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TOPOLOGICAL CLASSIFICATION IN EARLY CHILDHOOD

Allan L. Edelson and Karen Sanders Huie
University of California at Davis

ABSTRACT

Mathematically precise definitions of various topological properties were given. To investigate Piaget's theory that topological relationships are mastered before certain Euclidean ones, 58 subjects of 3 to 7 years old were asked to divide 14 items depicting topological and Euclidean properties into groups of "things they felt were alike." Younger children used significantly stronger topological classification techniques while older ones used significantly stronger Euclidean classification techniques. The suggestion was made that the topological intuition of 3-year-olds may be superseded by strongly reinforced Euclidean notions.

INTRODUCTION

Considering the preoccupation with arithmetic and geometric concept development in young children, as displayed on educational television and in early childhood educational materials, it is appropriate to note that there are other mathematical capabilities exhibited by young children which preceed certain commonly studied mathematical considerations. For example, according to Piaget (1956; 1964), topological relationships are mastered long before Euclidean and projective geometry. Indeed, it generally is accepted that the earliest mathematical concepts to develop in children are related to shape, followed by numeric or arithmetic concepts. Nevertheless, little has been done to further explore or empirically affirm Piaget's statements and to determine the precise topological or geometric properties of interest to the young child. Work in this field occasionally uses vague and imprecise notions of mathematical concepts (see Kapadia (1974) for a criticism of Piaget's misuse of topological notions). Accurate definitions of these concepts and an investigation of their development in young children would aid early childhood educators in recognizing this development in young children. This recognition would facilitate communication between educator and child and minimize the unsubtle judgment that a child failing to put circles with circles and squares with squares is in error.

Those mathematical concepts of a geometric nature can be refined into those which are topological and those which are of a combinatorial or Euclidean nature. Euclidean properties are defined here to be those specific properties relating to similarity of polygonal shapes. Included are the associated combinatorial properties relating to numbers of sides and the obvious dissimilarity between shapes of curved lines and those of straight ones. Henceforth the term "geometric" will denote only those Euclidean geometric properties defined above. Several topological concepts which have not previously been studied in the context of early childhood development will be introduced. Since many of these ideas may be new to readers with non-mathematical backgrounds, the next section is devoted to a description of topological properties of objects and a comparison of those definitions with some of the more traditional, though occasionally imprecise, uses of topological vocabulary.

While Kapadia (1974) suggested that Piaget not only misused topological vocabulary, but that he also was wrong in his premise that young children display topological operations, the present study does support the view that some of the earliest developing mathematical processes are of a topological nature.

The purpose, then, is two fold: 1) to give simple, accurate definitions of the topological properties which are relevant to this investigation, and 2) to investigate the use of these properties by children, determining which are earliest to develop.

TOPOLOGICAL DIMENSION

A common source of confusion is the distinction between the dimension of an object and dimension of the space in which the object lies. Objects are assigned a dimension based on their intrinsic or local properties. A solid sphere, or ball, is three-dimensional because, having selected a reference point inside the sphere (notice that the reference point must be a point on a part of the object), three numbers are required to specify the location of any other point from the reference point. The hollow sphere is two-dimensional; having selected a reference point on its surface, two numbers suffice to locate other points. Dimension is then a local property, having to do with a small piece of the object, rather than a global property, having to do with the overall shape which the object takes, or the space in which it is located. In the same way, a rectangular strip of paper is two-dimensional, a fact which does not change if the two ends of the strip are glued together to form a cylinder, or given a half twist to form a Möbius strip.

PLANARITY

If the flat rectangular paper and the paper cylinder are both two-dimensional, we must find another description of the obvious distinction between the two — that the rectangle can be laid on a flat surface without causing it to fold over on itself, while this is not true for the cylinder. Objects such as the rectangle are said to be "planar." Examples of two-dimensional nonplanar objects include hollow objects and the Möbius strip. There also are nonplanar one-dimensional objects. If one takes a string, ties a loose overhand knot and fastens the ends together, the resulting object is still one-dimensional, but it cannot be flattened out to lie in a plane without folding over on itself. This distinction between dimension and planarity often is not made by laymen. Frequently two-dimensional objects are only those which are planar, while three-dimensional ones are those which are nonplanar.

CONNECTIVITY

There is another topological property which distinguishes the flat rectangle from the cylinder or the Möbius strip. It is possible to shade the inside of a circle drawn anywhere on the rectangle. This property is not shared by all circles drawn on the cylinder. For example, any circle drawn on the cylinder which goes around the hole does not specify an inside or outside. Objects, like the rectangle, on or in which any circle drawn can be shaded in, are called "simply connected," while objects like the cylinder, on which we can draw circles that cannot be shaded in, are said to be "not simply connected." Other examples of non-simply connected objects are the doughnut shape, or torus, and a string tied together at the ends. In simplest terms, these non-simply connected objects have holes going through them.

CONTRACTIBILITY

There is another property which distinguishes the hollow sphere from the solid one, or the rectangular piece of paper from the cylinder. If one imagines these objects to be made of soft, infinitely bendable, yet not breakable or tearable, rubber, then the solid sphere could be squashed down into a single point. This is not true of the hollow sphere, which always must retain the hole (space) inside, and therefore cannot be squashed down to a point. Similarly the rectangle can be so squashed, or "contracted," while the cylinder always must retain its hole. Objects like the solid sphere which can be squashed down to a point are called "contractible." Other non-contractible objects are both hollow (two-dimensional) and solid (three-dimensional) tori. All non-simply connected objects are also non-contractible.

CLOSURE

The final topological property under consideration here can be illustrated by running a finger around one of the circular edges of a cylinder. There are two such edges on a cylinder, each being one-dimensional, as compared to the cylinder itself, which is two-dimensional. It is a somewhat surprising observation that the edge or boundary of a Möbius strip consists of a single circle. The boundary of a solid sphere consists of a hollow sphere, while the hollow sphere has no boundary in mathematical terms at all. Objects having no boundary are called "closed," and other examples are the hollow torus and the circle. The rectangle is not closed, as its boundary consists of its four edges, which make up a figure topologically the same as a circle, if one recalls that these edges could be bent or stretched into the shape of a circle.

The term "closed" has another mathematical meaning unrelated to our study. A curve drawn so as to come back on itself, e.g., a circle, is said to be a closed curve. Conversely, a curve which begins at one point and ends at another, without crossing over itself, e.g., a straight line segment, is said to be open. Sauvy and Sauvy (1974, p. 23) use the term "closure" in this sense, but they appear to use it also to denote both connectivity (1974, p. 22) and concave-convex distinctions (1974, p. 24).

The only mathematical relationships which exist between the various topological properties are that if an object is planar, it cannot be three-dimensional, and if an object is contractible, it must be simply connected. However, one can have a simply connected object, such as a hollow sphere, which is not contractible. Otherwise, any relationship between these topological properties has its origins in our perceptual and physical relationships with our environment, rather than in any strict mathematical hierarchy.

Now that the topological phenomena under consideration in this study have been defined, the hypothesis is presented that younger children will exhibit stronger topological classification of objects than older children. We also are interested in determining which topological classifications are more likely to be used by children of different ages and perhaps to delineate the order of onset of these various classifying techniques.

METHOD

SUBJECTS

The subjects for this study were 30 boys and 28 girls distributed fairly evenly in the following age groups: 3 years, 4 years, 5 years, and 6-7 years. All subjects were white and from middle class homes.

MATERIALS

The 14 objects developed for the study exhibited both the various topological and Euclidean geometric properties delineated earlier. They included straight, circular, and loosely knotted strings; flat paper shapes of triangle, rectangle, and an irregular curve; solid and hollow spheres; a solid tetrahedron (pyramid) and a hollow tetrahedron with a hole in it; a solid block; and 3 strips in the forms of a cylinder, Möbius strip, and a knotted cylinder. Although of different materials, all objects were the same color. All were quite light in weight except the solid tetrahedron which was markedly heavier than the rest of the items.

PROCEDURE

The children were tested individually in their own homes. All 14 items were placed before the subjects in a standard display for each of 4 trials. The subjects were told first to make as many groups as they wished with the items, putting things together that seemed "like" other items — things that seemed to "belong" together. Older children were told to disregard the material of which the objects were made, but rather to concentrate on their "other characteristics." For the second trial the subjects were told to make only 3 groups. This instruction was to see if we could force a choice between Euclidean and topological dimension which were the only three-group classifications. For the third trial, all strings and planar shapes were removed, leaving 8 items, and the subjects again were told to make as many groups as they wanted. For the fourth trial they were asked to make only 2 groups, again to force a distinction between dimension and polygonal groupings.

SCORING AND STATISTICAL ANALYSIS

The various objects used in the experiment were grouped according to each of the mathematical properties under consideration. Each grouping resulted in a matrix, allowing for comparison with the subject's groupings. For example, if the 8 items used in trials 3 and 4 are numbered and grouped according to topological dimension, the following groups are obtained:

[solid ball (1), solid tetrahedron (2), solid block (3)]

[cylinder (4), Möbius strip (5), knotted strip (6), hollow ball (7), hollow tetrahedron (8)]

The model matrix representing this grouping and others is depicted in figure 1.

Topological Classification

	1	2	3	4	5	6	7	8
1		x	x	0	0	0	0	0
2			x	0	0	0	0	0
3				0	0	0	0	0
4					x	x	x	x
5						x	x	x
6							x	x
7								x
8								

a) Topological dimension

(trials 3 and 4)

	1	2	3	4	5	6	7	8
1		0	0	0	0	0	x	0
2			0	0	0	0	0	x
3				0	0	0	0	0
4					x	x	0	0
5						x	0	0
6							0	0
7								0
8								

b) Euclidean

(trials 3 and 4)

	1	2	3	4	5	6	7	8	9	10	11	12	13	14
1		x	x	x	x	x	x	x	0	0	x	0	0	0
2			x	x	x	x	x	x	0	0	x	0	0	0
3				x	x	x	x	x	0	0	x	0	0	0
4					x	x	x	x	0	0	x	0	0	0
5						x	x	x	0	0	x	0	0	0
6							x	x	0	0	x	0	0	0
7								x	0	0	x	0	0	0
8									0	0	x	0	0	0
9										x	0	x	x	x
10											0	x	x	x
11												0	0	0
12													x	x
13														x
14														

c) Planarity

(trials 1 and 2)

	1	2	3	4	5	6	7	8	9	10	11	12	13	14
1		x	x	x	0	0	x	x	x	0	0	x	x	x
2			x	x	0	0	x	x	x	0	0	x	x	x
3				x	0	0	x	x	x	0	0	x	x	x
4					0	0	x	x	x	0	0	x	x	x
5						x	0	0	0	x	x	0	0	0
6							0	0	0	x	x	0	0	0
7								x	x	0	0	x	x	x
8									x	0	0	x	x	x
9										0	0	x	x	x
10											x	0	0	0
11												0	0	0
12													x	x
13														x
14														

d) Connectivity

1. solid ball
2. solid tetrahedron
3. solid cube
4. cylinder
5. Möbius strip
6. Knotted strip
7. hollow ball, or sphere
8. hollow tetrahedron
9. straight string
10. circular string
11. knotted string
12. paper triangle
13. paper rectangle
14. irregular paper curve

FIGURE 1

Model matrices representing grouping according to several mathematical properties

x designates objects grouped

0 designates objects not grouped

The second and fourth trials were compared only with those models that were appropriate in having either 3 or 2 groups respectively. Each subject's grouping of the objects was scored for each of the mathematical properties by counting the number of mismatches between the subject's matrix and the model matrix representing that property. Higher scores, then, showed a greater disparity between the subject's grouping and the model groupings, and a perfect match with any model, a condition which did in fact occur, resulted in a score of zero.

The scores were grouped according to age. The data were analyzed by sex because there is evidence of sex differences in mathematical functioning for older children (Garai and Scheinfeld, 1968). Each of the mathematical properties being tested was scored separately, so that multivariate analysis was rejected in favor of univariate. Using one-tailed t tests, the mean scores of various age groups were compared both with each other, and with the entire remaining group of subjects of the same sex to determine whether different mathematical properties appeared at different ages. Only those comparisons which were statistically significant will be considered here.

Comparisons of the models with each other reflected their non-correlation with the following exceptions: the Euclidean grouping (curved, triangular, and rectangular groups) obviously correlated with the polygonal grouping (curved vs. straight-edged items), and for the third and fourth trials, dimension and contractibility were correlated perfectly for our particular choice of items.

RESULTS

The expectation that the younger subjects would display stronger topological classification than the older subjects was borne out (See Table 1). The process was most marked in the 3-year-old girls. On both the first and second trials, their arrays displayed significantly more topological dimension than the rest of the girls. This trend was displayed even more strongly on the first trial by the younger (3 - 3½ year) subjects. The 3-year-old girls also displayed classification by closure. There was a significant trend on both the first and second trials for these girls *not* to display Euclidean classification. The rest of the comparisons were non-significant.

TABLE 1

Comparison of Mean Scores by Age - Girls

FIRST TRIAL

AGE	$3\text{-}3\frac{1}{2}$(N=3)	*(N=25)	3(N=9)	*(N=19)	6-7(N=9)	*(N=19)
DIMENSION	25,00 (c)	35.80	30.33 (b)	36.68	39.22 (b)	32.47
CLOSURE	--	--	44.89 (a)	49.47	50.89 (a)	46.63
EUCLIDEAN	--	--	35.67 (b)	27.00	22.78 (c)	33.11
PLANARITY	--	--	--	--	41.89 (a)	37.84

SECOND TRIAL

AGE	3(N=9)	*(N=19)	6-7(N=9	*(N=19)
DIMENSION	30.67 (d)	39.78	--	--
EUCLIDEAN	35.11 (a)	25.39	17.22 (e)	34.33

THIRD TRIAL

AGE	*(N=12)	3(N=9)	4(N=7)	6-7(N=9)	*(N=19)
CLOSURE	16.44 (e)	12.44 (e)	17.00	--	--
PLANARITY	20.33 (e)	8.00 (c)	21.00	--	--
EUCLIDEAN	7.22 (e)	14.44 (d)	7.33	6.56 (b)	11.17
POLYGONAL	7.89 (d)	13.22 (b)	8.33	--	--
CONNECTIVITY	10.89 (d)	16.11 (a)	10.67	--	--

FOURTH TRIAL

AGE	3(N=9)	*(N=19)	6-7(N=9)	*(N=19)
POLYGONAL	12 (d)	5.44	2.33 (e)	10.28

(a): $p < .05$
(b): $p < .025$
(c): $p < .01$
(d): $p < .005$
(e): $p < .0005$

(*) denotes all remaining subjects
all deleted scores were statistically insignificant

On the third trial the 3-year-old girls no longer displayed dimension. The task here was to divide two- from three-dimensional items, all of which were nonplanar, and the children did not make that distinction. They did, however, show strong classification by closure and planarity when compared with the rest of the girls.

The scores again indicated that the 3-year-old girls were not operating with Euclidean or polygonal classifications, nor were they using connectivity. The fourth trial added no new information, repeating a significant trend for nonuse of polygonal classification.

The 4-year-old girls showed no significant differences when compared with the rest of the girls. When compared only with the 3-year-olds, they showed on the third trial that they had moved into using Euclidean, polygonal, and connectivity classifications. They showed also that they definitely had moved away from closure and planar considerations.

Five-year-old girls showed no differences when compared with the rest of the girls. By 6 - 7 years, the girls were displaying strong Euclidean and/or polygonal classifications on all 4 trials. There also were significant trends on the first trial for the older girls *not* to use dimension, planarity, or closure.

It appears from these results that dimension, closure, and planarity occur the earliest in the girls, with connectivity following at about the time that

TABLE 2

Comparison of Mean Scores by Age - Boys

FIRST TRIAL

AGE	3-3½(N=3)	3½-4(N=3)	4-7(N=24)	3(N=6)	4(N=6)	5½-7(N=10)	*(N=20)
DIMENSION	36.33 (c)	52	38.13 (a)	44.17	--	--	--
EUCLIDEAN	--	--	27.83 (d)	46.50 (b)	30.17	22.40 (c)	36.15
PLANARITY	--	--	40.92 (c)	46.50 (c)	39.50	--	--
CONNECTIVITY	--	--	37.00 (b)	46.17 (a)	35.50	--	--
POLYGONAL	--	--	--	46.50 (a)	37.50	--	--

SECOND TRIAL

AGE	3-3½(N=3)	*(N=27)	3(N=6)	*(N=24)	5½-7(N=10)	*(N=20)
DIMENSION	28.50 (d)	41.44	35.00 (b)	41.71	--	--
EUCLIDEAN	--	--	--	--	19.40 (d)	34.26

THIRD TRIAL

AGE	3½-4(N=3)	*(N=27)	3(N=6)	*(N=24)	5½-7(N=10)	*(N=20)
CLOSURE	--	--	12.8 (a)	14.83	--	--
PLANARITY	2.67 (b)	15.54	9.60	15.17	--	--
EUCLIDEAN	--	--	--	--	8.80	13.11

FOURTH TRIAL

AGE	3(N=6)	*(N=24)	5½-7(N=10)	*(N=20)
POLYGONAL	13.40 (a)	9.08	6.50 (d)	11.58

(a): $p < .05$
(b): $p < .025$
(c): $p < .01$
(d): $p < .005$

(*) denotes all remaining subjects
all deleted scores were statistically insignificant

geometric shape makes its appearance. Contractibility scores were uniformly high for all age groups, so that it never seemed to be exhibited.

The classification processes for the boys were not quite so clearly age-delineated as for the girls (See Table 2). The 3-year-old boys seemed to be a study in randomness, with several significant results on the first trial for what they were *not* doing. They were not classifying by dimension, Euclidean, planarity, or connectivity when compared with the rest of the boys. Dimension did appear on the second trial, particularly for the younger (3 - 3½-year-old) boys, duplicating the girls' results. When the 3 - 3½-year-old boys were compared with the older 3-year-old boys, they showed more dimension classification on the first trial also.

On the third trial, the 3-year-old boys, like the 3-year-old girls, no longer displayed dimension. They did classify by closure, however, and they showed only a tendency towards planarity. Planarity was more marked in the older (3½ - 4-year-old) subjects. All other comparisons were non-significant. Duplicating the girls' results, the boys' fourth trial simply showed nonuse of polygonal classification.

The 4-year-old boys, like the 4-year-old girls, showed no significant differences when compared with the rest of the boys. When compared with the 3-year-old boys, the 4-year-old boys showed on the first trial that they had moved into using Euclidean, polygonal, connectivity, and more planarity classifications. Unlike the girls, there were no significant differences in closure.

From the age of 5½ on, boys displayed strong and/or polygonal classifications on all 4 trials. They did not show, however, the strong moves away from dimension and planarity that the girls did, and their move away from closure only approached significance. The boys' results paralleled those for the girls in suggesting that dimension and closure occur the earliest, with connectivity and geometric considerations occuring later. In general, any sex differences seem to be in quantity rather than kind, with the girls' trends being stronger. The order of appearance of the various phenomena seemed to be the same for both sexes, as was generally the age of onset. There did appear to be sex differences in the disappearances of topological considerations, which occurred more completely for the girls.

DISCUSSION

It is clear that among the mathematical operations tested in our study, those developing earliest in children are of a topological nature, with awareness of the specific Euclidean properties tested here arising later. The present study supports Piaget's contention and provides more precise mathematical characterization of the properties involved than Piaget (1956; 1964) used in his observations. The authors here do make the distinction, however, that they were investigating only mathematical operations, i.e., how the children performed, and not conceptualization. Indeed, when asked why they made various groupings, 3-year-old subjects could answer only with "because they just are (alike)," and other typical 3-year-old responses.

Still, one can conclude from the findings in both boys and girls that dimension, closure, and planarity are among the earliest of the topological properties which children use to categorize objects. They are followed by connectivity and then Euclidean geometric phenomena. The finding that children clearly were moving into Euclidean and polygonal classification as early as 4 years of age contradicts Sauvy and Sauvy's (1974) suggestion that children don't distinguish straight from curved lines until 5 - 6 years of age.

The fact that 3-year-old children display a mathematically sophisticated method in their classification techniques bears recognition. Topological operations seem to appear spontaneously in very young children; yet this capability is neither recognized nor reinforced in standard introductory lessons or materials on mathematical shapes and phenomena. As a consequence, the present study shows that by the age of 6 years this topological intuition begins to disappear, giving way to the strongly reinforced Euclidean operations. While the authors are not advocating a disregard for Euclidean geometric concepts, which indeed are primary considerations for so many other areas of cognitive development, they do reiterate that topology, also, is a form of geometric thought. To encourage the fullest development of mathematical capacities, Euclidean concepts should be taught so as to broaden and expand those processes which seem to occur spontaneously, rather than entirely to replace them. In this way we can increase the child's geometric insights into the world of nature ". . . for geometry, you know, is the gate of science, and the gate is so low and small that one can only enter it as a child." — William Kingdon Clifford.

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