AIRCRAFT LOSS-OF-CONTROL: REAL-TIME ADAPTIVE PREDICTION OF SAFE
CONTROL MARGINS AND PILOT ADVISORY DISPLAYS

A Dissertation by

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AIRCRAFT LOSS-OF-CONTROL: REAL-TIME ADAPTIVE PREDICTION OF SAFE CONTROL MARGINS AND PILOT ADVISORY DISPLAYS

The following faculty members have examined the final copy of this dissertation for form and content, and recommend that it be accepted in partial fulfillment of the requirement for the degree of Doctor of Philosophy, with a major in Aerospace Engineering.

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ABSTRACT

Predictive systems with the ability to warn pilots of impending entry into loss-of-control have the potential to improve safety in flight. Along these lines, the research documented in this dissertation has sought to develop an early warning mechanism to predict an aircraft’s “receding-horizon” safe control envelope and warn pilots of future impending excursion of its safe flight envelope. The intent of the concept is to mitigate entry into loss-of-control by continually keeping the aircraft at a certain time-distance from the edge of the flight envelope. The adaptive prediction architecture estimates the aircraft's proximity to its control loss boundaries through two methods. The first employs a “deflection-to-go” methodology that calculates critical control deflection, rate, acceleration, and higher order input limits that would drive the aircraft to control loss at some point several seconds in the future, while the second employs a “trajectory-to-go” methodology that optimally derives critical control trajectories that should not be exceeded for the aircraft to stay within the safe flight envelope. These critical inputs and trajectories form the boundaries of a safe control envelope or safe control space, and are presented on a pilot advisory display implemented on head-up augmented-reality technologies. Using the display, the pilot is provided with pre-emptive warning of impending entry into control loss within the look-ahead prediction window. The prediction architectures are applied to a dynamically-coupled 8th order light business jet model in desktop simulation and pilot-in-the-loop simulated flight testing, and results demonstrate successful prediction of the critical control limits that should not be exceeded in order to avoid future entry into control loss.
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<td>--------------------------------------------------</td>
<td></td>
</tr>
<tr>
<td>AA</td>
<td>Adverse Aerodynamics</td>
<td></td>
</tr>
<tr>
<td>CAST</td>
<td>Commercial Aviation Safety Team</td>
<td></td>
</tr>
<tr>
<td>CDSE</td>
<td>Control Deficiency Signal Estimate</td>
<td></td>
</tr>
<tr>
<td>D2G</td>
<td>Deflection-To-Go</td>
<td></td>
</tr>
<tr>
<td>DPC</td>
<td>Dynamic Pitch Control</td>
<td></td>
</tr>
<tr>
<td>DRC</td>
<td>Dynamic Roll Control</td>
<td></td>
</tr>
<tr>
<td>FAA</td>
<td>Federal Aviation Administration</td>
<td></td>
</tr>
<tr>
<td>FSAM</td>
<td>Flight Safety Assessment and Management</td>
<td></td>
</tr>
<tr>
<td>GCS</td>
<td>Guidance, Control, and Systems</td>
<td></td>
</tr>
<tr>
<td>GTM</td>
<td>Generic Transport Model</td>
<td></td>
</tr>
<tr>
<td>LOC</td>
<td>Loss-of-Control</td>
<td></td>
</tr>
<tr>
<td>LQR</td>
<td>Linear Quadratic Regulator</td>
<td></td>
</tr>
<tr>
<td>LQT</td>
<td>Linear Quadratic Tracker</td>
<td></td>
</tr>
<tr>
<td>MRAC</td>
<td>Model Reference Adaptive Control</td>
<td></td>
</tr>
<tr>
<td>NASA</td>
<td>National Aeronautics and Space Administration</td>
<td></td>
</tr>
<tr>
<td>PE</td>
<td>Persistence of Excitation</td>
<td></td>
</tr>
<tr>
<td>PFD</td>
<td>Primary Flight Display</td>
<td></td>
</tr>
<tr>
<td>PIO</td>
<td>Pilot-Induced Oscillation</td>
<td></td>
</tr>
<tr>
<td>QLC</td>
<td>Quantitative Loss-of-Control Criteria</td>
<td></td>
</tr>
<tr>
<td>SAFE-Cue</td>
<td>Smart Adaptive Flight Effective Cue</td>
<td></td>
</tr>
<tr>
<td>SI</td>
<td>Structural Integrity</td>
<td></td>
</tr>
<tr>
<td>SUPRA</td>
<td>Simulation of Upset Recovery in Aviation</td>
<td></td>
</tr>
<tr>
<td>Acronym</td>
<td>Description</td>
<td></td>
</tr>
<tr>
<td>---------</td>
<td>--------------------------------------------------</td>
<td></td>
</tr>
<tr>
<td>SVS</td>
<td>Synthetic Vision System</td>
<td></td>
</tr>
<tr>
<td>T2G</td>
<td>Trajectory-To-Go</td>
<td></td>
</tr>
<tr>
<td>TEM</td>
<td>Threat Error Management</td>
<td></td>
</tr>
<tr>
<td>UA</td>
<td>Unusual Attitude</td>
<td></td>
</tr>
<tr>
<td>UPRT</td>
<td>Upset Prevention and Recovery Training</td>
<td></td>
</tr>
<tr>
<td>VSST</td>
<td>Vehicle Systems Safety Technologies</td>
<td></td>
</tr>
</tbody>
</table>
LIST OF SYMBOLS

<table>
<thead>
<tr>
<th>Symbol</th>
<th>Description</th>
</tr>
</thead>
<tbody>
<tr>
<td>$A, B$</td>
<td>Aircraft state-space characterization matrices</td>
</tr>
<tr>
<td>$A_0, B_0$</td>
<td>Nominal aircraft state-space characterization matrices</td>
</tr>
<tr>
<td>$\Delta A, \Delta B$</td>
<td>Aircraft uncertainty matrices</td>
</tr>
<tr>
<td>$\alpha$</td>
<td>Angle-of-attack</td>
</tr>
<tr>
<td>$\beta$</td>
<td>Sideslip angle</td>
</tr>
<tr>
<td>$\mathcal{C}$</td>
<td>State limit target matrix for critical trajectory prediction</td>
</tr>
<tr>
<td>$\delta_e, \delta_a, \delta_r, \delta_t$</td>
<td>Elevator deflection, aileron deflection, rudder deflection, and throttle input</td>
</tr>
<tr>
<td>$\hat{E}$</td>
<td>Parameter estimation cost function for adaptation component</td>
</tr>
<tr>
<td>$\eta$</td>
<td>Damping term for adaptation component</td>
</tr>
<tr>
<td>$\phi$</td>
<td>Bank angle</td>
</tr>
<tr>
<td>$\gamma_A, \gamma_B$</td>
<td>Learning rate for adaptation component</td>
</tr>
<tr>
<td>$\gamma$</td>
<td>Flight path angle</td>
</tr>
<tr>
<td>$J$</td>
<td>Cost function for critical trajectory prediction</td>
</tr>
<tr>
<td>$K$</td>
<td>Feedback gains for critical trajectory prediction</td>
</tr>
<tr>
<td>$\kappa$</td>
<td>System dynamics coefficients</td>
</tr>
<tr>
<td>$\lambda$</td>
<td>Lagrange multiplier for adaptation component</td>
</tr>
<tr>
<td>$M$</td>
<td>Peak time forced-response coefficients</td>
</tr>
<tr>
<td>$\mu$</td>
<td>System dynamics forced-response coefficients</td>
</tr>
<tr>
<td>$\bar{\mu}$</td>
<td>Initial condition forced-response coefficients</td>
</tr>
<tr>
<td>$N$</td>
<td>Peak time unforced-response coefficients</td>
</tr>
<tr>
<td>$\nu$</td>
<td>System dynamics unforced-response coefficients</td>
</tr>
<tr>
<td>$\theta$</td>
<td>Pitch angle</td>
</tr>
</tbody>
</table>
\( \omega_d, \omega_n \)  
Damped/undamped natural frequency

\( \dot{P} \)  
Solution of continuous-time Lyapunov Equation for adaptation component

\( p \)  
Roll rate

\( \bar{Q} \)  
State weighting matrix for critical trajectory prediction

\( \dot{Q} \)  
Weighting matrix for adaptation component

\( q \)  
Pitch rate

\( \bar{R} \)  
Input weighting matrix for critical trajectory prediction

\( r \)  
Yaw rate

\( \bar{r} \)  
Final state constraint for critical trajectory prediction

\( S \)  
Solution to Riccati Equation for critical trajectory prediction

\( \hat{T} \)  
Length of prediction window

\( \hat{t} \)  
Prediction window time scale

\( t \)  
Real-time time scale

\( U \)  
Control input

\( U_{\text{crit}} \)  
Critical control deflection

\( U^*_{\text{crit}} \)  
Critical control trajectory (Trajectory-To-Go)

\( \Delta U \)  
Deflection-To-Go

\( u \)  
Forward velocity

\( v \)  
Modified state transition vector for critical trajectory prediction

\( X \)  
States of aircraft model

\( \zeta \)  
Damping ratio
CHAPTER 1
INTRODUCTION

Statistics compiled by the Boeing Company over the last decade [1] have shown that the three most significant factors behind fatal air accidents are Loss-of-Control (LOC) In-flight, Runway Excursion (RE) during landing, and Controlled Flight Into Terrain (CFIT). Of these, loss-of-control is the single largest contributor to fatal air accidents worldwide, with 13 air accidents and almost 1,200 fatalities attributed to LOC over the decade spanning 2009 to 2018. LOC was deemed to be the primary cause behind 25% of fatal air accidents and 47% of fatalities over this time period. Figure 1.1 provides an illustration of the significance of LOC-attributed air accidents across the worldwide commercial jet fleet [1].

![Figure 1.1. Statistical Summary of Air Accidents – Worldwide Commercial Jet Fleet (2009-2018)](image)

<table>
<thead>
<tr>
<th>Cause</th>
<th>Definition</th>
<th>Fatalities (Accidents)</th>
</tr>
</thead>
<tbody>
<tr>
<td>LOC-I</td>
<td>Loss-of-Control In-Flight</td>
<td>1183 (13)</td>
</tr>
<tr>
<td>CFIT</td>
<td>Controlled Flight Into Terrain</td>
<td>568 (10)</td>
</tr>
<tr>
<td>RE</td>
<td>Runway Excursion (Landing) including Abnormal Runway Contact and Undershoot/overshoot</td>
<td>186 (7)</td>
</tr>
<tr>
<td>SCF-PP</td>
<td>Powerplant Component Failure or Malfunction</td>
<td>164 (2)</td>
</tr>
<tr>
<td>FUEL</td>
<td>Fuel-related Causes</td>
<td>71 (1)</td>
</tr>
<tr>
<td>OTHER</td>
<td>RE (Takeoff), Ground handling, Midair Collision, Fire/Smoke, Other Causes, Runway Incursion</td>
<td>31 (14)</td>
</tr>
<tr>
<td>UNK</td>
<td>Unknown or Undetermined</td>
<td>329 (4)</td>
</tr>
</tbody>
</table>

Despite technological progresses in recent times, there have continued to be notable air accidents attributed to LOC. Air France Flight 447 [2] (June 2009) was an Airbus A330 that had experienced a failure of its Air Data Inertial Reference Unit, leading to a malfunction in its airspeed
instrumentation. The aircraft entered an aerodynamic stall and crashed into the Atlantic Ocean, claiming 228 lives. Adam Air Flight 574 [3] (January 2007) was a Boeing 737 whose crew had lost situational awareness and suffered spatial disorientation while troubleshooting onboard equipment issues. The aircraft entered an upset state and crashed into a strait in Sulawesi, Indonesia, claiming 102 lives. American Airlines Flight 587 [4] (November 2001) was an Airbus A300 that suffered a catastrophic structural failure brought on by excessive reaction to a wake vortex encounter. The aircraft crashed into a neighborhood in Queens, New York, claiming 265 lives. Historical records will show many more high-impact LOC-related accidents that have occurred beyond these three examples. One may reasonably argue that LOC has been a prevalent issue for as long as aircraft have been in atmospheric flight.

1.1 Understanding the Aircraft Loss-of-Control Problem

Given the notion that LOC has continually posed a significant risk to aircraft, a subset of the aerospace research community has focused much effort in recent times on the mitigation of aircraft loss-of-control. These efforts have covered causal analysis of aircraft control loss, the development of methods to detect, prevent, and recover from LOC, and the development of improved cues to warn flight crews of impending control loss. Much of this research has been conducted by the National Aeronautics and Space Administration (NASA), a limited number of universities, and some private organizations.

In this dissertation, it is postulated that the LOC problem needs to be categorized and examined in the sequence in which it occurs, beginning with causation, leading to detection, then prevention, and finally recovery. In order for efforts at mitigating LOC to be effective, a good understanding and knowledge of its causation is a prerequisite. If its intrinsic causes are understood, methods aimed at its detection would have a higher level of effectiveness. If the detection or advance prediction of LOC is made with a reasonable amount of accuracy, then the
prevention of LOC stands a higher chance of success. Finally, if LOC is then deemed to be inevitable, the possibilities of recovery from an LOC situation need to be considered. All these inherently depend not only on a fundamental understanding of the causal factors of LOC, but also on the dynamics and control of the aircraft in a situation where departure from linear behavior is typical and expected. Figure 1.2 illustrates the progression and subsets of the aircraft loss-of-control problem:

![Figure 1.2. Progression and Subsets of the Aircraft Loss-of-Control Problem](image)

For each of the subsets of the LOC problem, progress has already been made to better understand and address the inherent underlying issues. In the determination of its causes, efforts have been made to develop a set of agreed-upon quantitative metrics that define LOC. Further, extensive analysis has been done to classify its causative factors. These quantitative metrics and understanding of its causal factors have allowed for the development of adaptive methods to detect LOC while the aircraft is in motion. Recent research has taken detection a step further and proposed methods to predict its onset ahead of time given the aircraft’s current states and pilot inputs. Given the ability to detect LOC, work has also been done on developing control intervention and recovery mechanisms to prevent and/or recover from its occurrence. Research
efforts have also proposed tactile and visual cues to alert flight crews to its onset and provide guidance to aid in recovery. Most recently, the use of augmented reality head-up technologies has also been proposed as a means of rapidly providing flight crews with these visual cues. All of this research comes together to form a full-spectrum approach towards combating the aircraft loss-of-control problem.

1.2 Loss-of-Control Research Presented in this Dissertation

The work proposed in this research slots into multiple locations of the detection and prevention subsets of the broader LOC mitigation effort. Along these directions, the research documented in this dissertation is a novel concept targeted at mitigating the onset of LOC through the use of an adaptive prediction architecture that predicts an aircraft’s safe control envelope or safe control space and warns pilots of future impending entry into loss-of-control. A simplified overview of the architecture’s concept is presented in Figure 1.3:

![Figure 1.3. Loss-of-Control Prediction Architecture Concept](image)

The prediction architecture continuously estimates the higher-order aircraft’s current dynamics and analytically determines critical control deflection, rate, and acceleration limits that would drive the aircraft to control loss several seconds ahead of time. These limits then form the margins of a safe control envelope or safe control space that should not be exceeded, if the aircraft were to be kept within its safe flight envelope. These safe control margins are then presented to the pilot through visual, aural, or tactile feedback, suggesting corrective action so as to avert entry into
LOC. While the predictive elements address the subset of prediction, the feedback elements address the subset of prevention. Together, they form a two-pronged approach at combating aircraft LOC.

The first prong centers on the **prediction** aspect. Through the course of this research, two prediction models were developed: The first determines a “deflection-to-go” (D2G), which is defined as the critical amount of control surface deflection that, if the pilot were to apply now, would cause the aircraft to enter an LOC condition at some specified point in the future. The second calculates a “trajectory-to-go” (T2G), which is defined as the critical control surface trajectory that, if the pilot were to apply, would again cause the aircraft to enter an LOC condition. In the latter, the calculations are based on optimal control techniques. The analytical LOC prediction methods described here consider not only the prototypical step input, but also consider the higher order ramp and parabolic inputs. Also taken into account in the prediction architecture are the effects of multiple control effectors operating within a single mode and the presence of non-zero initial conditions on the states and inputs. The complete architecture also incorporates an adaptive element in the form of real-time system model identification, which accounts for uncertainties (such as failures or modeling error) in the nominal aircraft model. Such uncertainties are representative of the conditions that might be present leading up to an LOC event.

The second prong centers on the **prevention** aspect. The critical control surface deflections or trajectories are then presented to the pilot through the use of head-up augmented reality (AR) technologies, such as Google Glass and Microsoft HoloLens, to provide the pilot with rapid situational awareness of the LOC condition. With this feedback, the pilot could then apply a corrective input to the aircraft, driving the aircraft away from the LOC margin towards the safe region of the flight envelope.
While existing research into LOC has broadly focused on the post-LOC recovery aspects of the problem, less work has focused on the detection and prevention aspects, specifically the notion of predicting the onset of LOC given current control inputs, and then providing passive real-time feedback and advance warning to the pilot. In the long term, the goal of this research is the implementation of these predictive technologies aboard production aircraft. The author aspires that such predictive elements would create aircraft that are ultimately easier and safer to fly, and therefore also create an advantage for the aircraft in its regulatory certification process.

With this in mind, this dissertation is structured as follows: The general definitions of LOC and its causes are reviewed in Chapter 2, a survey of advancements in LOC mitigation is presented in Chapter 3, and the LOC prediction concept proposed in this research is detailed in Chapter 4. A detailed look at the aircraft dynamic model is then presented in Chapter 5, the LOC margin prediction models are introduced in Chapter 6 and Chapter 7, the adaptive parameter estimation framework is detailed in Chapter 8, and the pilot advisory display component is presented in Chapter 9. The light business jet used in verification and validation of the prediction models, as well as results illustrating the performance of the prediction architecture, are then presented in Chapter 10. Results from verification and validation through simulated flight testing are next presented in Chapter 11. Finally, closing remarks are presented in Chapter 12.
CHAPTER 2
DEFINING AIRCRAFT LOSS-OF-CONTROL

In 1998, the Commercial Aviation Safety Team (CAST) was formed to identify methods
and strategies to increase public safety in commercial air travel. CAST had identified LOC as one
of the major areas of concern in commercial aviation [5]. A fundamental first step in the mitigation
of LOC lies in the ability to identify its traits and detect its presence. Towards this goal, the Boeing
Company and NASA Langley Research Center jointly developed a set of metrics [6] to
quantitatively define LOC events, and these metrics are known as the Quantitative Loss-of-Control
Criteria (QLC). To arrive at these quantitative metrics, one should first consider the qualitative
flight behaviors that are typical of an LOC event. In general, an LOC event is characterized by
aircraft motion that:

• is outside the normal operating flight envelopes
• does not change in a predictable manner with a given pilot input
• consists of nonlinear characteristics, such as oscillatory behavior or disproportionately
  large responses
• would likely result in high displacements and angular rates, and
• is characterized by the lack of ability to maintain heading, altitude, and wings-level flight

2.1 Quantification of LOC

Building upon the characteristics defined above, the CAST’s analysis of LOC had
identified its primary causes, in order of decreasing significance, as: stalls, sideslip-induced rolls,
rolls from other causes, pilot-induced oscillation (PIO), and yaw. Overspeed was also found to
have an ancillary role in LOC. These factors formed the basis behind the development of the QLC,
with significant flight dynamic parameters identified as: angle-of-attack (\(\alpha\)), bank angle (\(\phi\)),

7
equivalent airspeed ($V_E$), pitch angle ($\theta$), pitch control ($\delta_e$), pitch rate ($q$), roll control ($\delta_a$), roll rate ($p$), sideslip angle ($\beta$), and load factor ($n$). From this, five flight envelopes were derived that represent the essence and relationships between these significant parameters. These are the Adverse Aerodynamics (AA), Unusual Attitude (UA), Structural Integrity (SI), Dynamic Pitch Control (DPC), and Dynamic Roll Control (DRC) envelopes.

![Figure 2.1. Adverse Aerodynamics Envelope](image1)

![Figure 2.2. Unusual Attitude Envelope](image2)

![Figure 2.3. Structural Integrity Envelope](image3)

The Adverse Aerodynamics envelope (Figure 2.1) maps normalized angle-of-attack ($\alpha_{norm}$) against normalized sideslip ($\beta_{norm}$), and indicates stall and adverse sideslip angles, accounting for the effects of sideslip-induced roll. Here, the angle-of-attack and sideslip limits would be determined based on the physical characteristics of the aircraft. The Unusual Attitude envelope (Figure 2.2) maps bank angle ($\phi$) against pitch angle ($\theta$). This envelope is built around the industry’s agreed-upon characterization of unusual attitude, imposing limits of $\pm 45^\circ$ for bank angle and $+25^\circ/-10^\circ$ for pitch angle. The Structural Integrity envelope (Figure 2.3) maps normalized airspeed ($V_{norm}$) against load factor ($n$). This envelope incorporates requirements from the Federal Aviation Administration’s (FAA) Part 25 regulations for aircraft structural design, imposing limits of $-1.0g/+2.5g$ and $0.0g/+2.0g$ for flaps-up and flaps-down configurations, respectively.
The Dynamic Pitch Control (Figure 2.4) and Dynamic Roll Control (Figure 2.5) envelopes respectively show whether the trends (or changes) in pitch and bank angle are consistent and show positive correlation with given pitch and roll commands. Here, dynamic pitch attitude ($\theta'\theta$) is defined as the sum of pitch angle and pitch rate ($\theta + \dot{\theta}$). Likewise, dynamic roll attitude ($\phi'\phi$) is defined as the sum of roll angle and roll rate ($\phi + \dot{\phi}$). This notion indicates that if the given inputs are resulting in expected responses in the aircraft’s actual pitching and rolling motion, or if they are instead resulting in opposing responses. Each of the four quadrants, (I–IV) relates to a safe or unsafe flight configuration.

A flight trajectory that stays within the appropriate bounds of these envelopes constitutes one that promotes safe flight. Conversely, a trajectory that exceeds the bounds of one or more these envelopes would potentially result in a LOC event. Wilborn and Foster [6] mapped historical flight test and accident analysis data against these envelopes to illustrate their validity as LOC margins. In this sense, we now have a series of standardized metrics that allow for the quantification of LOC. Pfifer et. al. [7] extended this effort through the quantification of these envelopes using robust tracking analysis, wherein the closed-loop tracking performance of onboard flight controllers is also taken into account (in addition to the aircraft dynamics).
2.2 Analysis of Causal Factors

Thus far, this chapter has characterized LOC, considered flight parameters that are associated with such events, and shown how these parameters may be used to define and set margins quantifying LOC. At this point, the discussion takes a deeper look at the empirical causes of control loss in aircraft. Table 2.1 and Table 2.2 respectively illustrate the number of fatal accidents associated with various causal factors and phases of flight.

TABLE 2.1

CAUSAL FACTORS LEADING TO LOC, 1999-2008
(ADAPTED BASED ON JACOBSON [8])

<table>
<thead>
<tr>
<th>Causal Factor</th>
<th>Number of Fatal Accidents</th>
</tr>
</thead>
<tbody>
<tr>
<td><strong>Human-induced</strong></td>
<td></td>
</tr>
<tr>
<td>Improper procedure</td>
<td>10</td>
</tr>
<tr>
<td>Spatial disorientation</td>
<td>6</td>
</tr>
<tr>
<td>Poor energy management</td>
<td>6</td>
</tr>
<tr>
<td>Distraction</td>
<td>5</td>
</tr>
<tr>
<td>Improper training</td>
<td>5</td>
</tr>
<tr>
<td>Poor design</td>
<td>2</td>
</tr>
<tr>
<td><strong>Environmentally-induced</strong></td>
<td></td>
</tr>
<tr>
<td>Weather</td>
<td>3</td>
</tr>
<tr>
<td>Icing</td>
<td>2</td>
</tr>
<tr>
<td>Wake vortex</td>
<td>1</td>
</tr>
<tr>
<td><strong>Systems-induced</strong></td>
<td></td>
</tr>
<tr>
<td>Aircraft system failures</td>
<td>5</td>
</tr>
<tr>
<td>Poor design</td>
<td>2</td>
</tr>
</tbody>
</table>

It is commonly agreed upon that LOC is a complex event that is typically attributed to a variety of causal factors either occurring individually or in combination. NASA has done extensive reviews of historical air accidents in an effort to classify factors behind these events. In its 2010
study [8] analyzing historical data from 1999-2008, it was determined that causal and contributing factors leading to LOC could be approximately classified into human-induced, environmentally-induced, and systems-induced categories.

Human-induced factors were responsible for the greatest number of accidents, while systems-induced factors were responsible for the least. The high-risk phases of flight, during which LOC had occurred the most, involved the aircraft operating at low altitude, such as during takeoff, climb, and final approach.

| TABLE 2.2 |
| PHASES OF FLIGHT INVOLVED IN LOC, 1999-2008 |
| (ADAPTED BASED ON JACOBSON [8]) |

<table>
<thead>
<tr>
<th>Phase of Flight</th>
<th>Number of Fatal Accidents</th>
</tr>
</thead>
<tbody>
<tr>
<td>Climb</td>
<td>6</td>
</tr>
<tr>
<td>Takeoff</td>
<td>5</td>
</tr>
<tr>
<td>Final approach</td>
<td>3</td>
</tr>
<tr>
<td>Initial climb</td>
<td>2</td>
</tr>
<tr>
<td>Cruise</td>
<td>2</td>
</tr>
<tr>
<td>Initial approach</td>
<td>2</td>
</tr>
<tr>
<td>Landing</td>
<td>2</td>
</tr>
</tbody>
</table>

Placing an emphasis on the complicated and interdependent nature of LOC, the same report stated that no single category was singularly responsible for these accidents, and efforts toward mitigation had to be multi-faceted and simultaneously targeted at the different causes. Towards this notion, Belcastro and Foster [9] sought to identify the worst-case causal combinations and sequencing of contributing factors behind LOC events. In this study, 126 LOC accidents between 1979 and 2009 were analyzed, and the authors arrived at seven generalized sets of event sequences most responsible for leading to LOC. These sequences map the highest-risk combinations of causal...
factors (and their chronological progressions) that have historically caused aircraft to depart from normal flight and enter an LOC event. Determination of these causal factors is key to pinpointing factors that have to be addressed in LOC mitigation efforts. The top three event sequences leading to the greatest number of accidents/fatalities are presented in Figure 2.6:

![Diagram of event sequences leading to LOC](image)

Figure 2.6. Generalized Event Sequences Leading to LOC (adapted based on Belcastro and Foster [9])

In total, these three sequences were attributed to 60 accidents and 2,950 fatalities. An updated study conducted in 2014 by Belcastro et. al. [10] once again concluded that the System & Component Failures/Malfunctions, Inappropriate Crew Input/Action, and Vehicle Upset Conditions contributed the most to the worst-case combinations of factors behind LOC events. The next four sequences are presented in Figure 2.7. These sequences were attributed to 32 accidents and 1,672 fatalities.
An observation of Figure 2.6 and Figure 2.7 gives an insight into the key causal factors complicit in an LOC event. Five of the seven sequences leading to LOC involve a vehicle upset consisting of abnormal attitudes and/or stall conditions. An understanding of the nonlinear dynamics of flight under LOC is certainly crucial. Furthermore, and perhaps of greater significance, five of the seven sequences involved some form of inappropriate crew responses either prior to or after a vehicle problem or external hazard. It is evident that the inherent pilot-centric issues behind LOC themselves warrant greater discourse.

### 2.2.1 Human-Induced Factors

Human-induced causes leading to LOC contain an extended list of factors, including but not limited to improper training or procedures, poor pilot situational or spatial awareness, automation-related issues leading to operational confusion, and poor energy management.
Combined, these factors have been attributed to the greatest number of accidents stemming from LOC. A recent accident attributed to such causes is that of Asiana Airlines Flight 214 [11] in July 2013.

Human-induced factors encompass a wide range of issues, and a subset of research has been dedicated to this aspect of LOC. It is generally agreed that, while fly-by-wire aircraft are less susceptible to LOC, they are still susceptible to pilot-induced actions that could result in fatal accidents. One recent report [12] states that current assumptions made in training civil transport pilots, while valid for normal flight conditions, do not adequately prepare flight crews for handling airplane upset conditions. The same report postulated that changes to the training regimen to account for pilot competencies needed in actual LOC encounters was necessary to mitigate the threat presented by human-induced LOC.

In understanding the human aspect at the elemental level, theoretical studies have focused on pilot mental models and LOC. Mamessier et. al. [13] developed a computational human agent simulating cognitive constructs related to human-automation interaction (namely situational awareness and mental models) to study continuous and discrete aspects of aircraft and automation behavior. In the study, three facets of pilot mentality were deemed vital to the LOC problem:

- Pilot monitoring and awareness drives the pilot’s cognizance of aircraft orientation and energy management.
- Pilot knowledge and awareness of continuous dynamics drives pilot manual control behavior and interaction with autoflight systems.
- Pilot knowledge and learning of discrete dynamics could be used to highlight where pilot mental models of autoflight systems could be faulty or incomplete.
These conclusions tie in with other findings which indicate that overly complex human-automation interaction, especially if it includes poor understanding of automation systems or information overload from such systems, is a key contributor to LOC events. Analysis of historical accident data has also shown that human-induced factors leading to LOC may additionally be catalyzed by environmental- and system-induced factors [14], which make up the other two categories of causal factors. It was noted that changing environmental conditions and out-of-routine indications (necessitating human-automation interaction), especially in critical phases of flight, could translate into distractions leading to panic/confusion, performance overload, or task fixation. These are discussed in greater detail below.

2.2.2 Environmentally-Induced Factors

Environmental factors leading to LOC typically involve some type of adverse weather condition, with the phenomena most dangerous to aircraft being microburst/wind shear, wake vortex, and icing conditions. Microburst conditions often have sudden onsets accompanied by wind shear. The abrupt losses in airspeed often cause aircraft to rapidly lose altitude, tying in also to the energy management aspects of LOC events. One accident attributed to microburst conditions was the crash on approach of Delta Airlines Flight 191 [15] in August 1985. On the other hand, ice buildup on aircraft wing and control surfaces has been known to cause hard-to-detect changes in aerodynamic performance that are severe enough to lead to an LOC event. Research into icing effects on atmospheric aircraft has demonstrated increases in drag of over 3,500% on an iced wing versus a clean wing [16]. Two accidents which were icing-related are that of American Eagle Flight 4184 [17] in October 1994 and Colgan Air Flight 3407 [18] in February 2009. The causal relationship between icing and LOC has been recognized through recent research highlighting its propensity to unexpected stall conditions and proposing mitigations thorough updated skills training for flight crew [19, 20].
2.2.3 System-Induced Factors

With improved technological advancements and regulatory standards set forth in the last several decades, LOC stemming from system-induced factors has been attributed to the least number of accidents. Factors in this category include but are not limited to poor design, erroneous sensor data, aircraft system failures, and pilot-induced oscillation. One example of an accident related to system-induced factors, in this case failure of the propulsion system, is that of British Airways Flight 38 [21], which suffered fuel starvation and a subsequent lack of thrust produced by the engines while on approach to land.

2.3 Nonlinear Nature of Flight Under LOC

The potential nonlinear nature of aircraft dynamics when LOC occurs brings associated challenges in designing LOC mitigations. One flight regime implicitly related to LOC is the aerodynamic stall. An accurate understanding of the continually changing dynamics under these operational regimes is crucial for the development of successful prediction, prevention, and recovery schemes. In Kwatny et. al. [22], an analysis of the nonlinear dynamics and influences of an aircraft under LOC found that an aircraft’s set of designed equilibrium points and corresponding control properties translate to direct links between the pilot’s LOC experience and analytical flight mechanics; in other words, when operating around a stall condition, the aircraft’s control properties can significantly alter with small changes in the aircraft’s states or parameters. Viable trim points comprising admissible pairs of control and state values might or might not exist for such conditions, presenting additional challenges for the pilot.

It follows that, for the majority of the detection, prediction, and recovery schemes cited in this article to function properly, it is implicitly understood that accurate modeling and simulation of the aircraft’s behavior, specifically its nonlinearities, is a prerequisite for successful results. The lack of inclusion of such characteristics in flight simulation models used in pilot training further...
has the potential to lead to incorrect pilot responses to the onset of LOC events [23]. NASA has undertaken a significant multi-year effort to advance the difficult task of computational fluid dynamics (CFD) modeling of the highly nonlinear flows associated with aircraft LOC (due to airframe damage or wing stall). Frink et. al. [24] documented the development of these CFD tools, utilizing the Reynolds-averaged Navier-Stokes (RANS) formulation, and described its potential benefits for augmenting existing simulation models with accurate stall and post-stall data.

2.4 Bridging Causal Analysis and LOC Mitigation Efforts

In summarizing this section, we have established that sufficient understanding of the causal and contributing factors leading to an LOC event is a necessary prerequisite for development of successful LOC mitigation schemes.

While no single intervention strategy is independently effective at countering LOC, if the causal factors responsible for LOC are known, action can be taken to directly address those factors and prevent normal flight from transitioning to an LOC event. Belcastro et. al. had led multiple efforts in the development of technology solutions [25, 26] and future safety-critical systems [27] that aid in LOC detection, prevention, and recovery rely heavily on such causal analysis. The causal event sequences discussed above tie into the future system concepts proposed by Belcastro and Jacobson [28], which stresses the need for a holistic approach towards mitigating LOC. These concepts are summarized in Figure 2.8:

In the following chapter, an overview and discussion of the research efforts fulfilling various aspects of the future systems concept is presented.
Figure 2.8. Holistic Approach to LOC Mitigation Through Development of Future Systems (adapted based on Belcastro and Jacobson [28])
CHAPTER 3

AN OVERVIEW OF ADVANCEMENTS IN LOSS-OF-CONTROL MITIGATION

In fulfilling various aspects of the future systems concept introduced in the previous chapter, recent research efforts have addressed vehicle health management (automatic fault and damage system identification), flight safety management/resilient control (real time detection, prediction, and recovery from LOC), and safe flight deck systems/operations (situational awareness through visual/tactile cues).

In this chapter, these research efforts are concisely amalgamated and presented to the reader as a holistic primer on current advancements in addressing the LOC problem. The progression of the LOC problem is categorized into the detection, prevention, and recovery phases, and a comprehensive literature survey of research efforts in each of the above subsets is presented. In the survey methods proposed towards realtime adaptive margin estimation, prevention methodologies, recovery approaches, and pilot advisory cueing for avoiding LOC are reviewed. These research efforts come together to form the elements of the LOC mitigation effort.

3.1 Detection and Prediction of Loss-of-Control

Various research efforts have been made in the area of LOC detection and prediction. These efforts center around attempts to provide flight crews with a pre-emptive warning of an impending LOC event. Chongvisal et al. [29] proposed a prediction algorithm where the aircraft’s current behavior was monitored and compared against quantified LOC envelopes. If multiple envelopes were exceeded, a determination was then made as to whether the aircraft would enter an LOC event. Krishnakumar et. al. [30] and related work [31-33] paired adaptive parameter estimation with an algorithm that computed an estimate of the deficiencies present in the control input signal, it being a good indicator of proximity to LOC. Rafi et. al. [34-36] coupled adaptive parameter
estimation with continuous monitoring of the aircraft’s current behavior and pilot input to predict excursion of LOC margins, where the aircraft’s subsequent future-time state trajectory and a prediction of the critical control deflection leading to LOC were computed. Frequency-based methods have also been proposed in Thompson et. al. [37] and Rajaram et. al. [38]. The former proposed a wavelet transform method to predict the onset of LOC, where the technique was demonstrated to have a faster detection time compared to Fourier transform methods, while the latter proposed a Fast Fourier Transform technique to analyze the frequency content of given pilot inputs, then comparing those to critical frequencies related to LOC. A generalized LOC detection/prediction architecture is illustrated in Figure 3.1:

![Figure 3.1. Generalized LOC detection and prediction framework](image)

### 3.1.1 Parameter Estimation

The accuracy of the predictions made by the LOC models typically depend upon a priori knowledge of the aircraft’s dynamics – the mechanisms rely upon a known linear model of the aircraft which is used to determine proximity to LOC. While small deviations from the nominal model do not usually cause significant inaccuracies, larger deviations would potentially invalidate the predictions made by the LOC architectures. In making valid LOC predictions, the intermediate
goal then, is to ensure that the prediction models receive good and current knowledge of the aircraft’s dynamics as the aircraft is in motion. Adaptive elements are often employed to fulfil this requirement. This is especially critical because of the nonlinear nature of flight under which LOC occurs.

In the matched uncertainty estimation method as proposed by Krishnakumar et. al. [30], the actual uncertainties are assumed to be structured and are required to follow a form determined by pre-defined parameter matrices. The unmatched uncertainty estimation method proposed in Rafi et. al. [34] provides an alternative technique capable of producing accurate estimates without the need for structured uncertainties to be present. Both utilize first-order gradient descent optimization to obtain the derivatives of the estimated uncertainty matrices, which are then integrated to obtain estimates of the uncertainties.

3.1.2 LOC Prediction Methodologies

The primary component of the Detection and Prediction subset comprises the methodologies developed to estimate control margins or compute future-time predictions of potential entry into an LOC condition. The following is a discussion of several of these methods.

3.1.2.1 Control Deficiency Estimation

Krishnakumar et. al. [30] proposed a type of passive adaptive control (akin in structure to Model Reference Adaptive Control) where the closed-loop system’s output is compared to a desired response from a reference model. The error between the derived adaptive control signal and the aircraft control signal is used to compute a control deficiency signal estimate (CDSE), which could either be used to control the aircraft and keep it within the safe flight envelope or correlated with system stability margins and used as a visual feedback mechanism for the pilot. This method is suited to a closed-loop setup where feedback to the controller is desired as a means
to actively maintain the aircraft’s position within the safe flight envelope (see accompanying discussion on critical control trajectory estimation).

In Trujillo and Gregory [39], experimental data was used to develop an indicator of pilot behavior changes from predicted estimates of pilot control input. This was achieved using gradient descent estimation and least-squares-based estimation algorithms. The prediction algorithms were found useful for detecting systemic behavioral changes, and by extension, provided the ability to detect unexpected changes which may be attributed to an off-nominal aircraft or to atmospheric causes. Given such real-time knowledge, it was proposed that onboard automation could be used to assist the pilot through workload alleviation, attention focusing, or tailoring flight director commands to specific pilot behaviors that could better resolve an upset condition.

3.1.2.2 Critical Control Deflection Estimation

A “deflection-to-go” methodology was proposed in Rafi et. al. [35], wherein the time-domain analytical solution of the aircraft dynamic modes was utilized to perform real-time calculations of the critical control surface deflection that, if the pilot were to apply at the current moment, would cause the aircraft to enter an LOC condition within some (pre-specified) time horizon in the near-term future. This method demonstrated good performance characteristics in a variety of cases, including the aircraft operating in steady and unsteady states, multiple control effectors for a single mode (ie. the prediction model factored in current deflections of the secondary control effector when calculating the critical deflection of the primary control effector, and vice versa), and uncertainties and failures arising from non-linearities in the aircraft.

3.1.2.3 Critical Control Trajectory Estimation

A slightly different approach towards the same end was presented in [34, 35], where a “trajectory-to-go” methodology was proposed. Instead of calculating a single critical deflection at each time, this method used optimal control methods to calculate a critical control surface
trajectory that, if the pilot were to apply over a time-horizon at any given time, would again cause the aircraft to enter an LOC condition at some point in the near-term future. Initially, methods were developed, one with an open-loop optimal controller and another with a closed-loop optimal controller in the form of a linear quadratic regulator. Both methods performed well, but lacked the ability to control the manner in which the state trajectory progressed over the course of the prediction time window. A second iteration in the form of a linear quadratic tracker (LQT) addressed this and allowed for critical trajectories to be calculated that better represented a pilot’s control interaction with the aircraft.

3.1.2.4 Critical Control Input Frequency Limit

A unique concept was presented in Rajaram et. al. [38], where the idea of calculating a critical input frequency was proposed. This method would be suitable in cases of pilot-induced oscillation, where the input to the aircraft would resemble a sinusoid. A fast fourier transform of the pilot-commanded input was computed online, and the magnitude of the dominant frequency of the input was compared against the calculated critical elevator needed to exceed the LOC margin within some pre-defined time window (along the lines presented in [34]).

3.1.3 Commentary

The methods presented for determining critical control envelopes rely on having current knowledge of the aircraft’s dynamics as the aircraft is in motion. While the nominal dynamics are known quantities derived through design, off-nominal dynamics may manifest in flight due to uncertainties or failures. In the methods presented, parameter estimation is used as a means of determining these off-nominal dynamics in realtime. One caveat of these estimation algorithms is the need for persistence of excitation (PE) [40]. In the above parameter estimation frameworks, the quality of the estimates is dependent upon the states and inputs of the system – no useful estimates of the parameters are obtainable if all signals are zero. The need for a persistently
exciting input is central to the success of the automatic system identification framework. With this in mind, it may be said that the unmatched uncertainty estimation framework is applicable to a wider variety of failure scenarios, as compared to the matched uncertainty framework, since it is not common that a failure or uncertainty in the aircraft would always structure itself “neatly” in the form of a matched uncertainty.

Fundamentally, these prediction architectures are still based upon linear aircraft models. While this is because the intent is to prevent LOC (thus implying that the aircraft stays within the linear region of the flight envelope), the prediction architectures could undoubtedly be made more robust through considerations of nonlinear flight characteristics (for example, by incorporating a stall model). One question that needs to be addressed is, “How should the information about the critical control envelope be utilized?” The architecture proposed by Krishnakumar et. al. leaves open the option of feedback to a controller, while that of Rafi et. al. presents an open-loop architecture where corrective action is dependent upon the pilot – the pilot utilizes the information presented and takes action to avoid entering LOC. Whether a controller should or should not intervene in such a complex situation needs to be studied more carefully. In a situation which could be rapidly progressing, the pilot may see more pertinent information that a computer does not have knowledge of, and thus be able to gauge an appropriate response. On the other hand, pilot error or incorrect inputs may also exacerbate the problem. Intervention schemes should perhaps have an arbitration mechanism which can decide when to take over if initial pilot inputs do not correct the entry into LOC.
3.2 Loss-of-Control Prevention

Accompanying its analysis of causal factors leading to LOC [9, 10], NASA proposed a wide-spectrum integrated system approach to LOC prevention (Belcastro, [41]) as part of the Vehicle Systems Safety Technologies (VSST) Project, which leverages a combination of guidance, control, and systems (GCS) technologies to aid in mitigating LOC.

![Vehicle Systems Safety Technologies Approach](image)

The GCS architecture (Figure 3.2) incorporates the subsets of flight safety assessment, upset prevention, LOC prevention, and LOC recovery. The former two subsets are focused on detection of failures in control and sensing systems, identification of vehicle impairment, prediction of LOC, and envelope constraint estimation and limitation. Research fulfilling these
subsets was discussed in Chapter 3. The latter two subsets are focused on enabling resilience in control systems to mitigate potential hazards and recover from an upset condition.

The LOC prevention approaches surveyed comprise a broad spectrum of methodologies and generally align with NASA’s proposed framework. The majority of these efforts are focused on the use of resilient control (in the form of control system reconfiguration) to alleviate the severity of an aircraft impairment, while others have proposed flight safety assessment schemes and crew training for LOC-specific situations to aid in prevention of LOC.

3.2.1 Fault-Tolerant and Resilient Control Methods

The use of resilient control towards LOC mitigation forms a large part of the LOC prevention effort. Kim and Calise [42] proposed Model Reference Adaptive Control (MRAC) as a viable assistance mechanism for pilots in the presence of off-nominal conditions stemming from uncertainties and failures in the aircraft. The ability of MRAC to adapt quickly to uncertainties in or changes to the plant dynamics or operating environment and modify control laws with adaptive gains tailored to the off-nominal dynamics makes it an effective mechanism for maintaining control of an impaired aircraft, reducing the potential for excursion of the safe flight envelope. Practical implementations of MRAC also allow for limits to be placed on certain aircraft states, effectively preventing the pilot from bringing the aircraft outside a pre-defined safe flight envelope. Steck et. al. adapted the MRAC framework to a 3 Degree-of-Freedom general aviation aircraft configuration [43-45], and subsequent efforts expanded the architecture to the 6 Degree-of-Freedom aircraft with an adaptive bias corrector and a dynamic inverse controller [46, 47]. With the addition of specific flight envelope protection schemes, the MRAC was also shown to reduce the potential for entry into an upset condition (and in the case of entry, a higher chance of safe recovery) in the presence of severe atmospheric disturbances such as turbulence [48], microburst [49] and wake vortex conditions [50]. Disturbances in the form of sensor noise present unique
challenges in control system design, and the additional implementation of a Modified State Observer [51] and Kalman filter [52] allowed for robust operation in real-world operating conditions, as demonstrated through flight test efforts on the Textron Aviation CJ-144 Fly-By-Wire test bed [53-57]. Various efforts towards implementing optimal control modifications [58, 59] to the MRAC architecture have also demonstrated improved control performance within the degraded aircraft. Piloted evaluations of MRAC have generally showed measurable benefits towards advancing fault-tolerant flight control.

Kwatny et. al. [60] proposed the application of safe set theory towards LOC prevention. Here, the plant controls are restricted to a bounded set of inputs such that there is an admissible control resulting in a trajectory that causes the aircraft to remain within some desired subset of the state space. With the presence of uncertainties or failures, this safe set would shrink accordingly. To maintain controlled flight, there is the additional requirement for the existence of a suitable set of steady motions and the ability to regulate around the particular trim points.

Along a similar vein, Bošković et. al. [61] proposed a multi-modal LOC prevention scheme which sought to rapidly identify failures and reconfigure the control law to compensate for the impairment. This scheme incorporated online prediction of trim points that were achievable with the remaining control authority, such that a new control law could be derived to facilitate a gradual degradation of the closed-loop performance. Using an F/A-18 fighter aircraft model with stabilator failure, simulation examples were presented demonstrating that closed-loop stability could be maintained.

Chang et. al. [62] utilized switching control and H2 control theory as solutions to the actuator jam problem, wherein it was demonstrated that an appropriately reconfigured optimal controller could compensate for the adverse effects of a jammed elevator, contingent upon the
presence of sufficient control authority in remaining control effectors (in this case, engine thrust). Addressing the notion that LOC is inherently a situation that is a result of uncertainties and nonlinearities, it was noted that the reconfigured controllers could further be made robust and adaptive to account for these conditions. The use of engine thrust commands to prevent LOC-related accidents was also proposed in Urnes and Smith [63], where it was demonstrated that thrust augmentation could be used to halt an excursion of an aircraft’s angle-of-attack limit through effecting changes in the aircraft’s airspeed. A further example demonstrated that, by utilizing asymmetric thrust commands, such a system could also limit bank angle excursions.

Towards mitigating engine actuation deficiencies and sensor failures, an adaptive reconfigurable control scheme was proposed in Bošković and Jackson [64], which estimated actuator and engine gains and generated an adaptive control input that tracked a reference model in the presence of slow-engine dynamics. A “virtual sensor” in the form of a state observer was used to compensate for deficiencies in air-data sensing. The combined approaches were tested in simulation using the NASA GTM, and favorable performance was observed in the presence of simultaneous actuator, engine, and sensor failures.

An adaptive control augmentation strategy to enable tolerance to severe winds was proposed in Zhao and Zhu [65, 66]. Building on a trajectory linearization control architecture, a bandwidth adaptation law was implemented based on singular perturbation theory that enabled real-time tradeoffs between tracking performance and maximum wind tolerance. It was demonstrated that the bandwidth adaptation modification could improve the maximum tolerable wind amplitude by up to five times in specific cases, working automatically to reduce the risk of LOC occurrence.
3.2.2 Flight Safety Assessment

An automated decision-based approach to LOC prevention was proposed in Balachandran and Atkins [67], wherein a flight safety assessment and management (FSAM) system incorporating logic- and physics-based models was developed to assist flight crews in the LOC avoidance. Based on Deterministic Timed Büchi Automata, the FSAM system continually monitored flight conditions and performed risk assessments for current flight conditions, providing suggested actions to prevent or recover from LOC, even having the ability to override crew commands in high-risk scenarios. In demonstrating the potential benefits of the proposed approach, the FSAM system was applied in a simulation replicating the incident involving Continental Airlines Flight 1404 [68] (December 2008), where a Boeing 737 aircraft encountered severe crosswinds on takeoff and was unable to maintain directional control, resulting in a runway excursion. The FSAM system was able to recognize when flight crew was no longer able to maintain directional control, and override crew inputs through appropriate transfers of control authority to accompanying onboard controllers. In another example, when onboard controllers were unable to maintain directional control, the FSAM system demonstrated the ability to autonomously decide on entering a takeoff abort state. In both cases, the system successfully prevented runway excursions. In subsequent research, the FSAM system was formulated as a Markov Decision Process [69] and applied as a mitigation strategy against LOC due to in-flight icing [70]. The updated system moved beyond just maintaining safe flight envelopes by working together with envelope-aware flight management systems to formulate alternatives to high-risk scenarios (eg. proposing alternate flight plans around areas with known icing conditions). Di Donato et. al. further proposed an envelope-aware flight management scheme [71] for LOC prevention in the event of rudder actuator jams that combined the FSAM approach with envelope estimation, flight planning, system identification, and adaptive control that could perform “just-
in-time” overrides of crew or flight management system commands prior to LOC to facilitate safe recovery of the aircraft.

3.2.3 Crew Awareness and Handling of LOC

While resilient control and automated flight assessment strategies have demonstrated the ability to reduce or prevent LOC, there remains no substitute for adequate training to enable flight crews to recognize and respond to various upsets that could trigger LOC conditions. Towards countering human-factors induced LOC events, Carroll [72] proposed the integration of expanded Upset Prevention and Recovery Training (UPRT) into the flight crew’s Threat Error Management (TEM) decision process. While commercial licensing training familiarizes pilots with operating in and slightly around the typical operational flight envelope, UPRT is a specialized addition that expands training to the full potential flight envelope. It seeks to provide flight crews with knowledge pertaining to escalation and recovery flight paths and skills required to bring an aircraft back to a safe flight configuration even in severe upsets. Nooij et. al. [73] performed an assessment of the SUPRA (Simulation of Upset Recovery in Aviation) aerodynamic model in simulated flight testing, and demonstrated its benefits for UPRT. UPRT is to be integrated with TEM, a human factors concept which seeks to have pilots consider not only external influences that compromise safety (ie. threats), but also their own reactions and/or mistakes to said influences (ie. errors). Integrated with each other and applied to LOC, the combined notion of UPRT and TEM seeks to prevent an LOC event from transpiring through improved crew awareness and effective skills to aid in LOC recovery.

3.2.4 Commentary

The various methods presented to aid in the prevention of LOC predominantly involve reconfiguration of control laws or activation of alternate control laws specific to the adverse flight configuration. These methods have shown good promise in preventing the onset of control loss,
demonstrated primarily through results from simulation. It would be beneficial to further validate the effectiveness of these methods through pilot-in-the-loop simulated (or perhaps actual) flight testing.

Further, there is a need to consider potential coupling between longitudinal and lateral/directional aircraft dynamics. While this coupling is typically negligible for normal flight configurations, adverse flight configurations with combinations of high angles-of-attack and large sideslip angles (such as those associated with LOC) may indeed exhibit coupled dynamics. The methods surveyed should consider the suitability of the proposed prevention mechanisms in the presence such coupled dynamics.

Ultimately, while increasing crew awareness of the threats and symptoms of impending LOC is important, it is also crucial to increase crew awareness of how these automated mechanisms are working to prevent LOC. The unified integration of all these mitigation efforts – resilient control, safety assessment, and crew awareness training – is needed, such that recovery actions initiated by the crew and those initiated by the flight management systems do not contradict, and so that they indeed resolve towards the goal of preventing the onset of an LOC event.
3.3 Loss-of-Control Recovery

If prediction or prevention of LOC fail, or if an LOC event is triggered despite these efforts, the mitigation effort has to shift to recovery from LOC. An overview of recovery approaches is given in Figure 3.3:

![Figure 3.3. Overview of LOC Recovery Approaches](image)

While linear analysis techniques may be sufficient for the former two subsets (since the prevention of LOC implies that the aircraft stays mostly within the safe “linear” portion of the flight envelope), recovery approaches have to rely more on nonlinear analysis techniques, since the aircraft would theoretically be in the nonlinear region of the flight envelope for a recovery strategy to be warranted. Given this, recovery strategies may utilize approaches such as sliding
mode control or adaptive methods using artificial neural networks to account for nonlinearities in the aircraft dynamics.

3.3.1 High Order Sliding Mode Control

Various research efforts have proposed the use of sliding mode control to facilitate autonomous recovery from an LOC event. Dongmo [74] demonstrated the use of high order sliding mode control to assist with post-LOC recovery, with the intent to regulate key aircraft states to acceptable levels of performance after an upset. Simulations using the NASA Generic Transport Model showed successful autonomous recovery post-stall with thrust and elevator commands. Subsequent work [75] added feedback linearization for input-output decoupling and dynamic extension to mitigate singularities in the decoupling matrix. The addition of high-order [76] and exponential observers [77] expanded the capability of the autonomous recovery system by permitting switching of controllers for specific flight modes. The observers facilitated precise decisions for choosing the best controller for the particular flight regime, further tailoring the responses of the recovery system to complex nonlinear behaviors. Along similar lines, accompanying work also promoted the use of nonlinear smooth feedback regulators [78] and nonlinear smooth trackers [79] towards autonomous recovery from LOC. Through solution of the Hamilton-Jacobi Bellman Partial Differential Equation, this effort improved the autonomous recovery system by enabling safe maneuvering of the aircraft around the LOC condition through the tracking of specific trim points. By regulation of the control rates, it also sought to alleviate structural loads on the aircraft while maneuvering around LOC. Zhang and Chen [80] proposed a similar strategy utilizing high order sliding mode control that sought to optimally determine the safe trim condition, using an optimization criterion that minimized the weighted distance between the new and initial trim points. The resulting optimal trim state would be the targeted safe flight condition, with which the sliding mode controller would then seek to restore.
3.3.2 Fast Computation of Recoverable Sets

The notion of online computation of recoverable sets for post-LOC recovery was proposed in McDonough and Atkins [81, 82]. Rapid computation of permissible transitions between trim points allowed for the generation of recovery sequences that could be used to restore a safe flight condition post-LOC. To make practical implementation feasible (since LOC events may progress rapidly), the method focused on discrete-time linear models, reducing the programmatic problem to that of a quadratic and facilitating rapid computation. By further chaining together subsets of the full recoverable set, constrained admissible transitions between recoverable sets for different trim points were also obtained. Simulations on both linear and nonlinear aircraft models demonstrated accurate predictions of feasible transitions between closely-located trim points. The authors envisioned that knowledge of these recovery sequences could then be applied towards an automated or pilot-commanded post-LOC recovery strategy.

3.3.3 Open Final Time Optimal Control with Receding Horizon

Garcia et. al. [83] devised a LOC recovery framework using an open final time dynamic optimization method that utilized a receding horizon approach, which iteratively calculated a recovery sequence at each point in time to optimally guide the aircraft from an upset condition to a safe wings-level flight condition. The algorithm accounted for unsteady and nonlinear aircraft dynamics through the use of online-trained artificial neural networks with sliding time windows. The receding horizon component allowed for an adaptive final time horizon, since the authors postulate that the fastest recovery sequence from LOC does not necessarily constitute the safest sequence. Simulations using the Meridian Uninhabited Aircraft System demonstrated smooth convergence on a set of recovery control sequences that brought the vehicle safely back from an off-bank and off-pitch configuration within a 15-second horizon length.
3.3.4 Model-Predictive-Based Automatic Recovery

Incorporating philosophies from flight safety assessment, Litt et. al. [84] developed a model-predictive automatic recovery system specifically addressing LOC on final approach. The system looked ahead over a 20-second time horizon to determine whether the aircraft, on its current trajectory, would make premature contact with the ground. If such a situation was deemed to occur, an automatic go-around would be initiated based on calculations from a Flight-Path Predictor. The predictor utilized a simplified, three degree-of-freedom rigid-body model to determine if the altitude loss over the following 20 seconds remained above a certain minimum. The output from the predictor was used by the Trigger System to decide on overriding crew commands and initiating a go-around. In piloted-simulation, this system was shown to work well in a variety of stabilized (but deficient) and unstabilized approaches, allowing consistent normal operation but intervening as expected when unrecoverable situations were imminent.

3.3.5 Reinforcement Learning-Based Recovery

Kim et. al. [85] proposed an intelligent reinforcement learning-based optimal controller for upset recovery from flat spin conditions. Bifurcation analysis was utilized to identify spin mode characteristics in the aircraft dynamics, and Q-learning and a Markov-based decision process was implemented to obtain an optimal recovery strategy from the spin condition through arrest of the angular rate and recovery from the unusual attitude. Numerical analysis and simulation on the F/A-18 High Alpha Research Vehicle showed that the proposed controller was able to rapidly search for the optimal recovery input that could successfully recover the aircraft from a flat spin and restore level flight conditions.

3.3.6 Commentary

Flight configurations after an LOC event has occurred typically invoke nonlinear aircraft behavior. Such nonlinearities may arise from high angles-of-attack past stall. Dynamic coupling
between the longitudinal and lateral/directional modes, resulting from combined high angles-of-attack and large sideslip angles, may also lead to instabilities and nonlinearities. In this regard, it is positive that a majority of the methods surveyed for post-LOC recovery have given consideration to nonlinear aircraft dynamic models. Such dynamics bring about various additional considerations that need to be factored into the said methods – accurate knowledge of these post-linear aircraft-specific behaviors is needed for the recovery methods to function as intended, and unpredictability and the lack of empirical data of aircraft behavior in this realm remains a challenge.

The methods surveyed predominantly prove their effectiveness through simulation results in a controlled environment. Further, there is generally an implied reliance on known aircraft dynamics. Nonlinear dynamics, which can stem from uncertainties and unknowns in the dynamics, may only manifest in physical and not simulated environments. These results could be potentially positively aided through implementation and simulated or experimental flight test. Validation and verification would allow the methods to be robustly augmented to handle “real-world” conditions. A caveat of nonlinear methods is their potential to require more intensive (and thus longer) computational times. For these methods to be successfully implemented in an applied scenario, considerations should also be given to the need for realtime computability.
3.4 Pilot Advisory Cueing and Safety Systems Validation

Feedback and advisory mechanisms are a crucial component in addressing the LOC problem. Even with accurate prediction, prevention, or autonomous recovery mechanisms in place, the flight crew is the ultimate entity providing inputs to the aircraft, whether in safe operation within the flight envelope or recovering the aircraft in a LOC event. Rapid, concise, and accurate advisories are thus beneficial as an aid to flight crews in maintaining safe flight to stay out of LOC, or restoring safe flight in the event of LOC. Several advisory technologies have been proposed to-date, including those of a visual, tactile, or aural nature (or a combination thereof), as depicted in Figure 3.4:

![Diagram of Pilot Advisory Cueing Objectives and Guidance Technologies]

Figure 3.4. Mitigating LOC through Pilot Advisory Cueing
These advisory mechanisms base their output on some form of predictive or preventive framework, akin to those discussed in prior sections, and some have further undergone piloted evaluations to validate their effectiveness. A survey of these technologies is presented below.

3.4.1 Visual Guidance Technologies

Conner et al. [86] described various visual and aural alert technologies developed by Honeywell International to mitigate LOC due to erroneous pilot inputs. One of these is a “roll arrow” that presents itself on the Primary Flight Display (PFD) when an adverse bank angle is detected, as depicted in Figure 3.5. This visual cue is accompanied by an aural alert in the form of “roll right” or “roll left” for excessive left or right conditions respectively.

![Corrective Roll Action Visual Cue](image)

**Figure 3.5. Illustration of Corrective Roll Arrow Alerting Technology**  
(concept proposed by Conner et al. [86])

This technology has undergone testing in simulation and has been implemented on hardware. In a separate effort, an aural callout for the Boeing 737 aircraft was also developed to warn flight crews of a low airspeed conditions potentially leading to in-flight stall. Based on the current angle-of-attack, a maneuvering speed was computed and compared with the current
airspeed, prompting an “airspeed low” callout and accompanying visual warnings on the airspeed indicator if both values were within a particular threshold of each other.

In addition, the research and development efforts also promote the benefits of synthetic vision systems (SVS) in maintaining flight crew situational awareness. An SVS includes the main instrumentation elements of a conventional head-down PFD, but additionally superimposes those elements atop a three-dimensional rendering of the outside environment, incorporating sky, terrain, water, and obstacle features. SVSs are typically incorporated into newer, large-format PFDs (the Garmin G1000 is a common example), and provide the flight crew with continuous awareness of the surroundings regardless of weather conditions or time of day. The advantages of such systems in mitigating accidents due to LOC were demonstrated through piloted evaluations conducted by NASA and Lockheed Martin in Glaab & Takallu [87]. An enhanced attitude indicator (EAI) system and an SVS were evaluated against the “standard” attitude indicator in common use. Results showed the subject pilots making far less errors when maneuvering with the EAI and SVS to avoid low-visibility-related LOC.

Schuet et. al. [88] proposed a nonlinear-dynamics-based autonomous flight envelope estimation approach utilizing system identification and Bayesian formulations to determine online the aerodynamic performance capabilities of the aircraft, generating an estimated robust trim envelope. In accompanying work by Lombaerts et. al. [89] and Shish et. al [90], the estimated trim envelope was mapped onto the airspeed, vertical speed, and attitude indicators on the PFD, and the projected flight path of the aircraft onto the Navigation Display. This work utilized aircraft energy-state prediction to predict a multi-dimensional flight path using system identification and online estimates of aerodynamic coefficients to rapidly compute the aircraft’s trim envelope. This accompanying display presented the pilot with a set of dynamically-updated limits on airspeed,
flight path angle, bank angle, and pitch. As an example, the bank angle limit (illustrated in Figure 3.6) would shrink as the aircraft approached its stall speed, at which point there would be little to no bank angle authority left available. Piloted evaluations in simulation noted positive effects from usage of the visual cues. Similar visual cues superimposed on the attitude indicator were also proposed in Richards et. al. [91], wherein pitch, roll, and throttle deflection cues were implemented as guidance mechanisms for pilots to use as during upset recovery maneuvers.

![Dynamic Bank Angle Limits](image)

Figure 3.6. Illustration of Dynamic Bank Angle Limits mapped onto attitude indicator (concept proposed by Lombaerts et. al. [89])

In Tekles et. al. [92], a command-limiting flight envelope protection system based on a total energy control law was proposed that sought to actively calculate changing limits of a safe flight envelope and to limit exceedances of flight parameters such as load factor, angle-of-attack, pitch angle, and airspeed. The visual output of the envelope protection system, in the form of an in-cockpit display, was evaluated in simulated flight testing on NASA’s Transport Class Model [93]. Validation efforts suggested that the system had the potential to reduce excursions of the flight envelope and contributed to better pilot situational awareness of command inputs, thus mitigating the occurrence of an LOC event. Further work implemented an $L_1$ adaptive control law.
(of which the risk-mitigating benefits for loss-of-control were demonstrated in [94]) to the flight envelope protection system [95], and this proposed method was deemed able to improve safety and performance even in the presence of unmodeled dynamics, system damage, and external disturbances.

Another subset of visual cues seeks to provide flight crews with predictive estimations of control margins that, if the pilots were to exceed, would not result in LOC immediately, but instead at some several seconds ahead of the present time. Building on the LOC detection & prediction methodologies described earlier in this chapter ([30-33] and [34, 38]), such visual cues were proposed in Krishnakumar et. al. [96], Stepanyan et. al. [97], and Rafi et. al. [35]. This safe control envelope is depicted in Figure 3.7.

![Figure 3.7. Predictive “Look-Ahead” Safe Control Envelope Display](image)

These visual cues have been evaluated on various head-down and head-up display formats. In Krishnakumar et. al. [96] and Stepanyan et. al. [97], the visual cue was superimposed on a HUD.
(similar to that shown in Figure 3.8) and evaluated through simulated flight testing with favorable feedback from pilots.

![Figure 3.8. Illustration of Safe Control Envelope Display integrated into HUD](image)

In Rafi et. al. [98], the safe control envelope depicted in Figure 3.7 was developed as a head-down conventional instrument and a head-up augmented reality display on the Microsoft HoloLens.

### 3.4.2 Tactile Guidance Technologies

In [99] and [100], Klyde & McRuer proposed tactile-feedback-based smart-cue and smart gain concepts towards alleviating LOC through control surface actuator rate limiting. The smart-cue concept provided a force-feedback cue to the pilot based on a measure of the “dynamic distortion”, which is defined by a departure of the actual primary manual control system from an idealized version of the system. This “position error” in the actual versus commanded control surface position was used to graduate the amount of force-feedback feel applied to the control column, thereby providing the flight crew with tactile awareness in off-nominal conditions. The
smart-gain concept, also based on a measure of the dynamic distortion, was used to attenuate the pilot input as a function of position error. Piloted evaluations in simulation showed that the combined smart-cue/smart-gain concept significantly reduced the incidences of LOC related to pilot-vehicle system interactions.

Building on the above, an updated concept called the SAFE-Cue (Smart Adaptive Flight Effective Cue) [101, 102] was developed as a potential solution to pilot-vehicle system LOC in the presence of an active adaptive control system. The SAFE-Cue based its feedback to the pilot on the measured system error between the response of an adaptive controller and that of the nominal system, alerting/guiding the pilot through force-feedback cues and attenuating commands to maintain pilot-vehicle system stability and performance in the presence of failures. Flight test evaluations of the SAFE-Cue system [103, 104] demonstrated positive LOC-mitigating effects with use of the cueing system. A later addition to the system came in the form of a visual display to accompany the existing tactile cues [105]. This was demonstrated in simulated flight testing to further reduce tendencies for pilot-induced oscillation leading to control loss through increased pilot awareness of the aircraft’s state.

### 3.4.3 Commentary

The guidance technologies surveyed adequately provide some form of visual, aural, or tactile guidance to the pilot about the aircraft’s proximity to LOC and also present suggested corrective actions to steer away from entering an LOC event. A number of these feedback mechanisms have been validated through pilot-in-the-loop simulated and/or actual flight testing, demonstrating positive benefits in increasing pilot awareness and reducing the risk of LOC. While a shared trait of these guidance technologies is that the pilot is left to “close the loop”, they broadly fall into two distinct subsets: one which presents the pilot with information pertaining to the current safe flight envelope, and one which presents the pilot with information pertaining to the current
safe control envelope. It is worthwhile to note the differences between these fundamentally different concepts. While the former allows the pilot to mentally translate the given information to a corresponding control input in the decision-making process, the latter more directly informs the pilot of what control input should (or can) be taken to stay outside of an LOC event. Presenting the pilot with information regarding how much control authority is remaining is beneficial to avoid input overcorrections or other complications such as pilot-induced oscillations.

The proliferation of head-up display technology may be a positive factor in the global effort to mitigate LOC. The prime benefits of such visual guidance mechanisms lie their ability to convey pertinent information directly and rapidly to the pilot, eliminating the time needed to glance at conventional head-down displays for information – this could be potentially crucial in a rapidly-deteriorating situation. The ability to implement these visual guidance mechanisms with readily-available head-up devices would open up these LOC avoidance mechanisms to more pilots who might otherwise not have onboard head-up displays or avionics that natively provide them with such information. More consideration should be given to designing guidance mechanisms for the head-up format.

Some of the guidance technologies surveyed were conceived as “standalone” displays. While reasonable as a proof-of-concept, these would have much better potential if integrated to function within existing industry-standard primary flight display formats. Human-factors considerations, such as design of symbology or aural alerts, need also be given more consideration to better optimize the advisory mechanisms. There is further potential in more thoroughly evaluating each guidance technology through piloted testing to arrive at an optimized advisory that amalgamates the strengths of each proposed method.
3.5 Concluding Perspectives

Considering the material presented thus far, one realizes that loss-of-control in-flight is a complex problem for aircraft that requires a multi-faceted approach in its solution. In part, this is due to the plethora of factors that contribute and eventually lead to an LOC event.

The LOC problem may be divided into its causal, detection, prevention, recovery, and feedback subsets. From the causal aspect, there is a need to study the sequences of human and non-human factors that lead to LOC. In the detection and prediction paradigm, various methods towards on-the-fly “future-time” prediction of safe control envelopes have been devised. In the prevention subset, resilient control methods and flight safety assessment frameworks were presented as mitigation solutions, and in the recovery subset, strategies utilizing nonlinear analysis to restore safe flight were proposed. Visual and tactile cueing technologies seeking to provide flight crews with guidance to avoid and maneuver out of LOC have also been developed.

Applied individually, each of the approaches in these subsets adequately tackle and successfully address a few of the hazards leading to LOC. Historical and anecdotal evidence, however, suggests that a full-blown LOC event leading to an accident is the result of a culmination of these hazards. The “Swiss cheese model” of accident causation may be used to analogize this concept. While several layers of defense mechanisms may lie between a hazard and an accident, no single layer of defense is “perfect”; each layer, while protecting against some hazards, may contain flaws (or holes) that permit other hazards to pass unguarded. With the assumption that these flaws are not all triggered by the same hazards (in other words, the “holes” do not “line up”), stacking these layers of defense ideally results in a mitigation mechanism that can guard against most of the hazards leading to LOC, sufficient enough to protect an aircraft and its occupants from an accident due to LOC.
The loss-of-control prediction architecture proposed in this research seeks to contribute to the ecosystem of safety mechanisms that keep aircraft and aviation safe. The LOC margin prediction concept is introduced in the following chapter, and the chapters thereafter detail the theoretical aspects and development of each of the components in the prediction architecture.
CHAPTER 4

CONCEPT OVERVIEW: REAL-TIME ADAPTIVE PREDICTION OF SAFE CONTROL MARGINS AND PILOT ADVISORY DISPLAYS

4.1 Overview

Figure 4.1 presents a functional overview of the loss-of-control prediction architecture proposed in this research:

![Figure 4.1. Functional Flow Diagram of Loss-of-Control Prediction Architecture](image)

The top-level architecture consists of the pilot, aircraft, adaptive parameter estimation, modal superposition or modal transformation component, and pilot advisory display. These operate in the “real-time” dimension, \( t \in [t_0, T] \). The second-level architecture consists of the loss-of-control margin prediction model. This operates in the “virtual-time” dimension, \( \hat{t} \in [0, \hat{T}] \), where \( \hat{T} \) is the length of the prediction time window.

4.1.1 Pilot Advisory Display

The pilot advisory display is the ultimate component of the prediction architecture. The display is implemented on conventional head-down and augmented-reality head-up formats, and the intent of the visual guidance technology is to rapidly inform the pilot of the changing bounds.
of the safe control envelope or safe control space. The \textit{control envelope} refers to a two-dimensional display that uses critical \textit{deflections} to form the bounds of the graphical safe regions, while the \textit{control space} refers to a three-dimensional display that uses critical \textit{trajectories} to form the bounds of the safe regions. The control space presents the pilot with the additional notion of a receding horizon time aspect of the critical input, akin to the idea of a “highway-in-the-sky”.

4.1.1.1 Safe Control Envelope

A conceptual illustration of the safe control envelope is presented in Figure 4.2:

![Figure 4.2. Conceptual Illustration of Pilot Advisory Display Depicting Safe Control Envelope](image)

The display presents the pilot with colored regions representing safe control envelopes corresponding to the 2-second and 5-second prediction windows. In addition, the current pilot-commanded deflection of the various control surfaces is superimposed on these colored regions. These are presented relative to the maximum control surface travel, which is denoted by the outer edges of the red regions. If the controls are manipulated such that the current deflection indicator meets the edge of the 2- or 5-second regions, the aircraft would respectively reach the LOC margin two or five seconds from the current time.
4.1.1.2 Safe Control Space

A conceptual illustration of the safe control space is presented in Figure 4.3 and Figure 4.4:

Figure 4.3. Conceptual Illustration of Safe Control Space (Front View on Pilot Advisory Display)

Figure 4.4. Conceptual Illustration of Safe Control Space (Three-Dimensional View)
The safe control space is an extension of the safe control envelope concept. With the safe control space, the pilot is additionally presented with a receding horizon time aspect of the critical input. The “immediate control space” provides the pilot with information about the amount of control authority remaining at the current moment, while the “latent control space” provides information about the critical (optimal) trajectories over the next several seconds that should not be exceeded, in order for the aircraft to stay within the safe flight envelope. Together, the immediate and latent components form a three-dimensional volume that becomes the “safe region” of the control input over the coming several seconds.

Eight optimal trajectories form the bounds of the safe control space. These trajectories are named after cardinal directions for ease of understanding and visualization. For example, the North Trajectory targets an angle-of-attack limit of +15°, while the NorthEast Trajectory targets a simultaneous angle-of-attack limit of +15° and bank angle limit of +45°. The display also contains a “depth” aspect, representing time, and a “motion” aspect, representing the notion of “flying through” a control space in the sky. More details on the display and on these concepts are presented in Chapter 9.

4.1.1.3 The Notion of Time-Distance

It is important to note that moving the controls to the edge of the control envelope or control space does not place the aircraft in LOC right now. Instead, doing so would cause the aircraft to enter LOC two or five seconds later. This is the basis for the “early warning” function of the predictive architecture – the idea is that the pilot would have an additional two or five seconds from the current time to bring the aircraft back into the safe region of the control envelope or control space, thereby also keeping the aircraft within the safe region of the flight envelope. In other words, always keeping the control surfaces within the safe control envelope or control space would also mean always keeping the aircraft from exceeding its LOC margins. The bounds of the
safe control envelope or control space are determined at each time in $t$ by the LOC Margin Prediction Models.

The pilot advisory displays are discussed further in Chapter 9.

4.1.2 Loss-of-Control Margin Prediction Models

The loss-of-control prediction models are tasked with determining the critical control deflections or critical control trajectories that, if applied as actual inputs at the current time, would exactly drive the aircraft to the edge of the safe flight envelope some several seconds in the future. This several second period makes up the prediction time window, and is taken to be 2-seconds or 5-seconds in this research. The critical deflections or trajectories form the boundaries of the safe control envelope, and is the metric that is presented to the pilot.

4.1.2.1 Deflection-To-Go Methods

The Deflection-To-Go models seek to determine critical control deflections that will drive the aircraft to the edge of the safe flight envelope within the given prediction time window. At every timestep in $t$, the prediction model solves a series of analytical expressions (the “loss-of-control equations”) along $\hat{t} = 0 \to \hat{T}$, and determines the control surface deflections required to reach the LOC margin in the next $\hat{T}$ seconds. The output from the prediction model is the “deflection-to-go” metric, $\Delta\hat{U}(t)$. This “deflection-to-go” is dependent on the bounds of the safe flight envelope, the aircraft dynamics, and the current pilot-commanded input.

The “deflection-to-go” represents the allowable amount of control deflection remaining, while the “critical deflection”, $U_{\text{crit}}(t)$, is the total critical deflection (including the current pilot-commanded input) and is the final metric that is presented to the pilot. The final critical deflection metric is presented to the pilot at every timestep in $t$, and is considered to be the critical control
surface deflection that would cause the aircraft to exactly reach the LOC margin at $T$ seconds in the future.

The Deflection-To-Go methods are further categorized into the Modal Superposition method and the Modal Transformation method. Each of these two loss-of-control prediction models are respectively discussed further in Chapter 6. A brief primer of the two models is presented below.

**4.1.2.1.1 Modal Superposition / Modal Transformation**

For the loss-of-control prediction models to function practically and in real-time, the higher-order aircraft dynamics need to be expressed as a function of elemental lower-order subsets. As such, to facilitate mathematically- and computationally-efficient analysis of the higher order aircraft dynamics, the modal superposition and modal transformation methods were derived and incorporated into the prediction architecture.

**4.1.2.1.2 Modal Superposition**

The modal superposition component expresses the higher order aircraft transfer functions as a series of *equivalent* elemental 2nd order transfer functions which individually contain the characteristics of each aircraft mode present in the parent dynamics. The elemental 2nd order transfer functions may then be expressed explicitly as a function of standard Laplace transforms, simplifying analysis of the aircraft’s time-domain system response. These superposed dynamics are provided to the prediction model, which in turn computes the predicted safe control envelope.

**4.1.2.1.3 Modal Transformation**

The modal transformation component transforms the higher order aircraft state space model into a set of block-diagonalized 2nd order state space sub-matrices which individually contain the *approximate* characteristics of each aircraft mode present in the parent dynamics. Each of the transformed 2nd order state space systems may be expressed as a 2nd order transfer function,
and consequently as a function of standard Laplace transforms, simplifying analysis of the aircraft’s time-domain system response. These decomposed dynamics are provided to the prediction model, which then computes the predicted safe control envelope.

4.1.2.1.4 Superposition versus Transformation

While both methods achieve similar goals, their functions vary greatly and the consequence of their usage on the accuracy of the predicted safe control envelope is highly apparent.

The “modal transformation” method analyzes the transformed aircraft dynamics separately by decomposing the $n^{th}$ order dynamics into multiple decoupled $2^{nd}$ order block-diagonalized subsystems in a transformed $Z$-domain, while the “modal superposition” method analyzes the $n^{th}$ order coupled aircraft dynamics *as a whole* in the native $X$-domain. Consequently, the modal superposition method allows for the computation of the *exact* critical control deflection required to reach the LOC margin, while the modal decomposition method often computes an *approximate* critical control deflection. Both methods are presented in this dissertation for the reader’s benefit.

4.1.2.2 Trajectory-To-Go Methods

The Trajectory-To-Go models seek to determine critical control *trajectories* that will drive the aircraft to the edge of the safe flight envelope within the given prediction time window. At every timestep in $t$, the prediction model utilizes optimal control methods to determine the control surface trajectory required to reach the LOC margin in the next $T$ seconds. The output from the prediction model is the “trajectory-to-go” metric, $\Delta U(t)$. This “trajectory-to-go” is dependent on the bounds of the safe flight envelope, the aircraft dynamics, the current pilot-commanded input, and the desired state trajectory.
The Trajectory-To-Go methods are further categorized into the open-loop method and the closed-loop methods. Each of these two loss-of-control prediction models are discussed further in Chapter 7.

4.1.3 Parameter Estimation

The need to maintain knowledge of the current aircraft dynamics is crucial to the accurate prediction of the aircraft’s safe control envelope. Under normal flight conditions, with the aircraft operating well within the linear portion of its flight envelope, knowledge of the nominal dynamics would be sufficient to produce accurate estimates of the safe control envelope. When these nominal dynamics change due to physical failures in the aircraft, the resulting uncertainties need to be factored into the mathematical aircraft model, and the need to adaptively identify and quantify these system deficiencies arises.

These uncertainties are accounted for through the parameter estimation algorithms, and are modeled through the introduction of $\Delta A$ and $\Delta B$ “uncertainty matrices” to the aircraft state space model, with the former addressing uncertain dynamics of the aircraft, and the latter addressing changes in control effectiveness. The numerical values of the uncertainty matrices $\Delta A$ and $\Delta B$ are unknown to the overall predictive architecture and are estimated online through the adaptive identification process.

The various parameter estimation algorithms are discussed further in Chapter 8.

4.1.4 Aircraft Dynamics

In a practical implementation, the aircraft dynamics would comprise of known values determined through design. These numerical values would then be fed into the prediction architecture as an external input. To simulate the aircraft for the purposes of this research, a dedicated subset of work is focused on deriving the aircraft equations of motion for a very general 8th order aircraft with dynamic coupling between the longitudinal and lateral/directional states, the
potential for asymmetry, and with the assumption of non-zero steady state values for aerodynamic angles, velocities, Euler angles, and rotation rates. The resulting algebraic aircraft state space model, when populated with numerical values of aerodynamic and thrust derivatives, serves as the nominal aircraft dynamic model.

The development of the higher-order, asymmetric, and dynamically-coupled aircraft model is detailed in Chapter 5.

4.2 Descriptive Example

As an example, let the “length” of the real-time dimension be $T = 20$ sec, and let the length of the prediction time window be $\hat{T} = 2$ sec. Suppose that real-time is currently at $t = 10$ sec. At this point, the current states $X$ and current inputs $U$ are fed into the prediction model as its initial conditions. The prediction model then runs for 2 virtual seconds, determining the critical control deflection required for the aircraft to reach the LOC margin in the next 2 real seconds. This critical control input, if applied by the pilot at the current time $t = 10$ sec, would then drive the aircraft to the LOC margin 2 seconds later at $t = 12$ sec. If the pilot were to give an input of greater magnitude than this critical input, the aircraft would exceed the LOC margin. In other words, to keep the aircraft within the LOC margin, the pilot should not exceed $u_{\text{crit}}(t)$ as calculated by the prediction model.
4.3 Research Objectives

Given the above description and intent of the prediction architecture, the overarching research questions that need to be answered are as follows:

*Can one predict if a pilot’s control input now at \( t \) seconds will cause the aircraft to exceed its LOC margins later at \( (t + \tilde{T}) \) seconds?*

*Knowing this, can one visually inform the pilot clearly and rapidly of impending entry into LOC?*

In setting out to accomplish the above, it is beneficial to define the roles and purposes of each component in the prediction architecture. An objective description of each component of the prediction architecture, the associated tasks needed in developing the component, and a description of each component’s purpose is summarized in Figure 4.5:
<table>
<thead>
<tr>
<th>Component</th>
<th>Objective</th>
<th>Purpose</th>
</tr>
</thead>
<tbody>
<tr>
<td>Aircraft Dynamics</td>
<td>Develop Equations of Motion for Asymmetric, Coupled, Higher Order Aircraft. Derive State Space and Transfer Function Models for Higher Order Aircraft.</td>
<td>Enables dynamic stability and response analysis of the aircraft operating in abnormal flight configurations associated with LOC, through a very general mathematical model derived with minimal steady-state assumptions. Integral to computing estimates of the safe control envelope.</td>
</tr>
<tr>
<td>Parameter Estimation</td>
<td>Derive adaptive system identification algorithms to handle matched and unmatched uncertainties. Apply parameter estimation algorithms to prediction architecture.</td>
<td>Provides a mechanism for real-time system identification, which is crucial for accurate prediction of the safe control envelope.</td>
</tr>
<tr>
<td>Modal Superposition/Transformation</td>
<td>Express higher order aircraft dynamics as a function of elemental lower order subsets, through superposition or transformation methods.</td>
<td>Facilitates mathematically- and computationally-efficient analysis of aircraft dynamic response by expressing the higher order aircraft dynamics as a series of 2nd order elemental subsets through superposition or decomposition.</td>
</tr>
<tr>
<td>Loss-of-Control Safe Margin Prediction</td>
<td>Develop models capable of computing estimates of critical control deflections or trajectories.</td>
<td>Enables the analytical determination of critical control inputs or trajectories that drive the aircraft to the edge of the safe flight envelope over a specified future time-period. Governs the boundaries of the safe control envelope.</td>
</tr>
<tr>
<td>Pilot Advisory Display</td>
<td>Design guidance technologies for visual or aural representations of safe control envelope or safe control space.</td>
<td>Functions as the mechanism through which the pilot is rapidly informed of the dynamically-changing boundaries of the safe control envelope/space. Promotes safe maneuvering of the aircraft and reduces chances of exceeding the safe flight envelope.</td>
</tr>
<tr>
<td>Prediction Architecture</td>
<td>Implement prediction architecture through desktop simulation. Implement prediction architecture and guidance technologies on head-down and head-up displays.</td>
<td>Validates effectiveness and accuracy of prediction models via &quot;simulated ground testing&quot; and &quot;simulated flight testing&quot;. Evaluates the merits of the proposed prediction architecture in applied scenarios that mimic real-world conditions.</td>
</tr>
</tbody>
</table>

Figure 4.5. Objective Flow Diagram of Loss-of-Control Prediction Architecture
CHAPTER 5
THE HIGHER-ORDER ASYMMETRIC AND DYNAMICALLY-COUPLED AIRCRAFT MODEL

5.1 Key Development Objectives

The analysis of aircraft configurations relating to straight-and-level flight typically allow for many assumptions that eventually lead to a simplified set of equations describing aircraft motion. These assumptions are commonly referred to as the “special steady state conditions”, and assume no initial side velocity, initial bank angle, or initial angular rates exist. Analysis in this domain also separates the longitudinal dynamics from the lateral/directional dynamics, and the two are studied independently as decoupled systems.

![Figure 5.1. The Rigid Aircraft and Associated Coordinate Systems](image)

In analysis of aircraft configurations associated with loss-of-control, these assumptions do not necessarily hold true. Such configurations may have combined high angles-of-attack and sideslip, may involve steep bank angles, and may start with high angular rates. Inherent dynamic coupling may exist between the aircraft’s longitudinal and lateral/directional dynamics. Furthermore, loss-of-control may also arise due to physical damage to the aircraft, wherein the assumption of symmetry no longer applies.
As a consequence, there arose the need for a very general set of expressions describing aircraft motion, formulated with the least possible number of assumptions and simplifications. This requirement thus necessitated the derivation of the aircraft equations of motion for a very general rigid-body aircraft with considerations for non-zero steady-state conditions on all states. The final product of this endeavor was a higher-order state space model for a generalized, asymmetric, fixed-wing, constant-mass, rigid aircraft with dynamic coupling between the longitudinal and lateral/directional modes.

5.2 Derivation of the Aircraft Equations of Motion

The six aircraft equations of motion and three kinematic equations presented in equations (5.1) to (5.9) form a starting point for obtaining the state space model of the higher-order, asymmetric, and dynamically-coupled aircraft. These equations are fundamentally derived using Newtonian Dynamics – specifically, through Newton’s 2\textsuperscript{nd} Law and conservation of linear and angular momentum – and this process, though lengthy, is crucial to understanding the origins of the equations of motion.

Consequently, a portion of this research effort was spent revisiting these concepts in detail, starting with Newton’s 2\textsuperscript{nd} Law and ending with the equations of motion, so as to arrive at equations (5.1) to (5.9). The author has endeavored to succinctly summarized the mathematical steps involved in this derivation process, while also accompanying it with qualitative explanations to aid in understanding. This effort is presented in Appendix A. In undertaking this endeavor, the following references were indispensable: Roskam [106] and Marcello [107]. It is the author’s hope that the material presented in Appendix A will present itself a useful resource to the reader.
5.3 The Aircraft Equations of Motion

A summarized overview of the equations of motion for a generalized aircraft, as was developed in Appendix A, is given in equations (5.1) to (5.9). The terms highlighted in red apply specifically to an asymmetric aircraft. For an aircraft with symmetry (or negligible asymmetry), where $I_{xy} = I_{yz} = 0$, these terms would be eliminated from the equations of motion.

**Force Equations:**

$$m(\ddot{U} - VR + WQ) = -mg\sin\Theta + F_{Ax} + F_{Tx}$$  \hspace{1cm} (5.1)

$$m(\ddot{V} + UR - WP) = mg\sin\Phi \cos\Theta + F_{Ay} + F_{Ty}$$  \hspace{1cm} (5.2)

$$m(\ddot{W} - UQ + VP) = mg\cos\Phi \cos\Theta + F_{Az} + F_{Tz}$$  \hspace{1cm} (5.3)

**Moment Equations:**

$$I_{xx}\ddot{p} - I_{xy}\ddot{q} - I_{xz}\ddot{r} + I_{xy}PR + I_{yz}(R^2 - Q^2) - I_{xz}PQ + (I_{zz} - I_{yy})RQ = L_A + L_T$$  \hspace{1cm} (5.4)

$$I_{yy}\ddot{q} - I_{xy}\ddot{p} - I_{yz}\ddot{r} + (I_{xx} - I_{zz})PR + I_{xz}(P^2 - R^2) - I_{xy}QR + I_{yz}PQ = M_A + M_T$$  \hspace{1cm} (5.5)

$$I_{zz}\ddot{r} - I_{xz}\ddot{p} - I_{yz}\ddot{q} + (I_{yy} - I_{xx})PQ + I_{xy}(Q^2 - P^2) + I_{xz}QR - I_{yz}PR = N_A + N_T$$  \hspace{1cm} (5.6)

**Kinematic Equations:**

$$P = \dot{\Phi} - \Psi \sin\Theta$$  \hspace{1cm} (5.7)

$$Q = \dot{\Theta} \cos\Phi + \Psi \cos\Theta \sin\Phi$$  \hspace{1cm} (5.8)

$$R = \Psi \cos\Theta \cos\Phi - \dot{\Theta} \sin\Phi$$  \hspace{1cm} (5.9)
5.4 The Steady State Aircraft Equations of Motion

The equations of motion for a generalized aircraft in equations (5.1) to (5.9) may be further simplified for a steady state flight condition. A steady state flight condition is defined as one for which all motion variables remain constant with time relative to the body-fixed axis system. This consequently translates to the assumptions that there are no linear or angular accelerations, such that $\ddot{V} = 0$ and $\ddot{\omega} = 0$. Using the subscript 1 to denote variables in the steady state, the general aircraft equations of motion for steady state flight are then given as follows:

**Force Equations:**

$$m(-V_1 R_1 + W_1 Q_1) = -mg \sin \Theta_1 + F_{Ax_1} + F_{Tx_1} \quad (5.10)$$

$$m(U_1 R_1 - W_1 P_1) = mg \sin \Phi_1 \cos \Theta_1 + F_{Ay_1} + F_{Ty_1} \quad (5.11)$$

$$m(-U_1 Q_1 + V_1 P_1) = mg \cos \Phi_1 \cos \Theta_1 + F_{Ax_1} + F_{Tx_1} \quad (5.12)$$

**Moment Equations:**

$$I_{xy} P_1 R_1 + I_{yz} (R_1^2 - Q_1^2) - I_{xz} P_1 Q_1 + (I_{zz} - I_{yy}) R_1 Q_1 = L_{A_1} + L_{T_1} \quad (5.13)$$

$$(I_{xx} - I_{zz}) P_1 R_1 + I_{xz} (P_1^2 - R_1^2) - I_{xy} Q_1 R_1 + I_{yz} P_1 Q_1 = M_{A_1} + M_{T_1} \quad (5.14)$$

$$(I_{yy} - I_{xx}) P_1 Q_1 + I_{xy} (Q_1^2 - P_1^2) + I_{xz} Q_1 R_1 - I_{yz} P_1 R_1 = N_{A_1} + N_{T_1} \quad (5.15)$$
Kinematic Equations:

\[
P_1 = \dot{\Phi}_1 - \dot{\Psi}_1 \sin \Theta_1
\]

\[
Q_1 = \dot{\Theta}_1 \cos \Phi_1 + \dot{\Psi}_1 \cos \Theta_1 \sin \Phi_1
\]

\[
R_1 = \dot{\Psi}_1 \cos \Theta_1 \cos \Phi_1 - \dot{\Theta}_1 \sin \Phi_1
\]

These equations of motion are useful for analysis of aircraft motion with respect to steady state control, and may be further simplified to facilitate analysis for specific flight conditions, such as steady state rectilinear or turning flight. For analysis of aircraft motion with respect to dynamic stability and response, the so-called “perturbed state” equations of motion are instead needed. The development of these equations of motion is discussed in the next section.
5.5 The Perturbed Aircraft Equations of Motion

A perturbed state flight condition is defined as one for which all motion variables are defined relative to a known steady state flight condition. These equations of motion are useful for analysis of aircraft dynamic stability and response. Arriving at the equations of motion for perturbed state flight involves several intermediate steps:

- Substitution of motion, force, and moment variables with a linear superposition of steady and perturbed-state representations.
- Application of trigonometric assumptions to linearization angular variables.
- Elimination of steady state terms and application of small perturbation assumptions to linearize remaining non-linear variables.

A qualitative analogy to this concept would be the taring of a weighing scale prior to measuring a mass. Say that one has a container of some mass. Setting this container on the scale would lead to a non-zero reading on scale. Now, say that one desires to fill the container with various amounts of liquid. Without taring the scale, the reading would reflect the mass of the container and the liquid. The mass of the container, however, is of no particular interest. Taring the scale before filling the container with liquid would allow one to directly obtain the mass of the liquid without having to subtract the mass of the container each time.

Applying this analogy to the aircraft: One might already know that the aircraft is trimmed at some known non-zero steady state angle-of-attack. This steady state value may not be of particular interest. The variations from this steady state value, however, are of interest. These variations would be the perturbations from the steady state.
The full development of the *aircraft equations of motion for perturbed state flight* is detailed in Appendix B, as this process involves multiple steps. The end result of this effort – the perturbed force and moment equations, represented in terms of the dimensional stability derivatives – are given in equations (5.19) to (5.24). As a recap, the kinematic equations are also once again presented in equations (5.25) to (5.27).

**Force Equations:**

\[
(u - V_1 r - R_1 v + W_1 q + Q_1 w) = -g \theta \cos \Theta_1 + \\
(X_u + X_{\tau u}) u + (X_\alpha + X_{\tau \alpha}) \alpha + X_\alpha \dot{\alpha} + X_q q + X_\delta_e \delta_e + X_\delta_t \delta_t + \\
(X_\beta + X_{\tau \beta}) \beta + X_\beta \dot{\beta} + X_p p + X_r r + X_\delta_a \delta_a + X_\delta_r \delta_r
\]  
\hspace{1cm} (5.19)

\[
\dot{v} + U_1 r + R_1 u - W_1 p - P_1 w = -g \theta \sin \Phi_1 \sin \Theta_1 + g \phi \cos \Phi_1 \cos \Theta_1 + \\
(Y_u + Y_{\tau u}) u + (Y_\alpha + Y_{\tau \alpha}) \alpha + Y_\alpha \dot{\alpha} + Y_q q + Y_\delta_e \delta_e + Y_\delta_t \delta_t + \\
(Y_\beta + Y_{\tau \beta}) \beta + Y_\beta \dot{\beta} + Y_p p + Y_r r + Y_\delta_a \delta_a + Y_\delta_r \delta_r
\]  
\hspace{1cm} (5.20)

\[
\dot{w} - U_1 q - Q_1 u + V_1 p + P_1 v = -g \theta \cos \Phi_1 \sin \Theta_1 - g \phi \sin \Phi_1 \cos \Theta_1 + \\
(Z_u + Z_{\tau u}) u + (Z_\alpha + Z_{\tau \alpha}) \alpha + Z_\alpha \dot{\alpha} + Z_q q + Z_\delta_e \delta_e + Z_\delta_t \delta_t + \\
(Z_\beta + Z_{\tau \beta}) \beta + Z_\beta \dot{\beta} + Z_p p + Z_r r + Z_\delta_a \delta_a + Z_\delta_r \delta_r
\]  
\hspace{1cm} (5.21)
Moment Equations:

\[
\dot{p} - \frac{l_{xy}}{l_{xx}} \dot{q} - \frac{l_{xz}}{l_{xx}} \dot{r} - \frac{l_{xz}}{l_{xx}} (P_1q + Q_1p) + \left( \frac{l_{zz} - l_{yy}}{l_{xx}} \right) (R_1q + Q_1r) + \\
\frac{l_{xy}}{l_{xx}} (P_1r + R_1p) + \frac{l_{yz}}{l_{xx}} (2R_1r - 2Q_1q) = \]

\[(L_u + L_{T_u})u + (L_\alpha + L_{T_\alpha})\alpha + L_\delta \ddot{\delta} + L_\delta \delta_t + \]

\[
(L_\beta + L_{T_\beta}) \beta + L_\delta \dot{\beta} + L_\delta \ddot{p} + L_r \beta + L_\delta \delta_a + L_\delta \delta_r
\]

\[
\dot{q} - \frac{l_{xy}}{l_{yy}} \dot{p} - \frac{l_{xz}}{l_{yy}} \dot{r} + \left( \frac{l_{xx} - l_{zz}}{l_{xy}} \right) (P_1r + R_1p) + \frac{l_{xz}}{l_{yy}} (2P_1p - 2R_1r) - \\
\frac{l_{xy}}{l_{yy}} (Q_1r + R_1q) + \frac{l_{yz}}{l_{yy}} (P_1q + Q_1p) = \]

\[(M_u + M_{T_u})u + (M_\alpha + M_{T_\alpha})\alpha + M_\delta \dot{\alpha} + M_\delta \delta_q + M_\delta \delta_e + M_\delta \delta_t + \]

\[
(M_\beta + M_{T_\beta}) \beta + M_\delta \dot{\beta} + M_\delta \ddot{p} + M_r \beta + M_\delta \delta_a + M_\delta \delta_r
\]

\[
\dot{r} - \frac{l_{yz}}{l_{zz}} \dot{q} - \frac{l_{xz}}{l_{zz}} \dot{p} + \left( \frac{l_{yy} - l_{xx}}{l_{xz}} \right) (P_1q + Q_1p) + \frac{l_{xz}}{l_{zz}} (Q_1r + R_1q) + \\
\frac{l_{xy}}{l_{zz}} (2Q_1q - 2P_1p) - \frac{l_{yz}}{l_{zz}} (P_1r + R_1p) = \]

\[(N_u + N_{T_u})u + (N_\alpha + N_{T_\alpha})\alpha + N_\delta \dot{\alpha} + N_\delta \delta_q + N_\delta \delta_e + N_\delta \delta_t + \]

\[
(N_\beta + N_{T_\beta}) \beta + N_\delta \dot{\beta} + N_\delta \ddot{p} + N_r \beta + N_\delta \delta_a + N_\delta \delta_r
\]
Kinematic Equations:

\[ p = \dot{\phi} - \dot{\psi}_1 \theta \cos \Theta_1 - \dot{\psi} \sin \Theta_1 \]  
\[ (5.25) \]

\[ q = -\dot{\Theta}_1 \phi \sin \Phi_1 + \dot{\theta} \cos \Phi_1 + \dot{\psi}_1 \phi \cos \Theta_1 \cos \Phi_1 - \dot{\psi}_1 \theta \sin \Theta_1 \sin \Phi_1 + \dot{\psi} \cos \Theta_1 \sin \Phi_1 \]  
\[ (5.26) \]

\[ r = -\dot{\psi}_1 \phi \cos \Theta_1 \sin \Phi_1 - \dot{\psi}_1 \theta \sin \Theta_1 \cos \Phi_1 + \dot{\psi} \cos \Theta_1 \cos \Phi_1 - \dot{\Theta}_1 \phi \cos \Phi_1 - \dot{\theta} \sin \Phi_1 \]  
\[ (5.27) \]
5.6 The Perturbed Aircraft Equations of Motion in State Space Form

Equations (5.19) to (5.27) form a starting point for the development of the aircraft’s state space model with coupled longitudinal and lateral/directional dynamics. The final derived state space system may be expressed either in terms of the kinematic states, which would be ideal for simulating motion of the aircraft, or in terms of the aerodynamic states, which would be ideal for analysis of the aircraft’s dynamic stability and response.

The states for the **kinematic** form are defined as:

\[ X_K = [u \ v \ w \ p \ q \ r \ \phi \ \theta \ \psi]^T \]  
(5.28)

Similarly, the states for the **aerodynamic** form are defined as:

\[ X_A = [u \ \alpha \ \theta \ q \ \beta \ \phi \ p \ r \ \psi]^T \]  
(5.29)

The inputs for both forms are defined as:

\[ U = [\delta_e \ \delta_a \ \delta_r \ \delta_t]^T \]  
(5.30)

5.6.1 Kinematic, Aerodynamic, and Moment of Inertia Constants

To aid in the development of a concise solution, it is beneficial to use constants to represent the kinematic, aerodynamic, and moment of inertia terms. This requires making several substitutions into equations (5.19) to (5.27). These substitutions also involve algebraically relating the aerodynamic states (angle-of-attack and sideslip angle) to the kinematic states (forward, side, and vertical velocities). The full development of these algebraic relationships is presented in **Appendix C**.
For the moment of inertia terms, these constants are:

\[
\begin{align*}
\bar{I}_p_1 &= \frac{l_{xz}}{l_{xx}} \\
\bar{I}_p_2 &= \frac{l_{zz} - l_{yy}}{l_{xx}} \\
\bar{I}_p_3 &= \frac{l_{xy}}{l_{xx}} \\
\bar{I}_p_4 &= \frac{l_{yz}}{l_{xx}} \\
\bar{I}_q_1 &= \frac{l_{xz}}{l_{yy}} \\
\bar{I}_q_2 &= \frac{l_{xx} - l_{zz}}{l_{yy}} \\
\bar{I}_q_3 &= \frac{l_{xy}}{l_{yy}} \\
\bar{I}_q_4 &= \frac{l_{yz}}{l_{yy}} \\
\bar{I}_r_1 &= \frac{l_{xz}}{l_{zz}} \\
\bar{I}_r_2 &= \frac{l_{yy} - l_{xx}}{l_{zz}} \\
\bar{I}_r_3 &= \frac{l_{xy}}{l_{zz}} \\
\bar{I}_r_4 &= \frac{l_{yz}}{l_{zz}}
\end{align*}
\]  
(5.31)

For the aerodynamic terms in the \textit{kinematic} form of the state space model, these constants are:

\[
\begin{align*}
\alpha &= -\bar{K}_\alpha u + \bar{K}_\alpha w \\
\dot{\alpha} &= -\bar{K}_\alpha \dot{u} + \bar{K}_\alpha \dot{w} \\
\beta &= -\bar{K}_\beta u + \bar{K}_\beta v - \bar{K}_\beta w \\
\dot{\beta} &= -\bar{K}_\beta \dot{u} + \bar{K}_\beta \dot{v} - \bar{K}_\beta \dot{w}
\end{align*}
\]  
(5.32)

Finally, for the kinematic terms in the \textit{aerodynamic} form of the state space model, these constants are:

\[
\begin{align*}
w &= \bar{A}_{w_d} \alpha + \bar{A}_{w_u} u \\
\dot{w} &= \bar{A}_{w_d} \dot{\alpha} + \bar{A}_{w_u} \dot{u} \\
v &= \bar{A}_{v_d} \alpha + \bar{A}_{v_p} \beta + \bar{A}_{v_u} u \\
\dot{v} &= \bar{A}_{v_d} \dot{\alpha} + \bar{A}_{v_p} \dot{\beta} + \bar{A}_{v_u} \dot{u}
\end{align*}
\]  
(5.33)
The Rigid-Body Aircraft State Space Model in Kinematic Form

With the substitutions defined in equations (5.31) and (5.32), the perturbed equations of motion become:

**Force Equations:**

\[
\begin{align*}
(\overline{K}_{\alpha u}X_a + \overline{K}_{\beta u}X_\beta + 1)\ddot{u} + (\overline{K}_{\alpha v}X_\beta)\dot{v} + (\overline{K}_{\alpha w}X_a + \overline{K}_{\beta w}X_\beta)\dot{w} &= \\
(\overline{K}_{\alpha u}(X_a + X_{Ta}) - \overline{K}_{\beta u}(X_\beta + X_{T\beta}) + (X_a + X_{Ta})\ddot{u} + \left(\overline{K}_{\beta u}(X_\beta + X_{T\beta}) + R_1\right)\dot{v} &+ \left(\overline{K}_{\alpha w}(X_a + X_{Ta}) - \overline{K}_{\beta w}(X_\beta + X_{T\beta}) - Q_1\right)w + (X_y)p + (-W_1 + X_q)q \\
&+ (V_1 + X_r)r + (-g \cos \Theta_1)\theta + (X_{\delta_x})\delta_e + (X_{\delta_a})\delta_t + (X_{\delta_a})\delta_a + (X_{\delta_r})\delta_r
\end{align*}
\]

\[5.34\]

\[
\begin{align*}
(\overline{K}_{\alpha u}Y_a + \overline{K}_{\beta u}Y_\beta)\ddot{u} + (1 - \overline{K}_{\beta w}Y_\beta)\dot{v} + (\overline{K}_{\alpha w}Y_a + \overline{K}_{\beta w}Y_\beta)\dot{w} &= \\
(\overline{K}_{\alpha u}(Y_a + Y_{Ta}) - \overline{K}_{\beta u}(Y_\beta + Y_{T\beta}) + (Y_a + Y_{Ta})R_1\ddot{u} + \left(\overline{K}_{\beta u}(Y_\beta + Y_{T\beta})\right)\dot{v} &+ \left(\overline{K}_{\alpha w}(Y_a + Y_{Ta}) - \overline{K}_{\beta w}(Y_\beta + Y_{T\beta})\right)w + (W_1 + Y_p)p + (Y_q)q \\
&+ (-U_1 + Y_r)r + (g \cos \Phi_1 \cos \Theta_1)\phi + (-g \sin \Phi_1 \sin \Theta_1)\theta + (Y_{\delta_x})\delta_e \\
&+ (Y_{\delta_a})\delta_t + (Y_{\delta_a})\delta_a + (Y_{\delta_r})\delta_r
\end{align*}
\]

\[5.35\]

\[
\begin{align*}
(\overline{K}_{\alpha u}Z_a + \overline{K}_{\beta u}Z_\beta)\ddot{u} + (1 - \overline{K}_{\beta u}Z_\beta)\dot{v} + (1 - \overline{K}_{\alpha w}Z_a + \overline{K}_{\beta w}Z_\beta)\dot{w} &= \\
(\overline{K}_{\alpha u}(Z_a + Z_{Ta}) - \overline{K}_{\beta u}(Z_\beta + Z_{T\beta}) + (Z_a + Z_{Ta})Q_1\ddot{u} + \left(\overline{K}_{\beta u}(Z_\beta + Z_{T\beta}) - P_1\right)\dot{v} &+ \left(\overline{K}_{\alpha w}(Z_a + Z_{Ta}) - \overline{K}_{\beta w}(Z_\beta + Z_{T\beta})\right)w + (-V_1 + Z_p)p + (U_1 + Z_q)q \\
&+ (Z_r)r + (-g \sin \Phi_1 \cos \Theta_1)\phi + (-g \cos \Phi_1 \sin \Theta_1)\theta + (Z_{\delta_x})\delta_e \\
&+ (Z_{\delta_a})\delta_t + (Z_{\delta_a})\delta_a + (Z_{\delta_r})\delta_r
\end{align*}
\]

\[5.36\]

\[
\begin{align*}
(\overline{K}_{\alpha u}L_a + \overline{K}_{\beta u}L_\beta)\ddot{u} + (\overline{K}_{\alpha w}L_\beta + L_{T\beta})\dot{v} + (\overline{K}_{\alpha u}L_a + \overline{K}_{\beta w}L_\beta)\dot{w} &= (1)\ddot{p} + (-\overline{I}_{p_3})q + (-\overline{I}_{p_1})r \\
(\overline{K}_{\alpha u}(L_a + L_{Ta}) - \overline{K}_{\beta u}(L_\beta + L_{T\beta}) + (L_a + L_{Ta})\ddot{u} + \left(\overline{K}_{\beta u}(L_\beta + L_{T\beta})\right)\dot{v} &+ \left(\overline{K}_{\alpha w}(L_a + L_{Ta}) - \overline{K}_{\beta w}(L_\beta + L_{T\beta})\right)w + (\overline{I}_{p_1}Q_1 - \overline{I}_{p_3}R_1 + L_p)p \\
&+ (\overline{I}_{p_1}P_1 + 2\overline{I}_{p_3}Q_1 - \overline{I}_{p_2}R_1 + L_q)q + (\overline{I}_{p_3}P_1 - \overline{I}_{p_2}Q_1 - 2\overline{I}_{p_4}R_1 + L_r)r \\
&+ (\overline{L}_{\delta_x})\delta_e + (\overline{L}_{\delta_t})\delta_t + (\overline{L}_{\delta_a})\delta_a + (\overline{L}_{\delta_r})\delta_r
\end{align*}
\]

\[5.37\]
Kinematic Equations:

\[
(\vec{K}_{\alpha u}M_{\alpha} + \vec{K}_{\beta u}M_{\beta})\dot{u} + (-\vec{K}_{\alpha u}M_{\beta})\dot{v} + (-\vec{K}_{\alpha u}M_{\alpha} + \vec{K}_{\beta u}M_{\beta})\dot{w} + (-\vec{q}_{\alpha})\dot{p} + (1)\dot{q} + (-\vec{q}_{\alpha})\dot{r} = \\
(\vec{K}_{\alpha u}(M_{\alpha} + M_{Ta}) - \vec{K}_{\beta u}(M_{\beta} + M_{Tb} + (M_{u} + M_{Ta}))u + (\vec{K}_{\beta u}(M_{\beta} + M_{Tb})))v \\
+ (\vec{K}_{\alpha w}(M_{\alpha} + M_{Ta}) - \vec{K}_{\beta w}(M_{\beta} + M_{Tb}))w \\
+ (-2\vec{q}_{\beta}p_1 - \vec{q}_{\alpha}q_1 - \vec{q}_{\beta}r_1 + M_p)p + (-\vec{q}_{\beta}p_1 + \vec{q}_{\beta}q_1 + M_p)q \\
+ (-\vec{q}_{\alpha}p_1 + \vec{q}_{\alpha}q_1 + 2\vec{q}_{\beta}R_1 + M_{r})r + (M_{\delta_e})\delta_e + (M_{\delta_r})\delta_r + (M_{\delta_a})\delta_a \\
+ (M_{\delta_e})\delta_e + (M_{\delta_r})\delta_r \\
\]

\[
(\vec{K}_{\alpha u}N_{\alpha} + \vec{K}_{\beta u}N_{\beta})\dot{u} + (-\vec{K}_{\alpha u}N_{\beta})\dot{v} + (-\vec{K}_{\alpha u}N_{\alpha} + \vec{K}_{\beta u}N_{\beta})\dot{w} + (-\vec{r}_{\alpha})\dot{p} + (-\vec{r}_{\alpha})\dot{q} + (1)\dot{r} = \\
(\vec{K}_{\alpha u}(N_{\alpha} + N_{Ta}) - \vec{K}_{\beta u}(N_{\beta} + N_{Tb}) + (N_{u} + N_{Ta}))u + (\vec{K}_{\beta u}(N_{\beta} + N_{Tb})))v \\
+ (\vec{K}_{\alpha w}(N_{\alpha} + N_{Ta}) - \vec{K}_{\beta w}(N_{\beta} + N_{Tb}))w + (2\vec{r}_{\beta}p_1 - \vec{r}_{\alpha}q_1 + \vec{r}_{\beta}R_1 + N_p)p \\
+ (-\vec{r}_{\beta}p_1 - 2\vec{r}_{\beta}q_1 - \vec{r}_{\alpha}R_1 + N_q)q + (\vec{r}_{\beta}p_1 - \vec{r}_{\alpha}q_1 + N_{r})r + (N_{\delta_e})\delta_e \\
+ (N_{\delta_r})\delta_r + (N_{\delta_a})\delta_a + (N_{\delta_e})\delta_e + (N_{\delta_r})\delta_r \\
\]

\[
(-1)\phi + (0)\dot{\theta} + (\sin \Theta_1)\dot{\psi} = (-1)p + (-\psi_1 \cos \Theta_1)\theta \\
(0)\phi + (-\cos \Phi_1)\dot{\theta} + (-\sin \Phi_1 \cos \Theta_1)\dot{\psi} \\
= (-1)q + (-\dot{\Theta}_1 \sin \Phi_1 + \psi_1 \cos \Phi_1 \cos \Theta_1)\phi + (-\psi_1 \sin \Phi_1 \sin \Theta_1)\theta \\
(0)\phi + (\sin \Phi_1)\dot{\theta} + (-\cos \Phi_1 \cos \Theta_1)\dot{\psi} \\
= (-1)r + (-\dot{\Theta}_1 \cos \Phi_1 - \psi_1 \sin \Phi_1 \cos \Theta_1)\phi + (-\psi_1 \cos \Phi_1 \sin \Theta_1)\theta \\
\]

Now, let the states of the system be given by \( X = X_K = [u \ v \ w \ p \ q \ r \ \phi \ \theta \ \psi]^T \) and the inputs be given by \( U = [\delta_e \ \delta_a \ \delta_r \ \delta_l]^T \). Eq. Equations (5.34) to (5.42) may then be cast in the following descriptor (implicit) state space representation:

\[
E\dot{x} = Ax + Bu \\
\]
The state matrix $A$ is given by:

\[
A = \begin{bmatrix}
-R_{a_u} (x_a + x_{\tau a}) - R_{p_a} (x_p + x_{\tau p}) + (x_u + x_{\tau u}) & R_{p_a} (x_p + x_{\tau p}) + R_{l_p} & R_{a_u} (x_a + x_{\tau a}) - R_{p_u} (x_p + x_{\tau p}) - Q_1 & x_p \\
-R_{a_u} (y_a + y_{\tau a}) - R_{p_u} (y_p + y_{\tau p}) + (y_u + y_{\tau u}) & R_{p_u} (y_p + y_{\tau p}) + R_{l_u} & R_{a_u} (y_a + y_{\tau a}) - R_{p_u} (y_p + y_{\tau p}) + P_1 & W_1 + y_p \\
-R_{a_u} (z_a + z_{\tau a}) - R_{p_u} (z_p + z_{\tau p}) + (z_u + z_{\tau u}) + Q_1 & R_{p_u} (z_p + z_{\tau p}) + R_{l_u} & R_{a_u} (z_a + z_{\tau a}) - R_{p_u} (z_p + z_{\tau p}) - P_1 & -V_1 + z_p \\
-R_{a_u} (l_a + l_{\tau a}) - R_{p_u} (l_p + l_{\tau p}) + (l_u + l_{\tau u}) & R_{p_u} (l_p + l_{\tau p}) + R_{l_u} & R_{a_u} (l_a + l_{\tau a}) - R_{p_u} (l_p + l_{\tau p}) & I_{p_a} Q_1 - I_{p_a} R_1 + L_p \\
-R_{a_u} (m_a + m_{\tau a}) - R_{p_u} (m_p + m_{\tau p}) + (m_u + m_{\tau u}) & R_{p_u} (m_p + m_{\tau p}) & R_{a_u} (m_a + m_{\tau a}) - R_{p_u} (m_p + m_{\tau p}) & -2I_{q_a} P_1 - I_{q_a} Q_1 - I_{q_a} R_1 + M_p \\
-R_{a_u} (n_a + n_{\tau a}) - R_{p_u} (n_p + n_{\tau p}) + (n_u + n_{\tau u}) & R_{p_u} (n_p + n_{\tau p}) & R_{a_u} (n_a + n_{\tau a}) - R_{p_u} (n_p + n_{\tau p}) & 2I_{r_a} P_1 - I_{r_a} Q_1 + I_{r_a} R_1 + N_p \\
-W_1 + x_q & V_1 + x_r & 0 & -g \cos \theta_1 \\
-Y_q & -U_1 + y_r & g \cos \Phi_1 \cos \theta_1 & -g \sin \Phi_1 \sin \theta_1 \\
-U_1 + z_q & Z_r & -g \sin \Phi_1 \cos \theta_1 & -g \cos \Phi_1 \sin \theta_1 \\
-I_{p_a} P_1 + 2I_{q_a} Q_1 - I_{p_a} R_1 + L_p & -I_{q_a} P_1 - I_{p_a} Q_1 - 2I_{q_a} R_1 + L_r & 0 & 0 \\
-I_{q_a} P_1 + I_{q_a} R_1 + M_q & -I_{q_a} P_1 + I_{q_a} Q_1 + 2I_{q_a} R_1 + M_r & 0 & 0 \\
-I_{r_a} P_1 - 2I_{r_a} Q_1 - I_{r_a} R_1 + N_q & I_{q_a} P_1 - I_{r_a} Q_1 + N_r & 0 & 0 \\
-1 & 0 & -\theta_1 \sin \Phi_1 + \Psi_1 \cos \Phi_1 \cos \theta_1 & -\Psi_1 \sin \Phi_1 \sin \theta_1 \\
0 & -1 & -\theta_1 \cos \Phi_1 - \Psi_1 \sin \Phi_1 \cos \theta_1 & -\Psi_1 \cos \Phi_1 \sin \theta_1 \\
0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 \\
\end{bmatrix}
\]

(5.44)
The state matrix $B$ is given by:

$$
B = \begin{bmatrix}
X_d e & X_d a & X_d r & X_d t \\
Y_d e & Y_d a & Y_d r & Y_d t \\
Z_d e & Z_d a & Z_d r & Z_d t \\
L_d e & L_d a & L_d r & L_d t \\
M_d e & M_d a & M_d r & M_d t \\
N_d e & N_d a & N_d r & N_d t \\
0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0
\end{bmatrix}
$$

And the matrix $E$ is given by:

$$
E = \begin{bmatrix}
-\frac{K_{ae}}{J_{ap}} Y_a e + \frac{K_{aw}}{J_{ap}} Z_a e & 0 & 0 & 0 & 0 \\
0 & -\frac{K_{ae}}{J_{ap}} Y_a e + \frac{K_{aw}}{J_{ap}} Z_a e & 0 & 0 & 0 \\
0 & 0 & -\frac{K_{aw}}{J_{ap}} Y_a e + \frac{K_{aw}}{J_{ap}} Z_a e & 0 & 0 \\
0 & 0 & 0 & -\frac{K_{aw}}{J_{ap}} Y_a e + \frac{K_{aw}}{J_{ap}} Z_a e & 0 \\
0 & 0 & 0 & 0 & -\frac{K_{aw}}{J_{ap}} Y_a e + \frac{K_{aw}}{J_{ap}} Z_a e \\
0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0
\end{bmatrix}
$$
5.6.3 The Rigid-Body Aircraft State Space Model in Aerodynamic Form

With the substitutions defined in equations (5.31) and (5.33), the perturbed equations of motion become:

**Force Equations:**

\[
(1)\ddot{u} + (-X_a)\dot{a} + (-X_\beta)\dot{\beta} = \\
\left(\vec{A}_{vu}R_1 - \vec{A}_{wQ_1} + (X_u + X_{\tau_a})\right)u + \left(\vec{A}_{vu}R_1 - \vec{A}_{wQ_1} + (X_u + X_{\tau_a})\right)\alpha + (-g \cos \Theta_1)\theta \\
+ (-W_1 + X_\eta)q + \left(\vec{A}_{v_\eta}R_1 + (X_{\eta} + X_{T_\eta})\right)\beta + (X_p)p + (V_1 + X_r)r \\
+ (X_{\delta_e})\delta_e + (X_{\delta_t})\delta_t + (X_{\delta_a})\delta_a + (X_{\delta_r})\delta_r
\]  (5.47)

\[
(\vec{A}_{va})\ddot{u} + (\vec{A}_{va} - Y_a)\dot{a} + (\vec{A}_{v_\beta} - Y_\beta)\dot{\beta} = \\
\left(\vec{A}_{va}P_1 - R_1 + (Y_u + Y_{\tau_a})\right)u + \left(\vec{A}_{va}P_1 + (Y_a + Y_{\tau_a})\right)\alpha + (-g \sin \Phi_1 \sin \Theta_1)\theta + (Y_\eta)q \\
+ \left((Y_\beta + Y_{T_\beta})\right)\beta + (g \cos \Phi_1 \cos \Theta_1)\phi + (W_1 + Y_p)p + (U_1 + Y_r)r \\
+ (Y_{\delta_e})\delta_e + (Y_{\delta_t})\delta_t + (Y_{\delta_a})\delta_a + (Y_{\delta_r})\delta_r
\]  (5.48)

\[
(\vec{A}_{wa})\ddot{u} + (\vec{A}_{wa} - Z_a)\dot{a} + (-Z_\beta)\dot{\beta} = \\
\left(-\vec{A}_{wa}P_1 + Q_1 + (Z_u + Z_{\tau_a})\right)u + \left(-\vec{A}_{wa}P_1 + (Z_a + Z_{\tau_a})\right)\alpha + (-g \cos \Phi_1 \sin \Theta_1)\theta \\
+ (U_1 + Z_\eta)q + \left(-\vec{A}_{v_\eta}P_1 + (Z_\beta + Z_{T_\eta})\right)\beta + (-g \sin \Phi_1 \cos \Theta_1)\phi \\
+ (-V_1 + Z_p)p + (Z_r)r + (Z_{\delta_e})\delta_e + (Z_{\delta_t})\delta_t + (Z_{\delta_a})\delta_a + (Z_{\delta_r})\delta_r
\]  (5.49)

**Moment Equations:**

\[
(-L_u)\dot{a} + (-L_\eta)q + (-L_\beta)\dot{\beta} + (1)p + (-L_p)\dot{r} = \\
(L_u + L_{T_a})u + (L_a + L_{T_a})\alpha + (I_{\eta}P_1 + 2I_{p_4}Q_1 - I_{p_2}R_1 + L_\eta)q + (L_\eta + L_{T_\eta})\beta \\
+ (I_{p_1}Q_1 - I_{p_2}R_1 + L_\eta)p + (-I_{p_3}P_1 - I_{p_2}Q_1) + 2I_{p_4}R_1 + L_r)r + (L_{\delta_e})\delta_e \\
+ (L_{\delta_t})\delta_t + (L_{\delta_a})\delta_a + (L_{\delta_r})\delta_r
\]  (5.50)
Now, let the states of the system be given by 
\[ X = X_a = [u \ a \ \theta \ q \ \beta \ \phi \ p \ r \ \psi]^T \] 
and the inputs be given by 
\[ U = [\delta_e \ \delta_a \ \delta_r \ \delta_t]^T \].

Equations (5.47) to (5.55) may then be cast in the following descriptor (implicit) state space representation:

\[
E \dot{x} = Ax + Bu
\]
The state matrix $A$ is given by:

$$
A = \begin{bmatrix}
\overline{A}_{vu} R_1 - \overline{A}_{wu} Q_1 + (X_u + X_{Tu}) & \overline{A}_{va} R_1 - \overline{A}_{wa} Q_1 + (X_a + X_{Ta}) & \cdots & -g \cos \theta_1 & -W_1 + X_q \\
\overline{A}_{wu} P_1 - R_1 + (Y_u + Y_{Tu}) & \overline{A}_{wa} P_1 + (Y_a + Y_{Ta}) & \cdots & -g \sin \Phi_1 \sin \theta_1 & Y_q \\
-\overline{A}_{vu} P_1 + Q_1 + (Z_u + Z_{Tu}) & \overline{A}_{va} P_1 + (Z_a + Z_{Ta}) & \cdots & -g \cos \Phi_1 \sin \theta_1 & U_1 + Z_q \\
(L_u + L_{Tu}) & (L_a + L_{Ta}) & \cdots & 0 & -\overline{L}_p \overline{p}_1 + 2 \overline{L}_p \overline{q}_1 - \overline{L}_q \overline{r}_1 + L_q \\
(M_u + M_{Tu}) & (M_a + M_{Ta}) & \cdots & 0 & -\overline{M}_p \overline{p}_1 + \overline{M}_q \overline{q}_1 + M_q \\
(N_u + N_{Tu}) & (N_a + N_{Ta}) & \cdots & 0 & -\overline{N}_p \overline{p}_1 - 2 \overline{N}_q \overline{q}_1 - \overline{N}_r \overline{r}_1 + N_q \\
0 & 0 & \cdots & -\Psi_1 \cos \theta_1 & 0 \\
0 & 0 & \cdots & -\Psi_1 \sin \Phi_1 \sin \theta_1 & -1 \\
0 & 0 & \cdots & -\Psi_1 \cos \Phi_1 \sin \theta_1 & 0 \\
0 & 0 & \cdots & 0 & 0 \\
0 & 0 & \cdots & 0 & 0 \\
0 & 0 & \cdots & 0 & 0 \\
0 & 0 & \cdots & 0 & 0 \\
0 & 0 & \cdots & 0 & 0 \\
0 & 0 & \cdots & 0 & 0 \\
0 & 0 & \cdots & 0 & 0 \\
0 & 0 & \cdots & 0 & 0 \\
\end{bmatrix}
$$

(5.57)
The state matrix $B$ is given by:

$$
\begin{bmatrix}
X_{\delta_e} & X_{\delta_a} & X_{\delta_r} & X_{\delta_t} \\
Y_{\delta_e} & Y_{\delta_a} & Y_{\delta_r} & Y_{\delta_t} \\
Z_{\delta_e} & Z_{\delta_a} & Z_{\delta_r} & Z_{\delta_t} \\
L_{\delta_e} & L_{\delta_a} & L_{\delta_r} & L_{\delta_t} \\
\end{bmatrix}
$$

$$
\begin{align*}
B = 
& M_{\delta_e} & M_{\delta_a} & M_{\delta_r} & M_{\delta_t} \\
& N_{\delta_e} & N_{\delta_a} & N_{\delta_r} & N_{\delta_t} \\
& 0 & 0 & 0 & 0 \\
& 0 & 0 & 0 & 0 \\
& 0 & 0 & 0 & 0 
\end{align*}
$$

(5.58)

And the matrix $E$ is given by:

$$
\begin{bmatrix}
1 & -X_\alpha & 0 & 0 & -X_\beta & 0 & 0 & 0 & 0 \\
A_{v_\alpha} & A_{v_\alpha} - Y_\alpha & 0 & 0 & A_{v_\beta} - Y_\beta & 0 & 0 & 0 & 0 \\
A_{w_\alpha} & A_{w_\alpha} - Z_\alpha & 0 & 0 & -Z_\beta & 0 & 0 & 0 & 0 \\
0 & -L_\alpha & 0 & -I_{p_3} & -L_\beta & 0 & 1 & -I_{p_1} & 0 \\
0 & -M_\alpha & 0 & 1 & -M_\beta & 0 & -I_{q_3} & -I_{q_4} & 0 \\
0 & -N_\alpha & 0 & -I_{r_4} & -N_\beta & 0 & -I_{r_1} & 1 & 0 \\
0 & 0 & 0 & 0 & -1 & 0 & 0 & \sin \Theta_1 & 0 \\
0 & 0 & -\cos \Phi_1 & 0 & 0 & 0 & 0 & 0 & -\sin \Phi_1 \cos \Theta_1 \\
0 & 0 & \sin \Phi_1 & 0 & 0 & 0 & 0 & 0 & -\cos \Phi_1 \cos \Theta_1 
\end{bmatrix}
$$

(5.59)
CHAPTER 6

LOSS-OF-CONTROL MARGIN PREDICTION
(DEFLECTION-TO-GO METHODS)

6.1 Overview

Figure 6.1 presents a functional outline of the Deflection-To-Go prediction model. The prediction model analytically determines, at each real-time \( t \), the critical control surface step, ramp, parabolic, or higher order input magnitude required to reach the LOC margins within the given prediction time window \( \hat{t} \in [0, T] \), accounting for non-zero initial conditions on both the states and inputs.

![Figure 6.1. Loss-of-Control Deflection-To-Go Prediction Model Concept](image)

The prediction model functions within the “virtual time” dimension \( \hat{t} \), and solves for a set of critical deflections \( u_{\text{crit}}(t) \) at every timestep in the real-time dimension \( t \). These critical deflections make up the bounds of the dynamically-changing safe control envelope, which is the
metric presented to the pilot through the advisory display. The critical deflections at the current time \( t \) for the first three orders of input are defined in Table 6.1:

**TABLE 6.1**

**COMPUTATION OF CRITICAL DEFLECTION**

<table>
<thead>
<tr>
<th>Input Type</th>
<th>Deflection-To-Go</th>
<th>Critical Deflection</th>
</tr>
</thead>
<tbody>
<tr>
<td>Step ((\hat{q} = 1))</td>
<td>(\Delta \hat{u}_j = \Delta u_j)</td>
<td>(u_{\text{crit}_j}(t) = u_j(t) + \Delta u_j)</td>
</tr>
<tr>
<td>Ramp ((\hat{q} = 2))</td>
<td>(\Delta \hat{u}_j = \Delta \dot{u}_j)</td>
<td>(u_{\text{crit}_j}(t) = u_j(t) + \int_0^\hat{t} \Delta \dot{u}_j , d\hat{t})</td>
</tr>
<tr>
<td>Parabolic ((\hat{q} = 3))</td>
<td>(\Delta \hat{u}_j = \Delta \ddot{u}_j)</td>
<td>(u_{\text{crit}_j}(t) = u_j(t) + \int_0^\hat{t} \Delta \ddot{u}_j , d\hat{t})</td>
</tr>
</tbody>
</table>

The critical deflection is composed of the current control surface deflection and the deflection-to-go. In order to compute these critical deflections, an analytical model of the current aircraft dynamic response in the time-domain must first be established and known. The derivation of these analytical expressions is discussed in this chapter.

### 6.2 The Aircraft Dynamic Model

Let a generalized aircraft have \( i = 1, \ldots, n \) states where \( \{n \in 2\mathbb{Z} : n \geq 2\} \), \( j = 1, \ldots, m \) inputs where \( \{m \in \mathbb{Z} : m \geq 1\} \), \( k = 1, \ldots, p^c \) complex modes where \( \{p^c \in \mathbb{Z} : p^c \leq n\} \), and \( k = 1, \ldots, p^r \) real modes where \( \{p^r \in \mathbb{Z} : p^r \leq n\} \). Also, let the critical deflection \( \Delta \hat{U} \) be of order \( \hat{q} \), where \( \{\hat{q} \in \mathbb{Z} : \hat{q} \geq 1\} \). For the 8th order aircraft with coupled longitudinal and lateral/directional dynamics, the system has \( n = 8 \) states, which are \([u \ a \ \theta \ q \ \beta \ \phi \ p \ r]^T\), and \( m = 4 \) inputs, which are \([\delta_e \ \delta_a \ \delta_r \ \delta_t]^T\).
6.2.1 The General $n^{th}$ Order State Space System

Consider the aircraft dynamic model in state space form, as presented in equation (6.1). The state space matrices $A$ and $B$ are respectively given by $A = A_0 + \Delta A$ and $B = B_0 + \Delta B$, encompassing the complete known and estimated dynamics of the aircraft at the current time $t$.

$$\dot{X}(t) = AX(t) + BU(t) \quad (6.1)$$

To account for non-zero initial conditions on the inputs, the input vector $U(t)$ may be written as the sum of the current input at the start of time $t$, denoted by $U(t_0)$, and the predicted “deflection-to-go”, denoted by $\Delta \hat{U}(t)$. Equation (6.1) may then be written as follows:

$$\dot{X}(t) = AX(t) + B\left(\Delta \hat{U}(t) + U(t_0)\right) \quad (6.2)$$

For the generalized aircraft, equation (6.2) may be expanded to become:

$$\begin{bmatrix}
\dot{x}_1(t) \\
\dot{x}_2(t) \\
\vdots \\
\dot{x}_n(t)
\end{bmatrix} =
\begin{bmatrix}
a_{1,1} & a_{1,2} & \cdots & a_{1,n} \\
a_{2,1} & a_{2,2} & \cdots & a_{2,n} \\
\vdots & \vdots & \ddots & \vdots \\
a_{n,1} & a_{n,2} & \cdots & a_{n,n}
\end{bmatrix}
\begin{bmatrix}
x_1(t) \\
x_2(t) \\
\vdots \\
x_n(t)
\end{bmatrix} +
\begin{bmatrix}
b_{1,1} & b_{1,2} & \cdots & b_{1,m} \\
b_{2,1} & b_{2,2} & \cdots & b_{2,m} \\
\vdots & \vdots & \ddots & \vdots \\
b_{n,1} & b_{n,2} & \cdots & b_{n,m}
\end{bmatrix}
\begin{bmatrix}
\Delta \hat{u}_1(t) + u_1(t_0) \\
\Delta \hat{u}_2(t) + u_2(t_0) \\
\vdots \\
\Delta \hat{u}_m(t) + u_m(t_0)
\end{bmatrix} \quad (6.3)$$

6.2.2 The General $n^{th}$ Order Transfer Function

Taking the Laplace Transform of equation (6.2) while accounting for non-zero initial conditions on the states and inputs yields:

$$\dot{X}(t) = AX(t) + B\left(\Delta \hat{U}(t) + U\right) \xrightarrow{\mathcal{L}} sX(s) = AX(s) + B\left(\Delta \hat{U}(s) + \frac{U}{s}\right) + X_0 \quad (6.4)$$

The variable of interest is $X(s)$. Rearranging equation (6.4) to make $X(s)$ the subject yields:
\[(sI - A)X(s) = B \left( \Delta \tilde{U}(s) + \frac{U}{s} \right) + X_0 \] (6.5)

\[X(s) = (sI - A)^{-1}B \left( \Delta \tilde{U}(s) + \frac{U}{s} \right) + (sI - A)^{-1}X_0 = H(s)\Delta \tilde{U} + \bar{H}(s)U + G(s) \] (6.6)

The matrix \((sI - A)^{-1}\) may be found by evaluating:

\[(sI - A)^{-1} = \frac{\text{adj}(sI - A)}{\det(sI - A)} \] (6.7)

The classical adjoint (or adjugate) of the \(n \times n\) matrix \([sI - A]\) is an \(n \times n\) matrix where the \((j,i)^{th}\) element of the adjoint matrix is the \((i,j)^{th}\) cofactor of \([sI - A]\), as defined by:

\[\text{adj}(sI - A)]_{j,i} = [\text{cofactor}(sI - A)]_{i,j} = (-1)^{i+j} \det([sI - A]_{i',j'}) \] (6.8)

Wherein \([sI - A]_{i',j'}\) is a submatrix of \([sI - A]\), formed by removing the \(i^{th}\) row and \(j^{th}\) column of \([sI - A]\). Ultimately, for each of the \(i\) states of the system, equation (6.6) may be expressed in the following form:

\[x_i(s) = \sum_{j=1}^{m} \left[ H_{i,j}(s) \left( \Delta \tilde{u}_j(s) + \frac{u_j}{s} \right) \right] + G_i(s) \] (6.9)

In equation (6.9), the transfer function \(H_{i,j}(s)\) contains the forced response of the system, the transfer function \(G_i(s)\) contains the unforced response of the system (for non-zero initial conditions on the states), and the deflection-to-go \(\Delta \tilde{u}_j(s)\) is the variable of interest.
In the most general case, the transfer functions $H_{i,j}(s)$ and $G_i(s)$ take on the forms given in equations (6.11) and (6.12), wherein the numerical coefficients $a_n$, $c_n$, and $d_n$ describe the characteristics of the aircraft:

$$H_{i,j}(s) = [(sI - A)^{-1}B]_{i,j}$$

$$G_i(s) = [(sI - A)^{-1}X_0]_i$$ (6.10)

$$\Delta \hat{u}_j(s) = \Delta \hat{u}_j \frac{1}{s^q}$$

In the most general case, the transfer functions $H_{i,j}(s)$ and $G_i(s)$ take on the forms given in equations (6.11) and (6.12), wherein the numerical coefficients $a_n$, $c_n$, and $d_n$ describe the characteristics of the aircraft:

$$H_{i,j}(s) = \frac{c_{n-1}s^{n-1} + \cdots + c_3s^3 + c_2s^2 + c_1s + c_0}{a_ns^n + \cdots + a_4s^4 + a_3s^3 + a_2s^2 + a_1s + a_0}$$ (6.11)

$$G_i(s) = \frac{d_{n-1}s^{n-1} + \cdots + d_3s^3 + d_2s^2 + d_1s + d_0}{a_ns^n + \cdots + a_4s^4 + a_3s^3 + a_2s^2 + a_1s + a_0}$$ (6.12)

6.3 Method of Modal Superposition

6.3.1 Overview

To facilitate mathematically- and computationally-efficient analysis of the $n^{th}$ order transfer functions in the time-domain, it is prudent to first simplify equations (6.11) and (6.12). This idea of “modal superposition” is illustrated in Figure 6.2. To maintain conciseness, an abbreviated presentation of the modal superposition method is presented in this section. For a full and thorough discourse of its theoretical development, the reader is encouraged to refer to Appendix D.
It may be shown that the $n^{\text{th}}$ order polynomial may be written as the sum of a series of 1$^{\text{st}}$ order and 2$^{\text{nd}}$ order polynomials, as in equations (6.13) and (6.14):

$$H_{i,j}(s) = \sum_{k=1}^{p^C} \frac{m_{2i,k}s + m_{1i,k}}{s^2 + 2\xi_k \omega_n s + \omega_n^2} + \sum_{k=1}^{p^R} \frac{m_{0i,k}}{s + \tau_k^{-1}} \quad (6.13)$$

$$G_{i}(s) = \sum_{k=1}^{p^C} \frac{n_{2i,k}s + n_{1i,k}}{s^2 + 2\xi_k \omega_n s + \omega_n^2} + \sum_{k=1}^{p^R} \frac{n_{0i,k}}{s + \tau_k^{-1}} \quad (6.14)$$

Each elemental 2$^{\text{nd}}$ or 1$^{\text{st}}$ order transfer function then respectively contains the characteristics of each of the $p^C$ complex modes and $p^R$ real modes in the $n^{\text{th}}$ order system, such as damping ratio, natural frequency, time constant, and system poles/zeros. Recalling equation (6.9), in the frequency domain, the complete response for the $i^{\text{th}}$ state of the system is then given by:
Each permutation of the step, ramp, parabolic, or higher order inputs $\Delta u_{ij}(s)$ with the elemental transfer function $H_{ij}(s)$ fundamentally consists of some linear combination of a transfer function of Type $\mathbb{Q}$ and Type $\mathbb{Q}^{-}$ multiplied by the coefficients $m_2$ and $m_1$, where $\mathbb{Q} = \hat{q}$ and $\mathbb{Q}^{-} = \hat{q} - 1$. The elemental 2\textsuperscript{nd} and 1\textsuperscript{st} order transfer functions in equation (6.15) may thus be further expanded and expressed explicitly as a function of standard Laplace transforms. This simplifies the inverses from the frequency domain back into the time domain and facilitates the derivation of an efficient solution for the time-domain system response for the $n$\textsuperscript{th} order system, which is in turn used to compute the critical control deflections. This is discussed in more detail in Appendix D.

### 6.3.2 Computation of Aircraft Dynamic Response

Now, recall equation (6.6) and (6.15) – taking the inverse Laplace transform, the projected time-domain response for the states of the aircraft may be given by:

$$X(\hat{t}) = \mathcal{L}^{-1}\{X(s)\} = X_H(\hat{t}) + X_{\bar{H}}(\hat{t}) + X_G(\hat{t}) = H(\hat{t})\Delta \bar{U} + \bar{H}(\hat{t})U + G(\hat{t}) \quad (6.16)$$

Note the use of the notation $\hat{t}$, which implies that the state response of the aircraft is being evaluated in the “virtual time” dimension. At this juncture, the technical analyses of the time-domain responses of the complex and real modes are conducted separately. Equation (6.16) may thus be recast as follows:
\[ X(\hat{t}) = X^C(\hat{t}) + X^R(\hat{t}) = \left[ X^C_H(t) + X^C_H(t) + X^C_G(t) \right] + \left[ X^R_H(t) + X^R_H(t) + X^R_G(t) \right] \]

\[ = [H^C(\hat{t})\Delta \hat{U} + \hat{H}^C(t)U + G^C(\hat{t})] + [H^R(\hat{t})\Delta \hat{U} + \hat{H}^R(t)U + G^R(\hat{t})] \] (6.17)

### 6.3.2.1 Complex Modes

Ultimately, for each of the \(i\) states of the aircraft, the term \(X^C(\hat{t})\) in equation (6.17) may be expressed generally in scalar form as follows:

\[ x^C_i(\hat{t}) = \sum_{j=1}^{m} \sum_{k=1}^{p} \left[ \Delta \hat{u}_j \sum_{l=1}^{\hat{q}} \left( \mu^C_{\hat{i},j,l} \left( \frac{\hat{t}^{l-1}}{(l-1)!} \right) + \mu^C_{\hat{g},j,l} \left( e^{-\xi_k \omega_d \hat{t}} \cos(\omega_d \hat{t}) \right) + \mu^C_{\hat{s},j,l} \left( e^{-\xi_k \omega_d \hat{t}} \sin(\omega_d \hat{t}) \right) \right) \right] + u_j \sum_{k=1}^{p} \left( \mu^C_{\hat{g},j,k} \left( e^{-\xi_k \omega_d \hat{t}} \cos(\omega_d \hat{t}) \right) + \mu^C_{\hat{s},j,k} \left( e^{-\xi_k \omega_d \hat{t}} \sin(\omega_d \hat{t}) \right) \right) \]

\[ + \sum_{k=1}^{p} \left( v^C_{\hat{g},k} \left( e^{-\xi_k \omega_d \hat{t}} \cos(\omega_d \hat{t}) \right) + v^C_{\hat{s},k} \left( e^{-\xi_k \omega_d \hat{t}} \sin(\omega_d \hat{t}) \right) \right) \] (6.18)

The coefficients \(\mu^C\) are associated with the forced response of the system, and they contain the system zeros (within \(m_2\) and \(m_1\)) and the system dynamics (within \(\kappa^C\)). The reader is asked to refer to Appendix D for further details on the development of the complex system response. Recall that, for an input of order \(\hat{q}\), each of the elemental complex transfer functions in equation (6.15) consists of a linear combination of transfer functions of Type \(Q\) and Type \(Q^-\). For example, for a parabolic input of order \(\hat{q} = 3\) (ie. \(Q = III\), \(\kappa^C_Q\) should be interpreted as \(\kappa_3\) for a Type III transfer function, and \(\kappa^C_{Q^-}\) should be interpreted as \(\kappa_2\) or \(\kappa_1\) for a Type II transfer function.

### 6.3.2.2 Real Modes

Similarly, for each of the \(i\) states of the aircraft, the term \(X^R(\hat{t})\) in equation (6.17) may be expressed generally in scalar form as follows:
As before, the coefficients $\mu_R$ are associated with the forced response of the system, and they contain the system zeros (within $m_0$) and the system dynamics (within $\kappa_{Q,R}$). The reader is asked to refer to Appendix D for further details on the development of the real system response.

For an input of order $q$, each of the elemental real transfer functions in equation (6.15) is of Type $Q$. For example, for a parabolic input of order $q = 3$ (ie. $Q = III$), $\kappa_{l,k}^{R,Q}$ should be interpreted as $\kappa_3$, $\kappa_2$ or $\kappa_1$ for a Type III transfer function.

### 6.3.3 Computation of Loss-of-Control Margins

At this point, an analytical model of the aircraft’s time-domain response has been established. This model forms the basis of the analytical prediction of the safe control envelope. In preparing for the discussion presented here, the reader is asked to recall equation (6.17) (the generalized state equation matrix), equation (6.18) (the state equation for complex modes), and equation (6.19) (the state equation for real modes).

Equations (6.18) and (6.19), when evaluated in the context of equation (6.17), provide the time-domain response for each of the $i$ states of the aircraft, given the presence of multiple inputs and non-zero initial conditions on the states and inputs. The objective is then to determine the set of inputs $\Delta \hat{U}$ that would drive the aircraft to the pre-defined LOC margin ($x_{LOC}$) at the end of the prediction time window $\hat{T}$. The solution procedure to accomplish this depends on the type of
dynamic response for the particular state. Typically, each state displays either a monotonic or oscillatory growth/decay response type. Thus, the computation method of the safe control envelope for each case varies accordingly.

### 6.3.3.1 Monotonic State Responses

For monotonic state responses, the objective is to find the exact critical input that causes the state response to pass through the intersection of the prediction time window \( \hat{T} \) and the state limit \( x_{LOC} \). Consider again equation (6.17), recast in the form of equation (6.20) for each state:

\[
x_i(\hat{t}) = \sum_{j=1}^{m} \left[ \left( H_{i,j}^C(\hat{t}) + H_{i,j}^R(\hat{t}) \right) \Delta \hat{u}_j \right] + \sum_{j=1}^{m} \left[ \left( \bar{H}_{i,j}^C(\hat{t}) + \bar{H}_{i,j}^R(\hat{t}) \right) u_j \right] + \left( G_i^C(\hat{t}) + G_i^R(\hat{t}) \right)
\]  

(6.20)

The left-hand-side of equation (6.20) is set to the fixed value of the state limit and the dependent variable \( \hat{t} \) is replaced with the length of the prediction window \( \hat{T} \). This leaves a set of unknowns \( \Delta \hat{u}_j \) for each control effector, as in equation (6.21):

\[
x_{LOC_i} = \sum_{j=1}^{m} \left[ \left( H_{i,j}^C(\hat{T}) + H_{i,j}^R(\hat{T}) \right) \Delta \hat{u}_j \right] + \sum_{j=1}^{m} \left[ \left( \bar{H}_{i,j}^C(\hat{T}) + \bar{H}_{i,j}^R(\hat{T}) \right) u_j \right] + \left( G_i^C(\hat{T}) + G_i^R(\hat{T}) \right)
\]  

(6.21)

By inverting equation (6.21), one then obtains the explicit analytical expression for the critical input, as in equation (6.22). The output of this expression is the magnitude of the input set that causes the aircraft to reach LOC at the end of time \( \hat{T} \) – in other words, the critical deflections that form the bounds of the safe control envelope.

\[
\Delta \hat{u}_j = \left[ x_{LOC_i} - \sum_{j=1}^{m} \left[ \left( \bar{H}_{i,j}^C(\hat{T}) + \bar{H}_{i,j}^R(\hat{T}) \right) u_j \right] - \left( G_i^C(\hat{T}) + G_i^R(\hat{T}) \right) \right] \cdot \left[ H_{i,j}^C(\hat{T}) + H_{i,j}^R(\hat{T}) \right]^{-1}
\]  

(6.22)
For monotonic state responses (eg. a step aileron input causing an increase in bank angle, or a ramp or parabolic elevator input causing an increase in angle-of-attack), equation (6.22) may be evaluated explicitly to obtain the exact magnitude of the input set that causes the aircraft to reach LOC at the end of time $T$.

6.3.3.2 Oscillatory State Responses

For oscillatory state responses (eg. a step elevator input causing an increase in angle-of-attack), which typically contain an underdamped response with a characteristic overshoot, the approach employed to compute the safe control envelope differs slightly. The objective in this case is to determine the input magnitude that would cause the peak amplitude to exactly reach (and not overshoot) the LOC margin within the prediction time window $\hat{T}$. The time in the $\hat{t}$ dimension at which the peak amplitude occurs is independent of the magnitude of the predicted critical input and is a function of the system dynamics and initial conditions on the states and inputs at that particular time in $t$.

An intermediate step towards computing the safe control envelope for oscillatory state responses involves determining the peak time of the system response. The time-derivative of the system response is given by:

$$\dot{X}(\hat{t}) = \dot{X}^C(\hat{t}) + \dot{X}^R(\hat{t}) = \left[ \dot{H}^C(\hat{t})(\Delta U + U) + \dot{G}^C(\hat{t}) \right] + \left[ \dot{H}^R(\hat{t})(\Delta U + U) + \dot{G}^R(\hat{t}) \right]$$  (6.23)

For complex modes, taking the time derivative of equation (6.18) for a step input ($\hat{q} = 1$) results in the expression given in equation (6.24) for each of the $i$ states of the aircraft:
For real modes, taking the time derivative of equation (6.19) for a step input results in the expression given in equation (6.25) for each of the $i$ states of the aircraft:

$$
\dot{x}^c_i(\hat{t}) = \sum_{j=1}^{m} \left( \Delta u_j + u_j \right) \sum_{k=1}^{p^c} \left( e^{-\zeta_k \omega_n k \hat{t}} \left[ M_{C_{i,k,j}} \cdot \cos(\omega d_k \cdot \hat{t}) + M_{S_{i,k,j}} \cdot \sin(\omega d_k \cdot \hat{t}) \right] \right)
+ \sum_{k=1}^{p^c} \left( e^{-\zeta_k \omega_n k \hat{t}} \left[ N_{C_{i,k}} \cdot \cos(\omega d_k \cdot \hat{t}) + N_{S_{i,k}} \cdot \sin(\omega d_k \cdot \hat{t}) \right] \right)
$$

(6.24)

For real modes, taking the time derivative of equation (6.19) for a step input results in the expression given in equation (6.25) for each of the $i$ states of the aircraft:

$$
\dot{x}^r_i(\hat{t}) = \sum_{j=1}^{m} \left( \Delta u_j + u_j \right) \sum_{k=1}^{p^r} \left( M_{E_{i,k,j}} e^{-\tau_{k-1} \hat{t}} \right) + \sum_{k=1}^{p^r} \left( N_{E_{i,k}} e^{-\tau_{k-1} \hat{t}} \right)
$$

(6.25)

The reader is asked to refer to Appendix D for further details on the development of the expressions for the real and complex time derivatives. Next, for each state $i$, the times at which the minima and maxima occur (ie. peak time) are found by equating the superposition of equations (6.24) and (6.25) to zero and solving for $\hat{t}_p$, as in equation (6.26):

$$
0 = x_i = \sum_{j=1}^{m} \left[ \left( \dot{h}_i^c(\hat{t}_p) + \dot{h}_i^r(\hat{t}_p) \right) \left( \Delta \hat{u}_j + u_j \right) \right] + \left( \dot{g}_i^c(\hat{t}_p) + \dot{g}_i^r(\hat{t}_p) \right)
$$

(6.26)

Note that $\hat{t}_p$ depends on the initial condition of the states and inputs at each time in $t$ (ie. at $\hat{t} = 0$), and is quantitatively the same for a given instance regardless of the amplitude of the predicted critical control deflection $\Delta \hat{u}_j$. The time at which the first peak amplitude occurs is then used in place of the fixed time window $\hat{t}$ in equation (6.22), and the critical control deflection is calculated according to equation (6.27):
For systems with monotonic state responses, two fixed prediction time windows are applied, \( \hat{T} = 2 \text{ sec} \) and \( \hat{T} = 5 \text{ sec} \). For systems with oscillatory and underdamped responses, the prediction time window varies as a function of the peak time, which is in turn a function of the aircraft dynamics and initial conditions.

In the case of the light business jet used for concept validation in this article, the peak time occurs before the end of the time window \( \hat{T} = 2 \text{ sec} \). For other aircraft systems where the peak time may occur outside of these prediction time windows, the intersection of the state trajectory with the LOC margin is considered, and the deflection required to reach that point at exactly the 2- or 5-second mark is taken as the critical deflection.

\[
\Delta \hat{u}_j = \left[ x_{\text{LOC}_i} - \sum_{j=1}^{m} \left( \left( \bar{H}_{i,j}^C(\hat{t}_p) + \bar{H}_{i,j}^R(\hat{t}_p) \right) u_j \right) - \left( G_i^C(\hat{t}_p) + G_i^R(\hat{t}_p) \right) \right] \cdot \left[ \bar{H}_{i,j}^C(\hat{t}_p) + \bar{H}_{i,j}^R(\hat{t}_p) \right]^{-1}
\]

(6.27)
6.4 Method of Modal Transformation

6.4.1 Overview

An alternative to the modal superposition method is the modal transformation method. In this case, the higher-order aircraft dynamics are “transformed” into decoupled lower-order systems, and then analyzed individually. Figure 6.3 presents an overview of the modal transformation concept. To maintain conciseness, an abbreviated presentation of the modal transformation method is presented in this section. For a full and thorough discourse of its theoretical development, the reader is encouraged to refer to Appendix E.

Consider again the coupled, generalized, higher order aircraft state space system given in equation (6.2):

\[
\dot{X}(t) = AX(t) + B \left( \Delta \tilde{U}(t) + U(t_0) \right)
\]  (6.28)

Figure 6.3. Modal Superposition Concept
Using a predefined matrix transformation, the higher order aircraft state space model described in equations (6.28) may be transformed into a series of 2nd order block diagonals. This process facilitates the analytical analysis of each of the aircraft modes as a 2nd order system, which is mathematically and computationally efficient. Each block diagonal contains the approximate dynamics of each of the modes originally in the higher order dynamics, namely the short period, phugoid, dutch roll, roll, and spiral. Of interest in the context of the prediction framework described here are the first four modes. The linear transformation is given by:

\[ X = T_D Z \]  
(6.29)

Applying equation (6.29) to the state space system in the \( X \)-domain results in the following transformation:

\[
\dot{X}(t) = AX(t) + B \left( \Delta U(t) + U(t_0) \right) \quad \Rightarrow \quad \dot{Z}(t) = A_Z Z(t) + B_Z \left( \Delta U(t) + U(t_0) \right)
\]  
(6.30)

The transformed \( A \) matrix is given by:

\[
A_Z = T_D^{-1} A T_D
\]  
(6.31)

The transformed \( B \) matrix is given by:

\[
B_Z = T_D^{-1} B
\]  
(6.32)

For a system with a mixture of complex and real eigenvalues, as is the case with the aircraft here, the transformation matrix \( T_D \) is then given by equation (6.33), which constitutes an amalgamation of the real and imaginary parts of the eigenvectors, \( v_i \), of the \( A \) matrix.
The resulting transformed state space system in the \( Z \)-domain takes on the form given in equations (6.34) and (6.35):

\[
T_D = \begin{bmatrix} \text{Re}(v_1) & \text{Im}(v_2) & \text{Re}(v_3) & \text{Im}(v_4) & \text{Re}(v_5) & \text{Im}(v_6) & \text{Re}(v_7) & \text{Re}(v_8) \end{bmatrix} \tag{6.33}
\]

Each of the 2\textsuperscript{nd} order block diagonals (or 1\textsuperscript{st} order diagonals) are then analyzed as decoupled, independent systems, using a solution methodology adapted from the modal superposition method.

\[
A_z = \begin{bmatrix}
  a_{x1,1} & a_{x1,2} & 0 & 0 & 0 \\
  a_{x2,1} & a_{x2,2} & 0 & 0 & 0 \\
  0 & a_{x3,3} & a_{x3,4} & 0 & 0 \\
  0 & a_{x4,3} & a_{x4,4} & 0 & 0 \\
  0 & 0 & a_{x5,5} & a_{x5,6} & 0 \\
  0 & 0 & a_{x6,5} & a_{x6,6} & 0 \\
  0 & 0 & 0 & a_{x7,7} & 0 \\
  0 & 0 & 0 & 0 & a_{x8,8}
\end{bmatrix} \tag{6.34}
\]

\[
B_z = \begin{bmatrix}
  b_{xz,1} & \cdots & b_{xz,1,m} \\
  b_{xz,2} & \cdots & b_{xz,2,m} \\
  b_{xz,3} & \cdots & b_{xz,3,m} \\
  b_{xz,4} & \cdots & b_{xz,4,m} \\
  b_{xz,5} & \cdots & b_{xz,5,m} \\
  b_{xz,6} & \cdots & b_{xz,6,m} \\
  b_{xz,7} & \cdots & b_{xz,7,m} \\
  b_{xz,8} & \cdots & b_{xz,8,m}
\end{bmatrix} \tag{6.35}
\]
6.4.2 Computation of Aircraft Dynamic Response

As with the modal superposition method, the projected time-domain response for the states of the aircraft may be given by:

\[ Z(\hat{t}) = Z_H(\hat{t}) + Z_H(\hat{t}) + Z_G(\hat{t}) = H(\hat{t}) \Delta \hat{U} + \bar{H}(\hat{t}) U + G(\hat{t}) \]  

(6.36)

Note the use of the notation \( \hat{t} \), which implies that the state response of the aircraft is being evaluated in the “virtual time” dimension. At this juncture, the technical analyses of the time-domain responses of the complex and real modes are conducted separately and independently.

6.4.2.1 Complex Modes

For complex modes, equation (6.36) may be recast as follows:

\[ Z(\hat{t}) = Z^C(\hat{t}) = [Z_H^C(\hat{t}) + Z_H^C(\hat{t}) + Z_G^C(\hat{t})] = [H^C(\hat{t}) \Delta \hat{U} + \bar{H}^C(\hat{t}) U + G^C(\hat{t})] \]  

(6.37)

Ultimately, for each of the \( i \) states of the aircraft, the term \( Z^C(\hat{t}) \) in equation (6.37) may be expressed generally in scalar form as follows:

\[
Z^C_i(\hat{t}) = \sum_{j=1}^{m} \left[ \Delta \hat{u}_j \left( \sum_{i=1}^{q} \left( \mu^C_{ij} \left( \frac{\hat{t}^{l-1}}{(l-1)!} \right) + \mu^C_{ij} \left( e^{-\zeta \omega_n t} \cos(\omega_d \hat{t}) \right) + \frac{\zeta \omega_n t \sin(\omega_d \hat{t})} \right) \right) 
+ u_j \left( \bar{\mu}^C_{ij} + \bar{\mu}^C_{ij} \left( e^{-\zeta \omega_n t} \cos(\omega_d \hat{t}) \right) + \bar{\mu}^C_{ij} \left( e^{-\zeta \omega_n t} \sin(\omega_d \hat{t}) \right) \right) 
+ \nu^C \left( e^{-\zeta \omega_n t} \cos(\omega_d \hat{t}) \right) + \nu^C \left( e^{-\zeta \omega_n t} \sin(\omega_d \hat{t}) \right) \right] 
\]

(6.38)

The coefficients \( \mu^C \) are associated with the forced response of the system, and they contain the system zeros (within \( m_2 \) and \( m_4 \)) and the system dynamics (within \( \kappa^C \)). The reader is asked to refer to Appendix E for further details on the development of the complex system response.

6.4.2.2 Real Modes

Similarly, for real modes, equation (6.36) may be recast as follows:
Then, for each of the $i$ states of the aircraft, the term $Z^R(\hat{t})$ in equation (6.39) may be expressed generally in scalar form as follows:

$$Z(\hat{t}) = Z^R(\hat{t}) = [Z_H^R(\hat{t}) + Z_J^R(\hat{t}) + Z_G^R(\hat{t})] = [H^R(\hat{t})\Delta \bar{U} + H^R(\hat{t})U + G^R(\hat{t})]$$ (6.39)

As before, the coefficients $\mu^R$ are associated with the forced response of the system, and they contain the system zeros (within $m_0$) and the system dynamics (within $\kappa^R$). The reader is asked to refer to Appendix E for further details on the development of the real system response.

6.4.3 Computation of Loss-of-Control Margins

At this point, an analytical model of the aircraft’s time-domain response in the $Z$-domain has been established. This model forms the basis of the analytical prediction of the safe control envelope. As with the modal superposition method, the solution procedure to accomplish this depends on the type of dynamic response for the particular state. Typically, each state displays either a monotonic or oscillatory growth/decay response type. Thus, the computation method of the safe control envelope for each case varies accordingly.

In the $Z$-domain, the additional step of transforming the state limits is also required. This is accomplished by applying the following transformation:

$$z_{LOC} = T_D^{-1} x_{LOC}$$ (6.41)
6.4.3.1 Monotonic State Responses (Real Modes)

For the real modes, the objective is to find the exact critical input that causes the state response to pass through the intersection of the prediction time window $\hat{T}$ and the state limit ($z_{LOC}$). Following the same processes detailed for the modal superposition method, the critical deflections that form the bounds of the safe control envelope are obtained according to equation (6.42):

$$\Delta \hat{u}_j = \left[ z_{LOC_i} - \sum_{j=1}^{m} \left[ \hat{H}_{i,j}^{\Re}(\hat{T}) u_j \right] - \hat{G}_{i}^{\Re}(\hat{T}) \right] \cdot \left[ H_{i,j}^{\Re}(\hat{T}) \right]^{-1} $$  (6.42)

For monotonic state responses (eg. a step aileron input causing an increase in bank angle, or a ramp or parabolic elevator input causing an increase in angle-of-attack), equation (6.42) may be evaluated explicitly to obtain the magnitude of the input set that causes the aircraft to reach LOC at the end of time $\hat{T}$.

6.4.3.2 Oscillatory State Responses (Complex Modes)

For the complex modes (eg. a step elevator input causing an increase in angle-of-attack), which typically contain an underdamped response with a characteristic overshoot, the approach employed to compute the safe control envelope differs slightly. The objective in this case is to determine the input magnitude that would cause the peak amplitude to exactly reach (and not overshoot) the LOC margin within the prediction time window $\hat{T}$.

An intermediate step towards computing the safe control envelope for complex modes involves determining the peak time of the system response. The time-derivative of the system response is given by:

$$\dot{Z}(\hat{t}) = \dot{Z}^C(\hat{t}) = \left[ \dot{H}^C(\hat{t})(\Delta \hat{U} + U) + \dot{G}^C(\hat{t}) \right]$$  (6.43)
For complex modes, taking the time derivative of equation (6.38) for a step input ($\dot{q} = 1$) results in the expression given in equation (6.44) for each of the $i$ states of the aircraft:

$$
\dot{z}_i^C(\acute{t}) = \sum_{j=1}^{m} \left[ (\Delta u_j + u_j) \left( e^{-\zeta \omega_n \acute{t}} \left[ M_{ci,j} \cdot \cos(\omega_d \cdot \acute{t}) + M_{si,j} \cdot \sin(\omega_d \cdot \acute{t}) \right] \right) \right] + e^{-\zeta \omega_n \acute{t}} \left[ N_{ci} \cdot \cos(\omega_d \cdot \acute{t}) + N_{si} \cdot \sin(\omega_d \cdot \acute{t}) \right]
$$

(6.44)

The reader is asked to refer to Appendix E for further details on the development of the complex time derivative. Next, for each state $i$, the times at which the minima and maxima occur (i.e. peak time) are found by solving for $\acute{t}_p$ through equation (6.45):

$$
0 = \dot{z}_i^C(\acute{t}) = \sum_{j=1}^{m} \left[ \dot{H}_{i,j}(\acute{t}_p)(\Delta \dot{u}_j + u_j) \right] + \dot{G}_i^C(\acute{t}_p)
$$

(6.45)

The time at which the first peak amplitude occurs is then used in place of the fixed time window $\hat{T}$ in equation (6.42), and the critical control deflection is calculated according to equation (6.46):

$$
\Delta \dot{u}_j = \left[ z_{LOC} - \sum_{j=1}^{m} \left[ \dot{H}_{i,j}(\acute{t}_p) u_j \right] - G_i^C(\acute{t}_p) \cdot [H_{i,j}(\acute{t}_p)]^{-1} \right]
$$

(6.46)
CHAPTER 7
LOSS-OF-CONTROL MARGIN PREDICTION
(TRAJECTORY-TO-GO METHODS)

7.1 Overview

Figure 7.1 presents a functional outline of the Trajectory-To-Go (T2G) prediction model. The prediction model analytically determines, at each real-time \( t \), a set of critical control surface trajectories required to reach the various combinations of LOC margins within the given prediction time window \( \hat{t} \in [0, T] \), accounting for non-zero initial conditions on both the states \textit{and} inputs.

![Figure 7.1. Loss-of-Control Trajectory-To-Go Prediction Model Concept](image)

The prediction model functions within the “virtual time” dimension \( \hat{t} \), and solves for a set of critical trajectories \( U_{\text{crit}}^{*} \) at every timestep in the real-time dimension \( t \). These critical trajectories make up the bounds of the dynamically-changing safe control space.
While the Deflection-To-Go methods allow calculating a single critical control surface deflection at each $t$ for each individual state limit, the Trajectory-To-Go methods allow calculating a set of critical control surface trajectories at each $t$ that simultaneously satisfies the various combinations of state limits. These trajectories, if applied as inputs by the pilot, would cause the aircraft to reach the LOC margin (or combined LOC margins) in exactly $T^*$ seconds.

7.1.1 Optimal Control Formulations

In the optimal control formulations, the assumption is made that the pilot, when interacting with the aircraft, is doing so in an “optimal” manner. The physical interpretation of “optimal” depends on the type of optimal control law used in the prediction model. Three T2G prediction models were developed. These utilize differing optimal control formulations to determine the estimated critical control trajectory, as follows:

- **Open Loop Fixed-Final-State Optimal Control**
  
The fixed-final-state open loop optimal control formulation is used to predict the least-energy control trajectory required to reach the predefined state limit. This method does not take into account feedback from the states and input as the critical trajectory is being applied.

- **Closed-Loop Linear Quadratic Regulator**
  
The fixed-final-state linear quadratic regulator is used to predict the least-energy control trajectory required to reach the predefined state limit. The “closed-loop” formulation assumes that the pilot, when giving inputs to the aircraft, is doing so in an “optimal” manner and allowing his/her perception of the states of the aircraft to alter the input.
Closed-Loop Linear Quadratic Tracker

The linear quadratic tracker is used to predict the control trajectory required to reach the predefined state limit. The LQT formulation adds the ability to control the manner in which the state trajectory progresses over the course of the prediction time window $T$. The ability to control the state trajectory may be used to adapt the prediction architecture to the individual pilot’s behavior; the reference trajectory may be adaptively changed to follow the manner in which the individual pilot interacts with the aircraft.

7.1.2 Mapping Trajectories to a Safe Control Space

For each optimal control method, a set of trajectories is calculated at each time in $t$, targeting individual and combined LOC margins as the final state. For example, trajectories targeting individual LOC margins present the critical input needed to reach the angle-of-attack or bank angle margin independently, while trajectories targeting combined LOC margins present the critical input needed to reach both the angle-of-attack and bank angle margins simultaneously.

In this sense, the T2G methods present a true safe control space for the pilot, in that simultaneously manipulating the controls to the “corner edges” of the predicted safe control space would theoretically not allow the aircraft to exceed the safe flight envelope. These critical trajectories are mapped to the boundaries of the safe regions on the pilot advisory display, as shown in Figure 7.2.

This chapter details the derivation process of the Trajectory-To-Go methods. The chapter begins chronologically from the aircraft state space system derived in Chapter 5, and culminates in a series of optimal control laws used to determine the critical control trajectories. These critical trajectories are then handed over to the pilot advisory display detailed in Chapter 9, which graphically presents the safe control regions to the pilot.
7.2 The Aircraft Dynamic Model

As before, let a generalized aircraft have \( i = 1, ..., n \) states where \( \{ n \in 2\mathbb{Z} : n \geq 2 \} \), \( j = 1, ..., m \) inputs where \( \{ m \in \mathbb{Z} : m \geq 1 \} \). The aircraft dynamic model in state space form is then given by equation (7.1), and the state space matrices \( A \) and \( B \) are respectively given by \( A = A_0 + \Delta A \) and \( B = B_0 + \Delta B \), encompassing the complete known and estimated dynamics of the aircraft at the current time \( t \).

\[
\dot{X}(t) = AX(t) + BU(t) \tag{7.1}
\]

The current, known system dynamics are then passed into the various optimal control formulations to determine the critical control trajectories.
7.3 Closed-Loop Optimal Trajectory Prediction

7.3.1 Linear Quadratic Tracker (LQT)

Figure 7.3 presents a procedural overview of the T2G prediction model using the linear quadratic tracker.

![Figure 7.3. Trajectory-To-Go Prediction Model Using Linear Quadratic Tracker](image)

**7.3.1.1 Compute Final State Constraint**

In computing the final state constraint, it is important to consider that, over the course of the prediction time window $\hat{T}$, the states of the aircraft could potentially be changing due to non-zero initial conditions on the states and inputs. For example, suppose that $\hat{T} = 2$ sec and that the aircraft commences a climb at $t = 10$ sec. The final state constraint calculated at $t = 10$ sec has to account for the fact that the current control surface positions $U(t)$ are presently causing the aircraft to get closer to/further away from the state limits over the course of the following 2 seconds. To
factor in these initial condition dynamics, a trajectory projection model is utilized to compute the projected final state of the aircraft at the end of the prediction time window. The trajectory projection model has its system defined by equation (7.2):

\[
\dot{X}_p(\hat{t}) = AX_p(\hat{t}) + BU(t) \tag{7.2}
\]

The model uses the current states \( X(t) \) and current elevator deflection \( U(t) \) as initial conditions, and runs forward in time for \( \hat{T} \) seconds. The final values of the projected states \( X_p(\hat{T}) \) are assumed to be an estimate of the aircraft’s states at the end of the following \( \hat{T} \) seconds. The value of the final state constraint is then determined as the additional amount the limit is beyond the projected state value at \( \hat{T} \), according to:

\[
\tilde{r}(\hat{T}) = \tilde{C} \left( X_{LOC} - X_p(\hat{T}) \right) \tag{7.3}
\]

The matrix \( \tilde{C} \) is used to specify which combination of states are to be targeted as state limits.

**7.3.1.2 Compute State Trajectory**

The reference state trajectory \( \tilde{r}(\hat{t}) \) represents the assumed manner in which the pilot interacts with the aircraft when effecting changes to the aircraft’s states. In this research, the reference state trajectory is taken as a ramp function between the current state value and the value of the state limit, according to equation (7.4).

\[
\tilde{r}(\hat{t}) = \frac{\tilde{r}(\hat{T})}{\hat{T}} \hat{t} \tag{7.4}
\]
The practical implication of this assumption is that the pilot seeks to manipulate the aircraft in a linear manner when effecting changes to the aircraft’s states. Since this trajectory is arbitrary, \( \tilde{r}(\hat{t}) \) could, in practice, be altered to better represent the behaviors of the individual pilot. For example, a parabolic or sigmoidal function could be used in place of a ramp function. It would also be possible to update \( \tilde{r}(\hat{t}) \) in real-time based on changes in the pilot’s behavior, allowing the T2G prediction model to produce even more accurate estimates of the safe control space.

7.3.1.3 Compute Optimal Control Parameters

The cost function for the linear quadratic tracker [108] is given by equation (7.5), where the weight \( \bar{Q} \) is placed on regulating the states and the weight \( \bar{R} \) is placed on the control input, such that:

\[
J(\hat{t}_0) = \frac{1}{2} \left( \bar{C}X(\hat{T}) - \tilde{r}(\hat{T}) \right)^T \bar{P} \left( \bar{C}X(\hat{T}) - \tilde{r}(\hat{T}) \right) + \frac{1}{2} \int_{\hat{t}}^{\hat{T}} \left( \bar{C}X(\hat{t}) - \tilde{r}(\hat{t}) \right)^T \bar{Q} \left( \bar{C}X(\hat{t}) - \tilde{r}(\hat{t}) \right) + U(\hat{t})^T \bar{R} U(\hat{t}) \ d\hat{t}
\]  

(7.5)

In determining the optimal control, the parameters in equations (7.6) to (7.8) are first computed backwards in time from \( \hat{t} = \hat{T} \to 0 \), starting with the final state constraint. To begin with, the Riccati Equation given in equation (7.6) is solved for the parameter \( S \):

\[
-\dot{S} = A^T S + SA - SB\bar{R}^{-1}B^T S + \bar{C}^T \bar{Q} \bar{C}
\]  

(7.6)

The feedback gains \( \bar{K} \) are then calculated according to equation (7.7) and the solution to the modified state transition vector is integrated according to equation (7.8). As above, these expressions are computed backwards along the prediction time window.

\[
\bar{K}(\hat{t}) = \bar{R}^{-1}B^T S(\hat{t})
\]  

(7.7)
\[ -\dot{v}(\hat{t}) = (A - B\bar{K})^T v(\hat{t}) + \bar{C}^T \bar{Q}r(\hat{t}) \quad (7.8) \]

### 7.3.1.4 Compute Trajectory-To-Go

With these parameters determined, the optimal closed-loop control surface trajectory required to reach the respective state limits within the prediction time window is then calculated according to:

\[
U^*_{crit}(\hat{t}) = -\bar{K}X(\hat{t}) - \bar{R}^{-1}B^Tv(\hat{t}) 
\]

(7.9)

This process is repeated for each desired state limit combination, using the matrix \(\bar{C}\). The optimal control \(U^*_{crit}(\hat{t})\) is then the set of safe control trajectories that make up the three-dimensional safe control space.
7.3.2 Linear Quadratic Regulator (LQR)

Figure 7.4 presents a procedural overview of the T2G prediction model using the linear quadratic regulator.

![Figure 7.4. Trajectory-To-Go Prediction Model Using Linear Quadratic Regulator](image)

7.3.2.1 Compute Final State Constraint

Once again, the value of the final state constraint, being a function of the current states $X(t)$ and current control surface positions $U(t)$, is first calculated using the same process described previously. The final state constraint is taken to be:

$$\bar{r}(\hat{t}) = \bar{C} \left( X_{LOC} - X_p(\hat{t}) \right)$$  \hspace{1cm} (7.10)

The matrix $\bar{C}$ is used to specify which combination of states are to be targeted as state limits.
7.3.2.2  Compute Optimal Control Parameters

The cost function for the linear quadratic regulator [108] is given by equation (7.11), where the weight $\bar{Q}$ is placed on regulating the states and the weight $\bar{R}$ is placed on the control input, such that:

$$J(\hat{t}_0) = \frac{1}{2} X(\hat{t}) S(\hat{t}) X(\hat{t}) + \frac{1}{2} \int_0^{\hat{t}} (X(t)\bar{Q}(t) + U(t)^T \bar{R} U(t)) \, dt$$  \hspace{1cm} (7.11)

In determining the optimal control, the parameters in equations (7.12) to (7.15) are first computed backwards in time from $\hat{t} = \hat{T} \rightarrow 0$, starting with the final state constraint. To begin with, the Riccati Equation given in equation (7.12) is solved for the parameter $S$:

$$-\dot{S} = A^T S + S A - S B \bar{R}^{-1} B^T S + \bar{Q}$$  \hspace{1cm} (7.12)

The feedback gains $\bar{K}$ are then calculated according to equation (7.13), the solution to the modified state transition matrix is integrated according to equation (7.14), and the auxiliary quadrature is determined according to equation (7.15). As above, these expressions are computed backwards along the prediction time window.

$$\bar{K} = \bar{R}^{-1} B^T S$$  \hspace{1cm} (7.13)

$$-\dot{\bar{V}} = (A - B \bar{K})^T \bar{V}$$  \hspace{1cm} (7.14)

$$\dot{\bar{P}} = \bar{V}^T B \bar{R}^{-1} B^T \bar{V}$$  \hspace{1cm} (7.15)
7.3.2.3 Compute Trajectory-To-Go

With these parameters determined, the optimal closed-loop control surface trajectory required to reach the respective state limits within the prediction time window is then calculated according to:

\[
U^*_\text{crit}(\hat{t}) = -(\bar{K} - \bar{R}^{-1}B^T\bar{V}\bar{P}^{-1}\bar{V}^T)X(\hat{t}) - \bar{R}^{-1} B^T\bar{V}\bar{P}^{-1}\bar{r}\]  \tag{7.16}

This process is repeated for each desired state limit combination, using the matrix \( \bar{C} \). The optimal control \( U^*_\text{crit}(\hat{t}) \) is then the set of safe control trajectories that make up the three-dimensional safe control space.
7.4 Open-Loop Optimal Trajectory Prediction

Figure 7.5 presents a procedural overview of the T2G prediction model using fixed-final-state open loop optimal control.

Figure 7.5. Trajectory-To-Go Prediction Model Using Open Loop Optimal Control

7.4.1 Compute Final State Constraint

Once again, the value of the final state constraint, being a function of the current states $X(t)$ and current control surface positions $U(t)$, is first calculated using the same process described previously. The final state constraint is taken to be:

$$\bar{r}(\bar{t}) = X_{LOC} - X_p(\bar{t})$$  \hspace{1cm} (7.17)

7.4.2 Compute Optimal Control Parameters

The cost function for the open-loop optimal control [108] is given by equation (7.18), where the weight $\bar{R}$ is placed on the control input. Note that, being an open loop formulation, the closed-loop regulation term $XQX$ is absent:
The general form of the optimal control, expressed in terms of the costate equation, is then:

$$J(\hat{t}_0) = \frac{1}{2} \int_0^T U(\hat{t})^T \hat{R} U(\hat{t}) \, d\hat{t}$$  \hfill (7.18)$$

The costate equation, for a linear system of the form described in equation (7.1), has its solution given by:

$$U_{\text{crit}}(\hat{t}) = -\hat{R}^{-1} B^T \hat{\lambda}(\hat{t})$$  \hfill (7.19)$$

To determine \(\hat{\lambda}(\hat{T})\), equations (7.19) and (7.20) are substituted into the state equation given in equation (7.1), and the solution to the state equation is evaluated at time \(\hat{T}\). Together with the final state constraint given in equation (7.17), \(\hat{\lambda}(\hat{T})\) may then be expressed according to:

$$\hat{\lambda}(\hat{T}) = e^{A(T-\hat{T})} \hat{\lambda}(\hat{T})$$  \hfill (7.20)$$

$$\hat{\lambda}(\hat{T}) = -G^{-1}(0, \hat{T}) [\hat{r}(\hat{T}) - e^{A(T-0)} \hat{x}(t_0)]$$  \hfill (7.21)$$

Where \(G(0, \hat{T})\) is the weighted continuous reachability gramian, which is obtained through integrating the Lyapunov Equation:

$$\dot{\hat{P}} = A\hat{P} + \hat{P}A^T + B\hat{R}^{-1}B^T$$  \hfill (7.22)$$

For \(\hat{P}(0) = 0\), the gramian is defined to be \(G(0, \hat{T}) = \hat{P}(\hat{T})\).
7.4.3 Compute Trajectory-To-Go

With \( \tilde{G}(0, \hat{T}) \) is determined, the optimal open-loop control surface trajectory required to reach the respective state limits within the prediction time window is then calculated according to:

\[
U_{\text{crit}}^{*}(\hat{t}) = \bar{R}^{-1}B^T e^{A^T(\hat{T} - t)} \bar{G}^{-1}(0, \hat{T})[\bar{f}(\hat{T}) - e^{A(\hat{T} - 0)}X(t_0)]
\]

(7.23)

The optimal control \( U_{\text{crit}}^{*}(\hat{t}) \) is then the set of safe control trajectories that make up the safe control space.
8.1 Overview

The accuracy of the calculations made by the Deflection-To-Go and Trajectory-To-Go prediction models depends upon *a priori* knowledge of the aircraft’s dynamics – the mechanisms rely upon a known linear model of the aircraft to project several seconds ahead of time, in making a determination if current-time control inputs would drive the aircraft to the LOC margin at the end of a future-time period. While small deviations from the nominal model do not cause significant inaccuracies, larger deviations could potentially invalidate the predictions made by the prediction models. Such deviations might result from failures or uncertainties arising in the aircraft.

In making valid predictions of the safe control envelope, the intermediate goal then, is to ensure that the prediction models receive good and *current* knowledge of the aircraft’s dynamics *as the aircraft is in motion*. To this extent, an adaptive online parameter estimation algorithm was derived to quantify, in real-time, any potential uncertainties in the aircraft’s dynamics. Two parameter estimation algorithms were derived, with one designed to accommodate matched/structured uncertainties, and the other designed to accommodate both matched and unmatched uncertainties. While the unmatched uncertainty estimation method represents a more ideal and flexible technique capable of producing more accurate estimates [34], the derivation of both methods are presented below for the reader’s benefit.
8.2 Unmatched Uncertainty Parameter Estimation

Recalling equation (6.1), the aircraft’s system may be expressed in form given by equation (8.1), where \( A_0 \) and \( B_0 \) represent the nominal aircraft dynamics, and \( \Delta A \) and \( \Delta B \) represent some uncertainties in the aircraft dynamics:

\[
\dot{X}(t) = A_0 X(t) + B_0 U(t) + \Delta A X(t) + \Delta B U(t) \quad X(0) = X_0 \quad (8.1)
\]

The actual numerical components of the \( \Delta A \) and \( \Delta B \) matrices are unknown to the parameter estimation algorithm, and the objective is to derive estimates of these matrices in the form of \( \hat{\Delta A} \) and \( \hat{\Delta B} \). The estimated uncertain aircraft dynamics are then written as:

\[
\dot{\hat{X}}(t) = A_0 \hat{X}(t) + B_0 U(t) + \hat{\Delta A} X(t) + \hat{\Delta B} U(t) \quad \hat{X}(0) = \hat{X}_0 \quad (8.2)
\]

The error between the actual and estimated uncertain aircraft dynamics is given by \( \bar{X}(t) = X(t) - \hat{X}(t) \), and the cost function to be minimized is then given by equation (8.3):

\[
E(t) = \bar{X}^T(t) \bar{Q} \bar{X}(t)\bigg|_T + \frac{1}{2} \int_0^T \bar{X}^T(t) \bar{Q} \bar{X}(t) \, dt + \int_0^T \lambda \left[ \dot{\hat{X}}(t) - A_0 \hat{X}(t) - B_0 U(t) - \hat{\Delta A} X(t) - \hat{\Delta B} U(t) - \eta \| \bar{X} \| \right] dt \quad (8.3)
\]

Equation (8.3) introduces \( \lambda \), which serves as the Lagrange multiplier, \( \eta \), which serves as the adaptation damping term, and \( \bar{Q} \), which is a symmetric positive definite weighting matrix. Gradient descent optimization is applied to the cost function to estimate \( \hat{\Delta A} \) and \( \hat{\Delta B} \), adding the inherent ability to adapt to uncertainties not governed by a particular structure. Taking the partial derivative of the cost function with respect to \( \hat{\Delta A} \), \( \hat{\Delta B} \), and \( \hat{\bar{X}} \) yields equations (8.4) to (8.6):
Note that $X(t)$ represents the states of the actual system, as determined by equation (8.1). Thus, the variation of $X$ is zero in the minimization process. The assumption is made that the Lagrange multiplier $\lambda$ is related to the error $\tilde{X}(t)$ through some matrix $\hat{P}$, such that $\lambda^\top(t) = \hat{P}\tilde{X}(t)$ and $\dot{\lambda}^\top(t) = \hat{P}\dot{\tilde{X}}(t)$. Together with equation (8.6), and assuming that $\eta$ is numerically small in a relative context, it may then be demonstrated through the minimization of $\frac{\delta E}{\delta \tilde{X}} = 0$ that $\hat{P}$ must be a solution of the continuous-time Lyapunov Equation, as follows:

$$
\hat{P}A_0 + A_0^\top\hat{P} = -\hat{Q}
$$

(8.7)

Through first-order gradient descent optimization, the derivatives of the estimated parameter matrices are then obtained according to equations (8.8) and (8.9), where $\gamma_A$ and $\gamma_B$ are the adaptation learning rates for estimating $\Delta A$ and $\Delta B$, respectively:

$$
\Delta \hat{A}(t) = -\gamma_A \frac{\delta \hat{E}}{\delta (\Delta A)} = \gamma_A \hat{P}\tilde{X}(t)X\top(t)
$$

(8.8)

$$
\Delta \hat{B}(t) = -\gamma_B \frac{\delta \hat{E}}{\delta (\Delta B)} = \gamma_B \hat{P}\tilde{X}(t)U\top(t)
$$

(8.9)
Finally, $\Delta \hat{A}(t)$ and $\Delta \hat{B}(t)$ are obtained by respectively integrating $\Delta \dot{A}(t)$ and $\Delta \dot{B}(t)$, directly leading to real-time estimates of the actual uncertainty matrices:

$$\Delta \hat{A}(t) = \int \Delta \dot{A}(t) \, dt \quad (8.10)$$

$$\Delta \hat{B}(t) = \int \Delta \dot{B}(t) \, dt \quad (8.11)$$

In optimizing the performance of the parameter estimation component, the adaptation behavior of the system was tuned to achieve rapid convergence on the numerical values of the uncertainty matrices with minimal oscillation and overshoot, as would be desired in a real-world scenario where a failure has occurred on an aircraft. The numerical values of the learning rates ($\gamma_A$ and $\gamma_B$) and damping term ($\eta$) were thus chosen to be as high as possible while still satisfying these qualitative characteristics.

From the process outlined above, it is also apparent that the quality of the estimates is dependent upon the states and inputs of the system – no useful estimates of the parameters are obtainable if all signals are zero. The need for persistence of excitation [40] (PE) is central to the success of the automatic system identification framework, in the context of the adaptive system proposed herein.

### 8.3 Matched Uncertainty Parameter Estimation

Once again, recalling equation (6.1), the aircraft’s system may be expressed in form given by equation (8.1), where $A_0$ and $B_0$ represent the nominal aircraft dynamics, and $\Delta A$ and $\Delta B$ represent some uncertainties in the aircraft dynamics:

$$\dot{X}(t) = A_0X(t) + B_0U(t) + \Delta AX(t) + \Delta BU(t) \quad X(0) = X_0 \quad (8.12)$$
As before, the actual numerical components of the $\Delta A$ and $\Delta B$ matrices are unknown to the parameter estimation algorithm, and the objective is to derive estimates of these matrices in the form of $\hat{\Delta A}$ and $\hat{\Delta B}$. In the matched uncertainty estimation method [30], the actual uncertainty matrices $\Delta A$ and $\Delta B$ are assumed to be structured such that $\Delta A = B_0 K_1$ and $\Delta B = B_0 K_2$, where $K_1$ and $K_2$ are terms relating parameter matrices introduced in equation (8.13). Because changes to the $B$ matrix often translate to reductions in control effectivity, to ensure that the uncertain aircraft maintains controllability, an unknown parameter matrix $\Lambda$ is introduced such that $0 < \Lambda(t) = I + K_2(t) < I$. With this in mind, equation (8.12) may be expressed as:

$$\dot{X}(t) = A_0 X(t) + B_0 U(t) + B_0 \Lambda(t)[\Lambda^{-1}(t)K_1 X(t) + \Lambda^{-1}(t)K_2 U(t)]$$

(8.13)

Introducing the parameter matrices $\Theta_1(t) = \Lambda^{-1}(t)K_1(t)$ and $\Theta_2(t) = -\Lambda^{-1}(t)$, the uncertain aircraft dynamics may finally be written as:

$$\dot{X}(t) = A_0 X(t) + B_0 U(t) + B_0 \Lambda(t)[U(t) + \Theta_1 X(t) + \Theta_2 U(t)]$$

(8.14)

In the formulation presented in equation (8.14), the parameter matrices $\Lambda$, $\Theta_1$, and $\Theta_2$ comprise unknown quantities, and are estimated online. The estimated versions of these parameter matrices are denoted by $\hat{\Lambda}$, $\hat{\Theta}_1$, and $\hat{\Theta}_2$. The estimated uncertain aircraft dynamics are now introduced, and are given by equation (8.15):

$$\dot{\hat{X}}(t) = A_0 \hat{X}(t) + B_0 U(t) + B_0 \hat{\Lambda}(t)[U(t) + \hat{\Theta}_1 \hat{X}(t) + \hat{\Theta}_2 U(t)]$$

(8.15)

$$\hat{X}(0) = \hat{X}_0$$

The error between the actual and estimated uncertain aircraft dynamics is then denoted by $\delta X(t)$, where $\delta X(t) = X(t) - \hat{X}(t)$. The cost function to be minimized is then given by equation (8.16):
\[
\begin{align*}
\dot{E}(t) &= \frac{1}{2} \int_0^T \dot{X}^\top(t) \hat{Q} \dot{X}(t) \, dt + \dot{X}^\top(t) \hat{Q} \dot{X}(t) \bigg|_t dt \\
&+ \int_0^T \lambda^\top \left[ \dot{X}(t) - A_0 \dot{X}(t) - B_0 U(t) - B_0 \tilde{\Lambda}(t) \left[U(t) + \Theta_1 X(t) + \Theta_2 U(t)\right] - \eta \mathbb{I} \dot{X} \right] \, dt \\
\text{(8.16)}
\end{align*}
\]

Equation (8.16) introduces \( \lambda \), which serves as the Lagrange multiplier, \( \eta \), which serves as the adaptation damping term, and \( \hat{Q} \), which is a symmetric positive definite weighting matrix. Taking the variation of the cost function with respect to \( \tilde{\Lambda}, \Theta_1, \Theta_2 \), and \( \dot{X} \) yields equations (8.17) to (8.20):

\[
\begin{align*}
\frac{\delta \dot{E}}{\delta \tilde{\Lambda}} &= -\lambda^\top B_0 \left[U(t) + \Theta_1 x(t) + \Theta_2 U(t)\right] \\
\text{(8.17)} \\
\frac{\delta \dot{E}}{\delta \Theta_1} &= -\lambda^\top B_0 \tilde{\Lambda}(t) X(t) \\
\text{(8.18)} \\
\frac{\delta \dot{E}}{\delta \Theta_2} &= -\lambda^\top B_0 \tilde{\Lambda}(t) U(t) \\
\text{(8.19)} \\
\frac{\delta \dot{E}}{\delta \dot{X}} &= -\frac{1}{2} \dot{X}^\top(t) \hat{Q} - \frac{1}{2} \hat{Q} \dot{X}(t) - \dot{\lambda}^\top - \lambda^\top A_0 - \eta \\
\text{(8.20)}
\end{align*}
\]

The assumption is made that the Lagrange multiplier \( \lambda \) is related to the error \( \tilde{X}(t) \) through the matrix \( \tilde{P} \), such that \( \lambda^\top = \tilde{P} \tilde{X}(t) \) and \( \dot{\lambda}^\top = \tilde{P} \dot{X}(t) \). Together with equation (8.20), and assuming that \( \eta \) is numerically small, it may then be shown through the minimization of \( \delta \dot{E} / \delta \dot{X} = 0 \) that \( \tilde{P} \) must once again be the solution of the continuous-time Lyapunov Equation, as follows:

\[
\tilde{P} A_0 + A_0^\top \tilde{P} = -\hat{Q} \\
\text{(8.21)}
\]
Through first-order gradient descent optimization, the derivatives of the estimated parameter matrices are obtained according to equations (8.22) to (8.24):

$$\dot{\Lambda}(t) = -\gamma_{\Lambda} \frac{\delta \hat{E}}{\delta \Lambda} = \gamma_{\Lambda} B_0^\top \bar{p} \bar{X}(t)[U(t) + \tilde{\Theta}_1 X(t) + \tilde{\Theta}_2 U(t)]^\top \quad (8.22)$$

$$\dot{\Theta}_1(t) = -\gamma_{\theta_1} \frac{\delta \hat{E}}{\delta \Theta_1} = \gamma_{\theta_1} B_0^\top \bar{p} \bar{X}(t) \tilde{\Lambda}(t) X^\top(t) \quad (8.23)$$

$$\dot{\Theta}_2(t) = -\gamma_{\theta_2} \frac{\delta \hat{E}}{\delta \Theta_2} = \gamma_{\theta_2} B_0^\top \bar{p} \bar{X}(t) \tilde{\Lambda}(t) U^\top(t) \quad (8.24)$$

Equations (8.22) to (8.24) are integrated to obtain real-time estimates of the parameter matrices, where $\gamma_{\Lambda}$, $\gamma_{\theta_1}$, and $\gamma_{\theta_2}$ are the adaptation learning rates. Recalling the definitions of the uncertainty matrices $\Delta A$ and $\Delta B$, and knowing that $K_1(t) = \tilde{\Theta}_1(t) \Lambda(t)$ and $K_2(t) = -(\tilde{\Theta}_2^{-1}(t) + I)$, the estimates of the uncertainty matrices are finally determined according to:

$$\Delta \hat{A} = B_0 \Lambda \tilde{\Theta}_1 \quad (8.25)$$

$$\Delta \hat{B} = -B_0 (\tilde{\Theta}_2^{-1} + I) \quad (8.26)$$

As with the unmatched uncertainty parameter estimation method, it is apparent that the quality of the estimates is dependent upon the states and inputs of the system. Consequently, the need for persistence of excitation (PE) is central to the success of the automatic system identification framework.

### 8.4 Performance Comparison

In validating the adaptive parameter estimation methods, the 2nd order short period dynamics of the NASA Generic Transport Model (GTM) were used. The GTM is a transport-class
vehicle testbed with scaling, data gathering, and control characteristics, developed in part to
advance research into control upset prevention and recovery technologies for transport-class
aircraft [109].

The states in the short period dynamics are represented by the angle-of-attack and pitch
rate, such that $x(t) = [\alpha(t) \quad q(t)]^T$, and the control input is the pilot-controlled elevator, $u(t) = \delta_e(t)$. The complete system is described in equation (8.27), and models the linearized GTM about
a straight and level flight condition at an altitude of 30,000ft and cruise speed of Mach 0.8:

$$
\begin{bmatrix}
\dot{\alpha}(t) \\
\dot{q}(t)
\end{bmatrix} =
\begin{bmatrix}
-0.7018 & 0.9761 \\
-2.6923 & -0.7322
\end{bmatrix}
\begin{bmatrix}
\alpha(t) \\
q(t)
\end{bmatrix} +
\begin{bmatrix}
-0.0573 \\
-3.5352
\end{bmatrix}
\delta_e(t)
$$

(8.27)

The numerical values for the uncertainty matrices are given below. These values approximately correspond to a 59% loss of pitch control effectiveness, a 67% loss of pitch damping, and a 59% loss of pitch stiffness

$$
\Delta A = \begin{bmatrix} 0.0035 & 0.0158 \\ 0.2144 & 0.9734 \end{bmatrix}, \quad \Delta B = \begin{bmatrix} 0.0286 \\ 1.7676 \end{bmatrix}
$$

(8.28)

The responses of the nominal and uncertain aircraft models to an elevator step input of -
10.75° are shown in Figure 8.1 and Figure 8.2. Whereas this elevator deflection was sufficient to
drive the nominal aircraft to the edge of the safe control envelope, the same deflection resulted in
a lower peak angle-of-attack in the uncertain aircraft. Because of the reduced pitch damping, the
aircraft also demonstrated a tendency to oscillate more than in the nominal case.
As discussed previously, the quality of the estimates is dependent upon the states and inputs of the system and the need for persistence of excitation. In exciting the system, a square wave of amplitude $\pm 9^\circ$ and of period 5 was provided as an input. In the context of an aircraft, this would be equivalent to the pilot performing a pitch doublet maneuver. This is illustrated in Figure 8.3.

The performance of the **matched** uncertainty parameter estimation method is depicted in Figure 8.4. Values of $\gamma = 1000$ and $\eta = \sqrt{2\gamma}$ were used in the adaptive element. Uncertainties were introduced at $t = 5$ sec, with numerical values as defined in equation (8.28). We assume that the pilot is concurrently made aware of some deficiency in the aircraft, potentially requiring
significant correction and input to the controls. At this time, the pilot provides an input of the form shown in Figure 8.3.

While the adaptation algorithm successfully and rapidly identified $\Delta A$ in approximately 30 seconds, it was unable to estimate accurate values of $\Delta B$, despite there being persistent excitation. The fundamental issue behind this was the inability of the matched uncertainty formulation to handle unmatched uncertainties, of which the numerical $\Delta A$ and $\Delta B$ matrices specified in equation (8.28) are. In reality, it is unlikely that a failure in the aircraft or uncertainty in operating conditions would fall into place “neatly” enough to be represented by a matched uncertainty. There is thus a need for a adaptation algorithm capable of handling unmatched uncertainties.

Figure 8.5 illustrates the performance of the unmatched uncertainty parameter estimation method. Values of $\gamma = 1250$ and $\eta = 18$ were used in the adaptive element. These values were shown to provide rapid convergence with minimal overshoot and oscillation. As before, uncertainties were introduced at $t = 5$ sec, with numerical values as defined in equation (8.28). We yet again assume that the pilot is concurrently made aware of some deficiency in the aircraft,
potentially requiring significant correction and input to the controls, and at this same time, the pilot provides a PE input of the form shown in Figure 8.3.

As depicted in Figure 8.5, the adaptation algorithm successfully and rapidly identified both $\Delta A$ and $\Delta B$ in approximately 40 seconds. While this is several seconds longer than in the matched uncertainty case, the adaptation algorithm was able to reduce the error between $X(t)$ and $\hat{X}(t)$ to zero. This is key for accurate output from the LOC prediction models further along in the signal path.

The unmatched uncertainty method also equips the LOC prediction models to handle a more diverse and realistic range of failure scenarios. As mentioned, it is not necessarily common that a failure or uncertainty in the aircraft would structure itself “neatly” in the form of a matched uncertainty.
CHAPTER 9

PILOT ADVISORY DISPLAYS

9.1 Overview

The ultimate component of the LOC prediction architecture is the guidance mechanism through which the pilot is informed of the aircraft’s safe control envelope and its proximity to the LOC margin. This information should be given to the pilot in a quick, clear, and concise manner. Following and expanding upon the work of Krishnakumar et. al. [30], an advisory display was developed to graphically depict the safe control envelope bounded by the set of critical deflections, as computed by the prediction models. The display plays its role in the “prevention” aspect of the LOC problem.

9.1.1 Two-Dimensional Safe Control Envelope

An illustration of the display depicting the safe control envelope is presented in Figure 9.1:

![Figure 9.1. Six Degree-of-Freedom Triple-Axis 2D Pilot Advisory Display displaying Safe Control Envelope](image-url)
The 2D display shows the current pilot-commanded deflection of the control surfaces, superimposed over colored regions representing the control envelopes corresponding to the 2-second and 5-second prediction windows. These are presented relative to the maximum control surface travel, which is denoted by the outer edges of the red region. If the controls are manipulated such that the current deflection indicator meets the edge of the 5-second region, the aircraft would reach the LOC margin five seconds from the current time. Likewise, if the current deflection indicator meets the edge of the 2-second region, the aircraft would reach the LOC margin two seconds from the current time.

It is important to note that moving the controls to the edge of the control envelope does not place the aircraft in LOC right now. Instead, doing so would cause the aircraft to enter LOC two or five seconds later. This is the basis for the “early warning” function of the prediction architecture – the idea is that the pilot would have an additional two or five seconds from the current time to bring the aircraft back into the safe portion of the control envelope, thereby also keeping the aircraft within the safe portion of the flight envelope. In other words, always keeping the indicator within the “safe regions” would also mean always keeping the aircraft from exceeding its LOC margins.

Figure 9.2. Single- and Dual-Axis 2D Pilot Advisory Displays
For each of the head-up and head-down formats, single-, dual, and triple-axis versions exist. The single-axis version is used for modes with only one control effector (such as the short period, where the critical elevator deflection is displayed), the dual-axis version is used for modes with two control effectors (such as the roll or dutch roll, where the critical aileron and rudder are displayed), and the triple-axis version is used for the fully-coupled higher-order aircraft (where the critical elevator, aileron, and rudder are *interdependent* and interacting simultaneously).

The dual- and triple-axis versions may also be used to simultaneously display the prediction model’s output for *decoupled* independent modes (such as the short period on the vertical axis, roll on the horizontal axis, and dutch roll on the third axis).

### 9.1.2 Three-Dimensional Safe Control Space

An illustration of the display depicting the safe control space is presented in Figure 9.3:

![Figure 9.3. Four Degree-of-Freedom Dual-Axis 3D Pilot Advisory Display displaying Safe Control Space](image)

The 3D display shows the control-to-go (or amount of control authority remaining) over the next several seconds by presenting pilot with the immediate control space and the latent control space. The “immediate control space” provides the pilot with information about the amount of control authority remaining at the *current moment*, while the “latent control space” provides
information about the critical (optimal) trajectories over the *next several seconds* that should not be exceeded, in order for the aircraft to stay within the safe flight envelope. Together, the immediate and latent components form a three-dimensional volume that becomes the “safe region” of the control input over the coming several seconds.

Eight optimally-derived control trajectories form the bounds of the safe control space. These trajectories are comprised of the calculated outputs from the Trajectory-To-Go prediction models, and are named after cardinal directions for ease of understanding and visualization. For example, the North Trajectory targets an angle-of-attack limit of $+15^\circ$, while the NorthEast Trajectory targets a simultaneous angle-of-attack limit of $+15^\circ$ and bank angle limit of $+45^\circ$. Table 9.1 describes these trajectories and their corresponding state limit targets:

**TABLE 9.1**

SAFE CONTROL SPACE TRAJECTORY TARGETS

<table>
<thead>
<tr>
<th>Trajectory</th>
<th>Direction</th>
<th>Angle-of-Attack Limit</th>
<th>Bank Angle Limit</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>North</td>
<td>$+15^\circ$</td>
<td>$0^\circ$</td>
</tr>
<tr>
<td>2</td>
<td>NorthEast</td>
<td>$+15^\circ$</td>
<td>$+45^\circ$</td>
</tr>
<tr>
<td>3</td>
<td>East</td>
<td>$0^\circ$</td>
<td>$+45^\circ$</td>
</tr>
<tr>
<td>4</td>
<td>SouthEast</td>
<td>$-25^\circ$</td>
<td>$+45^\circ$</td>
</tr>
<tr>
<td>5</td>
<td>South</td>
<td>$-25^\circ$</td>
<td>$0^\circ$</td>
</tr>
<tr>
<td>6</td>
<td>SouthWest</td>
<td>$-25^\circ$</td>
<td>$-45^\circ$</td>
</tr>
<tr>
<td>7</td>
<td>West</td>
<td>$0^\circ$</td>
<td>$-45^\circ$</td>
</tr>
<tr>
<td>8</td>
<td>NorthWest</td>
<td>$+15^\circ$</td>
<td>$-45^\circ$</td>
</tr>
</tbody>
</table>

The amount of control-to-go is represented by how much “green” is available in each of the four quadrants shown in Figure 9.3. This concept is further illustrated in Figure 9.4, wherein the aircraft is simultaneously at the positive angle-of-attack and positive bank angle limits. In such
a scenario, the point marked “2” (in Figure 9.3) will meet the control position indicator, which is a crosshair fixed to the center of the display. This indicates that there is no more control authority remaining to go “up and right”. Instead, the pilot should maneuver “down and left” to bring the aircraft back to the safe portion of the flight envelope.

Figure 9.4. Safe Control Space Depicting Aircraft at the Edge of the Positive Angle-of-Attack and Positive Bank Angle Limits

Notice also that the North, NorthEast, and East trajectories have converged. This is likewise the case for the North and NorthWest trajectories (since the positive angle-of-attack limit is saturated), as well as the South and SouthEast trajectories (since the positive bank angle limit is saturated).

With the safe control space, the pilot is further presented with a receding horizon time aspect of the critical input. This “depth” aspect, representing time, is given by the latent control space and assists the pilot with visually understanding how much control-to-go is available in the moments to come, and in which direction the control-to-go is available. This information could assist the pilot in making more informed control input decisions in keeping the aircraft in a safe flight configuration and steering it away from the edge of the flight envelope.
The display also features a “motion” aspect, depicting the appearance of “flying through” a control space in the sky. The wireframe of the three-dimensional control space continually moves towards the pilot, providing a sense of forward motion through a virtual tunnel (similar to a “highway-in-the-sky”). Flying out of this tunnel equates to an excursion of the safe control envelope, and hence an excursion of the safe flight envelope.

9.2 Implementation on Augmented-Reality Head-Up Displays

Head-up AR technologies have the potential to provide a pilot with important information in a quicker manner than conventional head-down displays, since the information is superimposed directly over the pilot’s field of view. The delay between processing information displayed on a conventional head-down display and subsequent interaction with the outside environment could be greatly reduced through the use of AR technologies.

Figure 9.5 and Figure 9.6 provide visual renderings of the 2D safe control envelope in the head-up format, illustrating the pilot’s perspective while wearing the Microsoft HoloLens. The projection prism is clear and the environment behind the prism is visible. The advisory display is presented to the pilot in a simple and concise manner, without obscuring the pilot’s outside view.
Figure 9.7 provides a visual rendering of the 3D safe control space in the head-up format, conceptually illustrating a pilot’s perspective while wearing a head-up wearable device.

Figure 9.7. 3D LOC Advisory Display on Head-Up Wearable Device
(conceptual visual render of pilot’s perspective)

Figure 9.8 and Figure 9.9 respectively depict the Google Glass and Microsoft HoloLens headsets. For purposes of simulated flight testing, the 2D pilot advisory displays (safe control envelopes) illustrated in Figure 9.1 were ported to these AR headsets, in addition to conventional head-down displays. These simulated flight testing efforts are detailed in Chapter 11. Work on porting the 3D pilot advisory displays (safe control spaces) illustrated in Figure 9.3 to the AR headsets is planned as a future endeavor.

Figure 9.8. Google Glass augmented-reality wearable glasses

Figure 9.9. Microsoft HoloLens augmented-reality headset
The HoloLens is an augmented-reality head-mounted display, giving the wearer the ability to see holographic objects interacting with their physical environment. In the context of the experiment, the device gives pilots the ability to see the pilot advisory display superimposed over the outside environment, without having to periodically look back and forth at the instrument panel. Figure 9.10 shows the inside view of the HoloLens. The projection prism may be seen, along with the surface of the table behind it.

![Figure 9.10. Inside View of Microsoft HoloLens](image)

The pilot advisory display was programmed such that the display constantly stayed in view of the pilot, even as the pilot moved his/her head to look around the environment. This is in contrast to the typical AR usage of the HoloLens, where an application is opened and then virtually “projected” against a physical surface in the wearer’s environment (such as a wall or table).

### 9.3 Implementation on Conventional Head-Down Instrumentation

To provide a standard for measuring the hypothetical potential benefits of the head-up formats, the pilot advisory displays were also implemented on conventional head-down instrumentation to facilitate obtaining a “control measurement”. Figure 9.11 and Figure 9.12 respectively depict the single- and triple-axis displays as conventional displays.
The head-down pilot advisory display is presented to the pilot using a Google Nexus 7 (2013) tablet. This device is depicted in Figure 9.13:

The device features a 7.02” screen with a 16:10 aspect ratio. Its specifications include a 323ppi pixel density with a 178° viewing angle, which allows the pilot to view the display clearly from various viewpoints. The tablet is mounted to the simulator’s instrument panel console and accompanies the aircraft’s existing instrumentation.

9.4 Implementation as an Aural Alert

In addition to testing a “visuals-only” format of the safe control envelope, an “aural-only” format was also developed. In this representation, no visual alerts are given to the pilot. Instead,
auditory warnings are sounded when the pilot reaches the edge of the safe control envelope, as in Figure 9.14. The warnings comprise the phrases “pitch up margin”, “pitch down margin”, “left roll margin”, and “right roll margin”, which each correspond to the appropriate outermost edges of the envelope.

Figure 9.14. Auditory alerts warning of Safe Control Envelope excursion

With the aural format, the pilot only receives an auditory warning at the point when the safe control envelope has been exceeded. With the visual format, the pilot is also given information about how close the aircraft is to the edge of the envelope. Tying in with the human factors considerations, one of the goals of the simulated flight testing effort is to determine whether the pilots respond better to warnings given via the visual or aural format. In addition, the pilots’ performance in keeping the aircraft within the bounds of the safe envelopes through use of the aural-only format are compared with that of the visuals-only format to determine which yields better results.
9.5 Potential Benefits & Human Factors Considerations

In addition to validating the effectiveness of the LOC prediction architecture, a secondary goal of the testing effort is to assess the effectiveness of head-up AR technologies in the cockpit. AR technologies have the potential to reduce a pilot’s cognitive workload by providing the pilot with important information in a quicker manner than conventional head-down displays, since the information is superimposed directly over the pilot’s field of view. The delay between processing information displayed on a conventional head-down display and subsequent interaction with the outside environment could be greatly reduced through the use of AR technologies.

To date, the use of head-up displays in the cockpit has been primarily restricted to military aircraft. Only recently have such displays been used on production commercial aircraft, such as the Boeing 787 and Boeing Next-Generation 737. Even so, the display hardware is typically built natively into the aircraft itself. A potential benefit of external units such as Google Glass or Microsoft HoloLens is that, with the appropriate hardware-software interface, portable head-up displays may be introduced into cockpits that do not natively feature them.

In their recent research [110, 111] into AR technologies for aerospace, researchers at the General Aviation Flight Lab (GAFL) at Wichita State University (WSU) developed a Google Glass-based three-dimensional (3D) visual cue designed to rapidly guide a pilot’s gaze towards air traffic intruding into the ownship’s airspace. The visual cue utilized a dynamically updating three-dimensional arrow superimposed over the environment to continually “point” the pilot towards the location of nearby aircraft. This was evaluated in simulated flight testing with significant reductions measured in the time taken to locate nearby aircraft when utilizing the visual cue, as compared to utilizing a conventional head-down display.

While research into AR for aerospace use has been limited, the opposite is true for the automobile industry, where using AR as part of driver warning systems has become commonplace.
Such systems attempt to give drivers enough time to react to hazards on the road, using both visual and auditory stimuli. These visual and auditory modalities are commonly used in alerting systems [112], and the pilot advisory display utilizes similar modalities to warn pilots of their proximity to edge of the safe control and flight envelopes. Studies have found that participants typically prefer to receive information through visual rather than auditory stimuli [113] within the context of AR systems, and that they react faster when additionally presented with the direction at which the hazards are located [114].
CHAPTER 10
CONCEPT VALIDATION

10.1 Overview

In this chapter, results from various desktop simulation runs are presented to demonstrate the performance of the Deflection-To-Go and Trajectory-To-Go prediction models in estimating the safe control envelope. The light business jet model used in concept validation is first presented, followed by a definition of the safe flight envelope, and then by the results of the D2G and T2G prediction models.

10.2 The Light Business Jet

The subject aircraft used in validation of the prediction models was a high-speed twin-engine light business jet with capacity to seat approximately six to eight passengers. The light business jet was derived from data given in [106]. An illustration of the light business jet is given in Figure 10.1:

Figure 10.1. Illustration of Light Business Jet used in Concept Validation
The basic characteristics of the light business jet are presented in Table 10.1.

**TABLE 10.1**

**CHARACTERISTICS OF LIGHT BUSINESS JET**

<table>
<thead>
<tr>
<th>Aircraft Property</th>
<th>Value</th>
</tr>
</thead>
<tbody>
<tr>
<td>Weight</td>
<td>13,000 lb</td>
</tr>
<tr>
<td>Wing Area</td>
<td>232 ft²</td>
</tr>
<tr>
<td>Wingspan</td>
<td>34 ft</td>
</tr>
<tr>
<td>Mean Aerodynamic Chord</td>
<td>7 ft</td>
</tr>
</tbody>
</table>

While the uncoupled longitudinal and lateral/directional stability derivatives were derived from data given in [106], the coupled stability derivatives relating the longitudinal and lateral/directional modes were obtained through analysis of a digital model of the light business jet in Athena Vortex Lattice (AVL), a program used for aerodynamic and flight-dynamic analysis of rigid aircraft.

![Figure 10.2. Digital Model of Light Business Jet in Athena Vortex Lattice](image-url)
The coupled and uncoupled stability derivatives were subsequently hybridized into a singular cross-coupled aircraft dynamic system. The model was linearized about a cruise condition configuration, with altitude of 40,000 ft and velocity of 400 kts. It was found that linearizing this particular aircraft model about an approach configuration led to unstable lateral/directional dynamics. While using approach dynamics would have been ideal, accounting for unstable dynamics was beyond the scope and intention of validating the prediction models. Consequently, the cruise condition dynamics were chosen for analysis.

The resulting 8th order aircraft is described by the state space system given in equation (10.1):

\[
\begin{bmatrix}
\dot{u}(t) \\
\dot{\alpha}(t) \\
\dot{\theta}(t) \\
\dot{\phi}(t) \\
\dot{\beta}(t) \\
\dot{\phi}(t) \\
\dot{p}(t) \\
\dot{r}(t)
\end{bmatrix} = 
\begin{bmatrix}
-0.007 & 9.09 & -32.2 & 0 & 0 & 0 & 0 & 0 \\
-0.0002 & -0.668 & 0 & 0.996 & 0.049 & 0 & -0.0001 & 0 \\
0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\
0.001 & -7.17 & 0 & -1.34 & 1.16 & 0 & -0.002 & 0.012 \\
0 & 0.003 & 0 & 0.0002 & -0.003 & 0.048 & 0 & -0.998 \\
0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\
0 & -0.056 & 0 & -0.013 & -4.05 & 0 & -0.437 & 0.152 \\
0 & 0.245 & 0 & -0.009 & 2.75 & 0 & -0.008 & -0.111
\end{bmatrix}
\begin{bmatrix}
u(t) \\
a(t) \\
\alpha(t) \\
\theta(t) \\
\phi(t) \\
\beta(t) \\
\phi(t) \\
\psi(t)
\end{bmatrix} + 
\begin{bmatrix}
0 & 0 & 0 & 0 \\
-0.063 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 \\
-17.6 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 \\
6.76 & 0.577 & 0 & 0
\end{bmatrix}
\begin{bmatrix}
\delta e(t) \\
\delta a(t) \\
\delta r(t) \\
\delta t(t)
\end{bmatrix} 
\]  

(10.1)

For the purposes of validating the prediction architecture, the coupled stability derivatives (off-block-diagonal terms) were intentionally designed to exaggerate the dynamic coupling between the longitudinal and lateral/directional modes. Since the performance of the prediction model in the presence of uncertainties was already demonstrated in prior work [34, 35], the validation results presented here are based on the nominal 8th order dynamics.

### 10.3 Safe Flight Envelope and Loss-of-Control Margins

The safe flight envelope was defined by upper and lower bounds placed on the states angle-of-attack (\(\alpha\)), bank angle (\(\phi\)), and sideslip angle (\(\beta\)). These are summarized in Table 10.2, and are delta values from the trim condition. While the architecture permits imposing limits on any state of the system, actuated by any input of the system, these three state/input combinations were
chosen as they represent critical aerodynamic and angular parameters relating to flight. Note also that the imposed state limits need not be constant values, but could be time-varying and dynamically-changing quantities based on specific flight conditions.

### TABLE 10.2
SAFE FLIGHT ENVELOPE AND LOC MARGINS

<table>
<thead>
<tr>
<th>State</th>
<th>Control Effector</th>
<th>Negative Limit</th>
<th>Positive Limit</th>
<th>Unit</th>
</tr>
</thead>
<tbody>
<tr>
<td>Angle-of-Attack ($\alpha$)</td>
<td>Elevator ($\delta_e$)</td>
<td>$-25$</td>
<td>$+25$</td>
<td>deg</td>
</tr>
<tr>
<td>Bank Angle ($\phi$)</td>
<td>Aileron ($\delta_a$)</td>
<td>$-45$</td>
<td>$+45$</td>
<td>deg</td>
</tr>
<tr>
<td>Sideslip Angle ($\beta$)</td>
<td>Rudder ($\delta_r$)</td>
<td>$-5$</td>
<td>$+5$</td>
<td>deg</td>
</tr>
</tbody>
</table>

### 10.4 Deflection-To-Go Models

In validating the effectiveness and accuracy of the deflection-to-go prediction models, various simulation runs were performed in MATLAB/Simulink with typical “pilot-commanded” critical inputs given to the aircraft. To demonstrate that the predicted margin is valid, the “pilot input” was chosen to deliberately match the critical control deflection determined by the prediction model at that particular time $t$. If the prediction model was functioning as intended, the input would be expected to drive the aircraft to the edge of the safe flight envelope in the $\hat{T}$ seconds following the current time $t$. 
10.4.1 Modal Superposition

10.4.1.1 Validation Scenario

Figure 10.3 to Figure 10.8 demonstrate the accuracy of the predicted critical deflections of the deflection-to-go LOC prediction model in the presence of coupled dynamics and under an array of initial conditions on the states and inputs. Figure 10.3 illustrates the critical elevator required to reach the angle-of-attack limits, Figure 10.5 illustrates the critical rudder required to reach the sideslip angle limits, and Figure 10.7 illustrates the critical aileron required to reach the bank angle limits.

First Input – Critical Elevator Targeting Angle-of-Attack Limit: At $t = 1$ sec, with zero initial conditions on the states and inputs, the predicted critical elevator deflection required to reach the positive angle-of-attack limit was approximately $-5.0^\circ$. This is depicted by the dashed orange line in Figure 10.3. Applying this exact amount of elevator as a step input at $t = 1$ sec then caused the angle-of-attack to increase from an initial condition of $\alpha = 0^\circ$ at $t = 1$ sec to $\alpha = \alpha_{limit} = 15^\circ$ within the prediction time window $T = 2$ sec, as shown in Figure 10.4.

Following the first critical input to the elevator, the aircraft remained at a positive angle-of-attack of $\alpha = 10^\circ$. Additionally, the inherent cross-coupled dynamics had caused changes to the sideslip and bank angle states as a result of this first critical elevator input, as shown in Figure 10.6 and Figure 10.8, introducing non-zero initial conditions on the aircraft states. Note also the changes in the predicted critical rudder and aileron as a direct result of applying the critical elevator and the resulting state change in angle-of-attack, as shown in Figure 10.5 and Figure 10.7.

Second Input – Critical Rudder Targeting Sideslip Limit: At $t = 3$ sec, with non-zero initial conditions on the states and elevator input, the predicted critical rudder deflection required to reach the negative sideslip angle limit was approximately $-4.4^\circ$. This is depicted by the dashed
orange line in Figure 10.5. Applying this exact amount of rudder as a step input at $t = 3$ sec then caused the sideslip angle to increase from an initial condition of $\beta = -2.1^\circ$ at $t = 3$ sec to $\beta = \beta_{\text{limit}} = -5^\circ$ within the prediction time window $\hat{T} = 2$ sec, as shown in Figure 10.6. As before, the application of the critical rudder input had influenced and induced changes to the other states and to the predicted critical elevator and aileron.

**Third Input – Critical Aileron Targeting Bank Angle Limit:** At $t = 5$ sec, with non-zero initial conditions on the states and elevator and rudder inputs, the predicted critical aileron deflection required to reach the positive bank angle limit at the end of the prediction time window $\hat{T} = 5$ sec was approximately $0.9^\circ$. This is depicted by the dashed orange line in Figure 10.7. Applying this exact amount of aileron as a step input at $t = 5$ sec then caused the bank angle to increase from an initial condition of $\phi = -15^\circ$ at $t = 5$ sec to $\phi = \phi_{\text{limit}} = 45^\circ$ at the end of the prediction time window $\hat{T} = 5$ sec, as shown in Figure 10.8.
Figure 10.3. Validation Scenario: Critical vs Applied Elevator

Figure 10.4. Validation Scenario: Angle-of-Attack Response

Figure 10.5. Validation Scenario: Critical vs Applied Rudder

Figure 10.6. Validation Scenario: Sideslip Angle Response

Figure 10.7. Validation Scenario: Critical vs Applied Aileron

Figure 10.8. Validation Scenario: Bank Angle Response
10.4.1.2  **Practical Scenario 1: Slow Flight with Steep Turn**

A flight condition combining slow flight with a steep turn might be encountered by pilots in actual flight when attempting an approach to land. An aircraft might enter such a condition when making turns on the downwind-base-final legs. Aircraft typically operate closer to the stall speed region during these maneuvers, compounding the dangers of the situation. Figure 10.9 to Figure 10.14 demonstrate the ability of the LOC prediction model in assisting pilots with maintaining safe flight in such a scenario. Figure 10.15 depicts snapshots of the pilot advisory display at key times, illustrating a pilot using the display to “track” the edge of the control envelope.

**First Input – Critical Elevator Targeting Angle-of-Attack Limit:** At $t = 1$ sec, the pilot applies a critical elevator input of approximately $-5.0^\circ$, causing the angle-of-attack to rise and reach $\alpha = 15^\circ$ (Figure 10.9). Following a brief pause approximately equivalent to the length of $\hat{t}_p$, the pilot then tracks the predicted critical elevator, based on the changing bounds of the control envelope. At this point, the control envelope indicates no further control authority is available in the nose-up direction (middle row of Figure 10.15), and the aircraft continually flies on the edge of the angle-of-attack envelope without excursion of the state limit (Figure 10.10).

**Second Input – Critical Aileron Targeting Bank Angle Limit:** At $t = 3$ sec, with non-zero initial conditions on the states and inputs, the pilot applies a critical aileron input of approximately $7.5^\circ$, causing the bank angle to rise. Momentarily after, the pilot then tracks the predicted critical aileron, based on the changing bounds of the control envelope (Figure 10.11). The aircraft rolls towards the positive bank angle limit, reaching the limit approximately 5 sec later at $t \approx 9$ sec. At this point, the control envelope indicates no further control authority is available in the right roll direction (bottom row of Figure 10.15), and the aircraft continually flies on right the edge of the bank angle envelope without excursion of the state limit (Figure 10.12).
Figure 10.9. Slow Flight-Steep Turn: Critical vs Applied Elevator

Figure 10.10. Slow Flight-Steep Turn: Angle-of-Attack Response

Figure 10.11. Slow Flight-Steep Turn: Critical vs Applied Aileron

Figure 10.12. Slow Flight-Steep Turn: Bank Angle Response

Figure 10.13. Slow Flight-Steep Turn: Critical vs Applied Rudder

Figure 10.14. Slow Flight-Steep Turn: Sideslip Angle Response
Figure 10.15. Slow Flight-Steep Turn: Snapshots of Control Envelope as Pilot Advisory Display
10.4.1.3 Practical Scenario 2: Forward Slip

The forward slip is typically executed by pilots when an aircraft is high and fast on final approach. This maneuver combines a sideslip with a roll in the opposite direction, and creates sufficient drag to reduce airspeed. The dangers of this situation arise from the possibility of combined high angles-of-attack, high sideslip, and large roll angles occurring simultaneously. Effectively, the aircraft is being operated at the “triple-edge” of the safe flight envelope. Cross-coupling between the longitudinal and lateral/directional dynamics becomes more pronounced, increasing the potential for unexpected behavior and nonlinear tendencies from the aircraft. Figure 10.16 to Figure 10.21 demonstrate the ability of the LOC prediction model in assisting pilots with maintaining safe flight in such a scenario. Figure 10.22 depicts snapshots of the pilot advisory display at key times, illustrating a pilot using the display to “track” the edge of the control envelope, thereby also tracking the edge of the safe flight envelope.

First Input – Critical Elevator Targeting Angle-of-Attack Limit: At \( t = 1 \) sec, the pilot applies a critical elevator input of approximately \(-5.0^\circ\), causing the angle-of-attack to rise and reach \( \alpha = 15^\circ \) (Figure 10.16). Following a brief pause approximately equivalent to the length of \( \hat{t}_p \), the pilot then tracks the predicted critical elevator, based on the changing bounds of the control envelope. At this point, the control envelope indicates no further control authority is available in the nose-up direction (top row of Figure 10.22), and the aircraft continually flies on the upper edge of the angle-of-attack envelope (Figure 10.17).

Second Input – Critical Rudder Targeting Sideslip Limit: At \( t = 3 \) sec, with non-zero initial conditions on the states and inputs, the pilot applies a critical rudder input of approximately \(-3.7^\circ\), causing the sideslip angle to rise and reach \( \beta = -5^\circ \). Following a brief pause approximately equivalent to the length of \( \hat{t}_p \), at approximately \( t = 5 \) sec, the pilot then tracks the predicted critical
rudder, based on the changing bounds of the control envelope (Figure 10.18). At this point, the control envelope indicates no further control authority is available in the left yaw direction (middle row of Figure 10.22).

**Third Input – Critical Aileron Targeting Bank Angle Limit:** At the same time, the pilot applies a critical aileron input of approximately 7.1°, tracking the predicted critical aileron momentarily thereafter, based on the changing bounds of the control envelope (Figure 10.20). The control envelope indicates no further control authority is available in the right roll direction (bottom row of Figure 10.22). The combined application of both the critical rudder and aileron causes the sideslip angle and bank angle to rise, and the aircraft is made to continually fly on the edge of the sideslip and bank angle envelopes without excursion of the limits placed on the states. (Figure 10.19 and Figure 10.21).

### 10.4.1.4 Key Takeaways from Simulation Results

The premise of these simulation runs is that the pilot is using information presented on the pilot advisory display when applying these critical inputs. The key takeaway from these scenarios is that, simply by using the information presented on the safe control envelope, the pilot is able to continually keep the aircraft within the safe flight envelope and is able to maneuver the aircraft on the edge of the safe flight envelope. These findings were further validated and observed through simulated flight testing, and this is detailed in Chapter 11.
Figure 10.16. Forward Slip: Critical vs Applied Elevator

Figure 10.17. Forward Slip: Angle-of-Attack Response

Figure 10.18. Forward Slip: Critical vs Applied Rudder

Figure 10.19. Forward Slip: Sideslip Angle Response

Figure 10.20. Forward Slip: Critical vs Applied Aileron

Figure 10.21. Forward Slip: Bank Angle Response
Figure 10.22. Forward Slip: Snapshots of Control Envelope as Pilot Advisory Display
10.4.2 Modal Transformation

The subject aircraft used in validation of the modal transformation prediction model was a 4th order variant of the 8th order light business jet presented in Section 10.2. As before, in validating the effectiveness of the prediction models, various simulation runs were performed in MATLAB/Simulink with typical “pilot-commanded” inputs given to the aircraft model. In the modal transformation method, the first input was chosen as a critical step targeted to reach the positive LOC margin starting with zero initial conditions, the second input was a critical ramp targeted to reach the negative LOC margin with the aircraft having non-zero initial conditions on the states and inputs, and the third input was a critical parabola targeted to reach the positive LOC margin again with non-zero initial conditions. This set of results also demonstrates the ability of the deflection-to-go prediction architecture to calculate critical inputs of higher order than the step input.

10.4.2.1 4th Order Longitudinal Mode

The aircraft model for the 4th order longitudinal mode is given in equation (10.2):

\[
\begin{bmatrix}
\dot{u}(t) \\
\dot{\alpha}(t) \\
\dot{\theta}(t) \\
\dot{q}(t)
\end{bmatrix} =
\begin{bmatrix}
-0.007 & 9.098 & -32.2 & 0 \\
0 & -0.667 & 0 & 0.994 \\
0 & 0 & 0 & 1 \\
0.001 & -7.415 & 0 & -0.978
\end{bmatrix}
\begin{bmatrix}
u(t) \\
\alpha(t) \\
\theta(t) \\
q(t)
\end{bmatrix} +
\begin{bmatrix}
0 \\
-0.063 \\
0 \\
-17.671
\end{bmatrix}
\begin{bmatrix}
\delta_e(t) \\
\delta_t(t)
\end{bmatrix} \quad (10.2)
\]

The 4th order aircraft dynamics were then decomposed into the 2nd order block diagonal form through the transformation matrix \( T_D \). For the light business jet used in concept validation, the numerical transformation matrix for the longitudinal mode was given by equation (10.3):

\[
T_D =
\begin{bmatrix}
0.932 & 0 & -1 & 0 \\
0.069 & 0.096 & 0 & 0 \\
0.043 & 0.106 & 0 & -0.003 \\
0.250 & -0.204 & 0 & 0
\end{bmatrix} \quad (10.3)
\]
In the transformed $z$-domain, the original state space system given in equation (10.2) then became:

\[
\begin{bmatrix}
\dot{z}_1 \\
\dot{z}_2 \\
\dot{z}_3 \\
\dot{z}_4 
\end{bmatrix} =
\begin{bmatrix}
-0.822 & -2.709 & 0 & 0 \\
2.709 & -0.822 & 0 & 0 \\
0 & 0 & -0.004 & -0.093 \\
0 & 0 & 0.093 & -0.004
\end{bmatrix}
\begin{bmatrix}
z_1 \\
z_2 \\
z_3 \\
z_4 
\end{bmatrix} +
\begin{bmatrix}
-44.857 & -0.017 \\
31.753 & -0.002 \\
-41.814 & -14.629 \\
487.84 & -0.680
\end{bmatrix}
\begin{bmatrix}
\delta_e \\
\delta_t 
\end{bmatrix}
\]  

(10.4)

Once decomposed, it was necessary to establish the mode dominance of each block diagonal to identify which represented the short period mode and which represented the phugoid mode. The inverse linear transformation is given by $z = T_D^{-1}x$. By evaluating this expression, one obtained the transformed states as a function of the original states, as shown in equation (10.5):

\[
\begin{align*}
  z_1 &= -0.001u + 5.36\alpha - 0.003\theta + 2.52q \\
  z_2 &= -0.0001u + 6.59\alpha - 0.013\theta - 1.82q \\
  z_3 &= -u + 4.99\alpha - 0.003\theta + 2.35q \\
  z_4 &= -0.047u + 320\alpha - 346\theta - 28.8q
\end{align*}
\]  

(10.5)

At this point, it is apparent that the states $z_1$ and $z_2$ correspond to the short period mode, since the dominant terms are $\alpha$ and $q$ (as indicated by the respective underbars). The states $z_3$ and $z_4$ thus correspond to the phugoid mode. However, because of the coupling that inherently exists between the four states of the longitudinal dynamics, some dominance of $\alpha$ in $z_4$ was also observed. This led to the notion that placing a limit on flight path angle $\gamma$ was more feasible than placing a limit on pitch angle $\theta$. To demonstrate the validity of this notion, $\theta = \gamma + \alpha$ was substituted into equation (10.5) and the state $u$ was dimensionalized to give the following:
Forward velocity then showed a clear dominance in the transformed state $z_3$ and flight path angle then showed a clear dominance in the transformed state $z_4$. This indicated that the second block diagonal approximated the phugoid mode.

To obtain the state limits in the $z$-domain, a similar inverse linear transformation was applied to the limits placed on the original states (in Table 10.2) through $z_{limit} = T_L^{-1}x_{limit}$, where $T_L$ was derived from $T_D$ by the substitution of the state $\theta$ with $\gamma + \alpha$ (with reasons for doing so as explained prior). The resulting transformed state limits in the $z$-domain, along with their approximate equivalents in the $x$-domain, are shown in Table 10.3:

TABLE 10.3

<table>
<thead>
<tr>
<th>Transformed State ($z$)</th>
<th>Equivalent State ($x$)</th>
<th>Negative Limit ($z$)</th>
<th>Positive Limit ($z$)</th>
</tr>
</thead>
<tbody>
<tr>
<td>$z_1$</td>
<td>$\alpha$</td>
<td>$-2.73$</td>
<td>$1.80$</td>
</tr>
<tr>
<td>$z_2$</td>
<td>$q$</td>
<td>$-2.55$</td>
<td>$1.40$</td>
</tr>
<tr>
<td>$z_3$</td>
<td>$u$</td>
<td>$37.45$</td>
<td>$-38.33$</td>
</tr>
<tr>
<td>$z_4$</td>
<td>$\gamma$</td>
<td>$48.35$</td>
<td>$-43.85$</td>
</tr>
</tbody>
</table>

In the case of the modal transformation method, these transformed limits in the $z$-domain were imposed on the LOC prediction model, and the model in the $z$-domain was directed to target
these transformed limits. This is in contrast to the direct use of the un-transformed original state limits in the x-domain (Table 10.2) for the modal superposition method.

10.4.2.1.1 Short Period Diagonal

Figure 10.23 to Figure 10.26 demonstrate the performance of the modal transformation prediction model for the decomposed short period dynamics. The state in the x-domain upon which the limit was placed is the angle-of-attack, and the equivalent state in the z-domain was $z_1$. The prediction model was thus set to target $z_1$ as the state limit.

**First Input Set – Elevator Step Targeting Positive $z_1$/Angle-of-Attack Limit:** At $t = 5$ sec, with zero initial conditions, the predicted critical elevator deflection required to reach the positive transformed $z_1$ margin (angle-of-attack limit) was approximately $-4.1^\circ$, as depicted by the dashed green line in Figure 10.23. Applying this exact amount of elevator as a step input at $t = 5$ sec then caused the transformed state to change up from an initial condition of $z_1 = 0$ at $t = 5$ sec to $z_1 \approx z_{1\text{limit}} = 1.80$ within the prediction time window $\hat{T} = 2$ sec, as shown in the $z_1$ response in Figure 10.25. This corresponded to approximately $\alpha \approx 12.5^\circ < \alpha_{\text{limit}}$, as shown in Figure 10.24, presenting a conservative limit on angle-of-attack. Following the first critical input, a constant input of $\delta_e = -1.25^\circ$ was applied to bring the aircraft to a trimmed nose-up state of $\alpha = 3^\circ$, placing the aircraft in an approximate steady-state non-zero initial condition.

**Second Input Set – Elevator Rate Targeting Negative $z_1$/Angle-of-Attack Limit:** At $t = 15$ sec, with non-zero initial conditions on the states and inputs, the predicted critical elevator rate required to reach the negative transformed $z_1$ margin was approximately $2.2^\circ$/sec. Applying this
exact ramp input at $t = 15$ sec caused the transformed state to change from an initial condition of $z_1 = 0.33$ at $t = 15$ sec to $z_1 \approx z_{1\text{limit}} = -2.73$ at the end of the prediction time window $\hat{T} = 5$ sec. This corresponded to approximately $\alpha \approx -21^\circ < \alpha_{\text{limit}}$, as seen in Figure 10.24, again presenting a conservative limit. Following the second critical input, a constant input of $\delta_e = 1.25^\circ$ was applied to bring the aircraft to a trimmed nose-down state of $\alpha = -2.5^\circ$, placing the aircraft in an approximate steady-state non-zero initial condition.

**Third Input Set – Elevator Acceleration Targeting Positive $z_1$/Angle-of-Attack Limit:** At $t = 30$ sec, again with non-zero initial conditions on the states and inputs, the predicted critical elevator acceleration required to reach the positive transformed $z_1$ margin was approximately $-0.6^\circ/\text{sec}^2$. Applying this exact parabolic input at $t = 30$ sec caused the transformed state to change from an initial condition of $z_1 = -0.33$ at $t = 30$ sec to $z_1 \approx z_{1\text{limit}} = 1.8$ at the end of the prediction time window $\hat{T} = 5$ sec. This corresponded to approximately $\alpha \approx 12.5^\circ < \alpha_{\text{limit}}$, as shown in Figure 10.24, once again presenting a conservative limit.
Figure 10.23. 4th Order Longitudinal (Short Period Diagonal): Critical vs Applied Elevator

Figure 10.24. 4th Order Longitudinal (Short Period Diagonal): Angle-of-Attack Response

Figure 10.25. 4th Order Longitudinal (Short Period Diagonal): Transformed State $z_1$

Figure 10.26. 4th Order Longitudinal (Short Period Diagonal): Transformed State $z_2$
10.4.2.1.2 Phugoid Diagonal

Figure 10.27 to Figure 10.29 demonstrate the performance of the prediction model for the decomposed phugoid dynamics while targeting velocity as the state limit. The equivalent state in the $z$-domain was $z_3$, and the prediction model was thus set to target $z_3$ as the state limit.

At $t = 5$ sec, with zero initial conditions, the predicted critical elevator deflection required to reach the positive transformed $z_3$ margin (velocity limit) was approximately $0.25^\circ$, as and depicted by the dashed green line in Figure 10.27. Applying this exact amount of elevator as a step input at $t = 5$ sec then caused the transformed state to change from an initial condition of $z_3 = 0$ at $t = 5$ sec to $z_3 \approx z_{3\text{limit}} = -38.3$ within an extended prediction time window corresponding to the peak time of the phugoid mode, as shown by the $z_3$ response in Figure 10.29. This corresponded to approximately $V \approx 38 \text{ ft/sec} \approx V_{\text{limit}}$, as shown in Figure 10.28.

Next, Figure 10.30 to Figure 10.32 demonstrate the performance of the prediction model for the decomposed phugoid dynamics while targeting flight path angle as the state limit. The equivalent state in the $z$-domain was $z_4$, and the prediction model was thus set to target $z_4$ as the state limit.

At $t = 5$ sec, with zero initial conditions, the predicted critical elevator deflection required to reach the positive $z_4$ margin (flight path angle limit) was approximately $-0.55^\circ$, as depicted by the dashed green line in Figure 10.30. Applying this exact amount of elevator as a step input at $t = 5$ sec then caused the transformed state to change from an initial condition of $z_4 = 0$ at $t = 5$ sec to $z_4 \approx z_{4\text{limit}} = -43.8$ within an extended prediction time window corresponding to the peak time of the phugoid mode, as shown by the $z_4$ response in Figure 10.32. This corresponded to approximately $\gamma \approx 7.5^\circ > \gamma_{\text{limit}}$, as shown in Figure 10.31. In this case, the limit was non-conservative, and the flight path angle exceeded its limit in the $x$-domain.
Figure 10.27. 4th Order Longitudinal (Phugoid Diagonal – Velocity): Critical vs Applied Elevator

Figure 10.28. 4th Order Longitudinal (Phugoid Diagonal – Velocity): Forward Velocity Response

Figure 10.29. 4th Order Longitudinal (Phugoid Diagonal – Velocity): Transformed State $z_3$

Figure 10.30. 4th Order Longitudinal (Phugoid Diagonal – Flight Path Angle): Critical vs Applied Elevator

Figure 10.31. 4th Order Longitudinal (Phugoid Diagonal – Flight Path Angle): Flight Path Angle Response

Figure 10.32. 4th Order Longitudinal (Phugoid Diagonal – Flight Path Angle): Transformed State $z_4$
10.4.2.2 4th Order Lateral/Directional Mode

The aircraft model for the 4th order lateral/ directional mode is given in equation (10.7):

\[
\begin{bmatrix}
\dot{\beta}(t) \\
\dot{\phi}(t) \\
\dot{p}(t) \\
\dot{r}(t)
\end{bmatrix} =
\begin{bmatrix}
-0.083 & 0.048 & 0 & -0.999 \\
0 & 0 & 1 & 0 \\
-4.139 & 0 & -0.437 & 0.155 \\
2.825 & 0 & 0.0004 & -0.113
\end{bmatrix}
\begin{bmatrix}
\beta(t) \\
\phi(t) \\
p(t) \\
r(t)
\end{bmatrix}
+ 
\begin{bmatrix}
0 & 0.016 \\
0 & 0 \\
6.766 & 0.627 \\
-0.323 & -1.679
\end{bmatrix}
\begin{bmatrix}
\delta_a(t) \\
\delta_r(t)
\end{bmatrix}
\tag{10.7}
\]

The 4th order aircraft dynamics were then decomposed into the 2nd order block diagonal form. For the light business jet used in concept validation, the numerical transformation matrix for the lateral/ directional mode was as follows given by equation (10.8):

\[
T_D =
\begin{bmatrix}
-0.081 & 0.283 & 0.006 & 0.002 \\
-0.016 & 0.417 & -0.893 & 0.999 \\
0.705 & 0 & 0.448 & -0.001 \\
-0.477 & -0.122 & -0.040 & 0.048
\end{bmatrix}
\tag{10.8}
\]

In the transformed z-domain, the original state space system given in equation (10.7) then became:

\[
\begin{bmatrix}
\dot{z}_1 \\
\dot{z}_2 \\
\dot{z}_3 \\
\dot{z}_4
\end{bmatrix} =
\begin{bmatrix}
-0.065 & -1.687 & 0 & 0 \\
1.687 & -0.065 & 0 & 0 \\
0 & 0 & -0.501 & 0 \\
0 & 0 & 0 & -0.001
\end{bmatrix}
\begin{bmatrix}
z_1 \\
z_2 \\
z_3 \\
z_4
\end{bmatrix}
+ 
\begin{bmatrix}
0.781 & 3.192 \\
-0.129 & 1.066 \\
13.913 & -3.633 \\
12.508 & -3.642
\end{bmatrix}
\begin{bmatrix}
\delta_a \\
\delta_r
\end{bmatrix}
\tag{10.9}
\]

From equation (10.9), it is observed that the first block diagonal represents a complex oscillatory mode (the dutch roll), while the second block diagonal represents two real modes (the roll and spiral). Since the roll mode is of interest, and to fit the form of the expressions described in Section 6.4, the second block diagonal is written in the form:
As before, once decomposed, it became necessary to establish the mode dominance of each block diagonal to identify which represented the short period mode and which represented the phugoid mode. The inverse linear transformation is given by \( z = T_D^{-1}x \). By evaluating this expression, one obtained the transformed states as a function of the original states, as shown in equation (10.11):

\[
\begin{bmatrix}
\dot{z}_3 \\
\dot{z}_4
\end{bmatrix} =
\begin{bmatrix}
0 & 1 \\
0 & -0.501
\end{bmatrix}
\begin{bmatrix}
z_3 \\
z_4
\end{bmatrix}
\]

(10.10)

At this point, it is apparent that the states \( z_1 \) and \( z_2 \) correspond to the dutch roll mode, since the dominant terms are \( \beta \) and \( r \) (as indicated by the respective underbars), and states \( z_3 \) and \( z_4 \) correspond to the spiral and roll modes, since the dominant terms are \( \phi \) and \( p \) in the latter and \( r \) in the former.

Next, to obtain the state limits in the \( z \)-domain, an inverse linear transformation was applied to the limits placed on the original states (in Table 10.2) through \( z_{\text{limit}} = T_D^{-1}x_{\text{limit}} \). The resulting transformed state limits in the \( z \)-domain, along with their approximate equivalents in the \( x \)-domain, are shown in Table 10.4:

\[
\begin{align*}
z_1 &= -0.957\beta + 0.092\phi + 0.025p - 1.9r \\
z_2 &= 3.23\beta + 0.024\phi - 0.049p - 0.623r \\
z_3 &= 1.51\beta - 0.143\phi + 2.2p + 3.0r \\
z_4 &= -0.019\beta + 0.865\phi + 1.99p + 2.91r
\end{align*}
\]

(10.11)
Table 10.4

Transformed State Limits (Lateral/Directional Mode)

<table>
<thead>
<tr>
<th>Transformed State (z)</th>
<th>Equivalent State (x)</th>
<th>Negative Limit (z)</th>
<th>Positive Limit (z)</th>
</tr>
</thead>
<tbody>
<tr>
<td>z_1</td>
<td>r</td>
<td>0.175</td>
<td>–0.175</td>
</tr>
<tr>
<td>z_2</td>
<td>β</td>
<td>–0.242</td>
<td>0.242</td>
</tr>
<tr>
<td>z_3</td>
<td>p</td>
<td>–0.473</td>
<td>0.473</td>
</tr>
<tr>
<td>z_4</td>
<td>φ</td>
<td>–1.105</td>
<td>1.105</td>
</tr>
</tbody>
</table>

In the case of the modal transformation method, these transformed limits in the z-domain were imposed on the LOC prediction model, and the model was directed to target these transformed limits instead of the un-transformed original state limits in the x-domain (Table 10.2).

10.4.2.2.1 Dutch Roll Diagonal

Figure 10.33 to Figure 10.36 demonstrate the performance of the prediction model for the decomposed dutch roll dynamics. The state in the x-domain upon which the limit was placed is the sideslip angle, and the equivalent state in the z-domain was z_2. The prediction model was thus set to target z_2 as the state limit.

*First Input Set – Rudder Step Targeting Positive z_2/Sideslip Limit:* At t = 5 sec, with zero initial conditions, the predicted critical rudder deflection required to reach the positive transformed z_2 margin (sideslip angle limit) was approximately 3.8°, as depicted by the dashed green line in Figure 10.33. Applying this exact amount of rudder as a step input at t = 5 sec then caused the transformed state to change from an initial condition of z_2 = 0 at t = 5 sec to z_2 \approx z_2\text{limit} = 0.24 within the prediction time window \( \hat{T} = 2 \) sec, as shown in the z_2 response in Figure
10.36. This corresponded to approximately $\beta \approx 4.0^\circ < \beta_{\text{limit}}$, as shown in Figure 10.34, presenting a conservative limit on sideslip angle. Following the first critical input, a constant input of $\delta_r = 1.5^\circ$ was applied to keep the aircraft in an untrimmed sideslip condition, placing the aircraft in a non-steady-state and non-zero initial condition.

**Second Input Set – Rudder Rate Targeting Negative $z_2$/Sideslip Limit:** At $t = 30$ sec, with non-zero initial conditions on the states and inputs, the predicted critical rudder rate required to reach the negative transformed $z_2$ margin was approximately $-6.0^\circ$/sec. Applying this exact ramp input at $t = 30$ sec caused the transformed state to change from an average initial condition of $z_2 = 0.05$ at $t = 30$ sec to $z_2 \approx z_{2\text{limit}} = -0.24$ at the end of the prediction time window $\hat{T} = 2$ sec. This corresponded to approximately $\beta \approx -4.0^\circ < \beta_{\text{limit}}$, as seen in Figure 10.34, again presenting a conservative limit. Following the second critical input, a constant input of $\delta_r = 1.5^\circ$ was applied to again bring the aircraft to an untrimmed sideslip condition, placing the aircraft in a non-steady-state and non-zero initial condition.

**Third Input Set – Rudder Acceleration Targeting Positive $z_2$/Sideslip Limit:** At $t = 60$ sec, again with non-zero initial conditions on the states and inputs, the predicted critical rudder acceleration required to reach the positive transformed $z_2$ margin was approximately $10.0^\circ$/sec$^2$. Applying this exact parabolic input at $t = 60$ sec caused the transformed state to change from an average initial condition of $z_2 = -0.05$ at $t = 60$ sec to $z_2 \approx z_{2\text{limit}} = 0.24$ at the end of the prediction time window $\hat{T} = 2$ sec. This corresponded to approximately $\beta \approx 4.2^\circ < \beta_{\text{limit}}$, as shown in Figure 10.34, once again presenting a conservative limit.
Figure 10.33. 4th Order Lateral/Directional (Dutch Roll Diagonal): Critical vs Applied Rudder

Figure 10.34. 4th Order Lateral/Directional (Dutch Roll Diagonal): Sideslip Angle Response

Figure 10.35. 4th Order Lateral/Directional (Dutch Roll Diagonal): Transformed State $z_1$

Figure 10.36. 4th Order Lateral/Directional (Dutch Roll Diagonal): Transformed State $z_2$
10.4.2.2.2 Roll Diagonal

Figure 10.37 to Figure 10.40 demonstrate the performance of the prediction model for the decomposed roll dynamics. The state in the $x$-domain upon which the limit was placed is the bank angle, and the equivalent state in the $z$-domain was $z_4$. The prediction model was thus set to target $z_4$ as the state limit.

**First Input Set – Aileron Step Targeting Positive $z_4$/Bank Angle Limit:** At $t = 5$ sec, with zero initial conditions, the predicted critical aileron deflection required to reach the positive transformed $z_4$ margin (bank angle limit) for $\hat{T} = 2$ sec was approximately $4.0^\circ$, as depicted by the dashed green line in Figure 10.37. Applying this exact amount of aileron as a step input at $t = 5$ sec then caused the transformed state to change from an initial condition of $z_4 = 0$ at $t = 5$ sec to $z_4 \approx z_{4\text{limit}} = 1.1$ two seconds later at $t = 7$ sec, as shown in the $z_4$ response in Figure 10.40. This corresponded to approximately $\phi \approx 45^\circ \approx \phi_{\text{limit}}$, as shown in Figure 10.38. Following the first critical input, an input of $\delta_a = -2.8^\circ$ was applied to place the aircraft in a steady left bank of approximately $\phi \approx -15^\circ$, giving the aircraft a non-zero initial condition.

**Second Input Set – Aileron Rate Targeting Negative $z_4$/Bank Angle Limit:** At $t = 30$ sec, with non-zero initial conditions on the states, the predicted critical aileron rate required to reach the negative transformed $z_4$ margin was approximately $-31^\circ$/sec. Applying this exact ramp input at $t = 30$ sec caused the transformed state to change from an initial condition of $z_4 = 0$ at $t = 30$ sec to $z_4 \approx z_{4\text{limit}} = -1.1$ at $t = 32$ sec (ie. at the end of the prediction time window $\hat{T} = 2$ sec). This corresponded to approximately $\phi \approx -45^\circ \approx \phi_{\text{limit}}$, as seen in Figure 10.38.
Following the second critical input, an input of $\delta_a = 3.0^\circ$ was applied to place the aircraft in a steady right bank of approximately $\phi \approx 15^\circ$, giving the aircraft a non-zero initial condition.

**Third Input Set – Aileron Acceleration Targeting Positive $z_4$/Bank Angle Limit:** At $t = 55$ sec, again with non-zero initial conditions on the states, the predicted critical aileron acceleration required to reach the positive transformed $z_4$ margin was approximately $54^\circ$/sec$^2$. Applying this exact parabolic input at $t = 55$ sec caused the transformed state to change from an initial condition of $z_4 = 0$ at $t = 55$ sec to $z_4 \approx z_4_{limit} = 1.1$ at the end of the prediction time window $\hat{T} = 2$ sec. This corresponded to approximately $\phi \approx 45^\circ \approx \phi_{limit}$, as shown in Figure 10.38.
Figure 10.37. 4th Order Lateral/Directional (Roll Diagonal): Critical vs Applied Aileron

Figure 10.38. 4th Order Lateral/Directional (Roll Diagonal): Bank Angle Response

Figure 10.39. 4th Order Lateral/Directional (Roll Diagonal): Transformed State $z_3$

Figure 10.40. 4th Order Lateral/Directional (Roll Diagonal): Transformed State $z_4$
10.5 Trajectory-To-Go Models

In validating the effectiveness and accuracy of the trajectory-to-go prediction models, various simulation runs were performed in MATLAB/Simulink based on the premise of a pilot performing pitch & roll doublet maneuvers in an attempt to fly on the edge of the safe flight envelope, using only the safe control space to govern the applied control inputs. The hypothesis is that, if the safe control space was predicted accurately, flying on the edge of the control space would also result in flying on the edge of (and not exceeding) the flight envelope. The results shown are based on the Linear Quadratic Tracker prediction model.

10.5.1 Pitch Doublet

Figure 10.41 to Figure 10.44 illustrate the applied inputs and state responses of the light business jet for a pitch doublet maneuver. Snapshots of the safe control space are presented in Figure 10.45, and the individual control-to-go surface spaces for the eight cardinal trajectory targets are presented in Figure 10.46 and Figure 10.47.

First Input Set – Elevator Targeting Positive Pitch Edge of Control Space: At \( t = 1 \text{ sec} \), the pilot applies an elevator input of approximately \(-5.25^\circ\) (Figure 10.41), causing the angle-of-attack to rise and reach \( \alpha = 15^\circ \). Following a brief pause, the pilot then updates this elevator input to approximately \(-6.85^\circ\), based on the changing bounds of the control space. During this period, corrective aileron inputs are also given to keep the aircraft in an approximate wings-level configuration (Figure 10.43). At this point, the control space indicates no further control-to-go is available in the nose-up direction (middle row of Figure 10.45), and the aircraft continually flies on the upper edge of the angle-of-attack envelope (Figure 10.42).
Second Input Set – Elevator Targeting Negative Pitch Edge of Control Space: At \( t = 8 \) sec, the pilot applies an elevator input of approximately 6.25° (Figure 10.41), causing the angle-of-attack to reduce and reach \( \alpha = -25^\circ \). Following a brief pause, the pilot then updates this elevator input to approximately 10.75°, based on the changing bounds of the control space. As before, corrective aileron inputs are given to keep the aircraft in an approximate wings-level configuration (Figure 10.43). At this point, the control space indicates no further control-to-go is available in the nose-down direction (bottom row of Figure 10.45), and the aircraft continually flies on the lower edge of the angle-of-attack envelope (Figure 10.42).

Third Input Set – Elevator Targeting Straight-and-Level Flight Configuration: At \( t = 12 \) sec, elevator inputs are applied to return the aircraft to the center of the control space, thereby bringing the aircraft back to a safe straight-and-level flight configuration.
Figure 10.41. Pitch Doublet: Applied Elevator

Figure 10.42. Pitch Doublet: Angle-of-Attack Response

Figure 10.43. Pitch Doublet: Applied Aileron

Figure 10.44. Pitch Doublet: Bank Angle Response
Figure 10.45. Pitch Doublet: Snapshots of Control Space as Pilot Advisory Display
Figure 10.46. Pitch Doublet: Surface Space of Elevator and Aileron (Part 1)
Figure 10.47. Pitch Doublet: Surface Space of Elevator and Aileron (Part 2)
10.5.2 Roll Doublet

Figure 10.48 to Figure 10.51 illustrate the applied inputs and state responses of the light business jet for a roll doublet maneuver. Snapshots of the safe control space are presented in Figure 10.52, and the individual control-to-go surface spaces for the eight cardinal trajectory targets are presented in Figure 10.53 and Figure 10.54.

First Input Set – Aileron Targeting Negative Roll Edge of Control Space: At $t = 1$ sec, the pilot applies an aileron input of approximately $-5^\circ$ (Figure 10.50), causing the bank angle to rise and reach $\phi = -45^\circ$. Following a brief pause, updated aileron inputs are applied to keep the aircraft in an approximate steady state bank, based on the changing bounds of the control space. During this period, the angle-of-attack stays fairly close to zero, and no corrective elevator inputs are necessary (Figure 10.48). At this point, the control space indicates no further control-to-go is available in the left roll direction (middle row of Figure 10.52), and the aircraft continually flies on the left edge of the bank angle envelope (Figure 10.51).

Second Input Set – Aileron Targeting Positive Roll Edge of Control Space: At $t = 6$ sec, the pilot applies an aileron input of approximately $5^\circ$ (Figure 10.50), causing the bank angle to rise and reach $\phi = 45^\circ$. Following a brief pause, updated aileron inputs are applied to keep the aircraft in an approximate steady state bank, based on the changing bounds of the control space. Once again, the angle-of-attack stays fairly close to zero, and no corrective elevator inputs are necessary (Figure 10.48). At this point, the control space indicates no further control-to-go is available in the right roll direction (bottom row of Figure 10.52), and the aircraft continually flies on the right edge of the bank angle envelope (Figure 10.51).
**Third Input Set – Aileron Targeting Straight-and-Level Flight Configuration:** At $t = 12$ sec, aileron inputs are applied to return the aircraft to the center of the control space, thereby bringing the aircraft back to a safe straight-and-level flight configuration.

![Figure 10.48. Roll Doublet: Applied Elevator](image)

![Figure 10.49. Roll Doublet: Angle-of-Attack Response](image)

![Figure 10.50. Roll Doublet: Applied Aileron](image)

![Figure 10.51. Roll Doublet: Bank Angle Response](image)
Figure 10.52. Roll Doublet: Snapshots of Control Space as Pilot Advisory Display
Figure 10.53. Roll Doublet: Surface Space of Elevator and Aileron (Part 1)
Figure 10.54. Roll Doublet: Surface Space of Elevator and Aileron (Part 2)
10.5.3 Pitch & Roll Doublet

Figure 10.55 to Figure 10.58 illustrate the applied inputs and state responses of the light business jet for a simultaneous pitch and roll doublet maneuver. This scenario illustrates the dynamic coupling of the aircraft at high combined angles-of-attack and bank angles, and also demonstrates the capability of the prediction architecture in handling such conditions. Snapshots of the safe control space are presented in Figure 10.59, and the individual control-to-go surface spaces for the eight cardinal trajectory targets are presented in Figure 10.60 and Figure 10.61.

First Input Set – Elevator and Aileron Targeting Positive Pitch and Negative Roll Edges of Control Space: At $t = 1$ sec, the pilot applies an elevator input of approximately $-5.25^\circ$ (Figure 10.55), causing the angle-of-attack to rise and reach $\alpha = 15^\circ$. Simultaneously at $t = 1$ sec, the pilot applies an aileron input of approximately $-5^\circ$ (Figure 10.57), causing the bank angle to rise and reach $\phi = -45^\circ$. Following a brief pause, the pilot then updates the elevator input to approximately $-6.85^\circ$ to maintain $\alpha = 15^\circ$ and provides corrective aileron input to maintain $\phi = -45^\circ$, based on the changing bounds of the control space. At this point, the control space indicates no further control-to-go is available in the nose-up and left-roll directions (middle row of Figure 10.59), and the aircraft continually flies on the upper edge of the angle-of-attack envelope (Figure 10.56) and the left edge of the bank angle envelope (Figure 10.58).

Second Input Set – Elevator and Aileron Targeting Negative Pitch and Positive Roll Edges of Control Space: At $t = 6$ sec, the pilot applies an elevator input of approximately $6.25^\circ$ (Figure 10.55), causing the angle-of-attack to reduce and reach $\alpha = -25^\circ$. Simultaneously at $t = 6$ sec, the pilot applies an aileron input of approximately $8^\circ$ (Figure 10.57), causing the aircraft to
roll in the opposite direction and the bank angle to reach $\phi = 45^\circ$. Following a brief pause, the pilot then updates the elevator input to approximately $10.75^\circ$ to maintain $\alpha = -25^\circ$ and provides corrective aileron input to maintain $\phi = 45^\circ$, based on the changing bounds of the control space. At this point, the control space indicates no further control-to-go is available in the nose-down and right-roll directions (bottom row of Figure 10.59), and the aircraft continually flies on the lower edge of the angle-of-attack envelope (Figure 10.56) and the right edge of the bank angle envelope (Figure 10.58).

**Third Input Set – Elevator and Aileron Targeting Straight-and-Level Flight Configuration:** At $t = 11$ sec, elevator and aileron inputs are applied to return the aircraft to the center of the control space, thereby bringing the aircraft back to a safe straight-and-level flight configuration.
Figure 10.55. Pitch & Roll Doublet: 
Applied Elevator

Figure 10.56. Pitch & Roll Doublet: 
Angle-of-Attack Response

Figure 10.57. Pitch & Roll Doublet: 
Applied Aileron

Figure 10.58. Pitch & Roll Doublet: 
Bank Angle Response
Figure 10.59. Pitch & Roll Doublet: Snapshots of Control Space as Pilot Advisory Display
Figure 10.60. Pitch & Roll Doublet: Surface Space of Elevator and Aileron (Part 1)
Figure 10.61. Pitch & Roll Doublet: Surface Space of Elevator and Aileron (Part 2)
CHAPTER 11
SIMULATED FLIGHT TESTING

11.1 Overview

This chapter details the efforts at verification and validation of the LOC prediction architecture through pilot-in-the-loop simulated flight testing. The experimental setup and experiment design is first presented, followed by the results, observations, and findings from the test effort.

11.2 Experimental Setup

Figure 11.1 illustrates the experimental setup and software-hardware interaction/configuration used for the flight test effort. The research flight simulator is comprised of the groundstation and simulation computers, and the pilot advisory display is run on an Android-based tablet (for the head-down display) or the Microsoft HoloLens (for the head-up display).

Figure 11.1. Overview of Experimental Setup

Communication between all three components is handled by a server utilizing the Node.js runtime, which runs on the groundstation computer. The server is responsible for coordinating data transfer between all components and providing centralized control.
The Communications Server receives current aircraft states and pilot inputs from X-Plane via UDP (User Datagram Protocol). UDP is used because it is a built-in network interface in X-Plane. This state and input information is then continuously polled by the LOC Prediction Architecture that runs within Simulink. These states and inputs are then used to calculate the critical deflections/safe control envelope in real-time, and the bounds of the envelope are sent from Simulink to the Communications Server over HTTP (Hypertext Transfer Protocol). The Communications Server then relays the bounds of the envelope and the current pilot inputs over WebSockets to the Pilot Advisory Display on the HoloLens/tablet. Finally, the display is rendered on the HoloLens/tablet, presenting the safe control envelope to the pilot. All-inclusive, this system runs at 25 Hz and is near-instantaneous with a lag of less than 0.1 seconds from pilot input to update of the Pilot Advisory Display.

The Communications Server is also connected to a browser-based dashboard which is used as an “experiment control center”, allowing the experimenter to pause the flight simulation, insert environmental obstacles, reposition the aircraft for each repetition, monitor aircraft states/inputs, record data, and control the progression of the experiment.

11.2.1 Research Flight Simulator

Figure 11.2 to Figure 11.4 depict the research flight simulator at the General Aviation Flight Lab at Wichita State University. The simulator consists of five 55-inch monitors combined to form a single large display, giving the pilot a 1:1-scaled 180-degree panoramic view of the environment. The simulator may be driven using Lockheed Martin Prepar3D® or Laminar Research X-Plane® flight simulation software and has dual control yokes and rudder pedals for a pilot and copilot, as well as a functional center console. A touchscreen panel above the center console serves as a re-configurable instrument panel capable of adapting to suit the varying instrumentation of multiple aircraft types.
Figure 11.2. Research Flight Simulator at the General Aviation Flight Lab at Wichita State University

Figure 11.3. Cockpit Area of Research Flight Simulator

Figure 11.4. Close-up of Flight Control Setup with Display Instrumentation for Light Business Jet
11.3 Experiment Design

In testing the effectiveness of the display, certified pilots were tasked to fly a variety of scenarios in the simulator which were designed to inadvertently push the aircraft to the edge of the safe flight envelope. They were then tasked to maneuver the aircraft within the bounds of the safe flight envelope, with and without assistance from the prediction model.

11.3.1 Aircraft

The aircraft used in simulated flight testing was a 1st/2nd order variant of the 8th order light business jet presented in Section 10.2. The modes being evaluated in the simulated flight test effort were the decoupled short period and roll. At the time of the experiment, the full 8th order LOC prediction architecture was not yet developed. Table 11.1 summarizes the limits imposed on the states that make up the corresponding LOC margins/safe flight envelope, as used in the simulated flight test effort. Note that these are delta values from the trim condition.

**TABLE 11.1**

SAFE FLIGHT ENVELOPE AND LOC MARGINS
USED IN SIMULATED FLIGHT TESTING

<table>
<thead>
<tr>
<th>Mode</th>
<th>State</th>
<th>Negative Limit</th>
<th>Positive Limit</th>
<th>Unit</th>
</tr>
</thead>
<tbody>
<tr>
<td>Short Period</td>
<td>Angle-of-Attack (α)</td>
<td>-25</td>
<td>+25</td>
<td>deg</td>
</tr>
<tr>
<td></td>
<td>Pitch Rate (q)</td>
<td>-</td>
<td>-</td>
<td>-</td>
</tr>
<tr>
<td>Roll</td>
<td>Bank Angle (ϕ)</td>
<td>-45</td>
<td>+45</td>
<td>deg</td>
</tr>
<tr>
<td></td>
<td>Roll Rate (p)</td>
<td>-</td>
<td>-</td>
<td>-</td>
</tr>
</tbody>
</table>

For the short period, the upper and lower bounds of the safe flight envelope were enforced by the limits $-25° \leq \alpha_{safe} \leq +15°$. The limit on $q$ was not enforced. For the roll, the left and
right bounds of the safe flight envelope were enforced by the limits \(-45^\circ \leq \phi_{safe} \leq +45^\circ\). The limit on \(p\) was not enforced. For purposes of experimentation, these values were withheld from the pilot. This served to ensure that the pilots were relying only on the alerts from the prediction model (or on airmanship skills in the case of no advisory cues) when executing maneuvers to keep the aircraft in the safe flight envelope.

11.3.2 Pilot Census

Six pilots with a varied range of flight experience were selected to participate in the study. Their highest rating, approximate hours logged as of this research, and basic biographical data are summarized in

<table>
<thead>
<tr>
<th>Pilot</th>
<th>Highest Rating</th>
<th>Hours Logged</th>
<th>Age</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>Private Pilot</td>
<td>120</td>
<td>24</td>
</tr>
<tr>
<td>2</td>
<td>Private Pilot</td>
<td>55</td>
<td>20</td>
</tr>
<tr>
<td>3</td>
<td>Private Pilot</td>
<td>190</td>
<td>20</td>
</tr>
<tr>
<td>4</td>
<td>Private Pilot</td>
<td>60</td>
<td>23</td>
</tr>
<tr>
<td>5</td>
<td>Airline Transport Pilot</td>
<td>2000</td>
<td>36</td>
</tr>
<tr>
<td>6</td>
<td>Private Pilot</td>
<td>120</td>
<td>41</td>
</tr>
</tbody>
</table>

All experiments were performed with participant consent obtained. Pilots were also given a survey to collect demographic information.

11.3.3 Flight Test Scenarios

The pilots were tasked to evaluate the guidance technologies across three scenarios: The first involved an unexpected obstruction on takeoff, the second involved a high crosswind on a
base-to-final turn during landing, and the third involved piloting the aircraft in slow-flight. These scenarios were designed in part based on historical incidents in civil aviation, and were specifically designed to bring the aircraft close to the edge of the safe flight envelope (while yet allowing the pilot to successfully perform the maneuvers without excursion of the safe flight envelope). Each scenario was repeated for four conditions:

1) the airmanship-only control scenario (no guidance technologies),
2) the aural-only guidance technology (audible warning of excursion without a visual cue),
3) the head-down guidance technology (safe control envelope on the instrument panel), and
4) the head-up guidance technology (safe control envelope on the Microsoft HoloLens).

In Condition 1, the pilots were tasked to complete the maneuvers with no aural or visual advisories, instead relying solely on their own airmanship skills. In Condition 2, the pilots were tasked to complete the maneuvers while keeping the aircraft within the safe control envelope denoted by the aural alerts, using the alerts to accompany airmanship. In Conditions 3 and 4, the pilots were tasked to complete the maneuvers while staying within the safe control envelope denoted by the pilot advisory display, using the display to accompany airmanship.

Conditions 2, 3, and 4 presented alerts to the pilot based on the calculations from the prediction model. With the aural format in Condition 2, the pilot only received alerts at the point when the safe control envelope had been exceeded. With the visual format in Conditions 3 and 4, the pilot was also given information about how close the aircraft was to the edge of the safe control envelope.

11.3.3.1 Scenario 1 – Obstruction on Runway During Takeoff

Scenario 1 placed the pilot in a situation where an obstruction appeared during the takeoff roll, necessitating a large pitch up maneuver to clear the obstruction. This scenario was designed to test the short period mode on the single-axis display (see Figure 9.2). Accordingly, the display
presented the safe control envelope for angle-of-attack with elevator as the single control effector. As such, the pilot was instructed to only use the elevator and pitch maneuvers in attempting to clear the obstruction. Directional control on the ground was permitted through use of the rudder, while flap and trim settings were pre-configured.

Figure 11.5 presents a schematic overview of Scenario 1. The obstruction was placed at a location on the runway which, for the given aircraft configuration, rotation point, and atmospheric conditions, would require the angle-of-attack be brought close to the limit of 15° for successful clearance. Successful clearance, however, was achievable without exceeding the safe flight envelope. To simulate harsh performance conditions, the situation was set up at an airport with a high field elevation (Denver International Airport, 5,434 ft), a high ambient temperature (100°F), and low visibility (1,400 ft).

Figure 11.5. Schematic Overview of Scenario 1

The pilot was made aware during the pre-experiment briefing that an obstruction existed, but was not informed of when the obstruction would appear. This was controlled by the environmental visibility setting, which was chosen such that the obstruction would only appear shortly before reaching the rotation marker. These environmental settings mimicked Instrument Flight Rules (IFR) conditions. At the start of the takeoff roll, the obstruction was thus not visible to the pilot (Figure 11.6). Upon visual contact of the obstruction, after passing the $V_1$ "commit to fly" speed and coincident with the aircraft reaching a pre-defined rotation marker, the pilot was
instructed to use as much elevator as necessary to pitch the aircraft up and bring it airborne with just enough altitude to clear the obstruction (Figure 11.7). To ensure consistency and to keep variables constant for experimentation, a rotation marker (as opposed to a rotation speed) was used to denote the point at which the pilot would bring the aircraft airborne. The situation was designed to necessitate a large amount of elevator input for successful clearance. This translated to a higher probability of excursion of the safe flight envelope (with or without advisory cues), and required a conscious effort on the part of the pilot to stay within the envelope.

11.3.3.2 Scenario 2 – Approach with High Crosswind on Base-to-Final Turn

Scenario 2 placed the pilot in a situation where a large direct crosswind was present on a base-to-final turn in the approach pattern, necessitating a large amount of left bank to align the aircraft with the runway while keeping overshoot of the extended runway centerline to a minimum. This scenario was designed to test the independent short period and roll modes on the dual-axis display (see Figure 9.2). Accordingly, the display presented the safe control envelope for angle-of-attack on the upper/lower bounds and bank angle on the left/right bounds, with elevator and aileron as the control effectors. The pilot was instructed to use both the elevator and aileron in
unison to perform pitch and roll maneuvers in attempting the approach. In addition, airspeed control was permitted through throttle adjustments. Landing gear, flap, and trim settings were pre-configured, and the pilot was instructed not to utilize the rudder.

Figure 11.8 presents a schematic overview of Scenario 2. The approach was configured for Runway 36 at Wichita Colonel James Jabara Airport (magnetic heading of 001°) with a crosswind from the west (35 kts at 270°). The pilot was inserted at the start of the downwind leg and was tasked to pilot the aircraft through a standard rectangular downwind-base-final pattern to bring the aircraft across the threshold of the runway, with touchdown not required.

![Figure 11.8. Schematic Overview of Scenario 2](image)

Commencement of the downwind-to-base turn was fixed by a “floating” base turn marker, which was a visual cue placed in the simulation environment notifying pilots to turn onto the base leg. The position of this marker was determined based on the standard approach pattern spacing for the airfield, and was intended to prevent pilots from flying an extended downwind (which would in turn lead to an extended final), ensuring consistency between trials and keeping variables as constant as possible for experimentation.

In completing the base-to-final turn, the pilot was tasked to align the aircraft with the runway’s extended centerline with minimal overshoot. The scenario was designed such that, for the given aircraft configuration/approach speeds, the high crosswind would require the bank angle
be brought \textit{close} to the limit of -45° for minimal overshoot, assuming the pilot had properly planned the approach and factored in the need to start the turn early due to the crosswind. As before, minimal overshoot was inherently achievable without exceeding the safe flight envelope.

\begin{figure}[h]
\centering
\includegraphics[width=\textwidth]{figure11_9.png}
\caption{Pilot’s Perspective of Base Turn Marker}
\end{figure}

\begin{figure}[h]
\centering
\includegraphics[width=\textwidth]{figure11_10.png}
\caption{Light business jet performing steep bank on base-to-final turn due to crosswind}
\end{figure}

\textbf{11.3.3.3 Scenario 3 – Slow Flight}

Scenario 3 tasked the pilot to initiate and maintain a slow-flight maneuver, while maintaining constant altitude and wings-level flight. This scenario was designed to test the short period, but on the dual-axis display (see Figure 9.2). Accordingly, the display presented the safe control envelope for angle-of-attack on the upper/lower bounds and bank angle on the left/right bounds, with elevator being the control effector of interest. The pilot was instructed to use throttle to maintain a fixed airspeed and elevator to perform the pitch maneuver, with aileron solely to keep wings level. Landing gear, flap, and trim settings were pre-configured, and the pilot was instructed not to utilize the rudder.

Figure 11.11 presents a schematic overview of Scenario 3. Pilots were inserted into a trimmed straight-and-level flight condition with calm winds, and were tasked to reduce airspeed
and maintain straight-and-level flight at approximately 82 knots. In maintaining the resulting slow-flight configuration, the aircraft would be brought close to the angle-of-attack limit of 15°. Upon reaching a stabilized slow-flight configuration, pilots were tasked with maintaining the configuration for one minute.

Figure 11.11. Schematic Overview of Scenario 3

Figure 11.12 depicts the light business jet in the slow-flight configuration:

Figure 11.12. Light business jet in slow flight
11.3.4 Test Procedure

11.3.4.1 Experimental Progression

The experimental progression and order of the conditions/repetitions is given in TABLE 11.3

<table>
<thead>
<tr>
<th>Item</th>
<th>Task</th>
<th>Time (hr:min)</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>Experimental Briefing</td>
<td>0:10</td>
</tr>
<tr>
<td>2</td>
<td>Familiarization Flight</td>
<td>0:15</td>
</tr>
<tr>
<td>3</td>
<td>Scenario 1 / Conditions 1-4 (in varying order)</td>
<td>0:10</td>
</tr>
<tr>
<td>4</td>
<td>Scenario 2 / Conditions 1-4 (in varying order)</td>
<td>0:20</td>
</tr>
<tr>
<td>5</td>
<td>Scenario 3 / Conditions 1-4 (in varying order)</td>
<td>0:10</td>
</tr>
<tr>
<td>6</td>
<td>Debrief</td>
<td>0:05</td>
</tr>
</tbody>
</table>

At the start of the experiment, an experimental briefing and familiarization flight was given to the pilot. This allowed the pilot to obtain a feel for the aircraft’s handling characteristics. In addition, pilots were trained on use of and interaction with the aural alert and pilot advisory display. Each Scenario/Condition combination was then performed and repeated once, to obtain two measurements for each Condition.

A four-per-block counterbalancing order was also implemented, which had different pilots perform the Scenarios with a different order of conditions. This variation of experimental order facilitated the analysis of how much the pilots were relying on the guidance technologies versus how much they were “learning” the situation.

At the end, a debrief and survey was given to each pilot, collecting qualitative feedback for the experiment and the pilot advisory display.
11.3.4.2 Measured Data and Post-Experiment Analysis

A total of 76 variables were recorded, encompassing aircraft states, control inputs/deflections, outputs from the prediction model, aircraft position, simulator states, and various associated data. This data was utilized for post-experiment analysis of pilot performance.

11.4 Results

11.4.1 Scenario 1 – Obstruction on Runway During Takeoff

Figure 11.13 shows the angle-of-attack responses of each pilot as a function of distance along the runway. The green vertical dotted line indicates the location of the rotation marker, while the red vertical dotted line indicates the location of the obstruction. Upon visual appearance of the obstruction and passing the rotation marker, most pilots were observed to pitch up as much as possible to bring the aircraft airborne to clear the obstruction, often times more than the minimum necessary.

![Figure 11.13. Flight Test Scenario 1: Angle-of-Attack versus Distance along Runway](image)

In the majority of cases, the pilots had indicated that the guidance technologies had not factored much into their control inputs for the initial pitch maneuver (between 2,500 ft and 3,000 ft) to get airborne. This was in part because of the rapidly-progressing nature of the scenario,
wherein the pilots had only several seconds between the obstruction appearing and having to provide sudden inputs to achieve successful clearance.

The benefits of the visual guidance technologies, however, are seen once the aircraft is airborne (between 3,000 ft and 3,500 ft). Figure 11.14 provides a closer look at the aircraft angle-of-attack post-rotation. Figure 11.15 shows the mean angle-of-attack as a function of distance along the runway. In trying not to exceed the bounds of the safe control envelope, one observes that the mean angle-of-attack is approximately two to five degrees less in the head-down (green) and head-up (magenta) conditions than in the no-assistance (dashed) and aural (blue) conditions. This suggests that the pilots were able to stay further away from a potential stall condition with the visual guidance technologies. There is also much less pilot-induced oscillation with the visual guidance technologies, which is another positive indicator.

![Figure 11.14](image1.png)
*Figure 11.14. Flight Test Scenario 1: Angle-of-Attack versus Distance along Runway Post-Rotation*

![Figure 11.15](image2.png)
*Figure 11.15. Flight Test Scenario 1: Mean Angle-of-Attack*

Figure 11.16 illustrates the climb-out trajectories of each pilot. One observes that the trajectories associated with the head-up display (magenta) were much more consistent than for the other guidance technologies, and also led to greater clearance (altitude-wise) of the obstruction. Since the scenario was designed to progress rapidly, this is perhaps because the pilots had little
time to “glance” downwards at the head-down display. Instead, having the safe control envelope
directly in their field-of-view allowed them to better factor the information into their interactions
with the aircraft.

Figure 11.16. Flight Test Scenario 1: Aircraft Climb Trajectory

This trend is also observed in Figure 11.17 and Figure 11.18, which shows the standard
deviation of altitude as a function of distance along the runway. On average, the standard deviation
is approximately 5 ft less in the head-up condition versus the head-down, aural, and no-assistance
conditions.

Figure 11.17. Flight Test Scenario 1: Mean Altitude Trajectory

Figure 11.18. Flight Test Scenario 1: Standard Deviation of Altitude Trajectory
11.4.2 Scenario 2 – Approach with High Crosswind on Base-to-Final Turn

Figure 11.19 illustrates the approach trajectories of each pilot for Scenario 2. The green X denotes the insertion point, the yellow circle denotes the base turn marker, the red X denotes the runway threshold, and the gray strip denotes the runway. From observation, one notes that the trajectories corresponding to the head-down, head-up, and aural conditions appear much more consistent than those of the no-assistance condition. In general, pilots were able to perform the base-to-final turn with less overshoot of the extended runway centerline. The majority of pilots had indicated that they were able to factor the guidance technologies into their control inputs much more in Scenario 2 than in Scenario 1, since the nature of this scenario allowed for more forward planning on the part of the pilots.

![Figure 11.19. Flight Test Scenario 2: Aircraft Approach Trajectory](image)

The benefits of the guidance technologies are also seen in the bank angle responses of the pilots, as shown in Figure 11.20. The initial downwind-to-base turn occurred between 45 sec and 65 sec, and the second base-to-final turn occurred between 70 sec and 90 sec. In addition to flying a more rectangular approach pattern with the guidance technologies, the visual guidance technologies also led to far less excursions of the bank angle limit, as compared to the aural
guidance technology and the no-assistance condition. Furthermore, one notes that pilots also stayed further away from the bank angle limit in the head-up condition than in the head-down condition.

Figure 11.20. Flight Test Scenario 2: Bank Angle versus Time Elapsed

Figure 11.21 and Figure 11.22 respectively show the mean and standard deviation of bank angle during the maneuver. From the standard deviation, one observes that the head-up condition corresponded to much more consistent bank angles than each of the other conditions, and this is especially so for the crucial second turn from base to final.

Figure 11.21. Flight Test Scenario 2: Mean Bank Angle

Figure 11.22. Flight Test Scenario 2: Standard Deviation of Bank Angle
This trend is also observed in Figure 11.20, and the notion is also reinforced when looking at the minimum and maximum bank angles of all the pilots, as shown in Figure 11.23. The visual guidance technologies demonstrated a much tighter range of bank angles. Further, the smallest excursion of the bank angle limit was associated with the head-up condition (magenta), while the largest excursion was associated with the no-assistance condition (dashed). All of these suggest that the visual guidance technologies did indeed have a positive impact on the pilots’ performances.

![Figure 11.23. Flight Test Scenario 2: Range of Bank Angle](image)

### 11.4.3 Scenario 3 – Slow Flight

Figure 11.24 illustrates the angle-of-attack time responses of each pilot for Scenario 3. While an initial observation of Figure 11.24 does not provide immediate insight or trends, an analysis of the mean and standard deviation of the angle-of-attack, as shown in Figure 11.25 and Figure 11.26 respectively, yields more meaningful information about the pilots’ performances. Generally, the mean angle-of-attack was observed to be lower by approximately half to one degree with use of the guidance technologies than without. The exception to this was an anomalous spike in the angle-of-attack in the latter third of the maneuver for the head-up condition. Observing the
standard deviation, pilots also demonstrated generally better consistency in performing the maneuver with use of the guidance technologies than in the no-assistance condition.

![Figure 11.24. Flight Test Scenario 3: Angle-of-Attack versus Time Elapsed](image)

![Figure 11.25. Flight Test Scenario 3: Mean Angle-of-Attack](image)

![Figure 11.26. Flight Test Scenario 3: Standard Deviation of Angle-of-Attack](image)

While these differences in the angle-of-attack may not be numerically large, the data suggests that the guidance technologies were enabling pilots to stay further away from the angle-of-attack limit. This is noteworthy, since one of the goals of the guidance technologies is to reduce the risk of occurrence of an LOC event.
11.5 Qualitative Survey

Table 11.4 and Table 11.5 present results from the post-experiment survey. The pilots were asked to rate various aspects of the experiment, ranging from experimental procedure to utility and perceived benefits of the various guidance technologies. A “1” represents the lowest rating (ie. No/Poor/Disagree) and a “5” represents the highest rating (ie. Yes/Good/Agree). Each “I” represents an entry from a pilot.

**TABLE 11.4**

**POST-EXPERIMENT QUALITATIVE SURVEY (PART A)**

<table>
<thead>
<tr>
<th>Question</th>
<th>1</th>
<th>2</th>
<th>3</th>
<th>4</th>
<th>5</th>
</tr>
</thead>
</table>
| A1 Did you find the pre-experiment briefing and familiarization with the simulator and pilot advisory displays to be adequate for completing the required tasks? |   |   | I | III | I
| A2 Did you find the objective of the tasks to be clear?                 |   | I | III |   |   |
| A3 Did you feel that you had sufficient information to satisfactorily perform the required tasks? | I | III |   |   |   |
| A4 How would you rate your performance on Scenario 1? [runway obstruction] | I | III | II |   |   |
| A5 How would you rate your performance on Scenario 2? [crosswind approach] | I | II | II | I |   |
| A6 How would you rate your performance on Scenario 3? [slow flight]      | II | I | III |   |   |
| A7 Did you feel that you were getting used to the situations as the experiment progressed? | I | III | I |   |   |
| A8 How fatigued were you at the end of the experiment?                   | I | II | II | I |   |
| A9 How would you rate the **effectiveness** of the aural-alert guidance type as a pilot advisory mechanism? | II | III | I |   |   |
| A10 How would you rate the **effectiveness** of the head-down guidance type as a pilot advisory display? | II | I | I | II |   |
| A11 How would you rate the **effectiveness** of the head-up guidance type as a pilot advisory display? | II | III |   | I | I |
| A12 How useful do you think the aural-alert guidance type would be in a real-world aircraft? | III | I | I |   |   |
| A13 How useful do you think the head-down guidance type would be in a real-world aircraft? | III | II | I |   |   |
| A14 How useful do you think the head-up guidance type would be in a real-world aircraft? | I | II | III |   |   |
| A15 How much would you like such pilot advisory displays to eventually become a norm in aviation? | I | I | III |   |   |
### TABLE 11.5
POST-EXPERIMENT QUALITATIVE SURVEY (PART B)

<table>
<thead>
<tr>
<th>Question</th>
<th>None</th>
<th>Aural Head Down</th>
<th>Aural Head Up</th>
</tr>
</thead>
<tbody>
<tr>
<td>B1 Which guidance type did you find most beneficial for Scenario 1?</td>
<td>II</td>
<td>I</td>
<td>II</td>
</tr>
<tr>
<td>B2 Which guidance type did you find most beneficial for Scenario 2?</td>
<td>II</td>
<td>III</td>
<td>I</td>
</tr>
<tr>
<td>B3 Which guidance type did you find most beneficial for Scenario 3?</td>
<td>III</td>
<td>I</td>
<td>I</td>
</tr>
<tr>
<td>B4 Which guidance type did you find the least effective?</td>
<td>III</td>
<td>I</td>
<td>I</td>
</tr>
<tr>
<td>B5 Which guidance type did you find the most effective?</td>
<td>II</td>
<td>III</td>
<td>I</td>
</tr>
<tr>
<td>B6 Which guidance type did you find to be the least distracting?</td>
<td>II</td>
<td>III</td>
<td></td>
</tr>
<tr>
<td>B7 Which guidance type did you find to be the most distracting?</td>
<td>II</td>
<td>III</td>
<td></td>
</tr>
<tr>
<td>B8 Which guidance type required the least amount of effort to use?</td>
<td>III</td>
<td>I</td>
<td>I</td>
</tr>
<tr>
<td>B9 Which guidance type required the most amount of effort to use?</td>
<td>I</td>
<td>I</td>
<td>III</td>
</tr>
<tr>
<td>B10 Which guidance type do you think requires most improvement?</td>
<td>I</td>
<td>II</td>
<td>III</td>
</tr>
<tr>
<td>B11 If you had to choose one guidance type for your airplane, which would you pick?</td>
<td>II</td>
<td>III</td>
<td></td>
</tr>
</tbody>
</table>

Listed below are some general observations and comments gathered from the survey results:

- [A4–A6] Pilots perceived their own performance as average for Scenario 1, fair to average for Scenario 2, and average to good for Scenario 3.
- [A7] Pilots strongly agreed that they were learning the scenarios as the experiment progressed. Noting that the experiment conditions were conducted in varying order, this lends credence to the effectiveness of the guidance technologies, in that the observed trends still showed benefits with use of the guidance technologies even with a counter-balancing scheme in place.
• [A8] Pilots did not perceive themselves as extremely fatigued at the end of the experiment, suggesting that performance degradation did not factor largely into the measurements.

• [A9–A11] The visual guidance technologies were rated as more effective than the aural guidance technology.

• [A12–A14] The aural and head-down guidance technologies were rated as more useful than the head-up guidance technology.

• [A15] In general, pilots favored the eventual implementation of such guidance technologies in actual aircraft.

• [B1–B3] For Scenario 1, pilots did not find any particular guidance technology to be more beneficial than another. This appears to correlate with the findings from the measured data, which suggest that the rapidly-progressing nature of the scenario led to less incorporation of the guidance technologies in performing the maneuver. For Scenario 2, pilots found the guidance technologies to be beneficial over having no assistance. The measured data suggests much improved performance with guidance as opposed to without. For Scenario 3, pilots did not find the guidance technologies to be beneficial over having no assistance. This somewhat correlates with the measured data, since the angle-of-attack improvements with the guidance technologies were present but relatively minimal.

• [B4–B5] Pilots perceived the aural guidance technology to be less effective, while the visual guidance technologies to be more effective. This is possibly because the aural guidance technology only presents alerts when the margins are about to be exceeded, while the visual guidance technologies also inform pilots of how close the aircraft is to the margins.
• [B6–B7] Pilots perceived the aural and head-down guidance technologies to be less distracting, while the head-up guidance technology to be more distracting (even though measured performance indices were better in the head-up condition). This is, however, not a fault of the concept of head-up guidance technologies. In debriefing sessions, pilots stated that the placement of the visual cue in this particular experiment was not ideal, being located at the front and slightly upper part of their visual field. This made it challenging at times to “look past” the visual cue at the instrumentation or the outside environment.

• [B8–B9] The aural guidance technology was perceived to require less effort to use (no “looking up-and-down” required), while the head-down guidance technology was perceived to require more effort to use (a lot of “looking up-and-down” required).

• [B10] Pilots indicated that the head-up guidance technology required most improvement, mostly due to the placement of the visual cue in their field-of-view.

11.6 Key Observations

While the guidance technologies showed greater quantitative advantages in some types of scenarios as compared to others, general trends were observed that lend credence to the effectiveness of such technologies in promoting safe flight. Notably:

• The guidance technologies helped pilots to avoid exceeding the safe flight envelope more frequently, and excursions were often of much smaller magnitude and duration as compared to the no-assistance condition.

• In some cases, the visual guidance technologies allowed pilots to not only stay within the safe flight envelope, but also stay further away from the edge of the flight envelope. Measured data also suggests the potential for the visual guidance technologies to reduce the potential for pilot-induced oscillation.
• The pilots’ inputs were much more consistent and had much less variation with use of the visual guidance technologies. In particular, the head-up display resulted in much lower standard deviations for the measured state.

• Qualitative feedback from pilots was generally positive when rating the perceived benefits and utility of the guidance technologies

11.7 Experimental Challenges

As with most subject-in-the-loop experiments, a plethora of variabilities and factors exists, and these can sometimes create challenges in the execution of the experiment and the analysis/interpretation of the measured data. Summarized below are some of the challenges encountered by the research team, and the measures taken to counter them:

• **Adequacy of Training.** While pilots were familiarized with the aircraft and guidance technologies over a 15-20 minute period, one cannot consider them adequately trained in such a short amount of time. In reality, pilots would have undergone a proper training regimen that would certify them to operate a particular aircraft and the instrumentation within. Implementing an extended training session would have increased fatigue on the pilots prior to the start of the actual experiment, potentially affecting their performance. As such, each run of the experiment would have, in itself, been a learning process for the pilot participants in use of the guidance technologies.

• **Learning of Scenarios.** The pilots were observed to “learn” the scenarios with each run (this point is reinforced by the results of the post-experiment survey). For example, by the third or fourth run in Scenario 2, pilots knew that the base-to-final turn had to be commenced early to avoid overshoot of the extended runway centerline. To counterbalance the learning effect as much as possible, the order of the guidance technologies was changed
between pilots. It is worth noting that, even with the counterbalance order in place, the measured data still generally showed positive benefits from the guidance technologies.

11.8 Potential Improvements

Based on observations by the research team and feedback from the pilots, the following potential improvements were identified:

- **Aural Alert**: A series of beeps of varying intensity was suggested as more useful than a vocalized alert. In addition to providing an alert when the safe control envelope was exceeded, this would also provide information of proximity to the edge of the envelope.

- **Head-Up Display**: It was suggested that the placement of the visual cue would be better if it were moved further to the periphery of the pilot’s field-of-view, allowing unencumbered visuals of the cockpit instrumentation and outside environment, while still providing information of the safe control envelope as necessary.
CHAPTER 12
CONCLUSION

In this research, an adaptive loss-of-control (LOC) prediction architecture was developed to provide real-time estimates of an aircraft’s safe control envelope and safe control space by calculating the aircraft’s remaining control authority at any particular time, given the aircraft’s current states and inputs. This safe control margin was then presented to the pilot using aural and visual guidance technologies, such as through an augmented-reality head-up pilot advisory display. With this guidance, pilots are able to have better knowledge of the aircraft’s proximity to future impending entry into an LOC event, thereby facilitating more informed decisions in applying control inputs to the aircraft and promoting safer flight.

In the validation scenarios presented in this dissertation, the prediction architecture has shown the ability to predict, with good accuracy, the critical control inputs and trajectories that should not be exceeded if the pilot were to keep the aircraft within the safe flight envelope. The prediction architecture has demonstrated promise and good performance characteristics in a wide range of flight configurations consisting of real-world operating conditions, including those with:

- zero and non-zero initial conditions on the states and inputs,
- the aircraft operating in steady and unsteady states,
- full dynamic coupling between aircraft states,
- multiple control effectors influencing a single mode, and
- uncertainties and failures arising from non-linearities in the aircraft.

Through pilot-in-the-loop simulated flight testing, positive trends were observed that demonstrate benefits to pilots from these guidance technologies. In particular, the guidance technologies led to less occurrences of flight envelope excursion, and excursions that did occur
were comparatively small in magnitude and short in duration when compared to the no-Assistance case. Pilots were able to perform the tasked maneuvers more consistently with the visual guidance technologies, and in some cases, with less pilot-induced oscillation. Qualitative ratings of the guidance technologies were also generally positive.

The findings in this research show promising potential towards the concept of predicting an aircraft’s near-term loss-of-control margins towards the mitigation of LOC. The concepts presented thus far have encompassed several feasible approaches for online computation of an aircraft’s safe control envelope, which could change dynamically depending on multiple factors. Having this knowledge could better equip pilots during phases of flight where the aircraft is brought close to the edge of its flight envelope, such as during takeoff or landing. In the event of some failure occurring in the aircraft, the adaptive element of the architecture enables pilots to be rapidly aware of the changes to the aircraft’s flight and control envelopes, providing potentially crucial information in making operational decisions.

In closing, the author would like to emphasize the merits of LOC margin prediction towards the furtherance of safe flight.
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APPENDIX A

DERIVATION OF THE AIRCRAFT EQUATIONS OF MOTION FOR A HIGHER-ORDER ASYMMETRIC AIRCRAFT

Appendix A details the development of the equations of motion for a higher-order, fully-coupled, and asymmetric aircraft, and is intended to complement the discussion presented in Chapter 5.

A.1 The Aircraft as a Rigid Body under Newtonian Dynamics

A.1.1 Defining the Rigid Aircraft, its Coordinate Systems, and Development Assumptions

Consider the aircraft depicted in Figure A.1 below:

![Figure A.1. The Rigid Aircraft and Associated Coordinate Systems](image)

At this point, the first assumption is made – that is, the aircraft behaves as a rigid body:

**Assumption 1**: The aircraft is assumed to be a “rigid body”, which is defined to be one in which the distance between any two mass elements of the body do not change with time. Practically, this assumption suits most fixed-wing aircraft with small sizes and low weights. However, aircraft which may exhibit large amounts of elastic deformation (ie. Fuselage bending or wing deflection) in flight, or morphing aircraft which actively change in shape, would not be suited to this assumption.
APPENDIX A (continued)

Next, consider the associated reference frames used to describe the aircraft’s position and orientation. Two axis systems are depicted in the illustration. The first is the earth-fixed $X'Y'Z'$ axis system, which is non-rotating and functions as the inertial reference frame, while the second is the body-fixed $XYZ$ axis system, which is rotating and is affixed to the body of the aircraft.

The second and third assumptions are now made, that is – the aircraft is of a continuous mass distribution, and the mass of the aircraft is constant and unchanging with time.

**Assumption 2:** The aircraft is assumed to be made up of a continuous distribution of constant-density mass elements that do not change with time. Practically, this means that the effects of a shifting center-of-gravity (CG) are neglected. Causes of a shifting CG might include sloshing of fuel in the tanks, movement of passengers about the cabin, or deployment of payloads or ordnances.

**Assumption 3:** The aircraft is assumed to be of constant mass. In other words, $\dot{m} = 0$. In practical terms, this means that the effects of mass-changing events such as fuel burn is considered negligible. This is a reasonable assumption to make when considering a short period of time.

Assumptions 1, 2, and 3 fundamentally govern the derivation process of the aircraft equations of motion, and the reader is asked to keep these assumptions in mind when following the discussion below.

A.1.2 The Action of External Forces on the Aircraft

As per Assumption 2, the aircraft is assumed to be made up of a continuous distribution of mass elements, $dm$, that together make up the volume of the aircraft. For a rigid aircraft, these
elements are considered to be at a fixed distance relative to one another. Each mass element is subjected to gravitational acceleration, $\ddot{g}$, which is assumed to act along the positive $Z'$-axis. Consequently, a force of $\ddot{g} \, dm$ acts on each element. This force may be expressed in terms of the mass density of each element, according to:

$$\ddot{g} \, dm = \rho_m \ddot{g} \, dV \quad (A.1)$$

The mass elements located at the surface of the aircraft are also subjected to some combination of aerodynamic and thrust forces per unit area, according to:

$$\ddot{P} \, dA \quad (A.2)$$

Together, these two forces are assumed to make up the totality of forces that act externally on the aircraft.

**A.1.3 The Mass of the Aircraft**

Next, as per Assumption 2 once again, it is assumed that the mass elements making up the total volume of the rigid aircraft are of constant density, and that the distribution of the mass elements are constant with time. The total mass of the aircraft may then be found by integrating the mass density of each mass element over the entire volume of the aircraft:

$$m = \int_V \rho_m \, dV \quad (A.3)$$

As per Assumption 3, this total mass is assumed to stay constant (or relatively constant within a short period of time), such that $\dot{m} = 0$. With these assumptions, the total mass is said to
be located at a point defined to be the center-of-mass (CM) of the aircraft. Because the aircraft is assumed to be of continuous and uniform mass distribution, this point is also the center-of-gravity (CG) of the aircraft. This point is denoted by $C$ in Figure A.1, and is the origin of the body-fixed axis system.

### A.1.4 Newton’s Second Law

Newton’s second law states that the rate of change of an object’s linear or angular momentum is equal to the externally applied forces or moments. This change in momentum occurs in the direction of the applied force. Fundamentally, Newton’s second law may be written as:

\[
\vec{F} = \frac{d\vec{p}}{dt} = \frac{d(m\vec{v})}{dt} \tag{A.4}
\]

For physical problems where the mass of the object stays constant (or relatively constant as a function of time), mass may be treated as a constant. The variable $m$ may thus be taken out of the differentiation operation, and Newton’s second law may be expressed in terms of the object’s acceleration, $\ddot{a}$:

\[
\vec{F} = m \frac{d\vec{v}}{dt} = m\ddot{a} \tag{A.5}
\]

The familiar expression of equation (A.5) forms the fundamental starting point for the development of the aircraft equations of motion.
A.2 Development of the Aircraft Force and Moment Equations

A.2.1 Application of Newton’s Second Law to the Aircraft: Conservation of Linear and Angular Momentum

With equations (A.1) to (A.3) and equation (A.5), Newton’s second law may be used to express the aircraft equations of motion in a vector-integral form. Through the principles of conservation of linear and angular momentum, one arrives at two expressions governing the aircraft’s forces and moments.

**Forces (Conservation of Linear Momentum):**

The sum of the forces acting on the aircraft must be to equal the rate of change of linear momentum of the aircraft. This leads to equation (A.6):

\[
\frac{d}{dt} \int_V \rho_m \frac{d\vec{r}_m}{dt} \ dV = \int_V \rho_m \vec{g} \ dV + \int_A \vec{P} \ dA
\]  
(A.6)

**Moments (Conservation of Angular Momentum):**

The sum of the moments acting on the aircraft must be to equal the rate of change of angular momentum of the aircraft. This leads to equation (A.7):

\[
\frac{d}{dt} \int_V \vec{r}_m \times \rho_m \frac{d\vec{r}_m}{dt} \ dV = \int_V \vec{r}_m \times \rho_m \vec{g} \ dV + \int_A \vec{r}_m \times \vec{P} \ dA
\]  
(A.7)

The left-hand-sides of equations (A.6) and (A.7) respectively represent the time-derivative of the linear momentum and the angular momentum, while the right-hand-sides respectively represent the sum of the forces and the sum of moments acting on the aircraft. These expressions
are, in essence, a very general form of the aircraft equations of motion, and form the starting point for the derivation of the force and moment equations.

A.2.2 Cartesian Vector Definitions

The development of the force and moment equations is done in vector form to aid in conciseness. Eventually, the force and moment equations are expressed in Cartesian form as separate expressions for each of the $X$, $Y$, and $Z$ axes. The Cartesian vector definitions are thus first outlined below in Table A.1:

**TABLE A.1**

**CARTESIAN VECTOR DEFINITIONS**

<table>
<thead>
<tr>
<th>Forces</th>
<th>Moments</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\vec{F}<em>A = F</em>{A_x}\vec{i} + F_{A_y}\vec{j} + F_{A_z}\vec{k}$ \hspace{1cm} (A.8)</td>
<td>$\vec{M}_A = L_A\vec{i} + M_A\vec{j} + N_A\vec{k}$ \hspace{1cm} (A.9)</td>
</tr>
<tr>
<td>$\vec{F}<em>T = F</em>{T_x}\vec{i} + F_{T_y}\vec{j} + F_{T_z}\vec{k}$ \hspace{1cm} (A.10)</td>
<td>$\vec{M}_T = L_T\vec{i} + M_T\vec{j} + N_T\vec{k}$ \hspace{1cm} (A.11)</td>
</tr>
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<tr>
<th>Linear and Angular Velocities</th>
<th>Linear and Angular Accelerations</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\vec{V}_C = U\vec{i} + V\vec{j} + W\vec{k}$ \hspace{1cm} (A.12)</td>
<td>$\dot{\vec{V}}_C = \dot{U}\vec{i} + \dot{V}\vec{j} + \dot{W}\vec{k}$ \hspace{1cm} (A.13)</td>
</tr>
<tr>
<td>$\vec{\omega} = P\vec{i} + Q\vec{j} + R\vec{k}$ \hspace{1cm} (A.14)</td>
<td>$\dot{\vec{\omega}} = \dot{P}\vec{i} + \dot{Q}\vec{j} + \dot{R}\vec{k}$ \hspace{1cm} (A.15)</td>
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</table>

<table>
<thead>
<tr>
<th>Distances</th>
<th>Constants</th>
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<tbody>
<tr>
<td>$\vec{r} = X\vec{i} + Y\vec{j} + Z\vec{k}$ \hspace{1cm} (A.16)</td>
<td>$\vec{g} = g_x\vec{i} + g_y\vec{j} + g_z\vec{k}$ \hspace{1cm} (A.17)</td>
</tr>
</tbody>
</table>
A.2.3 Moment and Product of Inertia Definitions

The moment equations will eventually be partly expressed in terms of the aircraft moment and product of inertias. This greatly simplifies the presentation and form of the expressions. The moment and product of inertia definitions are thus first outlined below in Table A.2:

**TABLE A.2**

<table>
<thead>
<tr>
<th>Moments of Inertia</th>
<th>Products of Inertia</th>
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<tr>
<td>$I_{xx} = \int_V (Y^2 + Z^2) \rho_m , dV$ (A.18)</td>
<td>$I_{xz} = \int_V (XZ) \rho_m , dV$ (A.19)</td>
</tr>
<tr>
<td>$I_{yy} = \int_V (X^2 + Z^2) \rho_m , dV$ (A.20)</td>
<td>$I_{xy} = \int_V (XY) \rho_m , dV$ (A.21)</td>
</tr>
<tr>
<td>$I_{zz} = \int_V (X^2 + Y^2) \rho_m , dV$ (A.22)</td>
<td>$I_{yz} = \int_V (YZ) \rho_m , dV$ (A.23)</td>
</tr>
</tbody>
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A.2.4 The Force Equations

Recall again the conservation of linear momentum expression given in equation (A.6) (presented once again in equation (A.24)):

$$\frac{d}{dt} \int_V \rho_m \frac{d\vec{r}_m}{dt} \, dV = \int_V \rho_m \vec{g} \, dV + \int_A \vec{P} \, dA$$ (A.24)

Through the steps outlined below, the reader is asked to keep in mind that the ultimate goal is to express equation (A.24) in the form specified in equation (A.25). In doing this, the position
vectors on the left-hand-side of equation (A.24) need to be expressed with reference to the aircraft’s CG (as opposed to the origin of the inertial reference frame).

\[ m \frac{d\vec{V}_C}{dt} = m\vec{g} + \vec{F}_A + \vec{F}_T \]  \hspace{1cm} (A.25)

The expression in equation (A.25) may then be further expanded to obtain the conservation of linear momentum nonlinear ordinary differential equations (ie. “the force equations”).

**A.2.4.1 Right-Hand-Side**

In expanding the right-hand-side, note that the integral of \( \int_V \rho_m \ dV \) is equivalent to the aircraft’s mass. The right-hand-side of equation (A.24) is thus equivalent to: 1) the force applied on the mass as a result of gravitational acceleration \( (m\vec{g}) \), and 2) aerodynamic and thrust forces \( (\vec{F}_A \text{ and } \vec{F}_T \text{ respectively}) \):

\[ \int_V \rho_m\vec{g} \ dV = m\vec{g} \] \hspace{1cm} (A.26)

\[ \int_A \vec{P} \ dA = \vec{F}_A + \vec{F}_T \] \hspace{1cm} (A.27)

**A.2.4.2 Left-Hand-Side**

An intermediate step in obtaining the force equations involves expressing the left-hand-side of equation (A.24) in terms of the position vector \( \vec{r}_C \), as opposed to \( \vec{r}_m \). In expanding the left-hand-side of equation (A.24), one therefore needs to obtain an expression for \( \vec{r}_C \). The derivative of this expression is then, \( \dot{\vec{r}}_C \) which is \( d\vec{V}_C/dt \).
From Figure A.1, recall that the origin of the body-fixed axis coincides with the CG of the aircraft. Noting that the position vector \( \vec{r} \) is the distance of a generic point of the aircraft with respect to the aircraft’s center of gravity, the integral specified in equation (A.28) must be equivalent to zero (when the integral is evaluated over the volume of the entire aircraft and subject to Assumptions 2 and 3 being satisfied):

\[
\int_V \vec{r} \rho_m \, dV = 0 \tag{A.28}
\]

This is an important property that will be used in the derivation of both the force and moment equations. Next, realize from Figure A.1 that \( \vec{r} = \vec{r}_m - \vec{r}_c \). Therefore, equation (A.28) may be written as:

\[
\int_V (\vec{r}_m - \vec{r}_c) \rho_m \, dV = 0 \tag{A.29}
\]

\[
\int_V \vec{r}_m \rho_m \, dV = \int_V \vec{r}_c \rho_m \, dV = \int_V \rho_m \, dV = m \vec{r}_c \tag{A.30}
\]

Rearranging equation (A.30) yields the following:

\[
\vec{r}_c = \frac{1}{m} \int_V \vec{r}_m \rho_m \, dV \tag{A.31}
\]

Now, returning to the left-hand-side of equation (A.24), and considering the implication of equation (A.28), one may express the left-hand-side as:
Finally, considering the definition of mass given in equation (A.3), one arrives at the following expression for the left-hand-side of equation (A.24). Described practically, the left-hand-side is equivalent to the aircraft mass multiplied by the time-derivative of the mass’ velocity at the origin of the body-fixed axis.

\[
\frac{d}{dt} \int_V \rho_m \frac{d\vec{r}_m}{dt} \, dV = \frac{d}{dt} \int_V \rho_m \vec{r}_m \, dV = \frac{d}{dt} \int_V \rho_m (\vec{r}_c + \vec{r}) \, dV = \frac{d}{dt} \int_V \rho_m \vec{r}_c \, dV
\] (A.32)

A.2.4.3 The Assembled Force Equation

Equating equation (A.33) to equations (A.26) and (A.27), the force equation may finally be written as in equation (A.34):

\[
m \frac{d\vec{V}_c}{dt} = m\vec{g} + \vec{F}_A + \vec{F}_T
\] (A.34)

The derivation has thus far used the earth-fixed axis as a frame of reference – the velocity component in equation (A.33) is that of the aircraft with respect to the inertial reference frame. Because the goal is to describe the aircraft’s motion with respect to the body-fixed axis, the relative motion of the body-fixed axis in relation to the earth-fixed axis must be considered. Therefore:

\[
\frac{d\vec{V}_c}{dt} = \frac{\partial \vec{V}_c}{\partial t} + \vec{\omega} \times \vec{V}_c
\] (A.35)
Substituting equation (A.35) into equation (A.34), the force equation then takes on the form:

\[ m \left( \frac{\partial \vec{V}_C}{\partial t} + \vec{\omega} \times \vec{V}_C \right) = m \vec{g} + \vec{F}_A + \vec{F}_T \]  \hspace{1cm} (A.36)

Evaluating the cross product of the angular velocity and linear velocity, one obtains:

\[ \vec{\omega} \times \vec{V}_C = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ P & Q & R \\ U & V & W \end{vmatrix} = (WQ - VR)\vec{i} - (PW - UR)\vec{j} + (VP - UQ)\vec{k} \]  \hspace{1cm} (A.37)

Finally, together with the definitions specified in Table A.1, the force equations may be written as:

\[ m(\dot{U} - VR + WQ) = mg_x + F_{Ax} + F_{Tx} \]  \hspace{1cm} (A.38)

\[ m(\dot{V} + UR - WP) = mg_y + F_{Ay} + F_{Ty} \]  \hspace{1cm} (A.39)

\[ m(\dot{W} - UQ + VP) = mg_z + F_{Az} + F_{Tz} \]  \hspace{1cm} (A.40)

Equations (A.38) to (A.40) are also known as the conservation of linear momentum equations, and represent a system of nonlinear ordinary differential equations (ODE) that may be solved for the component velocities of the aircraft. The solution of these ODEs also requires
knowledge of the angular velocities of the aircraft – resolving for the angular velocities requires
the development of the aircraft moment equations.

A.2.5 The Moment Equations

Recall again the conservation of angular momentum expression given in equation (A.7)
(presented once again in equation (A.41)):

\[
\frac{d}{dt} \int_V \vec{r}_m \times \rho_m \frac{d\vec{r}_m}{dt} \ dV = \int_V \vec{r}_m \times \rho_m \vec{g} \ dV + \int_A \vec{r}_m \times \vec{P} \ dA \quad (A.41)
\]

The intermediate goal is to write equation (A.41) in terms of the moment arm from the
aircraft CG to each mass element \(dm\) (as opposed to the moment arm from the origin of the inertial
reference frame to each mass element, as is the case in equation (A.41)). As such, the preliminary
expansion seeks to obtain an expression of the form given in equation (A.42):

\[
\frac{d}{dt} \int_V \vec{r} \times \rho_m \frac{d\vec{r}}{dt} \ dV = \int_A \vec{r} \times \vec{P} \ dA \quad (A.42)
\]

The expression in equation (A.42) may then be expanded into the form given by equation
(A.43):

\[
\int_V \left( \left[ \vec{r} \times (\vec{\omega} \cdot \vec{r}) \right]_1 + \left[ \vec{\omega} \cdot (\vec{r} \cdot \vec{r}) \right]_2 - \left[ \vec{r} \cdot \vec{\omega} \right] \right) \rho_m \ dV = \vec{M}_A + \vec{M}_T \quad (A.43)
\]

Finally, the cross-products in equation (A.43) may be expanded to obtain the conservation
of angular momentum nonlinear ordinary differential equations (ie. “the moment equations”).

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A.2.5.1 Preliminary Expansion

To begin with, note from Figure A.1 the vector relationship \( \vec{r}_m = \vec{r}_C + \vec{r} \). Introducing this relationship into equation (A.41) leads to:

\[
\frac{d}{dt} \int_V \left( (\vec{r}_C + \vec{r}) \times \rho_m \frac{d(\vec{r}_C + \vec{r})}{dt} \right) dV = \int_V (\vec{r}_C + \vec{r}) \times \rho_m \vec{g} \ dV + \int_A (\vec{r}_C + \vec{r}) \times \vec{P} \ dA \quad (A.44)
\]

Now, the left-hand-side of equation (A.44) may be expanded to become:

\[
\frac{d}{dt} \int_V (\vec{r}_C + \vec{r}) \times \rho_m \frac{d(\vec{r}_C + \vec{r})}{dt} \ dV = \left[ \frac{d}{dt} \int_V (\vec{r}_C \times \frac{d\vec{r}_C}{dt}) \rho_m \ dV \right]_{LHS1} + \left[ \frac{d}{dt} \int_V (\vec{r}_C \times \frac{d\vec{r}}{dt}) \rho_m \ dV \right]_{LHS2} + \left[ \frac{d}{dt} \int_V (\vec{r} \times \frac{d\vec{r}_C}{dt}) \rho_m \ dV \right]_{LHS3} + \left[ \frac{d}{dt} \int_V (\vec{r} \times \frac{d\vec{r}}{dt}) \rho_m \ dV \right]_{LHS4} \quad (A.45)
\]

Also, the right-hand-side of equation (A.44) may be expanded to become:

\[
\int_V (\vec{r}_C + \vec{r}) \times \rho_m \vec{g} \ dV + \int_A (\vec{r}_C + \vec{r}) \times \vec{P} \ dA = \left[ \int_V (\vec{r}_C \times \vec{g}) \rho_m \ dV \right]_{RHS1} + \left[ \int_V (\vec{r} \times \vec{g}) \rho_m \ dV \right]_{RHS2} + \left[ \int_A \vec{r}_C \times \vec{P} \ dA \right]_{RHS3} + \left[ \int_A \vec{r} \times \vec{P} \ dA \right]_{RHS4} \quad (A.46)
\]

Through the steps outlined below, it will be shown that LHS1 will be equivalent to RHS1 and RHS 3 added together, and that LHS2, LHS 3, and RHS2 will be equivalent to zero. The net result will be that LHS4 is equivalent to RHS4.

A.2.5.1.1 Bracket LHS1

Using the product rule, Bracket LHS1 may be expanded as follows:
APPENDIX A (continued)

\[
\frac{d}{dt} \int_V \left( \vec{r}_c \times \frac{d\vec{r}_c}{dt} \right) \rho_m \, dV = \int_V \left( \frac{d\vec{r}_c}{dt} \times \frac{d\vec{r}_c}{dt} \right) \rho_m \, dV + \int_V \left( \vec{r}_c \times \frac{d}{dt} \left( \frac{d\vec{r}_c}{dt} \right) \right) \rho_m \, dV
\]

(A.47)

\[
= \int_V \left( \vec{r}_c \times \frac{d\vec{V}_c}{dt} \right) \rho_m \, dV
\]

Since the position vector \( \vec{r}_c \) is independent of the volume and mass of the aircraft, it may be removed from the integrand and evaluated outside the integral. Noting also that the integral of \( \int_V \rho_m \, dV \) is equivalent to the aircraft’s mass, equation (A.47) then becomes:

\[
\int_V \left( \vec{r}_c \times \frac{d\vec{V}_c}{dt} \right) \rho_m \, dV = \vec{r}_c \times \int_V \frac{d\vec{V}_c}{dt} \rho_m \, dV = \vec{r}_c \times m \frac{d\vec{V}_c}{dt}
\]

(A.48)

Now, note from conservation of linear momentum and from equation (A.27) and (A.34) the following:

\[
m \frac{d\vec{V}_c}{dt} = m \vec{g} + \int_A \vec{P} \, dA
\]

(A.49)

Substituting equation (A.49) into equation (A.48) allows Bracket LHS1 to be expressed as follows:

\[
\vec{r}_c \times m \frac{d\vec{V}_c}{dt} = \vec{r}_c \times \left( m \vec{g} + \int_A \vec{P} \, dA \right) = \vec{r}_c \times m \vec{g} + \vec{r}_c \times \int_A \vec{P} \, dA
\]

(A.50)

A.2.5.1.2 Bracket LHS2

Using the product rule, Bracket LHS2 may be expanded as follows:

\[
\frac{d}{dt} \int_V \left( \vec{r}_c \times \frac{d\vec{r}_c}{dt} \right) \rho_m \, dV = \int_V \left( \frac{d\vec{r}_c}{dt} \times \frac{d\vec{r}_c}{dt} \right) \rho_m \, dV + \int_V \left( \vec{r}_c \times \frac{d}{dt} \left( \frac{d\vec{r}_c}{dt} \right) \right) \rho_m \, dV
\]

(A.51)
Once again, the position vector \( \mathbf{r}_c \) is independent of the volume and mass of the aircraft, and may thus be evaluated outside the integral. As before, note also that the position vector \( \mathbf{r} \) is the distance of a generic point of the aircraft with respect to the aircraft’s center of gravity. Thus, the integral of \( \int_V \mathbf{r} \rho_m \, dV \) must be equivalent to zero, when the integral is evaluated over the volume of the entire aircraft. This is, of course, subject to Assumptions 2 and 3 being satisfied with regards to the aircraft’s mass and density distribution. With these in mind, equation (A.51) then becomes:

\[
\frac{d\mathbf{r}_c}{dt} \times \int_V \frac{d\mathbf{r}}{dt} \rho_m \, dV + \mathbf{r}_c \times \int_V \frac{d}{dt} \left( \frac{d\mathbf{r}}{dt} \right) \rho_m \, dV = \frac{d\mathbf{r}_c}{dt} \times \frac{d}{dt} \left( \int_V \mathbf{r} \rho_m \, dV \right) + \mathbf{r}_c \times \frac{d}{dt} \left( \int_V \mathbf{r} \rho_m \, dV \right) = 0
\] (A.52)

### A.2.5.1.3 Bracket LHS3

Using the product rule, Bracket LHS3 may be expanded as follows:

\[
\frac{d}{dt} \int_V \left( \mathbf{r} \times \frac{d\mathbf{r}_c}{dt} \right) \rho_m \, dV = \int_V \left( \frac{d\mathbf{r}}{dt} \times \frac{d\mathbf{r}_c}{dt} \right) \rho_m \, dV + \int_V \left( \mathbf{r} \times \frac{d}{dt} \left( \frac{d\mathbf{r}_c}{dt} \right) \right) \rho_m \, dV
\] (A.53)

Now, recall the non-commutative property of vector multiplication, which states that

\( \mathbf{R}_1 \times \mathbf{R}_2 = -\mathbf{R}_2 \times \mathbf{R}_1 \). Also, recall that \( \int_V \mathbf{r} \rho_m \, dV = 0 \). With this in mind, equation (A.53) may be expanded to show that Bracket LHS3 reduces to zero:

\[
-\frac{d\mathbf{r}_c}{dt} \times \int_V \frac{d\mathbf{r}}{dt} \rho_m \, dV - \frac{d}{dt} \left( \frac{d\mathbf{r}_c}{dt} \right) \times \int_V \mathbf{r} \rho_m \, dV = -\frac{d\mathbf{r}_c}{dt} \times \frac{d}{dt} \left( \int_V \mathbf{r} \rho_m \, dV \right) - \frac{d}{dt} \left( \frac{d\mathbf{r}_c}{dt} \right) \times \int_V \mathbf{r} \rho_m \, dV = 0
\] (A.54)
A.2.5.1.4 Brackets RHS1 and RHS3

Recall that the position vector \( \vec{r}_c \) is independent of the volume and mass of the aircraft, and may thus be evaluated outside the integral. Bracket RHS1 and RHS3 may thus be expressed as follows:

\[
\int_V (\vec{r}_c \times \vec{g}) \rho_m \, dV + \int_A \vec{r}_c \times \vec{P} \, dA = \vec{r}_c \times \int_V \vec{g} \rho_m \, dV + \vec{r}_c \times \int_A \vec{P} \, dA \\
= \vec{r}_c \times m \vec{g} + \vec{r}_c \times \int_A \vec{P} \, dA \\
\]

(A.55)

Observe that Brackets RHS1 and RHS3 are equivalent to Bracket LHS1. These terms thus cancel out.

A.2.5.1.5 Bracket RHS2

In expanding Bracket RHS2, observe the non-commutative property of vector multiplication, and also recall that \( \int_V \vec{r} \rho_m \, dV = 0 \). With this in mind, it may be shown that Bracket RHS2 reduces to zero:

\[
\int_V (\vec{r} \times \vec{g}) \rho_m \, dV = \int_V (-\vec{g} \times \vec{r}) \rho_m \, dV = -\vec{g} \times \int_V \vec{r} \rho_m \, dV = 0 \\
\]

(A.56)

A.2.5.2 The Moment Equations

The remaining terms in the expression show that Bracket LHS4 is equivalent to Bracket RHS4. Equating the two terms leads to:

\[
\frac{d}{dt} \int_V \vec{r}_c \times \rho_m \frac{d\vec{r}_c}{dt} \, dV = \int_A \vec{r}_c \times \vec{P} \, dA \\
\]

(A.57)
A.2.5.2.1 Right-Hand-Side

With equation (A.27) in mind, the right-hand-side of equation (A.57) may be expanded as follows:

\[ \int_A \vec{r} \times \vec{P} \ dA = \vec{r} \times \int_A \vec{P} \ dA = \vec{r} \times (\vec{F}_A + \vec{F}_T) = \vec{M}_A + \vec{M}_T \]  
(A.58)

A.2.5.2.2 Left-Hand-Side

As with the force equation, the derivation has thus far used the earth-fixed axis as a frame of reference. Because the goal is to describe the aircraft motion with respect to the body-fixed axis, the relative motion of the body-fixed axis in relation to the earth-fixed axis must be considered. Additionally, with the moment equation, there is a need to address the time-dependency of the integral. Using the product rule, the external time-derivative may be “absorbed” into the integral, as follows:

\[ \frac{d}{dt} \int_V \vec{r} \times \rho_m \frac{d\vec{r}}{dt} \ dV = \int_V \vec{r} \times \rho_m \frac{d\vec{r}}{dt} \ dV + \int_V \vec{r} \times \rho_m \frac{d}{dt} \left( \frac{d\vec{r}}{dt} \right) \ dV \]  
(A.59)

Since the cross product of a vector and itself amounts to zero, the left-hand-side of equation (A.57) then reduces to:

\[ \frac{d}{dt} \int_V \vec{r} \times \rho_m \frac{d\vec{r}}{dt} \ dV = \int_V \vec{r} \times \rho_m \frac{d}{dt} \left( \frac{d\vec{r}}{dt} \right) \ dV = \int_V \left( \vec{r} \times \frac{d}{dt} \left( \frac{d\vec{r}}{dt} \right) \right) \rho_m \ dV \]  
(A.60)

Equation (A.60) presents itself as an integral with two vector-based time-derivatives within the integrand. The evaluation of these two derivatives will be performed in succession. For the first derivative:
Next, for the second derivative:

\[
\frac{d\ddot{r}}{dt} = \frac{\partial \ddot{r}}{\partial t} + \dddot{\omega} \times \dddot{r} = \dddot{r} + \dddot{\omega} \times \dddot{r} \tag{A.61}
\]

Using the vector triple product identity (ie. The “BAC-CAB” rule), the double cross product in equation (A.64) may be written as \(\dddot{\omega} \times (\dddot{\omega} \times \dddot{r}) = \dddot{\omega}(\dddot{\omega} \cdot \dddot{r}) - \dddot{r}(\dddot{\omega} \cdot \dddot{\omega})\), leading to:

\[
\frac{d}{dt} \left( \frac{d\ddot{r}}{dt} \right) = \dddot{r} + \dddot{\omega} \times \dddot{r} + 2\dddot{\omega} \times \dddot{\omega} \times \dddot{r} + \dddot{\omega} \times (\dddot{\omega} \times \dddot{r}) \tag{A.62}
\]

\[
\frac{d}{dt} \left( \frac{d\ddot{r}}{dt} \right) = \dddot{r} + \dddot{\omega} \times \dddot{r} + 2\dddot{\omega} \times \dddot{\omega} \times \dddot{r} + \dddot{\omega} \times (\dddot{\omega} \times \dddot{r}) \tag{A.63}
\]

\[
\frac{d}{dt} \left( \frac{d\ddot{r}}{dt} \right) = \dddot{r} + \dddot{\omega} \times \dddot{r} + 2\dddot{\omega} \times \dddot{\omega} \times \dddot{r} + \dddot{\omega} \times (\dddot{\omega} \times \dddot{r}) \tag{A.64}
\]

Using the vector triple product identity (ie. The “BAC-CAB” rule), the double cross product in equation (A.64) may be written as \(\dddot{\omega} \times (\dddot{\omega} \times \dddot{r}) = \dddot{\omega}(\dddot{\omega} \cdot \dddot{r}) - \dddot{r}(\dddot{\omega} \cdot \dddot{\omega})\), leading to:

\[
\frac{d}{dt} \left( \frac{d\ddot{r}}{dt} \right) = \dddot{r} + \dddot{\omega} \times \dddot{r} + 2\dddot{\omega} \times \dddot{\omega} \times \dddot{r} + \dddot{\omega} \times (\dddot{\omega} \times \dddot{r}) \tag{A.65}
\]

Next, recall Assumption 2, which states that the aircraft is made up of a continuous distribution of constant-density mass elements that are invariant with time. This implies that \(\dddot{r} = \dddot{\omega} = 0\). Equation (A.65) then simplifies to the following:

\[
\frac{d}{dt} \left( \frac{d\ddot{r}}{dt} \right) = \dddot{\omega} \times \dddot{r} + \dddot{\omega}(\dddot{\omega} \cdot \dddot{r}) - \dddot{r}(\dddot{\omega} \cdot \dddot{\omega}) \tag{A.66}
\]
Now, consider again equation (A.60). The cross product of \( \mathbf{r} \) with the double time-derivative of \( \mathbf{r} \) next needs to be evaluated.

\[
\mathbf{r} \times \left( \frac{d}{dt} \frac{d\mathbf{r}}{dt} \right) = \mathbf{r} \times \left( \dot{\mathbf{\omega}} \times \mathbf{r} + \mathbf{\omega} (\mathbf{r} \cdot \dot{\mathbf{\omega}}) - \dot{\mathbf{r}} (\mathbf{r} \cdot \dot{\mathbf{\omega}}) \right)
\]

\[= \left[ \mathbf{r} \times \left( \dot{\mathbf{\omega}} \times \mathbf{r} \right) \right]_1 + \left[ \mathbf{r} \times \left( \mathbf{\omega} (\mathbf{r} \cdot \dot{\mathbf{\omega}}) \right) \right]_2 - \left[ \mathbf{r} \times \left( \dot{\mathbf{r}} (\mathbf{r} \cdot \dot{\mathbf{\omega}}) \right) \right]_3 \tag{A.67}
\]

Using the vector triple product identity, the terms in bracket 1 may be written as
\[\mathbf{r} \times \left( \dot{\mathbf{\omega}} \times \mathbf{r} \right) = \dot{\mathbf{\omega}} (\mathbf{r} \cdot \dot{\mathbf{\omega}}) - \dot{\mathbf{r}} (\mathbf{r} \cdot \dot{\mathbf{\omega}}).\]

The terms in bracket 3 reduce to zero, because of the cross product of \( \mathbf{r} \) with itself. Thus, equation (A.67) reduces in complexity, and the left-hand-side of equation (A.57) may now be expressed as follows:

\[
\frac{d}{dt} \int_V \mathbf{r} \times \rho_m \frac{d\mathbf{r}}{dt} \ dV = \int_V \left( \left[ \mathbf{r} \times \left( \mathbf{\omega} (\mathbf{r} \cdot \dot{\mathbf{\omega}}) \right) \right]_1 + \left[ \mathbf{r} \times \left( \mathbf{\omega} (\mathbf{r} \cdot \dot{\mathbf{\omega}}) \right) \right]_2 - \left[ \mathbf{r} \times \left( \dot{\mathbf{r}} (\mathbf{r} \cdot \dot{\mathbf{\omega}}) \right) \right]_2 \right) \rho_m \ dV \tag{A.68}
\]

### A.2.5.2.3 The Assembled Moment Equation

Equating equation (A.68) to equation (A.58), the moment equation may finally be written as in equation (A.69):

\[
\int_V \left( \left[ \mathbf{r} \times \left( \mathbf{\omega} (\mathbf{r} \cdot \dot{\mathbf{\omega}}) \right) \right]_1 + \left[ \mathbf{r} \times \left( \mathbf{\omega} (\mathbf{r} \cdot \dot{\mathbf{\omega}}) \right) \right]_2 - \left[ \mathbf{r} \times \left( \dot{\mathbf{r}} (\mathbf{r} \cdot \dot{\mathbf{\omega}}) \right) \right]_2 \right) \rho_m \ dV = \mathbf{M}_A + \mathbf{M}_T \tag{A.69}
\]

At this point, the cross and dot products in equation (A.69) may be evaluated. The expansions are performed individually for the terms in each of the three brackets. For bracket 1:

\[
\mathbf{r} \times \left( \mathbf{\omega} (\mathbf{r} \cdot \dot{\mathbf{\omega}}) \right) = \begin{vmatrix}
\mathbf{i} & \mathbf{j} & \mathbf{k} \\
X & Y & Z \\
P^2X + PQY + PRZ & PQX + Q^2Y + QRZ & PRX + QRY + R^2Z
\end{vmatrix} \tag{A.70}
\]
APPENDIX A (continued)

\[
\mathbf{r} \times (\mathbf{\ddot{\omega}}(\mathbf{\dot{\omega}} \cdot \mathbf{r})) \\
= [QRY^2 - QRZ^2 + PRXY - PQXZ + (-Q^2 + R^2)YZ]\mathbf{i} \\
+ [-PRX^2 + PRZ^2 - QRXY + (P^2 - R^2)XZ + PQYZ]\mathbf{j} \\
+ [PQX^2 - PQY^2 + (-P^2 + Q^2)XY + QRXZ - PRYZ]\mathbf{k}
\]  \hspace{1cm} \text{(A.71)}

For bracket 2:

\[
\mathbf{\ddot{\omega}}(\mathbf{\dot{r}} \cdot \mathbf{r}) = (\dot{P}\mathbf{i} + \dot{Q}\mathbf{j} + \dot{R}\mathbf{k}) \left( (X\mathbf{i} + Y\mathbf{j} + Z\mathbf{k}) \cdot (X\mathbf{i} + Y\mathbf{j} + Z\mathbf{k}) \right)
\]  \hspace{1cm} \text{(A.72)}

\[
\mathbf{\ddot{\omega}}(\mathbf{\dot{r}} \cdot \mathbf{r}) = \left[ \dot{P}(X^2 + Y^2 + Z^2) \right]\mathbf{i} + \left[ \dot{Q}(X^2 + Y^2 + Z^2) \right]\mathbf{j} + \left[ \dot{R}(X^2 + Y^2 + Z^2) \right]\mathbf{k}
\]  \hspace{1cm} \text{(A.73)}

For bracket 3:

\[
\mathbf{\dot{r}}(\mathbf{\dot{r}} \cdot \mathbf{\ddot{\omega}}) = (X\mathbf{i} + Y\mathbf{j} + Z\mathbf{k}) \left( (X\mathbf{i} + Y\mathbf{j} + Z\mathbf{k}) \cdot (\dot{P}\mathbf{i} + \dot{Q}\mathbf{j} + \dot{R}\mathbf{k}) \right)
\]  \hspace{1cm} \text{(A.74)}

\[
\mathbf{\dot{r}}(\mathbf{\dot{r}} \cdot \mathbf{\ddot{\omega}}) = \left[ \dot{P}X^2 + QXY + RXZ \right]\mathbf{i} + \left[ \dot{P}XY + QY^2 + RYZ \right]\mathbf{j} + \left[ \dot{P}XZ + QYZ + RZ^2 \right]\mathbf{k}
\]  \hspace{1cm} \text{(A.75)}

Now, consider the moment and product of inertia definitions given in Table A.2. The goal is now to express the \(X\), \(Y\), and \(Z\) terms as a function of the aircraft rigid body inertias. As an intermediate step towards this goal, equation (A.68) may be written as follows:
Finally, to match the definitions of $I_{xx}$, $I_{yy}$, and $I_{zz}$, the coefficients of $X^2$, $Y^2$, and $Z^2$ need to be reformulated as functions of $(Y^2 + Z^2)$, $(X^2 + Z^2)$, and $(X^2 + Y^2)$. This then leads to equation (A.77):

\[
\begin{align*}
\int_V (Y^2)\rho_m \ dV \left[ (+QR - QR)\hat{i} + (-PR + Q)\hat{j} + (PQ + \hat{R})\hat{k} \right] + \\
\int_V (Z^2)\rho_m \ dV \left[ (QR + \hat{P})\hat{i} + (+PR - PR)\hat{j} + (-PQ + \hat{R})\hat{k} \right] + \\
\int_V (X^2)\rho_m \ dV \left[ (-QR + \hat{P})\hat{i} + (PR + \hat{Q})\hat{j} + (+PQ - PQ)\hat{k} \right] + \\
\int_V (XY)\rho_m \ dV \left[ (PR - \hat{Q})\hat{i} + (-QR - \hat{P})\hat{j} + (-P^2 + Q^2)\hat{k} \right] + \\
\int_V (XZ)\rho_m \ dV \left[ (-PR - \hat{R})\hat{i} + (P^2 - R^2)\hat{j} + (QR - \hat{P})\hat{k} \right] + \\
\int_V (YZ)\rho_m \ dV \left[ (-Q^2 + R^2)\hat{i} + (PQ - \hat{R})\hat{j} + (-PR - \hat{Q})\hat{k} \right] = \vec{M}_A + \vec{M}_T
\end{align*}
\]

Finally, to match the definitions specified in Table A.2, the moment equations may be written as:

\[
\begin{align*}
\int_V (Y^2 + Z^2)\rho_m \ dV \left[ (\hat{P})\hat{i} + (PR)\hat{j} + (-PQ)\hat{k} \right] + \\
\int_V (X^2 + Z^2)\rho_m \ dV \left[ (-QR)\hat{i} + (\hat{Q})\hat{j} + (PQ)\hat{k} \right] + \\
\int_V (X^2 + Y^2)\rho_m \ dV \left[ (QR)\hat{i} + (-PR)\hat{j} + (\hat{R})\hat{k} \right] + \\
\int_V (XY)\rho_m \ dV \left[ (PR - \hat{Q})\hat{i} + (-QR - \hat{P})\hat{j} + (-P^2 + Q^2)\hat{k} \right] + \\
\int_V (XZ)\rho_m \ dV \left[ (-PQ - \hat{R})\hat{i} + (P^2 - R^2)\hat{j} + (QR - \hat{P})\hat{k} \right] + \\
\int_V (YZ)\rho_m \ dV \left[ (-Q^2 + R^2)\hat{i} + (PQ - \hat{R})\hat{j} + (-PR - \hat{Q})\hat{k} \right] = \vec{M}_A + \vec{M}_T
\end{align*}
\]
Equations (A.78) to (A.80) are also known as the conservation of angular momentum equations, and represent a system of nonlinear ordinary differential equations (ODE) that may be solved for the component angular velocities of the aircraft.

A.3 Development of the Aircraft Kinematic and Gravity Equations

While the six equations of motion developed in the previous section describe the translational and rotational motion of the aircraft, a further set of expressions is needed to describe its angular orientation. These expressions are known as the Kinematic Equations, and their development is discussed in this section.

A.3.1 Defining the Angular Orientation of the Aircraft

Recall the earth-fixed $X'Y'Z'$ and body-fixed $XYZ$ axis systems introduced in the previous section. As is implied in its name, the $X'Y'Z'$ axis is fixed to the earth, and is thus inertial and non-rotating. The $XYZ$ axis is fixed to the body of the aircraft, and thus rotates with the kinematic
motion of the aircraft. The angular orientation of the body-fixed axis is described using the Euler Angles $\Psi$, $\Theta$, and $\Phi$, wherein $\Psi$ represents the heading angle, $\Theta$ represents the pitch angle, and $\Phi$ represents the bank angle. The sequential rotation that is used for the earth-to-body axis transformation is described in Figure A.2:

Figure A.2. Defining the Angular Orientation of the Aircraft

The angular orientation of the body-fixed axis is described through three sequential rotations, which are the:

1) First rotation over heading angle $\Psi$ about unit axis $\vec{k}_1$.

2) Second rotation over pitch angle $\Theta$ about unit axis $\vec{j}_2$.

3) Third rotation over bank angle $\Phi$ about unit axis $\vec{i}_3$.

Note that the unrotated body-fixed axis system is defined to be parallel to the earth-fixed axis system. Thus, to quantify the angular orientation of the body-fixed axis with respect to the earth-fixed axis is equivalent to quantifying the angular orientation of the body-fixed axis itself. The unity axis $(\vec{i}, \vec{j}, \vec{k})_1$ is equivalent in orientation to the body-fixed axis $XYZ$. 

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A.3.1.1 First Rotation

The body-fixed axis \( XYZ \), or \((\vec{i}, \vec{j}, \vec{k})_1\), is first rotated over heading angle \( \Psi \) about unit axis \( \vec{k}_1 \). The resulting intermediate axis is labeled \((\vec{i}, \vec{j}, \vec{k})_2\), as illustrated in Figure A.3:

![Figure A.3. First Euler Rotation (\( \Psi \)) about \( \vec{k}_1 \)](image)

The first rotation is mathematically described by equation (A.81):

\[
\begin{bmatrix}
\vec{i}_1 \\
\vec{j}_1 \\
\vec{k}_1
\end{bmatrix} =
\begin{bmatrix}
\cos \Psi & -\sin \Psi & 0 \\
\sin \Psi & \cos \Psi & 0 \\
0 & 0 & 1
\end{bmatrix}
\begin{bmatrix}
\vec{i}_2 \\
\vec{j}_2 \\
\vec{k}_2
\end{bmatrix}
\]

(A.81)

A.3.1.2 Second Rotation

The intermediate axis \((\vec{i}, \vec{j}, \vec{k})_2\), is then rotated over pitch angle \( \Theta \) about unit axis \( \vec{j}_2 \). The resulting intermediate axis is labeled \((\vec{i}, \vec{j}, \vec{k})_3\), as illustrated in Figure A.4:
Figure A.4. Second Euler Rotation ($\Theta$) about $\vec{j}_2$

The second rotation is mathematically described by equation (A.82):

$$
\begin{bmatrix}
\vec{i}_2 \\
\vec{j}_2 \\
\vec{k}_2
\end{bmatrix} =
\begin{bmatrix}
\cos \theta & 0 & \sin \theta \\
0 & 1 & 0 \\
-\sin \theta & 0 & \cos \theta
\end{bmatrix}
\begin{bmatrix}
\vec{i}_3 \\
\vec{j}_3 \\
\vec{k}_3
\end{bmatrix}
$$

(A.82)

By substituting equation (A.82) into equation (A.81), the first two rotations may be pre-multiplied as follows:

$$
\begin{bmatrix}
\vec{i}_1 \\
\vec{j}_1 \\
\vec{k}_1
\end{bmatrix} =
\begin{bmatrix}
\cos \Psi & -\sin \Psi & 0 \\
\sin \Psi & \cos \Psi & 0 \\
0 & 0 & 1
\end{bmatrix}
\begin{bmatrix}
\cos \theta & 0 & \sin \theta \\
0 & 1 & 0 \\
-\sin \theta & 0 & \cos \theta
\end{bmatrix}
\begin{bmatrix}
\vec{i}_3 \\
\vec{j}_3 \\
\vec{k}_3
\end{bmatrix}
$$

(A.83)

A.3.1.3 Third Rotation

The intermediate axis ($\vec{i}, \vec{j}, \vec{k}$), is finally rotated over bank angle $\Phi$ about unit axis $\vec{i}_3$. The final rotated axis is labeled ($\vec{i}, \vec{j}, \vec{k}$), as illustrated in Figure A.5:
APPENDIX A (continued)

Figure A.5. Third Euler Rotation (Φ) about →i₃

The third rotation is mathematically described by equation (A.84):

\[
\begin{bmatrix}
→i₃ \\
→j₃ \\
→k₃
\end{bmatrix} =
\begin{bmatrix}
1 & 0 & 0 \\
0 & \cos Φ & -\sin Φ \\
0 & \sin Φ & \cos Φ
\end{bmatrix}
\begin{bmatrix}
→i \\
→j \\
→k
\end{bmatrix}
\] (A.84)

By substituting equation (A.84) into equation (A.83), the three rotations may be expressed as follows:

\[
\begin{bmatrix}
→i₁ \\
→j₁ \\
→k₁
\end{bmatrix} =
\begin{bmatrix}
\cos Ψ & -\sin Ψ & 0 \\
\sin Ψ & \cos Ψ & 0 \\
0 & 0 & 1
\end{bmatrix}
\begin{bmatrix}
\cos Θ & 0 & \sin Θ \\
0 & 1 & 0 \\
-\sin Θ & 0 & \cos Θ
\end{bmatrix}
\begin{bmatrix}
1 & 0 & 0 \\
0 & \cos Φ & -\sin Φ \\
0 & \sin Φ & \cos Φ
\end{bmatrix}
\begin{bmatrix}
→i \\
→j \\
→k
\end{bmatrix}
\] (A.85)

Equation (A.85) represents the rotational relationship between the body-fixed axis \((→i, →j, →k)\) and the earth-fixed axis \((→i, →j, →k)₁\), expressed using the unit axes. Through the appropriate pre-multiplications, this general relationship may be used to derive the flight path equations, kinematic equations, and gravity equations.
A.3.2 Defining the Angular Rates of the Aircraft

Through the rotation sequence described in Section A.3.1, it follows that the angular velocity may be expressed in terms of the body-axis component angular velocities $P, Q, \text{ and } R$, as well as the rate of change of the Euler angles $\dot{\Psi}, \dot{\Theta}, \text{ and } \dot{\Phi}$. Thus, one arrives at the following equivalency:

$$\vec{\omega} = P\vec{i} + Q\vec{j} + R\vec{k} = \dot{\Psi} + \dot{\Theta} + \dot{\Phi} \quad (A.86)$$

The intermediate goal is now to express the Euler angle rates, $\dot{\Psi}, \dot{\Theta}, \text{ and } \dot{\Phi}$, in terms of the body-fixed unit axis $(\vec{i}, \vec{j}, \vec{k})$.

A.3.2.1 Heading Angle

The heading angle $\Psi$ is a rotation about the $\vec{k}_1$ axis. Since the $\vec{k}_1$ axis is equivalent to the $\vec{k}_2$ axis, the following is true for the yaw rate $\dot{\Psi}$:

$$\dot{\Psi} = \dot{\psi}\vec{k}_1 = \dot{\psi}\vec{k}_2 \quad (A.87)$$

From equation (A.82) and equation (A.84), one notes that the unit axis $\vec{k}_2$ may be expressed as follows:

$$\vec{k}_2 = -\sin \Theta \vec{i}_3 + \cos \Theta \vec{k}_3 = -\sin \Theta [\vec{i}] + \cos \Theta [\sin \Phi \vec{j} + \cos \Phi \vec{k}] \quad (A.88)$$

$$\vec{k}_2 = -\sin \Theta \vec{i} + \cos \Theta \sin \Phi \vec{j} + \cos \Theta \cos \Phi \vec{k} \quad (A.89)$$

Equation (A.89) may then be written as:
APPENDIX A (continued)

\[ \dot{\Psi} = -\Psi \sin \Theta \hat{i} + \Psi \cos \Theta \sin \Phi \hat{j} + \Psi \cos \Theta \cos \Phi \hat{k} \]  
\text{(A.90)}

A.3.2.2 Pitch Angle

The pitch angle \( \Theta \) is a rotation about the \( \hat{j}_2 \) axis. Since the \( \hat{j}_2 \) axis is equivalent to the \( \hat{j}_3 \) axis, the following is true for the pitch rate \( \dot{\Theta} \):

\[ \dot{\Theta} = \Theta \hat{j}_2 = \Theta \hat{j}_3 \]  
\text{(A.91)}

From equation (A.84), one notes that the unit axis \( \hat{j}_3 \) may be expressed as follows:

\[ \hat{j}_3 = \cos \Phi \hat{j} - \sin \Phi \hat{k} \]  
\text{(A.92)}

Equation (A.92) may then be written as:

\[ \dot{\Theta} = \dot{\Theta} \cos \Phi \hat{j} - \dot{\Theta} \sin \Phi \hat{k} \]  
\text{(A.93)}

A.3.2.3 Bank Angle

The bank angle \( \Phi \) is a rotation about the \( \hat{i}_3 \) axis. Since the \( \hat{i}_3 \) axis is equivalent to the \( \hat{i} \) axis, the following is true for the roll rate \( \dot{\Phi} \):

\[ \dot{\Phi} = \Phi \hat{i}_3 = \Phi \hat{i} \]  
\text{(A.94)}

No further expansion of equation (A.94) is needed, since the roll rate is already expressed in terms of the body-fixed unit axis.
A.3.3 The Kinematic Equations

All the elements needed to obtain the kinematic equations are now ready. Recall from equation (A.86) the relationship between the body-fixed component angular rates and the Euler angle rates. Substituting equations (A.90), (A.93), and (A.94) into equation (A.86) yields the following:

\[ P\ddot{\text{i}} + Q\ddot{\text{j}} + R\ddot{\text{k}} = \dot{\Psi} + \dot{\Theta} + \dot{\Phi} \]
\[ = \left[ -\Psi \sin \Theta \ddot{\text{i}} + \Psi \cos \Theta \sin \Phi \ddot{\text{j}} + \Psi \cos \Theta \cos \Phi \ddot{\text{k}} \right] \]
\[ + \left[ \dot{\Theta} \cos \Phi \ddot{\text{j}} - \dot{\Theta} \sin \Phi \ddot{\text{k}} \right] + \left[ \Phi \ddot{\text{i}} \right] \]  

\[ (A.95) \]

Grouping terms according to the unit vectors then gives:

\[ P\ddot{\text{i}} + Q\ddot{\text{j}} + R\ddot{\text{k}} = \left( \dot{\Phi} - \Psi \sin \Theta \right)\ddot{\text{i}} + \left( \dot{\Theta} \cos \Phi + \Psi \cos \Theta \sin \Phi \right)\ddot{\text{j}} \]
\[ + \left( \Psi \cos \Theta \cos \Phi - \dot{\Theta} \sin \Phi \right)\ddot{\text{k}} \]  

\[ (A.96) \]

Each of the body-fixed component angular rates \( P, Q, \) and \( R \) are then given by equations (A.97) to (A.99), as follows:

\[ P = \dot{\Phi} - \Psi \sin \Theta \]  

\[ Q = \dot{\Theta} \cos \Phi + \Psi \cos \Theta \sin \Phi \]  

\[ R = \Psi \cos \Theta \cos \Phi - \dot{\Theta} \sin \Phi \]  

\[ (A.97) \]

\[ (A.98) \]

\[ (A.99) \]
A.3.4 The Gravity Equations

One final set of expressions is needed to represent the gravitational terms \(g_x\), \(g_y\), and \(g_z\) in equations (A.38) to (A.40). The derivation of these expressions is done in a similar fashion to that of the Kinematic Equations. The gravity vector is defined according to equation (A.100):

\[
\vec{g} = g_x \vec{i} + g_y \vec{j} + g_z \vec{k}
\]  
(A.100)

Since gravity is assumed to act “straight-down” in the context of the earth-fixed axis (with the assumption of a spherical earth), the following equivalency should hold:

\[
\vec{g} = g \vec{k}_1 = g \vec{k}_2
\]  
(A.101)

As with the Kinematic Equations, the goal is to express the gravity vector in terms of the body-fixed unit axis \((\vec{i}, \vec{j}, \vec{k})\). Noting the expression for \(\vec{k}_2\) in equation (A.89), equations (A.100) and (A.101) then become:

\[
g_x \vec{i} + g_y \vec{j} + g_z \vec{k} = g [- \sin \Theta \vec{i} + \cos \Theta \sin \Phi \vec{j} + \cos \Theta \cos \Phi \vec{k}]
\]  
(A.102)

Each of the body-fixed components of the gravity vector are then given by equations (A.103) to (A.105) as follows:

\[
g_x = -g \sin \Theta
\]  
(A.103)

\[
g_y = g \sin \Phi \cos \Theta
\]  
(A.104)

\[
g_z = g \cos \Phi \cos \Theta
\]  
(A.105)
Appendix B details the development of the perturbed state equations of motion, and is intended to complement the discussion presented in Chapter 5.

B.1 Overview

A perturbed state flight condition is defined as one for which all motion variables are defined relative to a known steady state flight condition. These equations of motion are useful for analysis of aircraft dynamic stability and response. Arriving at the equations of motion for perturbed state flight involves several intermediate steps:

- Substitution of motion, force, and moment variables with a linear superposition of steady and perturbed-state representations.
- Application of trigonometric assumptions to linearization angular variables.
- Elimination of steady state terms and application of small perturbation assumptions to linearize remaining non-linear variables.

B.1.1 Perturbation Substitutions

The first step in deriving the perturbed equations of motion involves making perturbation substitutions into the general aircraft equations of motion (equations (5.1) to (5.9)). This step algebraically introduces the system’s states through the substitutions defined in equations (B.1) to (B.3). These are the forward velocity $u$, angle-of-attack $\alpha$ (via forward velocity $u$ and vertical velocity $w$), pitch angle $\theta$, pitch rate $q$, sideslip angle $\beta$ (via forward velocity $u$ and side velocity $v$), bank angle $\phi$, roll rate $p$, and yaw rate $r$. The derivatives of these states will ultimately become the dependent variables in the state space equations that describe the aircraft’s coupled longitudinal
APPENDIX B (continued)

and lateral/directional dynamic response. These perturbation substitutions (see equations 1.66 to 1.68 in [106]) are as follows:

**Motion Variables:**

\[
\begin{align*}
U &= U_1 + u \\
V &= V_1 + v \\
W &= W_1 + w \\
\end{align*}
\]

\[
\begin{align*}
P &= P_1 + p \\
Q &= Q_1 + q \\
R &= R_1 + r \\
\end{align*}
\]

\[
\begin{align*}
\Psi &= \Psi_1 + \psi \\
\Theta &= \Theta_1 + \theta \\
\Phi &= \Phi_1 + \phi \\
\end{align*}
\]

\[(B.1)\]

**Forces:**

\[
\begin{align*}
F_{Ax} &= F_{Ax_1} + f_A \:
F_{Ay} &= F_{Ay_1} + f_A \\
F_{Az} &= F_{Az_1} + f_A \\
F_{Tx} &= F_{Tx_1} + f_T \:
F_{Ty} &= F_{Ty_1} + f_T \\
F_{Tz} &= F_{Tz_1} + f_T \\
\end{align*}
\]

\[(B.2)\]

**Moments:**

\[
\begin{align*}
L_A &= L_{A_1} + L_A \\
M_A &= M_{A_1} + m_A \\
N_A &= N_{A_1} + n_A \\
L_T &= L_{T_1} + L_T \\
M_T &= M_{T_1} + m_T \\
N_T &= N_{T_1} + n_T \\
\end{align*}
\]

\[(B.3)\]

Substituting equations (B.1) to (B.3) into the general aircraft equations of motion gives the expressions in equations (B.4) to (B.12).
APPENDIX B (continued)

**Force Equations:**

\[
m[\ddot{u} - (V_1 + v)(R_1 + r) + (W_1 + w)(Q_1 + q)] = -mg \sin(\theta_1 + \theta) + (F_{A1} + f_{Ax}) + (F_{T1} + f_{Tx})
\]

\[
m[\ddot{v} + (U_1 + u)(R_1 + r) - (W_1 + w)(P_1 + p)] = mg \sin(\Phi_1 + \phi) \cos(\theta_1 + \theta) + (F_{A1} + f_{Ay}) + (F_{T1} + f_{Ty})
\]

\[
m[\ddot{w} - (U_1 + u)(Q_1 + q) + (V_1 + v)(P_1 + p)] = mg \cos(\Phi_1 + \phi) \cos(\theta_1 + \theta) + (F_{A1} + f_{Az}) + (F_{T1} + f_{Tz})
\]

**Moment Equations:**

\[
l_{xx}\ddot{p} - l_{xy}\ddot{q} - l_{xz}\ddot{r} + l_{xy}(P_1 + p)(R_1 + r) + l_{yz}[(R_1 + r)^2 - (Q_1 + q)^2] - l_{xz}(P_1 + p)(Q_1 + q) + (l_{xx} - l_{yy})(R_1 + r)(Q_1 + q)
\]

\[
= (L_{A1} + I_A) + (L_{T1} + L_T)
\]

\[
l_{yy}\ddot{q} - l_{xy}\ddot{p} - l_{yz}\ddot{r} + l_{yy}(Q_1 + q)(R_1 + r) + l_{yz}(P_1 + p)(Q_1 + q) - l_{xy}(Q_1 + q)(R_1 + r) + l_{yz}(P_1 + p)(Q_1 + q)
\]

\[
= (M_{A1} + m_A) + (M_{T1} + m_T)
\]

\[
l_{zz}\ddot{r} - l_{xz}\ddot{p} - l_{yz}\ddot{q} + (l_{yy} - l_{xz})(P_1 + p)(Q_1 + q) + l_{xy}[(Q_1 + q)^2 - (P_1 + p)^2] + l_{xz}(Q_1 + q)(R_1 + r) - l_{yz}(P_1 + p)(R_1 + r)
\]

\[
= (N_{A1} + n_A) + (N_{T1} + n_T)
\]

**Kinematic Equations:**

\[
P_1 + p = (\dot{\Phi} + \dot{\phi}) - (\dot{\Psi} + \dot{\psi}) \sin(\theta_1 + \theta)
\]

\[
Q_1 + q = (\dot{\Theta} + \dot{\theta}) \cos(\Phi_1 + \phi) + (\dot{\Psi} + \dot{\psi}) \cos(\theta_1 + \theta) \sin(\Phi_1 + \phi)
\]

\[
R_1 + r = (\dot{\Psi} + \dot{\psi}) \cos(\theta_1 + \theta) \cos(\Phi_1 + \phi) - (\dot{\Theta} + \dot{\theta}) \sin(\Phi_1 + \phi)
\]
B.1.2 Trigonometric Approximations

The next step in deriving the perturbed equations of motion involves making several trigonometric approximations. These linearize the Euler angle states $\theta$ and $\phi$ through the assumptions given in equation (B.13), and are valid since the analysis is concerned with small perturbations about a general non-zero steady state condition.

$$\cos \theta = \cos \phi \approx 1 \quad \sin \theta \approx \theta \quad \sin \phi \approx \phi \quad (B.13)$$

The following trigonometric expansions are also be utilized:

$$\sin(\Theta_1 + \theta) \approx \sin \Theta_1 + \theta \cos \Theta_1 \quad (B.14)$$

$$\sin(\Phi_1 + \phi) \cos(\Theta_1 + \theta) \approx \sin \Phi_1 \cos \Theta_1 - \theta \sin \Phi_1 \sin \Theta_1 + \phi \cos \Phi_1 \cos \Theta_1 - \phi \theta \cos \Phi_1 \sin \Theta_1 \quad (B.15)$$

$$\cos(\Phi_1 + \phi) \cos(\Theta_1 + \theta) \approx \cos \Phi_1 \cos \Theta_1 - \theta \cos \Phi_1 \sin \Theta_1 - \phi \sin \Phi_1 \cos \Theta_1 + \phi \theta \sin \Phi_1 \sin \Theta_1 \quad (B.16)$$

Expanding the force and moment equations in equations (B.4) to (B.12) with the above trigonometric approximations, and denoting the steady state terms with single-underbars and the non-linear terms with double-underbars, the perturbed equations of motion then become:
APPENDIX B (continued)

**Force Equations:**

\[
\begin{align*}
\dot{m}(V_r + W_q) + m(\dot{v} - V_r - R_i v + W_i q + Q_i w) + m(-vr + wq) \\
= -mg \sin \theta_1 + F_{A_{x_1}} + F_{r_2} - mg \theta \cos \theta_1 + f_{A_x} + f_{r_x} \\
\end{align*}
\]

(B.17)

\[
\begin{align*}
m(U_r + W_i p) + m(\dot{v} + U_r v + R_i u - W_i p - P_i w) + m(uv - wp) \\
= mg \Phi_1 \cos \theta_1 + F_{A_{y_1}} + F_{r_{y_1}} - mg \theta \sin \Phi_1 \sin \theta_1 \\
+ mg \Phi_1 \cos \Phi_1 \cos \theta_1 + f_{A_y} + f_{r_y} - mg \Phi_1 \cos \Phi_1 \sin \theta_1 \\
\end{align*}
\]

(B.18)

\[
\begin{align*}
m(-U_v + V_v p) + m(\dot{w} - U_v q - Q_i u + V_i p + P_i v) + m(-uv + vp) \\
= mg \cos \Phi_1 \cos \theta_1 + F_{A_{z_1}} + F_{r_{z_1}} - mg \Phi_1 \cos \Phi_1 \sin \theta_1 \\
- mg \Phi_1 \sin \Phi_1 \cos \theta_1 + f_{A_z} + f_{r_z} + mg \Phi_1 \sin \Phi_1 \sin \theta_1 \\
\end{align*}
\]

(B.19)

**Moment Equations:**

\[
\begin{align*}
-l_{xx} P_1 + (l_{zz} - l_{yy})R_i Q_1 + l_{xy} P_2 R_i + l_{xz}(R_i^2 - Q_i^2) + l_{xy} \dot{p} - l_{xy} \dot{q} - l_{xx} \dot{r} \\
= L_{A_1} + L_{r_1} + \tau + l_r \\
\end{align*}
\]

(B.20)

\[
\begin{align*}
(l_{xx} - l_{zz}) P_1 R_i + l_{xz}(P_1^2 - R_i^2) + l_{yz} P_1 Q_1 - l_{xy} Q_i R_i + l_{xy} \dot{q} - l_{xy} \dot{p} - l_{xz} \dot{r} \\
+ (l_{xx} - l_{zz})(P_1 R_i + R_i p) + l_{xz}(2P_1 p - 2R_i r) - l_{xy}(Q_i r + R_i q) \\
+ l_{yz}(P_1 q + Q_i p) + (l_{xx} - l_{zz})pr + l_{xz}(p^2 - r^2) - l_{xy} qr + l_{yz} pq \\
= M_{A_1} + M_{r_1} + m_t + \tau \\
\end{align*}
\]

(B.21)

\[
\begin{align*}
(l_{yy} - l_{xx}) P_1 Q_1 + l_{xy} Q_i Q_1 + l_{xy}(Q_i^2 - P_i^2) - l_{yz} P_1 R_i + l_{xx} \dot{r} - l_{yy} \dot{q} - l_{zz} \dot{p} \\
+ (l_{yy} - l_{xx})(P_1 q + Q_i p) + l_{xy}(Q_i r + R_i q) + l_{yz}(2Q_i q - 2P_i p) \\
- l_{yz}(P_1 r + R_i p) + (l_{yy} - l_{xx})pq + l_{xx} qr + l_{xy}(q^2 - p^2) - l_{yz} pr \\
= N_{A_1} + N_{r_1} + n_t + n_r \\
\end{align*}
\]

(B.22)
Kinematic Equations:

\[ P_1 + p = \Phi_1 + \phi - \Psi_1 \sin \Theta_1 - \Psi_1 \theta \cos \Theta_1 - \psi \sin \Theta_1 - \psi \theta \cos \Theta_1 \]  
\( (B.23) \)

\[ Q_1 + q = \dot{\Phi}_1 \cos \Phi_1 - \dot{\Theta}_1 \phi \sin \Phi_1 + \dot{\Theta}_1 \phi \sin \Phi_1 + \dot{\Phi}_1 \cos \Theta_1 \sin \Phi_1 + \Psi_1 \phi \sin \Theta_1 \sin \Phi_1 \sin \Phi_1 + \Psi_1 \theta \phi \sin \Theta_1 \cos \Phi_1 \]  
\( (B.24) \)

\[ R_1 + r = \Psi_1 \cos \Theta_1 \cos \Phi_1 - \Psi_1 \phi \cos \Theta_1 \sin \Phi_1 - \Psi_1 \theta \sin \Theta_1 \sin \Phi_1 - \Psi_1 \phi \theta \sin \Theta_1 \sin \Phi_1 \]  
\( (B.25) \)

B.2 The Perturbed Equations of Motion

B.2.1 Perturbed Equations of Motion for General Steady State Conditions

The final step in arriving at the perturbed equations of motion involves the elimination of steady state terms and application of small perturbation assumptions.

The single-underbar terms in equations (B.17) to (B.25) represent the steady-state equations of motion and kinematic equations. Since the steady state equations are inherently satisfied, they may be eliminated without loss of generality. The double-underbar terms contain the products or cross-products of perturbed motion variables (or in other words, the non-linear terms). To remove these terms, the small perturbation assumption, which implies that the non-linear terms are negligible compared to the linear terms, may be made.

The end result is the set of perturbed equations of motion relative to a very general steady state, as given in equations (B.26) to (B.34) – one for which all motion variables are allowed to have non-zero steady state values.


**APPENDIX B (continued)**

\textbf{Force Equations:}

\[
m(\dot{u} - V_1 r - R_1 v + W_1 q + Q_1 w) = -mg\theta \cos \Theta_1 + f_{Ax} + f_{Rx}
\]

(B.26)

\[
m(\dot{v} + U_1 r + R_1 u - W_1 p + P_1 w)
= -mg\theta \sin \Phi_1 \sin \Theta_1 + mg\phi \cos \Phi_1 \cos \Theta_1 + f_{Ay} + f_{Ry}
\]

(B.27)

\[
m(\dot{w} - U_1 q - Q_1 u + V_1 p + P_1 v)
= -mg\theta \cos \Phi_1 \sin \Theta_1 - mg\phi \sin \Phi_1 \cos \Theta_1 + f_{Ax} + f_{Rz}
\]

(B.28)

\textbf{Moment Equations:}

\[
l_{xx}\ddot{p} - l_{xy}\ddot{q} - l_{xz}\ddot{r} - l_{xz}(P_1 q + Q_1 p) + (l_{xx} - l_{yy})(R_1 q + Q_1 r)
+ l_{xy}(P_1 r + R_1 p) + l_{yz}(2R_1 r - 2Q_1 q) = I_A + I_T
\]

(B.29)

\[
l_{yy}\ddot{q} - l_{xy}\ddot{p} - l_{yz}\ddot{r} + (l_{xx} - l_{zz})(P_1 r + R_1 p) + l_{xz}(2P_1 p - 2R_1 r)
- l_{xy}(Q_1 r + R_1 q) + l_{yz}(P_1 q + Q_1 p) = m_a + m_T
\]

(B.30)

\[
l_{zz}\ddot{r} - l_{yz}\ddot{q} - l_{xz}\ddot{p} + (l_{yy} - l_{xx})(P_1 q + Q_1 p) + l_{xz}(Q_1 r + R_1 q)
+ l_{xy}(2Q_1 q - 2P_1 p) - l_{yz}(P_1 r + R_1 p) = n_a + n_T
\]

(B.31)

\textbf{Kinematic Equations:}

\[
p = \dot{\phi} - \dot{\Psi}_1 \theta \cos \Theta_1 - \dot{\psi} \sin \Theta_1
\]

(B.32)

\[
q = -\dot{\Theta}_1 \phi \sin \Phi_1 + \dot{\theta} \cos \Phi_1 + \dot{\Psi}_1 \phi \cos \Theta_1 \cos \Phi_1 - \dot{\psi} \sin \Theta_1 \sin \Phi_1
+ \dot{\psi} \cos \Theta_1 \sin \Phi_1
\]

(B.33)

\[
r = -\dot{\Psi}_1 \phi \cos \Theta_1 \sin \Phi_1 - \dot{\psi}_1 \theta \sin \Theta_1 \cos \Phi_1 + \ddot{\psi} \cos \Theta_1 \cos \Phi_1
- \dot{\theta}_1 \phi \cos \Phi_1 - \dot{\theta} \sin \Phi_1
\]

(B.34)
B.2.2 A Special Case: Perturbed Equations of Motion for Special Steady State Conditions

For the case of airplane dynamic stability problems dealing with perturbed motion relative to a wings level, steady state, straight line flight condition with a relatively small flight path angle, the following assumptions may be made to further simplify the perturbed Equations of Motion.

For such a condition, the following hold:

1) No initial steady state side velocity exists: \( V_1 = 0 \)

2) No initial bank angle exists: \( \Phi_1 = 0 \)

3) No initial angular velocities exist: \( P_1 = Q_1 = R_1 = \dot{\Psi}_1 = \dot{\Theta}_1 = \dot{\Phi}_1 = 0 \)

With these special steady state conditions applied, the perturbed equations of motion in equations (B.26) to (B.34) reduce to the following:

**Force Equations:**

\[
m(\dot{u} + W_1 q) = -mg\theta \cos \Theta_1 + f_{Ax} + f_{Tx} \quad (B.35)
\]

\[
m(\dot{v} + U_1 r - W_1 p) = mg\phi \cos \Theta_1 + f_{Ay} + f_{Ty} \quad (B.36)
\]

\[
m(\dot{w} - U_1 q) = -mg\theta \sin \Theta_1 + f_{Az} + f_{Tz} \quad (B.37)
\]

**Moment Equations:**

\[
l_{xx}\dot{p} - l_{xy}\dot{q} - l_{xz}\dot{r} = l_A + l_T \quad (B.38)
\]

\[
l_{yy}\dot{q} - l_{xy}\dot{p} - l_{yz}\dot{r} = m_a + m_T \quad (B.39)
\]

\[
l_{zz}\dot{r} - l_{yz}\dot{q} - l_{xz}\dot{p} = n_A + n_T \quad (B.40)
\]
APPENDIX B (continued)

**Kinematic Equations:**

\[ p = \dot{\phi} - \dot{\psi} \sin \Theta_1 \]  \hspace{1cm} (B.41)

\[ q = \dot{\theta} \]  \hspace{1cm} (B.42)

\[ r = \dot{\psi} \cos \Theta_1 \]  \hspace{1cm} (B.43)
APPENDIX B (continued)

B.3 Development of the Perturbed State Forces and Moments

Consider the force and moment terms on the right-hand-side of equations (B.26) to (B.34). These force and moment terms need to be expanded into a set of equations through a “force build-up” process. Expressions for the partial derivatives that comprise the resulting force and moment equations then need to be derived.

B.3.1 Non-Dimensional Quasi-Steady Aerodynamic Forces & Moments

\[
\begin{align*}
\dot{f}_A &= \frac{\partial F_{Ax}}{\partial \left( \frac{u}{U_1} \right)} + \frac{\partial F_{Ax}}{\partial \alpha} + \frac{\partial F_{Ax}}{\partial \left( \frac{\alpha \dot{c}}{2U_1} \right)} + \frac{\partial F_{Ax}}{\partial \left( \frac{q \dot{c}}{2U_1} \right)} \frac{\partial \delta_e}{\partial \delta_t} + \frac{\partial F_{Ax}}{\partial \delta_t} \\
&\quad + \frac{\partial F_{Ax}}{\partial \beta} + \frac{\partial F_{Ax}}{\partial \left( \frac{\beta b}{2U_1} \right)} + \frac{\partial F_{Ax}}{\partial \left( \frac{pb}{2U_1} \right)} \frac{\partial rb}{\partial \delta_a} + \frac{\partial F_{Ax}}{\partial \delta_a}
\end{align*}
\]

\[\text{(B.44)}\]

\[
\begin{align*}
\dot{f}_A &= \frac{\partial F_{Ay}}{\partial \left( \frac{u}{U_1} \right)} + \frac{\partial F_{Ay}}{\partial \alpha} + \frac{\partial F_{Ay}}{\partial \left( \frac{\alpha \dot{c}}{2U_1} \right)} + \frac{\partial F_{Ay}}{\partial \left( \frac{q \dot{c}}{2U_1} \right)} \frac{\partial \delta_e}{\partial \delta_t} + \frac{\partial F_{Ay}}{\partial \delta_t} \\
&\quad + \frac{\partial F_{Ay}}{\partial \beta} + \frac{\partial F_{Ay}}{\partial \left( \frac{\beta b}{2U_1} \right)} + \frac{\partial F_{Ay}}{\partial \left( \frac{pb}{2U_1} \right)} \frac{\partial rb}{\partial \delta_a} + \frac{\partial F_{Ay}}{\partial \delta_a}
\end{align*}
\]

\[\text{(B.45)}\]

\[
\begin{align*}
\dot{f}_A &= \frac{\partial F_{Az}}{\partial \left( \frac{u}{U_1} \right)} + \frac{\partial F_{Az}}{\partial \alpha} + \frac{\partial F_{Az}}{\partial \left( \frac{\alpha \dot{c}}{2U_1} \right)} + \frac{\partial F_{Az}}{\partial \left( \frac{q \dot{c}}{2U_1} \right)} \frac{\partial \delta_e}{\partial \delta_t} + \frac{\partial F_{Az}}{\partial \delta_t} \\
&\quad + \frac{\partial F_{Az}}{\partial \beta} + \frac{\partial F_{Az}}{\partial \left( \frac{\beta b}{2U_1} \right)} + \frac{\partial F_{Az}}{\partial \left( \frac{pb}{2U_1} \right)} \frac{\partial rb}{\partial \delta_a} + \frac{\partial F_{Az}}{\partial \delta_a}
\end{align*}
\]

\[\text{(B.46)}\]
APPENDIX B (continued)

\[ l_a = \frac{\partial L_A}{\partial (\frac{u}{U_1})} \left( \frac{a}{U_1} \right) + \frac{\partial L_A}{\partial \alpha} \alpha + \frac{\partial L_A}{\partial \left( \frac{\alpha c}{2U_1} \right)} \left( \frac{\alpha c}{2U_1} \right) + \frac{\partial L_A}{\partial \left( \frac{q c}{2U_1} \right)} \left( \frac{q c}{2U_1} \right) + \frac{\partial L_A}{\partial \delta_e} \delta_e + \frac{\partial L_A}{\partial \delta_t} \delta_t \]

\[ + \frac{\partial L_A}{\partial \beta} \beta + \frac{\partial L_A}{\partial \left( \frac{\beta b}{2U_1} \right)} \left( \frac{\beta b}{2U_1} \right) \]

\[ m_a = \frac{\partial M_A}{\partial (\frac{u}{U_1})} \left( \frac{a}{U_1} \right) + \frac{\partial M_A}{\partial \alpha} \alpha + \frac{\partial M_A}{\partial \left( \frac{\alpha c}{2U_1} \right)} \left( \frac{\alpha c}{2U_1} \right) + \frac{\partial M_A}{\partial \left( \frac{q c}{2U_1} \right)} \left( \frac{q c}{2U_1} \right) + \frac{\partial M_A}{\partial \delta_e} \delta_e + \frac{\partial M_A}{\partial \delta_t} \delta_t \]

\[ + \frac{\partial M_A}{\partial \beta} \beta + \frac{\partial M_A}{\partial \left( \frac{\beta b}{2U_1} \right)} \left( \frac{\beta b}{2U_1} \right) \]

\[ n_a = \frac{\partial N_A}{\partial (\frac{u}{U_1})} \left( \frac{a}{U_1} \right) + \frac{\partial N_A}{\partial \alpha} \alpha + \frac{\partial N_A}{\partial \left( \frac{\alpha c}{2U_1} \right)} \left( \frac{\alpha c}{2U_1} \right) + \frac{\partial N_A}{\partial \left( \frac{q c}{2U_1} \right)} \left( \frac{q c}{2U_1} \right) + \frac{\partial N_A}{\partial \delta_e} \delta_e + \frac{\partial N_A}{\partial \delta_t} \delta_t \]

\[ + \frac{\partial N_A}{\partial \beta} \beta + \frac{\partial N_A}{\partial \left( \frac{\beta b}{2U_1} \right)} \left( \frac{\beta b}{2U_1} \right) \]

Where the aerodynamic forces and moments are non-dimensionalized according to:

\[ F_{Ax} = C_x \bar{q} S \quad L_A = C_l \bar{q} S b \]

\[ F_{Ay} = C_y \bar{q} S \quad M_A = C_m \bar{q} S \bar{c} \]

\[ F_{Az} = C_z \bar{q} S \quad N_A = C_n \bar{q} S b \]
APPENDIX B (continued)

B.3.2 Non-Dimensional Quasi-Steady Thrust Forces & Moments

\[ f_{Tx} = \frac{\partial F_{Tx}}{\partial \left( \frac{u}{U_1} \right)} \left( \frac{u}{U_1} \right) + \frac{\partial F_{Tx}}{\partial \alpha} \alpha + \frac{\partial F_{Tx}}{\partial \beta} \beta \]  \hspace{1cm} \text{(B.51)}

\[ f_{Ty} = \frac{\partial F_{Ty}}{\partial \left( \frac{u}{U_1} \right)} \left( \frac{u}{U_1} \right) + \frac{\partial F_{Ty}}{\partial \alpha} \alpha + \frac{\partial F_{Ty}}{\partial \beta} \beta \]  \hspace{1cm} \text{(B.52)}

\[ f_{Tz} = \frac{\partial F_{Tz}}{\partial \left( \frac{u}{U_1} \right)} \left( \frac{u}{U_1} \right) + \frac{\partial F_{Tz}}{\partial \alpha} \alpha + \frac{\partial F_{Tz}}{\partial \beta} \beta \]  \hspace{1cm} \text{(B.53)}

\[ l_T = \frac{\partial L_T}{\partial \left( \frac{u}{U_1} \right)} \left( \frac{u}{U_1} \right) + \frac{\partial L_T}{\partial \alpha} \alpha + \frac{\partial L_T}{\partial \beta} \beta \]  \hspace{1cm} \text{(B.54)}

\[ m_T = \frac{\partial M_T}{\partial \left( \frac{u}{U_1} \right)} \left( \frac{u}{U_1} \right) + \frac{\partial M_T}{\partial \alpha} \alpha + \frac{\partial M_T}{\partial \beta} \beta \]  \hspace{1cm} \text{(B.55)}

\[ n_T = \frac{\partial N_T}{\partial \left( \frac{u}{U_1} \right)} \left( \frac{u}{U_1} \right) + \frac{\partial N_T}{\partial \alpha} \alpha + \frac{\partial N_T}{\partial \beta} \beta \]  \hspace{1cm} \text{(B.56)}

Where the thrust forces and moments are non-dimensionalized according to:

\[ F_{Tx} = C_{Tx} \bar{q}S \hspace{1cm} L_T = C_{tT} \bar{q}Sb \]

\[ F_{Ty} = C_{Ty} \bar{q}S \hspace{1cm} M_T = C_{mT} \bar{q}Sc \]  \hspace{1cm} \text{(B.57)}

\[ F_{Tz} = C_{Tz} \bar{q}S \hspace{1cm} N_T = C_{nT} \bar{q}Sb \]
### B.3.3 Aerodynamic Force & Moment Derivatives

#### TABLE B.1

<table>
<thead>
<tr>
<th></th>
<th>Longitudinal Forces &amp; Moments</th>
<th>Lateral/Directional Forces &amp; Moments</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>$\frac{\partial F_{Ax}}{\partial \frac{u}{U_1}} = -(C_{D_{u}} + 2C_{D_{1}})\bar{q}_1S$</td>
<td>$\frac{\partial F_{Ay}}{\partial \frac{u}{U_1}} = (C_{Y_{u}} + 2C_{Y_{1}})\bar{q}_1S$</td>
</tr>
<tr>
<td>$\frac{\partial F_{Az}}{\partial \frac{u}{U_1}} = -(C_{L_{u}} + 2C_{L_{1}})\bar{q}_1S$</td>
<td>$\frac{\partial L_A}{\partial \frac{u}{U_1}} = (C_{t_{u}} + 2C_{t_{1}})\bar{q}_1Sb$</td>
<td></td>
</tr>
<tr>
<td>$\frac{\partial M_A}{\partial \frac{u}{U_1}} = (C_{m_{u}} + 2C_{m_{1}})\bar{q}_1S\bar{c}$</td>
<td>$\frac{\partial N_A}{\partial \frac{u}{U_1}} = (C_{n_{u}} + 2C_{n_{1}})\bar{q}_1Sb$</td>
<td></td>
</tr>
</tbody>
</table>

#### TABLE B.2

<table>
<thead>
<tr>
<th></th>
<th>Longitudinal Forces &amp; Moments</th>
<th>Lateral/Directional Forces &amp; Moments</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>$\frac{\partial F_{Ax}}{\partial \alpha} = (-C_{D_{\alpha}} + C_{L_{\alpha}})\bar{q}_1S$</td>
<td>$\frac{\partial F_{Ay}}{\partial \alpha} = C_{y_{\alpha}}\bar{q}_1S$</td>
</tr>
<tr>
<td>$\frac{\partial F_{Az}}{\partial \alpha} = -(C_{L_{\alpha}} + C_{D_{\alpha}})\bar{q}_1S$</td>
<td>$\frac{\partial L_A}{\partial \alpha} = C_{t_{\alpha}}\bar{q}_1Sb$</td>
<td></td>
</tr>
<tr>
<td>$\frac{\partial M_A}{\partial \alpha} = C_{m_{\alpha}}\bar{q}_1S\bar{c}$</td>
<td>$\frac{\partial N_A}{\partial \alpha} = C_{n_{\alpha}}\bar{q}_1Sb$</td>
<td></td>
</tr>
</tbody>
</table>
### APPENDIX B (continued)

#### TABLE B.3

**AERODYNAMIC FORCE AND MOMENT DERIVATIVES WITH RESPECT TO ANGLE-OF-ATTACK RATE**

<table>
<thead>
<tr>
<th>Longitudinal Forces &amp; Moments</th>
<th>Lateral/Directional Forces &amp; Moments</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\frac{\partial F_{Ax}}{\partial \left(\frac{\alpha c}{2U_1}\right)} = \frac{\partial C_x}{\partial \left(\frac{\alpha c}{2U_1}\right)} \bar{q}<em>1 S = C</em>{\alpha x} \bar{q}<em>1 S = -C</em>{D\alpha} \bar{q}_1 S$</td>
<td>$\frac{\partial F_{Ay}}{\partial \left(\frac{\alpha c}{2U_1}\right)} = \frac{\partial C_y}{\partial \left(\frac{\alpha c}{2U_1}\right)} \bar{q}<em>1 S = C</em>{y\alpha} \bar{q}_1 S$</td>
</tr>
<tr>
<td>$\frac{\partial F_{Az}}{\partial \left(\frac{\alpha c}{2U_1}\right)} = \frac{\partial C_z}{\partial \left(\frac{\alpha c}{2U_1}\right)} \bar{q}<em>1 S = C</em>{\alpha z} \bar{q}<em>1 S = -C</em>{L\alpha} \bar{q}_1 S$</td>
<td>$\frac{\partial L_A}{\partial \left(\frac{\alpha c}{2U_1}\right)} = \frac{\partial C_l}{\partial \left(\frac{\alpha c}{2U_1}\right)} \bar{q}<em>1 Sb = C</em>{l\alpha} \bar{q}_1 Sb$</td>
</tr>
<tr>
<td>$\frac{\partial M_A}{\partial \left(\frac{\alpha c}{2U_1}\right)} = \frac{\partial C_m}{\partial \left(\frac{\alpha c}{2U_1}\right)} \bar{q}<em>1 S \bar{c} = C</em>{m\alpha} \bar{q}_1 S \bar{c}$</td>
<td>$\frac{\partial N_A}{\partial \left(\frac{\alpha c}{2U_1}\right)} = \frac{\partial C_n}{\partial \left(\frac{\alpha c}{2U_1}\right)} \bar{q}<em>1 Sb = C</em>{n\alpha} \bar{q}_1 Sb$</td>
</tr>
</tbody>
</table>

#### TABLE B.4

**AERODYNAMIC FORCE AND MOMENT DERIVATIVES WITH RESPECT TO PITCH RATE**

<table>
<thead>
<tr>
<th>Longitudinal Forces &amp; Moments</th>
<th>Lateral/Directional Forces &amp; Moments</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\frac{\partial F_{Ax}}{\partial \left(\frac{q c}{2U_1}\right)} = \frac{\partial C_x}{\partial \left(\frac{q c}{2U_1}\right)} \bar{q}<em>1 S = C</em>{x\alpha} \bar{q}<em>1 S = -C</em>{D\alpha} \bar{q}_1 S$</td>
<td>$\frac{\partial F_{Ay}}{\partial \left(\frac{q c}{2U_1}\right)} = \frac{\partial C_y}{\partial \left(\frac{q c}{2U_1}\right)} \bar{q}<em>1 S = C</em>{y\alpha} \bar{q}_1 S$</td>
</tr>
<tr>
<td>$\frac{\partial F_{Az}}{\partial \left(\frac{q c}{2U_1}\right)} = \frac{\partial C_z}{\partial \left(\frac{q c}{2U_1}\right)} \bar{q}<em>1 S = C</em>{xq} \bar{q}<em>1 S = -C</em>{Lq} \bar{q}_1 S$</td>
<td>$\frac{\partial L_A}{\partial \left(\frac{q c}{2U_1}\right)} = \frac{\partial C_l}{\partial \left(\frac{q c}{2U_1}\right)} \bar{q}<em>1 Sb = C</em>{lq} \bar{q}_1 Sb$</td>
</tr>
<tr>
<td>$\frac{\partial M_A}{\partial \left(\frac{q c}{2U_1}\right)} = \frac{\partial C_m}{\partial \left(\frac{q c}{2U_1}\right)} \bar{q}<em>1 S \bar{c} = C</em>{m\alpha} \bar{q}_1 S \bar{c}$</td>
<td>$\frac{\partial N_A}{\partial \left(\frac{q c}{2U_1}\right)} = \frac{\partial C_n}{\partial \left(\frac{q c}{2U_1}\right)} \bar{q}<em>1 Sb = C</em>{n\alpha} \bar{q}_1 Sb$</td>
</tr>
</tbody>
</table>
## APPENDIX B (continued)

### TABLE B.5
AERODYNAMIC FORCE AND MOMENT DERIVATIVES WITH RESPECT TO SIDESLIP ANGLE

<table>
<thead>
<tr>
<th>Longitudinal Forces &amp; Moments</th>
<th>Lateral/Directional Forces &amp; Moments</th>
</tr>
</thead>
<tbody>
<tr>
<td>( \frac{\partial F_{Ax}}{\partial \beta} = \frac{\partial C_x}{\partial \beta} \bar{q}<em>1 S = C</em>{x_\beta} \bar{q}<em>1 S = -C</em>{D_\beta} \bar{q}_1 S )</td>
<td>( \frac{\partial F_{Ay}}{\partial \beta} = \frac{\partial C_y}{\partial \beta} \bar{q}<em>1 S = C</em>{y_\beta} \bar{q}_1 S )</td>
</tr>
<tr>
<td>( \frac{\partial F_{Az}}{\partial \beta} = \frac{\partial C_z}{\partial \beta} \bar{q}<em>1 S = C</em>{z_\beta} \bar{q}<em>1 S = -C</em>{L_\beta} \bar{q}_1 S )</td>
<td>( \frac{\partial L_A}{\partial \beta} = \frac{\partial C_l}{\partial \beta} \bar{q}<em>1 Sb = C</em>{l_\beta} \bar{q}_1 Sb )</td>
</tr>
<tr>
<td>( \frac{\partial M_A}{\partial \beta} = \frac{\partial C_m}{\partial \beta} \bar{q}<em>1 \bar{c} = C</em>{m_\beta} \bar{q}_1 \bar{c} )</td>
<td>( \frac{\partial N_A}{\partial \beta} = \frac{\partial C_n}{\partial \beta} \bar{q}<em>1 Sb = C</em>{n_\beta} \bar{q}_1 Sb )</td>
</tr>
</tbody>
</table>

### TABLE B.6
AERODYNAMIC FORCE AND MOMENT DERIVATIVES WITH RESPECT TO SIDESLIP RATE

<table>
<thead>
<tr>
<th>Longitudinal Forces &amp; Moments</th>
<th>Lateral/Directional Forces &amp; Moments</th>
</tr>
</thead>
<tbody>
<tr>
<td>( \frac{\partial F_{Ax}}{\partial (\frac{\beta b}{2U_1})} = \frac{\partial C_x}{\partial \beta} \bar{q}<em>1 S = C</em>{x_\beta} \bar{q}<em>1 S = -C</em>{D_\beta} \bar{q}_1 S )</td>
<td>( \frac{\partial F_{Ay}}{\partial (\frac{\beta b}{2U_1})} = \frac{\partial C_y}{\partial \beta} \bar{q}<em>1 S = C</em>{y_\beta} \bar{q}_1 S )</td>
</tr>
<tr>
<td>( \frac{\partial F_{Az}}{\partial (\frac{\beta b}{2U_1})} = \frac{\partial C_z}{\partial \beta} \bar{q}<em>1 S = C</em>{z_\beta} \bar{q}<em>1 S = -C</em>{L_\beta} \bar{q}_1 S )</td>
<td>( \frac{\partial L_A}{\partial (\frac{\beta b}{2U_1})} = \frac{\partial C_l}{\partial \beta} \bar{q}<em>1 Sb = C</em>{l_\beta} \bar{q}_1 Sb )</td>
</tr>
<tr>
<td>( \frac{\partial M_A}{\partial (\frac{\beta b}{2U_1})} = \frac{\partial C_m}{\partial \beta} \bar{q}<em>1 \bar{c} = C</em>{m_\beta} \bar{q}_1 \bar{c} )</td>
<td>( \frac{\partial N_A}{\partial (\frac{\beta b}{2U_1})} = \frac{\partial C_n}{\partial \beta} \bar{q}<em>1 Sb = C</em>{n_\beta} \bar{q}_1 Sb )</td>
</tr>
</tbody>
</table>
APPENDIX B (continued)

### TABLE B.7
AERODYNAMIC FORCE AND MOMENT DERIVATIVES WITH RESPECT TO ROLL RATE

<table>
<thead>
<tr>
<th>Longitudinal Forces &amp; Moments</th>
<th>Lateral/Directional Forces &amp; Moments</th>
</tr>
</thead>
<tbody>
<tr>
<td>( \frac{\partial F_{Ax}}{\partial \left( \frac{pb}{2U_1} \right)} = \frac{\partial C_x}{\partial \left( \frac{pb}{2U_1} \right)} \bar{q}<em>1 S = C</em>{xp} \bar{q}<em>1 S = -C</em>{Cp} \bar{q}_1 S )</td>
<td>( \frac{\partial F_{Ay}}{\partial \left( \frac{pb}{2U_1} \right)} = \frac{\partial C_y}{\partial \left( \frac{pb}{2U_1} \right)} \bar{q}<em>1 S = C</em>{yp} \bar{q}_1 S )</td>
</tr>
<tr>
<td>( \frac{\partial F_{Az}}{\partial \left( \frac{pb}{2U_1} \right)} = \frac{\partial C_z}{\partial \left( \frac{pb}{2U_1} \right)} \bar{q}<em>1 S = C</em>{zp} \bar{q}<em>1 S = -C</em>{Cp} \bar{q}_1 S )</td>
<td>( \frac{\partial L_A}{\partial \left( \frac{pb}{2U_1} \right)} = \frac{\partial C_l}{\partial \left( \frac{pb}{2U_1} \right)} - \bar{q}<em>1 Sb = C</em>{lp} \bar{q}_1 Sb )</td>
</tr>
<tr>
<td>( \frac{\partial M_A}{\partial \left( \frac{pb}{2U_1} \right)} = \frac{\partial C_m}{\partial \left( \frac{pb}{2U_1} \right)} \bar{q}<em>1 S \bar{c} = C</em>{mp} \bar{q}_1 S \bar{c} )</td>
<td>( \frac{\partial N_A}{\partial \left( \frac{pb}{2U_1} \right)} = \frac{\partial C_n}{\partial \left( \frac{pb}{2U_1} \right)} - \bar{q}<em>1 Sb = C</em>{np} \bar{q}_1 Sb )</td>
</tr>
</tbody>
</table>

### TABLE B.8
AERODYNAMIC FORCE AND MOMENT DERIVATIVES WITH RESPECT TO YAW RATE

<table>
<thead>
<tr>
<th>Longitudinal Forces &amp; Moments</th>
<th>Lateral/Directional Forces &amp; Moments</th>
</tr>
</thead>
<tbody>
<tr>
<td>( \frac{\partial F_{Ax}}{\partial \left( \frac{rb}{2U_1} \right)} = \frac{\partial C_x}{\partial \left( \frac{rb}{2U_1} \right)} \bar{q}<em>1 S = C</em>{x_r} \bar{q}<em>1 S = -C</em>{Dx} \bar{q}_1 S )</td>
<td>( \frac{\partial F_{Ay}}{\partial \left( \frac{rb}{2U_1} \right)} = \frac{\partial C_y}{\partial \left( \frac{rb}{2U_1} \right)} \bar{q}<em>1 S = C</em>{y_r} \bar{q}_1 S )</td>
</tr>
<tr>
<td>( \frac{\partial F_{Az}}{\partial \left( \frac{rb}{2U_1} \right)} = \frac{\partial C_z}{\partial \left( \frac{rb}{2U_1} \right)} \bar{q}<em>1 S = C</em>{z_r} \bar{q}<em>1 S = -C</em>{Dz} \bar{q}_1 S )</td>
<td>( \frac{\partial L_A}{\partial \left( \frac{rb}{2U_1} \right)} = \frac{\partial C_l}{\partial \left( \frac{rb}{2U_1} \right)} - \bar{q}<em>1 Sb = C</em>{l_r} \bar{q}_1 Sb )</td>
</tr>
<tr>
<td>( \frac{\partial M_A}{\partial \left( \frac{rb}{2U_1} \right)} = \frac{\partial C_m}{\partial \left( \frac{rb}{2U_1} \right)} \bar{q}<em>1 S \bar{c} = C</em>{mr} \bar{q}_1 S \bar{c} )</td>
<td>( \frac{\partial N_A}{\partial \left( \frac{rb}{2U_1} \right)} = \frac{\partial C_n}{\partial \left( \frac{rb}{2U_1} \right)} - \bar{q}<em>1 Sb = C</em>{n_r} \bar{q}_1 Sb )</td>
</tr>
</tbody>
</table>
TABLE B.9
AERODYNAMIC FORCE AND MOMENT DERIVATIVES
WITH RESPECT TO CONTROL SURFACE DEFLECTION

<table>
<thead>
<tr>
<th>Longitudinal Forces &amp; Moments</th>
<th>Lateral/Directional Forces &amp; Moments</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\frac{\partial F_{Ax}}{\partial \delta_i} = \frac{\partial C_x}{\partial \delta_i} \bar{q}<em>1 S = C</em>{x\delta_i} \bar{q}<em>1 S = -C</em>{D\delta_i} \bar{q}_1 S$</td>
<td>$\frac{\partial F_{Ay}}{\partial \delta_i} = \frac{\partial C_y}{\partial \delta_i} \bar{q}<em>1 S = C</em>{y\delta_i} \bar{q}_1 S$</td>
</tr>
<tr>
<td>$\frac{\partial F_{Az}}{\partial \delta_i} = \frac{\partial C_z}{\partial \delta_i} \bar{q}<em>1 S = C</em>{z\delta_i} \bar{q}<em>1 S = -C</em>{L\delta_i} \bar{q}_1 S$</td>
<td>$\frac{\partial L_A}{\partial \delta_i} = \frac{\partial C_l}{\partial \delta_i} \bar{q}<em>1 S_b = C</em>{l\delta_i} \bar{q}_1 S_b$</td>
</tr>
<tr>
<td>$\frac{\partial M_A}{\partial \delta_i} = \frac{\partial C_m}{\partial \delta_i} \bar{q}<em>1 S \bar{c} = C</em>{m\delta_i} \bar{q}_1 S \bar{c}$</td>
<td>$\frac{\partial N_A}{\partial \delta_i} = \frac{\partial C_n}{\partial \delta_i} \bar{q}<em>1 S_b = C</em>{n\delta_i} \bar{q}_1 S_b$</td>
</tr>
</tbody>
</table>
### APPENDIX B (continued)

#### B.3.4 Thrust Force & Moment Derivatives

**TABLE B.10**

<table>
<thead>
<tr>
<th>Longitudinal Forces &amp; Moments</th>
<th>Lateral/Directional Forces &amp; Moments</th>
</tr>
</thead>
<tbody>
<tr>
<td>( \frac{\partial F_{T_x}}{\partial \left( \frac{u}{U_1} \right)} ) = ( \frac{\partial C_{T_x}}{\partial \left( \frac{u}{U_1} \right)} \bar{q}<em>1 S + \frac{\partial \bar{q}}{\partial \left( \frac{u}{U_1} \right)} C</em>{T_x} S ) = ( (C_{T_{xu}} + 2C_{T_{x\alpha}}) \bar{q}_1 S )</td>
<td>( \frac{\partial F_{T_y}}{\partial \left( \frac{u}{U_1} \right)} ) = ( \frac{\partial C_{T_y}}{\partial \left( \frac{u}{U_1} \right)} \bar{q}<em>1 S + \frac{\partial \bar{q}}{\partial \left( \frac{u}{U_1} \right)} C</em>{T_y} S ) = ( (C_{T_{yu}} + 2C_{T_{y\alpha}}) \bar{q}_1 S )</td>
</tr>
<tr>
<td>( \frac{\partial F_{T_z}}{\partial \left( \frac{u}{U_1} \right)} ) = ( \frac{\partial C_{T_z}}{\partial \left( \frac{u}{U_1} \right)} \bar{q}<em>1 S + \frac{\partial \bar{q}}{\partial \left( \frac{u}{U_1} \right)} C</em>{T_z} S ) = ( (C_{T_{zu}} + 2C_{T_{z\alpha}}) \bar{q}_1 S )</td>
<td>( \frac{\partial L_T}{\partial \left( \frac{u}{U_1} \right)} ) = ( \frac{\partial C_{lT}}{\partial \left( \frac{u}{U_1} \right)} \bar{q}<em>1 Sb + \frac{\partial \bar{q}}{\partial \left( \frac{u}{U_1} \right)} C</em>{lT} Sb ) = ( (C_{lT_{u\alpha}} + 2C_{lT_{z\alpha}}) \bar{q}_1 Sb )</td>
</tr>
<tr>
<td>( \frac{\partial M_T}{\partial \left( \frac{u}{U_1} \right)} ) = ( \frac{\partial C_{mT}}{\partial \left( \frac{u}{U_1} \right)} \bar{q}<em>1 \bar{c} + \frac{\partial \bar{q}}{\partial \left( \frac{u}{U_1} \right)} C</em>{mT} \bar{c} ) = ( (C_{mT_{u\alpha}} + 2C_{mT_{z\alpha}}) \bar{q}_1 \bar{c} )</td>
<td>( \frac{\partial N_T}{\partial \left( \frac{u}{U_1} \right)} ) = ( \frac{\partial C_{nT}}{\partial \left( \frac{u}{U_1} \right)} \bar{q}<em>1 Sb + \frac{\partial \bar{q}}{\partial \left( \frac{u}{U_1} \right)} C</em>{nT} Sb ) = ( (C_{nT_{u\alpha}} + 2C_{nT_{z\alpha}}) \bar{q}_1 Sb )</td>
</tr>
</tbody>
</table>

**TABLE B.11**

<table>
<thead>
<tr>
<th>Longitudinal Forces &amp; Moments</th>
<th>Lateral/Directional Forces &amp; Moments</th>
</tr>
</thead>
<tbody>
<tr>
<td>( \frac{\partial F_{T_x}}{\partial \alpha} ) = ( \frac{\partial C_{T_x}}{\partial \alpha} \bar{q}<em>1 S = C</em>{T_{x\alpha}} \bar{q}_1 S )</td>
<td>( \frac{\partial F_{T_y}}{\partial \alpha} ) = ( \frac{\partial C_{T_y}}{\partial \alpha} \bar{q}<em>1 S = C</em>{T_{y\alpha}} \bar{q}_1 S )</td>
</tr>
<tr>
<td>( \frac{\partial F_{T_z}}{\partial \alpha} ) = ( \frac{\partial C_{T_z}}{\partial \alpha} \bar{q}<em>1 S = C</em>{T_{z\alpha}} \bar{q}_1 S )</td>
<td>( \frac{\partial L_T}{\partial \alpha} ) = ( \frac{\partial C_{lT}}{\partial \alpha} \bar{q}<em>1 Sb = C</em>{lT_{\alpha}} \bar{q}_1 Sb )</td>
</tr>
<tr>
<td>( \frac{\partial M_T}{\partial \alpha} ) = ( \frac{\partial C_{mT}}{\partial \alpha} \bar{q}<em>1 \bar{c} = C</em>{mT_{\alpha}} \bar{q}_1 \bar{c} )</td>
<td>( \frac{\partial N_T}{\partial \alpha} ) = ( \frac{\partial C_{nT}}{\partial \alpha} \bar{q}<em>1 Sb = C</em>{nT_{\alpha}} \bar{q}_1 Sb )</td>
</tr>
</tbody>
</table>
TABLE B.12
THRUST FORCE AND MOMENT DERIVATIVES
WITH RESPECT TO SIDESLIP ANGLE

<table>
<thead>
<tr>
<th>Longitudinal Forces &amp; Moments</th>
<th>Lateral/Directional Forces &amp; Moments</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\frac{\partial F_{T_x}}{\partial \beta} = \frac{\partial C_{T_x}}{\partial \beta} \bar{q}<em>1 S = C</em>{T_x \beta} \bar{q}_1 S$</td>
<td>$\frac{\partial F_{T_y}}{\partial \beta} = \frac{\partial C_{T_y}}{\partial \beta} \bar{q}<em>1 S = C</em>{T_y \beta} \bar{q}_1 S$</td>
</tr>
<tr>
<td>$\frac{\partial F_{T_z}}{\partial \beta} = \frac{\partial C_{T_z}}{\partial \beta} \bar{q}<em>1 S = C</em>{T_z \beta} \bar{q}_1 S$</td>
<td>$\frac{\partial L_T}{\partial \beta} = \frac{\partial C_{l_T}}{\partial \beta} \bar{q}<em>1 S b = C</em>{l_T \beta} \bar{q}_1 S b$</td>
</tr>
<tr>
<td>$\frac{\partial M_T}{\partial \beta} = \frac{\partial C_{m_T}}{\partial \beta} \bar{q}<em>1 S \bar{c} = C</em>{m_T \beta} \bar{q}_1 S \bar{c}$</td>
<td>$\frac{\partial N_T}{\partial \beta} = \frac{\partial C_{n_T}}{\partial \beta} \bar{q}<em>1 S b = C</em>{n_T \beta} \bar{q}_1 S b$</td>
</tr>
</tbody>
</table>
B.3.5 Expanded Derivations for Selected Aerodynamic Force & Moment Derivatives

B.3.5.1 Derivatives with respect to Forward Velocity

Before evaluating the force and moment derivatives, the partial derivative of dynamic pressure with respect to forward velocity first needs to be determined. Note that the partial derivative of $\bar{q}$ is evaluated in the steady state flight condition. Dynamic pressure is defined as:

$$\bar{q} = \frac{1}{2} \rho (U_1 + u)^2 + v^2 + w^2)$$  \hspace{1cm} (B.58)

Then, the partial derivative of dynamic pressure with respect to forward velocity may be evaluated as follows:

$$\frac{\partial \bar{q}}{\partial \left( \frac{u}{U_1} \right)} = U_1 \left. \frac{\partial \bar{q}}{\partial u} \right|_1 = U_1 \left. \frac{\partial}{\partial u} \left( \frac{1}{2} \rho ((U_1 + u)^2 + v^2 + w^2) \right) \right|_1$$

$$= \frac{1}{2} \rho U_1 \left. \frac{\partial}{\partial u} [U_1^2 + 2U_1 u + u^2 + v^2 + w^2] \right|_1$$

$$= \frac{1}{2} \rho U_1^2 \left. (2U_1 + 2u) \right|_1 = \rho U_1^2$$  \hspace{1cm} (B.59)
B.3.5.1.1 Aerodynamic Force in the \( x \)-direction (\( F_{Ax} \))

The partial derivative of force in the \( x \)-direction with respect to forward velocity may be written according to equation (B.60). Note that the partial derivatives of \( C_x \) and \( \bar{q} \) should be evaluated in the steady state flight condition. This implies that \[ \frac{\partial F_{Ax}}{\partial \left( \frac{u}{U_1} \right)} = \left. \frac{\partial F_{Ax}}{\partial \left( \frac{u}{U_1} \right)} \right|_1, \] such that:

\[
\frac{\partial F_{Ax}}{\partial \left( \frac{u}{U_1} \right)} = \frac{\partial C_x}{\partial \left( \frac{u}{U_1} \right)} \bar{q}S + \frac{\partial \bar{q}}{\partial \left( \frac{u}{U_1} \right)} C_x S
\]  \hspace{1cm} (B.60)

The coefficient of force in the \( x \)-direction, \( C_x \), may be defined according to equation (B.61) through application of the small angle assumption (see Figure 3.51 in [106]):

\[ C_x = -C_D + C_L \alpha \]  \hspace{1cm} (B.61)

In the steady state, equation (B.61) becomes \( C_{x_1} = -C_{D_1} \). Then, the partial derivative of \( C_x \) with respect to forward velocity may be evaluated in the steady state, as follows:

\[
\frac{\partial C_x}{\partial \left( \frac{u}{U_1} \right)} = -\frac{\partial C_D}{\partial \left( \frac{u}{U_1} \right)} + \frac{\partial C_L}{\partial \left( \frac{u}{U_1} \right)} \alpha = C_{xu} = -C_{Du}
\]  \hspace{1cm} (B.62)

Substituting equations (B.59) and (B.62) into equation (B.60) yields:

\[
\frac{\partial F_{Ax}}{\partial \left( \frac{u}{U_1} \right)} = -C_{Du} \bar{q}_1 S + \rho U_1^2 c_{x_1} S = -C_{Du} \bar{q}_1 S + \frac{1}{2} \rho U_1^2 c_{x_1} S
\]

\[
= -C_{Du} \bar{q}_1 S + 2 \bar{q}_1 c_{x_1} S = \left( -C_{Du} + 2 c_{x_1} \right) \bar{q}_1 S
\]

\[
= -(C_{Du} + 2 C_{D_1}) \bar{q}_1 S
\]  \hspace{1cm} (B.63)
APPENDIX B (continued)

### B.3.5.1.2 Aerodynamic Force in the z-direction \( (F_{Az}) \)

The partial derivative of force in the z-direction with respect to forward velocity may be written according to equation (B.64):

\[
\frac{\partial F_{Az}}{\partial \left( \frac{u}{U_1} \right)} = -\frac{\partial C_L}{\partial \left( \frac{u}{U_1} \right)} + \frac{\partial C_D}{\partial \left( \frac{u}{U_1} \right)} \alpha = C_{zu} = -C_{D_u} \tag{B.64}
\]

The coefficient of force in the z-direction, \( C_z \), may be defined according to equation (B.65) through application of the small angle assumption (see Figure 3.51 in [106]):

\[
C_z = -C_L - C_D \alpha \tag{B.65}
\]

In the steady state, equation (B.65) becomes \( C_{z_1} = -C_{L_1} \). Then, the partial derivative of \( C_z \) with respect to forward velocity may be evaluated in the steady state, as follows:

\[
\frac{\partial C_z}{\partial \left( \frac{u}{U_1} \right)} = -\frac{\partial C_L}{\partial \left( \frac{u}{U_1} \right)} - \frac{\partial C_D}{\partial \left( \frac{u}{U_1} \right)} \alpha = C_{zu} = -C_{L_u} \tag{B.66}
\]

Substituting equations (B.59) and (B.66) into equation (B.64) yields:

\[
\frac{\partial F_{Az}}{\partial \left( \frac{u}{U_1} \right)} = -C_{L_u} \bar{q}_{1_1} S + \rho U_1^2 C_{z_1} S = -\left( C_{L_u} + 2C_{L_1} \right) \bar{q}_{1_1} S \tag{B.67}
\]
APPENDIX B (continued)

B.3.5.1.3 Aerodynamic Pitching Moment ($M_A$)

The partial derivative of pitching moment with respect to forward velocity may be written according to equation (B.68):

$$\frac{\partial M_A}{\partial \left( \frac{u}{U_1} \right)} = \frac{\partial C_m}{\partial \left( \frac{u}{U_1} \right)} \bar{q} S \bar{\epsilon} + \frac{\partial q}{\partial \left( \frac{u}{U_1} \right)} C_m S \bar{\epsilon}$$  \hspace{1cm} (B.68)

Then, the partial derivative of $C_m$ with respect to forward velocity may be evaluated as follows:

$$\frac{\partial C_m}{\partial \left( \frac{u}{U_1} \right)} = C_{m_u}$$  \hspace{1cm} (B.69)

In the steady state, $C_m$ may be written as $C_{m_1}$. Substituting equations (B.59) and (B.69) into equation (B.68) and evaluating at the steady state yields:

$$\frac{\partial M_A}{\partial \left( \frac{u}{U_1} \right)} = C_{m_u} \bar{q}_1 S \bar{\epsilon} + \rho U_1^2 C_{m_1} S \bar{\epsilon} = (C_{m_u} + 2C_{m_1}) \bar{q}_1 S \bar{\epsilon}$$  \hspace{1cm} (B.70)
B.3.5.1.4 Aerodynamic Force in the \( y \)-direction (\( F_{Ay} \))

The partial derivative of force in the \( y \)-direction with respect to forward velocity may be written according to equation (B.71):

\[
\frac{\partial F_{Ay}}{\partial \left( \frac{u}{U_1} \right)} = \frac{\partial C_y}{\partial \left( \frac{u}{U_1} \right)} \bar{q} S + \frac{\partial \bar{q}}{\partial \left( \frac{u}{U_1} \right)} C_y S \tag{B.71}
\]

The coefficient of force in the \( y \)-direction, \( C_y \), may be defined according to equation (B.72):

\[
C_y = C_y \tag{B.72}
\]

In the steady state, equation (B.72) becomes \( C_{y_1} = -C_{y_1} \). Then, the partial derivative of \( C_y \) with respect to forward velocity may be evaluated in the steady state, as follows:

\[
\frac{\partial C_y}{\partial \left( \frac{u}{U_1} \right)} = \frac{\partial C_y}{\partial \left( \frac{u}{U_1} \right)} = C_{yu} \tag{B.73}
\]

Substituting equations (B.59) and (B.73) into equation (B.71) yields:

\[
\frac{\partial F_{Ay}}{\partial \left( \frac{u}{U_1} \right)} = C_{yu} \bar{q}_1 S + \rho U_1^2 C_{y_1} S = C_{yu} \bar{q}_1 S + 2 \cdot \frac{1}{2} \rho U_1^2 C_{y_1} S
\]

\[
= C_{yu} \bar{q}_1 S + 2 \bar{q}_1 C_{y_1} S = (C_{yu} + 2C_{y_1}) \bar{q}_1 S = (C_{yu} + 2C_{y_1}) \bar{q}_1 S \tag{B.74}
\]
B.3.5.1.5 Aerodynamic Rolling Moment \( (L_A) \)

The partial derivative of rolling moment with respect to forward velocity may be written according to equation (B.75):

\[
\frac{\partial L_A}{\partial \left( \frac{u}{U_1} \right)} = \frac{\partial C_i}{\partial \left( \frac{u}{U_1} \right)} \bar{q} S b + \frac{\partial \bar{q}}{\partial \left( \frac{u}{U_1} \right)} C_i S b
\]  

(B.75)

Then, the partial derivative of \( C_i \) with respect to forward velocity may be evaluated as follows:

\[
\frac{\partial C_i}{\partial \left( \frac{u}{U_1} \right)} = C_{i_u}
\]  

(B.76)

In the steady state, \( C_i \) may be written as \( C_{i_1} \). Substituting equations (B.59) and (B.76) into equation (B.75) and evaluating at the steady state yields:

\[
\frac{\partial L_A}{\partial \left( \frac{u}{U_1} \right)} = C_{i_u} \bar{q}_1 S b + \rho U_1^2 C_{i_1} S b = \left( C_{i_u} + 2 C_{i_1} \right) \bar{q}_1 S b
\]  

(B.77)
APPENDIX B (continued)

B.3.5.1.6 Aerodynamic Yawing Moment ($N_A$)

The partial derivative of yawing moment with respect to forward velocity may be written according to equation (B.78):

$$\frac{\partial N_A}{\partial \left( \frac{u}{U_1} \right)} = \frac{\partial C_n}{\partial \left( \frac{u}{U_1} \right)} \overline{q} S b + \frac{\partial \overline{q}}{\partial \left( \frac{u}{U_1} \right)} C_n S b \tag{B.78}$$

Then, the partial derivative of $C_n$ with respect to forward velocity may be evaluated as follows:

$$\frac{\partial C_n}{\partial \left( \frac{u}{U_1} \right)} = C_{n_u} \tag{B.79}$$

In the steady state, $C_n$ may be written as $C_{n_1}$. Substituting equations (B.59) and (B.79) into equation (B.78) and evaluating at the steady state yields:

$$\frac{\partial N_A}{\partial \left( \frac{u}{U_1} \right)} = C_{n_u} \overline{q}_1 S b + \rho U_1^2 C_{n_1} S b = (C_{n_u} + 2C_{n_1}) \overline{q}_1 S b \tag{B.80}$$
APPENDIX B (continued)

B.3.5.2 Derivatives with respect to Angle-of-Attack

B.3.5.2.1 Aerodynamic Force in the $x$-direction ($F_{Ax}$)

The partial derivative of force in the $x$-direction with respect to angle-of-attack may be written according to equation (B.81):

$$\frac{\partial F_{Ax}}{\partial \alpha} = \frac{\partial C_x}{\partial \alpha} q S$$

(B.81)

The coefficient of force in the $x$-direction, $C_x$, may be defined according to equation (B.82) through application of the small angle assumption (see Figure 3.51 in [106]):

$$C_x = -C_D + C_L \alpha$$

(B.82)

Then, the partial derivative of $C_x$ with respect to angle-of-attack may be evaluated as follows:

$$\frac{\partial C_x}{\partial \alpha} = -\frac{\partial C_D}{\partial \alpha} + \frac{\partial C_L}{\partial \alpha} \alpha + C_L \frac{\partial \alpha}{\partial \alpha}$$

(B.83)

When evaluated in the steady state, equation (B.83) becomes:

$$\frac{\partial C_x}{\partial \alpha} = C_{x,\alpha} = -C_{D,\alpha} + C_{L,\alpha}$$

(B.84)

Substituting equation (B.84) into equation (B.81) yields:

$$\frac{\partial F_{Ax}}{\partial \alpha} = \frac{\partial C_x}{\partial \alpha} q_1 S = (-C_{D,\alpha} + C_{L,\alpha}) q_1 S$$

(B.85)
B.3.5.2.2 Aerodynamic Force in the z-direction ($F_{Az}$)

The partial derivative of force in the z-direction with respect to angle-of-attack may be written according to equation (B.86):

$$\frac{\partial F_{Az}}{\partial \alpha} = \frac{\partial C_z}{\partial \alpha} \bar{q}S \quad \text{(B.86)}$$

The coefficient of force in the z-direction, $C_z$, may be defined according to equation (B.87) through application of the small angle assumption (see Figure 3.51 in [106]):

$$C_z = -C_L - C_D \alpha \quad \text{(B.87)}$$

Then, the partial derivative of $C_z$ with respect to angle-of-attack may be evaluated as follows:

$$\frac{\partial C_z}{\partial \alpha} = -\frac{\partial C_L}{\partial \alpha} - \frac{\partial C_D}{\partial \alpha} \alpha - C_D \frac{\partial \alpha}{\partial \alpha} \quad \text{(B.88)}$$

When evaluated in the steady state, equation (B.88) becomes:

$$\frac{\partial C_z}{\partial \alpha} = C_{z\alpha} = -C_L \alpha - C_D \quad \text{(B.89)}$$

Substituting equation (B.89) into equation (B.86) yields:

$$\frac{\partial F_{Az}}{\partial \alpha} = \frac{\partial C_z}{\partial \alpha} \bar{q}_1 S = -(C_L \alpha + C_D) \bar{q}_1 S \quad \text{(B.90)}$$
APPENDIX B (continued)

B.3.5.2.3 Aerodynamic Pitching Moment ($M_A$)

The partial derivative of pitching moment with respect to angle-of-attack may be written according to equation (B.91):

$$\frac{\partial M_A}{\partial \alpha} = \frac{\partial C_m}{\partial \alpha} q \bar{S} \bar{c}$$  \hspace{1cm} (B.91)

Then, the partial derivative of $C_m$ with respect to angle-of-attack may be evaluated as follows:

$$\frac{\partial C_m}{\partial \alpha} = C_{m\alpha}$$  \hspace{1cm} (B.92)

Substituting equation (B.92) into equation (B.91) and evaluating at the steady state yields:

$$\frac{\partial M_A}{\partial \alpha} = C_{m\alpha} \bar{q}_1 \bar{S} \bar{c}$$  \hspace{1cm} (B.93)
APPENDIX B (continued)

B.3.5.2.4 Aerodynamic Force in the \( y \)-direction \( (F_{Ay}) \)

The partial derivative of force in the \( y \)-direction with respect to angle-of-attack may be written according to equation (B.94):

\[
\frac{\partial F_{Ay}}{\partial \alpha} = \frac{\partial C_y}{\partial \alpha} \frac{q}{S}\tag{B.94}
\]

The coefficient of force in the \( y \)-direction, \( C_y \), may be defined according to equation (B.95):

\[
C_y = C_Y\tag{B.95}
\]

Then, the partial derivative of \( C_y \) with respect to angle-of-attack may be evaluated as follows:

\[
\frac{\partial C_y}{\partial \alpha} = \frac{\partial C_Y}{\partial \alpha}\tag{B.96}
\]

When evaluated in the steady state, equation (B.96) becomes:

\[
\frac{\partial C_y}{\partial \alpha} = C_{ya}\tag{B.97}
\]

Substituting equation (B.97) into equation (B.94) yields:

\[
\frac{\partial F_{Ay}}{\partial \alpha} = \frac{\partial C_y}{\partial \alpha} \frac{q}{S} = C_{ya} \frac{q}{S}\tag{B.98}
\]
B.3.5.2.5 Aerodynamic Rolling Moment ($L_A$)

The partial derivative of rolling moment with respect to angle-of-attack may be written according to equation (B.99):

$$\frac{\partial L_A}{\partial \alpha} = \frac{\partial C_l}{\partial \alpha} q S b$$  \hspace{1cm} (B.99)

Then, the partial derivative of $C_l$ with respect to angle-of-attack may be evaluated as follows:

$$\frac{\partial C_l}{\partial \alpha} = C_{l\alpha}$$  \hspace{1cm} (B.100)

Substituting equation (B.100) into equation (B.99) and evaluating at the steady state yields:

$$\frac{\partial L_A}{\partial \alpha} = C_{l\alpha} \bar{q}_1 S b$$  \hspace{1cm} (B.101)
B.3.5.2.6 Aerodynamic Yawing Moment ($N_A$)

The partial derivative of yawing moment with respect to angle-of-attack may be written according to equation (B.102):

$$\frac{\partial N_A}{\partial \alpha} = \frac{\partial C_n}{\partial \alpha} \bar{q} S_b$$  \hspace{1cm} (B.102)

Then, the partial derivative of $C_n$ with respect to angle-of-attack may be evaluated as follows:

$$\frac{\partial C_n}{\partial \alpha} = C_{n \alpha}$$ \hspace{1cm} (B.103)

Substituting equation (B.103) into equation (B.102) and evaluating at the steady state yields:

$$\frac{\partial N_A}{\partial \alpha} = C_{n \alpha} \bar{q} S_b$$ \hspace{1cm} (B.104)
APPENDIX B (continued)

B.3.6 Assembled Non-Dimensional Aerodynamic & Thrust Force & Moment Derivatives

B.3.6.1 Aerodynamic Derivatives

Assembling the non-dimensional quasi-steady aerodynamic forces and moments, as specified by equations (B.44) to (B.49) gives the following matrix:

\[
\begin{bmatrix}
\frac{f_{Ax}}{q_1 S} \\
\frac{f_{Ay}}{q_1 S} \\
m_A \frac{m}{q_1 S c} \\
\frac{f_{Az}}{q_1 S} \\
l_A \frac{l}{q_1 S b} \\
n_A \frac{n}{q_1 S b} \\
\end{bmatrix} = \begin{bmatrix}
-(C_{D_a} + 2C_{D_1}) & -(C_{D_a} + C_{L_a}) & -C_{D_a} & -C_{D_q} & -C_{D_{ae}} & -C_{D_{at}} \\
-(C_{L_a} + 2C_{L_1}) & -(C_{L_a} + C_{D_1}) & -C_{L_a} & -C_{L_q} & -C_{L_{ae}} & -C_{L_{at}} \\
(C_{m_a} + 2C_{m_1}) & C_{m_a} & C_{m_a} & C_{m_q} & C_{m_{ae}} & C_{m_{at}} \\
(C_{C_{Y_a}} + 2C_{C_{Y_1}}) & C_{C_{Y_a}} & C_{C_{Y_a}} & C_{C_{Y_q}} & C_{C_{Y_{ae}}} & C_{C_{Y_{at}}} \\
(C_{l_a} + 2C_{l_1}) & C_{l_a} & C_{l_a} & C_{l_q} & C_{l_{ae}} & C_{l_{at}} \\
(C_{n_a} + 2C_{n_1}) & C_{n_a} & C_{n_a} & C_{n_q} & C_{n_{ae}} & C_{n_{at}} \\
\end{bmatrix}
\]

\[
\begin{bmatrix}
u \\
u/\gamma \\
\alpha \\
\alpha r/2\gamma \\
\alpha r/2\gamma \\
\end{bmatrix} = \begin{bmatrix}
-C_{D_p} & -C_{D_\beta} & -C_{D_r} & -C_{D_{sa}} & -C_{D_{sr}} \\
-C_{L_p} & -C_{L_\beta} & -C_{L_r} & -C_{L_{sa}} & -C_{L_{sr}} \\
C_{m_p} & C_{m_\beta} & C_{m_r} & C_{m_{sa}} & C_{m_{sr}} \\
\ldots \\
C_{C_{Y_p}} & C_{C_{Y_\beta}} & C_{C_{Y_r}} & C_{C_{Y_{sa}}} & C_{C_{Y_{sr}}} \\
C_{l_p} & C_{l_\beta} & C_{l_r} & C_{l_{sa}} & C_{l_{sr}} \\
C_{n_p} & C_{n_\beta} & C_{n_r} & C_{n_{sa}} & C_{n_{sr}} \\
\end{bmatrix}
\]

\[
\begin{bmatrix}
\delta_e \\
\delta_t \\
\delta_r \\
\end{bmatrix}
\]

\[
\begin{bmatrix}
\frac{\dot{\beta}}{2\gamma} \\
\frac{\dot{\gamma}}{2\gamma} \\
\delta_a \\
\end{bmatrix}
\]

(B.105)
B.3.6.2 Thust Derivatives

Assembling the non-dimensional quasi-steady thrust forces and moments, as specified in equations (B.51) to (B.56) gives the following matrix:

\[
\begin{bmatrix}
\frac{f_{Tx}}{\bar{q}_1 S} \\
\frac{f_{Tz}}{\bar{q}_1 S} \\
\frac{m_T}{\bar{q}_1 S \bar{c}} \\
\frac{f_{Ty}}{\bar{q}_1 S} \\
\frac{l_T}{\bar{q}_1 \bar{S}b} \\
\frac{n_T}{\bar{q}_1 \bar{S}b}
\end{bmatrix}
\begin{bmatrix}
C_{Tx_0} + 2C_{Tx_1} \\
C_{Tz_0} + 2C_{Tz_1} \\
C_{Tz_0} + 2C_{Tz_1} \\
C_{Ty_0} + 2C_{Ty_1} \\
C_{nT_0} + 2C_{nT_1}
\end{bmatrix}
\begin{bmatrix}
C_{Ts_0} \\
C_{Ts_0} \\
C_{Ts_0} \\
C_{Ty_0} \\
C_{nT_0}
\end{bmatrix}
\begin{bmatrix}
u \\
U_1 \alpha \\
\beta
\end{bmatrix}
= \begin{bmatrix}
\frac{u}{U_1} \alpha \\
\frac{\bar{c}}{U_1} \alpha \\
\frac{\bar{c}}{U_1} \beta
\end{bmatrix}
\]

(B.106)
B.4 Development of the Dimensional Stability Derivatives

At this point, the goal is then to obtain the dimensional stability derivatives, which allow for better qualitative understanding of the forces and moments. The general process of this step involves substituting the aerodynamic and thrust derivatives given in equations (B.105) and (B.106) into the perturbed force and moment equations of motion given in equations (B.26) to (B.31). The force equations are then divided by mass and the moment equations by respective moments of inertia, and the non-dimensional derivatives then become “dimensionalized”.

While existing literature has developed these dimensional derivatives for the decoupled longitudinal and lateral/directional modes, the aircraft model utilized in the LOC prediction architecture accounts for full dynamic cross-coupling. It is therefore necessary to derive the dimensional derivatives for these cross-coupled modes. The dimensional stability derivatives for the forces in the $x$-, $y$-, and $z$- directions and for the moments $L$, $M$, and $N$ are presented below.
B.4.1 Force in $x$-direction

Recall the force equation in the $x$-direction, given by equation (B.26) and shown again in equation (B.107):

$$m(\ddot{u} - V_1 r - R_1 v + W_1 q + Q_1 w) = -mg \theta \cos \Theta_1 + f_{Ax} + f_{Tx} \quad \text{(B.107)}$$

Substituting $f_{Ax}$ and $f_{Tx}$ from equations (B.105) and (B.106) into equation (B.26) gives:

$$m(\ddot{u} - V_1 r - R_1 v + W_1 q + Q_1 w) = -mg \theta \cos \Theta_1 +$$

$$\bar{q}_1 \left[ -(C_{Du} + 2C_{D_1}) \frac{u}{U_1} + (-C_{Da} + C_{L_1}) \alpha - C_{D_n} \frac{\bar{\alpha}}{2U_1} - C_{D_q} \frac{q}{2U_1} - C_{D_e} \delta_e - C_{D_t} \delta_t \right.$$

$$-C_{D_\beta} \beta - C_{D_\theta} \frac{\bar{b}}{2U_1} - C_{D_p} \frac{\bar{p}}{2U_1} - C_{D_p} \frac{\bar{r}}{2U_1} - C_{D_\delta_a} \delta_a - C_{D_\delta_r} \delta_r$$

$$\left. + \bar{q}_1 \left[ (C_{T_{xu}} + 2C_{T_{x1}}) \frac{u}{U_1} + C_{T_{xa}} \alpha + C_{T_{x_\beta}} \beta \right] \right] \quad \text{(B.108)}$$

Then, dividing by the mass $m$ yields:

$$(\ddot{u} - V_1 r - R_1 v + W_1 q + Q_1 w) = -g \theta \cos \Theta_1 +$$

$$\frac{\bar{q}_1 S}{m} \left[ -(C_{Du} + 2C_{D_1}) \frac{u}{U_1} + (-C_{Da} + C_{L_1}) \alpha - C_{D_n} \frac{\bar{\alpha}}{2U_1} - C_{D_q} \frac{q}{2U_1} - C_{D_e} \delta_e - C_{D_t} \delta_t \right.$$

$$-C_{D_\beta} \beta - C_{D_\theta} \frac{\bar{b}}{2U_1} - C_{D_p} \frac{\bar{p}}{2U_1} - C_{D_p} \frac{\bar{r}}{2U_1} - C_{D_\delta_a} \delta_a - C_{D_\delta_r} \delta_r$$

$$\left. + \frac{\bar{q}_1 S}{m} \left[ (C_{T_{xu}} + 2C_{T_{x1}}) \frac{u}{U_1} + C_{T_{xa}} \alpha + C_{T_{x_\beta}} \beta \right] \right] \quad \text{(B.109)}$$

From equation (B.109), the dimensional stability derivatives in Table B.13 may be obtained:
### APPENDIX B (continued)

#### TABLE B.13

**DIMENSIONAL STABILITY DERIVATIVES IN THE $X$-DIRECTION**

<table>
<thead>
<tr>
<th>Derivatives with respect to Longitudinal Variables</th>
<th>Derivatives with respect to Lateral/Directional Variables</th>
</tr>
</thead>
<tbody>
<tr>
<td><strong>Aerodynamic Derivatives</strong></td>
<td></td>
</tr>
<tr>
<td>$X_u = \frac{-\bar{q}<em>1 S(C</em>{D_u} + 2C_{D_u})}{mU_1}$</td>
<td>$X_\beta = \frac{-\bar{q}<em>1 S(C</em>{D_\beta})}{m}$</td>
</tr>
<tr>
<td>$X_\alpha = \frac{-\bar{q}<em>1 S(-C</em>{D_\alpha} + C_{L_1})}{m}$</td>
<td>$X_\beta = \frac{-\bar{q}<em>1 Sb(C</em>{D_\beta})}{2mU_1}$</td>
</tr>
<tr>
<td>$X_\delta = \frac{-\bar{q}<em>1 S\bar{C}(C</em>{D_\delta})}{2mU_1}$</td>
<td>$X_p = \frac{-\bar{q}<em>1 Sb(C</em>{D_p})}{2mU_1}$</td>
</tr>
<tr>
<td>$X_q = \frac{-\bar{q}<em>1 S\bar{C}(C</em>{D_q})}{2mU_1}$</td>
<td>$X_r = \frac{-\bar{q}<em>1 Sb(C</em>{D_r})}{2mU_1}$</td>
</tr>
<tr>
<td><strong>Thrust Derivatives</strong></td>
<td></td>
</tr>
<tr>
<td>$X_{T_u} = \frac{-\bar{q}<em>1 S(C</em>{T_{x_u}} + 2C_{T_{x_1}})}{mU_1}$</td>
<td>$X_{T_\beta} = \frac{-\bar{q}<em>1 S(C</em>{T_{x_\beta}})}{m}$</td>
</tr>
<tr>
<td>$X_{T_\alpha} = \frac{-\bar{q}<em>1 S(C</em>{T_{x_\alpha}})}{m}$</td>
<td></td>
</tr>
<tr>
<td><strong>Control Derivatives</strong></td>
<td></td>
</tr>
<tr>
<td>$X_{\delta_e} = \frac{-\bar{q}<em>1 S(C</em>{D_{\delta_e}})}{m}$</td>
<td>$X_{\delta_a} = \frac{-\bar{q}<em>1 S(C</em>{D_{\delta_a}})}{m}$</td>
</tr>
<tr>
<td>$X_{\delta_t} = \frac{-\bar{q}<em>1 S(C</em>{D_{\delta_t}})}{m}$</td>
<td>$X_{\delta_r} = \frac{-\bar{q}<em>1 S(C</em>{D_{\delta_r}})}{m}$</td>
</tr>
</tbody>
</table>
B.4.2 Force in $y$-direction

Recall the force equation in the $y$-direction, given by equation (B.27) and shown again in equation (B.110):

$$m(\dot{u} - V_1 r - R_1 v + W_1 q + Q_1 w) = -mg \theta \cos \Theta_1 + f_{A_y} + f_{T_y} \quad \text{(B.110)}$$

Substituting $f_{A_y}$ and $f_{T_y}$ from equations (B.105) and (B.106) into equation (B.27) gives:

$$m(\dot{v} + U_1 r + R_1 u - W_1 p - P_1 w) = -mg \theta \sin \Phi_1 \sin \Theta_1 + mg \phi \cos \Phi_1 \cos \Theta_1 +$$

$$\bar{q}_1 S \left[ (C_{y_a} + 2C_{y_1}) \frac{u}{U_1} + C_{y_a} \alpha + C_{y_a} \frac{\dot{c}}{2U_1} + C_{y_2} \frac{q c}{2U_1} + C_{y_\delta} \delta_e + C_{y_\delta} \delta_t \right.$$  

$$+ C_{y_\beta} \beta + C_{y_p} \frac{\beta b}{2U_1} + C_{y_p} \frac{p b}{2U_1} + C_{y_{\delta}} \delta_a + C_{y_{\delta}} \delta_r \left.\right] \quad \text{(B.111)}$$

$$+ \bar{q}_1 S \left[ (C_{T_{y_a}} + 2C_{T_{y_1}}) \frac{u}{U_1} + C_{T_{y_a}} \alpha + C_{T_{y_\beta}} \beta \right]$$

Then, dividing by the mass $m$ yields:

$$(\dot{v} + U_1 r + R_1 u - W_1 p - P_1 w) = -g \theta \sin \Phi_1 \sin \Theta_1 + g \phi \cos \Phi_1 \cos \Theta_1 +$$

$$\frac{\bar{q}_1 S}{m} \left[ (C_{y_a} + 2C_{y_1}) \frac{u}{U_1} + C_{y_a} \alpha + C_{y_a} \frac{\dot{c}}{2U_1} + C_{y_2} \frac{q c}{2U_1} + C_{y_\delta} \delta_e + C_{y_\delta} \delta_t \right.$$  

$$+ C_{y_\beta} \beta + C_{y_p} \frac{\beta b}{2U_1} + C_{y_p} \frac{p b}{2U_1} + C_{y_{\delta}} \delta_a + C_{y_{\delta}} \delta_r \left.\right] \quad \text{(B.112)}$$

$$+ \frac{\bar{q}_1 S}{m} \left[ (C_{T_{y_a}} + 2C_{T_{y_1}}) \frac{u}{U_1} + C_{T_{y_a}} \alpha + C_{T_{y_\beta}} \beta \right]$$

From equation (B.112), the dimensional stability derivatives in Table B.14 may be obtained:
### TABLE B.14

**DIMENSIONAL STABILITY DERIVATIVES IN THE Y-DIRECTION**

<table>
<thead>
<tr>
<th>Derivatives with respect to Longitudinal Variables</th>
<th>Derivatives with respect to Lateral/Directional Variables</th>
</tr>
</thead>
<tbody>
<tr>
<td><strong>Aerodynamic Derivatives</strong></td>
<td></td>
</tr>
<tr>
<td>$Y_u = \frac{\bar{q}<em>1 S (C</em>{yu} + 2C_{y_i})}{mU_1}$</td>
<td>$Y_\beta = \frac{\bar{q}<em>1 S (C</em>{y\beta})}{m}$</td>
</tr>
<tr>
<td>$Y_\alpha = \frac{\bar{q}<em>1 S (C</em>{y\alpha})}{m}$</td>
<td></td>
</tr>
<tr>
<td>$Y_\alpha = \frac{\bar{q}<em>1 S (C</em>{y\alpha})}{2mU_1}$</td>
<td></td>
</tr>
<tr>
<td>$Y_q = \frac{\bar{q}<em>1 S \bar{c} (C</em>{y_d})}{2mU_1}$</td>
<td>$Y_p = \frac{\bar{q}<em>1 S b (C</em>{yp})}{2mU_1}$</td>
</tr>
<tr>
<td><strong>Thrust Derivatives</strong></td>
<td></td>
</tr>
<tr>
<td>$Y_{Tu} = \frac{\bar{q}<em>1 S (C</em>{Tu,y} + 2C_{Tu,y_1})}{mU_1}$</td>
<td>$Y_{T\beta} = \frac{\bar{q}<em>1 S (C</em>{Ty\beta})}{m}$</td>
</tr>
<tr>
<td>$Y_{Ta} = \frac{\bar{q}<em>1 S (C</em>{Ty\alpha})}{m}$</td>
<td></td>
</tr>
<tr>
<td><strong>Control Derivatives</strong></td>
<td></td>
</tr>
<tr>
<td>$Y_{\delta e} = \frac{\bar{q}<em>1 S (C</em>{\delta e})}{m}$</td>
<td>$Y_{\delta a} = \frac{\bar{q}<em>1 S (C</em>{\delta a})}{m}$</td>
</tr>
<tr>
<td>$Y_{\delta t} = \frac{\bar{q}<em>1 S (C</em>{\delta t})}{m}$</td>
<td>$Y_{\delta r} = \frac{\bar{q}<em>1 S (C</em>{\delta r})}{m}$</td>
</tr>
</tbody>
</table>
B.4.3 Force in $z$-direction

Recall the force equation in the $z$-direction, given by equation (B.28) and shown again in equation (B.113):

$$m(\dot{w} - U_1 q - Q_1 u + V_1 p + P_1 v) = -mg \theta \cos \Phi_1 \sin \Theta_1 - mg \phi \sin \Phi_1 \cos \Theta_1 + f_{A_x} + f_{T_x} \quad (B.113)$$

Substituting $f_{A_x}$ and $f_{T_x}$ from equations (B.105) and (B.106) into equation (B.28) gives:

$$m(\dot{w} - U_1 q - Q_1 u + V_1 p + P_1 v) = -mg \theta \cos \Phi_1 \sin \Theta_1 - mg \phi \sin \Phi_1 \cos \Theta_1 +$$

$$\bar{q}_1 S \left[ -(C_{l_u} + 2C_{L_1}) \frac{u}{U_1} - (C_{l_alpha} + C_{D_1}) \alpha - C_{l_a} \frac{\alpha \bar{c}}{2U_1} - C_{l_q} \frac{q \bar{c}}{2U_1} - C_{l_\delta e} \delta_e - C_{l_\delta_t} \delta_t 
- C_{L_\beta} \beta - C_{L_p} \frac{\dot{\beta}b}{2U_1} - C_{L_p} \frac{pb}{2U_1} - C_{L_r} \frac{rb}{2U_1} - C_{L_\alpha \delta_\alpha} - C_{L_\eta \delta_\eta} \right]$$

$$+ \bar{q}_1 S \left[ (C_{T_u} + 2C_{T_1}) \frac{u}{U_1} + C_{T_\alpha} \alpha + C_{T_\beta} \beta \right] \quad (B.114)$$

Then, dividing by the mass $m$ yields:

$$\frac{\bar{q}_1 S}{m} \left[ -(C_{l_u} + 2C_{L_1}) \frac{u}{U_1} - (C_{l_alpha} + C_{D_1}) \alpha - C_{l_a} \frac{\alpha \bar{c}}{2U_1} - C_{l_q} \frac{q \bar{c}}{2U_1} - C_{l_\delta e} \delta_e - C_{l_\delta_t} \delta_t 
- C_{L_\beta} \beta - C_{L_p} \frac{\dot{\beta}b}{2U_1} - C_{L_p} \frac{pb}{2U_1} - C_{L_r} \frac{rb}{2U_1} - C_{L_\alpha \delta_\alpha} - C_{L_\eta \delta_\eta} \right]$$

$$+ \frac{\bar{q}_1 S}{m} \left[ (C_{T_u} + 2C_{T_1}) \frac{u}{U_1} + C_{T_\alpha} \alpha + C_{T_\beta} \beta \right] \quad (B.115)$$

From equation (B.115), the dimensional stability derivatives in Table B.15 may be obtained:
TABLE B.15

DIMENSIONAL STABILITY DERIVATIVES IN THE Z-DIRECTION

<table>
<thead>
<tr>
<th>Derivatives with respect to Longitudinal Variables</th>
<th>Derivatives with respect to Lateral/Directional Variables</th>
</tr>
</thead>
<tbody>
<tr>
<td><strong>Aerodynamic Derivatives</strong></td>
<td></td>
</tr>
<tr>
<td>$Z_u = \frac{-\bar{q}<em>1S(c</em>{L_u} + 2c_{L_1})}{mU_1}$</td>
<td>$Z_\beta = \frac{-\bar{q}<em>1S(c</em>{L_\beta})}{m}$</td>
</tr>
<tr>
<td>$Z_\alpha = \frac{-\bar{q}<em>1S(c</em>{L_\alpha} + c_{D_1})}{m}$</td>
<td>$Z_\beta = \frac{-\bar{q}<em>1Sb(c</em>{L_\beta})}{2mU_1}$</td>
</tr>
<tr>
<td>$Z_\alpha = \frac{-\bar{q}<em>1S(c</em>{L_\alpha} + 2c_{L_1})}{2mU_1}$</td>
<td>$Z_p = \frac{-\bar{q}<em>1Sb(c</em>{L_p})}{2mU_1}$</td>
</tr>
<tr>
<td>$Z_q = \frac{-\bar{q}<em>1S\bar{c}(c</em>{Lq})}{2mU_1}$</td>
<td></td>
</tr>
<tr>
<td>$Z_r = \frac{-\bar{q}<em>1Sb(c</em>{L_r})}{2mU_1}$</td>
<td></td>
</tr>
</tbody>
</table>

| **Thrust Derivatives**                            |                                                          |
| $Z_{\tau_u} = \frac{-\bar{q}_1S(c_{\tau_{zu}} + 2c_{\tau_{z1}})}{mU_1}$ | $Z_{\tau_\beta} = \frac{-\bar{q}_1S(c_{\tau_{z\beta}})}{m}$ |
| $Z_{\tau_\alpha} = \frac{-\bar{q}_1S(c_{\tau_{z\alpha}})}{m}$ |                                                          |

| **Control Derivatives**                           |                                                          |
| $Z_{\delta_e} = \frac{-\bar{q}_1S(c_{L_\delta e})}{m}$ | $Z_{\delta_a} = \frac{-\bar{q}_1S(c_{L_\delta a})}{m}$ |
| $Z_{\delta_t} = \frac{-\bar{q}_1S(c_{L_\delta t})}{m}$ | $Z_{\delta_r} = \frac{-\bar{q}_1S(c_{L_\delta r})}{m}$ |
Recall the rolling moment equation given by equation (B.29) and shown again in equation (B.116):

\[
I_{xx} \ddot{p} - I_{xy} \dot{q} - I_{xz} \ddot{r} - I_{xz}(P_1q + Q_1p) + (I_{zz} - I_{yy})(R_1q + Q_1r) + I_{xy}(P_1r + R_1p) + I_{yz}(2R_1r - 2Q_1q) = l_A + l_T
\]  

(B.116)

Substituting \( l_A \) and \( l_T \) from equations (B.105) and (B.106) into equation (B.29) gives:

\[
I_{xx} \ddot{p} - I_{xy} \dot{q} - I_{xz} \ddot{r} - I_{xz}(P_1q + Q_1p) + (I_{zz} - I_{yy})(R_1q + Q_1r) + \\
I_{xy}(P_1r + R_1p) + I_{yz}(2R_1r - 2Q_1q) = \\
\frac{\bar{q}_1 S b}{I_{xx}} \left[ (C_{tu} + 2C_{t_1}) \frac{u}{U_1} + C_{t_0} \alpha + C_{t_0} \frac{\alpha c}{2U_1} + C_{t_0} \frac{q c}{2U_1} + C_{t_0} \delta_e + C_{t_0} \delta_t \right] \\
+ C_{t_0} \beta + C_{t_0} \frac{\beta b}{2U_1} + C_{t_0} \frac{p b}{2U_1} + C_{t_0} \frac{r b}{2U_1} + C_{t_0} \delta_a + C_{t_0} \delta_r \\
+ \frac{\bar{q}_1 S b}{I_{xx}} \left[ (C_{t_{ra}} + 2C_{t_{ra}}) \frac{u}{U_1} + C_{t_{ra}} \alpha + C_{t_{ra}} \beta \right]
\]  

(B.117)

Then, dividing by the moment of inertia \( I_{xx} \) yields:

\[
\dot{p} - \frac{I_{xy}}{I_{xx}} \dot{q} - \frac{I_{xz}}{I_{xx}} \ddot{r} - \frac{I_{xz}}{I_{xx}}(P_1q + Q_1p) + (\frac{I_{zz} - I_{yy}}{I_{xx}})(R_1q + Q_1r) + \\
\frac{I_{xy}}{I_{xx}}(P_1r + R_1p) + \frac{I_{yz}}{I_{xx}}(2R_1r - 2Q_1q) = \\
\frac{\bar{q}_1 S b}{I_{xx}} \left[ (C_{tu} + 2C_{t_1}) \frac{u}{U_1} + C_{t_0} \alpha + C_{t_0} \frac{\alpha c}{2U_1} + C_{t_0} \frac{q c}{2U_1} + C_{t_0} \delta_e + C_{t_0} \delta_t \right] \\
+ C_{t_0} \beta + C_{t_0} \frac{\beta b}{2U_1} + C_{t_0} \frac{p b}{2U_1} + C_{t_0} \frac{r b}{2U_1} + C_{t_0} \delta_a + C_{t_0} \delta_r \\
+ \frac{\bar{q}_1 S b}{I_{xx}} \left[ (C_{t_{ra}} + 2C_{t_{ra}}) \frac{u}{U_1} + C_{t_{ra}} \alpha + C_{t_{ra}} \beta \right]
\]  

(B.118)

From equation (B.118), the dimensional stability derivatives in Table B.16 may be obtained.
### APPENDIX B (continued)

**TABLE B.16**

**DIMENSIONAL STABILITY DERIVATIVES ABOUT ROLL AXIS**

<table>
<thead>
<tr>
<th>Derivatives with respect to Longitudinal Variables</th>
<th>Derivatives with respect to Lateral/Directional Variables</th>
</tr>
</thead>
<tbody>
<tr>
<td><strong>Aerodynamic Derivatives</strong></td>
<td></td>
</tr>
<tr>
<td>$L_u = \frac{q_1 S b (c_{iu} + 2c_{i_1})}{l_{xx} U_1}$</td>
<td>$L_\alpha = \frac{q_1 S b (c_{i_\alpha})}{l_{xx}}$</td>
</tr>
<tr>
<td>$L_\alpha = \frac{q_1 S b (c_{i_\alpha})}{l_{xx}}$</td>
<td>$L_\beta = \frac{q_1 S b^2 (c_{i_\beta})}{2l_{xx} U_1}$</td>
</tr>
<tr>
<td>$L_\dot{\alpha} = \frac{q_1 S \ddot{c} b (c_{i_\alpha})}{2l_{xx} U_1}$</td>
<td>$L_\dot{\beta} = \frac{q_1 S b^2 (c_{i_\beta})}{2l_{xx} U_1}$</td>
</tr>
<tr>
<td>$L_q = \frac{q_1 S \ddot{c} b (c_{i_q})}{2l_{xx} U_1}$</td>
<td></td>
</tr>
<tr>
<td>$L_r = \frac{q_1 S b (c_{i_r})}{l_{xx} U_1}$</td>
<td></td>
</tr>
<tr>
<td><strong>Thrust Derivatives</strong></td>
<td></td>
</tr>
<tr>
<td>$L_{T_u} = \frac{q_1 S b (c_{T_u} + 2c_{T_1})}{l_{xx} U_1}$</td>
<td>$L_{T_\beta} = \frac{q_1 S b (c_{T_\beta})}{l_{xx}}$</td>
</tr>
<tr>
<td>$L_{T_\alpha} = \frac{q_1 S b (c_{T_\alpha})}{l_{xx}}$</td>
<td></td>
</tr>
<tr>
<td><strong>Control Derivatives</strong></td>
<td></td>
</tr>
<tr>
<td>$L_{\delta_e} = \frac{q_1 S b (c_{i_{\delta_e}})}{l_{xx}}$</td>
<td>$L_{\delta_a} = \frac{q_1 S b (c_{i_{\delta_a}})}{l_{xx}}$</td>
</tr>
<tr>
<td>$L_{\delta_t} = \frac{q_1 S b (c_{i_{\delta_t}})}{l_{xx}}$</td>
<td>$L_{\delta_r} = \frac{q_1 S b (c_{i_{\delta_r}})}{l_{xx}}$</td>
</tr>
</tbody>
</table>
APPENDIX B (continued)

B.4.5 Pitching Moment $M$

Recall the pitching moment equation given by equation (B.30) and shown again in equation (B.119):

$$I_{yy} \dot{q} - I_{xy} \dot{p} - I_{yz} \dot{r} + (I_{xx} - I_{zz})(P_1 r + R_1 p) + I_{xz}(2P_1 p - 2R_1 r) - I_{xy}(Q_1 r + R_1 q) + I_{yz}(P_1 q + Q_1 p) = m_a + m_T$$  \hspace{1cm} (B.119)

Substituting $m_A$ and $m_T$ from equations (B.105) and (B.106) into equation (B.30) gives:

$$I_{yy} \dot{q} - I_{xy} \dot{p} - I_{yz} \dot{r} + (I_{xx} - I_{zz})(P_1 r + R_1 p) + I_{xz}(2P_1 p - 2R_1 r) -$$

$$I_{xy}(Q_1 r + R_1 q) + I_{yz}(P_1 q + Q_1 p) =$$

$$\bar{q}_1 SC \left[ \left( c_{m_a} + 2c_{m_1} \right) \frac{u}{U_1} + c_{m_a} \alpha + c_{m_a} \frac{\dot{\alpha}}{2U_1} + c_{m_q} \frac{q \dot{c}}{2U_1} + c_{m_\delta_e} \delta_e + c_{m_\delta_t} \delta_t \right]$$

$$+ c_{m_\beta} \beta + c_{m_\dot{\beta}} \frac{\dot{\beta}}{2U_1} + c_{m_\beta} \frac{p \dot{b}}{2U_1} + c_{m_r} \frac{r \dot{b}}{2U_1} + c_{m_\delta_a} \delta_a + c_{m_\delta_r} \delta_r \right]$$

$$+ \bar{q}_1 SC \left[ \left( c_{m_{\tau_a}} + 2c_{m_{\tau_1}} \right) \frac{u}{U_1} + c_{m_{\tau_a}} \alpha + c_{m_{\tau_\beta}} \beta \right]$$  \hspace{1cm} (B.120)

Then, dividing by the moment of inertia $I_{yy}$ yields:

$$\ddot{q} - \frac{l_{xy}}{I_{yy}} \ddot{p} - \frac{l_{yz}}{I_{yy}} \ddot{r} + \left( \frac{l_{xx} - l_{zz}}{I_{yy}} \right) (P_1 r + R_1 p) + \frac{l_{xz}}{I_{yy}}(2P_1 p - 2R_1 r) -$$

$$\frac{l_{xy}}{I_{yy}}(Q_1 r + R_1 q) + \frac{l_{yz}}{I_{yy}}(P_1 q + Q_1 p) =$$

$$\bar{q}_1 SC \left[ \left( c_{m_a} + 2c_{m_1} \right) \frac{u}{U_1} + c_{m_a} \alpha + c_{m_a} \frac{\dot{\alpha}}{2U_1} + c_{m_q} \frac{q \dot{c}}{2U_1} + c_{m_\delta_e} \delta_e + c_{m_\delta_t} \delta_t \right]$$

$$+ c_{m_\beta} \beta + c_{m_\dot{\beta}} \frac{\dot{\beta}}{2U_1} + c_{m_\beta} \frac{p \dot{b}}{2U_1} + c_{m_r} \frac{r \dot{b}}{2U_1} + c_{m_\delta_a} \delta_a + c_{m_\delta_r} \delta_r \right]$$

$$+ \bar{q}_1 SC \left[ \left( c_{m_{\tau_a}} + 2c_{m_{\tau_1}} \right) \frac{u}{U_1} + c_{m_{\tau_a}} \alpha + c_{m_{\tau_\beta}} \beta \right]$$  \hspace{1cm} (B.121)

From equation (B.121), the dimensional stability derivatives in Table B.17 may be obtained:
### TABLE B.17
DIMENSIONAL STABILITY DERIVATIVES ABOUT PITCH AXIS

<table>
<thead>
<tr>
<th>Derivatives with respect to Longitudinal Variables</th>
<th>Derivatives with respect to Lateral/Directional Variables</th>
</tr>
</thead>
<tbody>
<tr>
<td><strong>Aerodynamic Derivatives</strong></td>
<td></td>
</tr>
<tr>
<td>( M_u = \frac{\bar{q}<em>1 \bar{S} \bar{e} (C</em>{m_u} + 2C_{m_1})}{I_{yy} U_1} )</td>
<td>( M_\beta = \frac{\bar{q}<em>1 \bar{S} \bar{e} (C</em>{m_\beta})}{I_{yy}} )</td>
</tr>
<tr>
<td>( M_\alpha = \frac{\bar{q}<em>1 \bar{S} \bar{e} (C</em>{m_\alpha})}{I_{yy}} )</td>
<td>( M_\beta = \frac{\bar{q}<em>1 \bar{S} \bar{e} b (C</em>{m_\beta})}{2I_{yy} U_1} )</td>
</tr>
<tr>
<td>( M_\alpha = \frac{\bar{q}<em>1 \bar{S} \bar{e}^2 (C</em>{m_\alpha})}{2I_{yy} U_1} )</td>
<td>( M_\rho = \frac{\bar{q}<em>1 \bar{S} \bar{e} b (C</em>{m_\rho})}{2I_{yy} U_1} )</td>
</tr>
<tr>
<td>( M_q = \frac{\bar{q}<em>1 \bar{S} \bar{e}^2 (C</em>{m_q})}{2I_{yy} U_1} )</td>
<td>( M_r = \frac{\bar{q}<em>1 \bar{S} \bar{e} b (C</em>{m_r})}{2I_{yy} U_1} )</td>
</tr>
<tr>
<td><strong>Thrust Derivatives</strong></td>
<td></td>
</tr>
<tr>
<td>( M_{T_u} = \frac{\bar{q}<em>1 \bar{S} \bar{e} (C</em>{m_{T_u}} + 2C_{m_{T_1}})}{I_{yy} U_1} )</td>
<td>( M_{T_\beta} = \frac{\bar{q}<em>1 \bar{S} \bar{e} (C</em>{m_{T_\beta}})}{I_{yy}} )</td>
</tr>
<tr>
<td>( M_{T_\alpha} = \frac{\bar{q}<em>1 \bar{S} \bar{e} (C</em>{m_{T_\alpha}})}{I_{yy}} )</td>
<td></td>
</tr>
<tr>
<td><strong>Control Derivatives</strong></td>
<td></td>
</tr>
<tr>
<td>( M_{\delta_e} = \frac{\bar{q}<em>1 \bar{S} \bar{e} (C</em>{m_{\delta_e}})}{I_{yy}} )</td>
<td>( M_{\delta_\alpha} = \frac{\bar{q}<em>1 \bar{S} \bar{e} (C</em>{m_{\delta_\alpha}})}{I_{yy}} )</td>
</tr>
<tr>
<td>( M_{\delta_t} = \frac{\bar{q}<em>1 \bar{S} \bar{e} (C</em>{m_{\delta_t}})}{I_{yy}} )</td>
<td>( M_{\delta_r} = \frac{\bar{q}<em>1 \bar{S} \bar{e} (C</em>{m_{\delta_r}})}{I_{yy}} )</td>
</tr>
</tbody>
</table>
B.4.6 Yawing Moment $N$

Recall the yawing moment equation given by equation (B.31) and shown again in equation (B.122):

$$\dot{\gamma} - \frac{l_{zy}}{l_{xz}} \dot{q} - \frac{l_{xz}}{l_{xz}} \dot{p} + \left( l_{yy} - l_{xx} \right) \left( P_1 q + Q_1 p \right) + l_{x y} (Q_1 r + R_1 q) + l_{x y} (2Q_1 q - 2P_1 p) - l_{yz} (P_1 r + R_1 p) = n_A + n_T$$  \hspace{1cm} (B.122)

Substituting $n_A$ and $n_T$ from equations (B.105) and (B.106) into equation (B.31) gives:

$$\dot{\gamma} - \frac{l_{zy}}{l_{xz}} \dot{q} - \frac{l_{xz}}{l_{xz}} \dot{p} + \left( l_{yy} - l_{xx} \right) \left( P_1 q + Q_1 p \right) + l_{x y} (Q_1 r + R_1 q) + l_{x y} (2Q_1 q - 2P_1 p) - l_{yz} (P_1 r + R_1 p) = \overline{q}_1 S b \left[ \left( C_{n_u} + 2C_{n_1} \right) \frac{u}{U_1} + C_{n_a} \alpha + \left( \alpha \overline{C} + C_{n_{\alpha}} \frac{\alpha}{2U_1} + C_{n_{\tau \delta}} \delta_e \right) + C_{n_{\tau t}} \delta_t \right]

+ \frac{\beta b}{2U_1} + \frac{p b}{2U_1} + \frac{r b}{2U_1} + C_{n_{\tau a}} \delta_e + C_{n_{\tau r}} \delta_r \right]

+ \overline{q}_1 S b \left[ \left( C_{n_{\tau u}} + 2C_{n_{\tau 1}} \right) \frac{u}{U_1} + \right. \left. C_{n_{\tau a}} \alpha + C_{n_{\tau r}} \beta \right]$$  \hspace{1cm} (B.123)

Then, dividing by the moment of inertia $l_{xz}$ yields:

$$\frac{\dot{\gamma}}{l_{xz}} - \frac{l_{zy}}{l_{xz}} \frac{\dot{q}}{l_{xz}} - \frac{l_{xz}}{l_{xz}} \frac{\dot{p}}{l_{xz}} + \frac{\left( l_{yy} - l_{xx} \right)}{l_{xz}} \left( P_1 q + Q_1 p \right) + \frac{l_{x y}}{l_{xz}} (Q_1 r + R_1 q) + \frac{l_{x y}}{l_{xz}} (2Q_1 q - 2P_1 p) - \frac{l_{yz}}{l_{xz}} (P_1 r + R_1 p) = \frac{\overline{q}_1 S b}{l_{xz}} \left[ \left( C_{n_u} + 2C_{n_1} \right) \frac{u}{U_1} + C_{n_a} \alpha + \left( \alpha \overline{C} + C_{n_{\alpha}} \frac{\alpha}{2U_1} + C_{n_{\tau \delta}} \delta_e \right) + C_{n_{\tau t}} \delta_t \right]

+ \frac{\beta b}{2U_1} + \frac{p b}{2U_1} + \frac{r b}{2U_1} + C_{n_{\tau a}} \delta_e + C_{n_{\tau r}} \delta_r \right]

+ \frac{\overline{q}_1 S b}{l_{xz}} \left[ \left( C_{n_{\tau u}} + 2C_{n_{\tau 1}} \right) \frac{u}{U_1} + C_{n_{\tau a}} \alpha + C_{n_{\tau r}} \beta \right]$$  \hspace{1cm} (B.124)

From equation (B.124), the dimensional stability derivatives in Table B.18 may be obtained:

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### Table B.18

**Dimensional Stability Derivatives About Yaw Axis**

<table>
<thead>
<tr>
<th>Derivatives with respect to Longitudinal Variables</th>
<th>Derivatives with respect to Lateral/Directional Variables</th>
</tr>
</thead>
<tbody>
<tr>
<td><strong>Aerodynamic Derivatives</strong></td>
<td></td>
</tr>
<tr>
<td>$N_u = \frac{\bar{q}<em>1 S b (C</em>{n_u} + 2C_{n_1})}{I_{zz}U_1}$</td>
<td>$N_{\beta} = \frac{\bar{q}<em>1 S b (C</em>{n_{\beta}})}{I_{zz}}$</td>
</tr>
<tr>
<td>$N_{\alpha} = \frac{\bar{q}<em>1 S b (C</em>{n_{\alpha}})}{I_{zz}}$</td>
<td>$N_{\beta} = \frac{\bar{q}<em>1 S b^2 (C</em>{n_{\beta}})}{2I_{zz}U_1}$</td>
</tr>
<tr>
<td>$N_{\alpha} = \frac{\bar{q}<em>1 S \bar{c} b (C</em>{n_{\alpha}})}{2I_{zz}U_1}$</td>
<td>$N_p = \frac{\bar{q}<em>1 S b^2 (C</em>{n_p})}{2I_{zz}U_1}$</td>
</tr>
<tr>
<td>$N_q = \frac{\bar{q}<em>1 S \bar{c} b (C</em>{n_q})}{2I_{zz}U_1}$</td>
<td>$N_r = \frac{\bar{q}<em>1 S b^2 (C</em>{n_r})}{2I_{zz}U_1}$</td>
</tr>
<tr>
<td><strong>Thrust Derivatives</strong></td>
<td></td>
</tr>
<tr>
<td>$N_{Tu} = \frac{\bar{q}<em>1 S b (C</em>{n_{Tu}} + 2C_{n_{r1}})}{I_{zz}U_1}$</td>
<td>$N_{Tr_{\beta}} = \frac{\bar{q}<em>1 S b (C</em>{n_{Tr_{\beta}}})}{I_{zz}}$</td>
</tr>
<tr>
<td>$N_{Ta} = \frac{\bar{q}<em>1 S b (C</em>{n_{Ta}})}{I_{zz}}$</td>
<td></td>
</tr>
<tr>
<td><strong>Control Derivatives</strong></td>
<td></td>
</tr>
<tr>
<td>$N_{\delta e} = \frac{\bar{q}<em>1 S b (C</em>{n_{\delta e}})}{I_{zz}}$</td>
<td>$N_{\delta a} = \frac{\bar{q}<em>1 S b (C</em>{n_{\delta a}})}{I_{zz}}$</td>
</tr>
<tr>
<td>$N_{\delta r} = \frac{\bar{q}<em>1 S b (C</em>{n_{\delta r}})}{I_{zz}}$</td>
<td>$N_{\delta r} = \frac{\bar{q}<em>1 S b (C</em>{n_{\delta r}})}{I_{zz}}$</td>
</tr>
</tbody>
</table>

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B.5  The Perturbed Equations of Motion for the Generalized Aircraft

The perturbed force and moment equations, represented in terms of the dimensional stability derivatives, are finally given in equations (B.125) to (B.130). As a recap, the kinematic equations are also once again presented in equations (B.131) to (B.133).

**Force Equations:**

\[
(\ddot{u} - V_1 r - R_1 v + W_1 q + Q_1 w) = -g\theta \cos \Theta_1 + \\
(\dot{X}_u + X_{\alpha u})u + (X_{\alpha} + X_{\alpha a})\alpha + X_{\alpha a} \dot{\alpha} + X_q q + X_{\delta e} \delta_e + X_{\delta t} \delta_t + \\
(\dot{X}_{\beta} + X_{\beta a})\beta + X_{\beta a} \dot{\beta} + X_p p + X_r r + X_{\delta a} \delta_a + X_{\delta r} \delta_r
\]

(B.125)

\[
(\dot{v} + U_1 r + R_1 u - W_1 p - P_1 w) = -g\theta \sin \Phi_1 \sin \Theta_1 + g\phi \cos \Phi_1 \cos \Theta_1 + \\
(\dot{Y}_u + Y_{\alpha u})u + (Y_{\alpha} + Y_{\alpha a})\alpha + Y_{\alpha a} \dot{\alpha} + Y_q q + Y_{\delta e} \delta_e + Y_{\delta t} \delta_t + \\
(\dot{Y}_{\beta} + Y_{\beta a})\beta + Y_{\beta a} \dot{\beta} + Y_p p + Y_r r + Y_{\delta a} \delta_a + Y_{\delta r} \delta_r
\]

(B.126)

\[
(\dot{w} - U_1 q - Q_1 u + V_1 p + P_1 v) = -g\theta \cos \Phi_1 \sin \Theta_1 - g\phi \sin \Phi_1 \cos \Theta_1 + \\
(\dot{Z}_u + Z_{\alpha u})u + (Z_{\alpha} + Z_{\alpha a})\alpha + Z_{\alpha a} \dot{\alpha} + Z_q q + Z_{\delta e} \delta_e + Z_{\delta t} \delta_t + \\
(\dot{Z}_{\beta} + Z_{\beta a})\beta + Z_{\beta a} \dot{\beta} + Z_p p + Z_r r + Z_{\delta a} \delta_a + Z_{\delta r} \delta_r
\]

(B.127)
Moment Equations:

\[
\dot{p} - \frac{l_{xy}}{l_{xx}} \dot{q} - \frac{l_{xz}}{l_{xx}} \dot{r} - \frac{l_{xx}}{l_{xx}} (P_1 q + Q_1 p) + \left( \frac{l_{zz} - l_{yy}}{l_{xx}} \right) (R_1 q + Q_1 r) + \frac{l_{xy}}{l_{xx}} (P_1 r + R_1 p) + \frac{l_{yz}}{l_{yx}} (2R_1 r - 2Q_1 q) = \]

\[
(L_u + L_T u) + (L_a + L_T a) \alpha + L_\alpha \dot{\alpha} + L_q q + L_{\delta e} \delta_e + L_{\delta t} \delta_t + (L_\beta + L_T \beta) \beta + L_\beta \dot{\beta} + L_p p + L_r r + L_{\delta a} \delta_a + L_{\delta r} \delta_r
\]

\[
\dot{q} - \frac{l_{xy}}{l_{yy}} \dot{p} - \frac{l_{yz}}{l_{yy}} \dot{r} + \left( \frac{l_{xx} - l_{zz}}{l_{yy}} \right) (P_1 r + R_1 p) + \frac{l_{xx}}{l_{yy}} (2P_1 p - 2R_1 r) - \frac{l_{xy}}{l_{yy}} (Q_1 r + R_1 q) + \frac{l_{yz}}{l_{yy}} (P_1 q + Q_1 p) = \]

\[
(M_u + M_T u) + (M_a + M_T a) \alpha + M_\alpha \dot{\alpha} + M_q q + M_{\delta e} \delta_e + M_{\delta t} \delta_t + (M_\beta + M_T \beta) \beta + M_\beta \dot{\beta} + M_p p + M_r r + M_{\delta a} \delta_a + M_{\delta r} \delta_r
\]

\[
\dot{r} - \frac{l_{yz}}{l_{zz}} \dot{q} - \frac{l_{zx}}{l_{zz}} \dot{p} + \left( \frac{l_{yy} - l_{xx}}{l_{zz}} \right) (P_1 q + Q_1 p) + \frac{l_{xx}}{l_{zz}} (Q_1 r + R_1 q) + \frac{l_{xy}}{l_{zz}} (2Q_1 q - 2P_1 p) - \frac{l_{yz}}{l_{zz}} (P_1 r + R_1 p) = \]

\[
(N_u + N_T u) + (N_a + N_T a) \alpha + N_\alpha \dot{\alpha} + N_q q + N_{\delta e} \delta_e + N_{\delta t} \delta_t + (N_\beta + N_T \beta) \beta + N_\beta \dot{\beta} + N_p p + N_r r + N_{\delta a} \delta_a + N_{\delta r} \delta_r
\]
APPENDIX B (continued)

**Kinematic Equations:**

\[ p = \dot{\phi} - \dot{\Psi}_1 \theta \cos \Theta_1 - \dot{\psi} \sin \Theta_1 \quad \text{(B.131)} \]

\[ q = -\dot{\Theta}_1 \phi \sin \Phi_1 + \dot{\theta} \cos \Phi_1 + \dot{\Psi}_1 \phi \cos \Theta_1 \cos \Phi_1 - \dot{\Psi}_1 \theta \sin \Theta_1 \sin \Phi_1 + \dot{\psi} \cos \Theta_1 \sin \Phi_1 \quad \text{(B.132)} \]

\[ r = -\dot{\Psi}_1 \phi \cos \Theta_1 \sin \Phi_1 - \dot{\Psi}_1 \theta \sin \Theta_1 \cos \Phi_1 + \dot{\psi} \cos \Theta_1 \cos \Phi_1 - \dot{\theta} \cos \Phi_1 - \dot{\Theta}_1 \phi \cos \Phi_1 - \dot{\psi} \sin \Phi_1 \quad \text{(B.133)} \]
APPENDIX C

RELATIONSHIPS BETWEEN ANGLE-OF-ATTACK/SIDESLIP AND VELOCITY

Appendix C details the development of the perturbed state equations of motion, and is intended to complement the discussion presented in Chapter 5.

C.1 Overview

Depending on individual analysis requirements, the final derived state space system may be expressed either in terms of the kinematic states (ideal for simulating motion of the aircraft) or in terms of the aerodynamic states (ideal for analysis of dynamic stability and response). In either case, relationships between the perturbed velocities \((u, v, \text{ and } w)\) and angle-of-attack/sideslip need to be defined.

C.2 Steady State Aerodynamic Angles

Consider the definitions of angle-of-attack and sideslip given respectively in Figure C.1 and Figure C.2:

The steady state angle-of-attack \((\alpha_1)\) is defined as the angle between the steady state forward and vertical velocity vectors, as given by (C.1):

\[
v_{s1} = \frac{u_1}{\cos \alpha_1}
\]
APPENDIX C (continued)

\[ \alpha_1 = \tan \frac{W_1}{U_1} \quad (C.1) \]

The steady state sideslip angle \( (\beta_1) \) is defined as the angle between the steady state velocity vector \( (V_{true_1}) \) and the velocity vector along the stability axes \( (V_{s_1}) \) when viewed on a projected plane (shaded in red in Figure C.2) created by those two vectors, as given by (C.2):

\[ \beta_1 = \tan \frac{V_1}{V_{s_1}} = \tan \left( \frac{V_1}{U_1} \cos \alpha_1 \right) \quad (C.2) \]

C.3 Perturbed State Aerodynamic Angles

C.3.1 Angle-of-Attack

Now, consider a velocity perturbation in the forward and vertical axes through the superposition of the vectors \( u \) and \( w \), as shown in Figure C.3. The objective is to obtain an expression for the perturbed state angle-of-attack as an explicit function of \( u \) and \( w \).

Figure C.3. Perturbed State Angle-of-Attack

Recall the expression for \( \alpha_1 \) given in equation (C.1). Through trigonometric relationships, the following holds true for \( \alpha \):
Applying the tangent function to both sides of equation (C.3), expanding the tangent function on the left-hand-side, and realizing that the small angle approximation allows for \( \tan \alpha \approx \alpha \), one obtains:

\[
\tan(\alpha_1 + \alpha) = \frac{W_1 + w}{U_1 + u} \quad \Rightarrow \quad \frac{\tan \alpha_1 + \tan \alpha}{1 - \tan \alpha_1 \tan \alpha} = \frac{W_1 + w}{U_1 + u}
\]

Cross-multiplying both sides of equation (C.4), expanding the result, and neglecting the products of small perturbations then gives:

\[
U_1 \tan \alpha_1 + (U_1)\alpha + (\tan \alpha_1)u + u\alpha = W_1 + w - (W_1 \tan \alpha_1)\alpha - (\tan \alpha_1)w\alpha \quad \Rightarrow \quad C.5
\]

\[
U_1 \tan \alpha_1 + (U_1)\alpha + (\tan \alpha_1)u = W_1 + w - (W_1 \tan \alpha_1)\alpha
\]

With equation (C.1) in mind, note that \( W_1 = U_1 \tan \alpha_1 \). These terms in equation (C.5) are equivalent to each other and thus cancel out. This leads to:

\[
(U_1)\alpha + (\tan \alpha_1)u = w - (W_1 \tan \alpha_1)\alpha \quad \Rightarrow \quad C.6
\]

\[
(U_1 + W_1 \tan \alpha_1)\alpha = -(\tan \alpha_1)u + w
\]

At this stage, the derivation may continue in two directions – the first allows the equations of motion to be presented in a "kinematic form" (wherein the states of the system are composed of
the velocities, the rates, and the Euler angles), while the second allows the equations of motion to be presented in an "aerodynamic form" (wherein the states of the systems are composed of the forward velocity, the aerodynamic angles, the rates, and the Euler angles).

C.3.1.1 Kinematic Form:

The objective here is to express $\alpha$ in terms of $U_1$, $u$, $W_1$, and $w$. The term $\alpha_1$ thus needs to be eliminated from equation (C.6). To begin, note from equation (C.1) that $\tan \alpha_1 = \frac{W_1}{U_1}$.

Substituting this into equation (C.6) yields:

$$\left( U_1 + W_1 \frac{W_1}{U_1} \right) \alpha = -\left( \frac{W_1}{U_1} \right) u + w$$

$$\left( U_1 + \frac{W_1^2}{U_1} \right) \alpha = \left( \frac{U_1^2 + W_1^2}{U_1} \right) \alpha = -\left( \frac{W_1}{U_1} \right) u + w$$

$$\alpha = -\left( \frac{W_1}{U_1^2 + W_1^2} \right) u + \left( \frac{U_1}{U_1^2 + W_1^2} \right) w$$  \hspace{1cm} (C.7)

Alternatively, equation (C.7) may be expressed in a more compact form, as follows:

$$\alpha = -\overline{K}_{\alpha u} u + \overline{K}_{\alpha w} w$$  \hspace{1cm} (C.8)
C.3.1.2 Aerodynamic Form:

The objective here is to express $w$ in terms of $\alpha_1$, $\alpha$, $U_1$, and $u$. The term $W_1$ thus needs to be eliminated from equation (C.6). To begin, note from equation (C.1) that $W_1 = U_1 \tan \alpha_1$.

Substituting this into equation (C.6) yields:

$$(U_1 + (U_1 \tan \alpha_1) \tan \alpha_1)\alpha = -(\tan \alpha_1)u + w$$

$$(U_1(1 + \tan^2 \alpha_1))\alpha = -(\tan \alpha_1)u + w$$

$$(U_1 \sec^2 \alpha_1)\alpha = -(\tan \alpha_1)u + w$$

$$w = (U_1 \sec^2 \alpha_1)\alpha + (\tan \alpha_1)u$$ \hspace{1cm} (C.9)

Alternatively, equation (C.9) may be expressed in a more compact form, as follows:

$$w = \bar{A}_{w\alpha}\alpha + \bar{A}_{wu}u$$ \hspace{1cm} (C.10)

$$\bar{A}_{w\alpha} = U_1 \sec^2 \alpha_1 \hspace{1cm} \bar{A}_{wu} = \tan \alpha_1$$
C.3.2 **Sideslip Angle**

Now, consider a velocity perturbation in the forward, vertical, and side axes through the superposition of the vectors \( u, v, \) and \( w \), as shown in Figure C.4 and Figure C.5. The objective is to obtain an expression for the perturbed state sideslip angle as an explicit function of \( u, v, \) and \( w \).

Recall the expression for \( \beta_1 \) given in equation (C.2). The perturbed form of this expression may be written as:

\[
\beta_1 + \beta = \arctan\left(\frac{V_1 + v}{U_1 + u \cos(\alpha_1 + \alpha)}\right) \tag{C.11}
\]

Applying the tangent function to both sides of equation (C.11), expanding the tangent and cosine functions, and realizing that the small angle approximations allow for \( \tan \beta \approx \beta, \sin \alpha \approx \alpha, \) and \( \cos \alpha \approx 1, \) one obtains:

\[
\tan(\beta_1 + \beta) = \frac{V_1 + v}{U_1 + u \cos(\alpha_1 + \alpha)}
\]
APPENDIX C (continued)

\[
\frac{\tan \beta_1 + \tan \beta}{1 - \tan \beta_1 \tan \beta} = \frac{V_1 + v}{U_1 + u} (\cos \alpha_1 \cos \alpha - \sin \alpha_1 \sin \alpha)
\]

\[
\frac{\tan \beta_1 + \beta}{1 - \beta \tan \beta_1} = \frac{V_1 + v}{U_1 + u} (\cos \alpha_1 - \alpha \sin \alpha_1)
\] (C.12)

Cross-multiplying both sides of equation (C.12), expanding the result, and neglecting the products of small perturbations then gives:

\[
U_1 \tan \beta_1 + (U_1)\beta + (\tan \beta_1)u + u\beta
= (V_1 \cos \alpha_1 - (V_1 \sin \alpha_1)\alpha + (\cos \alpha_1)v - (\sin \alpha_1)\alpha)(1 - \beta \tan \beta_1)
\]

\[
U_1 \tan \beta_1 + (U_1)\beta + (\tan \beta_1)u + u\beta
= V_1 \cos \alpha_1 - (V_1 \sin \alpha_1)\alpha + (\cos \alpha_1)v - (\sin \alpha_1)\alpha
- (V_1 \cos \alpha_1 \tan \beta_1)\beta + (V_1 \sin \alpha_1 \tan \beta_1)\alpha \beta
- (\cos \alpha_1 \tan \beta_1)v \beta + (\sin \alpha_1 \tan \beta_1)v \alpha \beta
\]

\[
U_1 \tan \beta_1 + (U_1)\beta + (\tan \beta_1)u
= V_1 \cos \alpha_1 - (V_1 \sin \alpha_1)\alpha + (\cos \alpha_1)v - (V_1 \cos \alpha_1 \tan \beta_1)\beta
\] (C.13)

With equation (C.2) in mind, note that \( U_1 \tan \beta_1 = V_1 \cos \alpha_1 \). These terms in equation (C.13) are equivalent to each other and thus cancel out. This leads to:

\[
(U_1)\beta + (\tan \beta_1)u = -(V_1 \sin \alpha_1)\alpha + (\cos \alpha_1)v - (V_1 \cos \alpha_1 \tan \beta_1)\beta
\]

\[
(V_1 \sin \alpha_1)\alpha + (U_1 + V_1 \cos \alpha_1 \tan \beta_1)\beta = -(\tan \beta_1)u + (\cos \alpha_1)v
\] (C.14)

At this stage, the derivation may once again continue in two directions – towards that of the "kinematic form" or towards that of the "aerodynamic form".
C.3.2.1 Kinematic Form

The objective here is to express $\beta$ in terms of $U_1$, $u$, $V_1$, $v$, $W_1$, and $w$. The terms $\alpha_1$, $\alpha$, and $\beta_1$ thus need to be eliminated from equation (C.14). To begin, consider the following geometric substitutions for the trigonometric functions in equation (C.14):

$$\sin \alpha_1 = \frac{W_1}{\sqrt{U_1^2 + W_1^2}} \quad \cos \alpha_1 = \frac{U_1}{\sqrt{U_1^2 + W_1^2}} \quad \tan \beta_1 = \frac{V_1}{\sqrt{U_1^2 + W_1^2}} \quad (C.15)$$

Also, recall from equation (C.7) that:

$$\alpha = -\left(\frac{W_1}{U_1^2 + W_1^2}\right) u + \left(\frac{U_1}{U_1^2 + W_1^2}\right) w \quad (C.16)$$

Substituting these into equation (C.14) yields:

$$\left(V_1 \frac{W_1}{\sqrt{U_1^2 + W_1^2}}\right) \alpha + \left(U_1 + V_1 \frac{U_1}{\sqrt{U_1^2 + W_1^2}} \frac{V_1}{\sqrt{U_1^2 + W_1^2}}\right) \beta =$$

$$-\left(\frac{V_1}{\sqrt{U_1^2 + W_1^2}}\right) u + \left(\frac{U_1}{\sqrt{U_1^2 + W_1^2}}\right) v$$

$$U_1 \left(\frac{U_1^2 + V_1^2 + W_1^2}{U_1^2 + W_1^2}\right) \beta =$$

$$-\left(\frac{U_1^2 V_1}{(U_1^2 + W_1^2)^{3/2}}\right) u + \left(\frac{U_1}{(U_1^2 + W_1^2)^{1/2}}\right) v - \left(\frac{U_1 V_1 W_1}{(U_1^2 + W_1^2)^{3/2}}\right) w$$

$$\beta = -\left(\frac{U_1 V_1}{(U_1^2 + V_1^2 + W_1^2)\sqrt{U_1^2 + W_1^2}}\right) u + \left(\frac{U_1^2 + W_1^2}{(U_1^2 + V_1^2 + W_1^2)\sqrt{U_1^2 + W_1^2}}\right) v$$

$$-\left(\frac{V_1 W_1}{(U_1^2 + V_1^2 + W_1^2)\sqrt{U_1^2 + W_1^2}}\right) w \quad (C.17)$$
Alternatively, equation (C.17) may be expressed in a more compact form, as follows:

\[
\begin{align*}
\beta &= -\bar{K}_{\beta_u} u + \bar{K}_{\beta_v} v - \bar{K}_{\beta_w} w \\
\bar{K}_{\beta_u} &= \frac{U_1 V_1}{(U_1^2 + V_1^2 + W_1^2)\sqrt{U_1^2 + W_1^2}} \\
\bar{K}_{\beta_v} &= \frac{U_1^2 + W_1^2}{(U_1^2 + V_1^2 + W_1^2)\sqrt{U_1^2 + W_1^2}} \\
\bar{K}_{\beta_w} &= \frac{V_1 W_1}{(U_1^2 + V_1^2 + W_1^2)\sqrt{U_1^2 + W_1^2}}
\end{align*}
\]
C.3.2.2 Aerodynamic Form

The objective here is to express $v$ in terms of $\alpha$, $\beta$, and $u$. The term $V_1$ thus needs to be eliminated from equation (C.14). To begin, note from equation (C.2) that $V_1 = U_1 (\tan \beta_1 / \cos \alpha_1)$. Substituting this into equation (C.14) yields:

$$
\left( U_1 \left( \frac{\tan \beta_1}{\cos \alpha_1} \right) \sin \alpha_1 \right) \alpha + \left( U_1 + U_1 \left( \frac{\tan \beta_1}{\cos \alpha_1} \cos \alpha_1 \tan \beta_1 \right) \right) \beta
= -(\tan \beta_1) u + (\cos \alpha_1) v
$$

$$(U_1 \tan \alpha_1 \tan \beta_1) \alpha + (U_1 + U_1 \tan^2 \beta_1) \beta = -(\tan \beta_1) u + (\cos \alpha_1) v
$$

$$(U_1 \tan \alpha_1 \tan \beta_1) \alpha + (U_1 \sec \beta_1) \beta + (\tan \beta_1) u = (\cos \alpha_1) v
$$

$$
v = \left( \frac{U_1 \tan \alpha_1 \tan \beta_1}{\cos \alpha_1} \right) \alpha + \left( \frac{U_1 \sec^2 \beta_1}{\cos \alpha_1} \right) \beta + \left( \frac{\tan \beta_1}{\cos \alpha_1} \right) u
$$

(C.19)

Alternatively, equation (C.19) may be expressed in a more compact form, as follows:

$$
v = \overline{A}_{v\alpha} \alpha + \overline{A}_{v\beta} \beta + \overline{A}_{vu} u
$$

$$
\overline{A}_{v\alpha} = \frac{U_1 \tan \alpha_1 \tan \beta_1}{\cos \alpha_1}
$$

$$
\overline{A}_{v\beta} = \frac{U_1 \sec^2 \beta_1}{\cos \alpha_1}
$$

$$
\overline{A}_{vu} = \frac{\tan \beta_1}{\cos \alpha_1}
$$

(C.20)
Appendix D presents a detailed look at the development of the modal superposition method, and is intended to complement the abbreviated discussion presented in Section 6.3.

### D.1 Overview

To facilitate mathematically- and computationally-efficient analysis of the $n$th order transfer functions in the time-domain, it is prudent to first simplify equations (6.11) and (6.12). This idea of “modal superposition” is illustrated in Figure D.1. It may be shown that the $n$th order polynomial may be written as the sum of a series of 1st order and 2nd order polynomials, as in equations (D.1) and (D.2):
APPENDIX D (continued)

\[ H_{li}(s) = \frac{c_{n-1}s^{n-1} + \cdots + c_3s^3 + c_2s^2 + c_1s + c_0}{a_ns^n + \cdots + a_4s^4 + a_3s^3 + a_2s^2 + a_1s + a_0} \]

\[ = \sum_{k=1}^{p_c} \frac{m_{2_{ljk}}s + m_{1_{ljk}}}{s^2 + 2\zeta_k \omega_{n_k} s + \omega_{n_k}^2} + \sum_{k=1}^{p_R} \frac{m_{0_{lk}}}{s + \tau_k^{-1}} \]  \hspace{1cm} (D.1)

\[ G_l(s) = \frac{d_{n-1}s^{n-1} + \cdots + d_3s^3 + d_2s^2 + d_1s + d_0}{a_ns^n + \cdots + a_4s^4 + a_3s^3 + a_2s^2 + a_1s + a_0} \]

\[ = \sum_{k=1}^{p_c} \frac{n_{2_{lk}}s + n_{1_{lk}}}{s^2 + 2\zeta_k \omega_{n_k} s + \omega_{n_k}^2} + \sum_{k=1}^{p_R} \frac{n_{0_{lk}}}{s + \tau_k^{-1}} \]  \hspace{1cm} (D.2)

Each elemental 1\textsuperscript{st} or 2\textsuperscript{nd} order transfer function then encompasses the characteristics of each of the modes in the \( n \)\textsuperscript{th} order system. For example, for the 4\textsuperscript{th} order lateral/directional system which has one complex mode and two real modes, the transfer function \( H_{li}(s) \) may be superposed as:

\[ H_{li}(s) = \frac{c_3s^3 + c_2s^2 + c_1s + c_0}{a_4s^4 + a_3s^3 + a_2s^2 + a_1s + a_0} \]

\[ = \frac{m_{2_{li1}}s + m_{1_{li1}}}{s^2 + 2\zeta_1 \omega_{n_1} s + \omega_{n_1}^2} + \frac{m_{0_{li2}}}{s + \tau_1^{-1}} + \frac{m_{0_{li3}}}{s + \tau_3^{-1}} \]  \hspace{1cm} (D.3)

Likewise, for an 8\textsuperscript{th} order aircraft with three complex modes and two real modes, the transfer function \( H_{li}(s) \) may be superposed as:

\[ H_{li}(s) = \frac{c_8s^8 + c_7s^7 + c_6s^6 + c_5s^5 + c_4s^4 + c_3s^3 + c_2s^2 + c_1s + c_0}{a_4s^4 + a_3s^3 + a_2s^2 + a_1s + a_0} \]

\[ = \frac{m_{2_{li1}}s + m_{1_{li1}}}{s^2 + 2\zeta_1 \omega_{n_1} s + \omega_{n_1}^2} + \frac{m_{2_{li2}}s + m_{1_{li2}}}{s^2 + 2\zeta_2 \omega_{n_2} s + \omega_{n_2}^2} + \frac{m_{2_{li3}}s + m_{1_{li3}}}{s^2 + 2\zeta_3 \omega_{n_3} s + \omega_{n_3}^2} + \frac{m_{0_{li4}}}{s + \tau_4^{-1}} + \frac{m_{0_{li5}}}{s + \tau_5^{-1}} \]  \hspace{1cm} (D.4)
To complete the modal superposition process, what remains is to match the coefficients that exist in the \( n \)th order transfer function with those in the 2nd or 1st order transfer functions, when the transfer functions are multiplied with a higher-order input. The elemental 2nd and 1st order transfer functions in equations (6.13) and (6.14) may be further expanded and expressed explicitly as a function of standard Laplace transforms, simplifying the inverses from the frequency domain back into the time domain. Consequently, this facilitates the derivation of an efficient solution for the time-domain system response for the \( n \)th order system, which is in turn used to compute the critical control deflections. The expansion of the elemental transfer functions is discussed next.

### D.2 The Elemental 2nd Order Transfer Function

Consider again the form of the general solution in the frequency-domain, as given in equation (6.9) and shown again in equation (D.5):

\[
x_i(s) = \sum_{j=1}^{m} \left[ H_{l,j}(s) \left( \Delta \hat{u}_j(s) + \frac{u_j}{s} \right) \right] + G_i(s)
\]

(D.5)

Considering only the complex modes, writing equation (D.5) in more explicit terms yields:

\[
x_i(s) = \sum_{j=1}^{m} \left[ \sum_{k=1}^{p_c} \frac{m_{2i,j,k} s + m_{1i,j,k}}{s^2 + 2\zeta_k \omega_n s + \omega_n^2} \left( \Delta \hat{u}_j(s) + \frac{u_j}{s} \right) \right] + \sum_{k=1}^{p_c} \frac{n_{2i,j,k} s + n_{1i,j,k}}{s^2 + 2\zeta_k \omega_n s + \omega_n^2} 
\]

(D.6)

\[
\Delta \hat{u}_j(s) = \Delta \hat{u}_j \frac{1}{s q}
\]

Each permutation of the step, ramp, parabolic, or higher order inputs \( \Delta \hat{u}_j(s) \) with the elemental transfer function \( H_{l,j}(s) \) fundamentally consists of some linear combination of a transfer
function of Type $\mathbb{Q}$ and Type $\mathbb{Q}^-$ multiplied by the coefficients $m_2$ and $m_1$, where $\mathbb{Q} = \hat{q}$ and $\mathbb{Q}^- = \hat{q} - 1$. The elemental 2nd order transfer functions in equation (D.6) may thus be further expanded and expressed explicitly as a function of standard Laplace transforms. This simplifies the inverses from the frequency domain back into the time domain and facilitates the derivation of an efficient solution for the time-domain system response for the $n^{th}$ order system. Likewise, the same applies to the elemental transfer function $G_i(s)$. This is summarized in Table D.1. For conciseness, the subscripts $i$, $j$, and $k$ are omitted but implied throughout.

### Table D.1

#### COEFFICIENTS OF BASIC TRANSFER FUNCTIONS

(Complex Modes)

<table>
<thead>
<tr>
<th>Type</th>
<th>Basic Transfer Function</th>
<th>$H_{ij}(s) \cdot \Delta \hat{u}_j(s)$</th>
<th>$H_{ij}(s) \cdot u_j(s)$</th>
<th>$G_i(s)$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$O_1$</td>
<td>$\frac{s}{s^2 + 2\zeta \omega_n s + \omega_n^2}$</td>
<td>-</td>
<td>-</td>
<td>-</td>
</tr>
<tr>
<td>$O_2$</td>
<td>$\frac{1}{s^2 + 2\zeta \omega_n s + \omega_n^2}$</td>
<td>$m_2$</td>
<td>-</td>
<td>-</td>
</tr>
<tr>
<td>$Q = I$</td>
<td>$\frac{1}{s(s^2 + 2\zeta \omega_n s + \omega_n^2)}$</td>
<td>$m_1$</td>
<td>$m_2$</td>
<td>-</td>
</tr>
<tr>
<td>$Q = II$</td>
<td>$\frac{1}{s^2(s^2 + 2\zeta \omega_n s + \omega_n^2)}$</td>
<td>-</td>
<td>$m_1$</td>
<td>$m_2$</td>
</tr>
<tr>
<td>$Q = III$</td>
<td>$\frac{1}{s^3(s^2 + 2\zeta \omega_n s + \omega_n^2)}$</td>
<td>-</td>
<td>-</td>
<td>$m_1$</td>
</tr>
<tr>
<td>$Q = IV$</td>
<td>$\frac{1}{s^4(s^2 + 2\zeta \omega_n s + \omega_n^2)}$</td>
<td>-</td>
<td>-</td>
<td>-</td>
</tr>
<tr>
<td>$Q = V$</td>
<td>$\frac{1}{s^5(s^2 + 2\zeta \omega_n s + \omega_n^2)}$</td>
<td>-</td>
<td>-</td>
<td>-</td>
</tr>
</tbody>
</table>
The intermediate objective is then to express these basic transfer functions explicitly in terms of the following standard Laplace transforms:

\[ F_1(s) = \frac{1}{s} \quad \rightarrow \quad f_1(t) = 1 \quad (D.7) \]
\[ F_2(s) = \frac{1}{s^2} \quad \rightarrow \quad f_2(t) = t \quad (D.8) \]
\[ F_3(s) = \frac{1}{s^3} \quad \rightarrow \quad f_3(t) = \frac{t^2}{2} \quad (D.9) \]
\[ F_4(s) = \frac{1}{s^4} \quad \rightarrow \quad f_4(t) = \frac{t^3}{6} \quad (D.10) \]
\[ F_5(s) = \frac{1}{s^5} \quad \rightarrow \quad f_5(t) = \frac{t^4}{24} \quad (D.11) \]

\[ F_s(s) = \frac{\omega_n \sqrt{1 - \zeta^2}}{(s + \zeta \omega_n)^2 + \left(\omega_n \sqrt{1 - \zeta^2}\right)^2} \quad \rightarrow \quad f_s(t) = e^{-\zeta \omega_n t} \sin \left(\omega_n \sqrt{1 - \zeta^2} \cdot t\right) \quad (D.12) \]
\[ F_c(s) = \frac{s + \zeta \omega_n}{(s + \zeta \omega_n)^2 + \left(\omega_n \sqrt{1 - \zeta^2}\right)^2} \quad \rightarrow \quad f_c(t) = e^{-\zeta \omega_n t} \cos \left(\omega_n \sqrt{1 - \zeta^2} \cdot t\right) \quad (D.13) \]

This is done through partial fraction expansion, and the full expansions are detailed in Appendix F. Ultimately, the expanded transfer functions take on the form specified in Table D.2:
TABLE D.2
COEFFICIENTS OF STANDARD LAPLACE TRANSFORMS
(COMPLEX MODES)

<table>
<thead>
<tr>
<th>Type</th>
<th>Coefficients of Standard Laplace Transforms</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>$\kappa_5^{c,Q}$</td>
</tr>
<tr>
<td>$O_1$</td>
<td>-</td>
</tr>
<tr>
<td>$O_2$</td>
<td>-</td>
</tr>
<tr>
<td>$Q = I$</td>
<td>-</td>
</tr>
<tr>
<td>$Q = II$</td>
<td>-</td>
</tr>
<tr>
<td>$Q = III$</td>
<td>-</td>
</tr>
<tr>
<td>$Q = IV$</td>
<td>-</td>
</tr>
<tr>
<td>$Q = V$</td>
<td>1</td>
</tr>
</tbody>
</table>

An observation of the partial fraction expansions in Appendix F shows that, as each
Transfer Function increases in Type Order $Q$ (for $Q = I$ and above), the preceding expanded
transfer function cascades into the following expanded transfer function. From Table D.2, a pattern consequently emerges demonstrating that each coefficient of the monotonic Laplace Transforms, $\kappa_{C}^{\text{C,Q}}$, is a function of the preceding coefficients. For an input of order $\hat{q}$, where $\Delta \hat{u}_j(s) = \Delta \hat{u}_j \frac{1}{s^{\hat{q}}}$, the coefficients $\kappa_{\hat{q}}^{\text{C,Q}}$ and $\kappa_{l}^{\text{C,Q}}$ may thus be determined by equation (D.14):

$$
\begin{align*}
\kappa_{\hat{q}}^{\text{C,Q}} &= \frac{1}{\omega_n^2} \\
\kappa_{l}^{\text{C,Q}} &= -\kappa_{(l+2)}^{\text{C,Q}} \cdot \frac{1}{\omega_n^2} - \kappa_{(l+1)}^{\text{C,Q}} \cdot \frac{2\zeta}{\omega_n} \\
&= \frac{l = (\hat{q} - 1), \ldots, 1; \kappa_{(\hat{q}+1)}^{\text{C,Q}} = 0}
\end{align*}
$$

Similarly, the coefficients for the sinusoidal Laplace Transforms, $\kappa_{c}^{\text{C,Q}}$ and $\kappa_{s}^{\text{C,Q}}$, may be determined by the coefficients of the preceding monotonic Laplace Transforms, as in equation (D.15):

$$
\begin{align*}
\kappa_{c}^{\text{C,Q}} &= -\kappa_{1}^{\text{C,Q}} \\
\kappa_{s}^{\text{C,Q}} &= -\frac{\zeta}{\sqrt{1 - \zeta^2}} \kappa_{1}^{\text{C,Q}} - \frac{1}{\omega_n \sqrt{1 - \zeta^2}} \kappa_{2}^{\text{C,Q}}
\end{align*}
$$

Equations (D.14) and (D.15) allow for the dynamic computation of the coefficients of each Laplace Transform present for an input of order $\hat{q}$. At this point, all the elements required to build up the forced and unforced responses of the aircraft have been defined.
D.3 The Elemental 1st Order Transfer Function

Consider again the form of the general solution in the frequency-domain, as given in equation (6.9) and shown again in equation (D.16):

\[ x_i(s) = \sum_{j=1}^{m} \left[ H_{i,j}(s) \left( \Delta \hat{u}_j(s) + \frac{u_j}{s} \right) \right] + G_i(s) \tag{D.16} \]

Considering only the real modes, writing equation (D.16) in more explicit terms yields:

\[ x_i(s) = \sum_{j=1}^{m} \left[ \sum_{k=1}^{p_R} \frac{m_{0ijk}}{s + \tau_{k-1}} \left( \Delta \hat{u}_j(s) + \frac{u_j}{s} \right) \right] + \sum_{k=1}^{p_R} \frac{n_{0ik}}{s + \tau_{k-1}} \tag{D.17} \]

\[ \Delta \hat{u}_j(s) = \Delta \hat{u}_j \frac{1}{s \hat{q}} \]

Each permutation of the step, ramp, parabolic, or higher order inputs \( \Delta \hat{u}_j(s) \) with the elemental transfer function \( H_{i,j}(s) \) fundamentally consists of one of the following basic transfer functions of Type \( Q \) multiplied by the coefficient \( m_0 \), where \( Q = \hat{q} \). The elemental 1st order transfer functions in equation (D.17) may thus be further expanded and expressed explicitly as a function of standard Laplace transforms. Likewise, the same applies to the elemental transfer function \( G_i(s) \). This is summarized in Table D.3. For conciseness, the subscripts \( i, j, \) and \( k \) are omitted but implied throughout.
APPENDIX D (continued)

TABLE D.3  
COEFFICIENTS OF BASIC TRANSFER FUNCTIONS  
(REAL MODES)

<table>
<thead>
<tr>
<th>Type</th>
<th>Basic Transfer Function</th>
<th>$H_{l,j}(s) \cdot \Delta \hat{u}_j(s)$</th>
<th>$H_{l,j}(s) \cdot \hat{u}_j(s)$</th>
<th>$G_i(s)$</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td></td>
<td>Step</td>
<td>Ramp</td>
<td>Parabolic</td>
</tr>
<tr>
<td>O</td>
<td>$\frac{1}{s + \tau_{k-1}}$</td>
<td>-</td>
<td>-</td>
<td>-</td>
</tr>
<tr>
<td>$Q = I$</td>
<td>$\frac{1}{s(s + \tau_{k-1})}$</td>
<td>$m_0$</td>
<td>-</td>
<td>-</td>
</tr>
<tr>
<td>$Q = II$</td>
<td>$\frac{1}{s^2(s + \tau_{k-1})}$</td>
<td>-</td>
<td>$m_0$</td>
<td>-</td>
</tr>
<tr>
<td>$Q = III$</td>
<td>$\frac{1}{s^3(s + \tau_{k-1})}$</td>
<td>-</td>
<td>-</td>
<td>$m_0$</td>
</tr>
<tr>
<td>$Q = IV$</td>
<td>$\frac{1}{s^4(s + \tau_{k-1})}$</td>
<td>-</td>
<td>-</td>
<td>-</td>
</tr>
<tr>
<td>$Q = V$</td>
<td>$\frac{1}{s^5(s + \tau_{k-1})}$</td>
<td>-</td>
<td>-</td>
<td>-</td>
</tr>
</tbody>
</table>
APPENDIX D (continued)

The intermediate objective is then to express these basic transfer functions explicitly in terms of the following standard Laplace transforms:

\[
F_1(s) = \frac{1}{s} \quad \Rightarrow \quad f_1(t) = 1 \tag{D.18}
\]

\[
F_2(s) = \frac{1}{s^2} \quad \Rightarrow \quad f_2(t) = t \tag{D.19}
\]

\[
F_3(s) = \frac{1}{s^3} \quad \Rightarrow \quad f_3(t) = \frac{t^2}{2} \tag{D.20}
\]

\[
F_4(s) = \frac{1}{s^4} \quad \Rightarrow \quad f_4(t) = \frac{t^3}{6} \tag{D.21}
\]

\[
F_5(s) = \frac{1}{s^5} \quad \Rightarrow \quad f_5(t) = \frac{t^4}{24} \tag{D.22}
\]

\[
F_e(s) = \frac{1}{s + \tau^{-1}} \quad \Rightarrow \quad f_e(t) = e^{-\tau t} \tag{D.23}
\]

This is done through partial fraction expansion, and the full expansions are detailed in Appendix F. Ultimately, the expanded transfer functions take on the form specified in Table D.4:
### TABLE D.4

**COEFFICIENTS OF STANDARD LAPLACE TRANSFORMS**

(REAL MODES)

<table>
<thead>
<tr>
<th>Frequency Domain</th>
<th>$\frac{1}{s^5}$</th>
<th>$\frac{1}{s^4}$</th>
<th>$\frac{1}{s^3}$</th>
<th>$\frac{1}{s^2}$</th>
<th>$\frac{1}{s}$</th>
<th>$\frac{1}{s + \tau^{-1}}$</th>
</tr>
</thead>
<tbody>
<tr>
<td>Time Domain</td>
<td>$\frac{t^4}{24}$</td>
<td>$\frac{t^3}{6}$</td>
<td>$\frac{t^2}{2}$</td>
<td>$t$</td>
<td>$1$</td>
<td>$e^{-\tau t}$</td>
</tr>
</tbody>
</table>

↑ multiplied by ↓

<table>
<thead>
<tr>
<th>Type</th>
<th>$\kappa_5^{R, Q}$</th>
<th>$\kappa_4^{R, Q}$</th>
<th>$\kappa_3^{R, Q}$</th>
<th>$\kappa_2^{R, Q}$</th>
<th>$\kappa_1^{R, Q}$</th>
<th>$\kappa_F^{R, Q}$</th>
</tr>
</thead>
<tbody>
<tr>
<td>O</td>
<td>$-$</td>
<td>$-$</td>
<td>$-$</td>
<td>$-$</td>
<td>$-$</td>
<td>$-$</td>
</tr>
<tr>
<td>Q = I ($\bar{q} = 1$)</td>
<td>$-$</td>
<td>$-$</td>
<td>$-$</td>
<td>$-$</td>
<td>$\tau$</td>
<td>$-\tau$</td>
</tr>
<tr>
<td>Q = II ($\bar{q} = 2$)</td>
<td>$-$</td>
<td>$-$</td>
<td>$-$</td>
<td>$\tau$</td>
<td>$-\tau^2$</td>
<td>$\tau^2$</td>
</tr>
<tr>
<td>Q = III ($\bar{q} = 3$)</td>
<td>$-$</td>
<td>$-$</td>
<td>$\tau$</td>
<td>$-\tau^2$</td>
<td>$\tau^3$</td>
<td>$-\tau^3$</td>
</tr>
<tr>
<td>Q = IV ($\bar{q} = 4$)</td>
<td>$-$</td>
<td>$\tau$</td>
<td>$-\tau^2$</td>
<td>$\tau^3$</td>
<td>$-\tau^4$</td>
<td>$\tau^4$</td>
</tr>
<tr>
<td>Q = V ($\bar{q} = 5$)</td>
<td>$\tau$</td>
<td>$-\tau^2$</td>
<td>$\tau^3$</td>
<td>$-\tau^4$</td>
<td>$\tau^5$</td>
<td>$-\tau^5$</td>
</tr>
</tbody>
</table>
From Table D.4, a pattern emerges demonstrating that each coefficient of the monotonic Laplace Transforms, $\kappa_i^{\mathbb{R},\mathbb{Q}}$, is a function of the modal time constant. For an input of order $\hat{q}$, where $\Delta \hat{u}_j(s) = \Delta \hat{u}_j \frac{1}{s^\hat{q}}$, the coefficient $\kappa_i^{\mathbb{R},\mathbb{Q}}$ may thus be determined by equation (D.24):

$$
\kappa_i^{\mathbb{R},\mathbb{Q}} = (-1)^{\hat{q} - l} \cdot \tau^{\hat{q} - l + 1}
$$

\[ l = \hat{q}, \ldots, 1; \quad \kappa_{\hat{q} + 1}^{\mathbb{R},\mathbb{Q}} = 0 \]

Similarly, the coefficient for the exponential Laplace Transform, $\kappa_E^{\mathbb{R},\mathbb{Q}}$, may be determined according to equation (D.25):

$$
\kappa_E^{\mathbb{R},\mathbb{Q}} = (-1)^{\hat{q}} \cdot \tau^{\hat{q}}
$$

Equations (D.24) and (D.25) allow for the dynamic computation of the coefficients of each Laplace Transform present for an input of order $\hat{q}$. At this point, all the elements required to build up the forced and unforced responses of the aircraft have been defined.
D.4 The Complete System Response

Ultimately, the general form of the frequency-domain response for the \( i \)th state of the system is given by equation (D.26):

\[
x_i(s) = \sum_{j=1}^{m} \left( \sum_{k=1}^{p^c} \frac{m_{2i,j,k}}{s^2 + 2\zeta_k\omega_n s + \omega_n^2} + \sum_{k=1}^{p^r} \frac{m_{0i,j,k}}{s + \tau_k^{-1}} \right) \left( \Delta u_j(s) + \frac{u_f}{s} \right) \\
+ \left( \sum_{k=1}^{p^c} \frac{n_{2i,k}}{s^2 + 2\zeta_k\omega_n s + \omega_n^2} + \sum_{k=1}^{p^r} \frac{n_{0i,k}}{s + \tau_k^{-1}} \right)
\]  

(D.26)

The frequency response for all \( n \) states of the system may be assembled as a series of transfer function matrices. The entire system may be further generalized and expressed in the form defined in equation (D.27), where \( H(s) \) represents a matrix of transfer functions for the forced response, \( \bar{H}(s) \) represents a matrix of transfer functions for the initial conditions on the inputs, and \( G(s) \) represents a matrix of transfer functions for the initial conditions on the states.

\[
X(s) = X_H(s) + X_{\bar{H}}(s) + X_G(s) = H(s)\Delta \bar{U}(s) + \bar{H}(s)U(s) + G(s) 
\]  

(D.27)

In expanded form, equation (D.27) becomes:

\[
\begin{bmatrix}
    x_1(s) \\
    x_2(s) \\
    \vdots \\
    x_n(s)
\end{bmatrix} =
\begin{bmatrix}
    H_{1,1}(s) & H_{1,2}(s) & \cdots & H_{1,m}(s) \\
    H_{2,1}(s) & H_{2,2}(s) & \cdots & H_{2,m}(s) \\
    \vdots & \vdots & \ddots & \vdots \\
    H_{n,1}(s) & H_{n,2}(s) & \cdots & H_{n,m}(s)
\end{bmatrix}
\begin{bmatrix}
    \Delta \bar{u}_1(s) \\
    \Delta \bar{u}_2(s) \\
    \vdots \\
    \Delta \bar{u}_m(s)
\end{bmatrix} +
\begin{bmatrix}
    H_{1,1}(s) & \bar{H}_{1,2}(s) & \cdots & \bar{H}_{1,m}(s) \\
    \bar{H}_{2,1}(s) & \bar{H}_{2,2}(s) & \cdots & \bar{H}_{2,m}(s) \\
    \vdots & \vdots & \ddots & \vdots \\
    \bar{H}_{n,1}(s) & \bar{H}_{n,2}(s) & \cdots & \bar{H}_{n,m}(s)
\end{bmatrix}
\begin{bmatrix}
    u_1(s) \\
    u_2(s) \\
    \vdots \\
    u_m(s)
\end{bmatrix} +
\begin{bmatrix}
    G_{1}(s) \\
    G_{2}(s) \\
    \vdots \\
    G_{n}(s)
\end{bmatrix}
\]  

(D.28)

Taking the inverse Laplace of equation (D.27) yields the following general form for the system response in the time domain:
In expanded form, equation (D.29) becomes:

\[
\begin{bmatrix}
    x_1(\ell) \\
    x_2(\ell) \\
    \vdots \\
    x_n(\ell)
\end{bmatrix} =
\begin{bmatrix}
    H_{1,1}(\ell) & H_{1,2}(\ell) & \cdots & H_{1,m}(\ell) \\
    H_{2,1}(\ell) & H_{2,2}(\ell) & \cdots & H_{2,m}(\ell) \\
    \vdots & \vdots & \ddots & \vdots \\
    H_{n,1}(\ell) & H_{n,2}(\ell) & \cdots & H_{n,m}(\ell)
\end{bmatrix} \begin{bmatrix}
    \Delta \bar{u}_1 \\
    \Delta \bar{u}_2 \\
    \vdots \\
    \Delta \bar{u}_m
\end{bmatrix} +
\begin{bmatrix}
    \bar{H}_{1,1}(\ell) & \bar{H}_{1,2}(\ell) & \cdots & \bar{H}_{1,m}(\ell) \\
    \bar{H}_{2,1}(\ell) & \bar{H}_{2,2}(\ell) & \cdots & \bar{H}_{2,m}(\ell) \\
    \vdots & \vdots & \ddots & \vdots \\
    \bar{H}_{n,1}(\ell) & \bar{H}_{n,2}(\ell) & \cdots & \bar{H}_{n,m}(\ell)
\end{bmatrix} \begin{bmatrix}
    u_1 \\
    u_2 \\
    \vdots \\
    u_m
\end{bmatrix} +
\begin{bmatrix}
    G_1(\ell) \\
    G_2(\ell) \\
    \vdots \\
    G_n(\ell)
\end{bmatrix}
\]

This section will develop the expressions that make up the elements of \(H(s)\Delta \bar{U}(s)\) and \(H(\ell)\Delta \bar{U}(\ell)\) for step, ramp, parabolic, and inputs of order \(q\), the expressions for \(\bar{H}(s)U(s)\) and \(\bar{H}(\ell)U(\ell)\), and the expressions for \(G(s)\) and \(G(\ell)\).

At this juncture, having developed very general expressions for the elemental 2\(^{nd}\) and 1\(^{st}\) order transfer functions, the technical analyses of the time-domain responses of the complex and real modes are conducted separately. Consequently, equations (D.27) and (D.29) may be respectively recast as follows:

\[
X(s) = X^C(s) + X^R(s)
= \left[ X^C_H(s) + X^C_H(s) + X^C_G(s) \right] + \left[ X^R_H(s) + X^R_H(s) + X^R_G(s) \right]
= \left[ H^C(s) \Delta \bar{U} + \bar{H}^C(s)U + G^C(s) \right] + \left[ H^R(s) \Delta \bar{U} + \bar{H}^R(s)U + G^R(s) \right]
\]

\[
X(\ell) = X^C(\ell) + X^R(\ell)
= \left[ X^C_H(\ell) + X^C_H(\ell) + X^C_G(\ell) \right] + \left[ X^R_H(\ell) + X^R_H(\ell) + X^R_G(\ell) \right]
= \left[ H^C(\ell) \Delta \bar{U} + \bar{H}^C(\ell)U + G^C(\ell) \right] + \left[ H^R(\ell) \Delta \bar{U} + \bar{H}^R(\ell)U + G^R(\ell) \right]
\]
D.4.1 Complex Modes

D.4.1.1 The Forced Response $X^C_H$

The forced response for the $i^{th}$ state of the system is given by equation (D.33):

$$ x^C_{Hi}(s) = \sum_{j=1}^{m} \left[ \sum_{k=1}^{p} \left( \frac{m_{2i,j,k} s + m_{1i,j,k}}{s^2 + 2\zeta_k \omega_n s + \omega_n^2} \Delta \hat{u}_j(s) \right) \right] $$  \hspace{1cm} (D.33)

The input $\Delta \hat{u}_j(s)$ is given by $\Delta \hat{u}_j \frac{1}{s^{\hat{q}}}$, where $\hat{q}$ represents the order of the input and $\Delta \hat{u}_j$ represents the magnitude of the corresponding input order for the $j^{th}$ input. For example, if $\hat{q} = 1$, the input is a step and $\Delta \hat{u}_j$ is the magnitude of the step. If $\hat{q} = 2$, the input is a ramp and $\Delta \hat{u}_j$ is the magnitude of the input rate. If $\hat{q} = 3$, the input is a parabola and $\Delta \hat{u}_j$ is the magnitude of the input acceleration. The expressions for these three special cases are developed below, and this culminates in the development of the general expressions for an input of order $\hat{q}$. 
D.4.1.1.1 Step Input ($\bar{q} = 1$)

For the step input, equation (D.33) may be written as follows:

$$x_{H_l}^c(s) = \sum_{j=1}^{m} \Delta \bar{u}_j \sum_{k=1}^{p^c} \left[ \frac{s}{m_{2i,j,k} s^2 + 2 \zeta_k \omega_{nk} s + \omega_{nk}^2} + \frac{1}{s^2 + 2 \zeta_k \omega_{nk} s + \omega_{nk}^2} \right] \right] \quad (D.34)$$

Explicitly multiplying the step input through allows equation (D.34) to be expressed in terms of the Type O2 and Type I transfer functions:

$$x_{H_l}^c(s) = \sum_{j=1}^{m} \Delta \bar{u}_j \sum_{k=1}^{p^c} \left[ \frac{1}{m_{2i,j,k} s^2 + 2 \zeta_k \omega_{nk} s + \omega_{nk}^2} + \frac{1}{s^2 + 2 \zeta_k \omega_{nk} s + \omega_{nk}^2} \right] \right] \quad (D.35)$$

Then, expressing the Type O2 and Type I transfer functions in terms of the basic Laplace Transforms in Table D.2 leads to:

$$x_{H_l}^c(s) = \sum_{j=1}^{m} \Delta \bar{u}_j \sum_{k=1}^{p^c} \left[ \frac{1}{m_{2i,j,k} s^2 + 2 \zeta_k \omega_{nk} s + \omega_{nk}^2} + \frac{1}{s^2 + 2 \zeta_k \omega_{nk} s + \omega_{nk}^2} \right] \right] \quad (D.36)$$
Expressing the coefficients of the basic Laplace Transforms using the notation $\kappa$ gives:

$$x^C_{\bar{H}_l}(s) = \sum_{j=1}^{m} \Delta \bar{u}_j \sum_{k=1}^{p^C} \left( m_{1,l,j,k} \kappa^C_{1k} \left( \frac{1}{s} \right) + m_{1,l,j,k} \kappa^C_{2k} \left( \frac{s + \zeta_k \omega_n k}{(s + \zeta_k \omega_n k)^2 + \omega_n k^2 (1 - \zeta_k^2)} \right) \right)$$

$$+ \left( m_{2,l,j,k} \kappa^C_{S_k} + m_{1,l,j,k} \kappa^C_{S_k} \right) \left( \frac{\omega_n k \sqrt{1 - \zeta_k^2}}{(s + \zeta_k \omega_n k)^2 + \omega_n k^2 (1 - \zeta_k^2)} \right) \right]$$

(D.37)

Finally, using the notation $\mu$ to represent the coefficients $m$ and $\kappa$ results in the following expression for the forced frequency-domain response for the $i^{th}$ state of the system:

$$x^C_{\bar{H}_l}(s) = \sum_{j=1}^{m} \Delta \bar{u}_j \sum_{k=1}^{p^C} \left( \mu^C_{1,l,j,k} \left( \frac{1}{s} \right) + \mu^C_{1,l,j,k} \left( \frac{s + \zeta_k \omega_n k}{(s + \zeta_k \omega_n k)^2 + \omega_n k^2 (1 - \zeta_k^2)} \right) \right)$$

$$+ \mu^C_{S,l,j,k} \left( \frac{\omega_n k \sqrt{1 - \zeta_k^2}}{(s + \zeta_k \omega_n k)^2 + \omega_n k^2 (1 - \zeta_k^2)} \right) \right]$$

(D.38)

Taking the inverse Laplace transform of equation (D.38) leads to the following expression for the forced time-domain response for the $i^{th}$ state of the system:

$$x^C_{\bar{H}_l}(\hat{t}) = \sum_{j=1}^{m} \Delta \bar{u}_j \sum_{k=1}^{p^C} \left( \mu^C_{1,l,j,k} + \mu^C_{1,l,j,k} e^{-\zeta_k \omega_n k \hat{t}} \cos \left( \omega_n k \sqrt{1 - \zeta_k^2} \cdot \hat{t} \right) \right)$$

$$+ \mu^C_{S,l,j,k} \left( e^{-\zeta_k \omega_n k \hat{t}} \sin \left( \omega_n k \sqrt{1 - \zeta_k^2} \cdot \hat{t} \right) \right)$$

(D.39)

$$\mu^C_{1,l,j,k} = m_{1,l,j,k} \kappa^C_{1k} \quad \mu^C_{2,l,j,k} = m_{1,l,j,k} \kappa^C_{2k} \quad \mu^C_{S,l,j,k} = m_{2,l,j,k} \kappa^C_{S_k} + m_{1,l,j,k} \kappa^C_{S_k}$$
D.4.1.1.2  Ramp Input ($\dot{q} = 2$)

For the ramp input, equation (D.33) may be written as follows:

$$x_{H_i}(s) = \sum_{j=1}^{m} \Delta \tilde{u}_j \sum_{k=1}^{p_c} \left( \frac{1}{s^2 + 2\zeta_k \omega_{nk}s + \omega_{nk}^2} + \frac{1}{s^2 + 2\zeta_k \omega_{nk}s + \omega_{nk}^2} \right) \frac{1}{s^2}$$  (D.40)

Explicitly multiplying the step input through allows equation (D.40) to be expressed in terms of the Type I and Type II transfer functions:

$$x_{H_i}(s) = \sum_{j=1}^{m} \Delta \tilde{u}_j \sum_{k=1}^{p_c} \left( \frac{1}{s^2 + 2\zeta_k \omega_{nk}s + \omega_{nk}^2} + \frac{1}{s^2 + 2\zeta_k \omega_{nk}s + \omega_{nk}^2} \right) \frac{1}{s^2}$$  (D.41)

Then, expressing the Type I and Type II transfer functions in terms of the basic Laplace Transforms in Table D.2 leads to:

$$x_{H_i}(s) = \sum_{j=1}^{m} \Delta \tilde{u}_j \sum_{k=1}^{p_c} \left[ m_{2,i,j,k} \left( \frac{1}{s^2 + 2\zeta_k \omega_{nk}s + \omega_{nk}^2} + \frac{1}{s^2 + 2\zeta_k \omega_{nk}s + \omega_{nk}^2} \right) \frac{1}{s^2} \right]$$  (D.42)
Expressing the coefficients of the basic Laplace Transforms using the notation $\kappa$ gives:

$$x^c_{Hi}(s) = \sum_{j=1}^{m} \Delta \hat{u}_j \sum_{k=1}^{p^c} \left( m_{1i,j,k} \kappa_{ck} + m_{2i,j,k} \kappa_{cII} \right) \left( \frac{1}{s^2} + \frac{s + \zeta_k \omega_n}{(s + \zeta_k \omega_n)^2 + \omega_n^2(1 - \zeta_k^2)} \right)$$

$$+ \left( m_{2i,j,k} \kappa_{cI} + m_{1i,j,k} \kappa_{cII} \right) \left( \frac{\omega_n \sqrt{1 - \zeta_k^2}}{(s + \zeta_k \omega_n)^2 + \omega_n^2(1 - \zeta_k^2)} \right)$$  \hspace{1cm} (D.43)

Finally, using the notation $\mu$ to represent the coefficients $m$ and $\kappa$ results in the following expression for the forced frequency-domain response for the $i^{th}$ state of the system:

$$x^c_{Hi}(s) = \sum_{j=1}^{m} \Delta \hat{u}_j \sum_{k=1}^{p^c} \left( \mu_{2i,j,k}^c \left( \frac{1}{s^2} + \frac{s + \zeta_k \omega_n}{(s + \zeta_k \omega_n)^2 + \omega_n^2(1 - \zeta_k^2)} \right) \right)$$

$$+ \left( \mu_{2i,j,k}^c + \mu_{1i,j,k}^c \right) \left( \frac{\omega_n \sqrt{1 - \zeta_k^2}}{(s + \zeta_k \omega_n)^2 + \omega_n^2(1 - \zeta_k^2)} \right)$$  \hspace{1cm} (D.44)

Taking the inverse Laplace transform of equation (D.44) leads to the following expression for the forced time-domain response for the $i^{th}$ state of the system:

$$x^c_{Hi}(\hat{t}) = \sum_{j=1}^{m} \Delta \hat{u}_j \sum_{k=1}^{p^c} \left( \mu_{2i,j,k}^c \hat{t} + \mu_{1i,j,k}^c \mu_{cI,j,k}^c \left( e^{-\zeta_k \omega_n \hat{t}} \cos \left( \omega_n \sqrt{1 - \zeta_k^2} \cdot \hat{t} \right) \right) \right)$$

$$+ \mu_{S,i,j,k}^c \left( e^{-\zeta_k \omega_n \hat{t}} \sin \left( \omega_n \sqrt{1 - \zeta_k^2} \cdot \hat{t} \right) \right)$$  \hspace{1cm} (D.45)
D.4.1.1.3 Parabolic Input \((\hat{q} = 3)\)

For the parabolic input, equation (D.33) may be written as follows:

\[
x^c_{HI}(s) = \sum_{j=1}^{m} \sum_{k=1}^{p^c} \Delta \hat{u}_j \left( m_{2i,j,k} \frac{s^2 + 2\zeta_k \omega_n s + \omega_n^2}{s^2 + 2\zeta_k \omega_n s + \omega_n^2 + 1} \right) \frac{1}{s^3}
\]

(D.46)

Explicitly multiplying the step input through allows equation (D.46) to be expressed in terms of the Type II and Type III transfer functions:

\[
x^c_{HI}(s) = \sum_{j=1}^{m} \sum_{k=1}^{p^c} \Delta \hat{u}_j \left( m_{2i,j,k} \frac{1}{s^2 + 2\zeta_k \omega_n s + \omega_n^2} + m_{1i,j,k} \frac{1}{s^3 + 2\zeta_k \omega_n s + \omega_n^2} \right)
\]

(D.47)

Then, expressing the Type II and Type III transfer functions in terms of the basic Laplace Transforms in Table D.2 leads to:

\[
x^c_{HI}(s) = \sum_{j=1}^{m} \sum_{k=1}^{p^c} \Delta \hat{u}_j \left( m_{2i,j,k} \left[ \frac{1}{\omega_n} \left( \frac{1}{s^2} \right) + \left( - \frac{2\zeta_k}{\omega_n} \right) \left( \frac{1}{s} \right) \right]
+ \left( \frac{2\zeta_k}{\omega_n^3} \right) \left( \frac{s + \zeta_k \omega_n}{(s + \zeta \omega_n)^2 + \omega_n^2(1 - \zeta_k^2)} \right)
+ \left( - \frac{1 - 2\zeta_k^2}{\omega_n^3 \sqrt{1 - \zeta_k^2}} \right) \left( \frac{\omega_n k \sqrt{1 - \zeta_k^2}}{(s + \zeta \omega_n)^2 + \omega_n^2(1 - \zeta_k^2)} \right) \right]
+ \left( \frac{1 - 4\zeta_k^2}{\omega_n^4} \right) \left( \frac{s + \zeta_k \omega_n}{(s + \zeta \omega_n)^2 + \omega_n^2(1 - \zeta_k^2)} \right)
+ \left( \frac{3\zeta_k - 4\zeta_k^3}{\omega_n^4 \sqrt{1 - \zeta_k^2}} \right) \left( \frac{\omega_n k \sqrt{1 - \zeta_k^2}}{(s + \zeta \omega_n)^2 + \omega_n^2(1 - \zeta_k^2)} \right) \right)
+ m_{1i,j,k} \left[ \left( \frac{1}{\omega_n} \right) \left( \frac{1}{s^2} \right) + \left( - \frac{2\zeta_k}{\omega_n} \right) \left( \frac{1}{s} \right) + \left( - \frac{1 - 4\zeta_k^2}{\omega_n^4} \right) \left( \frac{1}{s} \right) \right]
+ \left( \frac{1 - 4\zeta_k^2}{\omega_n^4} \right) \left( \frac{s + \zeta_k \omega_n}{(s + \zeta \omega_n)^2 + \omega_n^2(1 - \zeta_k^2)} \right)
+ \left( \frac{3\zeta_k - 4\zeta_k^3}{\omega_n^4 \sqrt{1 - \zeta_k^2}} \right) \left( \frac{\omega_n k \sqrt{1 - \zeta_k^2}}{(s + \zeta \omega_n)^2 + \omega_n^2(1 - \zeta_k^2)} \right) \right]
\]

(D.48)
APPENDIX D (continued)

Expressing the coefficients of the basic Laplace Transforms using the notation $\kappa$ gives:

$$x^c_{H_i}(s) = \sum_{j=1}^{m} \Delta\hat{u}_j \sum_{k=1}^{p^c} \left( \mu^c_{3,j,k} \left( \frac{1}{S^2} \right) + \left( m_{2,j,k} \kappa_{2,k}^{c,III} + m_{1,j,k} \kappa_{3,k}^{c,III} \right) \left( \frac{1}{S^2} \right) \right) + \left( m_{2,j,k} \kappa_{2,k}^{c,III} + m_{1,j,k} \kappa_{3,k}^{c,III} \right) \left( \frac{s + \zeta_k \omega_n}{(s + \zeta_k \omega_n)^2 + \omega_n^2 (1 - \zeta_k^2)} \right) + \left( m_{2,j,k} \kappa_{2,k}^{c,III} + m_{1,j,k} \kappa_{3,k}^{c,III} \right) \left( \frac{\omega_n \sqrt{1 - \zeta_k^2}}{(s + \zeta_k \omega_n)^2 + \omega_n^2 (1 - \zeta_k^2)} \right)$$

(D.49)

Finally, using the notation $\mu$ to represent the coefficients $m$ and $\kappa$ results in the following expression for the forced frequency-domain response for the $i^{th}$ state of the system:

$$x^c_{H_i}(s) = \sum_{j=1}^{m} \Delta\hat{u}_j \sum_{k=1}^{p^c} \left( \mu^c_{3,j,k} \left( \frac{1}{S^2} \right) + \left( m_{2,j,k} \kappa_{2,k}^{c,III} + m_{1,j,k} \kappa_{3,k}^{c,III} \right) \left( \frac{1}{S^2} \right) \right) + \left( m_{2,j,k} \kappa_{2,k}^{c,III} + m_{1,j,k} \kappa_{3,k}^{c,III} \right) \left( \frac{s + \zeta_k \omega_n}{(s + \zeta_k \omega_n)^2 + \omega_n^2 (1 - \zeta_k^2)} \right) + \left( m_{2,j,k} \kappa_{2,k}^{c,III} + m_{1,j,k} \kappa_{3,k}^{c,III} \right) \left( \frac{\omega_n \sqrt{1 - \zeta_k^2}}{(s + \zeta_k \omega_n)^2 + \omega_n^2 (1 - \zeta_k^2)} \right)$$

(D.50)

Taking the inverse Laplace transform of equation (D.50) leads to the following expression for the forced time-domain response for the $i^{th}$ state of the system:

$$x^c_{H_i}(t) = \sum_{j=1}^{m} \Delta\hat{u}_j \sum_{k=1}^{p^c} \left( \mu^c_{3,j,k} \left( \frac{\dot{t}^2}{2} \right) + \left( m_{2,j,k} \kappa_{2,k}^{c,III} + m_{1,j,k} \kappa_{3,k}^{c,III} \right) \left( \frac{\dot{t}^2}{2} \right) \right) + \left( m_{2,j,k} \kappa_{2,k}^{c,III} + m_{1,j,k} \kappa_{3,k}^{c,III} \right) \left( e^{-\zeta_k \omega_n t} \cos \left( \omega_n \sqrt{1 - \zeta_k^2} \cdot \dot{t} \right) \right) + \left( m_{2,j,k} \kappa_{2,k}^{c,III} + m_{1,j,k} \kappa_{3,k}^{c,III} \right) \left( e^{-\zeta_k \omega_n t} \sin \left( \omega_n \sqrt{1 - \zeta_k^2} \cdot \dot{t} \right) \right)$$

(D.51)
APPENDIX D (continued)

D.4.1.1.4 Input of Order $\hat{q}$

For an input of order $\hat{q}$, equation (D.33) may be written as follows:

$$x_H^c(s) = \sum_{j=1}^{m} \Delta \hat{u}_j \sum_{k=1}^{p^c} \left( m_{2,j,k} \frac{s}{s^2 + 2\zeta_k \omega_{n_k} s + \omega_{n_k}^2} + m_{1,j,k} \frac{1}{s^{\hat{q}}} \right) \left( \frac{1}{s^{\hat{q}}} \right)$$  \hspace{1cm} (D.52)

Explicitly multiplying the $\hat{q}^{th}$ order input through allows equation (D.52) to be expressed in terms of transfer functions of Type $\mathbb{Q}^-$ and Type $\mathbb{Q}$:

$$x_H^c(s) = \sum_{j=1}^{m} \left[ \Delta \hat{u}_j \sum_{k=1}^{p^c} \left( \frac{1}{s^{\hat{q}}} \right) \left( \frac{1}{s^2 + 2\zeta_k \omega_{n_k} s + \omega_{n_k}^2} \right) \right]$$  \hspace{1cm} (D.53)

Then, using the notation $\kappa$, expressing the transfer functions of Type $\mathbb{Q}^-$ and Type $\mathbb{Q}$ in terms of the basic Laplace Transforms in Table D.2 leads to:

$$x_H^c(s) = \sum_{j=1}^{m} \left[ \Delta \hat{u}_j \sum_{k=1}^{p^c} \left( \sum_{l=1}^{q-1} \left( m_{2,l,j,k} \kappa_{l,k}^{c,Q^-} \frac{1}{s^l} \right) \right) + \sum_{l=1}^{q} \left( m_{1,l,j,k} \kappa_{l,k}^{c,Q} \frac{1}{s^l} \right) \right]$$  \hspace{1cm} (D.54)
Finally, using the notation $\mu$ to represent the coefficients $m$ and $\kappa$ results in the following expression for the forced frequency-domain response for the $i^{th}$ state of the system:

$$ x_{Hi}^C(s) = \sum_{j=1}^{m} \Delta \hat{u}_j \sum_{k=1}^{p^c} \left( \mu_{\hat{q}_{i,j,k}}^C \left( \frac{1}{s \hat{q}} \right) + \sum_{l=1}^{q-1} \left( \mu_{i,j,k,l}^C \left( \frac{1}{s l} \right) \right) \right) + \mu_{c_{i,j,k}}^C \left( \frac{s + \zeta_k \omega_n}{(s + \zeta_k \omega_n)^2 + \omega_n^2 (1-\zeta_k^2)} \right) + \mu_{s_{i,j,k}}^C \left( \frac{\omega_n \sqrt{1-\zeta_k^2}}{(s + \zeta_k \omega_n)^2 + \omega_n^2 (1-\zeta_k^2)} \right) $$

(D.55)

Taking the inverse Laplace transform of equation (D.55) leads to the following expression for the forced time-domain response for the $i^{th}$ state of the system:

$$ x_{Hi}^C(\hat{t}) = \sum_{j=1}^{m} \Delta \hat{u}_j \sum_{k=1}^{p^c} \left( \mu_{\hat{q}_{i,j,k}}^C \left( \frac{\hat{t} \hat{q}^{-1}}{(\hat{q} - 1)!} \right) + \sum_{l=1}^{q-1} \left( \mu_{i,j,k,l}^C \left( \frac{\hat{t}^{l-1}}{(l-1)!} \right) \right) \right) + \mu_{c_{i,j,k}}^C \left( e^{-\zeta_k \omega_n \hat{t}} \cos \left( \omega_n \sqrt{1-\zeta_k^2} \cdot \hat{t} \right) \right) + \mu_{s_{i,j,k}}^C \left( e^{-\zeta_k \omega_n \hat{t}} \sin \left( \omega_n \sqrt{1-\zeta_k^2} \cdot \hat{t} \right) \right) $$

(D.56)
D.4.1.2 The Forced Response due to Initial Conditions on the Inputs $X^c_{H_l}$

The forced response due to initial conditions on the inputs for the $i^{th}$ state of the system is given by equation (D.57):

$$x^c_{H_l}(s) = \sum_{j=1}^{m} \left[ \sum_{k=1}^{p^c} \frac{m_2_{i,j,k} s + m_1_{i,j,k}}{s^2 + 2\zeta_k \omega_{n_k} s + \omega_{n_k}^2} \left( \frac{u_j}{s} \right) \right]$$

(D.57)

The initial condition on the input is taken to be a step input of some magnitude at each time $t$. This initial condition is then superposed with the actual commanded step, ramp, parabolic, or higher order input. As such, the frequency- and time-domain responses are equivalent in form to the expressions in equations (D.38) and (D.39). The overbar notation is used to distinguish the coefficients associated with the initial conditions on the inputs.

Using the notation $\bar{\mu}$ to represent the coefficients $m$ and $\kappa$ results in the following expression for the forced frequency-domain response for the $i^{th}$ state of the system:

$$x^c_{H_l}(s) = \sum_{j=1}^{m} u_j \sum_{k=1}^{p^c} \left[ \bar{\mu}_{1_{i,j,k}} \left( \frac{1}{s} \right) + \bar{\mu}_{2_{i,j,k}} \left( \frac{s + \zeta_k \omega_{n_k}}{(s + \zeta_k \omega_{n_k})^2 + \omega_{n_k}^2 (1 - \zeta_k^2)} \right) ight]$$

$$+ \bar{\mu}_{3_{i,j,k}} \left( \frac{\omega_{n_k} \sqrt{1 - \zeta_k^2}}{(s + \zeta_k \omega_{n_k})^2 + \omega_{n_k}^2 (1 - \zeta_k^2)} \right)$$

(D.58)
Taking the inverse Laplace transform of equation (D.58) leads to the following expression for the forced time-domain response for the $i^{th}$ state of the system:

\[
x_{\tilde{H}_i}(\hat{t}) = \sum_{j=1}^{m} u_j \sum_{k=1}^{n} \left[ \mu_{1i,j,k}^c + \mu_{Ci,j,k}^c \left( e^{-\zeta_k \omega_{nk} \hat{t}} \cos \left( \omega_{nk} \sqrt{1 - \zeta_k^2} \cdot \hat{t} \right) \right) 
+ \mu_{Si,j,k}^c \left( e^{-\zeta_k \omega_{nk} \hat{t}} \sin \left( \omega_{nk} \sqrt{1 - \zeta_k^2} \cdot \hat{t} \right) \right) \right]
\]

\[
\mu_{1i,j,k}^c = m_{1i,j,k} \kappa_{1k}^c, \quad \mu_{Ci,j,k}^c = m_{1i,j,k} \kappa_{k}^c, \quad \mu_{Si,j,k}^c = m_{2i,j,k} \kappa_{Sk}^c, \quad m_{1i,j,k} \kappa_{Sk}^c
\]
The unforced response for the $i$th state of the system is given by equation (D.60):

$$x_{Gi}^c(s) = \sum_{k=1}^{p^c} \frac{n_{2,ik} s + n_{1,ik}}{s^2 + 2\zeta_k \omega_{nk} s + \omega_{nk}^2}$$ (D.60)

Expanding equation (D.60) and expressing it in terms of the Type O1 and Type O2 transfer functions gives:

$$x_{Gi}^c(s) = \sum_{k=1}^{p^c} \left( \frac{s}{n_{2,ik} s^2 + 2\zeta_k \omega_{nk} s + \omega_{nk}^2} + \frac{1}{n_{1,ik} s^2 + 2\zeta_k \omega_{nk} s + \omega_{nk}^2} \right)$$  \quad \text{(D.61)}$$

Then, expressing the Type O1 and Type O2 transfer functions in terms of the basic Laplace Transforms in Table D.2 leads to:

$$x_{Gi}^c(s) = \sum_{k=1}^{p^c} n_{2,ik} \left[ \left( \frac{s + \zeta_k \omega_{nk}}{(s + \zeta_k \omega_{nk})^2 + \omega_{nk}^2 \left(1 - \zeta_k^2\right)} \right) \right. \\
+ \left. \left( -\frac{\zeta_k}{\sqrt{1 - \zeta_k^2}} \right) \left( \frac{\omega_{nk} \sqrt{1 - \zeta_k^2}}{(s + \zeta_k \omega_{nk})^2 + \omega_{nk}^2 \left(1 - \zeta_k^2\right)} \right) \right] \\
+ n_{1,ik} \left[ \left( \frac{1}{\omega_{nk} \sqrt{1 - \zeta_k^2}} \right) \left( \frac{\omega_{nk} \sqrt{1 - \zeta_k^2}}{(s + \zeta_k \omega_{nk})^2 + \omega_{nk}^2 \left(1 - \zeta_k^2\right)} \right) \right]$$ \quad \text{(D.62)}}
APPENDIX D (continued)

Expressing the coefficients of the basic Laplace Transforms using the notation $\kappa$ gives:

$$x^c_{\tilde{G}_i}(s) = \sum_{k=1}^{p_c} \left[\left(n_{2,k} \kappa^c_{\tilde{G}_k} \right) \left( \frac{s + \zeta_k \omega_n}{(s + \xi_k \omega_n)^2 + \omega_n^2(\zeta_k^2)} \right) + \left(n_{2,k} \kappa_{S_k}^c + n_{1,k} \kappa_{S_k}^c \right) \left( \frac{\omega_n \sqrt{1 - \zeta_k^2}}{(s + \xi_k \omega_n)^2 + \omega_n^2(1 - \zeta_k^2)} \right) \right] \right] \quad (D.63)$$

Finally, using the notation $\nu$ to represent the coefficients $n$ and $\kappa$ results in the following expression for the unforced frequency-domain response for the $i^{th}$ state of the system:

$$x^c_{\tilde{G}_i}(s) = \sum_{k=1}^{p_c} \left[\left(n_{2,k} \kappa^c_{\tilde{G}_k} \right) \left( \frac{s + \zeta_k \omega_n}{(s + \xi_k \omega_n)^2 + \omega_n^2(\zeta_k^2)} \right) + \left(n_{2,k} \kappa_{S_k}^c + n_{1,k} \kappa_{S_k}^c \right) \left( \frac{\omega_n \sqrt{1 - \zeta_k^2}}{(s + \xi_k \omega_n)^2 + \omega_n^2(1 - \zeta_k^2)} \right) \right] \quad (D.64)$$

Taking the inverse Laplace transform of equation (D.64) leads to the following expression for the unforced time-domain response for the $i^{th}$ state of the system:

$$x^c_{\tilde{G}_i}(\hat{t}) = \sum_{k=1}^{p_c} \left[\left(n_{2,k} \kappa^c_{\tilde{G}_k} \right) \left( e^{-\zeta_k \omega_n \hat{t}} \cos \left( \omega_n \sqrt{1 - \zeta_k^2} \cdot \hat{t} \right) \right) + \left(n_{2,k} \kappa_{S_k}^c + n_{1,k} \kappa_{S_k}^c \right) \left( e^{-\zeta_k \omega_n \hat{t}} \sin \left( \omega_n \sqrt{1 - \zeta_k^2} \cdot \hat{t} \right) \right) \right] \quad (D.65)$$

$$\nu^c_{\tilde{G}_k} = n_{2,k} \kappa^c_{\tilde{G}_k} \quad \nu^c_{S_k} = n_{2,k} \kappa_{S_k}^c + n_{1,k} \kappa_{S_k}^c$$
APPENDIX D (continued)

D.4.1.4  The Assembled Complex System Response $X^C$

The complete forced and unforced response for the $i^{th}$ state of the system (for the complex modes) is now assembled according to equation (D.66):

\[ x_i^C(\hat{t}) = x_{Hi}^C(\hat{t}) + x_{Hi}^E(\hat{t}) + x_{Gi}^E(\hat{t}) \]  (D.66)

Written explicitly, equation (D.66) may be expanded to become:

\[
x_i^C(\hat{t}) = \sum_{j=1}^{m} \Delta u_j \sum_{k=1}^{p^c} \left[ \sum_{l=1}^{q} \left( \mu_{\varepsilon,k,l,j}^C \left( \frac{\hat{t}^l}{(l-1)!} \right) + \mu_{\varepsilon,k,l,j}^E \left( e^{-\zeta_k \omega_n k \hat{t}} \cos(\omega_d k \hat{t}) \right) + \mu_{\varepsilon,k,l,j}^S \left( e^{-\zeta_k \omega_n k \hat{t}} \sin(\omega_d k \hat{t}) \right) \right) \right] + \sum_{k=1}^{p^c} \left[ \nu_{\varepsilon,k,l,j}^E \left( e^{-\zeta_k \omega_n k \hat{t}} \cos(\omega_d k \hat{t}) \right) + \nu_{\varepsilon,k,l,j}^S \left( e^{-\zeta_k \omega_n k \hat{t}} \sin(\omega_d k \hat{t}) \right) \right] + \sum_{k=1}^{p^c} \left[ \nu_{\varepsilon,k,l,j}^E \left( e^{-\zeta_k \omega_n k \hat{t}} \cos(\omega_d k \hat{t}) \right) + \nu_{\varepsilon,k,l,j}^S \left( e^{-\zeta_k \omega_n k \hat{t}} \sin(\omega_d k \hat{t}) \right) \right] \] (D.67)
D.4.2 Real Modes

D.4.2.1 The Forced Response $X_H^R$

The forced response for the $i^{th}$ state of the system is given by equation (D.68):

$$
x_{H_i}^R(s) = \sum_{j=1}^{m} \left[ \sum_{k=1}^{p^R} \frac{m_{0,i,j,k}}{s + \tau_{k}^{-1}} \Delta \dot{u}_j(s) \right]
$$

(D.68)

The input $\Delta \dot{u}_j(s)$ is given by $\Delta \dot{u}_j \frac{1}{s^{\hat{q}}}$, where $\hat{q}$ represents the order of the input and $\Delta \dot{u}_j$ represents the magnitude of the corresponding input order for the $j^{th}$ input. For example, if $\hat{q} = 1$, the input is a step and $\Delta \dot{u}_j$ is the magnitude of the step. If $\hat{q} = 2$, the input is a ramp and $\Delta \dot{u}_j$ is the magnitude of the input rate. If $\hat{q} = 3$, the input is a parabola and $\Delta \dot{u}_j$ is the magnitude of the input acceleration. The expressions for these three special cases are developed below, and this culminates in the development of the general expressions for an input of order $\hat{q}$.
D.4.2.1.1  Step Input ($\tilde{q} = 1$)

For the step input, equation (D.68) may be written as follows:

$$x_{Rt}^\circ(s) = \sum_{j=1}^{m} \left[ \Delta \tilde{u}_j \sum_{k=1}^{p^R} \left( m_{0,l,j,k} \frac{1}{s + \tau_k^{-1}} \right) \right]$$  \hspace{1cm} (D.69)

Explicitly multiplying the step input through allows equation (D.69) to be expressed in terms of the Type I transfer function:

$$x_{Rt}^\circ(s) = \sum_{j=1}^{m} \left[ \Delta \tilde{u}_j \sum_{k=1}^{p^R} \left( m_{0,l,j,k} \frac{1}{s + \tau_k^{-1}} \right) \right]$$  \hspace{1cm} (D.70)

Then, expressing the Type I transfer function in terms of the basic Laplace Transforms in Table D.4 leads to:

$$x_{Rt}^\circ(s) = \sum_{j=1}^{m} \left[ \Delta \tilde{u}_j \sum_{k=1}^{p^R} \left( m_{0,l,j,k} \frac{1}{s + \tau_k^{-1}} \right) \right]$$  \hspace{1cm} (D.71)

Expressing the coefficients of the basic Laplace Transforms using the notation $\kappa$ gives:

$$x_{Rt}^\circ(s) = \sum_{j=1}^{m} \left[ \Delta \tilde{u}_j \sum_{k=1}^{p^R} \left( m_{0,l,j,k} \left[ \kappa_{1,k} \frac{1}{s} + \kappa_{E,k} \frac{1}{s + \tau_k^{-1}} \right] \right) \right]$$  \hspace{1cm} (D.72)
Finally, using the notation $\mu$ to represent the coefficients $m$ and $\kappa$ results in the following expression for the forced frequency-domain response for the $i^{th}$ state of the system:

$$x_{Hi}^R(s) = \sum_{j=1}^{m} \Delta \tilde{u}_j \sum_{k=1}^{p} \left( \mu_{1i,j,k}^R \left( \frac{1}{s} \right) + \mu_{Ei,j,k}^R \left( \frac{1}{s + \tau_k^{-1}} \right) \right)$$  \hspace{1cm} (D.73)

Taking the inverse Laplace transform of equation (D.73) leads to the following expression for the forced time-domain response for the $i^{th}$ state of the system:

$$x_{Hi}^R(\hat{t}) = \sum_{j=1}^{m} \Delta \tilde{u}_j \sum_{k=1}^{p} \left( \mu_{1i,j,k}^R + \mu_{Ei,j,k}^R (e^{-\tau_k^{-1} \hat{t}}) \right)$$  \hspace{1cm} (D.74)

$$\mu_{1i,j,k}^R = m_{0i,j,k} \kappa_{1k}^R \quad \mu_{Ei,j,k}^R = m_{0i,j,k} \kappa_{E_k}^R$$
D.4.2.1.2 Ramp Input ($\dot{q} = 2$)

For the ramp input, equation (D.68) may be written as follows:

$$x_{HI}^R(s) = \sum_{j=1}^{m} \Delta \hat{u}_j \sum_{k=1}^{p^R} \left( m_{0i,jk} \frac{1}{S + \tau_{k-1}} \right) \frac{1}{s^2}$$  \hspace{1cm} (D.75)

Explicitly multiplying the ramp input through allows equation (D.75) to be expressed in terms of the Type II transfer function:

$$x_{HI}^R(s) = \sum_{j=1}^{m} \Delta \hat{u}_j \sum_{k=1}^{p^R} \left( m_{0i,jk} \frac{1}{S + \tau_{k-1}} \right) \frac{1}{s^2(s + \tau_{k-1})}$$  \hspace{1cm} (D.76)

Then, expressing the Type II transfer function in terms of the basic Laplace Transforms in Table D.4 leads to:

$$x_{HI}^R(s) = \sum_{j=1}^{m} \Delta \hat{u}_j \sum_{k=1}^{p^R} \left( m_{0i,jk} \left[ \tau_k \left( \frac{1}{s^2} \right) + (-\tau_k^2) \left( \frac{1}{s} \right) + (\tau_k^2) \left( \frac{1}{s + \tau_{k-1}} \right) \right] \right)$$  \hspace{1cm} (D.77)

Expressing the coefficients of the basic Laplace Transforms using the notation $\kappa$ gives:

$$x_{HI}^R(s) = \sum_{j=1}^{m} \Delta \hat{u}_j \sum_{k=1}^{p^R} \left( m_{0i,jk} \left[ (\kappa_2^R) \left( \frac{1}{s^2} \right) + (\kappa_1^R) \left( \frac{1}{s} \right) + (\kappa_{E_k}^R) \left( \frac{1}{s + \tau_{k-1}} \right) \right] \right)$$  \hspace{1cm} (D.78)
Finally, using the notation \( \mu \) to represent the coefficients \( m \) and \( \kappa \) results in the following expression for the forced frequency-domain response for the \( i \)th state of the system:

\[
\tilde{x}_{H_i}(s) = \sum_{j=1}^{m} \left[ \Delta \tilde{u}_j \sum_{k=1}^{p} \left( \mu_{2i,j,k}^R \left( \frac{1}{S^2} \right) + \mu_{1i,j,k}^R \left( \frac{1}{S} \right) + \mu_{Ei,j,k}^R \left( \frac{1}{S + \tau_k^{-1}} \right) \right) \right]
\]  
(D.79)

Taking the inverse Laplace transform of equation (D.79) leads to the following expression for the forced time-domain response for the \( i \)th state of the system:

\[
\tilde{x}_{H_i}(\tilde{t}) = \sum_{j=1}^{m} \left[ \Delta \tilde{u}_j \sum_{k=1}^{p} \left( \mu_{2i,j,k}^R \tilde{t} + \mu_{1i,j,k}^R + \mu_{Ei,j,k}^R (e^{-\tau_k^{-1}\tilde{t}}) \right) \right]
\]  
(D.80)

\[
\mu_{2i,j,k}^R = m_{0i,j,k} \kappa_{2k}^R \quad \mu_{1i,j,k}^R = m_{0i,j,k} \kappa_{1k}^R \quad \mu_{Ei,j,k}^R = m_{0i,j,k} \kappa_{E_k}^R
\]
D.4.2.1.3 Parabolic Input (\(\hat{q} = 3\))

For the parabolic input, equation (D.68) may be written as follows:

\[
x_{H_1}^R(s) = \sum_{j=1}^{m} \Delta \hat{u}_j \sum_{k=1}^{p_R} \left( m_{0,j,k} \frac{1}{s + \tau_k^{-1}} \frac{1}{s^3} \right)
\]

(D.81)

Explicitly multiplying the parabolic input through allows equation (D.81) to be expressed in terms of the Type III transfer function:

\[
x_{H_1}^R(s) = \sum_{j=1}^{m} \Delta \hat{u}_j \sum_{k=1}^{p_R} \left( m_{0,j,k} \frac{1}{s^3(s + \tau_k^{-1})} \right)
\]

(D.82)

Then, expressing the Type III transfer function in terms of the basic Laplace Transforms in Table D.4 leads to:

\[
x_{H_1}^R(s) = \sum_{j=1}^{m} \Delta \hat{u}_j \sum_{k=1}^{p_R} \left( m_{0,j,k} \left[ (\tau_k) \left( \frac{1}{s^3} \right) + (-\tau_k^2) \left( \frac{1}{s^2} \right) + (\tau_k^3) \left( \frac{1}{s} \right) \right] 
\]

(D.83)

\[
+ (-\tau_k^3) \left( \frac{1}{s + \tau_k^{-1}} \right) \right)
\]

Expressing the coefficients of the basic Laplace Transforms using the notation \(\kappa\) gives:

\[
x_{H_1}^R(s) = \sum_{j=1}^{m} \Delta \hat{u}_j \sum_{k=1}^{p_R} \left( m_{0,j,k} \left[ (\kappa_{3,k}^{III}) \left( \frac{1}{s^3} \right) + (\kappa_{2,k}^{III}) \left( \frac{1}{s^2} \right) + (\kappa_{1,k}^{III}) \left( \frac{1}{s} \right) 
\]

(D.84)

\[
+ (\kappa_{E,k}^{III}) \left( \frac{1}{s + \tau_k^{-1}} \right) \right)
\]
Finally, using the notation \( \mu \) to represent the coefficients \( m \) and \( \kappa \) results in the following expression for the forced frequency-domain response for the \( i^{th} \) state of the system:

\[
\begin{align*}
    x_{Hi}^R(s) &= \sum_{j=1}^{m} \left[ \Delta \hat{u}_j \sum_{k=1}^{p^R} \left( \mu_{3i,j,k}^R \left( \frac{1}{s^3} \right) + \mu_{2i,j,k}^R \left( \frac{1}{s^2} \right) + \mu_{1i,j,k}^R \left( \frac{1}{s} \right) 
    
    + \mu_{Ei,j,k}^R \left( \frac{1}{s + \tau_k^{-1}} \right) \right) \right] \\
    &= \sum_{j=1}^{m} \left[ \Delta \hat{u}_j \sum_{k=1}^{p^R} \left( \mu_{3i,j,k}^R \left( \frac{1}{s^3} \right) + \mu_{2i,j,k}^R \left( \frac{1}{s^2} \right) + \mu_{1i,j,k}^R \left( \frac{1}{s} \right) 
    
    + \mu_{Ei,j,k}^R \left( \frac{1}{s + \tau_k^{-1}} \right) \right) \right] \\
    &= \sum_{j=1}^{m} \left[ \Delta \hat{u}_j \sum_{k=1}^{p^R} \left( \mu_{3i,j,k}^R \left( \frac{1}{s^3} \right) + \mu_{2i,j,k}^R \left( \frac{1}{s^2} \right) + \mu_{1i,j,k}^R \left( \frac{1}{s} \right) 
    
    + \mu_{Ei,j,k}^R \left( \frac{1}{s + \tau_k^{-1}} \right) \right) \right]
\end{align*}
\]  

(D.85)

Taking the inverse Laplace transform of equation (D.85) leads to the following expression for the forced time-domain response for the \( i^{th} \) state of the system:

\[
\begin{align*}
    x_{Hi}(\hat{t}) &= \sum_{j=1}^{m} \left[ \Delta \hat{u}_j \sum_{k=1}^{p^R} \left( \mu_{3i,j,k}^R \left( \frac{\hat{t}^2}{2} \right) + \mu_{2i,j,k}^R \hat{t} + \mu_{1i,j,k}^R + \mu_{Ei,j,k}^R \left( e^{-\tau_k^{-1}\hat{t}} \right) \right) \right]
\end{align*}
\]  

(D.86)

\[
\begin{align*}
    \mu_{3i,j,k}^R &= m_{0i,j,k}^R \kappa_3^R \\
    \mu_{2i,j,k}^R &= m_{0i,j,k}^R \kappa_2^R \\
    \mu_{1i,j,k}^R &= m_{0i,j,k}^R \kappa_1^R \\
    \mu_{Ei,j,k}^R &= m_{0i,j,k}^R \kappa_E^R
\end{align*}
\]
APPENDIX D (continued)

D.4.2.1.4 Input of Order $\hat{q}$

For an input of order $\hat{q}$, equation (D.68) may be written as follows:

$$x^{R}_{H_i}(s) = \sum_{j=1}^{m} \left[ \Delta \hat{u}_{j} \sum_{k=1}^{p^{R}} \left( m_{0\ell,j,k} \frac{1}{s + \tau_{k}^{-1}} \right) \frac{1}{s^{\hat{q}}} \right]$$

(D.87)

Explicitly multiplying the $\hat{q}^{th}$ order input through allows equation (D.52) to be expressed in terms of a transfer function of Type $\mathbb{Q}$:

$$x^{R}_{H_i}(s) = \sum_{j=1}^{m} \left[ \Delta \hat{u}_{j} \sum_{k=1}^{p^{R}} \left( m_{0\ell,j,k} \frac{1}{s + \tau_{k}^{-1}} \hat{q} \right) \right]$$

(D.88)

Then, using the notation $\kappa$, expressing the transfer function of Type $\mathbb{Q}$ in terms of the basic Laplace Transforms in Table D.4 leads to:

$$x^{R}_{H_i}(s) = \sum_{j=1}^{m} \left[ \Delta \hat{u}_{j} \sum_{k=1}^{p^{R}} \left( m_{0\ell,j,k} \left( \hat{q} \right) k_{l_{j,k}}^{R,Q} \left( s^{l_{j}} \right) + \left( k_{E_{j}}^{R,Q} \right) \left( s + \tau_{k}^{-1} \right) \right) \right]$$

(D.89)

Finally, using the notation $\mu$ to represent the coefficients $m$ and $\kappa$ results in the following expression for the forced frequency-domain response for the $i^{th}$ state of the system:

$$x^{R}_{H_i}(s) = \sum_{j=1}^{m} \left[ \Delta \hat{u}_{j} \sum_{k=1}^{p^{R}} \left( \sum_{l=1}^{\hat{q}} \left( \mu_{l_{j,k}}^{R} \frac{1}{s^{l}} \right) + \mu_{E_{j,k}}^{R} \frac{1}{s + \tau_{k}^{-1}} \right) \right]$$

(D.90)
APPENDIX D (continued)

Taking the inverse Laplace transform of equation (D.90) leads to the following expression for the forced time-domain response for the \( i^{th} \) state of the system:

\[
\begin{align*}
\mathbf{x}_{H,i}^{\mathbb{R}}(\hat{t}) &= \sum_{j=1}^{m} \left[ \Delta \hat{u}_j \sum_{k=1}^{p} \sum_{l=1}^{q} \left( \mu_{l,j,k}^{\mathbb{R}} \left( \frac{\hat{t}^{l-1}}{(l-1)!} \right) \right) + \mu_{E,i,j,k}^{\mathbb{R}} (e^{-\tau_k \hat{t}}) \right] \\
&\quad + \sum_{j=1}^{m} \left[ \Delta \hat{u}_j \sum_{k=1}^{p} \sum_{l=1}^{q} \left( \mu_{l,j,k}^{\mathbb{R}} \left( \frac{\hat{t}^{l-1}}{(l-1)!} \right) \right) + \mu_{E,i,j,k}^{\mathbb{R}} (e^{-\tau_k \hat{t}}) \right] \\
&= \sum_{j=1}^{m} \Delta \hat{u}_j \sum_{k=1}^{p} \sum_{l=1}^{q} \left( \mu_{l,j,k}^{\mathbb{R}} \left( \frac{\hat{t}^{l-1}}{(l-1)!} \right) \right) + \mu_{E,i,j,k}^{\mathbb{R}} (e^{-\tau_k \hat{t}})
\end{align*}
\]

(D.91)

\[
\begin{align*}
\mu_{l,j,k}^{\mathbb{R}} &= m_{0_{l,j,k}} \kappa_{l,k}^{\mathbb{R},Q} \\
\mu_{E,i,j,k}^{\mathbb{R}} &= m_{0_{l,j,k}} \kappa_{E,k}^{\mathbb{R},Q}
\end{align*}
\]
D.4.2.2 The Forced Response due to Initial Conditions on the Inputs $X^\mathbb{R} _H$

The forced response due to initial conditions on the inputs for the $i^{th}$ state of the system is given by equation (D.92):

$$x^\mathbb{R} _{H_i}(s) = \sum_{j=1}^{\mathbb{I}} \left[ \sum_{k=1}^{p} \frac{m_{0,j,k} \left( \frac{\mu_j}{s} \right)}{s + \tau_k^{-1}} \right]$$  \hspace{1cm} (D.92)

The initial condition on the input is taken to be a step input of some magnitude at each time $t$. This initial condition is then superposed with the actual commanded step, ramp, parabolic, or higher order input. As such, the frequency- and time-domain responses are equivalent in form to the expressions in equations (D.73) and (D.74). The overbar notation is used to distinguish the coefficients associated with the initial conditions on the inputs.

Using the notation $\bar{\mu}$ to represent the coefficients $m$ and $\kappa$ results in the following expression for the forced frequency-domain response for the $i^{th}$ state of the system:

$$x^\mathbb{R} _{H_i}(s) = \sum_{j=1}^{\mathbb{I}} u_j \sum_{k=1}^{p} \left( \bar{\mu}^\mathbb{R}_{1,i,j,k} \left( \frac{1}{s} \right) + \bar{\mu}^\mathbb{R}_{E,i,j,k} \left( \frac{1}{s + \tau_k^{-1}} \right) \right)$$  \hspace{1cm} (D.93)

Taking the inverse Laplace transform of equation (D.93) leads to the following expression for the forced time-domain response for the $i^{th}$ state of the system:

$$x^\mathbb{R} _{H_i} (t) = \sum_{j=1}^{\mathbb{I}} u_j \sum_{k=1}^{p} \left( \bar{\mu}^\mathbb{R}_{1,i,j,k} + \bar{\mu}^\mathbb{R}_{E,i,j,k} (e^{-\tau_k^{-1} t}) \right)$$  \hspace{1cm} (D.94)

$$\bar{\mu}^\mathbb{R}_{1,i,j,k} = m_{0,i,j,k} \kappa_{1k}^\mathbb{R}$$  \hspace{1cm} $$\bar{\mu}^\mathbb{R}_{E,i,j,k} = m_{0,i,j,k} \kappa_{E_k}^\mathbb{R}$$
APPENDIX D (continued)

D.4.2.3 The Unforced Response $X_G^R$

The unforced response for the $i^{th}$ state of the system is given by equation (D.95):

$$X_G^R(s) = \sum_{k=1}^{p^R} \frac{n_{0,i,k}}{s + \tau_k^{-1}}$$  \hspace{1cm} (D.95)

Expressing equation (D.95) in terms of the Type O transfer function leads to:

$$X_G^R(s) = \sum_{k=1}^{p^R} \left( n_{0,i,k} \frac{1}{s + \tau_k^{-1}} \right)$$  \hspace{1cm} (D.96)

Then, expressing the Type O transfer function in terms of the basic Laplace Transforms in Table D.4 leads to:

$$X_G^R(s) = \sum_{k=1}^{p^R} \left( n_{0,i,k} \left[ \left(1\left( \frac{1}{s + \tau_k^{-1}} \right) \right) \right) \right)$$  \hspace{1cm} (D.97)

Expressing the coefficients of the basic Laplace Transforms using the notation $\kappa$ gives:

$$X_G^R(s) = \sum_{k=1}^{p^R} \left( n_{0,i,k} \left[ \left( \kappa_E^{R,O} \left( \frac{1}{s + \tau_k^{-1}} \right) \right) \right) \right)$$  \hspace{1cm} (D.98)

Finally, using the notation $\nu$ to represent the coefficients $n$ and $\kappa$ results in the following expression for the unforced frequency-domain response for the $i^{th}$ state of the system:
APPENDIX D (continued)

\[ x_{G_i}^R(s) = \sum_{k=1}^{p^R} \left( v_{E_{i,k}}^R \left( \frac{1}{s + \tau_k^{-1}} \right) \right) \]  

(D.99)

Taking the inverse Laplace transform of equation (D.99) leads to the following expression for the unforced time-domain response for the \(i^{th}\) state of the system:

\[ x_{G_i}^R(\hat{t}) = \sum_{k=1}^{p^R} \left( v_{E_{i,k}}^R \left( e^{-\tau_k^{-1}\hat{t}} \right) \right) \]  

(D.100)

\[ v_{E_{i,k}}^R = n_{O_{i,k}} K_{E_k}^{R,0} \]
APPENDIX D (continued)

D.4.2.4 The Assembled Real System Response \(X^R\)

The complete forced and unforced response for the \(i^{th}\) state of the system (for the real modes) is now assembled according to equation (D.101):

\[
x^R_i(\hat{t}) = x^R_{H_i}(\hat{t}) + x^R_{\tilde{H_i}}(\hat{t}) + x^R_{\tilde{G_i}}(\hat{t})
\]  

(D.101)

Written explicitly, equation (D.101) may be expanded to become:

\[
x^R_i(\hat{t}) = \sum_{j=1}^{m} \Delta \hat{u}_j \sum_{k=1}^{p} \left( \sum_{l=1}^{q} \mu^R_{i,j,k} \left( \frac{\hat{t}^{l-1}}{(l-1)!} \right) + \mu^R_{E_i,j,k} (e^{-\tau_k^{-1}\hat{t}}) \right) \\
+ \hat{u}_j \sum_{k=1}^{p} \left( \hat{\mu}^R_{i,j,k} + \bar{\mu}^R_{E_i,j,k} (e^{-\tau_k^{-1}\hat{t}}) \right) \\
+ \sum_{k=1}^{p} \left[ \nu^R_{E_i,k} (e^{-\tau_k^{-1}\hat{t}}) \right]
\]  

(D.102)
D.5 The Complete System Time-Derivative

For the special case of the step input, which typically causes an underdamped and oscillatory response, an additional step is required to determine the input magnitude that would cause the peak amplitude to exactly reach (and not overshoot) the state limit. An intermediate step towards this goal involves determining the peak time of the system response, which is, in turn, determined by finding the maxima and minima of the system. Recalling equation (D.29), the general form of the time-derivative of the system response may be written as follows:

$$\dot{X}(\hat{t}) = \dot{H}(\hat{t})\Delta \hat{U} + \ddot{H}(\hat{t})u + \dot{G}(\hat{t}) = \dot{X}_H(\hat{t}) + \ddot{X}_H(\hat{t}) + \dot{X}_G(\hat{t})$$  \hspace{1cm} (D.103)

Because the critical input $\Delta \hat{U}$ and the initial condition on the input $U$ are both step inputs, equation (D.103) may be condensed to:

$$\dot{X}(\hat{t}) = \dot{H}(\hat{t})(\Delta \hat{U} + U) + \dot{G}(\hat{t}) = \dot{X}_H(\hat{t}) + \dot{X}_G(\hat{t})$$  \hspace{1cm} (D.104)

In expanded form, equation (D.104) becomes:

$$\begin{bmatrix} \dot{x}_1(\hat{t}) \\ \dot{x}_2(\hat{t}) \\ \vdots \\ \dot{x}_n(\hat{t}) \end{bmatrix} = \begin{bmatrix} \dot{H}_{1,1}(\hat{t}) & \dot{H}_{1,2}(\hat{t}) & \cdots & \dot{H}_{1,m}(\hat{t}) \\ \dot{H}_{2,1}(\hat{t}) & \dot{H}_{2,2}(\hat{t}) & \cdots & \dot{H}_{2,m}(\hat{t}) \\ \vdots & \vdots & \ddots & \vdots \\ \dot{H}_{n,1}(\hat{t}) & \dot{H}_{n,2}(\hat{t}) & \cdots & \dot{H}_{n,m}(\hat{t}) \end{bmatrix} \begin{bmatrix} \Delta u_1 + u_1 \\ \Delta u_2 + u_2 \\ \vdots \\ \Delta u_m + u_m \end{bmatrix} + \begin{bmatrix} \dot{G}_1(\hat{t}) \\ \dot{G}_2(\hat{t}) \\ \vdots \\ \dot{G}_n(\hat{t}) \end{bmatrix}$$  \hspace{1cm} (D.105)

This section will develop the expressions for $\dot{H}(\hat{t})(\Delta \hat{U} + U)$ and $\dot{G}(\hat{t})$. As before, the complex mode transfer functions and real mode transfer functions will be discussed separately. Consequently, equation (D.104) may be further expanded as follows:
D.5.1 Complex Modes

D.5.1.1 The Forced Time-Derivative $\dot{X}^c_H$

The forced time-derivative response for the $i^{th}$ state of the system may be written as:

$$\dot{X}^c_H(t) = \sum_{j=1}^{m} (\Delta u_j + u_j) \sum_{k=1}^{p^c} \left( \mu_{c,i,j,k}^c \frac{d}{d\hat{t}} \left( e^{-\zeta_k \omega_n \hat{t}} \cos \left( \omega_n \sqrt{1 - \zeta_k^2} \cdot \hat{t} \right) \right) + \mu_{s,i,j,k}^c \frac{d}{d\hat{t}} \left( e^{-\zeta_k \omega_n \hat{t}} \sin \left( \omega_n \sqrt{1 - \zeta_k^2} \cdot \hat{t} \right) \right) \right)$$

(D.107)

The derivative of the cosine term may be obtained using the product rule, which results in:

$$\frac{d}{d\hat{t}} \left[ e^{-\zeta_k \omega_n \hat{t}} \cdot \cos \left( \omega_n \sqrt{1 - \zeta_k^2} \cdot \hat{t} \right) \right] = f'(\hat{t}) g(\hat{t}) + f(\hat{t}) g'(\hat{t})$$

$$= \left[ -\zeta_k \omega_n e^{-\zeta_k \omega_n \hat{t}} \cdot \cos \left( \omega_n \sqrt{1 - \zeta_k^2} \cdot \hat{t} \right) \right]$$

(D.108)

$$-e^{-\zeta_k \omega_n \hat{t}} \cdot \omega_n \sqrt{1 - \zeta_k^2} \sin \left( \omega_n \sqrt{1 - \zeta_k^2} \cdot \hat{t} \right)$$
Similarly, the derivative of the sine term may be obtained using the product rule, which results in:

\[
\frac{d}{dt} \left[ e^{-\zeta_k \omega_n \hat{t}} \cdot \sin \left( \omega_n \sqrt{1 - \zeta_k^2} \cdot \hat{t} \right) \right] = f'(\hat{t})g(\hat{t}) + f(\hat{t})g'(\hat{t})
\]

\[
= \left[ -\zeta_k \omega_n e^{-\zeta_k \omega_n \hat{t}} \cdot \sin \left( \omega_n \sqrt{1 - \zeta_k^2} \cdot \hat{t} \right) \right]
\]

\[
+ e^{-\zeta_k \omega_n \hat{t}} \cdot \omega_n \sqrt{1 - \zeta_k^2} \cdot \cos \left( \omega_n \sqrt{1 - \zeta_k^2} \cdot \hat{t} \right)
\]

Substituting equations (D.108) and (D.109) into equation (D.107), and then grouping the coefficients of the sine and cosine terms, results in:

\[
\dot{x}_{Hi}^c(\hat{t}) = \sum_{j=1}^{m} \left( \Delta u_j + u_j \right) \sum_{k=1}^{p^c} e^{-\zeta_k \omega_n \hat{t}} \left[ \left( -\mu_{\bar{c},i,j,k}^c \zeta_k \omega_n + \mu_{\bar{s},i,j,k}^c \omega_n \sqrt{1 - \zeta_k^2} \cdot \cos \left( \omega_n \sqrt{1 - \zeta_k^2} \cdot \hat{t} \right) \right)
\]

\[
+ \left( -\mu_{\bar{s},i,j,k}^c \zeta_k \omega_n - \mu_{\bar{c},i,j,k}^c \omega_n \sqrt{1 - \zeta_k^2} \cdot \sin \left( \omega_n \sqrt{1 - \zeta_k^2} \cdot \hat{t} \right) \right) \right]
\]

Using the notation \( M \) to represent the coefficients of the sine and cosine terms, and noting that \( \omega_n \sqrt{1 - \zeta_k^2} = \omega_{dk} \), equation (D.110) may then be written as:

\[
\dot{x}_{Hi}^c(\hat{t}) = \sum_{j=1}^{m} \left( \Delta u_j + u_j \right) \sum_{k=1}^{p^c} e^{-\zeta_k \omega_n \hat{t}} \left[ M_{\bar{c},i,j,k} \cdot \cos(\omega_{dk} \cdot \hat{t}) + M_{\bar{s},i,j,k} \cdot \sin(\omega_{dk} \cdot \hat{t}) \right]
\]

\[
M_{\bar{c},i,j,k} = -\mu_{\bar{c},i,j,k}^c \zeta_k \omega_n + \mu_{\bar{s},i,j,k}^c \omega_{dk} \quad M_{\bar{s},i,j,k} = -\mu_{\bar{s},i,j,k}^c \zeta_k \omega_n - \mu_{\bar{c},i,j,k}^c \omega_{dk}
\]

\[
(D.111)
\]
APPENDIX D (continued)

D.5.1.2 The Unforced Time-Derivative $\dot{X}_G^C$

The unforced time-derivative response for the $i^{th}$ state of the system may be written as:

$$\dot{X}^C_{Gi}(\hat{t}) = \sum_{k=1}^{p} \left[ \nu^C_{Ci,k} \frac{d}{d\hat{t}} \left( e^{-\zeta_k \omega_{nk} \hat{t}} \cos \left( \omega_{nk} \sqrt{1 - \zeta_k^2} \cdot \hat{t} \right) \right) + \nu^C_{Si,k} \frac{d}{d\hat{t}} \left( e^{-\zeta_k \omega_{nk} \hat{t}} \sin \left( \omega_{nk} \sqrt{1 - \zeta_k^2} \cdot \hat{t} \right) \right) \right] \tag{D.112}$$

Substituting the derivatives defined in equations (D.108) and (D.109) into equation (D.112), and then grouping the coefficients of the sine and cosine terms, results in:

$$\dot{X}^C_{Gi}(\hat{t}) = \sum_{k=1}^{p} \left[ e^{-\zeta_k \omega_{nk} \hat{t}} \left( -\nu^C_{Ci,k} \zeta_k \omega_{nk} + \nu^C_{Si,k} \omega_{nk} \sqrt{1 - \zeta_k^2} \cdot \cos \left( \omega_{nk} \sqrt{1 - \zeta_k^2} \cdot \hat{t} \right) \right) + e^{-\zeta_k \omega_{nk} \hat{t}} \left( -\nu^C_{Si,k} \zeta_k \omega_{nk} - \nu^C_{Ci,k} \omega_{nk} \sqrt{1 - \zeta_k^2} \cdot \sin \left( \omega_{nk} \sqrt{1 - \zeta_k^2} \cdot \hat{t} \right) \right) \right] \tag{D.113}$$

Using the notation $N$ to represent the coefficients of the sine and cosine terms, and noting that $\omega_{nk} \sqrt{1 - \zeta_k^2} = \omega_{d_k}$, equation (D.113) may then be written as:

$$\dot{X}^C_{Gi}(\hat{t}) = \sum_{k=1}^{p} \left[ e^{-\zeta_k \omega_{nk} \hat{t}} \left( N_{C_{Ci,k}} \cdot \cos(\omega_{d_k} \cdot \hat{t}) + N_{S_{Si,k}} \cdot \sin(\omega_{d_k} \cdot \hat{t}) \right) \right] \tag{D.114}$$

$$N_{C_{Ci,k}} = -\nu^C_{Ci,k} \zeta_k \omega_{nk} + \nu^C_{Si,k} \omega_{d_k} \quad N_{S_{Si,k}} = -\nu^C_{Si,k} \zeta_k \omega_{nk} - \nu^C_{Ci,k} \omega_{d_k}$$
D.5.1.3  The Assembled Complex Time-Derivative $\dot{X}^C$

The complete forced and unforced time-derivative for the $i^{th}$ state of the system is assembled as given by equation (D.115):

$$\dot{x}^C_i(\hat{t}) = \dot{x}^C_{H_i}(\hat{t}) + \dot{x}^C_{G_i}(\hat{t})$$  \hspace{1cm} (D.115)$$

Written explicitly, equation (D.115) may be expanded to become:

$$\begin{align*}
\dot{x}^C_i(\hat{t}) &= \sum_{j=1}^{m} \left[ (\Delta u_j + u_j) \sum_{k=1}^{p^C} \left( e^{-\zeta_k \alpha n_k \hat{t}} \left[ M_{C_{i,j,k}} \cdot \cos(\omega_d k \cdot \hat{t}) + M_{S_{i,j,k}} \cdot \sin(\omega_d k \cdot \hat{t}) \right] \right) \right] \\
&\quad+ \sum_{k=1}^{p^C} \left( e^{-\zeta_k \alpha n_k \hat{t}} \left[ N_{C_{i,k}} \cdot \cos(\omega_d k \cdot \hat{t}) + N_{S_{i,k}} \cdot \sin(\omega_d k \cdot \hat{t}) \right] \right)
\end{align*}$$

$$\hspace{1cm} (D.116)$$
D.5.2 Real Modes

D.5.2.1 The Forced Time-Derivative $\dot{X}_H^R$

The forced time-derivative response for the $i^{th}$ state of the system may be written as:

$$\dot{X}_H^R(t) = \sum_{j=1}^{m} \left( \Delta u_j + u_j \right) \sum_{k=1}^{p} \left( \mu_{E_{i,j,k}}^R \frac{d}{dt} \left( e^{-\tau_{k-1} t} \right) \right)$$

(D.117)

The derivative of the exponent term is obtained according to equation (D.118):

$$\frac{d}{dt} \left[ e^{-\tau_{k-1} t} \right] = -\tau_{k-1} e^{-\tau_{k-1} t}$$

(D.118)

With this in mind, equation (D.117) becomes:

$$\dot{X}_H^R(t) = \sum_{j=1}^{m} \left( \Delta u_j + u_j \right) \sum_{k=1}^{p} \left( \mu_{E_{i,j,k}}^R \left[ -\tau_{k-1} e^{-\tau_{k-1} t} \right] \right)$$

(D.119)

Using the notation $M$ to represent the coefficients of the exponent term, equation (D.119) may then be written as:

$$\dot{X}_H^R(t) = \sum_{j=1}^{m} \left( \Delta u_j + u_j \right) \sum_{k=1}^{p} \left( M_{E_{i,j,k}} e^{-\tau_{k-1} t} \right)$$

(D.120)

$$M_{E_{i,j,k}} = -\mu_{E_{i,j,k}}^R \tau_{k}^{-1}$$
D.5.2.2 The Unforced Time-Derivative $\dot{X}_G$ 

With the derivative of the exponent term as defined in equation (D.118), the unforced time-derivative response for the $i^{th}$ state of the system may be written as:

$$\dot{X}^i_G(t) = \sum_{k=1}^{p^R} \left( \nu^i_{E_{l,k}} \frac{d}{dt} \left( e^{-\tau_k^{-1}i} \right) \right) = \sum_{k=1}^{p^R} \left( \nu^i_{E_{l,k}} \left[ -\tau_k^{-1} e^{-\tau_k^{-1}i} \right] \right)$$

(D.121)

Using the notation $N$ to represent the coefficients of the exponent term, equation (D.121) may then be written as:

$$\dot{X}^i_G(t) = \sum_{k=1}^{p^R} \left( N_{E_{l,k}} e^{-\tau_k^{-1}i} \right)$$

(D.122)

$$N_{E_{l,k}} = -\nu^i_{E_{l,k}} \tau_k^{-1}$$
The Assembled Real Time-Derivative $\dot{X}^R$

The complete forced and unforced time-derivative for the $i^{th}$ state of the system is assembled as given by equation (D.123):

$$\dot{x}^R_i(t) = \dot{x}^R_{H_i}(t) + \dot{x}^R_{G_i}(t)$$  \hspace{1cm} (D.123)

Written explicitly, equation (D.123) may be expanded to become:

$$\dot{x}^R_i(t) = \sum_{j=1}^{m} \left[ (\Delta u_j + u_j) \sum_{k=1}^{p^R} \left( M_{E_{i,j,k}} e^{-\tau_k^{-1}t} \right) \right] + \sum_{k=1}^{p^R} \left( N_{E_{i,k}} e^{-\tau_k^{-1}t} \right)$$  \hspace{1cm} (D.124)
Appendix E presents a detailed look at the development of the modal transformation method, and is intended to complement the abbreviated discussion presented in Section 6.4.

E.1 Overview

An alternative to the modal superposition method is the modal transformation method. In this case, the higher-order aircraft dynamics are “transformed” into decoupled lower-order systems, and then analyzed individually. Figure E.1 presents an overview of the modal transformation concept.

Figure E.1. Modal Superposition Concept

Consider again the coupled, generalized, higher order aircraft state space system given in equation (6.2):
When expanded, equation (E.1) takes on the form shown in equation (E.2):

\[
\begin{bmatrix}
\dot{x}_1(t) \\
\dot{x}_2(t) \\
\vdots \\
\dot{x}_n(t)
\end{bmatrix} = 
\begin{bmatrix}
a_{1,1} & a_{1,2} & \cdots & a_{1,n} \\
a_{2,1} & a_{2,2} & \cdots & a_{2,n} \\
\vdots & \vdots & \ddots & \vdots \\
a_{n,1} & a_{n,2} & \cdots & a_{n,n}
\end{bmatrix} 
\begin{bmatrix}
x_1(t) \\
x_2(t) \\
\vdots \\
x_n(t)
\end{bmatrix} + 
\begin{bmatrix}
b_{1,1} & b_{1,2} & \cdots & b_{1,m} \\
b_{2,1} & b_{2,2} & \cdots & b_{2,m} \\
\vdots & \vdots & \ddots & \vdots \\
b_{n,1} & b_{n,2} & \cdots & b_{n,m}
\end{bmatrix} 
\begin{bmatrix}
\Delta \bar{u}_1(t) + u_1(t_0) \\
\Delta \bar{u}_2(t) + u_2(t_0) \\
\vdots \\
\Delta \bar{u}_m(t) + u_m(t_0)
\end{bmatrix}
\]  

(E.2)

Using a predefined matrix transformation, the higher order aircraft state space model described in equations (E.1) and (E.2) may be transformed into a series of 2nd order block diagonals. This process facilitates the analytical analysis of each of the aircraft modes as a 2nd order system, which is mathematically and computationally efficient. Each block diagonal contains the \textit{approximate} dynamics of each of the modes originally in the higher order dynamics, namely the short period, phugoid, dutch roll, roll, and spiral. Of interest in the context of the prediction framework described here are the first four modes. The linear transformation is given by:

\[
X = T_D Z
\]  

(E.3)

Applying equation (E.3) to the state space system in the \(X\)-domain results in the following transformation:

\[
\dot{X}(t) = AX(t) + B \left( \Delta \bar{U}(t) + U(t_0) \right) \quad \Rightarrow \quad \dot{Z}(t) = A_Z Z(t) + B_Z \left( \Delta \bar{U}(t) + U(t_0) \right)
\]  

(E.4)
APPENDIX E (continued)

The transformed $A$ matrix is given by:

$$ A_Z = T_D^{-1} A T_D $$

(E.5)

The transformed $B$ matrix is given by:

$$ B_Z = T_D^{-1} B $$

(E.6)

For a system with a mixture of complex and real eigenvalues, as is the case with the aircraft here, the transformation matrix $T_D$ is then given by equation (E.7), which constitutes an amalgamation of the real and imaginary parts of the eigenvectors, $v_i$, of the $A$ matrix.

$$ T_D = [\text{Re}(v_1) \; \text{Im}(v_2) \; \text{Re}(v_3) \; \text{Im}(v_4) \; \text{Re}(v_5) \; \text{Im}(v_6) \; \text{Re}(v_7) \; \text{Re}(v_8)] $$

(E.7)

The resulting transformed state space system in the $Z$-domain takes on the form given in equations (E.17) and (E.18):

$$ A_Z = 
\begin{bmatrix}
    a_{z_{1,1}} & a_{z_{1,2}} & 0 & 0 & 0 \\
    a_{z_{2,1}} & a_{z_{2,2}} & 0 & 0 & 0 \\
    0 & a_{z_{3,3}} & a_{z_{3,4}} & 0 & 0 \\
    0 & a_{z_{4,3}} & a_{z_{4,4}} & 0 & 0 \\
    0 & 0 & a_{z_{5,5}} & a_{z_{5,6}} & 0 \\
    0 & 0 & a_{z_{6,5}} & a_{z_{6,6}} & 0 \\
    0 & 0 & 0 & a_{z_{7,7}} & 0 \\
    0 & 0 & 0 & 0 & a_{z_{8,8}}
\end{bmatrix} $$

(E.8)
APPENDIX E (continued)

\[
B_z = \begin{bmatrix}
    b_{z1,1} & \cdots & b_{z1,m} \\
    b_{z2,1} & \cdots & b_{z2,m} \\
    b_{z3,1} & \cdots & b_{z3,m} \\
    b_{z4,1} & \cdots & b_{z4,m} \\
    b_{z5,1} & \cdots & b_{z5,m} \\
    b_{z6,1} & \cdots & b_{z6,m} \\
    b_{z7,1} & \cdots & b_{z7,m} \\
    b_{z8,1} & \cdots & b_{z8,m}
\end{bmatrix}
\] (E.9)

Each of the 2nd order block diagonals (or 1st order diagonals) are then analyzed as decoupled, independent systems, using a solution methodology adapted from the modal superposition method.

**E.2 The Elemental 2nd Order Transfer Function in the Z-Domain**

As with the modal superposition method, each of the transformed block diagonals in the Z-domain may be expressed in terms of elemental 2nd order transfer functions. Consider the form of the general solution of each state in the frequency-domain:

\[
z_i(s) = \sum_{j=1}^{m} \left[ H_{i,j}(s) \left( \Delta \hat{u}_j(s) + \frac{u_j}{s} \right) \right] + G_i(s)
\] (E.10)

Because the modal transformation method decomposes the higher order aircraft dynamics into a series of 2nd order block diagonals, each element of the transfer function matrices \( H_{i,j}(s) \) and \( G_i(s) \) is comprised of a single 2nd order transfer function (as opposed to a “superposition” of transfer functions). Writing equation (E.10) in more explicit terms thus yields:
As with the modal superposition method, each permutation of the step, ramp, parabolic, or higher order inputs \( \Delta \hat{u}_j(s) \) with the elemental transfer function \( H_{i,j}(s) \) fundamentally consists of some linear combination of a transfer function of Type \( \mathbb{Q} \) and Type \( \mathbb{Q}^- \) multiplied by the coefficients \( m_2 \) and \( m_1 \), where \( \mathbb{Q} = \hat{q} \) and \( \mathbb{Q}^- = \hat{q} - 1 \). The coefficients of these basic transfer functions remain unchanged from the modal superposition method, and are given in Table D.1.

The intermediate objective, which is to express the basic transfer functions explicitly in terms of the standard Laplace transforms, also remains the same as with the modal superposition method. The expansions of the transfer functions are detailed in Appendix F, and the expanded transfer functions follow the form given in Table D.2.

E.3 The Elemental 1st Order Transfer Function in the Z-Domain

In the case of diagonals that only contain real modes, the transformed block diagonals in the Z-domain may be expressed in terms of elemental 1st order transfer functions. Consider again the form of the general solution of each state in the frequency-domain:

\[
z_i(s) = \sum_{j=1}^{m} \left[ \frac{m_{2i,j} s + m_{1i,j}}{s^2 + 2\zeta \omega_n s + \omega_n^2} \left( \Delta \hat{u}_j(s) + \frac{u_j}{s} \right) \right] + \frac{n_{2i} s + n_{1i}}{s^2 + 2\zeta \omega_n s + \omega_n^2} \Delta \hat{u}_j(s) \]  

(E.11)

\[
\Delta \hat{u}_j(s) = \Delta \hat{u}_j \frac{1}{s^{\frac{q}{q'}}} 
\]

Each element of the transfer function matrices \( H_{i,j}(s) \) and \( G_i(s) \) is comprised of a single 1st order transfer function. Writing equation (E.12) in more explicit terms thus yields:
As with the modal superposition method, each permutation of the step, ramp, parabolic, or higher order inputs $\Delta u_j(s)$ with the elemental transfer function $H_{c,j}(s)$ fundamentally consists of one of the basic transfer functions of Type $\mathbb{Q}$ multiplied by the coefficient $m_0$, where $\mathbb{Q} = \hat{q}$. The coefficients of these basic transfer functions remain unchanged from the modal superposition method, and are given in Table D.3.

The intermediate objective, which is to express the basic transfer functions explicitly in terms of the standard Laplace transforms, also remains the same as with the modal superposition method. The expansions of the transfer functions are detailed in Appendix F, and the expanded transfer functions follow the form given in Table D.4.
E.4 The Complete System Response in the \( \mathcal{Z} \)-Domain

E.4.1 Complex Modes

Consider again the general form of the frequency-domain response for the \( i \)th state of the system, given in equation (E.14):

\[
Z_i^\mathcal{C}(s) = \sum_{j=1}^{m} \left[ \frac{m_{2ij}s + m_{1ij}}{s^2 + 2\zeta\omega_n s + \omega_n^2} \left( \Delta u_j(s) + \frac{u_j}{s} \right) \right] + \frac{n_{2i}s + n_{1i}}{s^2 + 2\zeta\omega_n s + \omega_n^2} \quad (E.14)
\]

The frequency response for all states of the block diagonalized system may be assembled as a series of transfer function matrices. The entire system may be further generalized and expressed in the form defined in equation (E.15), where \( H(s) \) represents a matrix of transfer functions for the forced response, \( \bar{H}(s) \) represents a matrix of transfer functions for the initial conditions on the inputs, and \( G(s) \) represents a matrix of transfer functions for the initial conditions on the states.

\[
Z^\mathcal{C}(s) = Z_H^\mathcal{C}(s) + Z_{H}^\mathcal{C}(s) + Z_G^\mathcal{C}(s) = H^\mathcal{C}(s)\Delta \bar{U} + \bar{H}^\mathcal{C}(s)U + G^\mathcal{C}(s) \quad (E.15)
\]

In expanded form, equation (E.15) becomes equation (E.16):

\[
\begin{bmatrix}
Z_1^\mathcal{C}(s) \\
Z_2^\mathcal{C}(s)
\end{bmatrix} =
\begin{bmatrix}
H_{1,1}(s) & H_{1,2}(s) & \cdots & H_{1,m}(s) \\
H_{2,1}(s) & H_{2,2}(s) & \cdots & H_{2,m}(s)
\end{bmatrix}
\begin{bmatrix}
\Delta \bar{u}_1(s) \\
\Delta \bar{u}_2(s) \\
\vdots \\
\Delta \bar{u}_m(s)
\end{bmatrix}
+ \begin{bmatrix}
\bar{H}_{1,1}(s) & \bar{H}_{1,2}(s) & \cdots & \bar{H}_{1,m}(s) \\
\bar{H}_{2,1}(s) & \bar{H}_{2,2}(s) & \cdots & \bar{H}_{2,m}(s)
\end{bmatrix}
\begin{bmatrix}
\bar{u}_1(s) \\
\bar{u}_2(s) \\
\vdots \\
\bar{u}_m(s)
\end{bmatrix}
+ \begin{bmatrix}
G_1(s) \\
G_2(s)
\end{bmatrix} 
\quad (E.16)
\]

Taking the inverse Laplace of equation (E.15) yields the following general form for the system response in the time domain:
In expanded form, equation (E.17) becomes equation (E.18):

\[
Z^C(\hat{t}) = Z_H^C(\hat{t}) + Z_H^C(\hat{t}) + Z_G^C(\hat{t}) = H^C(\hat{t})\Delta \mathbf{U} + \mathbf{H}^C(\hat{t})\mathbf{U} + G^C(\hat{t}) \quad \text{(E.17)}
\]

Based on the derivations done for the modal superposition method, this section will adapt, for the \( Z \)-domain, the expressions that make up the elements of \( H(s)\Delta \mathbf{U}(s) \) and \( H(\hat{t})\Delta \mathbf{U}(t) \) for step, ramp, parabolic, and inputs of order \( \hat{q} \), the expressions for \( \tilde{H}(s)U(s) \) and \( \tilde{H}(\hat{t})U(t) \), and the expressions for \( G(s) \) and \( G(\hat{t}) \).
APPENDIX E (continued)

E.4.1.1 The Forced Response $Z_{H}^C$

The forced response for the $i^{th}$ state of the system is given by equation (E.19):

$$
Z_{H_i}^C(s) = \sum_{j=1}^{m} \left[ \left( \frac{m_{2_{ij}}}{s^2 + 2\zeta \omega_n s + \omega_n^2} + \frac{m_{1_{ij}}}{s^2 + 2\zeta \omega_n s + \omega_n^2} \right) \Delta \hat{u}_j(s) \right] \tag{E.19}
$$

The input $\Delta \hat{u}_j(s)$ is given by $\Delta \hat{u}_j \frac{1}{s^{\hat{q}}}$, where $\hat{q}$ represents the order of the input and $\Delta \hat{u}_j$ represents the magnitude of the corresponding input order for the $j^{th}$ input. For an input of order $\hat{q}$, equation (E.19) may be written as follows:

$$
Z_{H_i}^C(s) = \sum_{j=1}^{m} \left[ \Delta \hat{u}_j \left( \frac{s}{m_{2_{ij}} \frac{1}{s^2 + 2\zeta \omega_n s + \omega_n^2}} + \frac{1}{m_{1_{ij}} \frac{1}{s^2 + 2\zeta \omega_n s + \omega_n^2}} \right) \frac{1}{s^{\hat{q}}} \right] \tag{E.20}
$$

Explicitly multiplying the $\hat{q}^{th}$ order input through allows equation (E.20) to be expressed in terms of transfer functions of Type $\mathbb{Q}^-$ and Type $\mathbb{Q}$:

$$
Z_{H_i}^C(s) = \sum_{j=1}^{m} \left[ \Delta \hat{u}_j \left( \frac{1}{m_{2_{ij}} \frac{s}{s^2 + 2\zeta \omega_n s + \omega_n^2}} + \frac{1}{m_{1_{ij}} \frac{s}{s^2 + 2\zeta \omega_n s + \omega_n^2}} \right) \frac{1}{s^{\hat{q}}} \right] \tag{E.21}
$$

Finally, using the notation $\mu$ to represent the coefficients $m$ and $\kappa$ results in the following expression for the forced frequency-domain response for the $i^{th}$ state of the system:
Taking the inverse Laplace transform of equation (E.22) leads to the following expression for the forced time-domain response for the $i^{th}$ state of the system:

\[
\begin{align*}
\tilde{z}_H^c(i) &= \sum_{j=1}^{m} \left[ \Delta \hat{u}_j \left( \mu^C_{\hat{q}_l,i,j} \left( \frac{1}{s^{\hat{q}_l}} \right) + \sum_{l=1}^{\hat{q}_l-1} \left( \mu^C_{l,i,j} \left( \frac{1}{s^l} \right) \right) + \mu^C_{c,i,j} \left( \frac{s + \zeta \omega_n}{(s + \zeta \omega_n)^2 + \omega_n^2(1 - \zeta^2)} \right) 
\right. \\
&\quad \left. + \mu^C_{s,i,j} \left( \frac{\omega_n \sqrt{1 - \zeta^2}}{(s + \zeta \omega_n)^2 + \omega_n^2(1 - \zeta^2)} \right) \right] \\
\end{align*}
\]  

(E.22)

for the forced time-domain response for the $i^{th}$ state of the system:

\[
\begin{align*}
\tilde{z}_H^c(i) &= \sum_{j=1}^{m} \left[ \Delta \hat{u}_j \left( \mu^C_{\hat{q}_l,i,j} \left( \frac{\hat{t}^{\hat{q}_l-1}}{\hat{q}_l-1)!} \right) + \sum_{l=1}^{\hat{q}_l-1} \left( \mu^C_{l,i,j} \left( \frac{\hat{t}^{l-1}}{(l-1)!} \right) \right) \\
&\quad + \mu^C_{c,i,j} \left( e^{-\zeta \omega_n \hat{t}} \cos \left( \omega_n \sqrt{1 - \zeta^2} \cdot \hat{t} \right) \right) \\
&\quad + \mu^C_{s,i,j} \left( e^{-\zeta \omega_n \hat{t}} \sin \left( \omega_n \sqrt{1 - \zeta^2} \cdot \hat{t} \right) \right) \right] \\
\end{align*}
\]  

(E.23)

\[
\begin{align*}
\mu^C_{\hat{q}_l,i,j} &= m_{1,i,j} \kappa_{\hat{q},Q}^C \\
\mu^C_{l,i,j} &= m_{2,i,j} \kappa_{l,Q}^C + m_{1,i,j} \kappa_{l,Q}^C \\
\mu^C_{c,i,j} &= m_{2,i,j} \kappa_{c,Q}^C + m_{1,i,j} \kappa_{c,Q}^C \\
\mu^C_{s,i,j} &= m_{2,i,j} \kappa_{S,Q}^C + m_{1,i,j} \kappa_{S,Q}^C \\
\end{align*}
\]
E.4.1.2 The Forced Response due to Initial Conditions on the Inputs $Z^c_H$

The forced response due to initial conditions on the inputs for the $i^{th}$ state of the system is given by equation (E.24):

$$z^c_{H_i}(s) = \sum_{j=1}^{m} \left[ \left( \frac{m_{2l,i} s + m_{1l,i}}{s^2 + 2\zeta \omega_n s + \omega_n^2} \right) \left( u_j \cdot \frac{1}{s} \right) \right] \text{ (E.24)}$$

Using the notation $\bar{\mu}$ to represent the coefficients $m$ and $\kappa$ results in the following expression for the forced frequency-domain response for the $i^{th}$ state of the system:

$$z^c_{H_i}(s) = \sum_{j=1}^{m} \left[ u_j \left( \bar{\mu}_{1l,j}^c \left( \frac{1}{s} \right) + \bar{\mu}_{Cl,j}^c \left( \frac{s + \zeta \omega_n}{(s + \zeta \omega_n)^2 + \omega_n^2(1 - \zeta^2)} \right) \right)
+ \bar{\mu}_{Sl,j}^c \left( \frac{\omega_n \sqrt{1 - \zeta^2}}{(s + \zeta \omega_n)^2 + \omega_n^2(1 - \zeta^2)} \right) \right] \text{ (E.25)}$$

Taking the inverse Laplace transform of equation (E.25) leads to the following expression for the forced time-domain response for the $i^{th}$ state of the system:

$$z^c_{H_i}(\hat{t}) = \sum_{j=1}^{m} \left[ u_j \left( \bar{\mu}_{1l,j}^c + \bar{\mu}_{Cl,j}^c \left( e^{-\zeta \omega_n \hat{t}} \cos(\omega_n \sqrt{1 - \zeta^2} \cdot \hat{t}) \right) \right)
+ \bar{\mu}_{Sl,j}^c \left( e^{-\zeta \omega_n \hat{t}} \sin(\omega_n \sqrt{1 - \zeta^2} \cdot \hat{t}) \right) \right] \text{ (E.26)}$$

$$\bar{\mu}_{1l,j}^c = m_{1l,j} \kappa_{1l,j}^c \quad \bar{\mu}_{Cl,j}^c = m_{1l,j} \kappa_{Cl,j}^c \quad \bar{\mu}_{Sl,j}^c = m_{2l,j} \kappa_{Sl,j}^c + m_{1l,j} \kappa_{Sl,j}^c$$
APPENDIX E (continued)

E.4.1.3 The Unforced Response $Z_{G_i}^C$

The unforced response for the $i^{th}$ state of the system is given by equation (E.27):

$$z_{G_i}^C(s) = \frac{n_{2i}n + n_{1i}}{s^2 + 2\zeta \omega_n s + \omega_n^2}$$  \hspace{1cm} (E.27)

Expanding equation (E.27) and expressing it in terms of the Type O1 and Type O2 transfer functions:

$$Z_{G_i}^C(s) = \frac{s}{n_{2i} \frac{n}{2} + 2\zeta \omega_n + \omega_n^2} + \frac{1}{n_{1i} \frac{n}{2} + 2\zeta \omega_n + \omega_n^2}$$  \hspace{1cm} (E.28)

Then, using the notation $\nu$ to represent the coefficients $n$ and $\kappa$ results in the following expression for the unforced frequency-domain response for the $i^{th}$ state of the system:

$$z_{G_i}^C(s) = \nu_{C_i}^C \left( \frac{s + \zeta \omega_n}{(s + \zeta \omega_n)^2 + \omega_n^2(1 - \zeta^2)} \right) + \nu_{S_i}^C \left( \frac{\omega_n \sqrt{1 - \zeta^2}}{(s + \zeta \omega_n)^2 + \omega_n^2(1 - \zeta^2)} \right)$$  \hspace{1cm} (E.29)

Taking the inverse Laplace transform of equation (E.29) leads to the following expression for the unforced time-domain response for the $i^{th}$ state of the system:

$$z_{G_i}^C(t) = \sum_{k=1}^{p} \nu_{C_i}^C \left( e^{-\zeta \omega_n t} \cos \left( \omega_n \sqrt{1 - \zeta^2} \cdot t \right) \right) + \nu_{S_i}^C \left( e^{-\zeta \omega_n t} \sin \left( \omega_n \sqrt{1 - \zeta^2} \cdot t \right) \right)$$  \hspace{1cm} (E.30)

$$\nu_{C_i}^C = n_{2i} \kappa_{C_i}^{C,01} \quad \nu_{S_i}^C = n_{2i} \kappa_{S_i}^{C,01} + n_{1i} \kappa_{S_i}^{C,02}$$
APPENDIX E (continued)

E.4.1.4 The Assembled Complex System Response $Z^C$

The complete forced and unforced response for the $i^{th}$ state of the system (for the complex modes) is assembled according to equation (E.31):

$$z^C_i(\hat{t}) = z^C_{Hi}(\hat{t}) + z^C_{Li}(\hat{t}) + z^C_{Gi}(\hat{t})$$  \hspace{1cm} (E.31)

Written explicitly, equation (E.31) may be expanded to become:

$$z^C_i(\hat{t}) = \sum_{j=1}^{m} \left[ \Delta \hat{u}_j \left( \sum_{l=1}^{q} \left( \mu_{Hi,j} \left( \frac{\hat{t}^{l-1}}{(l-1)!} \right) \right) + \mu_{Li,j} \left( e^{-\zeta \omega_n \hat{t}} \cos(\omega_d \hat{t}) \right) + \mu_{Gi,j} \left( e^{-\zeta \omega_n \hat{t}} \sin(\omega_d \hat{t}) \right) \right) \right]$$

$$+ u_j \left( \bar{\mu}_{1i,j} \left( e^{-\zeta \omega_n \hat{t}} \cos(\omega_d \hat{t}) \right) + \bar{\mu}_{Ci,j} \left( e^{-\zeta \omega_n \hat{t}} \sin(\omega_d \hat{t}) \right) \right)$$

$$+ v_{Li} \left( e^{-\zeta \omega_n \hat{t}} \cos(\omega_d \hat{t}) \right) + v_{Gi} \left( e^{-\zeta \omega_n \hat{t}} \sin(\omega_d \hat{t}) \right)$$

(E.32)
E.4.2 Real Modes

Consider again the general form of the frequency-domain response for the \(i^{th}\) state of the system, given in equation (E.33):

\[
Z_i^R(s) = \sum_{j=1}^{m} \left[ \frac{m_{0i,j}}{s + \tau^{-1}} \left( \Delta \hat{u}_j(s) + \frac{u_j}{s} \right) \right] + \frac{n_{0i}}{s + \tau^{-1}} \tag{E.33}
\]

The frequency response for all states of the block diagonalized system may be assembled as a series of transfer function matrices. The entire system may be further generalized and expressed in the form defined in equation (E.34), where \(H(s)\) represents a matrix of transfer functions for the forced response, \(\bar{H}(s)\) represents a matrix of transfer functions for the initial conditions on the inputs, and \(G(s)\) represents a matrix of transfer functions for the initial conditions on the states.

\[
Z^R(s) = X^R_H(s) + X^R_H(s) + X^R_G(s) = H^R(s)\Delta \bar{U} + \bar{H}^R(s)U + G^R(s) \tag{E.34}
\]

In expanded form, equation (E.34) becomes equation (E.35):

\[
z^R(s) = [H_{1,1}(s) \ H_{1,2}(s) \ \cdots \ H_{1,m}(s)] \begin{bmatrix} \Delta \hat{u}_1(s) \\ \Delta \hat{u}_2(s) \\ \vdots \\ \Delta \hat{u}_m(s) \end{bmatrix} + [\bar{H}_{1,1}(s) \ \bar{H}_{1,2}(s) \ \cdots \ \bar{H}_{1,m}(s)] \begin{bmatrix} u_1(s) \\ u_2(s) \\ \vdots \\ u_m(s) \end{bmatrix} + G_i(s) \tag{E.35}
\]

Taking the inverse Laplace of equation (E.34) yields the following general form for the system response in the time domain:

\[
Z^R(t) = Z^R_H(t) + Z^R_H(t) + Z^R_G(t) = H^R(t)\Delta \bar{U} + \bar{H}^R(t)U + G^R(t) \tag{E.36}
\]
In expanded form, equation (E.36) becomes equation (E.37):

\[
z_1^E(\hat{t}) = [H_{1,1}(\hat{t}) \ H_{1,2}(\hat{t}) \ \cdots \ H_{1,m}(\hat{t})] \begin{bmatrix} \Delta \hat{u}_1 \\ \Delta \hat{u}_2 \\ \vdots \\ \Delta \hat{u}_m \end{bmatrix} + [\bar{H}_{1,1}(\hat{t}) \ \bar{H}_{1,2}(\hat{t}) \ \cdots \ \bar{H}_{1,m}(\hat{t})] \begin{bmatrix} \bar{u}_1 \\ \bar{u}_2 \\ \vdots \\ \bar{u}_m \end{bmatrix} + G_1(\hat{t}) \quad (E.37)
\]

Based on the derivations done for the modal superposition method, this section will adapt, for the \(Z\)-domain, the expressions that make up the elements of \(H(s)\Delta \bar{U}(s)\) and \(H(\hat{t})\Delta \bar{U}(t)\) for step, ramp, parabolic, and inputs of order \(\hat{q}\), the expressions for \(\bar{H}(s)U(s)\) and \(\bar{H}(\hat{t})U(t)\), and the expressions for \(G(s)\) and \(G(\hat{t})\).
APPENDIX E (continued)

E.4.2.1 The Forced Response $Z_{Ht}^{\mathbb{R}}$

The forced response for the $i^{th}$ state of the system is given by equation (E.38):

$$
Z_{Ht}^{\mathbb{R}}(s) = \sum_{j=1}^{m} \left[ \left( -\frac{m_{0_{ij}}}{s + \tau^{-1}} \right) \Delta \hat{u}_j(s) \right] \tag{E.38}
$$

The input $\Delta \hat{u}_j(s)$ is given by $\Delta \hat{u}_j \frac{1}{s^{\hat{q}}}$, where $\hat{q}$ represents the order of the input and $\Delta \hat{u}_j$ represents the magnitude of the corresponding input order for the $j^{th}$ input. For an input of order $\hat{q}$, equation (E.38) may be written as follows:

$$
Z_{Ht}^{\mathbb{R}}(s) = \sum_{j=1}^{m} \left[ \Delta \hat{u}_j \left( \frac{m_{0_{ij}}}{s + \tau^{-1}} \right) \frac{1}{s^{\hat{q}}} \right] \tag{E.39}
$$

Explicitly multiplying the $\hat{q}^{th}$ order input through allows equation (E.39) to be expressed in terms of a transfer function of Type $\mathbb{Q}$:

$$
Z_{Ht}^{\mathbb{R}}(s) = \sum_{j=1}^{m} \left[ \Delta \hat{u}_j \left( m_{0_{ij}} \frac{1}{s^{\hat{q}}(s + \tau^{-1})} \right) \right] \tag{E.40}
$$

Then, using the notation $\mu$ to represent the coefficients $m$ and $\kappa$ results in the following expression for the forced frequency-domain response for the $i^{th}$ state of the system:

$$
Z_{Ht}^{\mathbb{R}}(s) = \sum_{j=1}^{m} \left[ \Delta \hat{u}_j \left( \sum_{l=1}^{q} \left( \mu l_{ij} \frac{1}{s^{l}} \right) + \mu E_{i,j} \frac{1}{s + \tau \kappa^{-1}} \right) \right] \tag{E.41}
$$
Taking the inverse Laplace transform of equation (E.41) leads to the following expression for the forced time-domain response for the $i^{th}$ state of the system:

\[
\begin{align*}
    z_{i_l}^R(t) &= \sum_{j=1}^{m} \left[ \Delta \hat{u}_j \left( \sum_{l=1}^{q} \left( \mu_{i_{l,j}}^R \left( \frac{\hat{t}^{l-1}}{(l-1)!} \right) \right) + \mu_{E_{i_{l,j}}}^R \left( e^{-t \tau_{k_{l-1}} \hat{t}} \right) \right) \right] \\
    \mu_{i_{l,j}}^R &= m_{0_{l,j}} \kappa_{i_l}^{R, Q} \\
    \mu_{E_{i_{l,j}}}^R &= m_{0_{l,j}} \kappa_{E}^{R, Q}
\end{align*}
\] (E.42)
APPENDIX E (continued)

E.4.2.2 The Forced Response due to Initial Conditions on the Inputs $Z^{\mathbb{R}}_{H_i}$

The forced response due to initial conditions on the inputs for the $i^{th}$ state of the system is given by equation (E.43):

$$z_{H_i}^{\mathbb{R}}(s) = \sum_{j=1}^{m} \left[ \left( \frac{m_{0_i}}{s + \tau^{-1}} \right) \left( \frac{u_j}{s} \right) \right]$$

(E.43)

Using the notation $\bar{\mu}$ to represent the coefficients $m$ and $\kappa$ results in the following expression for the forced frequency-domain response for the $i^{th}$ state of the system:

$$z_{H_i}^{\mathbb{R}}(s) = \sum_{j=1}^{m} \left[ u_j \left( \bar{\mu}_{1_{ij}}^{\mathbb{R}} \left( \frac{1}{s} \right) + \bar{\mu}_{E_{ij}}^{\mathbb{R}} \left( \frac{1}{s + \tau^{-1}} \right) \right) \right]$$

(E.44)

Taking the inverse Laplace transform of equation (E.44) leads to the following expression for the forced time-domain response for the $i^{th}$ state of the system:

$$z_{H_i}^{\mathbb{R}}(t) = \sum_{j=1}^{m} \left[ u_j \left( \bar{\mu}_{1_{ij}}^{\mathbb{R}} + \bar{\mu}_{E_{ij}}^{\mathbb{R}} (e^{-\tau^{-1}t}) \right) \right]$$

(E.45)

$\bar{\mu}_{1_{ij}}^{\mathbb{R}} = m_{0_i} \kappa_{1}^{\mathbb{R},l}$

$\bar{\mu}_{E_{ij}}^{\mathbb{R}} = m_{0_i} \kappa_{E}^{\mathbb{R},l}$
E.4.2.3 The Unforced Response $Z_{G_i}^\mathbb{R}$

The unforced response for the $i^{th}$ state of the system is given by equation (E.46):

$$x_{G_i}^\mathbb{R}(s) = \frac{n_{0i}}{s + \tau^{-1}}$$  \hspace{1cm} (E.46)

Expressing equation (E.46) in terms of the Type O transfer function leads to:

$$x_{G_i}^\mathbb{R}(s) = n_{0i} \frac{1}{s + \tau^{-1}}$$ \hspace{1cm} (E.47)

Then, using the notation $\nu$ to represent the coefficients $n$ and $\kappa$ results in the following expression for the unforced frequency-domain response for the $i^{th}$ state of the system:

$$Z_{G_i}^\mathbb{R}(s) = \nu_{E_i}^\mathbb{R} \left( \frac{1}{s + \tau^{-1}} \right)$$  \hspace{1cm} (E.48)

Taking the inverse Laplace transform of equation (E.48) leads to the following expression for the unforced time-domain response for the $i^{th}$ state of the system:

$$z_{G_i}^\mathbb{R}(t) = \nu_{E_i}^\mathbb{R} (e^{-\tau^{-1}t})$$  \hspace{1cm} (E.49)

\[ \nu_{E_i}^\mathbb{R} = n_{0i} \kappa_{E_i}^\mathbb{O} \]
E.4.2.4 The Assembled Real System Response $Z^\mathbb{R}$

The complete forced and unforced response for the $i^{th}$ state of the system (for the real modes) is now assembled according to equation (E.50):

$$z^\mathbb{R}_i(\hat{t}) = z^\mathbb{R}_{H_i}(\hat{t}) + z^\mathbb{R}_{\overline{H}_i}(\hat{t}) + z^\mathbb{R}_{A_i}(\hat{t})$$  \hspace{1cm} (E.50)

Written explicitly, equation (E.50) may be expanded to become:

\[
z_i^\mathbb{R}(\hat{t}) = \sum_{j=1}^{m} \Delta \tilde{u}_j \left[ \sum_{l=1}^{\bar{q}} \left( \mu^\mathbb{R}_{l,j} \left( \frac{\hat{t}^{l-1}}{(l-1)!} \right) + \mu_{E_{l,j}}^\mathbb{R} e^{-\tau^{-1}\hat{t}} \right) \right] + \sum_{j=1}^{\bar{q}} u_j \left( \mu_{1,j}^\mathbb{R} + \mu_{E_{1,j}}^\mathbb{R} e^{-\tau^{-1}\hat{t}} \right) + \nu_{E_i}^\mathbb{R} e^{-\tau^{-1}\hat{t}} \tag{E.51}\]
E.5 The Complete System Time-Derivative in the Z-Domain

As with the modal superposition method, for the special case of the step input, which typically causes an underdamped and oscillatory response, an additional step is required to determine the input magnitude that would cause the peak amplitude to exactly reach (and not overshoot) the state limit. This involves determining the times at which the maxima and minima occur.

E.5.1 Complex Modes

Recalling equation (E.17), the general form of the time-derivative of the system response may be written as follows:

$$\dot{Z}^C(\hat{t}) = \dot{H}^C(\hat{t})\Delta U + \dot{H}^C(\hat{t})U + \dot{G}^C(\hat{t}) = \dot{Z}_H^C(\hat{t}) + \dot{Z}_G^C(\hat{t})$$  \hspace{1cm} (E.52)

Because the critical input $\Delta U$ and the initial condition on the input $U$ are both step inputs, equation (E.52) may be condensed to:

$$\dot{Z}^C(\hat{t}) = \dot{H}^C(\Delta U + U) + \dot{G}^C(\hat{t}) = \dot{Z}_H^C(\hat{t}) + \dot{Z}_G^C(\hat{t})$$  \hspace{1cm} (E.53)

In expanded form, equation (E.53) becomes:

$$\begin{bmatrix} \dot{z}_1^C(\hat{t}) \\ \dot{z}_2^C(\hat{t}) \end{bmatrix} = \begin{bmatrix} \dot{H}_{1,1}(\hat{t}) & \dot{H}_{1,2}(\hat{t}) & \cdots & \dot{H}_{1,m}(\hat{t}) \\ \dot{H}_{2,1}(\hat{t}) & \dot{H}_{2,2}(\hat{t}) & \cdots & \dot{H}_{2,m}(\hat{t}) \end{bmatrix} \begin{bmatrix} \Delta \hat{u}_1 + u_1 \\ \Delta \hat{u}_2 + u_2 \end{bmatrix} + \begin{bmatrix} \dot{\hat{u}}_1(\hat{t}) \\ \dot{\hat{u}}_2(\hat{t}) \end{bmatrix}$$  \hspace{1cm} (E.54)

This section will adapt the expressions for $\dot{H}(\hat{t})(\Delta U + U)$ and $\dot{G}(\hat{t})$ that were previously derived for the modal superposition method to the modal transformation method.
E.5.1.1 The Forced Time-Derivative $\dot{Z}_H^C$

The forced time-derivative response for the $i^{th}$ state of the system may be written as:

$$
\dot{z}_H^C (\hat{t}) = \sum_{j=1}^{m} \left[ (\Delta u_j + u_j) \left( \mu_{c_{ij}} \frac{d}{d\hat{t}} \left( e^{-\zeta \omega_n \hat{t}} \cos \left( \omega_n \sqrt{1 - \zeta^2} \cdot \hat{t} \right) \right) \right. \\
\left. + \mu_{s_{ij}} \frac{d}{d\hat{t}} \left( e^{-\zeta \omega_n \hat{t}} \sin \left( \omega_n \sqrt{1 - \zeta^2} \cdot \hat{t} \right) \right) \right]\right] 
$$

(E.55)

With the substitutions defined in equations (D.108) and (D.109), one eventually obtains:

$$
\dot{z}_H^C (\hat{t}) = \sum_{j=1}^{m} \left[ (\Delta u_j + u_j) \left( e^{-\zeta \omega_n \hat{t}} \left[ -\mu_{c_{ij}} \zeta \omega_n + \mu_{s_{ij}} \omega_n \sqrt{1 - \zeta^2} \cdot \cos \left( \omega_n \sqrt{1 - \zeta^2} \cdot \hat{t} \right) \right] \right. \\
\left. + \left( -\mu_{s_{ij}} \zeta \omega_n - \mu_{c_{ij}} \omega_n \sqrt{1 - \zeta^2} \cdot \sin \left( \omega_n \sqrt{1 - \zeta^2} \cdot \hat{t} \right) \right) \right]\right] 
$$

(E.56)

Using the notation $M$ to represent the coefficients of the sine and cosine terms, and noting that $\omega_n \sqrt{1 - \zeta^2} = \omega_d$, equation (E.56) may then be written as:

$$
\dot{z}_H^C (\hat{t}) = \sum_{j=1}^{m} \left[ (\Delta u_j + u_j) \left( e^{-\zeta \omega_n \hat{t}} \left[ M_{c_{ij}} \cdot \cos (\omega_d \cdot \hat{t}) + M_{s_{ij}} \cdot \sin (\omega_d \cdot \hat{t}) \right] \right) \right] 
$$

(E.57)

$$
M_{c_{ij}} = -\mu_{c_{ij}} \zeta \omega_n + \mu_{s_{ij}} \omega_d \quad M_{s_{ij}} = -\mu_{s_{ij}} \zeta \omega_n - \mu_{c_{ij}} \omega_d
$$
E.5.1.2 The Unforced Time-Derivative $\dot{Z}_G^c$

The unforced time-derivative response for the $i^{th}$ state of the system may be written as:

$$\dot{z}_G^c(t) = \psi_{ci}^c \frac{d}{dt} \left( e^{-\zeta \omega_n t} \cos \left( \omega_n \sqrt{1 - \zeta^2} \cdot t \right) \right) + \psi_{si}^c \frac{d}{dt} \left( e^{-\zeta \omega_n t} \sin \left( \omega_n \sqrt{1 - \zeta^2} \cdot t \right) \right) \quad (E.58)$$

With the substitutions defined in equations (D.108) and (D.109), one eventually obtains:

$$\dot{z}_G^c(t) = e^{-\zeta \omega_n t} \left[ \left( -\psi_{ci}^c \zeta \omega_n + \psi_{si}^c \omega_n \sqrt{1 - \zeta^2} \right) \cdot \cos \left( \omega_n \sqrt{1 - \zeta^2} \cdot t \right) 
+ \left( -\psi_{si}^c \omega_n - \psi_{ci}^c \omega_n \sqrt{1 - \zeta^2} \right) \cdot \sin \left( \omega_n \sqrt{1 - \zeta^2} \cdot t \right) \right] \quad (E.59)$$

Using the notation $N$ to represent the coefficients of the sine and cosine terms, and noting that $\omega_n \sqrt{1 - \zeta^2} = \omega_d$, equation (E.59) may then be written as:

$$\dot{z}_G^c(t) = e^{-\zeta \omega_n t} \left[ N_{ci} \cdot \cos(\omega_d \cdot t) + N_{si} \cdot \sin(\omega_d \cdot t) \right] \quad (E.60)$$

$$N_{ci} = -\psi_{ci}^c \zeta \omega_n + \psi_{si}^c \omega_d \quad N_{si} = -\psi_{si}^c \zeta \omega_n - \psi_{ci}^c \omega_d$$
E.5.1.3 The Assembled Complex Time-Derivative $\dot{Z}^C$

The complete forced and unforced time-derivative for the $i^{th}$ state of the system is assembled as given by equation:

$$\dot{Z}_i^C(\hat{t}) = \dot{Z}_{Hi}^C(\hat{t}) + \dot{Z}_{Oi}^C(\hat{t})$$ \hspace{1cm} (E.61)

Written explicitly, equation (E.61) may be expanded to become:

$$\dot{Z}_i^C(\hat{t}) = \sum_{j=1}^{m} \left[ (\Delta u_j + u_j) \left( e^{-\zeta \omega_n \hat{t}} \left[ M_{C_{i,j}} \cos(\omega d \cdot \hat{t}) + M_{S_{i,j}} \sin(\omega d \cdot \hat{t}) \right] \right) \right] + e^{-\zeta \omega_n \hat{t}} \left[ N_{C_{i}} \cos(\omega d \cdot \hat{t}) + N_{S_{i}} \sin(\omega d \cdot \hat{t}) \right]$$ \hspace{1cm} (E.62)
E.5.2 Real Modes

Recalling equation (E.17), the general form of the time-derivative of the system response may be written as follows:

$$\dot{Z}^\mathbb{R}(\hat{t}) = \dot{H}^\mathbb{R}(\hat{t})\Delta U + \dot{H}^\mathbb{R}(\hat{t})U + \dot{G}^\mathbb{R}(\hat{t}) = \dot{Z}_H^\mathbb{R}(\hat{t}) + \dot{Z}_H^\mathbb{R}(\hat{t}) + \dot{Z}_G^\mathbb{R}(\hat{t}) \quad (E.63)$$

Because the critical input $\Delta U$ and the initial condition on the input $U$ are both step inputs, equation (E.63) may be condensed to:

$$\dot{Z}^\mathbb{R}(\hat{t}) = \dot{H}^\mathbb{R}(\hat{t})(\Delta U + U) + \dot{G}^\mathbb{R}(\hat{t}) = \dot{Z}_H^\mathbb{R}(\hat{t}) + \dot{Z}_G^\mathbb{R}(\hat{t}) \quad (E.64)$$

In expanded form, equation (E.63) becomes:

$$\dot{z}_1^\mathbb{R}(\hat{t}) = \begin{bmatrix} \dot{H}_{1,1}(\hat{t}) & \dot{H}_{1,2}(\hat{t}) & \cdots & \dot{H}_{1,m}(\hat{t}) \end{bmatrix} \begin{bmatrix} \Delta \hat{u}_1 + u_1 \\ \Delta \hat{u}_2 + u_2 \end{bmatrix} + \dot{G}_1(\hat{t}) \quad (E.65)$$

This section will adapt the expressions for $\dot{H}(\hat{t})(\Delta U + U)$ and $\dot{G}(\hat{t})$ that were previously derived for the modal superposition method to the modal transformation method.
E.5.2.1 The Forced Time-Derivative $\dot{Z}_H^R$

The forced time-derivative response for the $i^{th}$ state of the system may be written as:

$$\dot{Z}_H^R(t) = \sum_{j=1}^{m} \left[ (\Delta u_j + u_j) \left( \mu_{E_{ij}}^R \frac{d}{dt} \left( e^{-\tau^{-1}t} \right) \right) \right]$$

(E.66)

Substituting the derivative of the exponent term defined in equation (D.118), one eventually obtains:

$$\dot{Z}_H^R(t) = \sum_{j=1}^{m} \left[ (\Delta u_j + u_j) \left( \mu_{E_{ij}}^R \left[ -\tau^{-1} e^{-\tau^{-1}t} \right] \right) \right]$$

(E.67)

Using the notation $M$ to represent the coefficients of the exponent term, equation (E.67) may then be written as:

$$\dot{Z}_H^R(t) = \sum_{j=1}^{m} \left[ (\Delta u_j + u_j) \left( M_{E_{ij}}^R e^{-\tau^{-1}t} \right) \right]$$

(E.68)

$$M_{E_{ij}} = -\mu_{E_{ij}}^R \tau^{-1}$$
E.5.2.2 The Unforced Time-Derivative $\dot{Z}_G^i$

The unforced time-derivative response for the $i^{th}$ state of the system may be written as:

$$\dot{Z}_G^i(t) = \nu_{E_i} \frac{d}{dt} \left(e^{-\tau^{-1}t}\right)$$  \hspace{1cm} (E.69)

Then, equation (E.69) becomes:

$$\dot{Z}_G^i(t) = \nu_{E_i} \left[-\tau^{-1}e^{-\tau^{-1}t}\right]$$  \hspace{1cm} (E.70)

Using the notation $N$ to represent the coefficients of the exponent term, equation (E.70) may then be written as:

\[
\begin{align*}
\dot{Z}_G^i(t) &= N_{E_i} e^{-\tau^{-1}t} \\
N_{E_i} &= -\nu_{E_i} \tau^{-1}
\end{align*}
\]

(E.71)
E.5.2.3 **The Assembled Real Time-Derivative $\dot{Z}^R$**

The complete forced and unforced time-derivative for the $i^{th}$ state of the system is assembled as given by equation (E.72):

$$\dot{Z}_i^R(\hat{t}) = \dot{Z}_{H_i}(\hat{t}) + \dot{Z}_{G_i}(\hat{t})$$  \hspace{1cm} (E.72)

Written explicitly, equation (E.72) may be expanded to become:

$$\dot{Z}_i^R(\hat{t}) = \sum_{j=1}^{m} \left[ (\Delta u_j + u_j) \left( M_{E_{ij}} e^{-r^{-1}\hat{t}} \right) \right] + N_{E_i} e^{-r^{-1}\hat{t}}$$  \hspace{1cm} (E.73)
APPENDIX F
PARTIAL FRACTION EXPANSIONS

Appendix F presents the partial fraction expansions used in developing the modal superposition and modal transformation methods, and is intended to complement the material presented in Appendices D and E.

F.1 Complex Modes
F.1.1 Type O₁

The transfer function may be expanded by adding and subtracting “1” in the form of $\zeta \omega_n$:

$$\frac{s}{s^2 + 2\zeta \omega_n s + \omega_n^2} = \frac{s + \zeta \omega_n}{s^2 + 2\zeta \omega_n s + \omega_n^2} - \frac{\zeta \omega_n}{s^2 + 2\zeta \omega_n s + \omega_n^2} \tag{F.1}$$

Expressing equation (F.1) using standard Laplace forms gives:

$$\frac{s + \zeta \omega_n}{s^2 + 2\zeta \omega_n s + \omega_n^2} - \left(\frac{\zeta}{\sqrt{1 - \zeta^2}}\right) \frac{\omega_n\sqrt{1 - \zeta^2}}{s^2 + 2\zeta \omega_n s + \omega_n^2} \tag{F.2}$$

$$\frac{s + \zeta \omega_n}{(s + \zeta \omega_n)^2 + \omega_n^2(1 - \zeta^2)} + \left(-\frac{\zeta}{\sqrt{1 - \zeta^2}}\right) \frac{\omega_n\sqrt{1 - \zeta^2}}{(s + \zeta \omega_n)^2 + \omega_n^2(1 - \zeta^2)} \tag{F.3}$$
F.1.2 Type O₂

The transfer function may be expanded by multiplying “1” in the form of

\[ \omega_n \sqrt{1 - \zeta^2} / \omega_n \sqrt{1 - \zeta^2} : \]

\[
\frac{1}{s^2 + 2\zeta \omega_n s + \omega_n^2} \quad (F.4)
\]

Expressing equation (F.4) using standard Laplace forms gives:

\[
\left( \frac{1}{\omega_n \sqrt{1 - \zeta^2}} \right) \frac{\omega_n \sqrt{1 - \zeta^2}}{(s + \zeta \omega_n)^2 + \omega_n^2(1 - \zeta^2)} \quad (F.5)
\]
F.1.3 Type I

Through partial fraction expansion, the transfer function may be expressed as follows:

\[
\frac{1}{s(s^2 + 2\zeta\omega_ns + \omega_n^2)} = \frac{\Pi_2}{s} + \frac{\Pi_1s + \Pi_0}{s^2 + 2\zeta\omega_ns + \omega_n^2}
\]  

Equation (F.6)

Multiplying equation (F.6) by its denominator gives:

\[
\frac{s(s^2 + 2\zeta\omega_ns + \omega_n^2)}{s(s^2 + 2\zeta\omega_ns + \omega_n^2)} = \frac{\Pi_2s(s^2 + 2\zeta\omega_ns + \omega_n^2)}{s} + \frac{\Pi_1ss(s^2 + 2\zeta\omega_ns + \omega_n^2)}{s^2 + 2\zeta\omega_ns + \omega_n^2} + \frac{\Pi_0s(s^2 + 2\zeta\omega_ns + \omega_n^2)}{s^2 + 2\zeta\omega_ns + \omega_n^2}
\]  

Equation (F.7)

This simplifies to:

\[
1 = \Pi_2(s^2 + 2\zeta\omega_ns + \omega_n^2) + \Pi_1s^2 + \Pi_0s
\]  

Equation (F.8)

\[
1 = (\Pi_2 + \Pi_1)s^2 + (2\Pi_2\zeta\omega_n + \Pi_0)s + (\Pi_2\omega_n^2)
\]  

Equation (F.9)

Now, equating the coefficients on the left and right sides of equation (F.9) gives:

\[
\begin{align*}
    s^2: \quad & 0 = \Pi_2 + \Pi_1 \\
    s^1: \quad & 0 = 2\Pi_2\zeta\omega_n + \Pi_0 \\
    s^0: \quad & 1 = \Pi_2\omega_n^2
\end{align*}
\]  

Equation (F.10)

Solving equation (F.10) simultaneously leads to:

\[
\begin{align*}
    \Pi_2 &= \frac{1}{\omega_n^2} \\
    \Pi_1 &= -\frac{1}{\omega_n^2} \\
    \Pi_0 &= -\frac{2\zeta}{\omega_n}
\end{align*}
\]  

Equation (F.11)
Substituting equation (F.11) into (F.6) leads to:

\[
\frac{1}{\omega_n^2} \frac{s}{s^2 + 2\zeta \omega_n s + \omega_n^2} + \frac{-2\zeta}{\omega_n} \tag{F.12}
\]

\[
\frac{1}{\omega_n^2} \left[ \frac{1}{s} + \frac{-s - \zeta \omega_n}{s^2 + 2\zeta \omega_n s + \omega_n^2} + \frac{-\zeta \omega_n}{s^2 + 2\zeta \omega_n s + \omega_n^2} \right] \tag{F.13}
\]

\[
\frac{1}{\omega_n^2} \left[ \frac{1}{s} - \frac{s + \zeta \omega_n}{(s + \zeta \omega_n)^2 + \omega_n^2(1 - \zeta^2)} - \frac{\zeta}{\sqrt{1 - \zeta^2}} \frac{\omega_n \sqrt{1 - \zeta^2}}{(s + \zeta \omega_n)^2 + \omega_n^2(1 - \zeta^2)} \right] \tag{F.14}
\]

Finally, grouping the terms that multiply each transform gives:

\[
\left( \frac{1}{\omega_n^2} \right) \frac{s}{s + \zeta \omega_n} \frac{1}{(s + \zeta \omega_n)^2 + \omega_n^2(1 - \zeta^2)} + \left( -\frac{\zeta}{\omega_n \sqrt{1 - \zeta^2}} \right) \frac{\omega_n \sqrt{1 - \zeta^2}}{(s + \zeta \omega_n)^2 + \omega_n^2(1 - \zeta^2)} \tag{F.15}
\]
F.1.4 Type II

Through partial fraction expansion, the transfer function may be expressed as follows:

\[
\frac{1}{s^2(s^2 + 2\zeta \omega_n s + \omega_n^2)} = \frac{\Pi_3}{s^2} + \frac{\Pi_2}{s} + \frac{\Pi_1 s + \Pi_0}{s^2 + 2\zeta \omega_n s + \omega_n^2}
\]  

(F.16)

Multiplying equation (F.16) by its denominator gives:

\[
\frac{s^2(s^2 + 2\zeta \omega_n s + \omega_n^2)}{s^2(s^2 + 2\zeta \omega_n s + \omega_n^2)} = \frac{\Pi_3 s^2(s^2 + 2\zeta \omega_n s + \omega_n^2)}{s^2} + \frac{\Pi_2 s^2(s^2 + 2\zeta \omega_n s + \omega_n^2)}{s} + \frac{\Pi_1 s s^2(s^2 + 2\zeta \omega_n s + \omega_n^2)}{s^2 + 2\zeta \omega_n s + \omega_n^2} + \frac{\Pi_0 s^2(s^2 + 2\zeta \omega_n s + \omega_n^2)}{s^2 + 2\zeta \omega_n s + \omega_n^2}
\]  

(F.17)

This simplifies to:

\[
1 = \Pi_3(s^2 + 2\zeta \omega_n s + \omega_n^2) + \Pi_2 s(s^2 + 2\zeta \omega_n s + \omega_n^2) + \Pi_1 s^3 + \Pi_0 s^2
\]  

(F.18)

\[
1 = (\Pi_2 + \Pi_1)s^3 + (\Pi_3 + 2\Pi_2 \zeta \omega_n + \Pi_0)s^2 + (2\Pi_3 \zeta \omega_n + \Pi_2 \omega_n^2)s + (\Pi_3 \omega_n^2)
\]  

(F.19)

Now, equating the coefficients on the left and right sides of equation (F.19) gives:

\[
\begin{align*}
s^3 &: 0 = \Pi_2 + \Pi_1 \\
s^2 &: 0 = \Pi_3 + 2\Pi_2 \zeta \omega_n + \Pi_0 \\
s^1 &: 0 = 2\Pi_3 \zeta \omega_n + \Pi_2 \omega_n^2 \\
s^0 &: 1 = \Pi_3 \omega_n^2
\end{align*}
\]  

(F.20)

Solving equation (F.20) simultaneously leads to:
Finally, substituting equation (F.21) into (F.16) leads to:

\[
\begin{align*}
\Pi_3 &= \frac{1}{\omega_n^2} \\
\Pi_2 &= -\frac{2\zeta}{\omega_n^3} \\
\Pi_1 &= \frac{2\zeta}{\omega_n^3} \\
\Pi_0 &= \frac{4\zeta^2 - 1}{\omega_n^2}
\end{align*}
\] (F.21)

Finally, grouping the terms that multiply each transform gives:

\[
\begin{align*}
\left(\frac{1}{\omega_n^2}\right) \frac{1}{s^2} + \left(\frac{2\zeta}{\omega_n^3}\right) \left(\frac{1}{s} + \frac{2\zeta}{\omega_n^3} s + \frac{s + \zeta \omega_n}{s^2 + 2\zeta \omega_n s + \omega_n^2} + \frac{1}{s^2 + 2\zeta \omega_n s + \omega_n^2}\right)
\end{align*}
\] (F.22)

\[
\begin{align*}
\frac{1}{\omega_n^3} \left[\frac{\omega_n}{s^2 + 2\zeta \omega_n s + \omega_n^2} + \frac{2\zeta s}{s^2 + 2\zeta \omega_n s + \omega_n^2} + \frac{(4\zeta^2 - 1)\omega_n}{s^2 + 2\zeta \omega_n s + \omega_n^2}\right]
\end{align*}
\] (F.23)

\[
\begin{align*}
\frac{1}{\omega_n^2} \left[\frac{1}{s^2} + \frac{2\zeta}{\omega_n} \left[\frac{1}{s} + \frac{s + \zeta \omega_n}{s^2 + 2\zeta \omega_n s + \omega_n^2} + \frac{1}{s^2 + 2\zeta \omega_n s + \omega_n^2}\right] - \frac{1}{s^2 + 2\zeta \omega_n s + \omega_n^2}\right]
\end{align*}
\] (F.24)

\[
\begin{align*}
\frac{1}{\omega_n^2} \left[\frac{1}{s^2} - \frac{1}{\omega_n \sqrt{1 - \zeta^2}} \left(\frac{1}{s} + \frac{s + \zeta \omega_n}{s^2 + 2\zeta \omega_n s + \omega_n^2} + \frac{1}{s^2 + 2\zeta \omega_n s + \omega_n^2}\right) - \frac{\zeta}{\omega_n \sqrt{1 - \zeta^2}} \left(\frac{1}{s} + \frac{s + \zeta \omega_n}{s^2 + 2\zeta \omega_n s + \omega_n^2} + \frac{1}{s^2 + 2\zeta \omega_n s + \omega_n^2}\right)\right]
\end{align*}
\] (F.25)

Finally, grouping the terms that multiply each transform gives:

\[
\begin{align*}
\left(\frac{1}{\omega_n^2}\right) \frac{1}{s^2} + \left(\frac{2\zeta}{\omega_n^3}\right) \left(\frac{1}{s} + \frac{2\zeta}{\omega_n^3} s + \frac{s + \zeta \omega_n}{s^2 + 2\zeta \omega_n s + \omega_n^2} + \frac{1}{s^2 + 2\zeta \omega_n s + \omega_n^2}\right)
\end{align*}
\] (F.26)
F.1.5 Type III

Through partial fraction expansion, the transfer function may be expressed as follows:

\[
\frac{1}{s^3(s^2 + 2\zeta \omega_n s + \omega_n^2)} = \frac{\Pi_4}{s^3} + \frac{\Pi_3}{s^2} + \frac{\Pi_2}{s} + \frac{\Pi_1 s + \Pi_0}{s^2 + 2\zeta \omega_n s + \omega_n^2}
\]

(F.27)

Multiplying equation (F.27) by its denominator gives:

\[
\frac{s^3(s^2 + 2\zeta \omega_n s + \omega_n^2)}{s^3(s^2 + 2\zeta \omega_n s + \omega_n^2)} = \frac{\Pi_4 s^3(s^2 + 2\zeta \omega_n s + \omega_n^2)}{s^3(s^2 + 2\zeta \omega_n s + \omega_n^2)} + \frac{\Pi_3 s^3(s^2 + 2\zeta \omega_n s + \omega_n^2)}{s^3(s^2 + 2\zeta \omega_n s + \omega_n^2)} + \frac{\Pi_2 s^3(s^2 + 2\zeta \omega_n s + \omega_n^2)}{s^3(s^2 + 2\zeta \omega_n s + \omega_n^2)} + \frac{\Pi_1 s^3(s^2 + 2\zeta \omega_n s + \omega_n^2)}{s^3(s^2 + 2\zeta \omega_n s + \omega_n^2)} + \frac{\Pi_0 s^3(s^2 + 2\zeta \omega_n s + \omega_n^2)}{s^3(s^2 + 2\zeta \omega_n s + \omega_n^2)}
\]

(F.28)

This simplifies to:

\[
1 = \Pi_4 (s^2 + 2\zeta \omega_n s + \omega_n^2) + \Pi_3 s^2(s^2 + 2\zeta \omega_n s + \omega_n^2) + \Pi_2 s^2(s^2 + 2\zeta \omega_n s + \omega_n^2) + \Pi_1 s^4 + \Pi_0 s^3
\]

(F.29)

\[
1 = (\Pi_2 + \Pi_1)s^4 + (\Pi_3 + 2\Pi_2 \zeta \omega_n + \Pi_0)s^3 + (\Pi_4 + 2\Pi_3 \zeta \omega_n + \Pi_2 \omega_n^2)s^2 + (2\Pi_4 \zeta \omega_n + \Pi_3 \omega_n^2)s + (\Pi_4 \omega_n^2)
\]

(F.30)

Now, equating the coefficients on the left and right sides of equation (F.30) gives:

\[
\begin{align*}
    s^4: & \quad 0 = \Pi_2 + \Pi_1 \\
    s^3: & \quad 0 = \Pi_3 + 2\Pi_2 \zeta \omega_n + \Pi_0 \\
    s^2: & \quad 0 = \Pi_4 + 2\Pi_3 \zeta \omega_n + \Pi_2 \omega_n^2 \\
    s^1: & \quad 0 = 2\Pi_4 \zeta \omega_n + \Pi_3 \omega_n^2 \\
    s^0: & \quad 1 = \Pi_4 \omega_n^2
\end{align*}
\]

(F.31)
Solving equation (F.31) simultaneously leads to:

\[
\begin{align*}
\Pi_4 &= \frac{1}{\omega_n^2} \quad \Pi_3 = -\frac{2\zeta}{\omega_n^3} \quad \Pi_2 = \frac{4\zeta^2 - 1}{\omega_n^4} \\
\Pi_1 &= -\frac{4\zeta^2 - 1}{\omega_n^4} \quad \Pi_0 = \frac{4\zeta - 8\zeta^3}{\omega_n^3}
\end{align*}
\] (F.32)

Finally, substituting equation (F.32) into (F.27) leads to:

\[
\frac{1}{\omega_n^2} \left[ \omega_n^2 \left( \frac{1}{s^3} + \frac{-2\zeta}{s^2} + \frac{4\zeta^2 - 1}{s} \right) - \frac{1}{s^2 + 2\zeta\omega_n s + \omega_n^2} \right] + \left( \frac{4\zeta - 8\zeta^3}{\omega_n^3} \right) \omega_n
\] (F.33)

\[
\frac{1}{\omega_n^2} \left[ \frac{\omega_n^2}{s^3} + \frac{-2\zeta\omega_n}{s^2} + (4\zeta^2 - 1) \right] \left[ \frac{1}{s} - \frac{s + \zeta\omega_n}{s^2 + 2\zeta\omega_n s + \omega_n^2} - \frac{s + \zeta\omega_n}{s^2 + 2\zeta\omega_n s + \omega_n^2} + \frac{2\zeta\omega_n}{s^2 + 2\zeta\omega_n s + \omega_n^2} \right]
\] (F.34)

\[
\frac{1}{\omega_n^2} \left[ \frac{1}{s^3} + \frac{-2\zeta}{s^2} - \frac{1}{\omega_n\sqrt{1 - \zeta^2}} \left( \frac{1}{s + \zeta\omega_n} + \omega_n^2(1 - \zeta^2) \right) \right]
\] (F.35)

Finally, grouping the terms that multiply each transform gives:

\[
\frac{1}{\omega_n^2} \left[ \frac{1}{s^3} + \frac{-2\zeta}{s^2} - \frac{1}{\omega_n\sqrt{1 - \zeta^2}} \left( \frac{1}{s + \zeta\omega_n} + \omega_n^2(1 - \zeta^2) \right) \right] + (4\zeta^2 - 1) \left[ \frac{1}{\omega_n^2} \left[ \frac{1}{s} - \frac{s + \zeta\omega_n}{s^2 + \omega_n^2(1 - \zeta^2)} - \frac{s + \zeta\omega_n}{s^2 + \omega_n^2(1 - \zeta^2)} \right] \right]
\] (F.36)

\[
\frac{1}{\omega_n^2} \left[ \frac{1}{s^3} + \frac{-2\zeta}{s^2} + \frac{1 - 4\zeta^2}{s^2} + \frac{1 - 4\zeta^2}{s} + \frac{4\zeta^2 - 1}{s} + \frac{1}{s + \zeta\omega_n} + \frac{4\zeta^2 - 1}{s + \zeta\omega_n} + \omega_n^2(1 - \zeta^2) \right]
\] (F.37)
APPENDIX F (continued)

F.2 Real Modes

F.2.1 Type I

Through partial fraction expansion, the transfer function may be expressed as follows:

\[
\frac{1}{s(s + \tau^{-1})} = \frac{\Pi_1}{s} + \frac{\Pi_0}{s + \tau^{-1}} \tag{F.38}
\]

Multiplying equation (F.38) by its denominator gives:

\[
\frac{s(s + \tau^{-1})}{s(s + \tau^{-1})} = \frac{\Pi_1 s(s + \tau^{-1})}{s} + \frac{\Pi_0 s(s + \tau^{-1})}{s + \tau^{-1}} \tag{F.39}
\]

This simplifies to:

\[
1 = \Pi_1 (s + \tau^{-1}) + \Pi_0 s \tag{F.40}
\]

\[
1 = (\Pi_1 + \Pi_0) s + (\Pi_1 \tau^{-1}) \tag{F.41}
\]

Now, equating the coefficients on the left and right sides of equation (F.41) gives:

\[
s^1: \quad 0 = \Pi_1 + \Pi_0 \tag{F.42}
\]

\[
s^0: \quad 1 = \Pi_1 \tau^{-1}
\]

Solving equation (F.42) simultaneously leads to:

\[
\Pi_1 = \frac{1}{\tau^{-1}} = \tau \quad \Pi_0 = -\frac{1}{\tau^{-1}} = -\tau \tag{F.43}
\]

Finally, substituting equation (F.43) into (F.38) leads to:
APPENDIX F (continued)

\[\tau \frac{1}{s} + (-\tau) \frac{1}{s + \tau^{-1}} \quad (F.44)\]
F.2.2 Type II

Through partial fraction expansion, the transfer function may be expressed as follows:

\[
\frac{1}{s^2(s + \tau^{-1})} = \frac{\Pi_2}{s^2} + \frac{\Pi_1}{s} + \frac{\Pi_0}{s + \tau^{-1}} \tag{F.45}
\]

Multiplying equation (F.45) by its denominator gives:

\[
\frac{s^2(s + \tau^{-1})}{s^2(s + \tau^{-1})} = \frac{s^2(s + \tau^{-1})}{s^2} + \frac{s^2(s + \tau^{-1})}{s} + \frac{s^2(s + \tau^{-1})}{s + \tau^{-1}} \tag{F.46}
\]

This simplifies to:

\[
1 = \Pi_2(s + \tau^{-1}) + \Pi_1(s + \tau^{-1}) + \Pi_0s^2 \tag{F.47}
\]

\[
1 = (\Pi_1 + \Pi_0)s^2 + (\Pi_2 + \Pi_1\tau^{-1})s + (\Pi_2\tau^{-1}) \tag{F.48}
\]

Now, equating the coefficients on the left and right sides of equation (F.48) gives:

\[
s^2: \quad 0 = \Pi_1 + \Pi_0 \\
s^1: \quad 0 = \Pi_2 + \Pi_1\tau^{-1} \tag{F.49}
\]

\[
s^0: \quad 1 = \Pi_2\tau^{-1}
\]

Solving equation (F.49) simultaneously leads to:

\[
\Pi_2 = \frac{1}{\tau^{-1}} = \tau \quad \Pi_1 = -\tau^2 \quad \Pi_0 = \tau^2 \tag{F.50}
\]
Finally, substituting equation (F.50) into (F.45) leads to:

\[
(\tau) \frac{1}{s^2} + (-\tau^2) \frac{1}{s} + (\tau^2) \frac{1}{s + \tau^{-1}}
\]  

(F.51)
F.2.3 Type III

Through partial fraction expansion, the transfer function may be expressed as follows:

\[
\frac{1}{s^3(s + \tau^{-1})} = \frac{\Pi_3}{s^3} + \frac{\Pi_2}{s^2} + \frac{\Pi_1}{s} + \frac{\Pi_0}{s + \tau^{-1}} \tag{F.52}
\]

Multiplying equation (F.52) by its denominator gives:

\[
\frac{s^3(s + \tau^{-1})}{s^3(s + \tau^{-1})} = \frac{\Pi_3 s^3(s + \tau^{-1})}{s^3} + \frac{\Pi_2 s^3(s + \tau^{-1})}{s^2} + \frac{\Pi_1 s^3(s + \tau^{-1})}{s} + \frac{\Pi_0 s^3(s + \tau^{-1})}{s + \tau^{-1}} \tag{F.53}
\]

This simplifies to:

\[
1 = \Pi_3 (s + \tau^{-1}) + \Pi_2 s(s + \tau^{-1}) + \Pi_1 s^2(s + \tau^{-1}) + \Pi_0 s^3 \tag{F.54}
\]

\[
1 = (\Pi_1 + \Pi_0) s^3 + (\Pi_2 + \Pi_1 \tau^{-1}) s^2 + (\Pi_3 + \Pi_2 \tau^{-1}) s + (\Pi_3 \tau^{-1}) \tag{F.55}
\]

Now, equating the coefficients on the left and right sides of equation (F.55) gives:

\[
\begin{align*}
    s^3 & : 0 = \Pi_1 + \Pi_0 \\
    s^2 & : 0 = \Pi_2 + \Pi_1 \tau^{-1} \\
    s^1 & : 0 = \Pi_3 + \Pi_2 \tau^{-1} \\
    s^0 & : 1 = \Pi_3 \tau^{-1}
\end{align*} \tag{F.56}
\]

Solving equation (F.56) simultaneously leads to:

\[
\Pi_3 = \frac{1}{\tau^{-1}} = \tau \quad \Pi_2 = -\tau^2 \quad \Pi_1 = \tau^3 \quad \Pi_0 = -\tau^3 \tag{F.57}
\]
Finally, substituting equation (F.57) into (F.52) leads to:

\[
\tau \frac{1}{s^3} + (-\tau^2) \frac{1}{s^2} + (\tau^3) \frac{1}{s} + (-\tau^3) \frac{1}{s + \tau^{-1}}
\]

(F.58)
F.2.4 Type IV

Through partial fraction expansion, the transfer function may be expressed as follows:

\[
\frac{1}{s^4(s + \tau^{-1})} = \frac{\Pi_4}{s^4} + \frac{\Pi_3}{s^3} + \frac{\Pi_2}{s^2} + \frac{\Pi_1}{s} + \frac{\Pi_0}{s + \tau^{-1}}
\] (F.59)

Multiplying equation (F.59) by its denominator gives:

\[
\frac{s^4(s + \tau^{-1})}{s^4(s + \tau^{-1})} = \frac{\Pi_4 s^4(s + \tau^{-1})}{s^4} + \frac{\Pi_3 s^4(s + \tau^{-1})}{s^3} + \frac{\Pi_2 s^4(s + \tau^{-1})}{s^2} + \frac{\Pi_1 s^4(s + \tau^{-1})}{s} + \frac{\Pi_0 s^4(s + \tau^{-1})}{s + \tau^{-1}}
\] (F.60)

This simplifies to:

\[
1 = \Pi_4(s + \tau^{-1}) + \Pi_3 s(s + \tau^{-1}) + \Pi_2 s^2(s + \tau^{-1}) + \Pi_1 s^3(s + \tau^{-1}) + \Pi_0 s^4
\] (F.61)

\[
1 = (\Pi_1 + \Pi_0) s^4 + (\Pi_2 + \Pi_1 \tau^{-1}) s^3 + (\Pi_3 + \Pi_2 \tau^{-1}) s^2 + (\Pi_4 + \Pi_3 \tau^{-1}) s + (\Pi_4 \tau^{-1})
\] (F.62)

Now, equating the coefficients on the left and right sides of equation (F.62) gives:

\[
s^4: \quad 0 = \Pi_1 + \Pi_0
\]

\[
s^3: \quad 0 = \Pi_2 + \Pi_1 \tau^{-1}
\]

\[
s^2: \quad 0 = \Pi_3 + \Pi_2 \tau^{-1}
\]

\[
s^1: \quad 0 = \Pi_4 + \Pi_3 \tau^{-1}
\]

\[
s^0: \quad 1 = \Pi_4 \tau^{-1}
\]
Solving equation (F.63) simultaneously leads to:

\[
\Pi_4 = \frac{1}{\tau^{-1}} = \tau \quad \Pi_3 = -\tau^2 \quad \Pi_2 = \tau^3 \quad \Pi_1 = -\tau^4 \quad \Pi_0 = \tau^4 \quad \text{(F.64)}
\]

Finally, substituting equation (F.64) into (F.59) leads to:

\[
(\tau) \frac{1}{s^4} + (-\tau^2) \frac{1}{s^3} + (\tau^3) \frac{1}{s^2} + (-\tau^4) \frac{1}{s} + (\tau^4) \frac{1}{s + \tau^{-1}} \quad \text{(F.65)}
\]
F.2.5 Type V

Through partial fraction expansion, the transfer function may be expressed as follows:

\[
\frac{1}{s^5(s + \tau^{-1})} = \frac{\Pi_5}{s^5} + \frac{\Pi_4}{s^4} + \frac{\Pi_3}{s^3} + \frac{\Pi_2}{s^2} + \frac{\Pi_1}{s} + \frac{\Pi_0}{s + \tau^{-1}} \tag{F.66}
\]

Multiplying equation (F.66) by its denominator gives:

\[
\frac{s^5(s + \tau^{-1})}{s^5(s + \tau^{-1})} = \frac{\Pi_5 s^5(s + \tau^{-1})}{s^5} + \frac{\Pi_4 s^5(s + \tau^{-1})}{s^4} + \frac{\Pi_3 s^5(s + \tau^{-1})}{s^3} + \frac{\Pi_2 s^5(s + \tau^{-1})}{s^2} + \frac{\Pi_1 s^5(s + \tau^{-1})}{s} + \frac{\Pi_0 s^5(s + \tau^{-1})}{s + \tau^{-1}} \tag{F.67}
\]

This simplifies to:

\[
1 = \Pi_5 (s + \tau^{-1}) + \Pi_4 s(s + \tau^{-1}) + \Pi_3 s^2(s + \tau^{-1}) + \Pi_2 s^3(s + \tau^{-1}) + \Pi_1 s^4(s + \tau^{-1}) + \Pi_0 s^5 \tag{F.68}
\]

\[
1 = (\Pi_1 + \Pi_0)s^5 + (\Pi_2 + \Pi_1 \tau^{-1})s^4 + (\Pi_3 + \Pi_2 \tau^{-1})s^3 + (\Pi_4 + \Pi_3 \tau^{-1})s^2 + (\Pi_5 + \Pi_4 \tau^{-1})s + (\Pi_5 \tau^{-1}) \tag{F.69}
\]

Now, equating the coefficients on the left and right sides of equation (F.69) gives:

\[
s^5: \quad 0 = \Pi_1 + \Pi_0
\]
\[
s^4: \quad 0 = \Pi_2 + \Pi_1 \tau^{-1}
\]
\[
s^3: \quad 0 = \Pi_3 + \Pi_2 \tau^{-1}
\]
\[
s^2: \quad 0 = \Pi_4 + \Pi_3 \tau^{-1}
\]
\[
s^1: \quad 0 = \Pi_5 + \Pi_4 \tau^{-1}
\]
\[
s^0: \quad 1 = \Pi_5 \tau^{-1} \tag{F.70}
\]
Solving equation (F.70) simultaneously leads to:

\[ \Pi_5 = \frac{1}{\tau^{-1}} = \tau, \quad \Pi_4 = -\tau^2, \quad \Pi_3 = \tau^3, \quad \Pi_2 = -\tau^4, \quad \Pi_1 = \tau^5, \quad \Pi_0 = -\tau^5 \quad (F.71) \]

Finally, substituting equation (F.71) into (F.66) leads to:

\[ (\tau) \frac{1}{s^5} + (-\tau^2) \frac{1}{s^4} + (\tau^3) \frac{1}{s^3} + (-\tau^4) \frac{1}{s^2} + (\tau^5) \frac{1}{s} + (-\tau^5) \frac{1}{s + \tau^{-1}} \quad (F.72) \]