OPTIMAL CONTROL DESIGN OF LARGE SCALE SYSTEM WITH PARAMETER UNCERTAINTY

A Thesis by

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OPTIMAL CONTROL DESIGN OF LARGE SCALE SYSTEM WITH PARAMETER UNCERTAINTY

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DEDICATION

To my family members and professors Dr. John Watkins and Dr. M. Edwin Sawan for their immense support, understanding and encouragement to do this research during these two years.
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It’s my privilege and pleasure to express my profound sense of respect, gratitude to my advisors Dr. John Watkins and Dr. Edwin Sawan, Wichita State University for their valuable inputs, guidance and constant motivation throughout this research and even before that. Moreover, I would like to thank all my staffs, friends and family members for their motivation and good wishes, which made me to complete my thesis successfully.
ABSTRACT

In this thesis, a model order reduction technique is used to design optimal control strategies with low sensitivity to model uncertainty. The source of uncertainty encountered here is called parameter variation and an aggregation methodology is used to obtain the reduced order model. Performance sensitivity of the system is reduced by adding a sensitivity measure to the performance index that represents the cost to be minimized. The sensitivity measure is defined as the variable given by the partial derivative of the state with respect to the uncertain parameter evaluated at the nominal value. This results in an augmented model that includes the new sensitivity variable, which has the same size as the state vector of the original system. As a result, the order of the dynamic constraint of the optimization procedure will be twice that of the original plant. Therefore, developing a reduced order model and using it in the design procedure will alleviate the problem of larger dimensions. The design is completed based on the reduced order model. Then, such a design is used to obtain the approximate design for the full-order system. Numerical examples are presented to illustrate the effectiveness of the approximated design in reducing the performance sensitivity of the full-order system.
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<th>Abbreviation</th>
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<tr>
<td>LQR</td>
<td>Linear Quadratic Regulator</td>
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<tr>
<td>LTI</td>
<td>Linear Time-Invariant</td>
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<td>ODE</td>
<td>Ordinary Differential Equation</td>
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<td>PDE</td>
<td>Partial Differential Equation</td>
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<td>DPS</td>
<td>Distributed Parameter System</td>
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<td>PGD</td>
<td>Proper Generalized Decomposition</td>
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<td>SP</td>
<td>Singular Perturbation</td>
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A control system manages, commands, directs and regulates the behavior of other devices or systems. It can range from home heating controller using a thermostat controlling a domestic boiler to large industrial control systems which are used for controlling process or machines. In most common form, the feedback control system is desired to control a process called the plant, so that its output follows a control signal, which may be fixed or changing often. More clearly, it can be said that the parameters of the system may change unexpectedly. This parameters include the value of the elements like resistors, capacitors and inductors. These changes may be due to environmental effects, over usage and other perturbations. And this changes are called parameter variations.

In general, to analyze the performance sensitivity issue in detail a sensitivity variable has been included in the cost function [10] which represents the cost to be minimized and the performance sensitivity is defined as the rate of change of response of the system with respect to the sensitivity parameter $\alpha$.

$$\sigma(t) = \frac{\partial x(t)}{\partial \alpha}$$

(1.1)

By the introduction of this performance sensitivity, the augmented model includes an additional vector which doubles the order of the dynamic constraint of the optimization procedure. In this case, reducing the large order system can be beneficial. However, this can also run the risk of reducing accuracy. Therefore, having some techniques that can do both is clearly advantageous and one such technique is called as aggregation. In order to obtain the trajectory response of the original system, it would be easier to reduce the order of the system at the initial procedure and then obtain the state feedback value of the reduced order model which in turn can be utilized to...
acquire the full order feedback gain. Once full order feedback is obtained then the trajectory responses can be plotted to see the effect of the parameter variations with the help of MATLAB programming.

1.1 BACKGROUND AND MOTIVATION

In control systems, the design of the original plant may not be valid after some time because of parameter variation. Due to this, the response of the system also changes from the nominal value where this nominal value is referred to as the set value assigned by the manufacturer during the design procedure. For example: Resistance of 1 $K\Omega$ is set for a particular resistor used in that model. This may change to 0.7 $K\Omega$ due to over usage or environmental conditions. Therefore, it would be a good idea to bring the trajectory responses closer to the nominal even when the parameter uncertainty is present.

Moreover, it was found that the large-scale systems consume more time and includes complexity during the analysis. Hence, model order reduction techniques can be used and then the procedure cited by the authors Peter and Philip can be followed to obtain the feedback gain of the reduced order model. Later, full order feedback gain can be calculated to plot the trajectory responses of the original system. By doing so, the simulation time is reduced and the reduced order model can also be reused repeatedly during the design process.

1.2 LITERATURE REVIEW

Singular Perturbation (SP) method is used as a tool to solve many problems in the control field for example: modelling is the first and foremost problem, that is, how to mathematically describe the system which is to be controlled. The modelling should be done in a such a way that it should not include more details than required by the control function [1]. This method has been
used in linear control systems which had certain constraints such as stable and unstable parts [2] and some sensitivity issues [9]. On the other hand, H. Jardon and Jacquelien in their research work they investigated a class of slow-fast systems for which the singular perturbation cannot be applied due to the lack of a normally hyperbolic critical manifold. So, they worked on this and finally they came up with a conclusion that it is possible to design composite controllers which can stabilize the (non-hyperbolic) origin [3]. Later, Mark J. Balas during his work he decided to analyze a system with infinite dimensional nature which is also called as Distributed Parameter System (DPS) [4]. For this system, he designed a controller using the singular perturbation method and when these controllers are used in the actual distributed parameter system the closed loop stability was not ensured. Therefore, he provided bounds on the smallness of the singular parameter to ensure stable operation. With this same type of system, implementation of low sensitivity control has been proceeded via an observer. During this implementation they introduced a new term in the cost function as a measure of insensitivity and this term is called as the sensitivity variable.

As years went by, few authors came up with different methodologies of model order reduction and one such method of doing it is called as Proper Generalized Decomposition (PGD) [6]. This was used for the analysis of Linear Time-invariant System with parameter uncertainty. In this method the state variables are expanded as a sum of separate functions of time and uncertain parameter. Later the random states are obtained using PGD and the algorithm is proposed for obtaining the basic PGD functions and control input gain to obtain the control objective. The second and straightforward approach for the model order reduction is called as the aggregation method. It has been utilized for the analysis and control of some large-scale dynamics systems, where it ensures reduction in the computational time and complexity of the system [7]. Moreover, the same methodology can be applied to different types of systems such as Hidden-Markov model and
deterministic systems to analyze the stability of the models [8]. Therefore, all these research gives us a general idea about the concepts of the model reduction techniques and also presents a view about its implementation on different types of systems. From this, we can conclude that one method or the other requires some properties of the system to be satisfied, to be able to perform the model order reduction. But for the aggregation methodology it has no such boundaries. Thus, we prefer to use the aggregation method to get the reduced order model in this thesis. In addition to this, a system with an uncertain parameter has been considered and ideas cited by Peter and Philip [10] has been used to minimize the optimal control strategies.

1.3 PROBLEM STATEMENT

In state space representation, the matrixes A, B and C may depend on an uncertain parameter named \( \alpha \) which has a nominal value equal to \( \alpha_0 \). When this happens, the trajectory response of the system is subject to change and the design of the system may become invalid. Therefore, in this thesis we try to reduce the dependency of \( x(t) \) on \( \alpha \). In other words, we reduce the trajectory responses of the system close to the nominal when the parameter changes unexpectedly. However, when we try to apply this to any system, it may not be easier because in this new system we have two states such as \( x \) and \( \sigma \) where \( \sigma \) is the newly introduced state denoting the sensitivity function. This sensitivity variable doubles the system’s size as twice as the original and makes the system’s design complicated. In order to avoid this complications, we convert the large scale system to a reduced order model using some of the techniques such as singular perturbation and aggregation. Later, few set of equations are solved iteratively [10] until we obtain the feedback gain of the reduced model which minimizes the sensitivity variable simultaneously.
1.4 DEFINITIONS

Optimal Control: An optimal control is a set of differential equations describing the paths of control variables that minimizes the cost function.

LQR (Linear Quadratic Regulator): The settings of a controller governing either a machine or a process that are found by using a mathematical algorithm which minimizes a cost function with weighting factors supplied by a human. The cost (function) is often defined as a sum of deviations of key measurements from their desired values.

Feedback Control: Feedback control is a control mechanism that uses information from measurements to manipulate a variable to achieve the desired results.

1.5 COMPUTATIONAL TOOLS

MATLAB will be sufficient for mathematical analysis. The approach will be to find the mathematical representation of the system which we achieve by converting current transfer function into state-space representation. By that, we will be able to find out the performance and cost function of LQR model.

1.6 THESIS REVIEW

In chapter 2, singular perturbation method is defined. Followed by this, introduction to aggregation is well written with some advantages in chapter 3. Then, chapter 4 will be consisting of the detailed explanations cited by Peter Stavroulakis and Philip E. Sarachik which includes a set of non-linear equations similar to the Lyapunov functions to obtain the feedback gain of the reduced order model. Finally, these are solved using MATLAB programming language.
In chapter 5, numerical and real-life examples are solved. Later in chapter 6, conclusion and future scope of this method is presented. At last in the appendix, MATLAB code is shown respectively.
CHAPTER 2

SINGULAR PERTURBATION

Model order reduction techniques are used to reduce the complexity of a system. There are
many methods to do that but the particular type of method to be used usually depends on the
structure and properties of the system being examined. For example: Singular perturbation method
is the one which can reduce the order of the system and has some certain properties that needs to
be satisfied by the system. This methodology helps us to deal with large scale systems having two
sets of eigen values. One is the fast and the other is the slow part. The decomposition into fast and
slow stages is dictated by a separation of time scales. Typically, the reduced order model represents
the slow part which in most applications are dominant. Applications include robotics,
communications, electronics etc.

To illustrate, let us consider an example of a slow-fast subsystem which is of the form,

\[
\dot{x}(t) = A_1 x(t) + A_2 z(t) + B_1 u(t) \tag{2.1}
\]

\[
\epsilon \dot{z}(t) = A_3 x(t) + A_4 z(t) + B_2 u(t) \tag{2.2}
\]

where \(x \in R^n, u \in R^r, z \in R^m\) and \(\epsilon\) is a singular perturbation parameter representing the ratio
between fast and the slow poles. Then, by setting \(\epsilon = 0\) equation (2.2) changes to algebraic
equation as shown below and also the variables will be different from the original variables.

Therefore, we denote them as \(\hat{x}, \hat{z}\) and \(\hat{u}\).

\[
0 = A_3 \hat{x}(t) + A_4 \hat{z}(t) + B_2 \hat{u}(t) \tag{2.3}
\]

\[
A_4 \hat{z}(t) = - [A_3 \hat{x}(t) + B_2 \hat{u}(t)] \tag{2.4}
\]
\[ \ddot{x}(t) = -A_4^{-1}[A_3 \dot{x}(t) + B_2 \hat{u}(t)] \]  

(2.5)

From (2.2), (2.3), (2.4) and (2.5) we get

\[ \dot{x} = (A_1 - A_2 A_4^{-1} A_3) \dot{x}(t) + (-A_2 A_4^{-1} B_2 + B_1) \hat{u}(t) \]  

(2.6)

where \( \hat{u}(t) = \hat{F} \hat{x}(t) \)

And the reduced order model parameters are given by

\begin{align*}
\hat{A} &= A_1 - A_2 A_4^{-1} A_3 \\
\hat{B} &= -A_2 A_4^{-1} B_2 + B_1 \\
\hat{C} &= C_1 - C_2 A_4^{-1} A_3 \\
\hat{D} &= -C_2 A_4^{-1} B_2 \\
\hat{y} &= \hat{C} \hat{x} + \hat{D} \hat{u}(t)
\end{align*}

(2.7) \quad \text{(2.8)} \quad \text{(2.9)} \quad \text{(2.10)}

By substituting (2.9) and (2.10) in equation (2.11) we get

\[ \hat{y} = (C_1 - C_2 A_4^{-1} A_3) \hat{x}(t) - C_2 A_4^{-1} B_2 \hat{u}(t) \]  

(2.12)

Thus, these equations of the reduced order model help us proceed with the design procedure. As mentioned before, in this method during the analysis only the slow part is considered which is dominant in most of the cases and the fast part is neglected. Hence this is also called as quasi-steady state model.

More clearly, it can be said that the quasi-steady state is a method by which fast variable modes are assumed to have reached a steady state which is of the form (2.6) and (2.12) where the hat in these equations represents the quasi-steady state. Since singular perturbation method needs a
system to exhibit these kind of properties and behavior we proceed with the aggregation methodology (which is explained in Section 3) to investigate the uncertainties present in a system.
CHAPTER 3
AGGREGATION

In a control system, the parameters such as values describing geometrical measurements, material properties, damping of the system or the component flow rate play an important role in practical applications. For instances, Integrated Circuit Designs, Micro-Electro-Mechanical systems design (MEMS), Chemical Engineering and so on. Now in these models of real-life processes they pose challenges when used in numerical simulations due to large complexity of such systems. For example: Simulating large scale systems. Therefore, by reducing the model’s associated state space dimensions and degrees of freedom, an approximation to the original model is computed and simulated thereby recovering the solution of the original ODE or PDE of the large-scale system. By doing this, the simulation time of the large-scale system is shortened by several orders of magnitude and it also replaces the original system which enables us to use them repeatedly during the design process. However, these reduced order models are constructed such that all parameters are preserved with acceptable accuracy.

The reduced order models can be obtained using different methods and one such methodology is aggregation. To illustrate this method, consider two dynamic systems $S_1$ and $S_2$ where the dimensions of $S_1$, $n$ is much larger than that of $S_2$, $l$. The state vectors of these two systems are denoted by $x$ and $\hat{x}$ respectively. The relationship between them is given by,

$$\hat{x} = Mx$$

(3.1)

where $M$ is the Aggregation matrix or Non-square matrix with the dimensions $l \times n$ and here $l < n$. An aggregation matrix is also defined as the matrix which gives us the linear combination of the original state variables. There are at least two, not necessarily mutually exclusive, viewpoints
on the way such a relationship arises. One is to regard $S_1$ as a given system, called plant and to regard $S_2$ as an observer of $S_1$ [7]. With this view point, the linear transformation $M$ is to be chosen, subject to certain constraints, such that the state vector $\hat{x}$, together with originally available measurement $y$ on $S_1$, is used to reconstruct $x$ exactly or approximately, hence the name observer $S_2$. In order to be clear, the dynamic structure of $S_2$, is chosen to reflect only that part of the dynamics of $S_1$, which is not carried by the independent information contained in $y$. With the other view point, $S_2$ is regarded as a model of $S_1$. The dynamic structure of $S_2$, is to reflect the significant portion of $S_1$ and not to complement the information carried by $y$. For example, $S_1$ is the description of a physical or non-physical object according to some classification of variables, and $S_2$ is the description of the same object using a coarser grid of classification, hence of lesser dimension where this view point is more appropriate i.e., for a mathematical model of economic systems. With latter view point $S_2$ is called an aggregated model of $S_1$ and this view is considered for this thesis. In order to construct the reduced model of the system we need to choose the aggregation matrix $M$ which is a non-square matrix and we also need to define the inverse of $M$ matrix which cannot be done normally with the basic method because the inverse does not exist for a non-square matrix. So, in order to obtain the inverse of a non-square matrix, pseudo inverse procedure is followed. For example, the large-scale system is given by

$$\dot{x}(t) = A(\alpha)x(t) + Bu(t)$$

$$y(t) = Cx(t) + Du(t)$$

where $x \in \mathbb{R}^n$ and the dimensions are given by $A(\alpha) = n \times n; \ B = n \times 1; \ C = 1 \times n; \ D = 1 \times 1$ and the uncertainty parameter $\alpha$ has a nominal value $\alpha_0$.
The reduced order model is given by

\[
\hat{x}(t) = \hat{A}(\alpha)\hat{x}(t) + \hat{B}\hat{u}(t) \tag{3.4}
\]

\[
\hat{y}(t) = \hat{C}\hat{x}(t) + \hat{D}\hat{u}(t) \tag{3.5}
\]

where \( \hat{x} \in \mathbb{R}^l \) with the dimensions are given by \( \hat{A}(\alpha) = l \times l; \hat{B} = l \times 1; \hat{C} = 1 \times l; \hat{D} = 1 \times 1 \).

And these two systems are related by the equation (3.1).

The parameters of the reduced order model are given by the equations

\[
\hat{A}(\alpha) = MA(\alpha)M^R \tag{3.6}
\]

\[
\hat{B} = MB \tag{3.7}
\]

\[
\hat{C} = CM^R \tag{3.8}
\]

where \( M^R \) is known as right pseudo inverse [3]. Since, \( M \) is not a square, the standard inverse does not exist. However, \( MM^R = I_m \) where

\[
M^R = M^T(MMT)^{-1}; \quad l \times n \tag{3.9}
\]

Finally, the parameters of the reduced model are obtained as follows

\[
\hat{A}(\alpha) = MA(\alpha)MT(MMT)^{-1} \tag{3.10}
\]

\[
\hat{B} = MB \tag{3.11}
\]

\[
\hat{C} = CM^T(MMT)^{-1} \tag{3.12}
\]
and the feedback gain of the full order model is related to that of the reduced order model as follows

\[ F = \hat{F}M \] (3.13)

The aggregation matrix \( M \) is chosen such that the uncertainty parameter is retained in the reduced order model parameters.
CHAPTER 4
SENSITIVITY REDUCTION TECHNIQUE

In a linear regulator problem with a quadratic cost criterion the optimal control \( \hat{u}(t) \) is given by a linear transformation on the state \( \hat{x}(t) \) of the system. This means that if we denote the optimal control in the feedback configuration by \( \hat{F}(\hat{x}, t) \) then

\[
\hat{u}(t) = \hat{F}(\hat{x}, t) = k(t)\hat{x}(t)
\]  (4.1)

where \( k(t) \) is the gain matrix. A few years ago, researchers were investigating whether the sensitivity of the system to parameter variations can be included in the cost along with the relationship similar to the optimal control equation. Later, Peter and Philip [10] has briefly obtained the outline of the approach where \( \hat{x}(t) \) as a function of the parameter variable \( \alpha \) is given by

\[
\hat{x}(t) = \varphi(t, \hat{x}_0, \alpha)
\]  (4.2)

And the sensitivity variable is given by

\[
\sigma(t) = \frac{\partial \varphi(t, \hat{x}_0, \alpha)}{\partial \alpha}
\]  (4.3)

where \( \alpha \) has a nominal value equal to \( \alpha_0 \) and the quadratic performance index which includes the sensitivity variable is as follows

\[
J(\hat{x}_0, \hat{u}) = \frac{1}{2} \int_{0}^{\infty} (\hat{x}^T(t)Q \hat{x}(t) + \hat{u}^T(t) R \hat{u}(t) + \sigma^T(t)S\sigma(t)) dt
\]  (4.4)

For this formulation \( \hat{u}(t) \) was assumed to be

\[
\hat{u}(t) = k_1\hat{x}(t) + k_2\sigma(t)
\]  (4.5)

But here \( \sigma(t) \) cannot be measured directly and even if substituted in equation (4.1)
\[ \dot{x} = \hat{A}(\alpha)\dot{x}(t) + \hat{B}(\alpha)\dot{u}(t) \]  

(4.6)

And the derivative of the resulting equation is obtained with respect to \( \alpha \), then the equation for \( \sigma(t) \) given by Kreindler (1986) is

\[ \dot{\sigma}(t) = (\hat{A}(\alpha_0) + \hat{B}(\alpha_0)k_1 + \hat{B}_\alpha(\alpha_0)k_2)\sigma(t) + (\hat{A}_\alpha(\alpha_0) + \hat{B}_\alpha(\alpha_0)k_1)\dot{x}(t) + \hat{B}k_2 \frac{\partial^2 \varphi}{\partial \alpha^2} \]  

(4.7)

where \( \hat{A}_\alpha \) and \( \hat{B}_\alpha \) are \( \frac{\partial \hat{A}}{\partial \alpha} \) and \( \frac{\partial \hat{B}}{\partial \alpha} \) respectively and the higher order term \( \frac{\partial^2 \varphi}{\partial \alpha^2} \) causes further difficulty because it is generally unknown. To avoid this, the higher order terms are always assumed to be zero.

And the feedback control \( \dot{u}(t) \) is given by the equation,

\[ \dot{u}(t) = \hat{F}\dot{x}(t) \]  

(4.8)

such that \( \varphi \) can be expressed as

\[ \varphi(t, \dot{x}_0, \alpha_0 + \delta \alpha) = \varphi(t, \dot{x}_0, \alpha_0) + \sigma(t)\delta \alpha + 0(\alpha) \]  

(4.9)

where \( \alpha \) is evaluated at \( \alpha_0 \).

By rewriting the equations (4.6) and (4.7) with respect to (4.8) gives the system

\[ \dot{x}(t) = \hat{A}(\alpha_0)\dot{x}(t) + \hat{B}(\alpha_0)\dot{F}\dot{x}(t) \]  

(4.10)

\[ \dot{\sigma}(t) = (\hat{A}(\alpha_0) + \hat{B}(\alpha_0)\hat{F})\sigma(t) + (\hat{A}_\alpha(\alpha_0) + \hat{B}_\alpha(\alpha_0)\hat{F})\dot{x}(t) \]  

(4.11)

Therefore, equation (4.10) and (4.11) can be written as

\[ \dot{x}(t) = (\hat{A}(\alpha) + \hat{B}\hat{F})\dot{x}(t) \]  

(4.12)

\[ \dot{\sigma}(t) = (\hat{A}(\alpha) + \hat{B}\hat{F})\sigma(t) + (\hat{A}_\alpha + \hat{B}_\alpha\hat{F})\dot{x}(t) \]  

(4.13)
where (4.12) and (4.13) are evaluated at \( \alpha = \alpha_0 \) and therefore parameter \( \alpha \) has been dropped.

Since \( \sigma(0) = 0 \) it can be written that

\[
\sigma(t) = \int_0^t \exp[(\hat{A}(\alpha) + \hat{B}\hat{F})(t - \tau)](\hat{A}_\alpha + \hat{B}_\alpha \hat{F})\hat{x}(\tau) d\tau \quad (4.14)
\]

And

\[
\hat{x}(t) = \exp[(\hat{A}(\alpha) + \hat{B}\hat{F})t] \hat{x}_0 \quad (4.15)
\]

where \( \hat{x}(0) = \hat{x}_0 \) \hspace{1cm} (4.16)

then

\[
\sigma(t) = \int_0^t \exp[(\hat{A}(\alpha) + \hat{B}\hat{F})(t - \tau)](\hat{A}_\alpha + \hat{B}_\alpha \hat{F}) \exp[(\hat{A}(\alpha) + \hat{B}\hat{F})\tau] \hat{x}_0 d\tau \quad (4.17)
\]

Using (4.14) and (4.15) in the cost performance index we obtain,

\[
J(\hat{x}_0, \hat{F}) = \frac{1}{2} \int_0^\infty \hat{x}_0^T \exp \left[ (\hat{A}(\alpha) + \hat{B}\hat{F})^T t \right] (Q + \hat{F}^T R\hat{F}) \exp[(\hat{A}(\alpha) + \hat{B}\hat{F})t] \hat{x}_0 + \\
\int_0^t \hat{x}_0^T \exp \left[ (\hat{A}(\alpha) + \hat{B}\hat{F})^T \lambda \right] (\hat{A}_\alpha^T + \hat{B}_\alpha^T \hat{F})^T \exp \left[ (\hat{A}(\alpha) + \hat{B}\hat{F})^T (t - \lambda) \right] d\lambda \int_0^t \exp(\hat{A}(\alpha) + \\
\hat{B}\hat{F})(t - \tau)](\hat{A}_\alpha + \hat{B}_\alpha \hat{F}) \exp(\hat{A}(\alpha) + \hat{B}\hat{F})\tau] \hat{x}_0 d\tau \] 
\] \hspace{1cm} (4.18)

And the performance index which does not depend on the initial state \( \hat{x}_0 \) can be eliminated in the equation by weighted averaging over some set \( \hat{x}_0 \) of initial states namely,

\[
J_1(\hat{F}) = \int_{\hat{x}_0} J_1(\hat{x}_0, \hat{F}) w(\hat{x}_0) d\hat{x}_0 \quad (4.19)
\]

where \( w(\cdot) \) is a normalized non-negative scalar weighting function or probability density

\[
\int_{\hat{x}_0} w(\hat{x}_0) d\hat{x}_0 = 1 \quad (4.20)
\]
Hence \( J_1(\hat{F}) \) takes the form

\[
J(\hat{F}) = \frac{1}{2} \text{trace} \left[ \int_0^\infty \left\{ \exp \left[ (\hat{A}(\alpha) + \hat{B}\hat{F})^T \right] \left( Q + \hat{F}^T R \hat{F} \right) \exp \left[ (\hat{A}(\alpha) + \hat{B}\hat{F}) \right]\right \} dt +
\int_0^t \exp \left[ (\hat{A}(\alpha) + \hat{B}\hat{F})^T \right] \left( \hat{A}_\alpha^T + \hat{F}^T \hat{B}_\alpha^T \right) \exp \left[ (\hat{A}(\alpha) + \hat{B}\hat{F})^T \right] (t - \lambda) \right] d\lambda \right] \int_0^t \exp(\hat{A}(\alpha) + \\
\hat{B}\hat{F})(t - \tau)(\hat{A}_\alpha + \hat{B}_\alpha\hat{F}) \exp(\hat{A}(\alpha) + \hat{B}\hat{F})\tau d\tau \} dt \Sigma
\]

where

\[
\Sigma = \int_{\hat{x}_0} \hat{x}_0^T \hat{w}(\hat{x}_0) d\hat{x}_0
\]

Under the assumption that \( \hat{F} \) exists which makes the \( \hat{A}_c = (\hat{A}(\alpha) + \hat{B}\hat{F}) \) asymptotically stable a necessary condition that \( \hat{F} \) must satisfy a minimum of \( J_1(\hat{F}) \) is that

\[
\frac{\partial J_1(\hat{F})}{\partial \hat{F}} = 0
\]

(4.22)

And after some manipulation shown in Stavroulakis and Sarachik [12] \( \frac{\partial J_1(\hat{F})}{\partial \hat{F}} \) is given by

\[
\frac{\partial J(\hat{F})}{\partial \hat{F}} = R\hat{F}T + \hat{B}'(HT + E_1) + \hat{B}_\alpha' E_2
\]

(4.23)

where the matrices \( H, T, E_1 \) and \( E_2 \) are defined by the equations

\[
\hat{A}_c^T H + H\hat{A}_c = -(Q + \hat{F}^T R \hat{F})
\]

(4.24)

\[
\hat{A}_c T + T\hat{A}_c^T = -\Sigma
\]

(4.25)

\[
E_1 = (H_1 T + L_{12} N_1 + L_{21} N_1^T + L_{22} N_2)
\]

(4.26)

\[
\hat{A}_c^T H_1 + H_1\hat{A}_c = -(L_{12} \hat{A}_\alpha + \hat{A}_\alpha^T L_{21})
\]

(4.27)

\[
E_2 = L_{21} T + L_{22} N_1
\]

(4.28)

\[
\hat{A}_c^T L_{12} + L_{12}\hat{A}_c = -\hat{A}_c L_{22}
\]

(4.29)
\[ \hat{A}_c^T L_{21} + L_{21} \hat{A}_c = -L_{22} \hat{A}_{ca}^T \]  \hspace{1cm} (4.30)

\[ \hat{A}_c^T L_{22} + L_{22} \hat{A}_c = -S \]  \hspace{1cm} (4.31)

\[ N_1 = (\hat{A}_{ca} T)_{\beta\beta^T} \]  \hspace{1cm} (4.32)

\[ N_2 = \left( (\hat{A}_{ca} T)_{\beta\beta^T} \hat{A}_{ca}^T \right)_{\beta\beta^T} + (\hat{A}_{ca} (T\hat{A}_{ca}^T)_{\beta\beta^T})_{\beta\beta^T} \]  \hspace{1cm} (4.33)

\[ \hat{A}_\alpha = \left. \frac{\partial \hat{A}_c}{\partial \alpha} \right|_{\alpha=\alpha_0} \]  \hspace{1cm} (4.34)

And for any matrix \( z \), the matrix \( z_{\beta\beta^T} \) is defined as the solution of the matrix equation

\[ \hat{A}_c [z]_{\beta\beta^T} + [z]_{\beta\beta^T} \hat{A}_c^T = -z \]  \hspace{1cm} (4.35)

Equation (4.23) is solved for \( \hat{F} \) to give

\[ \hat{F} = -R^{-1}[\hat{B}^T(HT + E_1) + \hat{B}_\alpha^T E_2][T]^{-1} \]  \hspace{1cm} (4.36)

When \( E_1 = E_2 = 0 \) equation (4.36) becomes

\[ \hat{F} = -R^{-1}[\hat{B}^T HT][T]^{-1} \]  \hspace{1cm} (4.37)

And it will result in the standard Riccati equation for a linear regulator with quadratic cost function.
CHAPTER 5
APPLICATIONS

In this chapter the proposed methodology of model order reduction has been applied to several examples.

5.1 Example 1

Consider a translational mechanical system [14] which is shown in Figure 5.1.1.

![Figure 5.1.1 Translational Mechanical System](image)

The parameters of this particular system is defined as follows

\[ M_1 = \text{Mass one attached to the spring } K, Kg \]

\[ M_2 = \text{Mass two attached to the spring } K, Kg \]

\[ K = \text{Spring constant, } N/m \]

\[ D = \text{Friction constant, } Ns/m \]

\[ F_{m1} = \text{Force acting on mass } M_1, \text{Newton} \]

\[ F_{m2} = \text{Force acting on mass } M_2, \text{Newton} \]

\[ x_1 = \text{The coordinate along which the mass } M_1 \text{ translates, } m \]

\[ x_2 = \text{The coordinate along which the mass } M_2 \text{ translates, } m \]
And the relationship between the masses and the forces acting on them are

\[ F_{m1} = M_1 \frac{d^2x_1}{dt^2} \]  \hspace{1cm} (5.1)

\[ F_{m2} = M_2 \frac{d^2x_2}{dt^2} \]  \hspace{1cm} (5.2)

where the velocities of the masses \( M_1 \) and \( M_2 \), \( m \) are given respectively by

\[ \frac{dx_1}{dt} = v_1 \]  \hspace{1cm} (5.3)

\[ \frac{dx_2}{dt} = v_2 \]  \hspace{1cm} (5.4)

The state equations for the system shown in figure 5.1.1 are given by

\[ M_1 \frac{d^2x_1}{dt^2} + D \frac{dx_1}{dt} + K x_1 - K x_2 = 0 \]  \hspace{1cm} (5.5)

\[ - K x_1 + M_2 \frac{d^2x_2}{dt^2} + K x_2 = f(t) \]  \hspace{1cm} (5.6)

By combining equation (5.1), (5.2), (5.3), (5.4), (5.5) and (5.6) we get

\[ \frac{dv_1}{dt} = - \frac{K}{M_1} x_1 - \frac{D}{M_1} v_1 + \frac{K}{M_1} x_2 \]  \hspace{1cm} (5.7)

\[ \frac{dv_2}{dt} = \frac{K}{M_2} x_1 - \frac{K}{M_2} x_2 + \frac{1}{M_2} f(t) \]  \hspace{1cm} (5.8)

In vector-matrix form,

\[
\begin{bmatrix}
    \dot{x}_1 \\
    \dot{v}_1 \\
    \dot{x}_2 \\
    \dot{v}_2
\end{bmatrix} =
\begin{bmatrix}
    0 & 1 & 0 & 0 \\
    -K/M_1 & -D/M_1 & K/M_1 & 0 \\
    0 & 0 & 0 & 1 \\
    K/M_2 & 0 & -K/M_2 & 0
\end{bmatrix}
\begin{bmatrix}
    x_1 \\
    v_1 \\
    x_2 \\
    v_2
\end{bmatrix} +
\begin{bmatrix}
    0 \\
    0 \\
    0 \\
    1/M_2
\end{bmatrix} f(t) \hspace{1cm} (5.9)
\]

For this particular example let us consider the friction constant \( D \) to be an uncertain parameter.

Therefore, the equation (5.9) can be represented as
\[
\begin{bmatrix}
\dot{x}_1 \\
\dot{v}_1 \\
\dot{x}_2 \\
\dot{v}_2
\end{bmatrix} =
\begin{bmatrix}
0 & 1 & 0 & 0 \\
-K/M_1 & -\alpha/M_1 & K/M_1 & 0 \\
0 & 0 & 0 & 1 \\
K/M_2 & 0 & -K/M_2 & 0
\end{bmatrix}
\begin{bmatrix}
x_1 \\
v_1 \\
x_2 \\
v_2
\end{bmatrix} +
\begin{bmatrix}
0 \\
0 \\
0 \\
1/M_2
\end{bmatrix}
\]
\[f(t) \quad (5.10)\]

By considering the nominal values of the parameters \[16\] as given below,

\[K = 4 \frac{N}{m}, D = 5 \frac{Ns}{m}, M_1 = M_2 = 1 Kg,\]

we obtain \(A\) and \(B\) matrices as follows

\[
A =
\begin{bmatrix}
0 & 1 & 0 & 0 \\
-4 & -5 & 4 & 0 \\
0 & 0 & 0 & 1 \\
4 & 0 & -4 & 0
\end{bmatrix}; \quad B =
\begin{bmatrix}
0 \\
0 \\
0 \\
1
\end{bmatrix} ;
\]

The aggregation matrix is taken as

\[
M =
\begin{bmatrix}
1 & 1 & 1 & 0 \\
1 & 1 & 1 & 1 \\
1 & 1 & 0 & 1
\end{bmatrix}
\]

In this example, mechanical system’s initial condition responses are plotted for different values of the uncertain parameter where the initial conditions are assumed to be as follows

\[
X(0) =
\begin{bmatrix}
1 \\
1 \\
1 \\
1
\end{bmatrix}
\]

and these curves are compared to the nominal curve. Initially the mechanical system is reduced from fourth order to a third order model with the help of the aggregation technique. Therefore, the reduced order model is obtained as follows

\[
\hat{A} =
\begin{bmatrix}
-5 & 9 & -8 \\
-3 & 3 & 2 \\
-2 & 2 & -2
\end{bmatrix}
\]
\[ \hat{B} = \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix} \]

And then with the help of the reduced order model the design is done using the basic approach [10]. Later by choosing \( R \) and \( Q \) as identity matrices, the analysis is done in two cases. First by setting \( S = 0 \), and then by increasing \( S \) to a larger value in order to reduce the sensitivity variable.

Case 1: When \( S = 0 \)

Under this case, the cost equation (4.4) turns to be a standard LQR problem due to the absence of the sensitivity variable and the value of feedback of the reduced model equals to the standard LQR results. Later when the initial condition responses with the value \( X(0) = [1 \ 1 \ 1]^T \) are plotted for the full order system by considering different values of the uncertain parameter, the curves seems to be far away from the nominal curve as shown in Figure 5.1.2. Thus, the value of feedback is obtained as

\[ \hat{F} = LQR F = [0.7517 \ -1.3314 \ -0.4461] \]

\[ F = [-1.0258 \ -1.0258 \ -0.5797 \ -1.7775] \]
Case 2: When $S = 10000 I$

In this case, the sensitivity variable has been reduced by increasing the weighting matrix $S$ to a higher value i.e., 10000 and when this happens the problem is no more the standard case. Hence, the feedback obtained for the reduced order model seems to converge from the standard guess value (LQR value). Then, full order feedback is obtained using the equation (3.13) and the initial condition responses for $X(0) = [1 \ 1 \ 1]^T$ are plotted for different values of the uncertain parameter as shown in Figure 5.1.3. Therefore, from the Figure 5.1.3 it clearly shows that the curves get closer to the nominal curve as expected and they settle sooner when compared to the previous case. Therefore, the effect of the uncertain parameter on the initial condition responses of the system is reduced. And the feedback values are obtained as

$$LQR \ F = [0.7517 \ -1.3314 \ -0.4461]$$
\[ \hat{F} = [-2.0459 - 29.0309 11.3008] \]


FIGURE 5.1.3 RESPONSES OF A FULL ORDER SYSTEM WHEN \( S = 10000I \)

Here are the plots for the individual states

State 1:

FIGURE 5.1.4 RESPONSES OF STATE ONE WHEN \( S = 0 \)
FIGURE 5.1.5 RESPONSES OF STATE ONE WHEN $S = 10000I$  

State 2:

FIGURE 5.1.6 RESPONSES OF STATE TWO WHEN $S = 0$
FIGURE 5.1.7 RESPONSES OF STATE TWO WHEN $S = 10000/l$

State 3:

FIGURE 5.1.8 RESPONSES OF STATE THREE WHEN $S = 0$
FIGURE 5.1.9 RESPONSES OF STATE THREE WHEN $S = 10000I$

State 4:

FIGURE 5.1.10 RESPONSES OF STATE FOUR WHEN S=0
FIGURE 5.1.11 RESPONSES OF STATE FOUR WHEN $S = 10000I$

Control Signal Plots

Here we obtained the control signal plots of the full order system for the two cases which we considered earlier. That is when $S = 0$ the system turns to be a standard problem and the responses are plotted for different values of the uncertain parameter. Thus, similar to the state trajectory responses, the input responses also appear farther away from the nominal curve as shown in figure 5.1.12. And later when $S$ has been increased to a higher value of $10^4 I$ the effect of the sensitivity variable on the system’s input has been reduced and thus the curves became closer to the nominal value as shown in figure 5.1.13.
Case 1: When $S = 0$

![Control Signal Plot](image1)

**FIGURE 5.1.12 CONTROL SIGNAL PLOT OF A FULL ORDER SYSTEM WHEN $S = 0$**

Case 2: When $S = 10000$

![Control Signal Plot](image2)

**FIGURE 5.1.13 CONTROL SIGNAL PLOT OF A FULL ORDER SYSTEM WHEN $S = 10^4$**
5.2 Example 2

Consider a fourth order system [15] with an uncertain parameter

\[
\dot{x}(t) = \begin{bmatrix}
-1 & 3 & \alpha & 0 \\
-3 & -1 & \alpha & 0 \\
0 & 0 & -5 & 0 \\
\alpha & 0 & 0 & -5 \\
\end{bmatrix} x(t) + \begin{bmatrix} 1 \\
1 \\
1 \\
1 \\
\end{bmatrix} u(t)
\]

\[
M = \begin{bmatrix} 1 & 0 & 1 & 1 \\
1 & 1 & 1 & 1 \\
1 & 1 & 0 & 1 \\
\end{bmatrix}
\]

Here, this full order system is reduced with the help of aggregation procedure and the value of feedback gain is obtained using the basic approach to plot the initial condition responses of the full order system with an initial condition of \( X(0) = [1 \ 1 \ 1 \ 1]^T \) and for different values of the uncertain parameters. In order to solve the set of equations listed in the basic approach we assume the guess value of the feedback to be equal to the standard LQR result. So, by selecting \( Q \) and \( R \) as identity matrices and by setting \( S = 0 \), the value of feedback is equal to the standard LQR gain value because, the sensitivity term becomes zero under this initialization condition and the responses appear to be far away from the nominal curve as shown in Figure 5.2.1. But the value of feedback gain seems to converge its value from the standard problem when \( S \) is increased to higher values such as 100, 200 etc., which means that the uncertainty is present in the system. Therefore, by increasing the weighting matrix \( S \) the sensitivity variable is reduced and hence the system’s responses appear closer to the nominal curve as shown in Figure 5.2.2. After the MATLAB execution the reduced order parameters are given by

\[
\hat{A} = \begin{bmatrix}
-6 & 1 & 2 \\
-6.5 & 1.5 & 0.5 \\
-6.5 & 6.5 & -4.5 \\
\end{bmatrix};
\]
\[
\hat{B} = \begin{bmatrix} 3 \\ 4 \\ 3 \end{bmatrix};
\]

To obtain the initial condition responses of a full order system

Case 1: When \( S = 0 \)

\( \hat{F} = LQR \ F = [0.3121 \ - 0.9180 \ - 0.2873] \)

\( F = [ -0.8931 \ - 1.2053 \ - 0.6058 \ - 0.8931] \)

FIGURE 5.2.1 RESPONSES OF A FULL ORDER SYSTEM WHEN \( S = 0 \)
Case 2: When $S = 1000$ \\

LQR $F = [0.3121 - 0.9180 - 0.2873]$ \\

$\hat{F} = [-3.7711 - 5.1284 - 4.9627]$ \\


**FIGURE 5.2.2 RESPONSES OF A FULL ORDER SYSTEM WHEN $S = 1000$**
Here the individual plots of each state are shown for this particular example.

State 1:

**FIGURE 5.2.3** RESPONSES OF STATE ONE WHEN $S = 0$

**FIGURE 5.2.4** RESPONSES OF STATE ONE WHEN $S = 1000$
State 2:

**FIGURE 5.2.5 RESPONSES OF STATE TWO WHEN $S = 0$**

**FIGURE 5.2.6 RESPONSES OF STATE TWO WHEN $S = 1000I$**
State 3:

FIGURE 5.2.7 RESPONSES OF STATE THREE WHEN $S = 0$

FIGURE 5.2.8 RESPONSES OF STATE THREE WHEN $S = 1000I$
State 4:

**FIGURE 5.2.9** RESPONSES OF STATE FOUR WHEN $S = 0$

**FIGURE 5.2.10** RESPONSES OF STATE FOUR WHEN $S = 1000I$
Control Signal Plots

The control signal plots of the full order system has been obtained here and it has been analyzed in two cases. That is by setting $S = 0$ the control signal responses has been plotted for different values of the uncertain parameter. During this case, the system turns to be a standard problem and similar to the state trajectory responses the input responses also appear farther away from the nominal curve as shown in figure 5.2.11. And later when $S$ has been increased to a higher value of $10^3 l$ the effect of the sensitivity variable on the system’s input has been reduced and thus the curves became closer to the nominal value as shown in figure 5.2.12.

Case 1: When $S = 0$

![Control Signal Plot](image)

**FIGURE 5.2.11 CONTROL SIGNAL PLOT OF A FULL ORDER SYSTEM WHEN $S = 0$**
Case 2: When $S = 1000I$

![Control Signal Plot](image)

**FIGURE 5.2.12 CONTROL SIGNAL PLOT OF A FULL ORDER SYSTEM WHEN $S = 1000I$**

5.3 Example 3

In this section another example of an aircraft has been taken where an autopilot is controlling the pitch of an aircraft and this aircraft pitch has been governed by the longitudinal dynamics [13]. Later, this linearized model of an aircraft with state feedback controller has been simulated and the model with numerical values is given below in the form of state space representation.

Thus,

$$\dot{x}(t) = \begin{bmatrix} -0.313 & 56.7 & 0 \\ -0.0139 & -0.426 & 0 \\ 0 & 56.7 & 0 \end{bmatrix} x(t) + \begin{bmatrix} 0.232 \\ 0.0203 \\ 0 \end{bmatrix} u(t)$$

The model with uncertainty is assumed to be as follows

$$\dot{x}(t) = \begin{bmatrix} -0.313 & \alpha & 0 \\ -0.0139 & -0.426 & 0 \\ 0 & \alpha & 0 \end{bmatrix} x(t) + \begin{bmatrix} 0.232 \\ 0.0203 \\ 0 \end{bmatrix} u(t)$$
\[ M = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 0 & 1 \end{bmatrix} \]

The reduced order models are obtained as given below

\[
\hat{A} = \begin{bmatrix} 112.9740 & -113.1374 \\ 113.4000 & -113.5565 \end{bmatrix}
\]

\[
\hat{B} = \begin{bmatrix} 0.2523 \\ 0.2320 \end{bmatrix}
\]

Now by selecting \( Q \) and \( R \) as identity matrices, the analysis is done for two cases:

Case 1: When \( S = 0 \)

\[
LQR F = \hat{F} = [ -77.4223 \quad 76.3700 ]
\]

\[
F = [ -1.0523 \quad -77.4223 \quad -1.0523 ]
\]

FIGURE 5.3.1 RESPONSES OF AN AIRCRAFT WHEN \( S = 0 \)

As we discussed earlier, here also the feedback gain of the reduced order model equals the standard case when sensitivity variable is set to zero. And then the system’s initial condition responses for
the value of $X(0) = [1 \ 1 \ 1]^T$ is plotted for different uncertain parameter values where it shows the curves appearing farther away from the nominal curve as shown in Figure 5.3.1.

Case 2: When $S = 1000000I$

In this case, when $S$ is increased to higher values such as $10^6I$, the sensitivity variable is reduced to a minimum value and thus the curves appear closer to the nominal curve as shown in Figure 5.3.2. Therefore, the dependency of the responses on the uncertain parameter is reduced. And the feedback gain values are obtained as follows

$LQR \ F = [-77.4223 \ 76.3700]$

$\hat{F} = [-236.2584 \ 157.5851]$

$F = [-78.6733 \ -236.2584 \ -78.6733]$

FIGURE 5.3.2 RESPONSES OF AN AIRCRAFT WHEN $S = 10^6I$
Plots for the individual states are shown below

State 1:

FIGURE 5.3.3 RESPONSES OF STATE ONE WHEN $S = 0$

FIGURE 5.3.4 RESPONSES OF STATE ONE WHEN $S = 10^6I$
State 2:

FIGURE 5.3.5 RESPONSES OF STATE TWO WHEN $S = 0$

FIGURE 5.3.6 RESPONSES OF STATE TWO WHEN $S = 10^6 I$
State 3:

**FIGURE 5.3.7** RESPONSES OF STATE THREE WHEN $S = 0$

**FIGURE 5.3.8** RESPONSES OF STATE THREE WHEN $S = 10^6 I$

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Control Signal Plots

Here we obtained the control signal plots of the full order system for the two cases which we considered earlier. That is when $S = 0$ the system turns to be a standard problem and the responses are plotted for different values of the uncertain parameter. Therefore, the input responses appear farther away from the nominal curve as shown in figure 5.3.9. And later when $S$ has been increased to a higher value of $10^6 I$ the effect of the sensitivity variable on the system’s input has been reduced and thus the curves became closer to the nominal value as shown in figure 5.3.10

Case 1: When $S = 0$

![Control Signal Plot](image)

**FIGURE 5.3.9 CONTROL SIGNAL PLOT OF A FULL ORDER SYSTEM WHEN $S = 0$**
Case 2: When $S = 10^6 I$

**FIGURE 5.3.10** CONTROL SIGNAL PLOT OF A FULL ORDER SYSTEM WHEN $S = 10^6 I$
CHAPTER 6
CONCLUSION AND FUTURE WORK

This work proposes a methodology of model order reduction techniques for systems with parameter uncertainty. In order to analyze this, we introduce a performance sensitivity measure in the cost function to achieve optimal control strategies. The performance sensitivity is defined as the rate of change of state with respect to the uncertainty parameter. Moreover, the weighting matrix $S$ represents our emphasis on reducing the effect of uncertainty. By doing this, the number of variables of the model becomes double the original model. Hence as an alternative solution we use model order reduction techniques to reduce the computational complexity of the system’s design. And then, the design procedure is done with the reduced order model which is used to obtain an approximate design for the full-order system. Finally, the initial condition responses of the full order system is plotted where it is observed that by fixing $Q$ and $R$ as identity matrices and by increasing the value of $S$, it reduces the dependency of the states further on the uncertainty.

But there can be other sources of uncertainties like time delay, unstructured uncertainties, and small non linearities. Thus, further improvements can be made to analyze these types of uncertainties present in a system and measures can be taken to minimize their effects on the responses.
REFERENCES
REFERENCES


APPENDIXES
MATLAB CODE FOR CHAPTER 5

clear all;
clc;
Q= R = [ 1 0 0 ; 0 1 0; 0 0 1];
x0=[1;1;1;1];
S_N= [ 0 0 0; 0 0 0; 0 0 0];
SIGMA= [ 1 0 0; 0 1 0; 0 0 1];
C=[ 1 0 0; 0 1 0; 0 0 1];

A_L=[-1 3 0 0; -3 -1 0 0; 0 0 -5 0; 0 0 0 -5];
B_L=[1;1;1;1];
C_L=[ 1 0 0 0; 0 1 0 0; 0 0 1 0; 0 0 0 1];
M_H= [1 0 1 1; 1 1 1 1; 1 1 0 1];
A_H= (M_H*A_L*(M_H')*(inv(M_H*M_H')));
B_H= (M_H*B_L);
C_H=C;
disp('A_H');
disp(A_H);
disp('B_H');
disp(B_H);
disp('C_H');
disp(C_H);

A_alpl=[0 0 1 0; 0 0 1 0; 0 0 0 0; 1 0 0 0];
A_alp= (M_H*A_alpl*(M_H')*(inv(M_H*M_H')));
A2=[-1 3 2 0; -3 -1 2 0; 0 0 -5 0; 2 0 0 -5];
A3=[-1 3 5 0; -3 -1 5 0; 0 0 -5 0; 5 0 0 -5];
A4=[-1 3 6 0; -3 -1 6 0; 0 0 -5 0; 6 0 0 -5];
A5=[-1 3 8 0; -3 -1 8 0; 0 0 -5 0; 8 0 0 -5];

% A2=[-10/(2.75*10^-6) 0 -9.96*10^-3; 0 0 1; 8.5*10^3 0 1.089];
% A3=[-20/(2.75*10^-6) 0 -9.96*10^-3; 0 0 1; 8.5*10^3 0 1.089];
% A4=[-70/(2.75*10^-6) 0 -9.96*10^-3; 0 0 1; 8.5*10^3 0 1.089];
% A5=[-100/(2.75*10^-6) 0 -9.96*10^-3; 0 0 1; 8.5*10^3 0 1.089];
B0=[ 0;0;0;0];
D0=[0];

B_alp=[0;0;0];
D=[0];
sys9= ss(A_H, B_H, C_H,D);
[F,S,E]=lqr(sys9, Q,R);
disp('S');
disp(S);
disp('E');
disp(E);
F = -F;
disp('lqr F');
disp(F);

disp(' Now Let guess be zero');
F = [0 0 0];
disp('F');

for i = 1:50
Fold = F;
A_c = (A_H + (B_H * F));
disp('A_c');
disp(A_c);

J = eig(A_c);
disp('J');
disp(J);

T = lyap(A_c, SIGMA);
disp('T');
disp(T);
H = lyap( A_c', (Q+(F'*R*F)));
disp('H');
disp(H);
L22 = lyap( A_c', S_N);
disp('L22');
disp(L22);
L21 = lyap( A_c', (L22*A_alp));
disp('L21');
disp(L21);
L2 = lyap( A_c', ((A_alp')*L22));
disp('L2');
disp(L2);
Nl = lyap(A_c, (A_alp*T));
disp('N1');
disp(Nl);
N21 = lyap(A_c, (N1* A_alp'));
disp('N21');
disp(N21);
N22 = lyap(A_c, (A_alp*N1'))
disp('N22');
disp(N22);
N2 = (N21+N22);
disp('N2');
disp(N2);
H1 = lyap(A_c', ((L12*A_alp) + (A_alp'*L21)));
disp('H1');
disp(H1);
E1 = ((H1*T)+(L12*N1)+(L21*N1')+(L22*N2))
disp('E1');
disp(E1);
E2 = ((L21*T)+(L22*N1))
disp('E2');
disp(E2);
F = -[inv(R)*[ (B_H'*(H*T)+E1)) + (B_alp'*E2)]*[inv(T)]];
disp('New F');
disp(F);
Error= (F - Fold);
NE= (Error* Error');
NF= (F * F');
Percentage= ((NE/NF)*100);
disp('Percentage');
disp(Percentage);

if ( Percentage==0.05)
    break;
else
    continue;
end

end
disp('Final F');
disp(F);
F_L= (F*M_H);
disp('F_L');
disp(F_L);

A_c1= (A_L + (B_L* F_L));
disp('A_c1');
disp(A_c1);
E1=eig(A_c1);
disp('Eig1');
disp(E1);

A_c2= (A2 + (B_L* F_L));
disp('A_c2');
disp(A_c2);
E2=eig(A_c2);
disp('Eig2');
disp(E2);

A_c3= (A3 + (B_L* F_L));
disp('A_c3');
disp(A_c3);
E3=eig(A_c3);
disp('Eig3');
disp(E3);

A_c4= (A4 + (B_L* F_L));
disp('A_c4');
disp(A_c4);
E4=eig(A_c4);
disp('Eig4');
disp(E4);

A_c5= (A5 + (B_L* F_L));
disp('A_c5');
disp(A_c5);
E5=eig(A_c5);
disp('Eig5');
disp(E5);

sys= ss(A_c1, B0, C_L,D0);
sys1=ss(A_c2, B0, C_L, D0);
sys2= ss(A_c3, B0, C_L, D0);
sys3= ss(A_c4, B0, C_L, D0);
sys4= ss(A_c5, B0, C_L, D0);

initial(sys, sys1, sys2, sys3, sys4, x0);

title(' State trajectory responses of a full order system');
xlabel('Time'); ylabel('States of the system');
legend('Alpha=0','Alpha=2','Alpha=5','Alpha=6','Alpha=8');