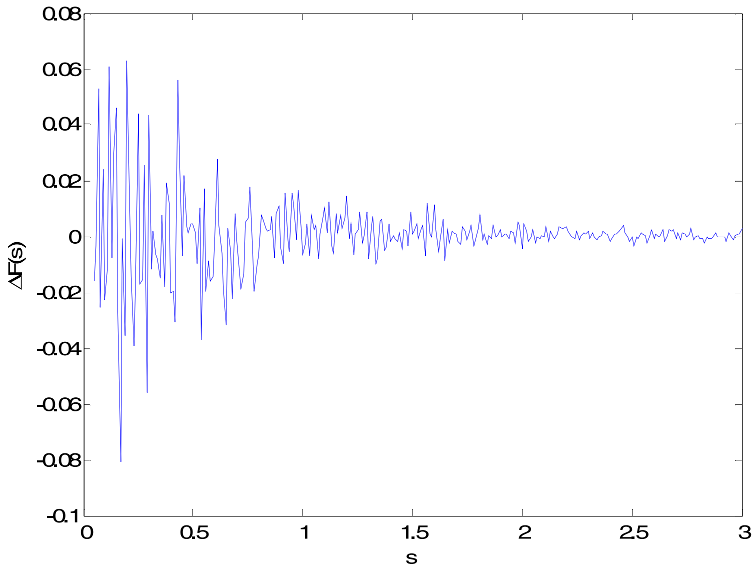

Figure 4.2: Delta between the calculated Laplace transform and constrained regression—κ=2

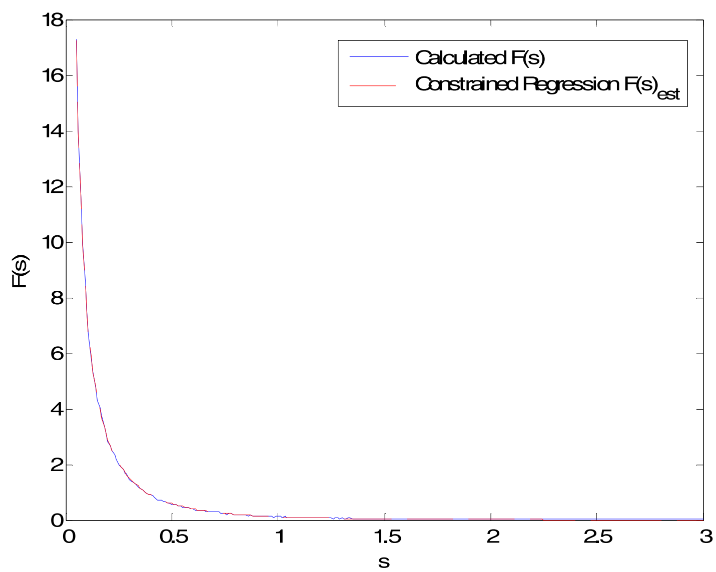

Figure 4.3: Calculated Laplace transform and constrained regression—κ=3

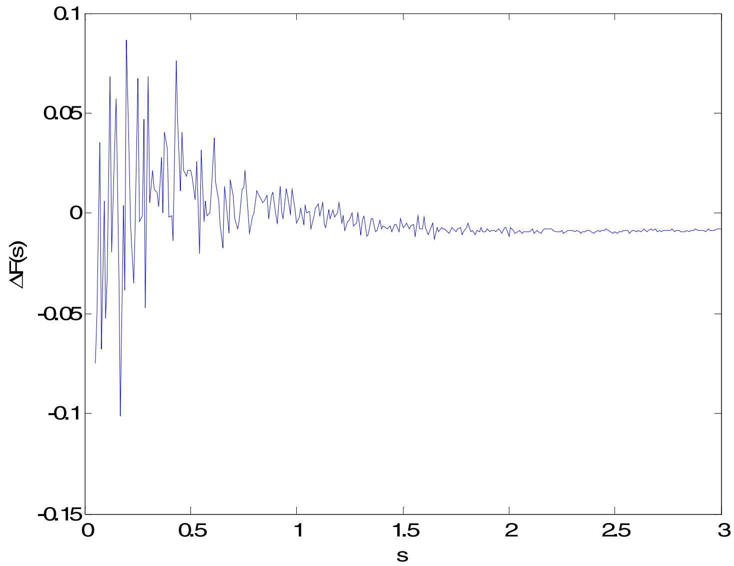

Figure 4.4: Delta between the calculated Laplace transform and constrained regression—κ=3

The MATLAB code proceeds to generated the inverse Laplace transform using Eq. (3.21) and the constrained regression results in Figure 4.1 and Figure 4.3. The inverse Laplace transform generates $F_{\varpi_{\kappa_i}}(t)$ which represents the *CDF* for the sum of κ failure times.

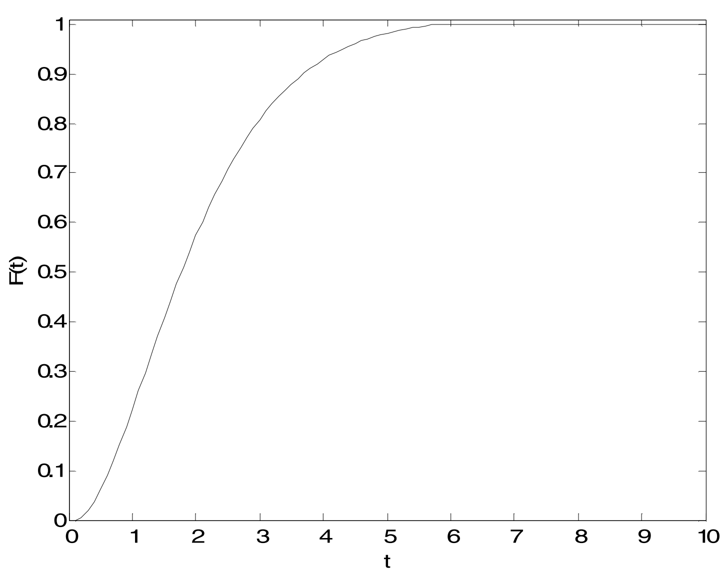

Figure 4.5: Jagerman *CDF* estimate—κ=2

Figure 4.5 and Figure 4.6 provide the results of Jagerman's method, $F_{\varpi_{\kappa_i}}(t)$, and are utilized to calculate the stockout probability values in Table 4.2. Eq. (3.31) calculates the stockout probabilities for $\kappa = 2$ and $\kappa = 3$ using the results in Figure 4.5 and Figure 4.6.

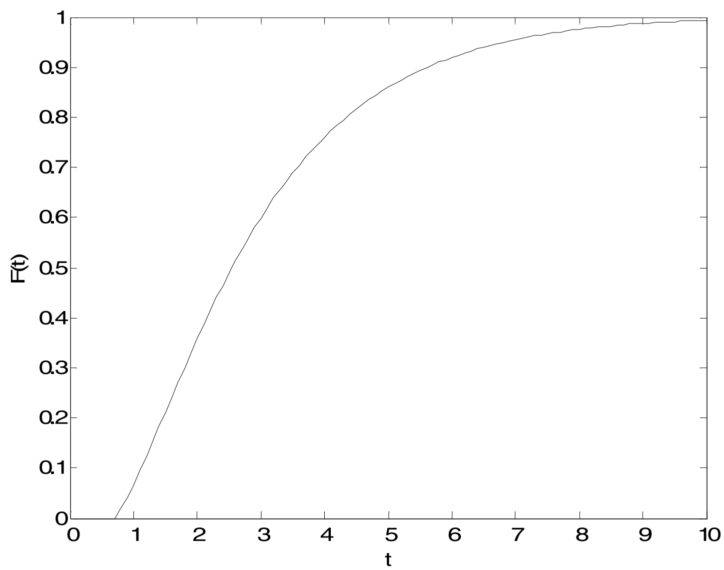

Figure 4.6: Jagerman *CDF* estimate—κ=3

4.4 Production Decision Model

Problem M, defined in Chapter 3, is solved to determine the optimal static preventive maintenance threshold given the model parameters in Table 4.1. MATLAB code, provided in the Appendix, is executed to generate the total average cost, the average failure cost, the average backorder cost, the average regular order cost, the average stockout probability, the average unavailability, and the cost standard deviation given each maintenance policy for processing an order. Once the model parameters are initialized, the MATLAB code executes according to the solution procedure defined in Figure 3.12. The optimal condition based maintenance policy is established from the model results.

Solving *Problem M*, using the parameter values in Table 4.1, generates the average total cost, cost standard deviation, average stockout probability, and average unavailability for incremental degradation thresholds which are provided in Table 4.3. The objective function and constraints are modeled in MATLAB and 2,000 iterations are calculated for each maintenance policy. The optimal degradation threshold that upholds the stockout probability is 13. Therefore, the component should be removed preventively from the system when the degradation is equal to 13 which results in a stockout probability of 0.098 and a system unavailability of 0.092.

Table 4.3: Results of Problem M

Degradation	Average Total Cost ($)	Cost Standard Deviation ($)	Average Stockout Probability	Average Unavailability
45	14048.80	3133.39	0.000	0.290
43	14052.31	3040.20	0.000	0.286
41	14049.24	3134.76	0.000	0.282
39	13940.98	3121.70	0.000	0.274
37	13865.64	2944.54	0.000	0.266
35	13705.77	3026.24	0.000	0.257
33	13640.13	2999.63	0.000	0.247
31	13448.94	2880.59	0.000	0.236
29	13323.83	2800.67	0.000	0.224
27	13111.51	2859.63	0.001	0.209
25	12916.57	2740.27	0.003	0.194
23	12692.88	2744.05	0.008	0.176
21	12423.20	2586.48	0.017	0.157
19	12121.26	2514.30	0.027	0.137
17	11902.30	2414.38	0.047	0.119
15	11755.16	2316.45	0.071	0.103
13	**11653.60**	**2341.95**	**0.098**	**0.092**
11	11759.53	2279.84	0.125	0.085
9	12027.56	2321.04	0.154	0.087
7	12646.05	2370.85	0.179	0.097

Figure 4.7 provides a pictorial representation of the optimal condition based maintenance policy considering the average total cost which is comprised of the average failure cost, the average

backorder cost, and the average regular order cost. The minimum average total cost that satisfies the stockout probability requirement represents the optimal condition based maintenance policy.

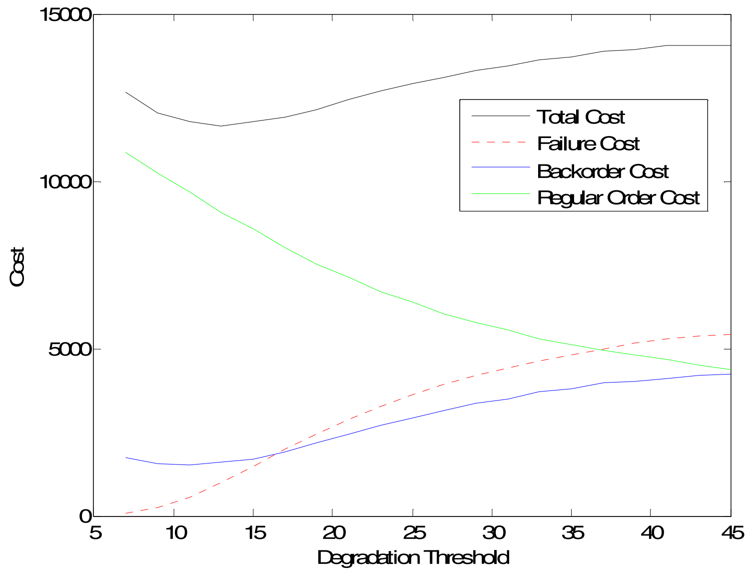

Figure 4.7: Cost curves in *Problem M*

4.5 Bayesian Analysis and Remaining Useful Life Update

Bayesian analysis allows for the use of actual degradation measurements to update the gamma process parameters in order to provide a better estimate of the degradation process for each component during processing. The degradation process can be updated to reflect the additional system state knowledge each time that a degradation reading is obtained. The Bayes update procedure, defined in Chapter 3, can be implemented given the prior hyperparameters in Table 4.1. The posterior hyperparameters are defined by Eq. (3.41) and Eq. (3.42) to characterize the posterior *pdf* of β expressed in Eq. (3.43).

MATLAB code, provided in the Appendix, is executed to define the posterior *pdf* of β given the gamma process variables α and β and the posterior hyperparameters α_0 and β_0. Consider that degradation readings of 0.0137 and 6.1162 are obtained from a part while processing an order. Utilizing Eq. (3.41) and Eq. (3.42), the posterior hyperparameter of β is obtained and the posterior *pdf* of β is shown in Figure 4.8.

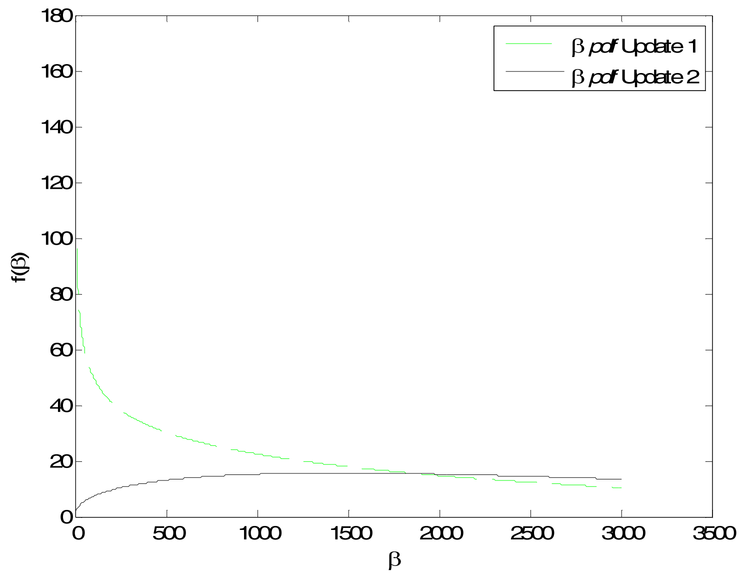

Figure 4.8: Posterior *pdf* update of β

The *RUL* is generated for each degradation point obtained from the component. Figure 4.9 provides the *RUL* of the component given the degradation data and the prior hyperparameters of β provided in Table 4.1. Each component will have a unique actual degradation process that is modeled from the shape and inverse scale parameters defined in Table 4.1. The degradation readings will be utilized to adjust and improve the degradation model for the component during processing and thus provide an updated *RUL* distribution. The degradation data obtained to generate Figure 4.9 reveals that the component has a slower rate of degradation than the initial estimate since the *RUL* is to the right of the initial *CDF* for the component. Therefore, the Bayes update procedure provides a powerful means to utilize additional information to refine and improve the degradation process for each component while processing a production order.

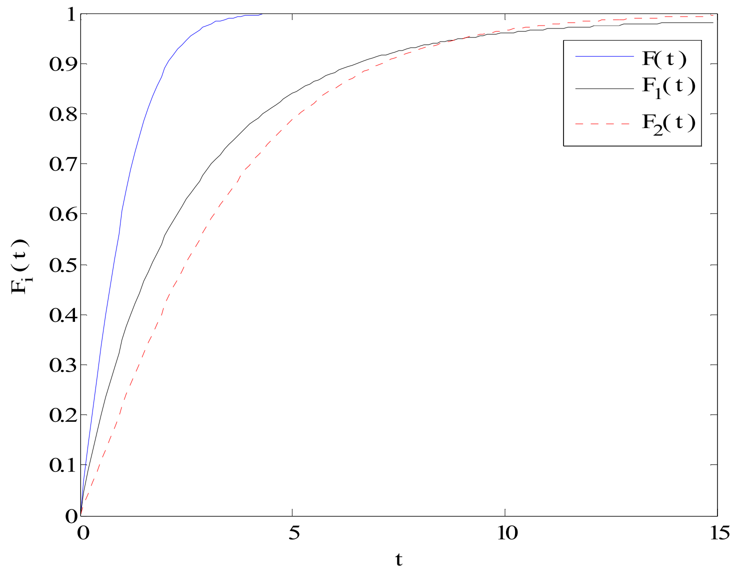

Figure 4.9: Updated *RUL* distribution

4.6 Summary and Discussion

Chapter 4 has demonstrated a comprehensive example that utilizes the methodology and framework provided in Chapter 3. This example provides a clear illustration of the results and the overall outcome of this research. Using the model parameters in Table 4.1, the optimal number of spare parts to ensure the stockout probability is $\kappa^*=3$ and the optimal degradation threshold is 13. There are many different cases that could be generated and this chapter provides one example to promote an overall understanding of the model development and execution. Chapter 5 will provide a summary of the model developments along with a summary of areas for future research extensions.

CHAPTER 5

CONCLUSION

5.1 Conclusion

This thesis addresses the relationship between condition based maintenance and spare part inventory control. Solution procedures have been developed to solve two optimization problems which utilize the gamma process to describe the degradation path. The results of the derivations include the optimal number of spare parts to hold in inventory to guarantee a specified stockout probability and the optimal condition based maintenance degradation threshold. Along with the optimization problems, Bayes analysis has been integrated into the overall model to address degradation parameter uncertainty. The results of this thesis provide a means for a production manager to effectively perform maintenance on a machine and at the same time manage the spare part inventories required for the maintenance activities.

One can see that the methodology developed in this thesis can provide a comprehensive condition based maintenance and spare part inventory control model to utilize in production. The current body of literature does not effectively address the relationship between continuous state/time degradation models, condition based maintenance, and spare part inventory control. This work is important in the area of reliability engineering since it integrates these approaches into a comprehensive model and provides solution procedures that reflect constraints in a production environment.

5.2 Future Research

There are a number of opportunities to extend the work presented in this thesis. Opportunities for extending this thesis include

1.) Generating maintenance policies with dynamic degradation thresholds, 2.) Allowing for expedited spare part orders,

3.) Multiple machines with one spare part inventory pool, and

4.) Optimizing the time between degradation observations in the Bayes update procedure.

The maintenance policy generated in this thesis results in a static preventive maintenance threshold for all spare parts utilized to produce a production order. Additional research could extend this model to allow for a dynamic degradation threshold that changed according to the system state. The dynamic threshold would still maintain the stockout probability constraint while minimizing the cost of performing maintenance. Figure 5.1 and Figure 5.2 provide the conceptual model for the dynamic threshold case.

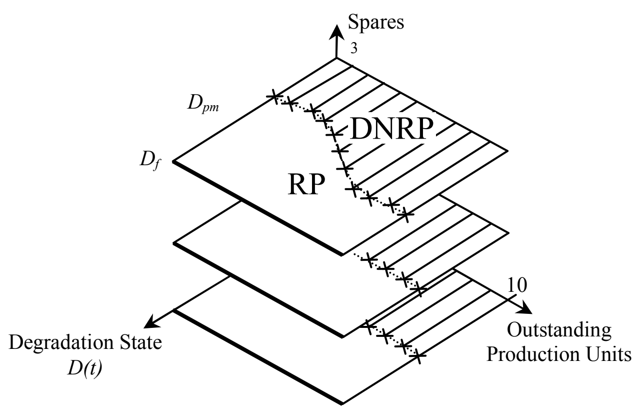

Figure 5.1: Dynamic degradation threshold conceptual model

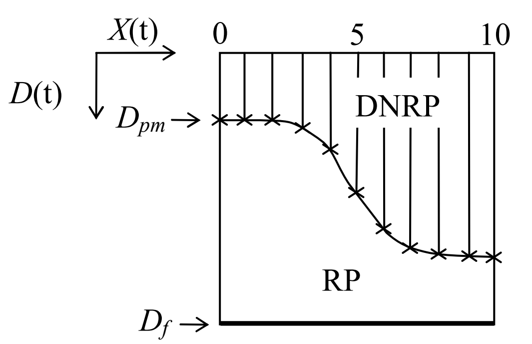

Figure 5.2: One surface from Figure 5.1

In many production environments, the production manager can initiate an expedited order given a stockout occurrence. The model developed in the thesis does not allow for expedited orders to occur. When a stockout occurs, the system is idle until a spare part is received. Further enhancements to the model could allow for expedited orders to occur to reduce the time to receive a spare part given a stockout occurrence. Given there is fee, c_{ex}, to initiate an expedited order, the conditional total expected ordering cost given $N_{D_{pm}}$ is defined by:

$$E\left[total\ ordering\ cost|\ N_{D_{pm}}\right] = E\left[expedite\ ordering\ cost|\ N_{D_{pm}}\right] + E\left[regular\ ordering\ cost|\ N_{D_{pm}}\right] \quad (5.1)$$

where:

$$E\left[expedite\ ordering\ cost|\ N_{D_{pm}}\right] = c_{ex}\sum_{i=0}^{N_{D_{pm}}}\binom{N_{D_{pm}}}{i}\alpha^i\left(1-\alpha\right)^{N_{D_{pm}}-i} i = c_{ex}N_{D_{pm}}\alpha \quad (5.2)$$

$$E\left[regular\ ordering\ cost|\ N_{D_{pm}}\right] = c_{reg}\sum_{i=0}^{N_{D_{pm}}}\binom{N_{D_{pm}}}{i}\alpha^i\left(1-\alpha\right)^{N_{D_{pm}}-i}\left(N_{D_{pm}}-i\right) = c_{reg}N_{D_{pm}}\left(1-\alpha\right) \quad (5.3)$$

Therefore, the conditional total expected ordering cost give $N_{D_{pm}}$ is defined by:

$$E\left[total\ ordering\ cost|\ N_{D_{pm}}\right] = c_{ex}N_{D_{pm}}\alpha + c_{reg}N_{D_{pm}}\left(1-\alpha\right) \quad (5.4)$$

The unconditional expected ordering cost is defined by:

$$E_{N_{D_{pm}}}\left[E\left(total\ ordering\ cost|\ N_{D_{pm}}\right)\right] = E_{N_{D_{pm}}}\left(c_{em}N_{D_{pm}}\alpha\right) + E_{N_{D_{pm}}}\left(c_{reg}N_{D_{pm}}\left(1-\alpha\right)\right) \quad (5.5)$$

Additional enhancements and model assumptions would need to be incorporated into *Problem M* to address expedited orders in the overall optimization approach.

The model developed in this thesis considers a single machine with a spare part inventory pool. Allowing multiple machines to share the same spare part inventory pool would enhance the overall applicability of the model to large industrial organizations. The optimization models would have to be redefined to address multiple machines pulling from the same stock along with additional model constraints.

The final enhancement to the overall model would be to optimize the time period between collecting degradation readings from the system. Each new degradation point updates the degradation path and *RUL* for that component. There is an economic balance between the cost of retrieving data from the system and the benefit provided through the Bayes update procedure. Further investigation and research would be required to define and develop the solution approach.

REFERENCES

LIST OF REFERENCES

Almeida, A. (2001). Multicriteria decision making on maintenance: Spares and contracts planning. *European Journal of Operational Research*, 129, 235-241.

Armstrong, M. & Atkins, D. (1996). Joint optimization of maintenance and inventory policies for a simple system. *IIE Transactions*, 28, 415-424.

Arnold, B., Castillo, E., & Sarabia, J. (1998). The use of conditionally conjugate priors in the study of ratios of gamma scale parameters. *Computational Statistics and Data Analysis*, 27, 125-139.

Aronis, K., Magou, I., Dekker, R., & Tagaras, G. (2004). Inventory control of spare parts using a Bayesian approach: A case study. *European Journal of Operational Research*, 154, 730-739.

Barone, S., Ambrosio, P., & Erto, P. (2007). A Statistical Monitoring Approach for Automotive On-board Diagnostic Systems. *Quality and Reliability Engineering International*, 23, 565-575.

Barnard, G. & Bayes, T. (1958). Studies in the History of Probability and Statistics: IX. Thomas Bayes' Essay Towards Solving a Problem in the Doctrine of Changes. *Biometrika*, 45, 293-315.

Blischke, W. R., & Murthy, D. N. (2003). *Case Studies in Reliability and Maintenance.* New Jersey: John Wiley & Sons.

Chelbi, A. & Ait-Kadi, D. (2001). Spare provisioning strategy for preventively replaced systems subjected to random failure. *International Journal of Production Economics*, 74, 183-189.

Chen, D. & Trivedi, K. (2002). Closed-form analytical results for condition-based maintenance. *Reliability Engineering and System Safety*, 76, 43-51.

Chinnam, R. (1999). On-Line Reliability Estimation of Individual Components, Using Degradation Signals. *IEEE Transactions on Reliability*, 48, 403-412.

Chinnam, R. (2002). Online Reliability Estimation for Individual Components Using Statistical Degradation Signal Models. *Quality and Reliability Engineering International*, 18, 53-73.

Damsleth, E. (1975). Conjugate Classes for Gamma Distribution. *Scandinavian Journal of Statistics*, 2, 80-84.

DeGroot, M. (1970). *Optimal Statistical Decisions*. New York: McGraw-Hill, Inc..

Dimli, D. (2002). The Correlation of Vibration Signal Features to Cutting Tool Wear in a Metal Turing Operation. *International Journal of Advanced Manufacturing Technology*, 19, 705-713.

Eisinger, S. & Rakowsky, U.K. (2001). Modeling of uncertainties in reliability centered maintenance—a probabilistic approach. *Reliability Engineering and System Safety*, 71, 159-164.

Endrenyi, J., Aboresheid, S., Allan, N., Anders, G.J., Asgarpoor, S., Billinton, R., et al. (2001). The Present Status of Maintenance Strategies and the Impact of Maintenance on Reliability. *IEEE Transactions on Power Systems*, 16, 638-646.

Gebraeel, N., Lawley, M., Li, R., & Ryan, J. (2005). Residual-life distributions from component degradation signals: A Bayesian approach. *IIE Transactions*, 37, 543-557.

Gelman, A., Carlin, J., Stern, H., & Rubin, D. (1995). *Bayesian Data Analysis*. London: Chapman & Hall.

Ghodrati, B. & Kumar, U. (2005). Reliability and operating environment-based spare parts estimation approach. *Journal of Quality in Maintenance Engineering*, 11, 169-184.

Giorgio, M., Guida, M., & Pulcini, G. (2007). A Wear Model for Assessing the Reliability of Cylinder Liners in Marine Diesel Engines. *IEEE Transactions on Reliability*, 56, 158-166.

Grall, A., Berenguer, C., & Dieulle, L. (2002). A condition-based maintenance policy for stochastically deteriorating systems. *Reliability Engineering and System Safety*, 76, 167-180.

Habetler, T., Harley, R., Tallman, R., Lee, S., Obaid, R., & Stack, J. (2002). Complete Current-Based Induction Motor Condition Monitoring: Stator, Rotor, Bearings, and Load. *IEEE Technical Proceedings from International Power Electronics Congress*. Guadalajara Mexico.

Hamada, M. (2005). Using Degradation Data to Assess Reliability. *Quality Engineering*, 17, 615-620.

Harihara, P., Kim, K., & Parlos, A. (2003). Signal-Based versus Model-Based Fault Diagnosis-A Trade-Off in Complexity and Performance. *2003 Symposium on Diagnostics for Electric Machines, Power, and Electronic Drives*. Atlanta, GA. August 2003, 277-282.

Hontelez, J., Burger, H., & Wijnmalen, D. (1996). Optimum condition-based maintenance policies for deteriorating systems with partial information. *Reliability Engineering and System Safety*, 51, 267-274.

Hopp, W. & Spearman, M. (2001). Factory Physics. New York: McGraw-Hill Higher Education.

Huiskonen, J. (2001). Maintenance spare parts logistics: Special characteristics and strategic choices. *International Journal of Production Economics*, 71, 125-133.

Iravani, S. & Duenyas, I. (2002). Integrated maintenance and production control of a deteriorating system. *IIE Transactions*, 34, 423-435.

Isermann, R. (1984). Process Fault Detection Based on Modeling and Estimation Methods—A Survey. *Automatica*, 20, 387-404.

Jagerman, D.L. (1978). An Inversion technique for the Laplace transform with application to approximation. *The Bell System Technical Journal*, 57, 669-710.

Jagerman, D.L. (1982). An Inversion technique for the Laplace transform. *The Bell System Technical Journal*, 61, 1995-2002.

Joseph, V., & Yu, I. (2006). Reliability Improvement Experiments with Degradation Data. *IEEE Transactions on Reliability*, 55, 149-157.

Kaufman, D., & Lewis, M. (2007). Machine Maintenance with Workload Considerations. *Naval Research Logistics*, 54, 750-766.

Kennedy, W.J., Patterson, J. & Fredendall, L. (2002). An overview of recent literature on spare parts inventories. *International Journal of Production Economics*, 76, 201-215.

Kharoufeh, J. & Cox, S. (2005). Stochastic models for degradation-based reliability. *IIE Transactions*, 37, 533-542.

Kim, Y. & Kolarik, W. (1992). Real-time conditional reliability prediction from on-line tool performance data. *International Journal of Production Research*, 30, 1831-1844.

Kumar, D. & Klefsjo, B. (1994). Proportional hazards model: a review. *Reliability Engineering and System Safety*, 44, 177-188.

Kuo, Y. & Chang, Z. (2007). Integrated Production Scheduling and Preventive Maintenance Planning for a Single Machine Under a Cumulative Damage Failure Process. *Naval Research Logistics*, 00, 1-13.

Lawless, J. & Crowder, M. (2004). Covariates and Random Effects in a Gamma Process Model with Application to Degradation and Failure. *Lifetime Data Analysis*, 10, 213-227.

Li, Y., Kurfess, T.R., & Liang, S.Y. (2000). Stochastic Prognostics for Rolling Element Bearings. *Mechanical Systems and Signal Processing*, 14, 747-762.

Liao, H.T., Lin, D.M., Qiu, H., Banjevic, D., Jardine, A. & Lee, J. (2005). A Predictive Tool for Remaining Useful Life Estimation of Rotating Machinery Components", *Proceedings of ASME International 20th Biennial Conference on Mechanical Vibration and Noise*, Long Beach, CA, Sep. 24-28.

Liao, H., Elsayed, E., & Chan, L. (2006). Maintenance of continuously monitored degrading systems. *European Journal of Operational Research*, 175, 821-835.

Lu, C., & Meeker, W. (1993). Using Degradation Measures to Estimate a Time-to-Failure Distribution. *Technometrics*, 35, 161-174.

Lu, J., Park, J., & Yang, Q. (1997). Statistical Inference of a Time-to-Failure Distribution Derived from Linear Degradation Data. *Technometrics*, 39, 391-400.

Lu, S., Tu, Y., & Lu, H. (2007). Predictive Condition-based Maintenance for Continuously Deteriorating Systems. *Quality and Reliability Engineering International*, 23, 71-81.

Lwin, T. & Singh, N. (1974). Bayesian Analysis of the Gamma Distribution Model in Reliability Estimation. *IEEE Transactions on Reliability*, 23, 314-319.

Ma, J. & Li, C.J. (1996). Gear Defect Detection through Model-Based Wideband Demodulation of Vibrations. *Mechanical Systems and Signal Processing*, 10, 653-665.

Maillart, L. (2006). Maintenance policies for systems with condition monitoring and obvious failures. *IIE Transactions*, 38, 463-475.

Mann, L., Saxena, A., & Knapp, G. (1995). Statistical-based or condition-based maintenance?. *Journal of Quality in Maintenance*, 1, 46-59.

Mannan, M. & Stone, J. (1998). The Use of Vibration Measurements for Quality Control of Machine Tool Spindles. *International Journal of Advanced Manufacturing Technology*, 14, 889-893.

Marseguerra, M., Zio, E., & Podofillini, L. (2005). Multiobjective spare part allocation by means of genetic algorithms and Monte Carlo simulation. *Reliability Engineering and System Safety*, 87, 325-335.

Martz, H. & Waller, R. (1982). *Bayesian Reliability Analysis*. New York: John Wiley & Sons, Inc..

Mathew, J. & Alfredson, R.J. (1984). The Condition Monitoring of Rolling Element Bearings Using Vibration Analysis. *Journal of Vibration, Acoustics, Stress, and Reliability in Design*, 106, 447-453.

Meeker, W. & Escobar, L. (1998). *Statistical Methods for Reliability Data*. New York: John Wiley & Sons, Inc.

Meeker, W., Doganaksoy, N., & Hahn, G. (2001). Using Degradation Data for Product Reliability Analysis. *Quality Progress*, 34, 60-65.

Meinhold, R. & Singpurwalla, N. (1983). Understanding the Kalman Filter. *The American Statistician*, 37, 123-127.

Merrick, J., Soyer, R., & Mazzuchi, T. (2003). A Bayesian Semiparametric Analysis of the Reliability and Maintenance of Machine Tools. *Technometrics*, 45, 58-69.

Miller, R. (1980). Bayesian Analysis of the Two-Parameter Gamma Distribution. *Technometrics*, 22, 65-69.

Monk, R. (1972). Vibration measurement gives early warning of mechanical faults. *Process Engineering*, 53, 135-137.

Moseler, O. & Iserman, R. (2000). Application of Model-Based Fault Detection to a Brushless DC Motor. *IEEE Transactions on Industrial Electronics*, 47, 1015-1020.

Onel, I., Dalci, K., Senol, I. (2005). Detection of outer raceway bearing defects in small induction motors using stator current analysis. *Sadhaha*, 30, 713-722.

Obi, W. (1990). Error Analysis of a Laplace Transform Inversion Procedure. *SIAM Journal of Numerical Analysis*, 27, 457-469.

Park, C. & Padgett, W. (2005). Accelerated Degradation Models for Failure Based on Geometric Brownian Motion and Gamma Process. *Lifetime Data Analysis*, 11, 511-527.

Pedregal, D. & Carnero, M. (2006). State space models for condition monitoring: a case study. *Reliability Engineering and System Safety*, 91, 171-180.

Rausand, M. (1998). Reliability Centered Maintenance. *Reliability Engineering and System Safety*, 60, 121-132.

Ray, A. & Tangirala, S. (1996). Stochastic Modeling of Fatigue Crack Dynamics for Online Failure Prognostics. *IEEE Transactions on Control Systems Technology*, 4, 443-451.

Rehmert, I. (2000). Availability Analysis for the Quasi-Renewal Process. (Doctoral Dissertation, Virginia Polytechnic Institute and State University, 2000).

Rustenburg, W.D., Houtum, G.J., & Zijm, W.H.M. (2000). Spare parts management for technical systems: resupply of spare parts under limited budgets. *IIE Transactions*, 32, 1013-1026.

Schoen, R., Habetler, T., Kamran, F., & Bartheld, R. (1995). Motor Bearing Damage Detection Using Stator Current Monitoring. *IEEE Transactions on Industry Applications*, 31, 1274-1279.

Sarang, H. & Knezevic, J. (2001). Reliability prediction from condition-based maintained systems. *Reliability Engineering and System Safety*, 71, 219-224.

Saranga, H. (2002). Relevant condition-parameter strategy for an effective condition-based maintenance. *Journal of Quality in Maintenance Engineering*, 8, 92-105.

Sarker, R. & Haque, A. (2000). Optimization of maintenance and spare provisioning policy using simulation. *Applied Mathematical Modeling*, 24, 751-760.

Shibuya, T., Dohi, T., & Osaki, S. (1998). Optimal continuous review policies for spare part provisioning with random lead times. *International Journal of Production Economics*, 55, 257-271.

Sloan, T. & Shanthikumar (2002). Using in-line equipment condition and yield information for maintenance scheduling and dispatching in semiconductor wafer fab. *IIE Transactions*, 34, 191-209.

Smidt-Destombes, K., Heijden, M. & Harten, A. (2006). On the interaction between maintenance, spare part inventories, and repair capacity for a k-out-of-N system with wear-out. *European Journal of Operational Research*, 174, 182-200.

Srinivasan, M. & Lee, H. (1996). Production-inventory systems with preventive maintenance. *IIE Transactions*, 28, 879-890.

Tsang, A. (1995). Condition-based maintenance: tools and decision making. *Journal of Quality in Maintenance Engineering*, 1, 3-17.

Vaughan, T. (2005). Failure replacement and preventive maintenance spare part ordering policy. *European Journal of Operational Research*, 161, 183-190.

Wang, H. (2002). A survey of maintenance policies of deteriorating systems. *European Journal of Operational Research*, 139, 469-489.

Wireman, T. (1990). *World Class Maintenance Management*. New York: Industrial Press.

Yam, R., Tse, P., Li, L., & Tu, P. (2001). Intelligent Predictive Support System for Condition Based Maintenance. *International Journal of Advanced Manufacturing Technology*, 17, 383-391.

Yang, S. (1999). State Estimation for predictive maintenance using Kalman filter. *Reliability Engineering and System Safety*, 66, 29-33.

Yang, S. (2002). An experiment of state estimation for predictive maintenance using Kalman filter on a DC motor. *Reliability Engineering and System Safety*, 75, 103-111.

Yang, S. (2003). A Condition-Based Failure Prediction and Processing-Scheme for Preventive Maintenance. *IEEE Transactions on Reliability*, 52, 373-383.

Zarchan, P. & Musoff, H. (2005). *Fundamentals of Kalman Filtering: A Practical Approach*. Virginia: American Institute of Aeronautics and Astronautics, Inc.

Zhan, Y., Makis, V., & Jardine, A. (2006). Adaptive state detection of gearboxes under varying load conditions based on parametric modeling. *Mechanical Systems and Signal Processing*, 20, 188-221.

Zhou, X., Xi, L., & Lee, J. (2007). Reliability-centered predictive maintenance scheduling for a continuously monitored system subject to degradation. *Reliability Engineering and System Safety*, 92, 530-534.

APPENDIXES

APPENDIX A

MATLAB gamma process degradation paths

```
%This code generates a gamma process degradation path according to alpha and beta with a time %interval dt
Var=[700,0.006;1200,0.009;2000,0.008]; %[alpha1,beta1;alpha2,beta2;alpha3,beta3]
dt=0.001; %time interval (t-s)
for i=1:3, %define alpha and beta to variables
alpha(i)=Var(i,1);
end
for j=1:3,
beta(j)=Var(j,2);
end

figure
for k=1:3,
D=gamrnd(alpha(k)*dt,beta(k),200,1); %generate gamma process D(i)-D(i-1)~gamma
sumD=0;
for l=1:200, %sum the degradation intervals to acquire degradation path
   sumD=sumD+D(l);
   ArraysumD(l,k)=sumD;
end
temp(1:l,k)=ArraysumD(1:l,k);
end
plot([0.001:0.001:0.2]',temp(:,1), 'k-');
ylabel('D(t)')
xlabel('t')
hold on
plot([0.001:0.001:0.2]',temp(:,2), 'k--');
plot([0.001:0.001:0.2]',temp(:,3), 'k-.');
legend('\alpha*\Deltat=0.7 \beta=0.006','\alpha*\Deltat=1.2 \beta=0.009','\alpha*\Deltat=2.0 \beta=0.008')
```

APPENDIX B

MATLAB failure time *CDF* for the gamma process

```
%This code will generate the CDF of the Gamma process
beta=0.006;
alpha=0.7;
Df=45;
t=0.001:0.001:6;
for i=1:6000;
D(i,1)=1-gammainc((Df*beta),alpha*t(i));
end
plot([0.001:0.001:6],D,'-k');
xlabel('t')
ylabel('F(t)')
```

APPENDIX C

MATLAB failure time *pdf* for the gamma process

```
%Gamma process pdf
%The genHyper function is required to execute this code-the function is
%available on MATLAB website for download
alpha=0.7;
beta=0.006;
D=45;
t=0.1:0.5:20;

for i=1:40,
   a=[alpha*t(i),alpha*t(i)];
   b=[alpha*t(i)+1;alpha*t(i)+1];
   z=-beta*D;
   g(i)=(alpha.*(psi(alpha*t(i))-log(beta*D)).*(1-(1-
gammainc((D*beta),alpha*t(i)))))+((alpha./gamma(alpha*t(i))).*(((D*beta).^(alpha*t(i)))./((alpha*t(i)).^2)).*genHy
per(a,b,z));
end

plot(t,g,'b-');
xlabel('t')
ylabel('f(t)')

%testing code for genHyper

alpha=0.7;
beta=0.006;
D=45;
t=0.1:0.5:20;

for j=1:40,
a=[alpha*t(j),alpha*t(j)];
b=[alpha*t(j)+1;alpha*t(j)+1];
z=-beta*D;
g(j,1)=genHyper(a,b,z,0);
end
plot([0.1:0.5:20],g);
```

APPENDIX D

MATLAB exponential *CDF* inverse Laplace transform

```
%Laplace Transform, Non-linear Regression, Laplace Inverse Estimation
%Model, & Probability Calculations for the exponential distribution

%This code will calculate the Laplace transform of the CDF for the Exponential Distribution

lambda=0.1;
s_LAP=0.05:0.01:3;
size_s=size(s_LAP,2);

LapEXPO=inline('exp(-s.*t).*(1-exp(-lambda.*t))','t','s','lambda'); %Laplace transform definition equation for
Exponential CDF

for i=1:size_s,
   Lap(i,1)=quad(@(t) LapEXPO(t,s_LAP(i),lambda),0.0001,5000,0.000000001);
end

coef=[0.1]; %found from Excel spreadsheet that minimizes MSE

model=inline('p(1)./(s.*(s+p(1)))','p','s');

%Plot the Laplace transform data and the non-linear regression model
figure
plot([0.05:0.01:3],Lap(:,1),'b-');
xlabel('s')
ylabel('F(s)')
title('Calculated Laplace Transform F(s) vs. Non-linear Regression F(s)_e_s_t')
hold on
plot([0.05:0.01:3],model(coef,s_LAP),'r--');
legend('Calculated F(s)','Non-linear regression F(s)_e_s_t')
hold off

delta_model=model(coef,s_LAP)'-Lap(:,1); %calculates the delta between the calculated curve and the regression
model

%Plot the delta between the regression model and the calculated F(s) values
figure
plot([0.05:0.01:3],delta_model);
xlabel('s')
ylabel('\DeltaF(s)')
title('Delta between Laplace Transform F(s) & Non-linear Regression F(s)_e_s_t')

%Calculate the estimate for the inverse Laplace transform
syms s
fs=coef(1)/(s*(coef(1)+s)); %Laplace transform regression model

n=1:1:25; %number (n) of times that the functional is calculated
szn=size(n,2); %calculate the size of n to be utilized in the for loop
a=0.1;
t=0.1:a:50; %t values for the function domain
szt=size(t,2); %calculate the size of t to be utilized in the for loop
```

```
for j=1:szt, %loop through time t

   for i=1:szn, %for each t value calculate n number of estimates
      se_n=((n(i)+1)/t(j));
      se_2n=((2*n(i)+1)/t(j));
      g_n=diff(fs,n(i)); %nth derivative of the Laplace transform
      g_2n=diff(fs,2*n(i)); %2*nth derivative of the Laplace transform
      deriv_n(i)=subs(g_n,se_n);
      deriv_2n(i)=subs(g_2n,se_2n);

      fnt_n(i,j,1)=(((-1)^n(i))/factorial(n(i)))*(se_n^(n(i)+1))*deriv_n(i);
      fnt_2n(i,j,1)=(((-1)^(2*n(i)))/factorial(2*n(i)))*(se_2n^(2*n(i)+1))*deriv_2n(i);
      myfun(i,j,1)=(2+1/n(i))*fnt_2n(i,j,1)-(1+1/n(i))*fnt_n(i,j,1);
   end
end

%compares the actual F(t) values to inverse Laplace transform results from Jagerman
figure
plot(t,myfun(szn,:),'b--');
xlabel('t')
ylabel('f(t)')
title('Jagerman CDF Estimate  L^-^1[F(s)] vs. Actual F(t)=1-exp(-\lambdat)')
hold on
plot(t,1-exp(-lambda.*t),'r:');
legend('Jagerman F(t)','Actual F(t)')
hold off

figure
plot(t,(1-exp(-lambda.*t))-myfun(szn,:),'k-');
xlabel('t')
ylabel('\DeltaF(t)')
title('Delta between Jagerman CDF Estimate  L^-^1[F(s)] and Actual F(t)=1-exp(-\lambdat)')
```

APPENDIX E

MATLAB spare part model

```
%Laplace Transform, Non-linear Regression, Laplace Inverse Estimation Model, & Probability Calculations

%This code will calculate the Laplace transform of the CDF for the Gamma process
beta=0.006;
%beta=0.6;
alpha=0.7;
%alpha=1.7;
Df=45;
s_LAP=0.05:0.01:3; %fix to 0
size_s=size(s_LAP,2);

mu=0.02; %lognormal lead time distribution parameters
sigma=0.05;

%Probabilty of stockout for the initial part spares=1 (initial part on the machine b=5000; for k=1:b, %generate many
trials to calculate the probability
    rhs=rand();
    q(k)=fzero(@(t)logncdf(t,mu,sigma)-rhs,[0,25]); %solves the CDF of the lead time distribution for t given a
random number between 0 and 1
    result(k)=1-gammainc((Df*beta),alpha*q(k));
end
probability(1)=mean(result(1,:));

stockout_prob=0.2;

if probability(1)>stockout_prob,

    for j=1:size_s,

        for i=1:5000,
            rhs=rand();
            x(i,1)=fzero(@(t) 1-gammainc((Df*beta),alpha*t)-rhs,[0,10]); %solves the CDF for t given a random
number between 0 and 1
            r(i,j)=exp(-s_LAP(j)*x(i,1)); %calculates one estimate for the Laplace transform curve for each i
        end
        fs(j,1)=mean(r(:,j)); %generates one point for the Laplace transform curve by averaging all of the Laplace
estimates
    end
end

count=1;
spares=1;
b_est(:,count)=[1,1,1,1];
while probability(count)>stockout_prob, %Recursion to determine the spare parts required
    spares=spares+1;
    count=count+1;

    fs(:,count)=(fs(:,1).^(spares))./(s_LAP');

    model=inline('(sum((fs-((b(1).*s.^2)+(b(2).*s))./((s.^4)+(b(3).*s.^3)+(b(4).*s.^2))).^2))./size_s','b','s','fs','size_s');
```

```
A=[0,1,0,-1];
b1=[0];
b_est(:,count)=fmincon(@(b) model(b,s_LAP',fs(:,count),size_s),b_est(:,count-1),[],[],A,b1);

 model2=inline('((p(1).*s.^2)+(p(2).*s))./((s.^4)+(p(3).*s.^3)+(p(4).*s.^2))','p','s');

figure
plot(s_LAP,fs(:,count),'b-');
xlabel('s')
ylabel('F(s)')
title('Calculated Laplace Transform F(s) vs. Constrained Regression F(s)_e_s_t')
hold on
plot(s_LAP,model2(b_est(:,count),s_LAP),'r--');
legend('Calculated F(s)','Constrained Regression F(s)_e_s_t')
hold off

delta_model=model2(b_est(:,count),s_LAP)'-fs(:,count); %calculates the delta between the calculated curve and
the regression model

%Plot the delta between the regression model and the calculated F(s) values
figure
plot(s_LAP,delta_model);
xlabel('s')
ylabel('\DeltaF(s)')
title('Delta between Laplace Transform F(s) & Constrained Regression F(s)_e_s_t')

%Jagerman Recursion
syms s
fs_sym=((b_est(1,count)*(s^2))+(b_est(2,count)*s))/((s^4)+(b_est(3,count)*(s^3))+(b_est(4,count)*(s^2)));

n2=1:1:8; %number (n) of times that the functional is calculated
szn2=size(n2,2); %calculate the size of n to be utilized in the for loop
a=0.1;
t=0.1:a:5; %t values for the function domain
szt=size(t,2); %calculate the size of t to be utilized in the for loop

for l=1:szt, %loop through time t

   for m=1:szn2, %for each t value calculate n number of estimates
      se_n=((n2(m)+1)/t(l));
      se_2n=((2*n2(m)+1)/t(l));
      g_n=diff(fs_sym,n2(m)); %nth derivative of the Laplace transform
      g_2n=diff(fs_sym,2*n2(m)); %2*nth derivative of the Laplace transform
      deriv_n(m)=subs(g_n,se_n);
      deriv_2n(m)=subs(g_2n,se_2n);

      fnt_n(m,l,count-1)=(((-1)^n2(m))/factorial(n2(m)))*(se_n^(n2(m)+1))*deriv_n(m);
      fnt_2n(m,l,count-1)=(((-1)^(2*n2(m)))/factorial(2*n2(m)))*(se_2n^(2*n2(m)+1))*deriv_2n(m);
      myfun(m,l,count-1)=(2+1/n2(m))*fnt_2n(m,l,count-1)-(1+1/n2(m))*fnt_n(m,l,count-1);
      myfun_noadj(m,l,count-1)=myfun(m,l,count-1);

      if myfun(m,l,count-1)>1,
```

```
            myfun(m,l,count-1)=1;
        end
        if myfun(m,l,count-1)<0,
            myfun(m,l,count-1)=0;
        end

    end
end

figure
plot(t,myfun(szn2,:,count-1),'k')
xlabel('t')
ylabel('F(t)')
title('CDF Estimate using Jagerman L^-^1[F(s)^(^1^+^s^p^a^r^e^s^)]')
hold on
plot(t,myfun_noadj(szn2,:,count-1),'b--')
legend('F(t)_a_d_j','F(t)')

for w=1:b, %generate many trials to calculate the probability
    rhs=rand();
    q(w)=fzero(@(t)logncdf(t,mu,sigma)-rhs,[0,25]); %solves the CDF for t given a random number between 0 and
1

    if q(w)<t(1),
        result(count,w)=0;
    elseif ((t(1)<=q(w)) && (q(w)<=t(szt))),
        p=floor(q(w)/a);
        result(count,w)=myfun(szn2,p,count-1);
    else
        result(count,w)=1;
    end
end

probability(count)=mean(result(count,:)); %generates probability

end
```

APPENDIX F

MATLAB gamma process Bayesian and *RUL* update code

```
%Assume Di~gamma(0.5,0.25) where 0.25 represents the scale parameter for the
%degradation process
%Assume alpa is known for the gamma process
%Assume beta~gamma(alpha0=2,beta0=1.3) where beta0 represents the inverse scale
%parameter
%Posterior pdf  ****The posterior distribution is derived using the inverse
%scale parameter****

figure
xlabel('Beta')
ylabel('f(beta)')
D=gamrnd(0.5,0.25,200,1); %gamma distributed random numbers for D values
alpha=0.5; %shape parameter for gamma process
alpha0=2; %shape parameter for conjugate prior
beta0=1.3; %inverse scale parameter of conjugate prior
MEAN=alpha0/beta0;
beta=(0.01:0.01:10); %generate beta values for posterior pdf plot
countj=0;
sumD=0;
for j=1:200,
alpha0=alpha+alpha0; %alpha0 is updated with each new observation of D
beta0=beta0+D(j); %beta0 is update with each new observation of D
sumD=sumD+D(j);
MEAN(j)=alpha0/beta0;
countj=countj+1;
Arrayalpha0(countj)=alpha0;
Arraybeta0(countj)=beta0;
ArraysumD(countj)=sumD;
counti=0;
for i=1:1000, %evaluate the posterior distribution for each beta value and store in an array
Posterior=((beta(i)^(alpha0-1))*(exp(-(beta0)*beta(i)))*((beta0)^alpha0))/gamma(alpha0);
if Posterior < 1.e-40
   Posterior=0;
end
counti=counti+1;
ArrayPosterior(counti,j)=Posterior;
end
temp=ArrayPosterior(1:counti,j);
plot([0.01:0.01:10],temp,'k-')
hold on
end
hold off

%Integrate the test.m file.  The unconditioned CDF is found by the integral of the CDF %multiplied by the pdf
being updated with each new degradation points

Df=15; %failure threshold
t=1:1:800; %time
for k=1:10, %updated values from Bayes analysis
countl=0;
```

```
for l=1:800, % time increments
Integral=quadl(@(x)test(x,alpha,Arrayalpha0(k),Arraybeta0(k),t(l),Df-
ArraysumD(k)),0.0001,1000,0.0000000000001);
countl=countl+1;
ArrayIntegral(countl,k)=Integral;
end
end

figure
plot(t,ArrayIntegral(:,1),'k-')
ylabel('F_i ( t )')
xlabel('t')
hold on
plot(t,ArrayIntegral(:,2),'r:')
plot(t,ArrayIntegral(:,3),'b-.')
plot(t,ArrayIntegral(:,4),'g--')
legend('F_1( t )','F_2( t )','F_3( t )','F_4( t )')
hold off

test.m file
function y = g(x,alpha,alpha0,beta0,t,a)
y = ((1-gammainc((a*x),alpha*t)).*((x.^(alpha0-1)).*exp(-beta0.*x).*(beta0.^alpha0)))./(gamma(alpha0));
```

APPENDIX G

MATLAB code for Section 4.3

```
%Laplace Transform, Non-linear Regression, Laplace Inverse Estimation Model, & Probability Calculations

%This code will calculate the Laplace transform of the CDF for the Gamma process
beta=0.006;
alpha=0.7;
Df=45;
s_LAP=0.05:0.01:3;
size_s=size(s_LAP,2);

mu=0.02; %lognormal lead time distribution parameters
sigma=0.05;


%Probabilty of stockout for the initial part spares=1 (initial part on the machine
b=1000;
for k=1:b, %generate many trials to calculate the probability
   q(k)=lognrnd(mu,sigma);
   result(k)=1-gammainc((Df*beta),alpha*q(k));
end
probability(1)=mean(result(1,:));

stockout_prob=0.1; %0.2,0.1 works

if probability(1)>stockout_prob,

   for j=1:size_s,

      for i=1:1000,
         rhs=rand();
         x(i,1)=fzero(@(t) 1-gammainc((Df*beta),alpha*t)-rhs,[0,10]); %solves the CDF for t given a random
number between 0 and 1  works
         r(i,j)=exp(-s_LAP(j)*x(i,1)); %calculates one estimate for the Laplace transform curve for each i
      end
      fs(j,1)=mean(r(:,j)); %generates one point for the Laplace transform curve by averaging all of the Laplace
estimates
   end
end

count=1;
spares=1;
b_est(:,count)=[1,1,1,1];
while probability(count)>stockout_prob, %Recursion to determine the spare parts required
   spares=spares+1;
   count=count+1;

   fs(:,count)=(fs(:,1).^(spares))./(s_LAP');


   model=inline('(sum((fs-((b(1).*s.^2)+(b(2).*s))./((s.^4)+(b(3).*s.^3)+(b(4).*s.^2))).^2))./size_s','b','s','fs','size_s');
   A=[0,1,0,-1];
```

```
b1=[0];
b_est(:,count)=fmincon(@(b) model(b,s_LAP',fs(:,count),size_s),b_est(:,count-1),[],[],A,b1);


model2=inline('((p(1).*s.^2)+(p(2).*s))./((s.^4)+(p(3).*s.^3)+(p(4).*s.^2))','p','s');

figure
plot(s_LAP,fs(:,count),'b-');
xlabel('s')
ylabel('F(s)')
title('Calculated Laplace Transform F(s) vs. Constrained Regression F(s)_e_s_t')
hold on
plot(s_LAP,model2(b_est(:,count),s_LAP),'r--');
legend('Calculated F(s)','Constrained Regression F(s)_e_s_t')
hold off

delta_model=model2(b_est(:,count),s_LAP)'-fs(:,count); %calculates the delta between the calculated curve and
the regression model

%Plot the delta between the regression model and the calculated F(s) values
figure
plot(s_LAP,delta_model);
xlabel('s')
ylabel('\DeltaF(s)')
title('Delta between Laplace Transform F(s) & Constrained Regression F(s)_e_s_t')


%Jagerman Recursion
syms s
fs_sym=((b_est(1,count)*(s^2))+(b_est(2,count)*s))/((s^4)+(b_est(3,count)*(s^3))+(b_est(4,count)*(s^2)));

n2=1:1:8; %number (n) of times that the functional is calculated
szn2=size(n2,2); %calculate the size of n to be utilized in the for loop
a=0.1;
t=0.1:a:10; %t values for the function domain
szt=size(t,2); %calculate the size of t to be utilized in the for loop

for l=1:szt, %loop through time t

  for m=1:szn2, %for each t value calculate n number of estimates
    se_n=((n2(m)+1)/t(l));
    se_2n=((2*n2(m)+1)/t(l));
    g_n=diff(fs_sym,n2(m)); %nth derivative of the Laplace transform
    g_2n=diff(fs_sym,2*n2(m)); %2*nth derivative of the Laplace %transform
    deriv_n(m)=subs(g_n,se_n);
    deriv_2n(m)=subs(g_2n,se_2n);

    fnt_n(m,l,count-1)=(((-1)^n2(m))/factorial(n2(m)))*(se_n^(n2(m)+1))*deriv_n(m);
    fnt_2n(m,l,count-1)=(((-1)^(2*n2(m)))/factorial(2*n2(m)))*(se_2n^(2*n2(m)+1))*deriv_2n(m);
    myfun(m,l,count-1)=(2+1/n2(m))*fnt_2n(m,l,count-1)-(1+1/n2(m))*fnt_n(m,l,count-1);
    myfun_noadj(m,l,count-1)=myfun(m,l,count-1);
```

```
            if myfun(m,l,count-1)>1,
                myfun(m,l,count-1)=1;
            end

            if myfun(m,l,count-1)<0,
                myfun(m,l,count-1)=0;
            end

        end
    end

    figure
    plot(t,myfun(szn2,:,count-1),'k')
    xlabel('t')
    ylabel('F(t)')
    title('CDF Estimate using Jagerman L^-^1[F(s)^(^1^+^s^p^a^r^e^s^)]')
    hold on
    plot(t,myfun_noadj(szn2,:,count-1),'b--')
    legend('F(t)_a_d_j','F(t)')

    for w=1:b, %generate many trials to calculate the probability
        q(w)=lognrnd(mu,sigma);
        if q(w)<t(1),
            result(count,w)=0;
        elseif ((t(1)<=q(w)) && (q(w)<=t(szt))),
            p=floor(q(w)/a);
            result(count,w)=myfun(szn2,p,count-1);
        else
            result(count,w)=1;
        end
    end

    probability(count)=mean(result(count,:)); %generates probability

end
```

APPENDIX H

MATLAB code for Section 4.4

```
Q=35; %total number of units that must be processed
lambda=3.25; %production processing rate
cost_fail=750; %cost of failure
cost_back=450; %cost of backlog
cost_reg=550; %cost of regular order

Tinv=0; %inventory order receive time variable
Df=45; %static failure threshold
alpha=0.7; %gamma process variable
beta=0.006; %gamma process variable
mu=0.02; %mean of lead time distribution for spare parts
sigma=0.05; %standard deviation of lead time distribution for spare parts
MT=0.4; %maximum maintenance time
Tc=10; %total clock time available to process an order

sum_delta=0;
sum_lead_time=0;
long_run_stockout=0;
delta_time=0;

x=1;
Dpm(x)=45+2;

iterations=2000;

for b=1:20,
   x=x+1;
   Dpm(x)=Dpm(x-1)-2;

   sum_delta=0;
   long_run_stockout=0;
   unavailability=0;

   for i=1:iterations,

      p=0; %count variable
      j=1; %count variable
      n=0; %count variable
      g=0; %count variable
      delta_time=0; %delta time variable
      count(i)=0; %number of spare parts utilized
      Tcum(:,i)=0; %cumulative time counter
      Tf(:,i)=0; %time to failure
      Tinv(:,i)=0;
      order_quan(i)=0; %count variable for spare part orders
      stockout_count(i)=0; %count of the number of stockout occurrences
      count_em(i)=0; %emergency order counter
      total_time(i)=0; %total time variable
      I(j,i)=4; %initial inventory position--from the initial spare parts model
      Temp(i)=0;
```

APPENDIX H (continued)

```
Temp2(i)=0;

rhs(j,i)=0.999*rand();
Tf(j,i)=fzero(@(t) 1-gammainc((Dpm(x)*beta),alpha*t)-rhs(j,i),[0,2000]);

if Tf(j,i)>Tc
   Tf(j,i)=0;
   break
else
   Tcum(j,i)=Tcum(j,i)+Tf(j,i)+MT*exp(1-Df/Dpm(x));
   I(j,i)=I(j,i)-1;
   count(i)=count(i)+1;
   LT(j,i)=lognrnd(mu,sigma);
   n=n+1;
   Tinv(n,i)=Tcum(j,i)-MT*exp(1-Df/Dpm(x))+LT(j,i);
   order_quan(i)=order_quan(i)+1;

   if Tinv(n,i)<Tcum(j,i),
      I(j,i)=I(j,i)+1;
   end

   while Tcum(j,i)<Tc && floor(Q-lambda*sum(Tf(:,i)))>0,
      j=j+1;
      rhs(j,i)=0.999*rand();
      Tf(j,i)=fzero(@(t) 1-gammainc((Dpm(x)*beta),alpha*t)-rhs(j,i),[0,2000]);
      count(i)=count(i)+1;
      Tcum(j,i)=Tcum(j-1,i)+Tf(j,i)+MT*exp(1-Df/Dpm(x));
      LT(j,i)=lognrnd(mu,sigma);
      n=n+1;
      Tinv(n,i)=Tcum(j,i)-MT*exp(1-Df/Dpm(x))+LT(j,i);
      order_quan(i)=order_quan(i)+1;
      I(j,i)=I(j-1,i)-1+size(find(Tinv(:,i)>Tcum(j-1,i) & Tinv(:,i)<=Tcum(j,i)),1);

      if Tcum(j,i)>Tc,
         if Tcum(j,i)-MT*exp(1-Df/Dpm(x))<Tc,
            Tcum(j,i)=Tcum(j,i)-MT*exp(1-Df/Dpm(x));
            break
         else
            count(i)=count(i)-1;
            Tcum(j,i)=Tc;
            Temp(i)=Tf(j,i);
            break
         end
      end

      if I(j,i)==0,
         j=j+1;
         p=p+1;
         stockout_count(i)=stockout_count(i)+1;
         position=find(Tinv(:,i)==min(Tinv(find(Tinv(:,i)>Tcum(j-1,i)),i)));
         delta_time(p)=Tinv(position,i)-Tcum(j-1,i);
         Tcum(j,i)=Tinv(position,i);
```

```
                if Tcum(j,i)>Tc,
                    Tcum(j,i)=Tcum(j-1,i);
                    Temp2(i)=delta_time(p);
                    break
                end

                I(j,i)=I(j-1,i)+size(find(Tinv(:,i)>Tcum(j-1,i) & Tinv(:,i)<=Tcum(j,i)),1);
            end

        end
    end

    total_time(i)=min(Tc,Tcum(j,i));

    sum_delta(i)=sum(delta_time)-Temp2(i);
    unavailability(i)=(sum_delta(i)+(MT*exp(1-Df/Dpm(x))*count(i)))/total_time(i);
    long_run_stockout(i)=stockout_count(i)/order_quan(i);

  end


  for l=1:iterations,

    if  floor(Q-lambda*(sum(Tf(:,l))-Temp(l)))>0,
        bin=1;
    else
        bin=0;
    end

    cost(l)=(cost_fail*exp(1-Df/Dpm(x)))*count(l) + cost_back*floor(Q-lambda*(sum(Tf(:,l))-Temp(l)))*bin +
cost_reg*order_quan(l);
    fail_cost(l)=cost_fail*exp(1-Df/Dpm(x))*count(l);
    back_cost(l)=cost_back*floor(Q-lambda*(sum(Tf(:,l))-Temp(l)))*bin;
    reg_cost(l)=cost_reg*order_quan(l);

  end

  variance_cost(x)=var(cost);
  expected_cost(x)=mean(cost);
  stockout_average(x)=mean(long_run_stockout);
  expected_fail(x)=mean(fail_cost);
  expected_back(x)=mean(back_cost);
  expected_reg(x)=mean(reg_cost);
  overall(x)=expected_fail(x)+expected_back(x)+expected_reg(x);

  expected_unavailability(x)=mean(unavailability);

end

figure
plot(Dpm(:,2:size(Dpm,2)),overall(2:size(overall,2)),'k-');
xlabel('Degradation Threshold')
```

```
ylabel('Cost')
hold on
plot(Dpm(:,2:size(Dpm,2)),expected_fail(2:size(expected_fail,2)),'r:');
plot(Dpm(:,2:size(Dpm,2)),expected_back(2:size(expected_back,2)),'b-');
plot(Dpm(:,2:size(Dpm,2)),expected_reg(2:size(expected_reg,2)),'g');
legend('Total Cost','Failure Cost','Backorder Cost','Regular Order Cost')

figure
plot(Dpm(:,2:size(Dpm,2)),stockout_average(2:size(stockout_average,2)));
xlabel('Degradation Threshold')
ylabel('Stockout Probability')

figure
plot(Dpm(:,2:size(Dpm,2)),expected_unavailability(2:size(expected_unavailability,2)));
xlabel('Degradation Threshold')
ylabel('Unavailability')

figure
plot(Dpm(:,2:size(Dpm,2)),variance_cost(2:size(variance_cost,2)));
xlabel('Degradation Threshold')
ylabel('Variance')
```

APPENDIX I

MATLAB code for Section 4.5

```
figure
D=[0.0137,6.1162];
alpha=0.7; %shape parameter for gamma process
alpha0=0.05; %shape parameter for conjugate prior
beta0=25; %inverse scale parameter of conjugate prior
MEAN=alpha0/beta0;
beta=(0:0.00001:0.03); %generate beta values for posterior pdf plot
countj=0;
sumD=0;
for j=1:size(D,2),
alpha0=alpha+alpha0; %alpha0 is updated with each new observation of D
beta0=beta0+D(j); %beta0 is update with each new observation of D
sumD=sumD+D(j);
MEAN(j)=alpha0/beta0;
countj=countj+1;
Arrayalpha0(countj)=alpha0;
Arraybeta0(countj)=beta0;
ArraysumD(countj)=sumD;
counti=0;
for i=1:size(beta,2), %evaluate the posterior distribution for each beta %value and store in an array
Posterior=((beta(i)^(alpha0-1))*(exp(-(beta0)*beta(i)))*((beta0)^alpha0))/gamma(alpha0);
if Posterior < 1.e-40
   Posterior=0;
end
counti=counti+1;
ArrayPosterior(counti,j)=Posterior;
end
temp=ArrayPosterior(1:counti,j);
plot(beta,temp,'k-')
hold on
end
xlabel('\beta')
ylabel('f(\beta)')
hold off

%Integrate the test.m file.  The unconditioned CDF is found by the integral %of the CDF multiplied by the pdf being updated with each new degradation %points

Df=45; %failure threshold
t=0.001:0.1:15; %time
for k=1:2, %updated values from Bayes analysis
countl=0;
for l=1:size(t,2), % time increments
Integral=quadl(@(x)test(x,alpha,Arrayalpha0(k),Arraybeta0(k),t(l),Df-
ArraysumD(k)),0.0001,1000,0.0000000000001);
countl=countl+1;
ArrayIntegral(countl,k)=Integral;
end
end
```

APPENDIX I (continued)

```
figure
plot(t,1-gammainc((Df*0.006),alpha.*t),'b')
ylabel('F_i ( t )')
xlabel('t')
hold on
plot(t,ArrayIntegral(:,1),'k-')
plot(t,ArrayIntegral(:,2),'r:')
legend('F( t )','F_1( t )','F_2( t )')
hold off
```

test.m file

```
function y = g(x,alpha,alpha0,beta0,t,a)
y = ((1-gammainc((a*x),alpha*t)).*((x.^(alpha0-1)).*exp(-beta0.*x).*(beta0.^alpha0)))./(gamma(alpha0));
```