ON THE KOLMOGOROV-ARNOLD REPRESENTATION THEOREM FOR CONTINUOUS FUNCTIONS

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The following faculty members have examined the final copy of this thesis for form and content, and recommend that it be accepted in partial fulfillment of the requirement for the degree of Master of Science with a major in Mathematics.

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DEDICATION

This thesis is dedicated to my advisor, Buma Fridman, for his dedication to the completion of my Master's thesis and to Phil Parker for the immeasurable support he gave me as an undergrad.

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ABSTRACT

In 1900 at the International Congress of Mathematicians in Paris, D. Hilbert posed 23 questions that later became known as Hilbert's 23 problems. Number 13 remained unresolved for over half a century until 1956 and 1957 when A. N. Kolmogorov and his student V. I. Arnold, in a series of three papers, provided the solution. In this paper, I present Hilbert's 13^{th} problem as well as give my interpretation of Kolmogorov's solution to this.

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1 Introduction

At the turn of the 20^{th} century on August 8^{th}, at the International Congress of Mathematicians in Paris, France, a young, German mathematician by the name of David Hilbert listed twenty-three problems. These problems, it was believed, would help mold mathematics for the next century. Hilbert's 13^{th} stems from the solution of polynomial equations. The equation $a_n x^n + \cdots + a_1 x + a_o = 0$ produces a multi-valued, multi-variate, complex algebraic function $x = x(a_n, \cdots, a_o)$ of $(n+1)$-variables. When $n \leq 4$ there are explicit formulas for these functions and they are compositions of a few arithmetic operations and roots. For example, for $n = 2$, $ax^2 + bx + c = 0$ yields $x(a, b, c) = \dfrac{-b \pm \sqrt{b^2 - 4ac}}{2a}$. The works of P. Ruffini, N. H. Abel, and E. Galois show that it is not possible to present a similar solution for $n \geq 5$. For $n = 5, 6$, one can present the solution by using arithmetic operations, roots, and a specific algebraic function of one variable for $n = 5$ or two variables for $n = 6$. For $n = 7$, the solution can be reduced to arithmetic operations, roots, and the following algebraic function of three variables: $x^7 + ax^3 + bx^2 + cx + 1 = 0$. Hilbert asked if this particular function can, locally, be a composition of functions of two variables. He posed his question the following way: Can this function be a composition of continuous functions of no more than two variables? He conjectured that it is not possible to find such a solution.

We generalize this question: Can any function $f(x_1, x_2, \cdots, x_n)$ of $n \geq 3$ variables be written as a composition of functions of no more than two variables? For precision, "a function of n variables" refers to a real function $f : E^n \to \mathbb{R}$ where $E^n = \{x \in \mathbb{R}^n \mid 0 \leq x_i \leq 1, i = 1, 2, \cdots, n\}$. If there are no restrictions placed on the class of functions then it is a simple exercise to show the answer is "yes". If in that question we consider continuous functions the answer remains affirmative and this will solve Hilbert's 13^{th} problem.

In section one of this paper, I prove the following theorem:

Theorem 1. *Any function F_n of n-variables, $n \geq 3$ can be written as a composition of* *functions of no more than two variables.*

In 1956, A. N. Kolmogorov showed that any continuous function of several variables can be constructed by a finite number of three-variable continuous functions. The following year, Hilbert's conjecture for continuous functions was disproven by the work of V. I. Arnold who proved that functions of less than three variables could be used. Shortly thereafter, Kolmogorov simplified the work of Arnold. The second section of this paper is my interpretation of Kolmogorov's proof,

Theorem 2 (A.N. Kolmogorov)**.** *For every integer $n \geq 2$ there exist continuous real functions $\varphi^q(x)$, defined on E^1, such that every continuous real function $f(x_1, x_2, \cdots, x_n)$, defined on E^n, is representable in the form*

$$f(x_1, x_2, \cdots, x_n) = \sum_{q=1}^{2n+1} \chi^q \left[\sum_{p=1}^{n} \varphi^q(x_p) \right]$$

where $\chi^q(t)$ are continuous functions of one variable.

A. G. Vitushkin and G. M. Henkin later proved that these inner functions, $\varphi^q(x_p)$ can not be made to be C^1 and in 1967, B. Fridman proved that a Lipschitz condition can be imposed on $\varphi^q(x_p)$.

A natural question to ask is whether Hilbert's function of three variables can be represented as a composition of *algebraic* functions of no more than two variables. The answer to this question is still unresolved.

2 General Functions

Consider a function of n variables $F_n : E^n \to \mathbb{R}$. In this section we will prove theorem 1.

Before proving this theorem, we must make some additional constructions. To do this, first consider the following lemma:

Lemma 1 (Schröder-Bernstein Theorem)**.** *Given sets A and B, if there exists injective maps $f : A \to B$ and $g : B \to A$ then there exists a bijection $h : A \to B$.*

This lemma allows us to prove an important base case:

Theorem 3. *There is a bijection $f : E^1 \to E^2$ and $g : E^2 \to E^3$.*

Proof. To begin, consider a function $f_1 : E^1 \to E^2$ such that $a \mapsto (a, 0)$ and another function $f_2 : E^2 \to E^1$ such that $(0.a_1a_2a_3\cdots, 0.b_1b_2b_3\cdots) \mapsto 0.a_1b_1a_2b_2a_3b_3\cdots$. In general, decimal representations for rational numbers are not unique. For example, $.02 = 0.01999\cdots$. For the purpose of this proof, we will choose the decimal representation of every rational number to be such that there are not infinitely many trailing zeros with the only exception being $0 = 0.000\cdots$. I.e. for 1, we use $0.999\cdots$ and we write $0.01999\cdots$ for 0.02. Now that this uniqueness has been established, both maps are injective and therefore, by Lemma 1, there is a bijection $f : E^1 \to E^2$. Next, consider a function $g_1 : E^2 \to E^3$ such that $(a, b) \mapsto (a, b, 0)$ and another function $g_2 : E^3 \to E^2$ such that

$$(0.a_1a_2a_3\cdots, 0.b_1b_2b_3\cdots, 0.c_1c_2c_3c_4\cdots) \mapsto (0.a_1c_1a_2c_3a_3\cdots, 0.b_1c_2b_2c_4b_3\cdots).$$

Both of these maps are injective and, again, by Lemma 1, there is a bijection $g : E^2 \to E^3$. $\square$

Define φ to be just such a bijective map from E^2 to E^1. With this base case in mind, it is a matter of extending the proof to the n-dimensional case to obtain:

Lemma 2. *There exists a bijection $\psi_n : E^n \to E^{n-1}$.*

Now that this is established, we may define $\psi_n : E^n \to E^{n-1}$ to be $\psi_n(x_1, x_2, \cdots, x_{n-1}, x_n) = (x_1, x_2, \cdots, \varphi(x_{n-1}, x_n))$. Finally, for $n \geq 3$, denote $\chi_n : E^n \to E^2$ to be

$$\chi_n(x_1, x_2, \cdots, x_{n-1}, x_n) = (x_1, \varphi(x_2, \varphi(x_3, \varphi(\cdots x_{n-2}, \varphi(x_{n-1}, x_n)))))$$

or in other words $\chi_n = \psi_3 \circ \psi_4 \circ \cdots \circ \psi_{n-1} \circ \psi_n$.

We now have the components necessary to prove Theorem 1. We shall do this by induction on n. For clarity, the base case and induction step will be split in two. We begin by proving the base case.

Theorem 4. *For any $F : E^3 \to \mathbb{R}$, there exists a function $f : E^2 \to \mathbb{R}$ such that , $F(x_1, x_2, x_3) = f(x_1, \varphi(x_2, x_3))$*

Proof. We wish to find a map $f : E^2 \to \mathbb{R}$. To do this, consider a point $(u_\circ, v_\circ) \in E^2$. By Lemma 2, $\psi_3^{-1} : E^2 \to E^3$ such that $(u_\circ, v_\circ) \mapsto (x_\circ, y_\circ, z_\circ) \in E^3$. Define $f(u_\circ, v_\circ) = F \circ \psi_3^{-1}(u_\circ, v_\circ)$. Since the point $(u_\circ, v_\circ)$ was chosen arbitrarily, this will work for all E^2. Finally, since $f = F \circ \psi_3^{-1}$

$$F = f \circ \psi_3$$
$$F(x_1, x_2, x_3) = f(\psi_3(x_1, x_2, x_3))$$
$$F(x_1, x_2, x_3) = f(x_1, \varphi(x_2, x_3)).$$

□

Because of the definition of χ_n, we may also say $\chi_3 = \psi_3$ and $F_3 = f \circ \chi_3$. Note also that χ_n is a composition of functions of two variables. We may finally finish the proof of Theorem 1 by establishing the inductive step.

Proof. Theorem 4 proves a base case for $n = 3$. Now assume the statement is true for all $j \geq 3$ up to and including n. So, for $F_n : E^n \to \mathbb{R}$, there exists $f : E^2 \to \mathbb{R}$ such that

$F_n = f \circ \chi_n$. Now consider $F_{n+1} : E^{n+1} \to \mathbb{R}$ and $\chi_{n+1} : E^{n+1} \to E^2$ where $\chi_{n+1} = \chi_n \circ \psi_{n+1}$. $\psi_{n+1} : E^{n+1} \to E^n$ is bijective so define $F_n : E^n \to \mathbb{R}$ as $F_n = F_{n+1} \circ \psi_{n+1}^{-1}$. Therefore,

$$F_{n+1} = F_n \circ \psi_{n+1}$$

$$F_{n+1} = f \circ \chi_n \circ \psi_{n+1}$$

$$F_{n+1} = f \circ \chi_{n+1}$$

And so any function of n-variables with $n \geq 3$ can be written as a composition of functions of no more than two variables proving Theorem 1. $\square$

3 Continuous Functions

In this section, we wish to prove theorem 2. For sake of ease, I will consider the case for $n = 2$, however for any $n \geq 3$ it is a direct extension. Therefore, we prove the following:

There exists real continuous functions $\varphi_1^q(x_1)$ and $\varphi_2^q(x_2)$, defined on E^1, $q = 1, \cdots, 5$, such that every continuous real function $f(x_1, x_2)$, defined on E^2, is representable in the form $f(x_1, x_2) = \sum_{q=1}^{5} \chi^q\left(\varphi_1^q(x_1) + \varphi_2^q(x_2)\right)$ where $\chi^q(t)$ are continuous functions.

The proof consists of three parts.

3.1 Constructions of Families of Rectangles in E^2

In this part, we will construct five families of unions of intervals $A_{k,p}^q = \bigcup_i A_{k,p_i}^q \subset E^1$ in which

$$\begin{cases} p = 1, 2 & \text{where } p \text{ represents the interval of each sub-dimension} \\ 1 \leq q \leq 5 & \text{where } q \text{ represents the individually familiy member for each } p \\ 1 \leq i \leq m_k & \text{where } i \text{ is the sequence of subintervals, } m_k \in \mathbb{N} \text{ the number of subintervals} \\ k = 1, 2, 3, \cdots & \end{cases}$$

and $p, q, i, k \in \mathbb{N}$. For each q and fixed k, the intervals A_{k,p_i}^q will be non-intersecting and as $k \to \infty$ the length of each of these intervals will decrease and the limit will be zero. Finally, any point $x \in E^1$ will be contained in at least four of the $A_{k,p}^q$ families. With these intervals, we may then construct families of rectangles $S_{k,1,2}^q \subset E^2$ such that, for fixed q, the rectangles $S_{k,1_i,2_j}^q$ are non-intersecting and that any point $(x_1, x_2) \in E^2$ is contained in no less than three families for fixed k.

We begin the series of constructions with the $A_{k,p}^q$ intervals. For this construction, $p = 1$

and $p = 2$ will be done similarly and thus we will only present $p = 1$.

Fix k and for each q construct the family member $A_{k,1}^q$ of unions of intervals the following way:

- Each $A_{k,1}^q \subset E^1$
- $A_{k,1}^q = \bigcup_i A_{k,1_i}^q = \bigcup_i [\alpha_i, \beta_i]$ where $\beta_i < \alpha_{i+1}$
- $\left|\left|A_{k,1_i}^q\right|\right| = |\beta_i - \alpha_i| < \dfrac{1}{k}$ for all i

One final condition that must be adhered to will be presented as a lemma for future reference.

Lemma 3. *For fixed k, every $x \in E^1$ will be contained in no less than four $A_{k,1}^q$.*

Proof. Divide E^1 into intervals of length less than $\delta = \dfrac{1}{k+1}$. Around each endpoint, consider another interval sufficiently small in length as to not allow any of the new intervals to intersect with one another (say δ^4). Inside each of these new intervals, choose four more points in such a way that they, along with the initial point, are equidistant from one another inside of their respective intervals. Finally, around each of the five points construct another set of non-intersecting intervals of sufficiently small length (say δ^{30}). Let each of these intervals be the gaps for the five q-families. By their construction, none of these gaps intersect anywhere on E^1. Therefore, should a point exist in a gap on any q-family, it would be impossible to exist in a gap in another q-family. Therefore, every $x \in E^1$ exists in no less that four $A_{k,1}^q$. □

With the construction of the $A_{k,p}^q$ intervals complete, we move now to the next construction; the rectangles $S_k^q \subset E^2$. Let $S_k^q = A_{k,1}^q \times A_{k,2}^q$. Given this, S_k^q will adhere to similar properties as that of $A_{k,p}^q$. Namely,

- Each $S_k^q \subset E^2$
- $S_k^q = \bigcup_{i,j} S_{k,1_i,2_j}^q = \bigcup_{i,j} \left(A_{k,1_i}^q \times A_{k,2_j}^q \right)$

- $S^q_{k,1_i,2_j}$ are non-intersecting

An important final condition will, once again, be presented as a lemma for future reference.

Lemma 4. *The system of all rectangles S^q_k, with constant k and variable q, cover the unit square E^2 so that every point in E^2 is covered at least 3 times.*

Proof. Consider a point $(x_1, x_2) \in E^2$. By Lemma 3, x_1 will belong to at least four $A^q_{k,1}$ $\left(\text{say } A^{2-5}_{k,1}\right)$ and x_2 will belong to at least four $A^q_{k,2}$ $\left(\text{say } A^{1,3-5}_{k,2}\right)$. Then the point (x_1, x_2) will miss only two of the five S^q_k families, in this case the point will belong to S^{3-5}_k, and thus any point $(x_1, x_2) \in E^2$ will belong to at least three S^q_k families. $\square$

3.2 Constructions of Inner Functions

Now that we have a sequence of families $\{S^q_k\}$, $k = 1, 2, \cdots$, we shall choose a subsequence of these families. Call this subsequence $\{S^q_r\}$, $r = 1, 2, \cdots$. We now establish and prove the following lemma

Lemma 5. *There exists continuous functions $\psi^q(x_1, x_2) = \varphi^q_1(x_1) + \varphi^q_2(x_2)$ on E^2 for $q = 1, \cdots, 5$ with the following properties:*

- *For all r, q, $\psi^q\left(S^q_{r,1_i,2_j}\right) \bigcap \psi^q\left(S^q_{r,1_m,2_n}\right) = \emptyset$ as long as $(i,j) \neq (m,n)$*
- *φ^q_1, φ^q_2 are non-decreasing continuous functions.*

To prove this lemma we proceed with the following. We begin the construction of ψ^q by first constructing $\varphi^q_p(x_p) = \lim_{r\to\infty} \varphi^q_{p,r}(x_p)$ with $\varphi^q_{p,r}$ constructed inductively on r for both $p = 1$ and $p = 2$. For $r = 1$, define

$$\varphi^q_{1,1}(x_1) = \begin{cases} c^q_{1_i,1} & \text{for } x_1 \in A^q_{1,1_i};\, c^q_{1_i,1} \in \mathbb{Q} \bigcap [0,1] \\ \text{connects linearly} & \text{for } x_1 \in E^1 \sim A^q_{1,1} \end{cases}$$

and

$$\varphi_{2,1}^{q}\left(x_2\right)=\begin{cases} c_{2_i,1}^{q}\sqrt{2} & \text{for } x_2\in A_{1,2_i}^{q};c_{2_i,1}^{q}\in\mathbb{Q}\bigcap[0,1] \\ \text{connects linearly} & \text{for } x_2\in E^1\sim A_{1,2}^{q}\end{cases}$$

where $\varphi_{p,1}^{q}$ are increasing step functions on $A_{1,p}^{q}$. Due to the choice of values $\psi_1^q\left(x_1,x_2\right)=\varphi_{1,1}^{q}\left(x_1\right)+\varphi_{2,1}^{q}\left(x_2\right)$ satisfies the first condition of the lemma for $r=1$.

Remark. If $c_{1_j,1}^{q}+c_{2_j,1}^{q}\sqrt{2}=c_{1_k,1}^{q}+c_{2_k,1}^{q}\sqrt{2}$ for $c_{p_i,1}^{q}\in\mathbb{Q}$ then $c_{p_j,1}^{q}=c_{p_k,1}^{q}$.

For $r=2$ the construction is as follows. Consider an ϵ_1-neighborhood of the graph of $\varphi_{1,p}^{q}$ where $0<\epsilon_1\le\dfrac{1}{4}\min\left|\psi_1^q\left(S_{1,1_i,2_j}^{q}\right)-\psi_1^q\left(S_{1,1_m,2_n}^{q}\right)\right|$, where $(i,j)\neq(m,n)$. Call this neighborhood of the graph of $\varphi_{1,p}^{q}$, U_1. Next consider horizontal lines L such that $L\bigcap U_1\neq\emptyset$ and consider δ_1 to be the infinum of the length of $L\bigcap U_1$ for all these lines. Next choose k large enough so that each $\left|\left|A_{k,1_i}^{q}\right|\right|<\delta_1$. Now, construct the functions $\varphi_{2,p}^{q}$, similarly to that of $\varphi_{1,p}^{q}$, such that

$$\varphi_{1,2}^{q}\left(x_1\right)=\begin{cases} c_{1_i,2}^{q} & \text{for } x_1\in A_{2,1_i}^{q};c_{1_i,2}^{q}\in\mathbb{Q}\bigcap[0,1] \\ \text{connects linearly} & \text{for } x_1\in E^1\sim A_{2,1}^{q}\end{cases}$$

and

$$\varphi_{2,2}^{q}\left(x_2\right)=\begin{cases} c_{2_i,2}^{q}\sqrt{2} & \text{for } x_2\in A_{2,2_i}^{q};c_{2_i,2}^{q}\in\mathbb{Q}\bigcap[0,1] \\ \text{connects linearly} & \text{for } x_2\in E^1\sim A_{2,2}^{q}\end{cases}$$

with the stipulation that $\left|\left|\varphi_{1,p}^{q}\left(x_p\right)-\varphi_{2,p}^{q}\left(x_p\right)\right|\right|<\dfrac{1}{2}\epsilon_1$. This will ensure that $\varphi_{p,2}^{q}$ is also inside the ϵ_1-neighborhood. Now, $\psi_2^q\left(x_1,x_2\right)=\varphi_{2,1}^{q}\left(x_1\right)+\varphi_{2,2}^{q}\left(x_2\right)$ satisfies the first condition of the lemma for $r=1,2$.

For $r=3$, choose $\epsilon_2>0$ such that $\epsilon_2\le\dfrac{1}{4}\min\left|\psi_2^q\left(S_{2,1_i,2_j}^{q}\right)-\psi_2^q\left(S_{2,1_m,2_n}^{q}\right)\right|$ for $(i,j)\neq(m,n)$ and $\epsilon_2<\dfrac{1}{2}\epsilon_1$, and construct $\varphi_{3,p}^{q}$ similarly to the previous two constructions so that $\left|\left|\varphi_{2,p}^{q}\left(x_p\right)-\varphi_{3,p}^{q}\left(x_p\right)\right|\right|<\dfrac{1}{2}\epsilon_2$. $\psi_3^q\left(x_1,x_2\right)=\varphi_{3,1}^{q}\left(x_1\right)+\varphi_{3,2}^{q}\left(x_2\right)$ now satisfies the first condition of the lemma for $r=1,2,3$.

Continue this process to obtain a sequence $\left\{\varphi_{r,p}^{q}\right\}_{r=1}^{\infty}$ of continuous functions inside of nested ϵ_r-neighborhoods where $\left|\left|\varphi_{r-1,p}^{q}-\varphi_{r,p}^{q}\right|\right| < \frac{1}{2^{r-1}}\epsilon_1$ and $\{\psi_r^q\}_{r=1}^{\infty}$ where $\psi_r^q(x_1,x_2) = \varphi_{r,1}^{q}(x_1)+\varphi_{r,2}^{q}(x_2)$. Given the construction, $\epsilon_r < \frac{1}{2^r}\epsilon_1$ and thus as $r\to\infty$, $\epsilon_r\to 0$ therefore $\varphi_{r,p}^{q}$ converges uniformly to a limit function φ_p^q and $\lim\limits_{r\to\infty}\varphi_{r,p}^{q} = \varphi_p^q$. We also get $\psi^q(x_1,x_2) = \lim\limits_{r\to\infty}\psi_r^q(x_1,x_2) = \lim\limits_{r\to\infty}\left(\varphi_{r,1}^{q}(x_1)+\varphi_{r,2}^{q}(x_2)\right) = \varphi_1^q(x_1)+\varphi_2^q(x_2)$. Due to the construction, the properties of φ_1^q, φ_2^q, and ψ^q, described in lemma 5, hold.

3.3 Construction of Outer Functions

We finally shift our focus to the desired parent function f. The following is the construction and proof.

In the previous section, we constructed a subsequence $\{S_r^q\}$ of $\{S_k^q\}$. We will now fix this subsequence and refer to it as S_k^q, $k=1,2,\cdots$. For our function f, we will construct a new subsequence of this $\{S_k^q\}$. We will refer to this new subsequence as $\{S_r^q\}$.

On each $S_{k,1_i,2_j}^{q}$, define $\left(x_{1c_k^q}, x_{2c_k^q}\right)$ to be the center of each rectangle. Define $M_o = \sup||g_o||$ where $g_o(x_1,x_2) = f(x_1,x_2)$. g_o is on a compact set and thus it is uniformly continuous on this set. For this, it is possible to find a $\delta>0$ such that when $d\left[(x_1',x_2'),(x_1'',x_2'')\right]<\delta$ then $|g_o(x_1',x_2')-g_o(x_1'',x_2'')| < \frac{1}{6}M_o$. Choose k such that the diameter of each $S_{k,1_i,2_i}^{q}$ is less than this δ. Label this value $r=1$. This will start a subsequence of k. Define $\Delta_{r,1_i,2_j}^{q} = \psi^q\left(S_{r,1_i,2_j}^{q}\right)$. Let $t\in\Delta_{r,1_i,2_i}^{q}$ and define $\chi_1^q(t) = \frac{1}{3}g_o\left(x_{1c_1^q}, x_{2c_1^q}\right)$. Finally, for the intervals outside of these, connect the pieces linearly. For any point $(x_1,x_2)\in E^2$ we can guarantee that it is covered by at least three of the S_1^q. The choice of which three cover the point is irrelevant so assume the three are $q=1,2,3$. To finish, consider the function $g_1 = g_o - f_1$ where $f_1(x_1,x_2) = \sum\limits_{q=1}^{5}\chi_1^q(\psi^q(x_1,x_2))$ and any point (x_1,x_2) and deduce it's norm. This is

$$||g_1|| = ||g_o - f_1|| = \max_{E^2}|g_o - f_1| = \max_{E^2}\left|g_o - \sum_{q=1}^{5}\chi_1^q\left(\psi^q(x_1,x_2)\right)\right|$$

$$\max_{E^2}\left|g_o-\sum_{q=1}^{3}\chi_1^q\left(\psi^q\left(x_1,x_2\right)\right)-\sum_{q=4}^{5}\chi_1^q\left(\psi^q\left(x_1,x_2\right)\right)\right|$$

$$\leq \max_{E^2}\left|g_o(x_1,x_2)-\sum_{q=1}^{3}\chi_1^q\left(\psi^q\left(x_1,x_2\right)\right)\right|+\max_{E^2}\left|\sum_{q=4}^{5}\chi_1^q\left(\psi^q\left(x_1,x_2\right)\right)\right|$$

$$=\max_{E^2}\left|g_o(x_1,x_2)-\sum_{q=1}^{3}\frac{1}{3}g_o\left(x_{1c_1^q},x_{2c_1^q}\right)\right|+\max_{E^2}\left|\sum_{q=4}^{5}\chi_1^q\left(\psi^q\left(x_1,x_2\right)\right)\right|$$

$$=\max_{E^2}\left|3\frac{1}{3}g_o(x_1,x_2)-\sum_{q=1}^{3}\frac{1}{3}g_o\left(x_{1c_1^q},x_{2c_1^q}\right)\right|+\max_{E^2}\left|\sum_{q=4}^{5}\chi_1^q\left(\psi^q\left(x_1,x_2\right)\right)\right|$$

$$<\frac{3}{18}M_o+\frac{2}{3}M_o=\frac{5}{6}M_o$$

Now consider $g_1=g_o-f_1$. This, again, is a continuous function on a compact set and therefore it will be uniformly continuous and attain its maximum. Due to the uniform continuity we can, similar to the construction of g_o, find a δ that satisfies $\epsilon=\frac{1}{6}M_1=\frac{1}{6}||g_1||$. Because of the construction of g_1, it is clear that $M_1\leq\frac{5}{6}M_o$. Again, choose k large enough to make the diameter of each $S^q_{k,1_i,2_i}$ less than the chosen δ. This will become $r=2$ of the subsequence of families of rectangles. Now, define a new function $\chi_2^q(t)=\frac{1}{3}g_1\left(x_{1c_2^q},x_{2c_2^q}\right)$ for $t\in\Delta^q_{2,1_i,2_i}$ with a similar structure to χ_1^q elsewhere. Again, any point $(x_1,x_2)\in E^2$ is covered by at least three S_2^q and so we get a similar estimate as before. Finally, define $f_2(x_1,x_2)=\sum_{q=1}^{5}\chi_2^q\left(\psi^q\left(x_1,x_2\right)\right)$ and $g_2=g_1-f_2$ then examine $||g_2||$. We get a similar set of inequalities. This is

$$\frac{1}{6}M_1+\frac{2}{3}M_1=\frac{1}{6}\left(\frac{5}{6}M_o\right)+\frac{2}{3}\left(\frac{5}{6}M_o\right)=\left(\frac{5}{6}\right)^2M_o.$$

Repeat this process to obtain sequences of functions $\{f_r\}_{r=1}^{\infty}$ where $f_r\left(x_1,x_2\right)=\sum_{q=1}^{5}\chi_r^q\left(\psi^q\left(x_1,x_2\right)\right)$, $\{g_r\}_{r=1}^{\infty}$ where $g_r=g_{r-1}-f_r$, and $\{\chi_r^q\}$ where $\chi_r^q\left(\psi^q\left(x_1,x_2\right)\right)=\frac{1}{3}g_{r-1}\left(x_1,x_2\right)$ inside of each $S^q_{r,1_i,2_j}$, and connected similarly to the previous iterations outside. Given the construction, $||g_r||\leq\left(\frac{5}{6}\right)^r||f||=\left(\frac{5}{6}\right)^rM_o$. Then, $||\chi_r^q||\leq\frac{1}{3}||g_{r-1}||\leq\frac{1}{3}\left(\frac{5}{6}\right)^{r-1}M$. Therefore, the series of continuous functions $\sum_{r=1}^{\infty} \chi_r^q \left(\psi^q \left(x_1, x_2\right)\right)$ on the closed, bounded interval E^1 will converge uniformly to the limit function $\chi^q \left(\psi^q \left(x_1, x_2\right)\right)$. Because q was never specified, we also get $\sum_{r=1}^{\infty} \sum_{q=1}^{5} \chi_r^q \left(\psi^q \left(x_1, x_2\right)\right) = \sum_{q=1}^{5} \chi^q \left(\psi^q \left(x_1, x_2\right)\right)$. Now, $g_r = g_{r-1} - f_r$ thus $f_r = g_{r-1} - g_r$. If we consider $||f_r||$ we will obtain

$$||f_r|| = ||g_{r-1} - g_r|| \leq \left(\frac{5}{6}\right)^{r-1} M_o + \left(\frac{5}{6}\right)^{r} M_o < 2\left(\frac{5}{6}\right)^{r-1} M_o.$$

This will ensure that the series $\sum_{r=1}^{\infty} f_r$ is absolutely convergent. Therefore, if we consider this series we get $\sum_{r=1}^{\infty} f_r = \sum_{r=1}^{\infty} \left(g_{r-1} - g_r\right) = \left(g_{r-1} - g_r\right) + \left(g_{r-2} - g_{r-1}\right) + \left(g_{r-3} - g_{r-2}\right) + \cdots$ because of the above estimates, we may rearrange the right side in any way we want to obtain $\sum_{r=1}^{\infty} f_r = g_o + g_1 - g + 1 + g_2 - g_2 + g_3 + g_3 + \cdots = g_o = f$. Thus, $\sum_{r=1}^{\infty} f_r = f$. Finally, $f_r \left(x_1, x_2\right) = \sum_{q=1}^{5} \chi_r^q \left(\psi^q \left(x_1, x_2\right)\right)$ thus we get $f\left(x, y\right) = \sum_{q=1}^{5} \chi^q \left(\psi^q \left(x, y\right)\right) = \sum_{q=1}^{5} \chi^q \left(\varphi_1^q \left(x\right) + \varphi_2^q \left(y\right)\right)$. This completes the proof of Theorem 2.

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