EPILEPTIC FOCI LOCALIZATION USING THE INVERSE SOURCE PROBLEM FOR MAXWELL'S EQUATIONS

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The following faculty members have examined the final copy of this thesis for form and content, and recommend that it be accepted in partial fulfillment of the requirement for the degree of Master of Science with a major in Mathematics.

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DEDICATION

To my mother, Charlotte.
Yours is the courage and faith by which I measure my own.

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ABSTRACT

Consider an application of the inverse source problem for Maxwell's equations to the matter of epileptic foci localization in the human brain. Using a current dipole to model the epileptic focus in the brain, and by considering those conditions particular to medical applications, we constructed an approximate inverse problem which we used to subsequently determine a unique location and orientation for the dipole. In addition, we developed a numerical experiment demonstrating practical efficacy at mulitple sensor positions.

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Epileptic Foci Localization using the Inverse Source Problem for Maxwell's Equations

Joshua Kehler

Introduction

Significance

Epilepsy affects over fifty million people worldwide with a global incidence of up to 51/100,000 people per year (Banerjee et al. 2009). 60% of epilepsy cases may be categorized within focal epilepsy syndrome. This subset is characterized by the generation of focal seizures from a small group of neurons, the epileptic focus. Focal epilepsy is distinguished from generalized epilepsy by origin of onset, where in the latter both hemispheres are involved (Kandel et al. 2000). For approximately 15% of those with focal epilepsy, the disorder is not adequately controlled by anticonvulsive drugs, informing the use of alternative treatments such as surgery (Rosenow et al. 2001). According to the definition agreed upon by the International League Against Epilepsy (ILAE) for drug resistant, or intractable epilepsy (Kwan et al 2010), conservatively 50% of patients with an intractable form of epilepsy may be potential candidates for surgical treatment implying that nearly 5% of all people with epilepsy may benefit from epilepsy surgery. In the case where the epileptogenic zone was clearly defined and resected, between 30-85% of patients remained seizure free post-operation (Engel 1992).

Biological Considerations

Considering the complexity of epilepsy, no one mechanism for seizure onset has been elucidated. However, it is generally agreed that seizures arise due to an imbalance between excitatory and inhibitory mechanisms within the nervous system, where normal controls for neurons fail to prevent excessive action potential discharge (Scharfman 2007). A net resting transmembrane potential of -60mV is attributed to the action of these control mechanisms, one such control is the sodium potassium pump which maintains a higher intracellular concentration of potassium versus a higher extracellular concentration of sodium (McCormick, et al. 1994). In the case of ionic perturbation, such as an abnormal elevation in extracellular potassium, depolarization may occur thus initiating a cascade of excitatory depolarizations downstream as surrounding inhibition breaks down (Somjen 2002). However, excessive discharge alone does not necessitate a seizure. Cortical epileptogenic events captured during seizure correspond to paroxysmal depolarization shifts (PDS) of neuron ensembles, these cortical pyramidal cells discharge synchronously propagating the seizure event (Matsumoto et al. 1964).

Inverse Problem

Let us examine the MEG inverse source problem for Maxwell's equations with respect to epileptic foci mapping during the presurgical evaluation process. Since the human brain exhibits enormous complexity, it will be modeled as a heterogenous medium and volume conductor where the material coefficients; magnetic permeability μ, electric permittivity ε, and electric conductivity σ will be treated as spatial functions . With respect to the frequency of detected signals, we will consider those below 100Hz, consistent with studies in MEG (Ammari et al. 2002). Per our previous discussion, focal seizures are attributed to the synchronous electrical discharge generated within a small volume of neural tissue, and therefore will be approximated as a current dipole. Taken together, the inverse source problem may be described as the localization of a current dipole from the boundary measurements of a volume conductor's eletromagnetic field .

Magnetoencephalography

Developed in the late 1960's by James Zimmerman, MEG is a noninvasive technique founded upon the superconducting quantum interference device (SQUID) and may be used to detect magnetic flux in various tissues (Zimmerman et al. 1970). MEG holds many advantages over alternative techniques, warranting its use in the *in vivo* investigation of cortical activity within the human brain (Hamalainen et al. 1993). Multichannel SQUIID gradiometers can detect the minute, magnetic fields generated by neurons on the order of 10 fT - 1 pT, along with time resolution of 1 ms and a spatial discrimination of 2-3 mm. MEG will be used to record the tangential component of the magnetic field at the boundary, from which the dipole will be uniquely localized assuming a low, fixed frequency. Since a current will produce an externally detectable field only if it has a component that is tangential to the surface of a spherically symmetric conductor, it follows that the radial components will remain undetected and the activity recorded by MEG is localized to cortical fissures (Ammari et al. 2002).

Following problem formulation, a uniqueness result will be presented for the approximate inverse problem, and finally we will treat the numerical experiment using artifical data.

Problem Formulation

Let (x_1, x_2, x_3) be the Cartesian coordinate system with an orthonormal basis, $(e(1), e(2), e(3))$ (Ammari et al. 2002). Let Ω be the open unit ball in $\mathbb{R}^3$ and define Ω^i as the interior of Ω. Define Ω^e as the complement of $\overline{\Omega^i}$ in $\mathbb{R}^3$. Let $\Gamma = \partial\Omega$ and ν is the unit outward normal.

Consider the propagation of time harmonic electromagnetic waves within a dielectric medium with constant electric permittivity ε, magnetic permeability μ, and electric conductivity σ which satisfy $\varepsilon > 0$, $\mu > 0$, and $\sigma > 0$.

We will examine wave progagation as determined by Maxwell's equations:

$$\text{curl } E = -i\omega\mu H \quad \text{in } \mathbb{R}^3,$$

$$\text{curl } H = i\omega\varepsilon E - \sigma E + \delta(x-y)P \quad in \quad \mathbb{R}^3, \tag{1}$$

which will be considered together with the exponential decay of E,H:

$$|E(x)| + |H(x)| < Ce^{-\delta|x|}, \tag{2}$$

for some positive constants C, δ.

Here $\delta(x-y)$ is the Dirac Delta function given at the source point y, and the orientation of the electric dipole is given by the constant vector P. Given y, P, the forward problem is to determine the electromagnetic field distributions E, H by fixing frequency ω and the material parameters ε, μ, and σ.

In particular, medical applications including electroencephalography and related bioimpedance studies use materials parameters $\varepsilon = 8.854 \times 10^{-12}$ farads/meter, $\mu = 4\pi \times 10^{-7}$ henrys/meter, $\omega = 100$ Hz and $\sigma = .33$ mhos/meter. (3)

As known, the solution to the forward problem yields for electric field:

$$E(x,y,P) = i\omega\mu \frac{e^{ik|x-y|}}{4\pi|x-y|}P + i\omega\mu\frac{1}{k^2}\text{grad div }\left(\frac{e^{ik|x-y|}}{4\pi|x-y|}P\right), \tag{4}$$

where

$$k = \omega\sqrt{\mu\left(\varepsilon + i\frac{\sigma}{\omega}\right)}. \tag{5}$$

The Inverse Problem

Determine y, P given the data $E(x,y,P) \times \nu$ for $x \in \partial\Omega$.

Since the material parameters (3) are sufficiently small we may simplify $E(x,y,P)$. For (simpler) approximate inverse problem, we will determine y, P given the data $E(x,y,P;0) \times \nu$ for $x \in \partial\Omega$.

To calculate $E(x,y,P;0)$, we pass through the limit as $\omega\mu \to 0$ and $\omega\varepsilon \to 0$.

Making the appropriate substitution for k^2,

$$E(x,y,P) = i\omega\mu \frac{e^{ik|x-y|}}{4\pi|x-y|}P + \frac{i\omega\mu}{\omega\mu(\omega\varepsilon + i\sigma)}\text{grad div }\left(\frac{e^{ik|x-y|}}{4\pi|x-y|}P\right),$$

and simplifying yields

$$E(x,y,P) = i\omega\mu \frac{e^{ik|x-y|}}{4\pi|x-y|}P + \frac{i}{\omega\varepsilon + i\sigma}\text{grad div}\left(\frac{e^{ik|x-y|}}{4\pi|x-y|}P\right).$$

There is

$$E(x,y,P;0) = \lim_{\omega\mu,\omega\varepsilon\to 0}\left(i\omega\mu \frac{e^{ik|x-y|}}{4\pi|x-y|}P + \frac{i}{\omega\varepsilon + i\sigma}\text{grad div}\left(\frac{e^{ik|x-y|}}{4\pi|x-y|}P\right)\right)$$

or

$$E(x,y,P;0) = \frac{1}{\sigma}\text{grad div}\left(\frac{1}{4\pi|x-y|}P\right) \quad , \qquad x \in \partial\Omega. \tag{6}$$

The Approximate Inverse Problem

Determine y, P given the data $E(x, y, P; 0) \times \nu(x)$ for $x \in \partial\Omega$.

Computing $E(x, y, P; 0)$ results in the following system of equations:

In coordinate form,

$$E_1(x,y,P;0) = \frac{\partial^2}{\partial x_1^2}\left(\frac{1}{4\pi\sigma|x-y|}\right)P_1 + \frac{\partial^2}{\partial x_1 \partial x_2}\left(\frac{1}{4\pi\sigma|x-y|}\right)P_2 + \frac{\partial^2}{\partial x_1 \partial x_3}\left(\frac{1}{4\pi\sigma|x-y|}\right)P_3,$$

$$E_2(x,y,P;0) = \frac{\partial^2}{\partial x_2 \partial x_1}\left(\frac{1}{4\pi\sigma|x-y|}\right)P_1 + \frac{\partial^2}{\partial x_2^2}\left(\frac{1}{4\pi\sigma|x-y|}\right)P_2 + \frac{\partial^2}{\partial x_2 \partial x_3}\left(\frac{1}{4\pi\sigma|x-y|}\right)P_3, \tag{7}$$

$$E_3(x,y,P;0) = \frac{\partial^2}{\partial x_3 \partial x_1}\left(\frac{1}{4\pi\sigma|x-y|}\right)P_1 + \frac{\partial^2}{\partial x_3 \partial x_2}\left(\frac{1}{4\pi\sigma|x-y|}\right)P_2 + \frac{\partial^2}{\partial x_3^2}\left(\frac{1}{4\pi\sigma|x-y|}\right)P_3.$$

Now that we have determined $E(x, y, P; 0)$, we may solve the inverse problem to find y, P given the data $E(x, y, P; 0) \times \nu(x)$ for $x \in \partial\Omega$.

Uniqueness Result for the Approximate Inverse Problem

We will show that the location, y and orientation, P of the dipole are uniquely determined by the tangential components of the electric field at the boundary.

Theorem. If $E(x, y(1), P(1); 0) \times \nu(x) = E(x, y(2), P(2); 0) \times \nu(x)$ for $x \in \partial\Omega$ and $P(1) \neq 0$, then $y(1) = y(2)$ and $P(1) = P(2)$.

Proof: Fully expand $E(x, y, P; 0)$:

$$E_1 = \frac{1}{4\pi\sigma}\left[\left(\frac{2(x_1-y_1)^2 - (x_2-y_2)^2 - (x_3-y_3)^2}{|x-y|^5}\right)P_1 + \left(\frac{3(x_1-y_1)(x_2-y_2)}{|x-y|^5}\right)P_2 + \left(\frac{3(x_1-y_1)(x_3-y_3)}{|x-y|^5}\right)P_3\right],$$

$$E_2 = \frac{1}{4\pi\sigma}\left[\left(\frac{3(x_1-y_1)(x_2-y_2)}{|x-y|^5}\right)P_1 + \left(\frac{2(x_2-y_2)^2 - (x_1-y_1)^2 - (x_3-y_3)^2}{|x-y|^5}\right)P_2 + \left(\frac{3(x_2-y_2)(x_3-y_3)}{|x-y|^5}\right)P_3\right], \tag{8}$$

$$E_3 = \frac{1}{4\pi\sigma}\left[\left(\frac{3(x_1-y_1)(x_3-y_3)}{|x-y|^5}\right)P_1 + \left(\frac{3(x_2-y_2)(x_3-y_3)}{|x-y|^5}\right)P_2 + \left(\frac{2(x_3-y_3)^2-(x_1-y_1)^2-(x_2-y_2)^2}{|x-y|^5}\right)P_3\right].$$

Now, rewritten for convenience:

$$E(x,y,P;0) = \frac{A(x,y,P)}{|x-y|^5}, \tag{9}$$

$$A(x,y) =$$

$$\begin{cases} \frac{1}{4\pi}\left[\left(2(x_1-y_1)^2-(x_2-y_2)^2-(x_3-y_3)^2\right)P_1 + (3(x_1-y_1)(x_2-y_2))P_2 + (3(x_1-y_1)(x_3-y_3))P_3\right] \\ \frac{1}{4\pi}\left[(3(x_1-y_1)(x_2-y_2))P_1 + \left(2(x_2-y_2)^2-(x_1-y_1)^2-(x_3-y_3)^2\right)P_2 + (3(x_2-y_2)(x_3-y_3))P_3\right] \\ \frac{1}{4\pi}\left[(3(x_1-y_1)(x_3-y_3))P_1 + (3(x_2-y_2)(x_3-y_3))P_2 + \left(2(x_3-y_3)^2-(x_1-y_1)^2-(x_2-y_2)^2\right)P_3\right]. \end{cases}$$

For purposes of proving uniqueness of the inverse problem, we introduce the function $h(x,y,P)$,

$$h(x,y,P) = \frac{\partial}{\partial x_1}\left(\frac{1}{4\pi\sigma|x-y|}\right)P_1 + \frac{\partial}{\partial x_2}\left(\frac{1}{4\pi\sigma|x-y|}\right)P_2 + \frac{\partial}{\partial x_3}\left(\frac{1}{4\pi\sigma|x-y|}\right)P_3. \tag{10}$$

We may rewrite $E(x,y,P;0)$ as

$$E(x,y,P;0) = \frac{1}{\sigma}\nabla h(x,y,P). \tag{11}$$

Observe from our definition of $h(x,y,P)$, that it is a harmonic function decaying at infinity. Now suppose we have two functions, $h(x;y(1),P(1))$ and $h(x;y(2),P(2))$,

$$h(x,y(1),P(1)) = \frac{\partial}{\partial x_1}\left(\frac{1}{4\pi\sigma|x-y|}\right)P_1(1) + \frac{\partial}{\partial x_2}\left(\frac{1}{4\pi\sigma|x-y|}\right)P_2(1) + \frac{\partial}{\partial x_3}\left(\frac{1}{4\pi\sigma|x-y|}\right)P_3(1), \quad x \neq y(1),$$

$$h(x,y(2),P(2)) = \frac{\partial}{\partial x_1}\left(\frac{1}{4\pi\sigma|x-y|}\right)P_1(2) + \frac{\partial}{\partial x_2}\left(\frac{1}{4\pi\sigma|x-y|}\right)P_2(2) + \frac{\partial}{\partial x_3}\left(\frac{1}{4\pi\sigma|x-y|}\right)P_3(2), \quad x \neq y(2),$$

such that $\nabla h(x; y(1), P(1)) \times \nu(x) = \nabla h(x; y(2), P(2)) \times \nu(x)$, for $x \in \partial\Omega$. First we show that $h(x; y(1), P(1))$ and $h(x; y(2), P(2))$ are indeed equal for $x \in \partial\Omega$.

Let $x(0) \in \partial\Omega$. We can choose orthogonal coordinates such that $x(0) = (1, 0, 0)$. Choose vector $\nu = (1, 0, 0)$, $|\nu| = 1$.

$$\nabla h(x; y(1), P(1)) \times \nu(x) = (0, -\partial_3 h(x; y(1), P(1)), -\partial_2 h(x; y(1), P(1)))\,,$$

$$\nabla h(x; y(2), P(2)) \times \nu(x) = (0, -\partial_3 h(x; y(2), P(2)), -\partial_2 h(x; y(2), P(2)))\,.$$

Hence, the tangential gradients at $x(0)$ are equal for $h(x; y(1), P(1))$ and $h(x; y(2), P(2))$ therefore the tangential gradients for $h(x; y(1), P(1))$ and $h(x; y(2), P(2))$ are equal for $x \in \partial\Omega$. Since the gradients are equal then $h(x; y(1), P(1)) = h(x; y(2), P(2)) + C$ for some constant C only for $x \in \partial\Omega$. Since $h(x; y(1), P(1)) = h(x; y(2), P(2)) + C$ it follows that $h = h(, y(1), P(1)) - h(, y(2), P(2)) = C$ on $\partial\Omega$. From our definition of $h(x, y, P)$, it is clear that

$$\lim_{|x| \to \infty} h(x) = 0.$$

Let

$$u(x) = \frac{C}{|x|}.$$

It follows that $u = C$ on $\partial\Omega$. Moreover,

$$\lim_{|x| \to \infty} u(x) = 0.$$

Therefore, $h = u$ outside Ω.

However,

$$|h(x)| \leq \frac{C_1}{|x|^2},$$

which implies that

$$\frac{C}{|x|} \le \frac{C_1}{|x|^2},$$

which may only occur if $C = 0$.

Hence

$$h(x; y(1), P(1)) = h(x; y(2), P(2)), \qquad x \in \partial\Omega.$$

Since $h(x; y(1), P(1)) = h(x; y(2), P(2))$ for $x \in \partial\Omega$, and their limits are zero at infinity, it follows by maximum principle that:

$$h(x; y(1), P(1)) = h(x; y(2), P(2)), \qquad x \in \mathbb{R}^3 \setminus \Omega. \tag{12}$$

By uniqueness of the continuation as applied to harmonic functions, since $h(x; y(1), P(1))$ and $h(x; y(2), P(2))$ are equal for $x \in \mathbb{R}^3 \setminus \Omega$, then they are equal for $x \in \mathbb{R}^3 \setminus (\{y(1)\} \cup \{y(2)\})$.

Now we will prove uniqueness of y, P by contradiction. Consider two functions $E(x, y(1), P(1); 0)$ and $E(x, y(2), P(2); 0)$, analytic on $\Omega_0 = \mathbb{R}^3 \setminus \ (\{y(1)\} \cup \{y(2)\})$. From (4), since σ is constant and the equality of $h(x; y(1), P(1))$ and $h(x; y(2), P(2))$ remains unchanged under the gradient operator, it follows from (9) that $E(x, y(1), P(1); 0) = E(x, y(2), P(2); 0)$ for $x \in \mathbb{R}^3 \setminus (\{y(1)\} \cup \{y(2)\})$.

$$E(x, y(1), P(1); 0) = \frac{A(x, y(1), P(1))}{|x - y(1)|^5} = \frac{1}{\sigma}\nabla h(x, y(1), P(1)),$$

$$E(x, y(2), P(2); 0) = \frac{A(x, y(2), P(2))}{|x - y(2)|^5} = \frac{1}{\sigma}\nabla h(x, y(2), P(2)). \tag{13}$$

Suppose $y(1) \ne y(2)$.

Let $x(\varepsilon) = y(1) + \varepsilon e$, where $e = (e_1, e_2, e_3)$ and $|e| = 1$, then

$$E_3(x(\varepsilon), y(1), P(1); 0) = \frac{3e_1e_3P_1(1) + 3e_2e_3P_2(1) + (2e_3^2 - e_1^2 - e_2^2)P_3(1)}{4\pi\varepsilon^3}.$$

Next, choose the vector e such that

$$\langle e_1, e_2, e_3 \rangle =$$

$$\begin{cases} \langle 1, 0, 0 \rangle & if \quad P_3(1) \neq 0 \\ \langle \frac{1}{\sqrt{2}}, 0, \frac{1}{\sqrt{2}} \rangle & if \quad P_3(1) = 0. \end{cases}$$

Taking $\varepsilon \to 0$ and choosing e such that $3e_1e_3P_1(1) + 3e_2e_3P_2(1) + (2e_3^2 - e_1^2 - e_2^2)P_3(1) \neq 0$ it follows from our definition of $E_3(x(\varepsilon), y(1), P(1); 0)$ that

$$E_3(x(\varepsilon), y(1), P(1); 0) =$$

$$\begin{cases} -\frac{1}{4\pi\varepsilon^3} P_3(1) & if \quad P_3(1) \neq 0 \quad and \quad \langle e_1, e_2, e_3 \rangle = \langle 1, 0, 0 \rangle \\ \frac{3}{8\pi\varepsilon^3} P_1(1) & if \quad P_3(1) = 0 \quad and \quad \langle e_1, e_2, e_3 \rangle = \langle \frac{1}{\sqrt{2}}, 0, \frac{1}{\sqrt{2}} \rangle, \end{cases}$$

and that $E_3(x(\varepsilon), y(1), P(1); 0) \to \infty$. Since $y(1) \neq y(2)$ then $E_3(x(\varepsilon), y(2), P(2); 0)$ remains bounded as $\varepsilon \to 0$. However, we know that $h(x, y(1), P(1)) = h(x, y(2), P(2))$ for $x \in \mathbb{R}^3 \setminus (\{y(1)\} \cup \{y(2)\})$ which implies $E(x, y(1), P(1); 0) = E(x, y(2), P(2); 0)$ for $x \in \mathbb{R}^3 \setminus (\{y(1)\} \cup \{y(2)\})$. Therefore, by contradiction we see that $y(1) = y(2)$.

Since $y(1) = y(2)$, we know that the denominators of $E(x, y(1), P(1); 0)$ and $E(x, y(2), P(2); 0)$ are equal. Let $x_1 = y_1(1)$, and recall that $y_1(1) = y_1(2)$ and that $E(x, y(1), P(1); 0) = E(x, y(2), P(2); 0)$ for $x \in \mathbb{R}^3 \setminus (\{y(1)\} \cup \{y(2)\})$. Moreover, from (9), we see that $A(x, y(1), P(1)) = A(x, y(2), P(2))$ for $x \in \mathbb{R}^3 \setminus (\{y(1)\} \cup \{y(2)\})$.

From (9), the first equations,

$$A_1(x, y(1), P(1)) = \frac{1}{4\pi} \left[\left(2(x_1 - y_1)^2 - (x_2 - y_2)^2 - (x_3 - y_3)^2 \right) P_1(1) + (3(x_1 - y_1)(x_2 - y_2)) P_2(1) + (3(x_1 - y_1)(x_3 - y_3)) P_3(1) \right],$$

$$A_1(x, y(2), P(2)) = \frac{1}{4\pi} \left[\left(2(x_1 - y_1)^2 - (x_2 - y_2)^2 - (x_3 - y_3)^2 \right) P_1(2) + (3(x_1 - y_1)(x_2 - y_2)) P_2(2) + (3(x_1 - y_1)(x_3 - y_3)) P_3(2) \right],$$

hence,

$$\left(-(x_2 - y_2(1))^2 - (x_3 - y_3(1))^2\right) P_1(1) = \left(-(x_2 - y_2(1))^2 - (x_3 - y_3(1))^2\right) P_1(2).$$

Rearranging yields,

$$\left(-(x_2 - y_2(1))^2 - (x_3 - y_3(1))^2\right) (P_1(1) - P_1(2)) = 0.$$

Choose a particular x_2, x_3 such that $\left(-(x_2 - y_2(1))^2 - (x_3 - y_3(1))^2\right) \neq 0$.

For ξ, $\eta > 0$, where $\xi \neq \eta$.

$$x_2(\xi) = y_2(1) + \xi,$$

$$x_3(\eta) = y_3(1) + \eta.$$

Making the appropriate substitutions,

$$-(y_2(1) + \xi - y_2(1))^2 - (y_3(1) + \eta - y_3(1))^2 = (-(\xi)^2 - (\eta)^2) = -\xi^2 - \eta^2 \neq 0.$$

Therefore, $P_1(1) = P_1(2)$.

Let $x_2 = y_2(1)$. From (9), the second equations,

$$A_2(x, y(1), P(1)) = \frac{1}{4\pi}\left[(3(x_1 - y_1)(x_2 - y_2))\, P_1(1) + \left(2(x_2 - y_2)^2 - (x_1 - y_1)^2 - (x_3 - y_3)^2\right) P_2(1) + (3(x_2 - y_2)(x_3 - y_3))\, P_3(1)\right],$$

$$A_2(x, y(2), P(2)) = \frac{1}{4\pi}\left[(3(x_1 - y_1)(x_2 - y_2))\, P_1(2) + \left(2(x_2 - y_2)^2 - (x_1 - y_1)^2 - (x_3 - y_3)^2\right) P_2(2) + (3(x_2 - y_2)(x_3 - y_3))\, P_3(2)\right],$$

hence,

$$\left(-(x_1 - y_1(1))^2 - (x_3 - y_3(1))^2\right) P_2(1) = \left(-(x_1 - y_1(1))^2 - (x_3 - y_3(1))^2\right) P_2(2).$$

By a similar argument, for a particular x_1, x_3 such that $(-(x_1 - y_1(1))^2 - (x_3 - y_3(1))^2) \neq 0$, for ξ, $\eta > 0$, where $\xi \neq \eta$.

$$x_1(\xi) = y_1(1) + \xi,$$

$$x_3(\eta) = y_3(1) + \eta,$$

we see that $P_2(1) = P_2(2)$.

Let $x_3 = y_3(1)$. From (9), the third equations,

$$A_3(x, y(1), P(1)) = \frac{1}{4\pi}\left[(3(x_1 - y_1)(x_3 - y_3))\, P_1(1) + (3(x_2 - y_2)(x_3 - y_3))\, P_2(1) + \left(2(x_3 - y_3)^2 - (x_1 - y_1)^2 - (x_2 - y_2)^2\right) P_3(1)\right],$$

$$A_3(x, y(2), P(2)) = \frac{1}{4\pi}\left[(3(x_1 - y_1)(x_3 - y_3))\, P_1(2) + (3(x_2 - y_2)(x_3 - y_3))\, P_2(2) + \left(2(x_3 - y_3)^2 - (x_1 - y_1)^2 - (x_2 - y_2)^2\right) P_3(2)\right],$$

hence,

$$\left(-(x_1 - y_1(1))^2 - (x_2 - y_2(1))^2\right) P_3(1) = \left(-(x_1 - y_1(1))^2 - (x_2 - y_2(1))^2\right) P_3(2).$$

By a similar argument, for a particular x_1, x_2 such that $(-(x_1 - y_1(1))^2 - (x_2 - y_2(1))^2) \neq 0$, for ξ, $\eta > 0$, where $\xi \neq \eta$.

$$x_1(\xi) = y_1(1) + \xi,$$

$$x_2(\eta) = y_2(1) + \eta,$$

we see that $P_3(1) = P_3(2)$.

Hence, $P(1) = P(2)$.

Numerical Experiment

Now we will consider a numerical experiment using data in accordance with Maxwell's equations in a homogeneous medium of infinite extent (Ammari et al. 2002). The material parameters include $\varepsilon = 8.854 \times 10^{-12}$ farads/meter, $\mu = 4\pi \times 10^{-7}$ henrys/meter, and $\sigma = .33$ mhos/meter. Consider the dipole current source

$$J(x, y, P) = \delta(x - y)P,$$

where y is the location vector and P is the orientation vector of the dipole. We will use the numerical experiment to determine the position and orientation of the dipole current within an artificial spherical conductor. From (8), we will use the complete electromagnetic field measured at the boundary.

Consider the functions $f(y, P) = E(x(1), y, P), g(y, P) = E(x(2), y, P), h(y, P) = E(x(3), y, P)$, where $x(j)$ are sensor postions. Using $f(y, P), g(y, P), h(y, P)$ to represent three distinct sensor positions on the boundary, we may compute the general Jacobian which will be a 9×6 matrix,

$$\mathbf{J}_{i,j} = \begin{bmatrix} \frac{\partial f_1}{\partial y_1} & \frac{\partial f_1}{\partial y_2} & \frac{\partial f_1}{\partial y_3} & \frac{\partial f_1}{\partial P_1} & \frac{\partial f_1}{\partial P_2} & \frac{\partial f_1}{\partial P_3} \\ \frac{\partial f_2}{\partial y_1} & \frac{\partial f_2}{\partial y_2} & \frac{\partial f_2}{\partial y_3} & \frac{\partial f_2}{\partial P_1} & \frac{\partial f_2}{\partial P_2} & \frac{\partial f_2}{\partial P_3} \\ \frac{\partial f_3}{\partial y_1} & \frac{\partial f_3}{\partial y_2} & \frac{\partial f_3}{\partial y_3} & \frac{\partial f_3}{\partial P_1} & \frac{\partial f_3}{\partial P_2} & \frac{\partial f_3}{\partial P_3} \\ \frac{\partial g_1}{\partial y_1} & \frac{\partial g_1}{\partial y_2} & \frac{\partial g_1}{\partial y_3} & \frac{\partial g_1}{\partial P_1} & \frac{\partial g_1}{\partial P_2} & \frac{\partial g_1}{\partial P_3} \\ \frac{\partial g_2}{\partial y_1} & \frac{\partial g_2}{\partial y_2} & \frac{\partial g_2}{\partial y_3} & \frac{\partial g_2}{\partial P_1} & \frac{\partial g_2}{\partial P_2} & \frac{\partial g_2}{\partial P_3} \\ \frac{\partial g_3}{\partial y_1} & \frac{\partial g_3}{\partial y_2} & \frac{\partial g_3}{\partial y_3} & \frac{\partial g_3}{\partial P_1} & \frac{\partial g_3}{\partial P_2} & \frac{\partial g_3}{\partial P_3} \\ \frac{\partial h_1}{\partial y_1} & \frac{\partial h_1}{\partial y_2} & \frac{\partial h_1}{\partial y_3} & \frac{\partial h_1}{\partial P_1} & \frac{\partial h_1}{\partial P_2} & \frac{\partial h_1}{\partial P_3} \\ \frac{\partial h_2}{\partial y_1} & \frac{\partial h_2}{\partial y_2} & \frac{\partial h_2}{\partial y_3} & \frac{\partial h_2}{\partial P_1} & \frac{\partial h_2}{\partial P_2} & \frac{\partial h_2}{\partial P_3} \\ \frac{\partial h_3}{\partial y_1} & \frac{\partial h_3}{\partial y_2} & \frac{\partial h_3}{\partial y_3} & \frac{\partial h_3}{\partial P_1} & \frac{\partial h_3}{\partial P_2} & \frac{\partial h_3}{\partial P_3} \end{bmatrix}.$$

The respective partials for the first row are as follows:

$$\frac{\partial f_1}{\partial y_1} = \left[\frac{5(x_1 - y_1)\left[(2(x_1 - y_1)^2 - (x_2 - y_2)^2 - (x_3 - y_3)^2)P_1 + 3(x_1 - y_1)(x_2 - y_2)P_2 + 3(x_1 - y_1)(x_3 - y_3)P_3\right]}{4\pi\sigma|x - y|^7} \right]$$

$$- \left[\frac{4(x_1 - y_1)P_1 + 3(x_2 - y_2)P_2 + 3(x_3 - y_3)P_3}{4\pi\sigma|x - y|^5} \right],$$

$$\frac{\partial f_1}{\partial y_2} = \left[\frac{5(x_2 - y_2)\left[(2(x_1 - y_1)^2 - (x_2 - y_2)^2 - (x_3 - y_3)^2)P_1 + 3(x_1 - y_1)(x_2 - y_2)P_2 + 3(x_1 - y_1)(x_3 - y_3)P_3\right]}{4\pi\sigma|x - y|^7}\right]$$

$$+ \left[\frac{2(x_2 - y_2)P_1 - 3(x_1 - y_1)P_2}{4\pi\sigma|x - y|^5}\right],$$

$$\frac{\partial f_1}{\partial y_3} = \left[\frac{5(x_3 - y_3)\left[(2(x_1 - y_1)^2 - (x_2 - y_2)^2 - (x_3 - y_3)^2)P_1 + 3(x_1 - y_1)(x_2 - y_2)P_2 + 3(x_1 - y_1)(x_3 - y_3)P_3\right]}{4\pi\sigma|x - y|^7}\right]$$

$$+ \left[\frac{2(x_3 - y_3)P_1 - 3(x_1 - y_1)P_3}{4\pi\sigma|x - y|^5}\right],$$

$$\frac{\partial f_1}{\partial P_1} = \frac{2(x_1 - y_1)^2 - (x_2 - y_2)^2 - (x_3 - y_3)^2}{4\pi\sigma|x - y|^5},$$

$$\frac{\partial f_1}{\partial P_2} = \frac{3(x_1 - y_1)(x_2 - y_2)}{4\pi\sigma|x - y|^5},$$

$$\frac{\partial f_1}{\partial P_3} = \frac{3(x_1 - y_1)(x_3 - y_3)}{4\pi\sigma|x - y|^5}.$$

Once we have constructed our Jacobian, we will apply Singular Value Decomposition.

SVD Applied to four distinct sensor positions,

$$f : x(1) = (1, 0, 0); y = (0.5, 0, 0); P = (1, 0, 0),$$

$$g : x(2) = (0, 1, 0); y = (0.5, 0, 0); P = (1, 0, 0),$$

$$h : x(3) = (0, 0, 1); y = (0.5, 0, 0); P = (1, 0, 0),$$

$$m : x(4) = (0, 0, -1); y = (0.5, 0, 0); P = (1, 0, 0),$$

generates the following matrix of singular values:

$$\mathbf{S}_4 = \begin{bmatrix} 7.7482 & 0 & 0 & 0 & 0 & 0 \\ 0 & 3.8821 & 0 & 0 & 0 & 0 \\ 0 & 0 & 3.8779 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0.1968 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0.1688 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0.1317 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}.$$

SVD Applied to three distinct sensor positions,

$$f : x(1) = (1, 0, 0); y = (0.5, 0, 0); P = (1, 0, 0),$$

$$g : x(2) = (0, 1, 0); y = (0.5, 0, 0); P = (1, 0, 0),$$

$$h : x(3) = (0, 0, 1); y = (0.5, 0, 0); P = (1, 0, 0),$$

generates the following matrix of singular values:

$$\mathbf{S}_3 = \begin{bmatrix} 7.7470 & 0 & 0 & 0 & 0 & 0 \\ 0 & 3.8775 & 0 & 0 & 0 & 0 \\ 0 & 0 & 3.8775 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0.1584 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0.1472 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0.0984 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}.$$

SVD Applied to two distinct sensor positions,

$$f : x(1) = (1, 0, 0); y = (0.5, 0, 0); P = (1, 0, 0),$$

$$g : x(2) = (-1, 0, 0); y = (0.5, 0, 0); P = (1, 0, 0),$$

generates the following matrix of singular values:

$$\mathbf{S}_2 = \begin{bmatrix} 7.7453 & 0 & 0 & 0 & 0 & 0 \\ 0 & 3.8726 & 0 & 0 & 0 & 0 \\ 0 & 0 & 3.8726 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0.0620 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0.0310 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0.0310 \end{bmatrix}.$$

Conclusion

In conclusion, we have formulated the approximate inverse problem by passing the formula for electric field $E(x, y, P)$ through the limit as $\omega\mu \to 0$ and $\omega\varepsilon \to 0$ which resulted in the approximate electric field $E(x, y, P; 0)$. Next, we determined a unique location y and orientation P for a dipole current using the $E(x, y, P; 0)$ thus solving the approximate inverse problem. Finally, we conducted a numerical experiment to determine position and orientation of the dipole current for up to four sensors at distinct electrode positions. Future work will address the tangential components and solve the complete nonlinear system for y, P. In addition, looking for two y, P as opposed to one and analyzing uniqueness from multiple sensors would add to the growing body of work on the EEG/MEG Inverse problem.

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