Multivariate Experimental Clinical Research, Volume 3, Number 2, 1977, 69-73

FACTORIAL VALIDITY BY TAUTOLOGY: METHODOLOGICAL COMMENTS ON STEWART AND STEWART[1]

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ABSTRACT

A recent attempt to assess the factorial validity of the Eight State Questionnaire used an unusual methodological procedure that is tautological in nature. Retention and rotation of all principal components to a trivial simple structure, in which each component is marked by one and only one scale, is guaranteed for any nonsingular correlation matrix. The certain existence of the transformation to this trivial structure precludes the use of its existence as scientific evidence of factorial validity. A verbal, geometric argument and a formal mathematical proof support the contention that this unusual methodology is tautological, and hence should be avoided in future research.

INTRODUCTION

Stewart and Stewart (1976) recently attempted to provide evidence supporting the factorial validity of Curran and Cattell's (1975) Eight State Questionnaire (8SQ), a recently developed scale for the assessment of emotional states and moods. In their effort, Stewart and Stewart used the admittedly unconventional procedure of retaining and rotating all eight components[2] from the 8SQ correlation matrix. Apparently, the Stewarts sought an ideal, yet trivial simple structure in which each component is marked by one and only one observed scale. At first glance, this type of simple structure, which can be characterized by a diagonal matrix, may have immense intuitive appeal. However, formal analysis leads to the conclusion that this trivial simple structure is tautologically guaranteed. Because this tautological guarantee may not be immediately obvious, and because it should be avoided in future research on factorial validity, the present paper briefly reviews the Stewarts' methodology and then formally exposes the nature of the tautology.

In their study, a 20-year-old subject completed the Eight State Questionnaire twice daily for a period of 30 days. The eight scales of the 8SQ were correlated across occasions. A principal components analysis was performed and all eight components were retained. The rationale for retaining all eight components was that previous research (Barton, Cattell, and Conner, 1972; Curran and Cattell, 1975) warranted the existence of eight factors. To validate these previous results, Stewart and Stewart apparently sought an ideal, yet trivial, simple structure form[3], for the components weight matrix, in which every component is marked by a single unique 8SQ scale. To attain this ideal structure, a promax ($k = 4$) rotation was performed. The desired structure was not obtained, but Stewart and Stewart thought the solution sufficiently confirmed the factorial validity of the 8SQ.

THE NATURE OF THE TAUTOLOGY

The retention and rotation of all the principal components of a correlation matrix to verify the factorial validity of any set of scales is improper, tautological methodology. Retention of all principal components merely results in a representation of the original test space in terms of a different set of axes. In other words, the test space is rotated to a principal axes orientation. All the information about the relationships among the original test vectors is unchanged. Only the reference system, within which this information is presented, has been changed. The 8SQ test vectors are the axes of the original reference system in the eight dimensional test space. The principal components define the new reference system in this same space.

Since the original test space is characterized by an ideal, yet trivial, simple structure, i.e., each scale represents itself, then it should be possible to transform the principal components structure into a trivial simple structure. In other words, it should be straightforward to rotate the test space back to its original orientation. In the derivations that follow, the tautological nature of this transformation is formally presented. All matrices utilized below are full rank square matrices of order n, where n is the number of original scales. For those not sufficiently comfortable with matrix algebra and the properties of eigensolutions, an introductory matrix algebra book, such as Hohn (1972), is a suggested companion.

A PROOF OF THE TAUTOLOGY

Any nonsingular, correlation matrix $\mathbf{R}$ can always be expressed as,

$$\mathbf{R} = \mathbf{V}\mathbf{D}^2\mathbf{V}' \tag{1}$$

where $\mathbf{D}^2$ is a positive definite diagonal matrix of eigenvalues and $\mathbf{V}$ is a matrix of corresponding eigenvectors. Note the equality in Equation 1: The relationship is exact. The eigensolution in Equation 1 is characterized by the row and columnwise orthogonality[4] of $\mathbf{V}$, i.e.,

$$\mathbf{V}'\mathbf{V} = \mathbf{V}\mathbf{V}' = \mathbf{I}\,. \tag{2}$$

Alternatively, this orthogonality can be expressed as $\mathbf{V}' = \mathbf{V}^{-1}$ and $\mathbf{V} = (\mathbf{V}')^{-1}$.

In practice, an unrotated components matrix is defined as

$$\mathbf{A} = \mathbf{V}\mathbf{D}\,, \tag{3}$$

such that the correlation matrix is represented in terms of orthogonal components with unit variance, i.e.,

$$\mathbf{R} = \mathbf{ACA}', \tag{4}$$

where **C**, the correlation matrix among the components, is an identity matrix.

A square, nonsingular transformation matrix **T** is sought to rotate **A** to a component weight matrix **B**,

$$\mathbf{B} = \mathbf{AT}, \tag{5}$$

which has an ideal, yet trivial simple structure form, that is, **B** = **I**. Examination of Equation 3 and a knowledge of the orthogonality properties of **V** suggests a direct solution,

$$\mathbf{T} = \mathbf{D}^{-1}\mathbf{V}'. \tag{6}$$

To verify that **T** is the desired transformation matrix, substitute Equations 6 and 3 into Equation 5 to get

$$\mathbf{B} = \mathbf{VDD}^{-1}\mathbf{V}', \tag{7}$$

which reduces to

$$\mathbf{B} = \mathbf{VV}'. \tag{8}$$

However, from Equation 2 we immediately recognize that the desired result has been obtained,

$$\mathbf{B} = \mathbf{I}. \tag{9}$$

In other words, Equation 6 defines the transformation matrix **T** that can be applied to **A** to obtain rotated components that are characterized by a trivial simple structure form, **B** = **I**.

Note that **T** will not alter the scale of the components. The component variances remained unchanged under this rotation. In fact, the intercorrelation matrix among the rotated components has a recognizable form. In order to maintain the equality in Equation 4, when **T** is applied to **A**, an inverse transformation is applied to the component scores, such that,

$$\mathbf{R} = \mathbf{AT}(\mathbf{T}^{-1}\mathbf{CT}'^{-1})\mathbf{T}'\mathbf{A}'. \tag{10}$$

By defining the intercorrelation matrix among rotated components as

$$\tilde{\mathbf{C}} = (\mathbf{T}^{-1}\mathbf{CT}'^{-1}) \tag{11}$$

and substituting it, with the expression for **B**, into Equation 10, we obtain the expected result,

$$\mathbf{R} = \mathbf{B}\tilde{\mathbf{C}}\mathbf{B}'. \tag{12}$$

Of interest is the matrix $\tilde{\mathbf{C}}$. Making use of the relationship in Equation 6 and the fact that the original **C** was an identity matrix, Equation 11 becomes

$$\tilde{\mathbf{C}} = (\mathbf{D}^{-1}\mathbf{V}')^{-1}\,(\mathbf{V}\mathbf{D}^{-1})^{-1} \tag{13}$$

which reduces to

$$\tilde{\mathbf{C}} = \mathbf{VDDV}', \tag{14}$$

or equivalently,

$$\tilde{\mathbf{C}} = \mathbf{R}\,. \tag{15}$$

In other words, the correlations among the rotated components are identical to the correlations among the observed scales.

The equality in Equation 15 reflects the fact that the rotated components are the original observed scales. The transformation **T** is that rotation which returns the orientation of the reference axes back to their original location in the test space. It is merely an inverse transformation that converts the presentation of original test space information from a principal components form back to its original form. Using a computerized analytic rotation, such as promax, to find this transformation, is somewhat analogous to using a computerized least squares regression program to find the regression equation that converts standardized scores back to raw scores. The existences of both the standard score to raw score transformation, and the rotation from the principal axes orientation back to the original test space orientation, are tautologically guaranteed. Thus, the existence of either transformation cannot be used as empirical proof of any hypothesis. In particular, the fact that principal components can be rotated to a trivial simple structure form, the original test vector orientation, does not constitute support for the factorial validity of any set of scales.

In sum, Equations 1—9 prove that whenever all principal components of a nonsingular correlation matrix are retained, it is always possible to rotate to an ideal, but trivial simple structure. In addition, Equations 10—15 verify the contention that each rotated component is actually one of the original scales. This equality is ensured by the tautological trap that Stewart and Stewart fell into in their well-intentioned effort to assess the factorial validity of the 8SQ. The most succinct mathematical expression of this tautology is obtained by substituting Equations 9 and 15 into Equation 12,

$$\mathbf{R} = \mathbf{IRI}\,. \tag{16}$$

This equation formally emphasizes the inevitable tautological conclusion: Each rotated component is one of the original tests. Hopefully, the comments and derivations presented above will minimize the recurrence of this tautology in future research aimed at the valuable and necessary goal of assessing the factorial validity of established or experimental factorial scales.

NOTES

[1]The author would like to thank Drs. Carl Finkbeiner and Samuel E. Krug for their useful comments and suggestions. Requests for reprints should be sent to Neil Dorans, Test Services Division, Institute for Personality and Ability Testing, Champaign, Illinois 61820.

[2]Although performing a components analysis, Stewart and Stewart referred to the dimensions they obtained as factors. The confusion of components analysis with factor analysis appears frequently in the applied multivariate behavioral literature. The common factor analysis model and the principal components model are fundamentally different. Unlike the components model, which expresses dimensions as linear combinations of the observed variables, the common factor model (Thurstone, 1947) postulates that the observed variables are linear combinations of common, specific and error factors. Consequently, the breakdown of observed variance into common, specific and error variance is a basic aspect of the common factor model. No corresponding breakdown exists in the principal components model. To avoid further propagation of the existing confusion, I have decided to call a component a component. For the reader interested in a more complete distinction between the two models, relevant sections of Thurstone (1947), Harman (1967), Mulaik (1972) and Lawley and Maxwell (1971) are suggested.

[3]The phrase, *ideal, yet trivial, simple structure,* is used repeatedly for a particular reason. A diagonal matrix is ideal because it satisfies four of Thurstone's five simple structure criteria (Thurstone, 1947). However, it clearly fails to meet Thurstone's third criterion. A diagonal matrix can never satisfy the requirement that for every pair of columns of the matrix there should be *several* tests whose entries vanish in one column, but not in the other column. This criterion states the need for an overdetermined solution. A diagonal components matrix represents a barely determined solution since the number of model parameters equals the number of available data points. For n observed tests and r retained components, the number of parameters of the component model is $nr - r(r-1)/2$, the number of independent component loadings (Thurstone, 1947; Mulaik, 1972). The quantity of data points equals the number of independent elements of the original correlation matrix, $n(n+1)/2$. In order to have a determinate solution, the number of independent correlations (data points) should be equal to or greater than the number of independent component loadings (model parameters), i.e., $n(n+1)/2 \geqslant nr - r(r-1)/2$. When all components are retained, $r = n$, then the equality holds, indicating a barely determined and hardly convincing solution. In order to obtain a diagonal simple structure form, all components are retained, and a barely determined solution is all that is possible. Hence, the word trivial is used to describe a simple structure form, that is indicative of a solution in which the number of data points equals the number of parameters of the model for the data.

[4]The orthogonality of **V** should not be confused with the orthogonality, or uncorrelatedness, of the component scores, which is reflected by the diagonality of $\mathbf{D}^2$.

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