SCALAR AND VECTOR TOMOGRAPHY FOR THE WEIGHED TRANSPORT EQUATION

A Dissertation by

Nathan Lee Thompson

Master of Science, Wichita State University, 2013

Bachelor of Science, Wichita State University, 2010

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The following faculty members have examined the final copy of this dissertation for form and content, and recommend that it be accepted in partial fulfillment of the requirement for the degree of Doctor of Philosophy with a major in Applied Mathematics.

Alexander Bukhgeym, Committee Chair

Susan Castro, Committee Member

Thomas Delillo, Committee Member

Jason Ferguson, Committee Member

Victor Isakov, Committee Member

Accepted for the College of Liberal Arts and Sciences

Ron Matson, Dean

Accepted for the Graduate School

Dennis Livesay, Dean

DEDICATION

I would like to dedicate this dissertation to my grandparents Clyde and Loyce Maddox, who fostered in me a curiosity and love of learning that has guided me my entire life.

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ABSTRACT

Beginning with the weighted transport equation,

$$Pu(x,\omega) = \langle \omega, \nabla_x u(x,\omega) \rangle + \mu(x)u(x,\omega) = \rho(x,\omega)a(x), \quad x \in \omega \subset \mathbb{R}^2,$$

we examine the properties of acoustic waves that travel below the surface of a solid, such as the Earth or the Sun. According to the laws of geometric optics, these acoustic signals travel along ray paths which obey Fermat's principle of least time. Every material has an optimal path and travel time and deviations from this travel time reveal features of the material hidden within. These optimal paths are encoded into our transport equation by the weight function $\rho(x,\omega)$. Using the Fourier series decomposition of u and ρ with respect to ω, we examine several approximations to an arbitrary weight function and solve for the perturbations to the expected travel time in both the scalar and vector tomography cases. Our results show that the solutions to the resulting systems of differential equations are unique and stable.

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Chapter 1

1 Introduction

1.1 Tomography

Tomography is the study of a body in sections using a penetrating wave. It is a way of examining complicated objects in small pieces or slices, usually with the goal of reconstructing an image of the interior. This approach is seen in many fields, such as medical radiology, biology, atmospheric and weather science, geophysics, plasma physics, material science, astrophysics and quantum information theory, among others. The type of wave chosen as the penetrator is dependent upon which application is being considered. For example, computerized axial tomography (CAT) scans make use of X-rays passing through the human body. By measuring the signal after it passes through the body and employing a mathematical inversion formula, it is possible to reconstruct an image of the inside of the human body, such as the brain, to assist in making a medical diagnosis.

This work focuses on inverting travel time data of a penetrating acoustic wave to recover the internal structure and behavior of large body. In general, the quickest path of an acoustic (pressure) wave from one point on the body's surface to another point is not along the surface. The optimal path is through the interior of the body, because the "speed of sound," or more precisely the acoustic propagation speed in the Earth or Sun, increases with depth. Fermat's principle of least time states that a signal passing through a body will always take the quickest path. The length and shape of this optimal path are functions of the properties of the material, and each signal passing through has an amount of time associated with it to travel from point to another. Deviations from this expected travel time reveal inhomogeneities within the material and be used to infer its interior structure. This gives rise to what is called the inverse kinematics problem, which has been studied for more than a century.

The history of this field is rooted in seismic tomography, and the one-dimensional techniques may be traced to G. Herglotz [16] and E. Weichert [29] who were inspired by the integral equation published by Abel in 1826 [3]. The two and three dimensional approaches originated from the work of J. Radon (1917) [26] who recovered a function in two dimensions by considering its integrals along all straight lines and P. Funk (1916) [14] who recovered a function on the unit sphere through its integrals along all great circles. The Radon transform is the primary basis for the mathematics powering CAT scan technology [26]. However, for physical applications ray paths are rarely straight lines or great circles and nonlinear aspects arise due to the rays depending on unknown velocity distributions. As such, more sophisticated mathematical models than the ones considered by Radon and Funk are necessary. Significant improvements to the theory of seismic tomography and inverse kinematics were made at Russia's Novosibirsk State University by M. M. Lavrentiev, V. G. Romanov, V. G. Vasiliev (1970) who considered the linearized inverse kinematic problem, R. G. Mukometov (1975, 1977) [23] who obtained global uniqueness results for the two-dimensional inverse kinematic problem with complete data, V. G. Romanov (1979) and R. G. Mukometov (1979) who obtained multidimensional global uniqueness results for the inverse kinematic problem, and A. L. Bukhgeĭm (1983) who obtained necessary and sufficient conditions for the solvability of the two-dimensional inverse kinematic problem with partial local data in the class of real analytic functions.

1.2 Time-Distance Helioseismology

The study of inverse kinematics took a new and exciting turn when Duvall et al. (1993) [10] applied the terrestrial seismic tomography techniques from above to the Sun. In *Time-Distance Helioseismology*, Duvall et al. interpreted the travel times between acoustic waves between two points on the Sun's surface and appealed to inverse kinematic theory to recover information about the interior structure. However, this is an inherently more difficult problem than seismic tomography, because the Earth is solid and mostly stationary within. On the Sun, travel times for acoustic signals depend on two factors: the change of acoustic propagation speed as depth increases and the velocity of subsurface flows [17]. So while

Fermat's principle applies, it is necessary to consider the solutions of two separate tomography problems, the scalar tomography case where the goal is to recover the acoustic speed profile and the vector tomography case to recover the subsurface flows.

Kosovichev (1996) developed an inversion technique based upon Fermat's principle and these facts, borrowing from work done in seismic tomography. From his paper, the travel time τ_i along a single ray path Γ_i can be expressed in a standard geometric optics approximation as a function of the acoustic speed $c(\boldsymbol{x}, t)$ and the flow velocity $\boldsymbol{v}(\boldsymbol{x}, t)$:

$$\tau_i(t) = \int_{\Gamma_i} \frac{ds}{c(\boldsymbol{x}, t) + \boldsymbol{v}(\boldsymbol{x}, t) \cdot \boldsymbol{n}}, \tag{1.1}$$

with $\boldsymbol{x}$ as the spatial coordinate, t as time, s the distance along the ray and $\boldsymbol{n}$ the unit vector tangent to the ray. The direction of propagation determines the sign of $\boldsymbol{v} \cdot \boldsymbol{n}$, so flow effects cause travel time differences when waves propagate in opposite directions. Variations in the travel times are small (less than 5% in the areas examined by Kosovichev), so inhomogeneity in the acoustic speed can be examined in a first approximation as perturbations to a static reference acoustic speed profile $c_0(\boldsymbol{x})$. For this static case, the reference travel time is $\tau_{0,i} = \int_{\Gamma_{0,i}} ds/c_0(\boldsymbol{x})$.

He then applies Fermat's principle of least time and expresses the travel times as

$$\delta\tau_i^{\pm} \equiv \tau_i^{\pm}(t) - \tau_{0,i} = -\int_{\Gamma_{0,i}} \frac{\delta c(\boldsymbol{x}, t) \pm \boldsymbol{v}(\boldsymbol{x}, t) \cdot \boldsymbol{n}}{c_0^2(\boldsymbol{x})} ds. \tag{1.2}$$

Here the plus and minus signs signify travel with and against the direction of the prevailing vector field, like a head or tailwind in an airplane. Since travel time varies due to the direction of propagation, it is possible to recover the two functions from (1.2) by splitting the travel time residuals into two equations. The average and difference between travel times are

$$\delta\tau_i^{\text{aver}}(t) \equiv \frac{1}{2}\left[\delta\tau_i^+(t) + \delta\tau_i^-(t)\right] = -\int_{\Gamma_{0,i}} \frac{\delta c(\boldsymbol{x},t)}{c_0^2(\boldsymbol{x})} ds, \tag{1.3}$$

$$\delta\tau_i^{\text{diff}}(t) \equiv \delta\tau_i^+(t) - \delta\tau_i^-(t) = -2\int_{\Gamma_{0,i}} \frac{\boldsymbol{v}(\boldsymbol{x},t)\cdot\boldsymbol{n}}{c_0^2(\boldsymbol{x})} ds \tag{1.4}$$

respectively. It is assumed that δc and $\boldsymbol{v}$ did not change during the observation. The separation of this problem into the recovery of two functions, the acoustic speed perturbation $\delta c(\boldsymbol{x}, t)$ and the tangent component of the ray path flow $\boldsymbol{v}(\boldsymbol{x}, t) \cdot \boldsymbol{n}$, shows how both the scalar and vector tomography problems arise. In this work, we treat both of these problems as separate cases so as to recover as much information about the acoustic speed profile and subsurface flows as possible.

To perform this type of analysis, it is necessary to determine the nature of $c_0(\boldsymbol{x})$, the unperturbed acoustic speed profile, for a solar problem. In seismic cases, acoustic speed $c_0(\boldsymbol{x})$ increases linearly with depth which results in an explicit inversion formula for the acoustic speed profile in the scalar tomography case [5], [7]. Unfortunately, the solar case does not allow the use of this solution because the geodesic ray paths that the acoustic information travels along are not semi-circular in nature. Duvall et al. (1997) states that the typical distance between surface reflections of waves is π times the depth of the ray and the length along the ray path is four times the depth. Geometrically, this defines a cycloid, the classical answer to the brachistochrone problem famously solved by Bernoulli in 1696 [27]. Cycloidal ray paths mean that the square of the acoustic speed $c_0^2(\boldsymbol{x})$ increases linearly with depth. With this understanding, we design an inversion method to recover the acoustic perturbation function $\delta c(\boldsymbol{x}, t)$ and show that the solution to this inversion method is unique and stable.

1.3 Important Past Research

Before considering the case specific to this dissertation, we will consider some classical results of inverse kinematics in more detail. Considering the stationary two-dimensional transport equation, seek to recover the unknown right hand side of

$$Pu(x,\omega) = \langle \omega, \nabla_x u(x,\omega)\rangle + \mu(x)u(x,\omega) = \rho(x,\omega)h(x,\omega) \tag{1.5}$$

where $x \in \Omega \subset \mathbb{R}^2$, $\omega = (\cos\alpha, \sin\alpha) \in S = \{\omega \in \mathbb{R}^2 : |\omega| = 1\}$, $\mu(x)$ is a given attenuation, $\rho(x,\omega)$ is a known weight function, and $h(x)$ is an unknown function that has the form:

$$h(x,\omega) = a(x) \qquad \text{scalar tomography,} \tag{1.6}$$

$$h(x,\omega) = \langle \omega, \boldsymbol{v}\rangle, \quad \boldsymbol{v} = (v_1, v_2) \qquad \text{vector tomography.} \tag{1.7}$$

To recover this function h, we assume that the trace of the function u is known on the boundary $\partial\Omega$

$$u|_{\partial\Omega} = f.$$

The most straightforward classical problem is the one considered by Radon in 1917, who studied (1.5) with $\mu(x) \equiv 0$, $\rho \equiv 1$, and $h(x,\omega) = a(x)$, which reduces exactly to inverting the Radon transform. Next, the first global uniqueness and stability results for $\mu \equiv 0$, $h(x,\omega) = a(x)$, and $\rho(x,\omega)$ so that $\left|\frac{\partial\rho}{\partial\alpha}\right| \leq q\rho$, $\rho > 0$, and $q < 1$ come from much more general work done by Mukometov [23]. Mukometov's method also works for small attenuation μ and $\rho \equiv 1$. This was followed by the first global uniqueness theorem and inversion formula for an arbitrarily smooth (say C^2) $\mu(x)$, $\rho \equiv 1$, and $h(x) = a(x)$ obtained by È. V. Arbuzov, A. L. Bukhgeim and S. G. Kazantsev [2]. A result for the vector case $h(x) = \langle\omega, \boldsymbol{v}\rangle$, $\rho \equiv 1$, $|\mu| \geq \mu_0 > 0$ obtained by A. A. Buhkgeĭm and S. G. Kazantsev shows that is possible to recover the entire vector field $\boldsymbol{v}$. Since this is the same approach as used in this paper, we will give some details about this method.

Suppose that $\rho \equiv 1$ and $h(x) = a(x)$. Replacing $x = (x_1, x_2)$ with $z = z_1 + iz_2$, we write down the Fourier series for $u(z,\omega)$:

$$u(z,\omega) = \sum_{n\in\mathbb{Z}} u_n(z)e^{-in\alpha} \tag{1.8}$$

Since $u(z,\omega)$ is a real-valued function, $\bar{u}_n = u_{-n}$ and hence it is sufficient to find the vector $\boldsymbol{u}(z) = (u_0(z), u_1(z), u_2(z), \ldots) \in l_2(\mathbb{Z}_+)$. For this vector we obtain from equation (1.5) that

$$\bar{\partial}_A \boldsymbol{u} + B(z)\boldsymbol{u} = 0, \qquad z \in \Omega \tag{1.9}$$

$$\boldsymbol{u}|_{\partial\Omega} = \boldsymbol{f} = (f_0, f_1, f_2, \ldots). \tag{1.10}$$

It follows from this that for a function $a(z)$ we have an inversion formula which depends on the functions u_0 and u_1:

$$a = 2\mathrm{Re}\,(\partial_1 u_1) + \mu u_0. \tag{1.11}$$

The operator $\bar{\partial}_A = \bar{\partial} - A\partial$ where $\bar{\partial} = \frac{1}{2}(\partial_1 + i\partial_2)$ and $\partial = \frac{1}{2}(\partial_1 - i\partial_2)$ are the usual complex derivatives. A and $B(z)$ are in a Hilbert space $l_2(\mathbb{Z}_+)$ and are given by

$$Ae_n = -H(n-2)e_{n-2}, \qquad n \in \mathbb{Z}_+ \tag{1.12}$$

$$B(z)e_n = \mu(x)H(n-1)e_n, \qquad n \in \mathbb{Z}_+ \tag{1.13}$$

for the nonnegative integers $\mathbb{Z}_+$. The set $\{e_n : n \in \mathbb{Z}_+\}$ is the standard orthonormal basis in $l_2(\mathbb{Z}_+)$, and

$$H(n) = \begin{cases} 1 & n \geq 0 \\ 0 & n < 0 \end{cases}$$

is the Heaviside function.

Since $[A, B(z)] = 0$, the operators A and $B(z)$ commute, and it is possible to write down the generalized Cauchy-Pompeiu formula for the operator $\bar{\partial}_A + B(z)$,

$$u(z) = e^{-T_A B} C_A e^{T_A B} u + e^{-T_A B} T_A e^{T_A B} (\bar{\partial}_A + B(z)) u \tag{1.14}$$

where we have the analogs to the Cauchy integral formula and Teodorescu transform for $\bar{\partial}_A$:

$$C_A u(z) = \frac{1}{2\pi} \int_{\partial\Omega} \nu_A (\zeta - z)_{-A}^{-1} u(\zeta)\, ds, \tag{1.15}$$

$$T_A u(z) = \frac{1}{\pi} \int_{\Omega} (\zeta - z)_{-A}^{-1} u(\zeta)\, d\zeta. \tag{1.16}$$

In this expression, $z_{-A}^{-1} = (z - \bar{z}A)^{-1}$ and $\nu_A = \nu - \bar{\nu}A$ for the outward unit normal $\nu = \nu_1 + i\nu_2$ to the boundary $\partial\Omega$. Details about how these formulas arise given in [5], [4].

With formula (1.14), we find the vector $\boldsymbol{u} = (u_0, u_1, u_2, \ldots)$ using equations (1.9) and (1.10),

$$\boldsymbol{u}(z) = e^{-T_A B} C_A e^{T_A B} \boldsymbol{f} \tag{1.17}$$

and recover $a(x)$ using formula (1.11). In the vector case $h(x, \omega) = \langle \omega, \boldsymbol{v} \rangle$ and $|\mu| \neq 0$ we have a similar purely algebraic formula in terms of μ and the vector $\boldsymbol{u}$.

1.4 General Outline

The situation changes dramatically when we replace the condition $\rho \equiv 1$ in equation (1.5) with a trigonometric polynomial. The formula (1.14) still applies for the vector $u = (u_{k+1}, u_{k+2}, \ldots)$ for $k \geq 0$ that depends on the degree of the polynomial ρ. However, for the first $k+1$ components $(u_0, u_1, \ldots, u_k)$ we have a system of partial differential equations with unknown right hand side. Our primary focus is to investigate this inverse problem. We show how to transform this inverse problem into the Cauchy problem for the system

$$Pu := \partial_1 u + A\partial_2 u + Bu = 0 \tag{1.18}$$

in domain Ω, where now $A(z)$ and $B(z)$ are given real-valued $n \times n$ matrices and the full Cauchy data for the vector u is known on the boundary $\partial\Omega$.

Assuming that all eigenvalues of the matrix $A(z)$ are distinct for all $z \in \Omega$ we prove the estimate

$$\|u\|_{L_2(\Omega;\mathbb{R}^n)} \leq C\left(\|Pu\|_{L_2(\Omega;\mathbb{R}^n)} + \|Tu\|_{W_2^1(\partial\Omega;\mathbb{R}^n)}\right) \qquad \forall u \in C^1(\bar{\Omega};\mathbb{R}^n) \tag{1.19}$$

where $Tu := u|_{\partial\Omega}$, the trace of u on the boundary.

We investigate two cases of the trigonometric polynomial

$$\rho(x,\omega) = \sum_{|n|\leq N} \check{\rho}_n(x) e^{-in\alpha}, \tag{1.20}$$

where

1. $N = 2$, $\check{\rho}_1 \equiv 0$
2. $N = 4$, $\check{\rho}_1 = \check{\rho}_3 \equiv 0$.

The motivation to investigate these two cases comes from the aforementioned applications of three dimensional seismology and helioseismology in a linearized setting. For this case,

$$\check{\rho}(x,\omega) = \sqrt{1 + (\varphi')^2 \frac{\langle x,\omega\rangle^2}{|x|^2}}. \tag{1.21}$$

where the function φ depends on a static acoustic speed profile $c_0(\boldsymbol{x})$, $\boldsymbol{x} \in \mathbb{R}^3$ which we assume has the form $c_0(\boldsymbol{x}) = c_0(x_3)$ where the acoustic speed only depends on depth and the true (measured) acoustic speed $c(\boldsymbol{x},t) \approx c(\boldsymbol{x})$ may be expressed as

$$c(\boldsymbol{x},t) \approx c_0(\boldsymbol{x}) + c_1(\boldsymbol{x}). \tag{1.22}$$

This is the stationary transport equation, so time is neglected, and $c_1(\boldsymbol{x})$ is the perturbation from the expected travel time $c_0(\boldsymbol{x})$. In the seismic case, the magnitude of the perturbation

$|c_1| << c_0$ as $c_0(x_3)$ increases solely with depth. For the helioseismic case $|c_1| \leq c_0$ where $c_1(\boldsymbol{x}, t)$ also depends on the velocity of a vector field $\boldsymbol{v}$ of moving plasma.

We show that for $\rho(x, \omega)$ of our chosen type (1.21), the odd coefficients $\check{\rho}_{2n+1} \equiv 0$, and

$$|\check{\rho}_0| \leq \sqrt{1+q^2} \tag{1.23}$$

$$|\check{\rho}_{2n}| \leq \sqrt{1+q^2}\left(\frac{q^2}{1+q^2}\right)^{2n} \frac{2}{\pi(4n^2-1)} \tag{1.24}$$

for $q = \varphi'(|x|)$ and $n = \{1, 2, \ldots\}$. From (1.24), we see that for $n \geq n_0 \geq 3$ that $\check{\rho}_{2n} \approx 0$ and so taking only a small number of terms in the trigonometric polynomial is sufficient. For the scalar tomography case, we assume that

$$u|_{\partial\Omega \times S} = f(x, \omega) \tag{1.25}$$

is a given function (measured from data in applications). In the vector tomography case, we have the data (1.25) and require the additional assumptions that

$$\left.\frac{\partial u_1}{\partial \nu}\right|_{\partial\Omega} = g_1 \qquad \text{when } N = 2 \tag{1.26}$$

$$\left.\frac{\partial u_j}{\partial \nu}\right|_{\partial\Omega} = g_j, \quad j = 1, 3 \qquad \text{when } N = 4 \tag{1.27}$$

where the g_j's are known in each case and the functions $u_n = \hat{u}_{-n}$ are the Fourier coefficients of the function u:

$$\hat{u}_n = \int_{-\pi}^{\pi} u(x, \omega) e^{-in\alpha}\, d\alpha. \tag{1.28}$$

The full recovery of the vector field $v(x)$ requires that the attenuation $|\mu(x)| \geq \mu_0 > 0$ for all $x \in \Omega$.

The estimate guaranteeing uniqueness and stability (1.19) is proven using a Carleman estimate with boundary terms for our operator P (1.18). We use the results and methods of classical works from T. Carleman [9] and A. P. Calderon [8]. As a final step, we verify using MATLAB® computations that the matrix A has distinct eigenvalues everywhere in a domain specific to the application of helioseismology.

Chapter 2

2 The Integral Geometry Problem

2.1 The Transport Equation in 2D

We consider a stationary two-dimensional transport equation given with the linear first order partial differential operator P [5]:

$$Pu(x,\omega) = \langle \omega, \nabla_x u(x,\omega)\rangle + \mu(x)u(x,\omega) = a(x), \qquad x \in \Omega \subset \mathbb{R}^2. \tag{2.1}$$

Here $\langle \cdot, \cdot \rangle$ is the usual scalar product in $\mathbb{R}^2$, $\nabla_x = (\partial_1, \partial_2)$, $u(x,\omega)$ is the density of particles at a point x moving in direction ω for $\omega = (\cos\alpha, \sin\alpha) \in S = \{\omega \in \mathbb{R}^2 : |\omega| = 1\}$, $\mu(x)$ is the attenuation in the body at point x and $a(x)$ is the acoustic source (perturbation) in the body at point x. The domain Ω is a strictly convex subset of $\mathbb{R}^2$ with a C^1 boundary $\partial\Omega$.

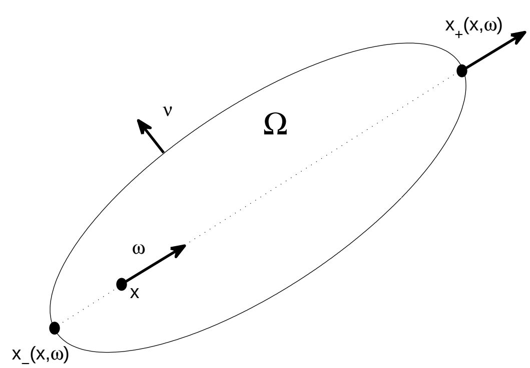


Figure 2.1

For $x \in \partial\Omega$, let

$$u(x,\omega)|_{\partial\Omega\times S} = f(x,\omega) = \begin{cases} f_-(x,\omega) & \langle x, \nu\rangle < 0 \\ f_+(x,\omega) & \langle x, \nu\rangle > 0 \end{cases}. \tag{2.2}$$

Here ν is the outward unit normal, as indicated in Figure 2.1. Physically, f_- represents the incoming particle flow into the domain Ω and f_+ is the outgoing particle flow from Ω. The direct problem for this system is to find the particle density $u(x, \omega)$ and hence f_+ given a, μ and f_-. The inverse problem is determining a given the trace of u on $\partial\Omega \times S$ and the attenuation μ. Taking the variables to be in the complex plane, $z = x_1 + ix_2$ and writing $u(x, \omega) = u(z, \alpha)$, $f(x, \omega) = f(z, \alpha)$, $\mu(x) = \mu(z)$ and $a(x) = a(z)$ for $z \in \mathbb{C}$ and $\omega = \omega(\alpha) = (\cos\alpha, \sin\alpha)$, we may apply Euler's formulae

$$\cos\alpha = \frac{e^{i\alpha} + e^{-i\alpha}}{2} \qquad\qquad \sin\alpha = \frac{e^{i\alpha} - e^{-i\alpha}}{2i}$$

to the equation (2.1) to obtain

$$e^{i\alpha} P u = \left(\bar{\partial} + e^{2i\alpha}\partial + e^{i\alpha}\mu\right) u = e^{i\alpha} a(x). \tag{2.3}$$

for the complex differential operators $\partial = \frac{1}{2}(\partial_1 - i\partial_2)$ and $\bar{\partial} = \frac{1}{2}(\partial_1 + i\partial_2)$. The Fourier series expansion of $u(z, \alpha)$ with respect to α is of the form

$$u(z, \alpha) = \sum_{n \in \mathbb{Z}} \hat{u}_n(z) e^{in\alpha} \qquad\qquad \hat{u}_n = \int_0^{2\pi} u(z, \alpha) e^{-in\alpha} d\alpha.$$

For notational convenience, we define $u_n(z) = \hat{u}_{-n}(z)$, the negative Fourier coefficients, and obtain

$$u(z, \alpha) = \sum_{n \in \mathbb{Z}} u_n e^{-in\alpha}. \tag{2.4}$$

Substituting this into (2.3) yields

$$e^{i\alpha} P u = \sum_{n \in \mathbb{Z}} \left(\bar{\partial} u_n e^{-in\alpha} + \partial u_n e^{-i(n-2)\alpha} + \mu u_n e^{i(n-1)\alpha}\right) = e^{i\alpha} a(x).$$

Finally, we change indices and the expression simplifies to

$$e^{i\alpha}Pu = \sum_{n\in\mathbb{Z}} \left(\bar{\partial}u_n + \partial u_{n+2} + \mu u_{n+1}\right) e^{-in\alpha} = e^{i\alpha}a(x). \tag{2.5}$$

The function u is real-valued, so $u_n = \bar{u}_{-n}$. Define the vector $\boldsymbol{u} = (u_0, u_1, u_2, \ldots) \in l_2(0, \infty)$. Using the shift operator

$$U : (u_0, u_1, u_2, \ldots) \mapsto (0, u_0, u_1, u_2, \ldots),$$

its adjoint U^* in $l_2(0, \infty)$, $A = -(U^*)^2$ and $\bar{\partial}_A = \bar{\partial} - A\partial$, we may cast this inverse problem as follows:

Problem 2.1. *Let $\boldsymbol{u}$ be the solution of the Cauchy problem*

$$\begin{aligned} \bar{\partial}_A \boldsymbol{u} + \mu U^* \boldsymbol{u} = 0 \quad &in\ \Omega, \\ \boldsymbol{u}|_{\partial\Omega} = \boldsymbol{f}, & \end{aligned} \tag{2.6}$$

where $\bar{\partial}_A$ is a Beltrami-type operator with operator coefficient A. Reconstruct the function $\boldsymbol{u}$ from its trace $\boldsymbol{f}$.

This problem is solved in [2], [5] using an analog for the standard Cauchy integral formula.

$$C_A u(z) = \frac{1}{2\pi} \int_{\partial\Omega} \nu_A \left(\zeta - z\right)^{-1}_{-A} u(\zeta) ds$$

Here $\nu_A = \nu - \bar{\nu}A$ where $\nu = \nu_1 + i\nu_2$ is the outward unit normal to $\partial\Omega$, and $z_A^{-1} = (z + \bar{z}A)^{-1}$. In our case it is not possible to fully recover $\boldsymbol{u}$, and the data in each case will dictate how many of its components may be determined.

2.2 Weighted Transport Equation

Problem 2.1 supposes that the domain the information is passing through is flat. Since we are examining acoustic waves that pass beneath the surface, it is necessary to introduce a way to make the physical problem compatible with the mathematical one. Choosing coordinates $x = (x_1, x_2)$ and $z = x_3$, we linearize about an acoustic "slowness" n_0 that only depends on the depth z. The ray path Γ_{n_0} for the acoustic "slowness" $n_0(z) = 1/c_0(z)$ that connects two points intersecting the plane perpendicular to the x_1, x_2 directions $p_1 := \langle x, \nu \rangle - h = 0$ and the circle $x_1^2 + x_2^2 = r^2$ on the plane $(x_1, x_2, 0)$ is given by the intersection of p_1 and the two-dimensional surface $p_2 := z - \varphi(|x|, r) = 0$. Surface p_2 is obtained by rotating the ray that connects points $(r, 0, 0)$ and $(-r, 0, 0)$ about the z axis.

We now introduce a weight function ρ that transforms integrals over Γ_{n_0} on the surface $p_2 = 0$ to integrals over straight lines in a disk with $|x| < r$,

$$\rho(x, \omega) = \sqrt{1 + (\varphi')^2 \frac{\langle x, \omega \rangle^2}{|x|^2}}. \tag{2.7}$$

Under this transformation, we have that the integral over a ray path Γ_{n_0} takes the form

$$\int_{\Gamma_{n_0}} f \, d\sigma = \int_{|x|<r} \delta(\langle x, \nu \rangle - h) f(x, \varphi(|x|, r)) \rho(x, \omega) \, dx$$

In practice, this expression for $\rho(x, \omega)$ is very difficult to work with. As an approximation, we will replace $\rho(x, \omega)$ with a finite Fourier series. A finite approximation is acceptable, as we will show the Fourier coefficients diminish rapidly.

Taking the Fourier coefficients of $\rho(x, \omega)$ with negative coefficients (denoted by $\check{\rho}$) yields

$$\check{\rho}_n(x, \omega) = \frac{1}{2\pi} \int_{-\pi}^{\pi} \rho(x, \omega) e^{in\alpha} \, d\alpha. \tag{2.8}$$

To simplify this expression, we make the following substitutions: $\nu = \dfrac{x}{|x|} = e^{i\beta}$ and $\omega = e^{i\alpha}$ which imply that $\langle \nu, \omega \rangle_{\mathbb{R}^2} = \mathrm{Re}(e^{i\beta}\overline{e^{i\beta}}) = \cos(\alpha - \beta)$. Along with setting $q = \varphi'(|x|)$, we achieve

$$\begin{aligned}\check{\rho}_n(x,\omega) &= \frac{1}{2\pi}\int_{-\pi}^{\pi}\sqrt{1+q^2\cos^2(\alpha-\beta)}e^{in\alpha}\,d\alpha \\ &= \frac{1}{2\pi}\int_{-\pi-\beta}^{\pi-\beta}\sqrt{1+q^2\cos^2\theta}e^{i\theta}e^{in\beta}\,d\theta \end{aligned} \tag{2.9}$$

after a change of variables $\alpha - \beta = \theta$.

To find the value of this integral, we express it in terms of elliptic integrals. The radicand above, $1 + q^2 \cos^2\theta$, can be rewritten using the trigonometric Pythagorean identity.

$$\begin{aligned}1+q^2\cos^2\theta &= 1+q^2(1-\sin^2\theta) \\ &= (1+q^2)\left(1-\frac{q^2}{1+q^2}\sin^2\theta\right)\end{aligned} \tag{2.10}$$

Taking the coefficient $k^2 = q^2/(1+q^2)$, where $0 \geq k^2 > 1$, and shifting the limits of integration to $-\pi$ and π (by 2π periodicity), the Fourier coefficients of $\rho(x,\omega)$ are expressed as

$$\check{\rho}_n(x,\omega) = \sqrt{1+q^2}\,\frac{e^{in\beta}}{2\pi}\underbrace{\int_{-\pi}^{\pi}\sqrt{1-k^2\sin^2\theta}e^{in\theta}\,d\theta}_{I_n}. \tag{2.11}$$

The term I_n has the elliptic form we seek. I_n is an even function, so the term $e^{in\theta}$ is replaced with $\cos n\theta$ and by symmetry is expressed as

$$I_n = \int_{-\pi}^{\pi}\cos n\theta\sqrt{1-k^2\sin^2\theta}\,d\theta = 2\int_0^{\pi}\cos n\theta\sqrt{1-k^2\sin^2\theta}\,d\theta.$$

For $n = 0$, $I_0 = 2\int_0^{\pi}\sqrt{1-k^2\sin^2\theta}\,d\theta = 4\int_0^{\frac{\pi}{2}}\sqrt{1-k^2\sin^2\theta}\,d\theta$ due to symmetry about $\frac{\pi}{2}$ for the integrand. This is precisely the complete elliptic integral of the second kind.

Traditional notation for the complete elliptic integral of the second kind is $E(k) = \int_0^{\frac{\pi}{2}}\sqrt{1-k^2\sin^2\theta}\,d\theta$. Since $E(0) = \frac{\pi}{2}$ and E(1) = 1 and our $k \in [0,1)$, we examine these elliptic integrals in this domain. Additionally, we examine incomplete elliptic integrals of the first kind to handle the terms I_n for $n \neq 0$.

$$F(\phi, k) = \int_0^{\phi} \frac{1}{\sqrt{1 - k^2 \sin^2 \theta}} \, d\theta$$

Setting $\phi = \frac{\pi}{2}$, we obtain the complete elliptic integral of the first kind $K(k) = F(\frac{\pi}{2}, k)$. The formula for this integral can be written explicitly as an infinite series:

$$\begin{aligned} K(k) &= \frac{\pi}{2} \sum_{m=0}^{\infty} \left[\frac{(2m)!}{2^{2m}(m!)^2} \right] \\ &= \frac{\pi}{2} \, {}_2F_1 \left(\frac{1}{2}, \frac{1}{2}; 1; k^2 \right). \end{aligned} \tag{2.12}$$

The notation for (2.12) is a hypergeometric series. In general, this notation is interpreted as

$${}_2F_1(a, b; c; z) = \sum_{m=0}^{\infty} \left[\frac{(a)_m (b)_m}{(c)_m} \frac{z^m}{m!} \right]. \tag{2.13}$$

The terms $(x)_m = \Gamma(x + m)/\Gamma(x)$ are the Pochhammer symbol for the rising factorial, most easily interpreted as the quotient of gamma functions. The derivative of $E(k)$,

$$\frac{dE(k)}{dk} = \frac{d}{dk} \int_0^{\frac{\pi}{2}} \sqrt{1 - k^2 \sin^2 \theta} \, d\theta = \int_0^{\frac{\pi}{2}} \frac{1}{2} \frac{-2k \sin^2 \theta}{\sqrt{1 - k^2 \sin^2 \theta}} \, d\theta < 0,$$

is negative, which implies that $E(k)$ is a decreasing function and hence bound by $\frac{\pi}{2}$. With this information, we draw the conclusion that $I_0 = 4E(k) \leq 4 \cdot \frac{\pi}{2} = 2\pi$. Hence, recalling (2.11) we have the following estimate for $\check{\rho}_0$:

$$|\check{\rho}_0| \leq \sqrt{1 + q^2} \tag{2.14}$$

We now move forward examining the odd terms I_{2n+1} for $n \in \mathbb{N}$. Looking at the original formula for our integrals with $n := 2n+1$ substituted in, we have

$$I_{2n+1} = 2\int_0^{\pi} \cos(2n+1)\theta\sqrt{1-k^2\sin^2\theta}\; d\theta.$$

Changing variables to $t = \theta - \frac{\pi}{2}$ yields

$$\begin{aligned} I_{2n+1} &= 2\int_{-\pi/2}^{\pi/2} \cos(2n+1)\left(t+\frac{\pi}{2}\right)\sqrt{1-k^2\sin^2\left(t+\frac{\pi}{2}\right)}\; dt \\ &= 2\int_{-\pi/2}^{\pi/2} \cos\left[(2n+1)t+\pi n+\frac{\pi}{2}\right]\sqrt{1-k^2\sin^2\left(t+\frac{\pi}{2}\right)}\; dt \\ &= 2\int_{-\pi/2}^{\pi/2} -\sin\left[(2n+1)t+\pi n\right]\sqrt{1-k^2\cos^2 t}\; dt \qquad (2.15) \end{aligned}$$

In the form given by (2.15), we can show that the integrand is the product of an even and odd function and hence an odd function with respect to the limits of integration. There are two cases, one for n even and another for n odd. If n is even, then $\sin(x+\pi n) = \sin x$ and if n is odd, $\sin(x+\pi n) = -\sin(x)$. In both cases, the resulting definite integral is zero:

$$I_{2n+1} = \mp\, 2\int_{-\pi/2}^{\pi/2} \underbrace{\sin(2n+1)t}_{\text{odd}}\,\underbrace{\sqrt{1-k^2\cos^2 t}}_{\text{even}}\; dt = 0.$$

Hence, all odd Fourier coefficients $\check{\rho}_{2n+1} = 0$ for our weight function $\varphi(x,\omega)$.

Lastly, we would like to investigate the terms I_{2n} for $n \in \mathbb{N},\ n \neq 0$. The formula for this integral appears in [24],

$$\begin{aligned} I_{2n} &= 2\int_0^{\pi} \cos 2n\theta\sqrt{1-k^2\sin^2\theta}\; d\theta \\ I_{2n} &= \frac{-\pi(2n-3)!!\,k^{2n}}{2^{3n-1}\,n!}\,{}_2F_1\left(\frac{2n-1}{2},\frac{2n+1}{2};2n+1;k^2\right). \qquad (2.16) \end{aligned}$$

Where $(2n-3)!!$ is the double factorial and the second term is a hypergeometric series (2.13). This series may be estimated by taking $k^2 = 1$, and by a standard formula provided in [21], we have

$$ {}_2F_1\left(\frac{2n-1}{2}, \frac{2n+1}{2}; 2n+1; k^2\right) \leq {}_2F_1\left(\ldots; 1\right) = \frac{\Gamma(1)\Gamma(2n+1)}{\Gamma\left(n+\frac{3}{2}\right)\Gamma\left(n+\frac{1}{2}\right)}. $$

Therefore, we have the following estimate for I_{2n}:

$$ |I_{2n}| \leq \frac{\pi(2n-3)!!\,k^{2n}}{2^{3n-1}\,n!} \frac{\Gamma(1)\Gamma(2n+1)}{\Gamma\left(n+\frac{3}{2}\right)\Gamma\left(n+\frac{1}{2}\right)}. \tag{2.17} $$

To simplify the estimate, the following identities are employed:

$$ \begin{aligned} \frac{\Gamma(2n+1)}{\Gamma(n+\frac{1}{2})} &= \frac{2^{2n}\Gamma(n+1)}{\sqrt{\pi}}, \\ (2n-3)!! &= \frac{2^{n-1}\Gamma(n-\frac{1}{2})}{\sqrt{\pi}}. \end{aligned} $$

Making these substitutions, we have

$$ \begin{aligned} |I_{2n}| &\leq \frac{\sqrt{\pi}(2n-3)!!\,k^{2n}}{2^{n-1}\,n!} \frac{\Gamma(n+1)}{\Gamma(n+\frac{3}{2})} \\ &= \frac{k^{2n}\Gamma(n-\frac{1}{2})}{\Gamma(n+\frac{3}{2})} \\ &= \frac{k^{2n}\Gamma(n-\frac{1}{2})}{(n+\frac{1}{2})(n-\frac{1}{2})\Gamma(n-\frac{1}{2})} \\ &= \frac{4k^{2n}}{(2n+1)(2n-1)} \\ &= \frac{4k^{2n}}{4n^2-1}. \end{aligned} \tag{2.18} $$

This yields an estimate for $\check{\rho}_{2n}$,

$$ |\check{\rho}_{2n}| \leq \sqrt{1+q^2}\,\frac{2}{\pi}\left(\frac{q^2}{1+q^2}\right)^{2n} \frac{1}{4n^2-1}. \tag{2.19} $$

Note that the coefficients decrease exponentially as n increases because $q^2/(1+q^2) < 1$ and is raised to the $2n$ power, and we have the term $4n^2-1$ in the denominator. In summary, the Fourier coefficients for the weight function (2.7) are

$$\check{\rho}_{2n+1} = 0 \tag{2.20}$$

$$\check{\rho}_0 = \sqrt{1+q^2}\,\frac{2E(k)}{\pi} \tag{2.21}$$

$$\check{\rho}_{2n} = -\sqrt{1+q^2}\,\frac{(2n-3)!!\,k^{2n}}{2^{3n-2}\,n!}\,{}_2F_1\left(\frac{2n-1}{2},\frac{2n+1}{2};2n+1;k^2\right) \tag{2.22}$$

2.3 Tomography in Three Dimensions

Fundamentally, there are two tomography problems to investigate using the approach we have developed, the scalar and vector tomography cases. In each case, we wish to determine a function or a vector field in a three dimensional region of some body. For the scalar case, we recover the acoustic source function $a(x)$ as it appears in equation (2.3). Vector tomography is more difficult, as our task is to recover a vector field $\boldsymbol{v} = (v_1(x), v_2(x))$ whose components v_1 and v_2 are scalar functions. Instead of having $a(x)$ on the right hand side as in equation (2.3), we have the term $\langle \omega, \boldsymbol{v} \rangle$. The specifics of these two cases will be handled later.

In addition, each time we solve the inverse problem for the transport equation as described here, we only recover the acoustic source or vector field for a single "shell" within our region. To recover the desired information in the entire region, it is necessary to solve our tomography problem on several of these shells. To cover our region, we create a foliation of shells to fill the space. The foliation closely resembles the famous *matryoshka* dolls from Russia. By solving the tomography problem on a series of concentric shells that fit our region, it is possible to reconstruct the scalar source and vector field in a 3D region by studying one section at a time and combining the results.

Figure 2.2: ©2011 http://jronaldlee.com/ J. Ronald Lee, Creative Commons License

The shells are nested concentrically because this is the most stable geometric configuration to utilize our data. Before this fact was known, studies were done using typical rectangular grids. Due to the work done by A. L. Bukhgeĭm, S. M. Zerkal, and V. V. Pikalov [7], it is known that latitudinal slices are numerically unstable and that a concentric foliation gives a stable algorithm for recovering the interior. This stems from the fact that the latitudinal slices are angle limited and do not take into account all possible data for the integral geometry problem.

Numerically Unstable Numerically Stable

Figure 2.3: Slices of 3D Region

This is the top down view of the measurement grid. We, of course, choose the numerically stable geometry to proceed.

In the application of helioseismology, measurements of the magnetic field are taken to determine the time required for a signal to pass from one point on a ring to another. These travel times are averaged to determine the flow rate at many points on the surface [13], and in turn these flows may interpreted as data for the integral geometry problem of the transport equation. The only missing element is an approximate shape for our shells so that we may have a baseline to compare the expected travel times to and hence determine the perturbations that lie beneath the surface. In general, the optimal ray path, as predicted by Fermat's principle of least time, is given as $\Gamma(|x|, r)$. The systems constructed in the next chapter are valid for any radially symmetric set of ray paths, but numerical experiments are conducted using the ray path followed by acoustic signals in the Sun.

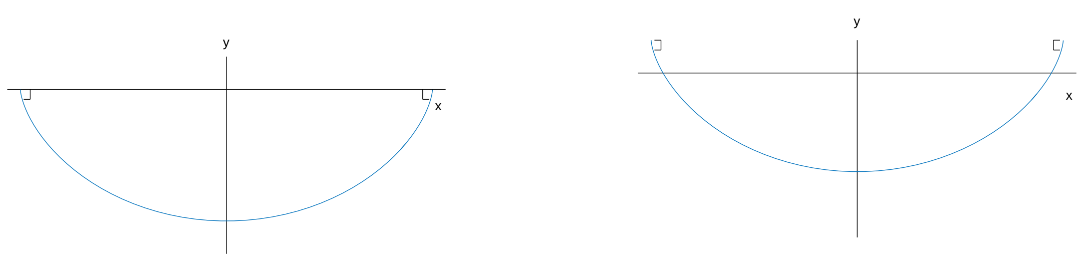
In the Sun, the square of the acoustic propagation speed $c^2(x, z)$ increases roughly linearly with depth z [12] (as opposed to the acoustic propagation speed $c(x, z)$ increasing linearly with depth as is the case with the Earth [20]). This means that the geodesics $\Gamma(|x|, r)$ along $\varphi(|x|, r)$ under the surface of the Sun are cycloids and must be defined as the relationship between $|x|$ and z with respect to parameter $t \in [\pi, t_0]$ [27], [12]:

$$\Gamma(|x|, r) = \begin{cases} |x| = R\,(t - \sin(t)) \\ z = R\,(1 - \cos(t)) - d \end{cases} \tag{2.23}$$

where R and t_0 are the solutions of the equations $|x(t_0)| = R$ and $z(t_0) = 0$ and d is a vertical shift (for the Earth, the path is a vertically shifted semicircle [20]).

Lastly, we note that the shells where we perform our inversion are not full cycloids, but truncated sections of them. A full cycloid has vertical tangent lines at the boundary, which

Cycloid Ray Path



Vertical Tangents at Boundary Physical Case Shifted Upward

would indicate a flow straight upwards on the surface. Our analysis is impossible in this case, so we assume that the shape is shifted upward as in the right figure so that no vertical tangents occur at the boundary. This vertical shift if encoded into (2.23) by d and r is the distance from the origin to the intersection of ray path with the x-axis, the radius of a measurement ring in our foliation. With all of the geometric details in place, we may move to the formulation to our problem using the transport equation an an approximation to one of these ray paths.

Chapter 3

3 Constructing the Systems of Partial Differential Equations

3.1 The Scalar Tomography Problem

Problem 3.1. *Scalar Tomography: Recover the scalar acoustic perturbation function $a(x)$ by solving Problem 2.1 for equation (2.5) with the right hand side weighed by $\rho(x,\omega)$:*

$$e^{i\alpha} Pu = \sum_{n\in\mathbb{Z}} \left(\bar{\partial} u_n + \partial u_{n+2} + \mu u_{n+1}\right) e^{-in\alpha} = e^{i\alpha}\rho(x,\omega)a(x). \tag{3.1}$$

For our approach, the function $\rho(x,\omega)$ will be replaced with a finite trigonometric approximation. We will investigate the first few approximations separately, then attempt to understand how to handle these cases in general as the number of terms increase. It should be noted that while we discussed a specific

$$\rho(x,\omega) = \sqrt{1 + (\varphi')^2 \frac{\langle x,\omega\rangle^2}{|x|^2}},$$

our development of systems of scalar and vector tomography problems is not limited to just this $\rho(x,\omega)$. Any weight function that can be represented by a finite trigonometric polynomial may be used in its place.

We begin with the first approximation $\rho(x,\omega) \approx \check{\rho}_0(x)$. This changes equation (3.1) to

$$\sum_{n\in\mathbb{Z}} \left(\bar{\partial} u_n + \partial u_{n+2} + \mu u_{n+1}\right) e^{-in\alpha} = e^{i\alpha}\check{\rho}_0(x)a(x). \tag{3.2}$$

Now we collect the finite system of equations with nonzero right hand sides by examining the equation associated with each integer n. For this case, there is only one such equation.

Overall, our system looks like

$$\bar{\partial}\bar{u}_1 + \partial u_1 + \mu u_0 = \check{\rho}_0(x)a(x) \qquad n = -1 \tag{3.3}$$

$$\bar{\partial}u_n + \partial u_{n+2} + \mu u_{n+1} = 0 \qquad n \in \mathbb{Z} \setminus \{-1\} \tag{3.4}$$

Using the theory of A-analytic functions and the solution developed for Problem 2.1, we know that our boundary data for the function u is such that $\boldsymbol{f} = \boldsymbol{u}|_{\partial\Omega \times S} = (f_0, f_1, \ldots)$. We know $\boldsymbol{f}$ starting for $n = 0$ because that is the lowest integer for which all larger n are associated with equations of zero right hand side. It also means that for this case, all Fourier components of the function $\boldsymbol{u} = (u_0, u_1, \ldots)$ can be reconstructed from measurements. Hence, finding our acoustic source $a(x)$ is as simple as rearranging equation (3.3), since $\check{\rho}_0(x) \neq 0$,

$$a(x) = \frac{\bar{\partial}\bar{u}_1 + \partial u_1 + \mu u_0}{\check{\rho}_0(x)}. \tag{3.5}$$

This case, however, is clearly trivial. We gain more insight into the nature of these systems by moving to the second approximation scalar case, where

$$\rho(x, \omega) \approx \check{\rho}_{-2}(x)e^{2i\alpha} + \check{\rho}_0(x) + \check{\rho}_2(x)e^{-2i\alpha}.$$

Problem 3.2. *Using the second-order approximation of the weight function*

$$\rho(x, \omega) \approx \check{\rho}_{-2}(x)e^{2i\alpha} + \check{\rho}_0(x) + \check{\rho}_2(x)e^{-2i\alpha},$$

determine the scalar acoustic source function $a(x)$ from the weighted transport equation

$$\sum_{n\in\mathbb{Z}} \left(\bar{\partial}u_n + \partial u_{n+2} + \mu u_{n+1}\right) e^{-in\alpha} = e^{i\alpha}\left[\check{\rho}_{-2}(x)e^{2i\alpha} + \check{\rho}_0(x) + \check{\rho}_2(x)e^{-2i\alpha}\right] a(x). \tag{3.6}$$

Replacing the weight function $\rho(x, \omega)$ with this approximation and collecting equations yields the following system:

$$\bar{\partial}\bar{u}_3 + \partial\bar{u}_1 + \mu\bar{u}_2 = \check{\rho}_{-2}(x)a(x) \qquad n = -3 \qquad (3.7)$$

$$\bar{\partial}\bar{u}_1 + \partial u_1 + \mu u_0 = \check{\rho}_0(x)a(x) \qquad n = -1 \qquad (3.8)$$

$$\bar{\partial}u_1 + \partial u_3 + \mu u_2 = \check{\rho}_2(x)a(x) \qquad n = 1 \qquad (3.9)$$

$$\bar{\partial}u_n + \partial u_{n+2} + \mu u_{n+1} = 0 \qquad n \in \mathbb{Z} \setminus \{-3, -1, 1\} \qquad (3.10)$$

Purely zero right hand sides begin for the system being for $n \geq 2$, so the functions $(u_2, u_3, \ldots)$ are known from data. Additionally, we see that the equation (3.5) appears as a rearrangement of equation (3.8). We have an explicit inversion formula for the acoustic source, one that requires knowledge of u_0 and u_1 to recover $a(x)$. Also, note that this precise formula for $a(x)$ will appear in all subsequent scalar tomography cases under this approximation scheme. This means that we will depend on the appearance of this expression for $a(x)$ to substitute back into our system to transform it into a configuration that more easily allows for the determination of the Fourier components of $\boldsymbol{u}$.

To guarantee the uniqueness of our Fourier components of $\boldsymbol{u}$, we are going to cast our system in a form studied by Torsten Carleman in 1939. In his paper, he examined systems of the form

$$\frac{\partial w_p}{\partial x} + \sum_{q=1}^{n} B_{pq}(x,y)\frac{\partial w_p}{\partial y} + \sum_{q=1}^{n} C_{pq}(x,y)w_q = 0, \qquad (3.11)$$

where w_p is a component of $\boldsymbol{w} = (w_1, w_2, \ldots, w_n)$, B_{pq} are a set of functions that are real and twice continuously differentiable, and the functions C_{pq} are real and continuous. To more easily understand this system, we express it using matrix notation.

$$\frac{\partial \boldsymbol{w}}{\partial x} + B(x,y)\frac{\partial \boldsymbol{w}}{\partial y} + C(x,y)\boldsymbol{w} = 0 \qquad (3.12)$$

By examining this system, Carleman determined that when the eigenvalues of the matrix B

are distinct, the solution of the system $\boldsymbol{w}$ is unique [9]. To solve our tomography problems, we will take advantage of this fact by rewriting our complex-valued systems in real coordinates to match this Carleman form. After placing our system in this form we will show that the eigenvalues are distinct, which guarantees that our acoustic source $a(x)$ may be recovered.

With this goal in mind, we return to the second approximation scalar system. From this system, we require the following equations:

$$a(x) = \frac{\bar{\partial}\bar{u}_1 + \partial u_1 + \mu u_0}{\check{\rho}_0(x)} \qquad n = -1$$

$$\bar{\partial} u_0 + \partial u_2 + \mu u_1 = 0 \qquad n = 0$$

$$\bar{\partial} u_1 + \partial u_3 + \mu u_2 = \check{\rho}_2(x) a(x) \qquad n = 1$$

These equations are chosen from our system because they are the ones involving the unknown components of $\boldsymbol{u}$, u_0 and u_1. Equation (3.7) is omitted because it is the complex conjugate of (3.9). The next step is to substitute the formula for $a(x)$ into the other equation so that our system only depends on $\boldsymbol{u}$.

$$\bar{\partial} u_0 + \partial u_2 + \mu u_1 = 0 \qquad n = 0 \qquad (3.13)$$

$$\bar{\partial} u_1 + \partial u_3 + \mu u_2 = \frac{\check{\rho}_2(x)}{\check{\rho}_0(x)} \left(\bar{\partial}\bar{u}_1 + \partial u_1 + \mu u_0\right) \qquad n = 1 \qquad (3.14)$$

At this point, we switch to real coordinates and derivatives. Replacing the complex derivatives with their definitions $\partial = \frac{1}{2}(\partial_1 - i\partial_2)$ and $\bar{\partial} = \frac{1}{2}(\partial_1 + i\partial_2)$ and defining the function $\boldsymbol{w} : \mathbb{R}^2 \to \mathbb{R}^3$ such that $u_0(z) = w_0(x, y)$ and $u_1(z) = w_1(x, y) + iw_2(x, y)$ allows us to collect the real and imaginary parts of equations (3.13) and (3.14) into a system of 4 equations and 3 unknowns (w_0, w_1, and w_2). This is an overdetermined system. After some more development, we will handle this by removing an unnecessary equation from the system. Furthermore, recall that u_0 only has one component function because it is real-valued. We set $\alpha = \check{\rho}_2/\check{\rho}_0$ for simplicity. This function α is a complex-valued, so we consider it as $\alpha = \alpha_1 + i\alpha_2$. Also, any terms involving known components of $\boldsymbol{u}$ (u_2, u_3, etc.) will be absorbed into other functions g_1, g_2 to reduce the complexity of the notation.

$$\frac{1}{2}(\partial_1 + i\partial_2)\, w_0 + \mu(w_1 + iw_2) + \underbrace{\partial_1 u_2}_{g_1} = 0 \tag{3.15}$$

$$\frac{1}{2}(\partial_1 + i\partial_2)(w_1 + iw_2) + \underbrace{\mu u_2 + \partial_1 u_3}_{g_2}$$
$$-\alpha\left[\frac{1}{2}(\partial_1 + i\partial_2)(w_1 - iw_2) + \frac{1}{2}(\partial_1 - i\partial_2)(w_1 + iw_2) + \mu w_0\right] = 0 \tag{3.16}$$

Next, we multiply through by the differential operators and rearrange the equations (3.15) and (3.16) to isolate the real and imaginary parts of each. For equation (3.15), we have

$$\partial_1 w_0 + 2\mu w_1 + 2\mathrm{Re}\, g_1 + i\,(\partial_2 w_0 + 2\mu w_2 + 2\mathrm{Im}\, g_1) = 0. \tag{3.17}$$

And for equation (3.16),

$$(1 - 2\alpha_1)\partial_1 w_1 - (1 + 2\alpha_1)\partial_2 w_2 - 2\alpha_1 \mu w_0 + 2\mathrm{Re}\, g_2 +$$
$$i\,(-2\alpha_2\partial_1 w_1 + \partial_1 w_2 + \partial_2 w_1 - 2\alpha_2\partial_2 w_2 - 2\alpha_2 \mu w_0 + 2\mathrm{Im}\, g_2) = 0. \tag{3.18}$$

Treating the above two equations each as two real valued equations, we obtain this system in $\boldsymbol{w}$:

$$\partial_1 w_0 + 2\mu w_1 + 2\mathrm{Re}\, g_1 = 0 \tag{3.19}$$

$$\partial_2 w_0 + 2\mu w_2 + 2\mathrm{Im}\, g_1 = 0 \tag{3.20}$$

$$(1 - 2\alpha_1)\partial_1 w_1 - (1 + 2\alpha_1)\partial_2 w_2 - 2\alpha_1 \mu w_0 + 2\mathrm{Re}\, g_2 = 0 \tag{3.21}$$

$$-2\alpha_2\partial_1 w_1 + \partial_1 w_2 + \partial_2 w_1 - 2\alpha_2\partial_2 w_2 - 2\alpha_2 \mu w_0 + 2\mathrm{Im}\, g_2 = 0 \tag{3.22}$$

We are close to obtaining a form compatible with Carleman's assumption, but first need to place our system in matrix form to be able to identify and remove the unnecessary equation.

The matrix A has a row of all zeros, so we delete the indicated row to obtain an invertible matrix.

$$\underbrace{\left[\begin{array}{ccc} 1 & 0 & 0 \\ \hline 0 & 0 & 0 \\ \hline 0 & 1-2\alpha_1 & 0 \\ 0 & -2\alpha_2 & 1 \end{array}\right]}_{A} \partial_1 \boldsymbol{w} + \underbrace{\left[\begin{array}{ccc} 0 & 0 & 0 \\ \hline 1 & 0 & 0 \\ \hline 0 & 0 & -1-2\alpha_1 \\ 0 & 1 & -2\alpha_2 \end{array}\right]}_{B} \partial_2 \boldsymbol{w} + \underbrace{\left[\begin{array}{ccc} 0 & 2\mu & 0 \\ \hline 0 & 0 & 2\mu \\ \hline -2\alpha_1\mu & 0 & 0 \\ -2\alpha_2\mu & 0 & 0 \end{array}\right]}_{C} \boldsymbol{w} + \underbrace{\left[\begin{array}{c} 2\mathrm{Re}\, g_1 \\ \hline 2\mathrm{Im}\, g_1 \\ \hline 2\mathrm{Re}\, g_2 \\ 2\mathrm{Im}\, g_2 \end{array}\right]}_{D} = \mathbf{0} \tag{3.23}$$

To utilize Carleman's hypothesis there is one more step. We require the form (3.12), where the matrix A is the identity. Therefore, our A must be invertible so that we may multiply our system by A^{-1} and then verify that we satisfy Carleman's hypothesis that the eigenvalues of B are distinct. Looking at the matrix A from (3.23), there is a row of zeros, which prevents A from having an inverse. To continue, we simply delete this row from all equations.

$$\underbrace{\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1-2\alpha_1 & 0 \\ 0 & -2\alpha_2 & 1 \end{bmatrix}}_{A} \partial_1 \boldsymbol{w} + \underbrace{\begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & -1-2\alpha_1 \\ 0 & 1 & -2\alpha_2 \end{bmatrix}}_{B} \partial_2 \boldsymbol{w} + \underbrace{\begin{bmatrix} 0 & 2\mu & 0 \\ -2\alpha_1\mu & 0 & 0 \\ -2\alpha_2\mu & 0 & 0 \end{bmatrix}}_{C} \boldsymbol{w} + \underbrace{\begin{bmatrix} 2\mathrm{Re}\, g_1 \\ 2\mathrm{Re}\, g_2 \\ 2\mathrm{Im}\, g_2 \end{bmatrix}}_{D} = \mathbf{0} \tag{3.24}$$

In this form, we see that

$$A^{-1} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \dfrac{1}{1-2\alpha_1} & 0 \\ 0 & \dfrac{-2\alpha_2}{1-2\alpha_1} & 1 \end{bmatrix}$$

and so our system after multiplying by this factor becomes

$$\partial_1 \boldsymbol{w} + \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & \dfrac{2\alpha_1 + 1}{2\alpha_1 - 1} \\ 0 & 1 & \dfrac{4\alpha_2}{2\alpha_1 - 1} \end{bmatrix} \partial_2 \boldsymbol{w} + \tilde{C}\boldsymbol{w} + \tilde{D} = \mathbf{0} \tag{3.25}$$

for $\tilde{C} = A^{-1}C,\ \tilde{D} = A^{-1}D$. The eigenvalues of the new matrix B are 0 and $\dfrac{2\alpha_2 \pm \sqrt{4\alpha_1 + 4\alpha_2 - 1}}{2\alpha_1 - 1}$, which are distinct when $4\alpha_1 + 4\alpha_2 - 1 \neq 0$. When the surface φ is defined by a cycloid, these eigenvalues are distinct. These computations were completed using MATLAB® and discussed in more detail in Chapter 4, with the routines used included in the appendix. Based on this result and the estimate obtained in Chapter 4, the functions w_0, w_1, and w_2 form a unique solution to the system (3.25).

Problem 3.3. *Using the third-order approximation of the weight function,*

$$\rho(x, \omega) \approx \check{\rho}_{-4}(x)e^{4i\alpha} + \check{\rho}_{-2}(x)e^{2i\alpha} + \check{\rho}_0(x) + \check{\rho}_2(x)e^{-2i\alpha} + \check{\rho}_4(x)e^{-4i\alpha},$$

determine the scalar acoustic source function $a(x)$ from the weighted transport equation

$$\begin{aligned} &\sum_{n\in\mathbb{Z}} \left(\bar{\partial} u_n + \partial u_{n+2} + \mu u_{n+1}\right) e^{-in\alpha} = \\ &e^{i\alpha}\left[\check{\rho}_{-4}(x)e^{4i\alpha} + \check{\rho}_{-2}(x)e^{2i\alpha} + \check{\rho}_0(x) + \check{\rho}_2(x)e^{-2i\alpha} + \check{\rho}_4(x)e^{-4i\alpha}\right] a(x). \end{aligned} \tag{3.26}$$

Constructing a system with a unique solution for this problem will follow the same process as in the second approximation scalar tomography case. Ultimately, we place the system into the form studied by Carleman and analyze the properties of the matrix B to determine if it has distinct eigenvalues. Placing the system into this form involves the following steps:

1. Find all complex valued equations from (3.26) with nonzero right hand side.

2. Use the theory of A-analytic functions to determine which components of the vector

$\boldsymbol{u}$ are known.

3. From the system in step 1, collect all equations involving *unknown* elements of $\boldsymbol{u}$, where $n \geq 0$ (since the remaining equations are conjugates of these)

4. Substitute the formula $a(x) = \dfrac{\bar{\partial}\bar{u}_1 + \partial u_1 + \mu u_0}{\check{\rho}_0(x)}$ (which is the equation for $n = -1$) into the system from step 3

5. Switch to real derivatives and coordinates by replacing ∂ and $\bar{\partial}$ with their definitions and the unknown components of $\boldsymbol{u}$ with real functions $\boldsymbol{w}$ so that $u_0 = w_0$, $u_1 = w_1 + iw_2$, $u_2 = w_3 + iw_4$, etc.

6. Create a new system in $\boldsymbol{w}$, taking the real and imaginary parts of the above system after the substitutions from step 5 as distinct equations. This new system will be overdetermined by one equation.

7. Identify the equation that holds the least amount of relevant information and delete it. This equation typically generates a row in the matrix A that is identical to another row in A. Delete the one has less or redundant information in B.

8. A should now be invertible. Multiply the system by A^{-1}.

9. Determine if $\tilde{B} = A^{-1}B$ has distinct eigenvalues. If it does, the solution for $\boldsymbol{w}$ and hence $a(x)$ is unique.

With our algorithm outlined, we apply it to (3.26). Step 1 yields the following system of complex differential equations:

$$\bar{\partial}\bar{u}_5 + \partial\bar{u}_3 + \mu\bar{u}_4 = \check{\rho}_{-4}(x)a(x) \qquad n = -5 \qquad (3.27)$$

$$\bar{\partial}\bar{u}_3 + \partial\bar{u}_1 + \mu\bar{u}_2 = \check{\rho}_{-2}(x)a(x) \qquad n = -3 \qquad (3.28)$$

$$\bar{\partial}\bar{u}_1 + \partial u_1 + \mu u_0 = \check{\rho}_0(x)a(x) \qquad n = -1 \qquad (3.29)$$

$$\bar{\partial}u_1 + \partial u_3 + \mu u_2 = \check{\rho}_2(x)a(x) \qquad n = 1 \qquad (3.30)$$

$$\bar{\partial}u_3 + \partial u_5 + \mu u_4 = \check{\rho}_4(x)a(x) \qquad n = 3 \qquad (3.31)$$

$$\bar{\partial}u_n + \partial u_{n+2} + \mu u_{n+1} = 0 \qquad n \in \mathbb{Z} \setminus \{-5, -3, -1, 1, 3\} \qquad (3.32)$$

Evaluating this system as in step 2, we see that $n \geq 4$ is where all equations have solely zero right hand sides. Therefore, the components $(u_4, u_5, \ldots)$ of $\boldsymbol{u}$ are known from data and found using the techniques discussed in Chapter 2. For step 3, the construction of our system also requires the equations for $n = 0$ and $n = 2$

$$\bar{\partial}u_0 + \partial u_2 + \mu u_1 = 0 \qquad n = 0$$

$$\bar{\partial}u_2 + \partial u_4 + \mu u_3 = 0 \qquad n = 2$$

as they involve unknown components of $\boldsymbol{u}$. Step 4, gather the equations with unknown component of $\boldsymbol{u}$ and make the substitution for $a(x)$:

$$\bar{\partial}u_0 + \partial u_2 + \mu u_1 = 0 \qquad n = 0 \qquad (3.33)$$

$$\bar{\partial}u_1 + \partial u_3 + \mu u_2 = \frac{\check{\rho}_2}{\check{\rho}_0}\left(\bar{\partial}\bar{u}_1 + \partial u_1 + \mu u_0\right) \qquad n = 1 \qquad (3.34)$$

$$\bar{\partial}u_2 + \partial u_4 + \mu u_3 = 0 \qquad n = 2 \qquad (3.35)$$

$$\bar{\partial}u_3 + \partial u_5 + \mu u_4 = \frac{\check{\rho}_4}{\check{\rho}_0}\left(\bar{\partial}\bar{u}_1 + \partial u_1 + \mu u_0\right) \qquad n = 3 \qquad (3.36)$$

Step 5 begins our conversion to real coordinates. The complex differentials ∂ and $\bar{\partial}$ will be replaced with their real analogs and the unknown components of $\boldsymbol{u}$ will be replaced by a set of new functions $\boldsymbol{w} = (w_0, w_1, \ldots, w_6)$ so that

$$u_0 = w_0 \qquad u_2 = w_3 + iw_4$$

$$u_1 = w_1 + iw_2 \qquad u_3 = w_5 + iw_6$$

Also, for notational simplicity, we set $\alpha = \check{\rho}_2/\check{\rho}_0 = \alpha_1 + i\alpha_2$ and $\beta = \check{\rho}_4/\check{\rho}_0 = \beta_1 + i\beta_2$, complex functions. With these substitutions and collecting the real and imaginary parts, we have the following system:

$$\partial_1 w_0 + \partial_1 w_3 + \partial_2 w_4 + 2\mu w_1 + \\ i\left[\partial_1 w_4 + \partial_2 w_0 - \partial_2 w_3 + 2\mu w_2\right] = 0 \tag{3.37}$$

$$(1-2\alpha_1)\partial_1 w_1 + \partial_1 w_5 - (1+2\alpha_1)\partial_2 w_2 + \partial_2 w_6 - 2\mu w_0\alpha_1 + 2\mu w_3 + \\ i\left[-2\alpha_2\partial_1 w_1 + \partial_1 w_2 + \partial_1 w_6 + \partial_2 w_1 - 2\alpha_2\partial_2 w_2 - \partial_2 w_6 - 2\mu w_0\alpha_2 + 2\mu w_4\right] = 0 \tag{3.38}$$

$$\partial_1 w_3 - \partial_2 w_4 + 2\mu w_5 + 2\mathrm{Re}\,\underbrace{\partial u_4}_{g_1} + \\ i\left[\partial_1 w_4 + \partial_2 w_3 + 2\mu w_6 + 2\mathrm{Im}\,g_1\right] = 0 \tag{3.39}$$

$$-2\beta_1\partial_1 w_1 + \partial_1 w_5 - 2\beta_1\partial_2 w_2 - \partial_2 w_6 - 2\mu w_0\beta_1 + 2\mathrm{Re}\,\underbrace{\partial u_5}_{g_2} + \\ i\left[-2\beta_2\partial_1 w_1 + \partial_1 w_6 - 2\beta_2\partial_2 w_2 + \partial_2 w_5 - 2\mu w_0\beta_2 + 2\mathrm{Im}\,g_2\right] = 0, \tag{3.40}$$

where again g_1, g_2 are known functions. For step 6, we condense the above equation into matrix form.

$$
\left[\begin{array}{ccccccc}
1 & 0 & 0 & 1 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 1 & 0 & 0 \\
0 & 1-2\alpha_1 & 0 & 1 & 0 & 1 & 0 \\
0 & -2\alpha_2 & 1 & 0 & 0 & 0 & 1 \\
0 & 0 & 0 & 1 & 0 & 0 & 0 \\
\hline
0 & 0 & 0 & 0 & 1 & 0 & 0 \\
\hline
0 & -2\beta_1 & 0 & 0 & 0 & 1 & 0 \\
0 & -2\beta_2 & 0 & 0 & 0 & 0 & 1
\end{array}\right] \partial_1 \boldsymbol{w} +
\left[\begin{array}{ccccccc}
0 & 0 & 0 & 0 & 1 & 0 & 0 \\
1 & 0 & 0 & \text{-1} & 0 & 0 & 0 \\
0 & 0 & -1-2\alpha_1 & 0 & 0 & 0 & 1 \\
0 & 1 & -2\alpha_2 & 0 & 0 & \text{-1} & 0 \\
0 & 0 & 0 & 0 & \text{-1} & 0 & 0 \\
\hline
0 & 0 & 0 & 1 & 0 & 0 & 0 \\
\hline
0 & 0 & -2\beta_1 & 0 & 0 & 0 & \text{-1} \\
0 & 0 & -2\beta_2 & 0 & 0 & 1 & 0
\end{array}\right] \partial_2 \boldsymbol{w} + C\boldsymbol{w} + D = \mathbf{0}
$$

where C and D are matrices of lower order terms and are included in the appendix.

Step 7 requires that we delete an extraneous equation so that the leftmost matrix A is invertible. Examining the matrix closely, we see that the third to last row of A matches the second row. Looking at matrix B on the right, we see that more information is contained in the second row than the third to last. This is because the second row is the only one in B with a nonzero entry in the first column. In our previous case, we also deleted the third to last row of each matrix. If we delete this row here, we obtain an A that is invertible for $2\alpha_1 - 2\beta_1 - 1 \neq 0$.

$$
A^{-1} = \begin{bmatrix}
1 & 0 & 0 & 0 & \text{-1} & 0 & 0 \\
0 & 0 & \dfrac{-1}{2\alpha_1 - 2\beta_1 - 1} & 0 & \dfrac{1}{2\alpha_1 - 2\beta_1 - 1} & \dfrac{1}{2\alpha_1 - 2\beta_1 - 1} & 0 \\
0 & 0 & \dfrac{-2\alpha_2 + 2\beta_2}{2\alpha_1 - 2\beta_1 - 1} & 1 & \dfrac{2\alpha_2 - 2\beta_2}{2\alpha_1 - 2\beta_1 - 1} & \dfrac{2\alpha_2 - 2\beta_2}{2\alpha_1 - 2\beta_1 - 1} & \text{-1} \\
0 & 0 & 0 & 0 & 1 & 0 & 0 \\
0 & 1 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & \dfrac{-2\beta_1}{2\alpha_1 - 2\beta_1 - 1} & 0 & \dfrac{2\beta_1}{2\alpha_1 - 2\beta_1 - 1} & \dfrac{-1 + 2\alpha_1}{2\alpha_1 - 2\beta_1 - 1} & 0 \\
0 & 0 & \dfrac{-2\beta_2}{2\alpha_1 - 2\beta_1 - 1} & 0 & \dfrac{2\beta_2}{2\alpha_1 - 2\beta_1 - 1} & \dfrac{2\beta_2}{2\alpha_1 - 2\beta_1 - 1} & 1
\end{bmatrix}
$$

Multiplying our system by A^{-1} places our matrix equation into its final form.

$$\partial_1 \boldsymbol{w} + \begin{bmatrix} 0 & 0 & 0 & 0 & 2 & 0 & 0 \\ 0 & 0 & \frac{2\alpha_1 - 2\beta_1 + 1}{2\alpha_1 - 2\beta_1 - 1} & 0 & \frac{-1}{2\alpha_1 - 2\beta_1 - 1} & 0 & \frac{-2}{2\alpha_1 - 2\beta_1 - 1} \\ 0 & 1 & \frac{4\alpha_2 - 4\beta_2}{2\alpha_1 - 2\beta_1 - 1} & 0 & \frac{-2\alpha_2 + 2\beta_2}{2\alpha_1 - 2\beta_1 - 1} & -2 & \frac{-4\alpha_2 + 4\beta_2}{2\alpha_1 - 2\beta_1 - 1} \\ 0 & 0 & 0 & 0 & -1 & 0 & 0 \\ 1 & 0 & 0 & -1 & 0 & 0 & 0 \\ 0 & 0 & \frac{4\beta_1}{2\beta_1 - 2\alpha_1 + 1} & 0 & \frac{-2\beta_1}{2\alpha_1 - 2\beta_1 - 1} & 0 & \frac{1 - 2\alpha_1 - 2\beta_1}{2\alpha_1 - 2\beta_1 - 1} \\ 0 & 0 & \frac{4\beta_2}{2\beta_1 - 2\alpha_1 + 1} & 0 & \frac{-2\beta_2}{2\alpha_1 - 2\beta_1 - 1} & 1 & \frac{-4\beta_2}{2\alpha_1 - 2\beta_1 - 1} \end{bmatrix} \partial_2 \boldsymbol{w} + \tilde{C}\boldsymbol{w} + \tilde{D} = 0$$

where again $\tilde{C} = A^{-1}C$ and $\tilde{D} = A^{-1}D$. The eigenvalues of this matrix are 0, $\pm\sqrt{3}$ and the roots of the quartic polynomial

$$(2\beta_1 - 2\alpha_1 + 1)z^4 + (4\alpha_2 - 8\beta_2)z^3 + (2 - 12\beta_1)z^2 + (8\beta_2 + 4\alpha_2)z + 2\beta_1 + 2\alpha_1 + 1 = 0.$$

The fact that they are distinct is evaluated numerically at several thousand grid points within the unit circle, under the assumption that the surface φ is defined by a cycloid (see Chapter 4).

3.2 The Vector Tomography Problem

The second principal problem to be studied is the vector tomography case. In this case, the goal is recover the projection of the vector field $\boldsymbol{v} = (v_1, v_2)$ under the weight function $\rho(x, \omega)$. From this vector field, we recover physical information concerning the interior of our region of study.

Problem 3.4. *Vector Tomography: Recover the two-dimensional projection of the vector field* $\boldsymbol{v} = (v_1(x), v_2(x))$ *under weight* $\rho(x, \omega)$ *by solving Problem 2.1 for the following system of equations.*

$$e^{i\alpha} Pu = \sum_{n \in \mathbb{Z}} \left(\bar{\partial} u_n + \partial u_{n+2} + \mu u_{n+1} \right) e^{-in\alpha} = e^{i\alpha} \rho(x, \omega) \langle \omega, \boldsymbol{v} \rangle. \tag{3.41}$$

The solution of this problem is inherently more difficult than the scalar tomography case.

It requires the recovery of two scalar functions $v_1(x)$ and $v_2(x)$ instead of just a single function like before. Additionally, we require two more assumptions than the scalar case to be able to solve for $\boldsymbol{v}$, that the attenuation term $\mu(x) \neq 0$ anywhere in its domain and that the normal derivative $\partial_\nu u_1$ is known. Having the attenuation $\mu \neq 0$ is a necessary condition for recovering the full vector field, determined by [6]. To prove that these systems have unique solutions, we will again appeal to the Carleman systems referenced in our discussion of the scalar tomography cases. We will place the vector tomography systems in the same form and evaluate the same hypotheses to determine if the solutions are unique. However, these more complicated systems require more delicate care and generate second order systems of differential equations for us to solve. Fortunately, it is possible to reduce these second order systems to larger systems of first order equations. After this, we will obtain a system in the desired form of equation (3.11) and examine the eigenvalues of matrix B to determine if they are distinct.

The first approximation vector case, with $\rho(x,\omega) \approx \check{\rho}_0$ is trivial and does not inform us on how to handle larger scale systems. It is included in the appendix. We will begin with the second order approximation, where $\rho(x,\omega) \approx \check{\rho}_{-2}(x)e^{2i\alpha} + \check{\rho}_0(x) + \check{\rho}_2(x)e^{-2i\alpha}$. Before making this substitution into (3.41), we note the following identity

$$
\begin{aligned}
e^{i\alpha}\langle \omega, \boldsymbol{v}\rangle &= e^{i\alpha}\left[v_1\cos\alpha + v_2\sin\alpha\right] \\
&= e^{i\alpha}\left[v_1\left(\frac{e^{i\alpha}+e^{-i\alpha}}{2}\right) + v_2\left(\frac{e^{i\alpha}-e^{i\alpha}}{2i}\right)\right] \\
&= \frac{1}{2}\left[v_1\left(e^{2i\alpha}+1\right) - iv_2\left(e^{2i\alpha}-1\right)\right] \\
&= \frac{1}{2}\left[\bar{v}e^{2i\alpha} + v\right]
\end{aligned}
$$

where the functions $v = v_1 + iv_2$ and $\bar{v} = v_1 - iv_2$. After this notational simplification, we have

$$e^{i\alpha}Pu = \sum_{n\in\mathbb{Z}} \left(\bar{\partial}u_n + \partial u_{n+2} + \mu u_{n+1}\right) e^{-in\alpha} = \frac{1}{2}\rho(x,\omega)\left[\bar{v}e^{2i\alpha} + v\right] \tag{3.42}$$

and are ready to proceed with the construction of our vector tomography system.

We will follow these steps:

1. Approximate $\rho(x,\omega)$ in (3.42) with an even trigonometric polynomial.
2. Find all complex valued equations from (3.42) after the substitution with a nonzero right hand side.
3. Applying the theory of A-analytic functions, determine which elements of vector $\boldsymbol{u}$ may be determined from the boundary conditions.
4. Use the equations from the system to find explicit formulas for the functions v and $\bar{v}$.
5. Find formulas for the unknown even u_{2n} in terms of the unknown odd u_{2n+1}.
6. Substitute for v, $\bar{v}$ and the u_{2n} using these formulas, resulting in a second order system in terms of the unknown u_{2n+1}. There should be as many equations as there are unknown u_{2n+1}.
7. Switch to real derivatives and coordinates by replacing all unknown u_{2n+1} with real-valued component functions $\boldsymbol{w}$ so that $u_1 = w_1 + iw_2$, $u_3 = w_3 + iw_4$, etc.
8. Define additional components of the vector $\boldsymbol{w}$ to represent the second-order terms in the system.
9. Simplify the equations and rewrite them in the form

$$A(x,y)\frac{\partial \boldsymbol{w}}{\partial x} + B(x,y)\frac{\partial \boldsymbol{w}}{\partial y} + C(x,y)\boldsymbol{w} + D(x,y) = \boldsymbol{0}$$

10. Check matrix $A(x,y)$ for invertibility, then multiply both sides by A^{-1} to achieve our desired form

$$\frac{\partial \boldsymbol{w}}{\partial x} + B(x,y)\frac{\partial \boldsymbol{w}}{\partial y} + C(x,y)\boldsymbol{w} + D(x,y) = \mathbf{0}$$

after relabeling $B(x,y) = A^{-1}B(x,y)$, etc.

Problem 3.5. *Vector Tomography Second Approximation: Reconstruct the function* $\boldsymbol{u} = (u_0, u_1, u_2, \dots)$ *for the second order vector tomography approximation of the weight* ρ.

Beginning with step 1, the second order approximation of our weight is $\rho(x,\omega) \approx \check{\rho}_{-2}(x)e^{2i\alpha} + \check{\rho}_0(x) + \check{\rho}_2(x)e^{-2i\alpha}$, so the system will be formed from

$$\sum_{n\in\mathbb{Z}} \left(\bar{\partial} u_n + \partial u_{n+2} + \mu u_{n+1}\right) e^{-in\alpha} = \frac{1}{2}\left[\check{\rho}_{-2}(x)e^{2i\alpha} + \check{\rho}_0(x) + \check{\rho}_2(x)e^{-2i\alpha}\right]\left[\bar{v}e^{2i\alpha} + v\right].$$

Collecting equations as instructed in step 2 yields

$$\bar{\partial}\bar{u}_4 + \partial\bar{u}_2 + \mu\bar{u}_3 = \frac{1}{2}\check{\rho}_{-2}\bar{v} \qquad n=-4 \qquad (3.43)$$

$$\bar{\partial}\bar{u}_2 + \partial u_0 + \mu\bar{u}_1 = \frac{1}{2}\left(\check{\rho}_{-2}v + \check{\rho}_0\bar{v}\right) \qquad n=-2 \qquad (3.44)$$

$$\bar{\partial}u_0 + \partial u_2 + \mu u_1 = \frac{1}{2}\left(\check{\rho}_2\bar{v} + \check{\rho}_0 v\right) \qquad n=0 \qquad (3.45)$$

$$\bar{\partial}u_2 + \partial u_4 + \mu u_3 = \frac{1}{2}\check{\rho}_2 v \qquad n=2 \qquad (3.46)$$

$$\bar{\partial}u_n + \partial u_{n+2} + \mu u_{n+1} = 0 \qquad n\in\mathbb{Z}\setminus\{-4,-2,0,2\} \qquad (3.47)$$

Next is step 3, where we note that all equation with $n \geq 3$ have 0 right hand side, so our data is $(u_3, u_4, \dots)$. Ultimately, our system will be a second order system where we try to determine the component functions of the unknown odd Fourier coefficient of $\boldsymbol{u}$, u_1. In general, there will be more than one but this case reduces to just one unknown u_{2n+1}. The rest of our efforts will involve making it so that this is the only unknown left in the system after several substitutions.

Step 4 requires explicit formulas for $\bar{v}$ and v, which are provided by rearranging (3.43) and (3.46) respectively.

$$v = \frac{2}{\check{\rho}_2}\left(\bar{\partial} u_2 + \partial u_4 + \mu u_3\right) \qquad \bar{v} = \frac{2}{\check{\rho}_{-2}}\left(\bar{\partial} \bar{u}_4 + \partial \bar{u}_2 + \mu \bar{u}_3\right)$$

We see that it is necessary that $\check{\rho}_2 \neq 0$ for all $x \in \bar{\Omega}$, which is true for cycloidal ray paths. Additionally, we have that the equations with zero right hand side for $n = -1$ and $n = 1$ are

$$\bar{\partial} \bar{u}_1 + \partial u_1 + \mu u_0 = 0 \tag{3.48}$$

$$\bar{\partial} u_1 + \partial u_3 + \mu u_2 = 0 \tag{3.49}$$

which may solved for the even Fourier terms of $\boldsymbol{u}$, giving

$$u_0 = \frac{-\bar{\partial} \bar{u}_1 - \partial u_1}{\mu} \qquad u_2 = \frac{-\bar{\partial} u_1 - \partial u_3}{\mu}, \tag{3.50}$$

completing step 5.

With these formula for v, $\bar{v}$, u_0, and u_1, the process moves to step 6. All of the formulas found in the last two steps are substituted into equation (3.45). Following multiplication by -1,

$$\bar{\partial}\left(\frac{\bar{\partial} \bar{u}_1}{\mu}\right) + \bar{\partial}\left(\frac{\partial u_1}{\mu}\right) + \partial\left(\frac{\bar{\partial} u_1}{\mu}\right) - \gamma \bar{\partial}\left(\frac{\bar{\partial} u_1}{\mu}\right) - \delta \partial\left(\frac{\partial \bar{u}_1}{\mu}\right) - \mu u_1 - g = 0 \tag{3.51}$$

where g is a known function, given by

$$g = \partial\left(\frac{\partial u_3}{\mu}\right) + \gamma\left[\partial u_4 + \mu u_3 - \bar{\partial}\left(\frac{\partial u_3}{\mu}\right)\right] + \delta\left[\bar{\partial} \bar{u}_4 + \mu u_3 - \partial\left(\frac{\bar{\partial} \bar{u}_3}{\mu}\right)\right] \tag{3.52}$$

and $\gamma = \dfrac{\check{\rho}_0}{\check{\rho}_2}$ and $\delta = \dfrac{\check{\rho}_2}{\check{\rho}_{-2}}$.

Equation (3.51) is a second order differential equation, one that must be reduced to first order before we may check for Carleman's uniqueness condition. First, we switch to real variables and then we will introduce additional equations to reduce this second order equation into a system of first order equations. Before beginning step 7 of our process, we require a large number of calculations to determine a usable form of the terms like $\partial\left(\frac{\bar{\partial}\bar{u}_1}{\mu}\right)$ in real variables. To ease this process, we develop a few lemmas to handle these terms. The first lemma deals with the μ factors appearing in (3.51). The attenuation μ is a real function, and for real derivatives ∂_p, ∂_q with $p, q = 1, 2$ we have

Lemma 3.1.

$$\begin{aligned}\partial_p\left(\frac{\partial_q w}{\mu}\right) &= \left(\frac{1}{\mu}\right)\partial_p\partial_q w + \partial_p\left(\frac{1}{\mu}\right)\partial_q w \\ &= \left(\frac{1}{\mu}\right)\partial_p\partial_q w - \partial_q w\left(\frac{\partial_p\mu}{\mu^2}\right) \\ &= \left(\frac{1}{\mu}\right)\left[\partial_p\partial_q w - l_p\partial_q w\right]\end{aligned}$$

where $l_p = \frac{\partial_p\mu}{\mu}$.

The next lemma will be for converting the second order complex valued terms into real coordinates. These terms have eight different forms, four pairs of conjugates. The computations involving the derivation of one of these identities will be shown and the others just stated for brevity. To begin, replace u_1 with $w_1 + iw_2$ (and $\bar{u}_1$ with $w_1 - iw_2$) and the operators $\partial = \frac{1}{2}(\partial_1 - i\partial_2)$ and $\bar{\partial} = \frac{1}{2}(\partial_1 + i\partial_2)$ and recall that μ is a real valued function. Thus,

$$\begin{aligned}\bar{\partial}\left(\frac{\bar{\partial}\bar{u}_1}{\mu}\right) &= \frac{1}{4}(\partial_1 + i\partial_2)\left(\frac{(\partial_1 + i\partial_2)(w_1 - iw_2)}{\mu}\right) \\ &= \frac{1}{4}(\partial_1 + i\partial_2)\left(\frac{\partial_1 w_1 + \partial_2 w_2 + i(-\partial_1 w_2 + \partial_2 w_1)}{\mu}\right) \\ &= \frac{1}{4}\left[\partial_1\left(\frac{\partial_1 w_1 + \partial_2 w_2}{\mu} + i\frac{-\partial_1 w_2 + \partial_2 w_1}{\mu}\right) + i\partial_2\left(\frac{\partial_1 w_1 + \partial_2 w_2}{\mu} + i\frac{-\partial_1 w_2 + \partial_2 w_1}{\mu}\right)\right] \\ &= \frac{1}{4}\left\{\partial_1\left(\frac{\partial_1 w_1 + \partial_2 w_2}{\mu}\right) + \partial_2\left(\frac{\partial_1 w_2 - \partial_2 w_1}{\mu}\right) + i\left[\partial_1\left(\frac{-\partial_1 w_2 + \partial_2 w_1}{\mu}\right) + \partial_2\left(\frac{\partial_1 w_1 + \partial_2 w_2}{\mu}\right)\right]\right\}\end{aligned}$$

At this point, it is convenient to multiply both sides by 4μ. After this, we apply Lemma 3.1 many times and collect terms.

$$4\mu\bar{\partial}\left(\frac{\bar{\partial}\bar{u}_1}{\mu}\right) = \partial_1\partial_1 w_1 + 2\partial_2\partial_1 w_2 - \partial_2\partial_2 w_1 - l_1\partial_1 w_1 - l_2\partial_1 w_2 + l_2\partial_2 w_1 - l_1\partial_2 w_2$$
$$+ i\left(-\partial_1\partial_1 w_2 + \partial_1\partial_2 w_1 + \partial_2\partial_1 w_1 + \partial_2\partial_2 w_2 - l_2\partial_1 w_1 + l_1\partial_1 w_2 - l_1\partial_2 w_1 - l_2\partial_2 w_2\right) \tag{3.53}$$

Here, the fact that the operators ∂_1 and ∂_2 commute was used. In later steps, we will prefer $\partial_2\partial_1 w_j$ expressions over having the differential operators in the opposite order due to some substitutions yet to be described. This expanded expression provides the form necessary to collect into the Carleman system we are seeking. The other expressions are derived similarly:

$$4\mu\bar{\partial}\left(\frac{\partial u_1}{\mu}\right) = \partial_1\partial_1 w_1 + \partial_2\partial_2 w_1 - l_1\partial_1 w_1 + l_2\partial_1 w_2 - l_2\partial_2 w_1 - l_1\partial_2 w_2$$
$$+ i\left(\partial_1\partial_1 w_2 + \partial_2\partial_2 w_2 - l_2\partial_1 w_1 - l_1\partial_1 w_2 + l_1\partial_2 w_1 - l_2\partial_2 w_2\right) \tag{3.54}$$
$$4\mu\partial\left(\frac{\bar{\partial}u_1}{\mu}\right) = \partial_1\partial_1 w_2 + \partial_2\partial_2 w_2 - l_1\partial_1 w_1 - l_2\partial_1 w_2 - l_2\partial_2 w_1 + l_1\partial_2 w_2$$
$$+ i\left(\partial_1\partial_1 w_2 + \partial_2\partial_2 w_2 + l_2\partial_1 w_1 - l_1\partial_1 w_2 - l_1\partial_2 w_1 - l_2\partial_2 w_2\right) \tag{3.55}$$
$$4\mu\bar{\partial}\left(\frac{\bar{\partial}u_1}{\mu}\right) = \partial_1\partial_1 w_1 - 2\partial_2\partial_1 w_2 - \partial_2\partial_2 w_1 - l_1\partial_1 w_1 + l_2\partial_1 w_2 + l_2\partial_2 w_1 + l_1\partial_2 w_2$$
$$+ i\left(\partial_1\partial_1 w_2 + \partial_1\partial_2 w_1 + \partial_2\partial_1 w_1 - \partial_2\partial_2 w_2 - l_2\partial_1 w_1 - l_1\partial_1 w_2 - l_1\partial_2 w_1 + l_2\partial_2 w_2\right) \tag{3.56}$$

Equations (3.53), (3.54), (3.55), and (3.56) are the conjugates of 4μ times $\partial\left(\frac{\partial u}{\mu}\right)$, $\partial\left(\frac{\bar{\partial}\bar{u}}{\mu}\right)$, $\bar{\partial}\left(\frac{\partial\bar{u}}{\mu}\right)$, and $\partial\left(\frac{\partial\bar{u}}{\mu}\right)$ respectively, meaning that we only need to determine four of the possible eight expressions. We now look back to completing step 7 of our process using these identities. We convert (3.51) into real coordinates and separate the real and imaginary parts (temporarily leaving γ and δ as complex valued).

$$\begin{aligned}(3-\gamma-\delta)\partial_1\partial_1 w_1 + (1+\gamma+\delta)\partial_2\partial_2 w_1 + (2+2\gamma+2\delta)\partial_2\partial_1 w_2 + (-3+\gamma+\delta)l_1\partial_1 w_1 + \\ (-3-\gamma-\delta)l_2\partial_2 w_1 + (1-\gamma-\delta)l_2\partial_1 w_2 + (-1-\gamma-\delta)l_1\partial_2 w_2 - 4\mu^2 w_1 - 4\mu \mathrm{Re}\, g + \\ i\Big[(1-\gamma+\delta)\partial_1\partial_1 w_2 + (2-2\gamma+2\delta)\partial_2\partial_1 w_1 + (3+\gamma-\delta)\partial_2\partial_2 w_2 + (-1+\gamma-\delta)l_1\partial_1 w_1 + \\ (-1+\gamma-\delta)l_2\partial_2 w_1 + (-1+\gamma-\delta)l_1\partial_1 w_2 + (-3-\gamma+\delta)l_2\partial_2 w_2 - 4\mu^2 w_2 - 4\mu \mathrm{Im}\, g\Big] = 0\end{aligned} \tag{3.57}$$

Now it is time to switch the system to be in terms of the real-valued component functions $\boldsymbol{w}$, introducing the following first order derivatives of w_1 and w_2.

$$\begin{aligned}\partial_1 w_1 &= w_3 \qquad & \partial_1 w_2 &= w_5 \\ \partial_2 w_1 &= w_4 \qquad & \partial_2 w_2 &= w_6\end{aligned} \tag{3.58}$$

After these substitutions, (3.57) becomes

$$\begin{aligned}(3-\gamma-\delta)\partial_1 w_3 + (1+\gamma+\delta)\partial_2 w_4 + (2+2\gamma+2\delta)\partial_2 w_5 + (-3+\gamma+\delta)l_1 w_3 + \\ (-3-\gamma-\delta)l_2 w_4 + (1-\gamma-\delta)l_2 w_5 + (-1-\gamma-\delta)l_1 w_6 - 4\mu^2 w_1 - 4\mu \mathrm{Re}\, g + \\ i\Big[(1-\gamma+\delta)\partial_1 w_5 + (2-2\gamma+2\delta)\partial_2 w_5 + (3+\gamma-\delta)\partial_2 w_6 + (-1+\gamma-\delta)l_1 w_3 + \\ (-1+\gamma-\delta)l_2 w_4 + (-1+\gamma-\delta)l_1 w_5 + (-3-\gamma+\delta)l_2 w_6 - 4\mu^2 w_2 - 4\mu \mathrm{Im}\, g\Big] = 0\end{aligned} \tag{3.59}$$

which will be fully separated into real and imaginary equations after applying the identities

$\gamma = \gamma_1 + i\gamma_2$ and $\delta = \delta_1 + i\delta_2$ to (3.59) and regrouping. Hence,

$$(3-\gamma_1-\delta_1)\partial_1 w_3 + (-\gamma_2+\delta_2)\partial_1 w_5 + (1+\gamma_1+\delta_1)\partial_2 w_4 + 2(1+\gamma_1-\gamma_2+\delta_1+\delta_2)\partial_2 w_5 + (\gamma_2-\delta_2)\partial_2 w_6 + (-3+\gamma_1+\gamma_2+\delta_1-\delta_2)l_1 w_3 + (-3-\gamma_1+\gamma_2-\delta_1-\delta_2)l_2 w_4 + \Big[(\gamma_2-\delta_2)l_1 + (1-\gamma_1-\delta_1)l_2\Big]w_5 + \Big[(-1-\gamma_1-\delta_1)l_1 + (-\gamma_2+\delta_2)l_2\Big]w_6 - 4\mu^2 w_1 - 4\mu \mathrm{Re}\, g = 0 \tag{3.60}$$

$$(-\gamma_2-\delta_2)\partial_1 w_3 + (1-\gamma_1+\delta_1)\partial_1 w_5 + (\gamma_2+\delta_2)\partial_2 w_4 + 2(1-\gamma_1+\gamma_2+\delta_1+\delta_2)\partial_2 w_5 + (3+\gamma_1-\delta_1)\partial_2 w_6 + (-1+\gamma_1+\gamma_2-\delta_1+\delta_2)l_1 w_3 + (-1+\gamma_1-\gamma_2-\delta_1-\delta_2)l_2 w_4 + \Big[(-1+\gamma_1-\delta_1)l_1 + (-\gamma_2-\delta_2)l_2\Big]w_5 + \Big[(-\gamma_2-\delta_2)l_1 + (-3-\gamma_1+\delta_1)l_2\Big]w_6 - 4\mu^2 w_2 - 4\mu \mathrm{Im}\, g = 0 \tag{3.61}$$

The remainder of our first order system will be formed by taking equations (3.60) and (3.61) and adding four more equations composed of identities (3.58). The first two of these equations will be formed by choosing a real factor λ, multiplying that factor times the second row of identities and adding and subtracting vertical pairs of the identities, respectively. This yields

$$\partial_1 w_1 - w_3 + \lambda\partial_2 w_1 - \lambda w_4 = 0 \tag{3.62}$$

$$\partial_1 w_2 - w_5 - \lambda\partial_2 w_2 + \lambda w_6 = 0 \tag{3.63}$$

The values $\pm\lambda$ will be eigenvalues of matrix B when our system is formed. In higher order approximations, more choices for factors will be necessary and each one must be distinct. The final two equations for our system are identities formed by the fact that the differential operators ∂_1 and ∂_2 commute.

$$\partial_1 w_4 = \partial_1\partial_2 w_1 = \partial_2\partial_1 w_1 = \partial_2 w_3 \qquad \Rightarrow \qquad \partial_1 w_4 - \partial_2 w_3 = 0$$

$$\partial_1 w_6 = \partial_1\partial_2 w_2 = \partial_2\partial_1 w_2 = \partial_2 w_5 \qquad \Rightarrow \qquad \partial_1 w_6 - \partial_2 w_5 = 0$$

Finally, we have all necessary equations to form our system. Substituting the identities for w_3, w_4, w_5, and w_6 into equations (3.57) and (3.59) and collecting and ordering the equations by the $\partial_1 w_j$ terms:

$$\partial_1 w_1 - w_3 + \lambda \partial_2 w_1 - \lambda w_4 = 0 \tag{3.64}$$

$$\partial_1 w_2 - w_5 - \lambda \partial_2 w_2 + \lambda w_6 = 0 \tag{3.65}$$

$$\begin{aligned}(3-\gamma_1-\delta_1)\partial_1 w_3 + (-\gamma_2+\delta_2)\partial_1 w_5 + (1+\gamma_1+\delta_1)\partial_2 w_4 + 2(1+\gamma_1-\gamma_2+\delta_1+\delta_2)\partial_2 w_5 + \\ (\gamma_2-\delta_2)\partial_2 w_6 + (-3+\gamma_1+\gamma_2+\delta_1-\delta_2) l_1 w_3 + (-3-\gamma_1+\gamma_2-\delta_1-\delta_2) l_2 w_4 + \\ \Big[(\gamma_2-\delta_2) l_1 + (1-\gamma_1-\delta_1) l_2\Big] w_5 + \Big[(-1-\gamma_1-\delta_1) l_1 + (-\gamma_2+\delta_2) l_2\Big] w_6 - 4\mu^2 w_1 - 4\mu \operatorname{Re} g = 0\end{aligned} \tag{3.66}$$

$$\partial_1 w_4 - \partial_2 w_3 = 0 \tag{3.67}$$

$$\begin{aligned}(-\gamma_2-\delta_2)\partial_1 w_3 + (1-\gamma_1+\delta_1)\partial_1 w_5 + (\gamma_2+\delta_2)\partial_2 w_4 + 2(1-\gamma_1+\gamma_2+\delta_1+\delta_2)\partial_2 w_5 + \\ (3+\gamma_1-\delta_1)\partial_2 w_6 + (-1+\gamma_1+\gamma_2-\delta_1+\delta_2) l_1 w_3 + (-1+\gamma_1-\gamma_2-\delta_1-\delta_2) l_2 w_4 + \\ \Big[(-1+\gamma_1-\delta_1) l_1 + (-\gamma_2-\delta_2) l_2\Big] w_5 + \Big[(-\gamma_2-\delta_2) l_1 + (-3-\gamma_1+\delta_1) l_2\Big] w_6 - 4\mu^2 w_2 - 4\mu \operatorname{Im} g = 0\end{aligned} \tag{3.68}$$

$$\partial_1 w_6 - \partial_2 w_5 = 0 \tag{3.69}$$

This system conforms to our first order differential equation with matrix coefficients,

$$A(x,y)\frac{\partial \boldsymbol{w}}{\partial x} + B(x,y)\frac{\partial \boldsymbol{w}}{\partial y} + C(x,y)\boldsymbol{w} + D(x,y) = 0$$

where

$$A = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 3-\gamma_1-\delta_1 & 0 & -\gamma_2+\delta_2 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & -\gamma_2-\delta_2 & 0 & 1-\gamma_1+\delta_1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix}$$

$$B = \begin{bmatrix} \lambda & 0 & 0 & 0 & 0 & 0 \\ 0 & \text{-}\lambda & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1+\gamma_1+\delta_1 & 2(1+\gamma_1-\gamma_2+\delta_1+\delta_2) & \gamma_2-\delta_2 \\ 0 & 0 & \text{-1} & 0 & 0 & 0 \\ 0 & 0 & 0 & \gamma_2+\delta_2 & 2(1-\gamma_1+\gamma_2+\delta_1+\delta_2) & 3+\gamma_1-\delta_1 \\ 0 & 0 & 0 & 0 & \text{-1} & 0 \end{bmatrix}$$

$$C = \begin{bmatrix} 0 & 0 & \text{-1} & \text{-}\lambda & 0 & 0 \\ 0 & 0 & 0 & 0 & \text{-1} & \lambda \\ 4\mu^2 & 0 & (-3+\gamma_1+\gamma_2+\delta_1-\delta_2)l_1 & (-3-\gamma_1+\gamma_2-\delta_1-\delta_2)l_2 & (\gamma_2-\delta_2)l_1+(1-\gamma_1-\delta_1)l_2 & (-1-\gamma_1-\delta_1)l_1+(-\gamma_2+\delta_2)l_2 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 4\mu^2 & (-1+\gamma_1+\gamma_2-\delta_1+\delta_2)l_1 & (-1+\gamma_1-\gamma_2-\delta_1-\delta_2)l_2 & (-1+\gamma_1-\delta_1)l_1+(-\gamma_2-\delta_2)l_2 & (-\gamma_2-\delta_2)l_1+(-3-\gamma_1+\delta_1)l_2 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

$$D = \begin{bmatrix} 0 \\ 0 \\ 4\mu \operatorname{Re} g \\ 0 \\ 4\mu \operatorname{Im} g \\ 0 \end{bmatrix}$$

The determinant of matrix A is $\gamma_1^2 - 4\gamma_1 - \gamma_2^2 - \delta_1^2 + 2\delta_1 + \delta_2^2 + 3$ and the matrix is invertible when this is not zero. After multiplication by A^{-1}, the system takes on the familiar form of

$$\partial_1 \boldsymbol{w} + \begin{bmatrix} \lambda & 0 & 0 & 0 & 0 & 0 \\ 0 & \text{-}\lambda & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & \frac{-\gamma_1^2+\gamma_2^2+\delta_1^2+2\delta_1-\delta_2^2+1}{\gamma_1^2-4\gamma_1-\gamma_2^2-\delta_1^2+2\delta_1+\delta_2^2+3} & \frac{2(-\gamma_1^2+\gamma_2^2+\delta_1^2+2\delta_1-\delta_2^2+1)}{\gamma_1^2-4\gamma_1-\gamma_2^2-\delta_1^2+2\delta_1+\delta_2^2+3} & \frac{4\gamma_2-4\delta_2}{\gamma_1^2-4\gamma_1-\gamma_2^2-\delta_1^2+2\delta_1+\delta_2^2+3} \\ 0 & 0 & \text{-1} & 0 & 0 & 0 \\ 0 & 0 & 0 & \frac{4\gamma_2+4\delta_2}{\gamma_1^2-4\gamma_1-\gamma_2^2-\delta_1^2+2\delta_1+\delta_2^2+3} & \frac{2(\gamma_1^2-4\gamma_1-\gamma_2^2-\delta_1^2+2\delta_1+4\delta_2+\delta_2^2+3}{\gamma_1^2-4\gamma_1-\gamma_2^2-\delta_1^2+2\delta_1+\delta_2^2+3} & \frac{-\gamma_1^2-\gamma_2^2-\delta_1^2+6\delta_1+\delta_2-9}{\gamma_1^2-4\gamma_1-\gamma_2^2-\delta_1^2+2\delta_1+\delta_2^2+3} \\ 0 & 0 & 0 & 0 & \text{-1} & 0 \end{bmatrix} \partial_2 \boldsymbol{w} + \tilde{C}\boldsymbol{w} + \tilde{D} = 0 \tag{3.70}$$

where again $\tilde{C} = A^{-1}C$ and $\tilde{D} = A^{-1}D$. The eigenvalues of this matrix are the free parameter

$\pm\lambda$ and the roots of the polynomial

$$
\begin{aligned}
&(4\gamma_1 - \gamma_1^2 + \gamma_2^2 + \delta_1^2 - 2\delta_1 - \delta_2^2 - 3)z^4 + (2\gamma_1^2 - 8\gamma_1 - 2\gamma_2^2 + 8\gamma_2 - 2\delta_1^2 + 4\delta_1 + 2\delta_2^2 + 8\delta_2 + 6)z^3 \\
&+ (2\gamma_1^2 - 2\gamma_2^2 - 2\delta_1^2 + 4\delta_1 + 2\delta_2^2 - 10)z^2 + (2\gamma_2^2 - 2\gamma_1^2 + 2\delta_1^2 + 4\delta_1 - 2\delta_2^2 + 2)z \\
&- \gamma_1^2 - 4\gamma_1 + \gamma_2^2 + \delta_1^2 - 2\delta_1 - \delta_2^2 - 3 = 0
\end{aligned}
$$

Problem 3.6. *Vector Tomography Third Approximation*

The last case to be examined here is the third order approximation for vector tomography. The 10 step process for constructing the first order Carleman system will be followed again, except the weight function will be approximated as $\rho(x,\omega) \approx \check{\rho}_{-4}e^{4i\alpha} + \check{\rho}_{-2}(x)e^{2i\alpha} + \check{\rho}_0(x) + \check{\rho}_2(x)e^{-2i\alpha} + \check{\rho}_4(x)e^{-4i\alpha}$.

Step 1:

Set $\rho(x,\omega) = \check{\rho}_{-4}e^{4i\alpha} + \check{\rho}_{-2}(x)e^{2i\alpha} + \check{\rho}_0(x) + \check{\rho}_2(x)e^{-2i\alpha} + \check{\rho}_4(x)e^{-4i\alpha}$.

Step 2:

Using the approximation for the weight function, we employ equation (3.42) to construct our system from

$$
\begin{aligned}
&\sum_{n\in\mathbb{Z}} \left(\bar{\partial}u_n + \partial u_{n+2} + \mu u_{n+1}\right) e^{-in\alpha} \\
&\qquad = \frac{1}{2}\left[\check{\rho}_{-4}(x)e^{4i\alpha} + \check{\rho}_{-2}(x)e^{2i\alpha} + \check{\rho}_0(x) + \check{\rho}_2(x)e^{-2i\alpha} + \check{\rho}_4(x)e^{-4i\alpha}\right]\left[\bar{v}e^{2i\alpha} + v\right].
\end{aligned}
$$

Collecting the equations with non-zero right hand sides,

$$\bar{\partial}\bar{u}_6 + \partial\bar{u}_4 + \mu\bar{u}_5 = \frac{1}{2}\check{\rho}_{-4}\bar{v} \qquad n = -6 \tag{3.71}$$

$$\bar{\partial}\bar{u}_4 + \partial\bar{u}_2 + \mu\bar{u}_3 = \frac{1}{2}\left(\check{\rho}_{-2}\bar{v} + \check{\rho}_{-4}v\right) \qquad n = -4 \tag{3.72}$$

$$\bar{\partial}\bar{u}_2 + \partial u_0 + \mu\bar{u}_1 = \frac{1}{2}\left(\check{\rho}_0\bar{v} + \check{\rho}_{-2}v\right) \qquad n = -2 \tag{3.73}$$

$$\bar{\partial}u_0 + \partial u_2 + \mu u_1 = \frac{1}{2}\left(\check{\rho}_2\bar{v} + \check{\rho}_0 v\right) \qquad n = 0 \tag{3.74}$$

$$\bar{\partial}u_2 + \partial u_4 + \mu u_3 = \frac{1}{2}\left(\check{\rho}_4\bar{v} + \check{\rho}_2 v\right) \qquad n = 2 \tag{3.75}$$

$$\bar{\partial}u_4 + \partial u_6 + \mu u_5 = \frac{1}{2}\check{\rho}_4 v \qquad n = 4 \tag{3.76}$$

$$\bar{\partial}u_n + \partial u_{n+2} + \mu u_{n+1} = 0 \qquad n \in \mathbb{Z} \setminus \{-6, -4, -2, 0, 2, 4\} \tag{3.77}$$

Step 3:

Examining (3.77), we see that $n = 5$ is the point from which all equations have a zero right hand side, meaning that the components $(u_5, u_6, \ldots)$ of $\boldsymbol{u} = (u_0, u_1, u_2, \ldots)$ are known from boundary conditions. This means that u_0, u_1, u_2, u_3, and u_4 are unknown (recalling that $u_{-j} = \bar{u}_j$). After substitutions, our system will second order in terms of u_1 and u_3, the unknown odd components of $\boldsymbol{u}$.

Step 4:

Using equations (3.71) and (3.76), explicit formulas for v and $\bar{v}$ are found:

$$v = \frac{2}{\check{\rho}_4}\left(\bar{\partial}u_4 + \partial u_6 + \mu u_5\right) \qquad \bar{v} = \frac{2}{\check{\rho}_{-4}}\left(\bar{\partial}\bar{u}_6 + \partial\bar{u}_4 + \mu\bar{u}_5\right).$$

In these expressions, u_4 is the only unknown.

Step 5:

We now require equations for the unknown even elements of $\boldsymbol{u}$; u_0, u_2, and u_4 in terms of u_1 and u_3. To obtain these, we look at the cases for $n = -1, 1, 3$ of equation (3.77).

$$\bar{\partial}\bar{u}_1 + \partial u_1 + \mu u_0 = 0 \qquad n = -1$$

$$\bar{\partial} u_1 + \partial u_3 + \mu u_2 = 0 \qquad n = 1$$

$$\bar{\partial} u_3 + \partial u_5 + \mu u_4 = 0 \qquad n = 3$$

Under the condition $\mu \neq 0$, it is possible to solve these for our desired functions:

$$u_0 = \frac{-\bar{\partial}\bar{u}_1 - \partial u_1}{\mu} \qquad u_2 = \frac{-\bar{\partial} u_1 - \partial u_3}{\mu} \qquad u_4 = \frac{-\bar{\partial} u_3 - \partial u_5}{\mu}$$

Step 6:

We have now reached the point where we will begin substituting our formulas from the previous steps into equations (3.74) and (3.76). After these substitutions, there will be two second order differential equations in terms of the complex functions u_1 and u_3.

$$\bar{\partial}\left[\frac{-1}{\mu}\left(\bar{\partial}\bar{u}_1 + \partial u_1\right)\right] + \partial\left[\frac{-1}{\mu}\left(\bar{\partial} u_1 + \partial u_3\right)\right] + \mu u_1$$
$$= \frac{\check{\rho}_0}{\check{\rho}_4}\left\{\partial u_6 + \bar{\partial}\left[\frac{-1}{\mu}\left(\bar{\partial} u_3 + \partial u_5\right)\right] + \mu u_5\right\} + \frac{\check{\rho}_2}{\check{\rho}_{-4}}\left\{\bar{\partial}\bar{u}_6 + \partial\left[\frac{-1}{\mu}\left(\partial\bar{u}_3 + \bar{\partial}\bar{u}_5\right)\right] + \mu\bar{u}_5\right\} \tag{3.78}$$

$$\bar{\partial}\left[\frac{-1}{\mu}\left(\bar{\partial} u_1 + \partial u_3\right)\right] + \partial\left[\frac{-1}{\mu}\left(\bar{\partial} u_3 + \partial u_5\right)\right] + \mu u_3$$
$$= \frac{\check{\rho}_2}{\check{\rho}_4}\left\{\partial u_6 + \bar{\partial}\left[\frac{-1}{\mu}\left(\bar{\partial} u_3 + \partial u_5\right)\right] + \mu u_5\right\} + \frac{\check{\rho}_4}{\check{\rho}_{-4}}\left\{\bar{\partial}\bar{u}_6 + \partial\left[\frac{-1}{\mu}\left(\partial\bar{u}_3 + \bar{\partial}\bar{u}_5\right)\right] + \mu\bar{u}_5\right\} \tag{3.79}$$

Before replacing u_1 and u_3 with component functions $\boldsymbol{w}$, we collect the lower order terms to simplify the expression. Also, we replace the Fourier coefficient ratios with

$$\alpha = \frac{\check{\rho}_0}{\check{\rho}_4}, \qquad \beta = \frac{\check{\rho}_2}{\check{\rho}_{-4}}, \qquad \gamma = \frac{\check{\rho}_2}{\check{\rho}_4}, \qquad \delta = \frac{\check{\rho}_4}{\check{\rho}_{-4}}. \tag{3.80}$$

After these changes equations (3.78) and (3.79) are

$$\begin{aligned} &\bar{\partial}\left[\frac{-1}{\mu}\left(\bar{\partial}\bar{u}_1 + \partial u_1\right)\right] + \partial\left[\frac{-1}{\mu}\left(\bar{\partial}u_1 + \partial u_3\right)\right] + \mu u_1 - \alpha\bar{\partial}\left[\frac{-1}{\mu}\left(\bar{\partial}u_3\right)\right] - \beta\partial\left[\frac{-1}{\mu}\left(\partial\bar{u}_3\right)\right] \\ &\quad \underbrace{-\alpha\left\{\partial u_6 + \bar{\partial}\left[\frac{-1}{\mu}\left(\partial u_5\right)\right] + \mu u_5\right\} - \beta\left\{\bar{\partial}\bar{u}_6 + \partial\left[\frac{-1}{\mu}\left(\bar{\partial}\bar{u}_5\right)\right] + \mu\bar{u}_5\right\}}_{d_1(x)+id_2(x)} \end{aligned} \tag{3.81}$$

$$\begin{aligned} &\bar{\partial}\left[\frac{-1}{\mu}\left(\bar{\partial}u_1 + \partial u_3\right)\right] + \partial\left[\frac{-1}{\mu}\left(\bar{\partial}u_3 + \partial u_3\right)\right] + \mu u_3 - \gamma\bar{\partial}\left[\frac{-1}{\mu}\left(\bar{\partial}u_3\right)\right] - \delta\partial\left[\frac{-1}{\mu}\left(\partial\bar{u}_3\right)\right] \\ &\quad \underbrace{-\gamma\left\{\partial u_6 + \bar{\partial}\left[\frac{-1}{\mu}\left(\partial u_5\right)\right] + \mu u_5\right\} - \delta\left\{\bar{\partial}\bar{u}_6 + \partial\left[\frac{-1}{\mu}\left(\bar{\partial}\bar{u}_5\right)\right] + \mu\bar{u}_5\right\}}_{d_3(x)+id_4(x)} \end{aligned} \tag{3.82}$$

where the $d_j(x)$ functions are the lower order terms indicated, separated into real and imaginary parts. In this form, we are prepared to proceed to the next step.

Step 7:

Here we multiply both (3.81) and (3.82) by 4μ in preparation to use equations (3.53), (3.54), (3.55), and (3.56) to expand the expressions. The computations involve using these lemmas several times and result in very large expressions to simplify as was seen in the second approximation case, rather than showing that intermediary step, we move to step 8 and show the results after introducing the component functions $\boldsymbol{w}$ which will replace u_1, u_3 and their derivatives.

Step 8:

Reducing the second order equations (3.81) and (3.82) to a first order system is accomplished by introducing the terms

$$
\begin{aligned}
\partial_1 w_1 &= w_3 & \partial_1 w_7 &= w_9 \\
\partial_2 w_1 &= w_4 & \partial_2 w_7 &= w_{10} \\
\partial_1 w_2 &= w_5 & \partial_1 w_8 &= w_{11} \\
\partial_2 w_2 &= w_6 & \partial_2 w_8 &= w_{12}.
\end{aligned}
\tag{3.83}
$$

After the calculations described in Step 7, we have

$$2\partial_1 w_3 + \partial_1 w_5 + (1-\alpha-\beta)\partial_1 w_9 + 2\partial_2 w_5 + \partial_2 w_6 + (\beta-\alpha-1)\partial_2 w_{10} + (2-2\alpha+2\beta)\partial_2 w_{11}$$
$$- 3l_1 w_3 - l_2 w_4 - l_1 w_5 + l_2 w_6 + (\alpha+\beta-1)l_1 w_9 + (1-\alpha-\beta)\left(l_2 w_{10} + l_2 w_{11} + l_1 w_{12}\right) + 4\mu d_1(x) +$$
$$i\Big\{\partial_1 w_4 + \partial_1 w_5 + (\beta-\alpha)\partial_1 w_{10} + (\beta-\alpha)\partial_1 w_{11} + \partial_2 w_3 + \partial_2 w_4 + 2\partial_2 w_6 + (\beta-\alpha)\partial_2 w_9 + (\alpha-\beta)\partial_2 w_{12}$$
$$- l_2 w_3 - l_1 w_4 - l_1 w_5 - 3l_2 w_6 + (\alpha-\beta)\left(l_2 w_9 + l_1 w_{10} + l_1 w_{11} - l_2 w_{12}\right) + 4\mu d_2(x)\Big\} = 0 \tag{3.84}$$

$$\partial_1 w_3 + (2-\gamma-\delta)\partial_1 w_9 + \partial_2 w_4 - \partial_2 w_5 + (-\gamma-\delta)\partial_2 w_{10} + (2+2\gamma+2\delta)\partial_2 w_{11} - l_1 w_3 + l_2 w_4$$
$$+ l_2 w_5 + l_1 w_6 + (\gamma+\delta-2)l_1 w_9 + (-\gamma-\delta)\left(l_2 w_{10} + l_2 w_{11}\right) + (-2-\gamma-\delta)l_1 w_{12} + 4\mu d_3(x) +$$
$$i\Big\{\partial_1 w_4 + \partial_1 w_5 + (\gamma-\delta-1)\partial_1 w_{10} + (2+\gamma-\delta)\partial_1 w_{11} + \partial_2 w_3 - \partial_2 w_6 + (\gamma-\delta-1)\partial_2 w_9 +$$
$$(\gamma-\delta)\partial_2 w_{12} - l_2 w_3 - l_1 w_4 - l_1 w_5 + l_2 w_6 + (\delta-\gamma)\left(l_2 w_9 - l_2 w_{12}\right) + (2-\gamma+\delta)l_1 w_{10} +$$
$$(-2-\gamma+\delta)l_1 w_{11} + 4\mu d_4(x)\Big\} = 0 \tag{3.85}$$

These equations form four of the twelve total equations in our first order system for $\boldsymbol{w}$. The remaining 8 equations are based upon the definitions of w_3 - w_6 and w_9 - w_{12} given in (3.83). Choose real numbers $\lambda_1 \neq \lambda_2$, and multiply $\partial_2 w_1 - w_4 = 0$ and $\partial_2 w_2 + w_6 = 0$ by λ_1. Add and subtract these equations from $\partial_1 w_1 - w_3 = 0$ and $\partial_1 w_2 - w_5 = 0$ respectively to obtain:

$$\partial_1 w_1 - w_3 + \lambda_1 \partial_2 w_1 - \lambda_1 w_4 = 0$$
$$\partial_1 w_2 - w_5 - \lambda_1 \partial_2 w_2 + \lambda_1 w_6 = 0$$

Similar steps are taken with w_9 - w_{12}'s definitions and λ_2 for

$$\partial_1 w_7 - w_9 + \lambda_2 \partial_2 w_7 - \lambda_2 w_{10} = 0$$
$$\partial_1 w_8 - w_{11} - \lambda_2 \partial_2 w_8 + \lambda_2 w_{12} = 0$$

The remaining equations are generated by using Clairaut's theorem for the reordering of partial derivatives. For example, $\partial_1 w_4 = \partial_1\partial_2 w_1 = \partial_2\partial_1 w_1 = \partial_2 w_3$. We rearrange to set this equal to zero, and collecting all similar equations yields

$$\begin{aligned}
\partial_1 w_4 - \partial_2 w_3 &= 0 \\
\partial_1 w_6 - \partial_2 w_5 &= 0 \\
\partial_1 w_{10} - \partial_2 w_9 &= 0 \\
\partial_1 w_{12} - \partial_2 w_{11} &= 0
\end{aligned}$$

which are the remaining equations necessary to construct our system.

Step 9: We are going to write the system in the matrix equations form

$$A(x,y)\frac{\partial \boldsymbol{w}}{\partial x} + B(x,y)\frac{\partial \boldsymbol{w}}{\partial y} + C(x,y)\boldsymbol{w} + D(x,y) = 0.$$

The matrices A and B are the most important, so they will shown here with C and D in the index. Forming A and B requires one last step, because we must account for the real and imaginary parts of our coefficients α, β, γ, and δ. They will be replaced by $\alpha = \alpha_1 + i\alpha_2$ and so forth in the matrices.

$$A = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 2 & 0 & 1 & 0 & 0 & 0 & 1-\alpha_1-\beta_1 & \alpha_2-\beta_2 & \alpha_2-\beta_2 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 1 & 0 & 0 & 0 & -\alpha_2-\beta_2 & \beta_1-\alpha_1 & \beta_1-\alpha_1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 2-\gamma_1-\delta_1 & \delta_2-\gamma_2 & \delta_2-\gamma_2 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 1 & 0 & 0 & 0 & -\gamma_2-\delta_2 & \gamma_1-\delta_1-1 & 2+\gamma_1-\delta_1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix}$$

$$B = \begin{bmatrix} \lambda_1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & -\lambda_1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 2 & 1 & 0 & 0 & \alpha_2-\beta_2 & \beta_1-\alpha_1-1 & 2-2\alpha_1-2\beta_1 & \beta_2-\alpha_2 \\ 0 & 0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 1 & 0 & 2 & 0 & 0 & \beta_1-\alpha_1 & \beta_2-\alpha_2 & -2\alpha_2-2\beta_2 & \alpha_1-\beta_1 \\ 0 & 0 & 0 & 0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & \lambda_2 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & -\lambda_2 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & -2 & 0 & 0 & 0 & \delta_2-\gamma_2 & -\gamma_1-\delta_1 & 2+2\gamma_1+2\delta_1 & \gamma_2-\delta_2 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & -1 & 0 & 0 & \gamma_1-\delta_1-1 & -\gamma_2-\delta_2-1 & 2\gamma_2+2\delta_2 & \delta_1-\gamma_1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & 0 \end{bmatrix}$$

With the complexity of the formulas increasing along with matrix size, $\tilde{B}$ will be given in sparse form, with only the nonzero elements listed. The notation (i, j) indicates the row and column where the nonzero entries appear.

$\tilde{B}$:

$$(1,1) = \lambda_1$$

$$(2,2) = -\lambda_1$$

$$(3,3) = \frac{-2}{d}[\alpha_1(\gamma_1 + 3\delta_1 - \alpha_1 - 5) - \alpha_2(\alpha_2 + \gamma_2 - 3\delta_2) + \beta_1(\beta_1 - 3\gamma_1 - \delta_1 + 1)$$
$$+ \beta_2(\beta_2 - 3\gamma_2 + \delta_2) + 2\gamma_1^2 + \gamma_1 + 2\gamma_2^2 - 2\delta_1^2 + 7\delta_1 - 2\delta_2^2 - 6]$$

$$(4,3) = -1$$

$$(5,3) = \frac{-2}{d}[2\alpha_1 - 2\beta_1 - 4\delta_1 - \alpha_1(\gamma_1 + \delta_1) + \alpha_2(\gamma_2 - \delta_2) + \beta_1(\gamma_1 + \delta_1) + \beta_2(\gamma_2 - \delta_2)$$
$$- \gamma_1^2 + \delta_1^2 + \delta_2^2 + 4]$$

$$(9,3) = \frac{2}{d}(\alpha_1 - \beta_1 + \gamma_1 - \delta_1 + 2)$$

$$(11,3) = \frac{2}{d}(\gamma_2 + \delta_2 - \alpha_2 - \beta_2)$$

$$(3,4) = \frac{1}{d}[\alpha_1(\alpha_1 + \gamma_1 - 2\gamma_2 - \delta_1 + 1) + \alpha_2(\alpha_2 - 2\gamma_1 - \gamma_2 - \delta_2) - \beta_1(\beta_1 - \gamma_1 + \delta_1 - 2\delta_2 - 3)$$
$$- \beta_2(\beta_2 - \gamma_2 - 2\delta_1 - \delta_2 + 4) + \gamma_1^2 - \gamma_1 + \gamma_2^2 + \gamma_2 - \delta_1^2 + 5\delta_1 - \delta_2^2 - \delta_2 - 6]$$

$$(5,4) = \frac{1}{d}[\alpha_1(2 + \gamma_1 + 2\gamma_2 - \delta_1 + 2\delta_2) + \alpha_2(4 + 2\gamma_1 - \gamma_2 - 2\delta_1 - \delta_2) + \beta_1(2 + \gamma_1 - 2\gamma_2 - \delta_1 - 2\delta_2)$$
$$+ \beta_2(4 + 2\gamma_1 + \gamma_2 - \delta_1 + \delta_2) - 2\gamma_1^2 - \gamma_1 + 2\delta_1^2 - 7\delta_1 + 2\delta_2^2 - 6]$$

$$(9,4) = \frac{1}{d}[2\alpha_1 - \alpha_2 - 2\beta_1 + \beta_2 + 3\gamma_1 - 2\gamma_2 - 3\delta_1 + 2\delta_2 + 6]$$

$$(11,4) = \frac{1}{d}[-\alpha_1 - 2\alpha_2 - \beta_1 - 2\beta_2 + 2\gamma_1 + 3\gamma_2 + 2\delta_1 + 3\delta_2]$$

$$(3,5) = \frac{2}{d}[-\alpha_1(\alpha_1 + 2\gamma_1 - \gamma_2 - 1) - \alpha_2(\alpha_2 - \gamma_1 - 2\gamma_2 + \delta_1 - 2) + \beta_1(\beta_1 - \gamma_2 + 2\delta_1 - \delta_2 - 5)$$
$$+ \beta_2(\beta_2 + \gamma_1 - \delta_1 - 2\delta_2 + 2) - \gamma_1^2 + \gamma_1 - \gamma_2^2 + \delta_1^2 - 5\delta_1 + \delta_2^2 + 6]$$

$$(5,5) = \frac{-6}{d}[2\alpha_2 + 2\beta_2 + \alpha_1\gamma_2 + \alpha_2\gamma_1 + \alpha_1\delta_2 - \alpha_2\delta_1 - \beta_1\gamma_2 + \beta_2\gamma_1 - \beta_1\delta_2 - \beta_2\delta_1)]$$

$$(6,5) = \frac{1}{d}[-\alpha_1(\alpha_1 - 4\gamma_1 + 3\gamma_2 + \delta_2 - 1) - \alpha_2(\alpha_2 + \gamma_1 + 4\gamma_2 - \delta_1 + 2) + \beta_1(\beta_1 + \gamma_2 - 4\delta_1 + \delta_2 + 7)$$
$$+ \beta_2(\beta_2 - \gamma_1 + \delta_1 + 4\delta_2 - 2) + 4\gamma_1^2 + 2\gamma_1 + 4\gamma_2^2 - 4\delta_1^2 + 14\delta_1 - 4\delta_2^2 - 12]$$
$$(9,5) = \frac{-6}{d}[\alpha_1 - \beta_1 + \gamma_1 - \delta_2 + 2]$$
$$(11,5) = \frac{6}{d}[\alpha_2 + \beta_2 - \gamma_2 - \delta_2]$$
$$(3,6) = \frac{1}{d}[\alpha_1(4 - 2\gamma_1 - 3\gamma_2 - 2\delta_1 + 3\delta_2) + \alpha_2(6 - 3\gamma_1 + 2\gamma_2 - 3\delta_1 - 2\delta_2) + \beta_1(2\gamma_1 - 3\gamma_2 + 2\delta_1$$
$$+ 3\delta_2 - 4) + \beta_2(2\gamma_2 + 3\delta_1 - 2\delta_2 - 6) + \gamma_1^2 + 3\gamma_2 + \gamma_2^2 - \delta_1^2 + 4\delta_1 - \delta_2^2 - 3\delta_2 - 4]$$
$$(5,6) = \frac{1}{d}[-\alpha_1(\alpha_1 - 4\gamma_1 + 3\gamma_2 + \delta_2 - 1) - \alpha_2(\alpha_2 + \gamma_1 + 4\gamma_2 - \delta_1 + 2) + \beta_1(\beta_1 + \gamma_2 - 4\delta_1 + \delta_2 + 7)$$
$$+ \beta_2(\beta_2 - \gamma_1 + \delta_1 + \delta_2 - 2) + 4\gamma_1^2 + 2\gamma_1 + 4\gamma_2^2 - 4\delta_1^2 + 14\delta_1 - 4\delta_2^2 - 12]$$
$$(9,6) = \frac{1}{d}[-2\alpha_1 - 3\alpha_2 + 2\beta_1 + 3\beta_2 + \gamma_1 - 6\gamma_2 - \delta_1 + 6\delta_2 + 2]$$
$$(11,6) = \frac{1}{d}[-3\alpha_1 + 2\alpha_2 - 3\beta_1 + 2\beta_2 + 6\gamma_1 + \gamma_2 + 6\delta_1 + \delta_2 - 9]$$
$$(7,7) = \lambda_2$$
$$(8,8) = -\lambda_2$$
$$(3,9) = \frac{2}{d}\Big[- \alpha_1(3\gamma_1 + 2\gamma_2 + 3\delta_1 - 2\delta_2 + \gamma_2^2 - \alpha_1\gamma_2 + \delta_2^2 + \alpha_1\delta_2 - \gamma_1\gamma_2 + \gamma_1\delta_2 + \gamma_2\delta_1 - \delta_1\delta_2 - 6)$$
$$- \alpha_2(3\gamma_1 - \gamma_2 + 3\delta_1 + \delta_2 + \gamma_2^2 - \alpha_2\gamma_2 - \delta_2^2 + \alpha_2\delta_2 + \gamma_1\gamma_2 - \gamma_1\delta_2 - \gamma_2\delta_1 + \delta_1\delta_2 - 6)$$
$$+ \beta_1(3\gamma_1 + 3\delta_1 + \gamma_2^2 - \beta_1\gamma_2 - \delta_2^2 + \beta_1\delta_2 + \gamma_1\gamma_2 - \gamma_1\delta_2 - \gamma_2\delta_1 + \delta_1\delta_2 - 6)$$
$$+ \beta_2(3\gamma_1 + \gamma_2 + 3\delta_1 - \delta_2 + \gamma_2^2 - \beta_2\gamma_2 - \delta_2^2 + \beta_2\delta_2 - \gamma_1\gamma_2 + \gamma_1\delta_2 + \gamma_2\delta_1 - \delta_1\delta_2 - 6)$$
$$- \gamma_1(\gamma_2 - \delta_2) + \gamma_2(\delta_1 + 1) - \delta_2(\delta_1 + 1)\Big]$$
$$(5,9) = \frac{-2}{d}\Big[\alpha_1(3\alpha_1 - 3\gamma_1 - 2\gamma_2^2 - 6\delta_1 + 2\delta_2^2 + 9) + \alpha_2(3\alpha_2 + 2\gamma_2 - 2\delta_2 - 2\gamma_1\gamma_2 + 2\gamma_1\delta_2 + 2\gamma_2\delta_1$$
$$- 2\delta_1\delta_2) - \beta_1(-3\beta_1 + 6\gamma_1 + 6\delta_1 + 2\gamma_2^2 - 2\delta_2^2 - 9)$$
$$+ \beta_2(-3\beta_2 + 2\gamma_2 - 2\delta_2 - 2\gamma_1\gamma_2 + 2\gamma_1\delta_2 + 2\gamma_2\delta_1 - 2\delta_1\delta_2)\Big]$$
$$(9,9) = \frac{-2}{d}\Big[\alpha_1(-2\gamma_2 + 2\delta_2 + 3) + 3\alpha_2 + \beta_1(2\gamma_2 - 2\delta_2 - 3) + \gamma_1(-2\gamma_2 + 2\delta_2) + \gamma_2(2\delta_2 + 2)$$
$$- \delta_2(2\delta_1 + 2)\Big]$$

$$(10,9) = -1$$

$$(11,9) = \frac{1}{d}\Big[2\alpha_1(\alpha_1 - \gamma_1 - \gamma_2 - 3\delta_1 - \delta_2 + 2) + \alpha_2(\alpha_2 - \gamma_1 - \gamma_2 + \delta_1 - \delta_2 + 1)$$
$$- \beta_1(\beta_1 - \gamma_1 - \delta_1 - \delta_2 + 4) + \beta_2(1 + \gamma_2 + \delta_1 + \delta_2 - \beta_2) - 2\gamma_1^2 + 5\gamma_1 + 2\delta_1^2 - \delta_1 - 3\Big]$$

$$(3,10) = \frac{-2}{d}\Big[\alpha_1(\alpha_1 - 2\beta_1 - \gamma_2 - 2\delta_1 - \delta_2 + \beta_1\gamma_1 + \beta_2\gamma_2 + \beta_1\delta_1 - \beta_2\delta_2 + 3)$$
$$+ \alpha_2(\alpha_2 - 2\beta_2 - \gamma_1 + \delta_1 - 2\delta_2 - \beta_1\gamma_2 + \beta_2\gamma_1 + \beta_1\delta_2 + \beta_2\delta_1 - 2) + \beta_1^2(1 - \gamma_1 - \gamma_1)$$
$$+ \beta_1(2\gamma_1 + \gamma_2 + \gamma_1^2 + \gamma_2^2 - \delta_1^2 - \delta_2^2 - 3) + \beta_2^2(1 - \gamma_1 - \delta_1) + \beta_2(-\gamma_1 + \gamma_2 - 3\delta_1$$
$$- \gamma_1^2 - \gamma_2^2 + \delta_1^2 + \delta_2^2 + 2) - \gamma_1^2 - \gamma_1 - \gamma_2^2 + \delta_1^2 - 3\delta_1 + \delta_2^2\Big]$$

$$(5,10) = \frac{-2}{d}\Big[\alpha_1(-2\beta_2 + \gamma_2 + \delta_2 - \beta_1\gamma_2 - \beta_2\gamma_1 + \beta_1\delta_2 + \beta_2\delta_1) + \alpha_2(2\beta_1 + \gamma_1 - \delta_1$$
$$+ \beta_1\gamma_1 + \beta_2\gamma_2 - \beta_1\delta_1 + \beta_2\delta_2 + 2) - \beta_1^2(\gamma_2 + \delta_2) - \beta_1(\gamma_2 + \delta_2) - \beta_2^2(\gamma_2 + \delta_2)$$
$$+ \beta_2(2\gamma_1 + 6\delta_1 + 2\gamma_1^2 + 2\gamma_2^2 - 2\delta_1^2 - 2\delta_2^2 - 4)\Big]$$

$$(9,10) = \frac{1}{d}\Big[\alpha_1^2 + \alpha_1(3 - 2\beta_1 - \gamma_1 - \gamma_2 - 3\delta_1 - \delta_2) + \alpha_2^2 - \alpha_2(2 - \beta_2 + \gamma_1 - \gamma_2 - \delta_1 - 3\delta_2) + \beta_1^2$$
$$+ \beta_1(\gamma_1 + \gamma_2 + 3\delta_1 + \delta_2) + \beta_2^2 + \beta_2(\gamma_1 - \gamma_2 - \delta_1 + 3\delta_2 + 2)$$
$$- 2\gamma_1^2 - 3\gamma_1 - 2\gamma_2^2 + 2\delta_1^2 - 5\delta_1 + 2\delta_2^2\Big]$$

$$(11,10) = \frac{2}{d}\Big[\alpha_2(\beta_1 + 1) + \beta_1(\alpha_2 - \gamma_2 - \delta_2) - \gamma_2 - \delta_2\Big]$$

$$(3,11) = \frac{-2}{d}\Big[\alpha_1^2(1 - 2\gamma_1 - 2\delta_1) + \alpha_1(1 - 2\gamma_1^2 - \gamma_1 - 2\gamma_2^2 + 3\gamma_2 + 2\delta_1^2 - 3\delta_1 + 2\delta_2^2 + 3\delta_2)$$
$$+ \alpha_2^2(-2 - 2\gamma_1 - 2\delta_1) + \alpha_1(1 + 2\gamma_1^2 + 3\gamma_1 + 2\gamma_2^2 + 3\gamma_2 - 2\delta_1^2 + 5\delta_1 - 2\delta_2^2 + 3\delta_2)$$
$$+ \beta_1^2(2\gamma_1 + 2\delta_1 - 1) + \beta_1(3 - 2\gamma_1^2 - 5\gamma_1 - 2\gamma_2^2 - 3\gamma_2 + 2\delta_1^2 - 7\delta_1 - 2\delta_2^2 - 3\delta_2)$$
$$+ \beta_2^2(2\gamma_1 + 2\delta_1 - 1) + \beta_2(2\gamma_1^2 + 3\gamma_1 + 2\gamma_2^2 - \gamma_2 - 2\delta_1^2 + 5\delta_1 - 2\delta_2^2 - 3\delta_2 - 2)$$
$$+ 2\gamma_1^2 + 3\gamma_1 + 2\gamma_2^2 - 2\delta_1^2 + 5\delta_1 - 2\delta_2^2$$

$$(5,11) = \frac{4}{d}\Big[\alpha_1^2(\gamma_2 + \delta_2) + 2\alpha_1(\gamma_2 + \delta_2) + \alpha_2^2(\gamma_2 + \delta_2) + \alpha_2(2\gamma_1^2 + 3\gamma + 2\gamma_2^2 - 2\delta_1^2 + 5\delta_1 - 2\delta_2^2 - 2)$$
$$- \beta_1^2(\gamma_2 + \delta_2) - 2\beta_1(\gamma_2 + \delta_2) - \beta_2^2(\gamma_2 + \delta_2) + \beta_2(2\gamma_1^2 + 3\gamma_1 + 2\gamma_2^2 - 2\delta_1^2 + 5\delta_1 - 2\delta_2^2 - 2)\Big]$$

$$
\begin{aligned}
(9,11) &= \frac{2}{d}\Big[\alpha_1(\alpha_1+3\gamma_1+\gamma_2+\delta_1+\delta_2+3)+\alpha_2(\alpha_2-\gamma_1+3\gamma_2+\delta_1-\delta_2-2) \\
&\quad+\beta_1(1-\beta_1-\gamma_1-\gamma_2-3\delta_1-\delta_2)-\beta_2(\beta_2+\gamma_1-\gamma_2-\delta_1+3\delta_2+2) \\
&\quad+2\gamma_1^2+5\gamma_1+2\gamma_2-2\delta_1^2+3\delta_1-2\delta_2^2\Big] \\
(11,11) &= \frac{4}{d}\Big[\alpha_1(\gamma_2+\delta_2)+\alpha_2(1-2\gamma_1-\gamma_2-2\delta_1-\delta_2)+\beta_1(\gamma_2+\delta_2) \\
&\quad+\beta_2(1-2\gamma_2-\gamma_2-2\delta_1-\delta_2)+2\gamma_2+2\delta_2\Big] \\
(12,11) &= -1 \\
(3,12) &= \frac{-2}{d}\Big[\alpha_1(2-\gamma_1-\gamma_2-\delta_1+\delta_2)+\alpha_2(2-\gamma_1+\gamma_2-\delta_1-\delta_2)+\beta_1(\gamma_1-\gamma_2+\delta_1+\delta_2-2) \\
&\quad+\beta_2(\gamma_1+\gamma_2+\delta_1-\delta_2-2)+\gamma_2-\delta_1\Big] \\
(5,12) &= \frac{-2}{d}\Big[\alpha_1(-\alpha_1+2\gamma_1+2\delta_1-3)-\alpha_2(\alpha_2+2\gamma_2-2\delta_2)+\beta_1(\beta_1-2\gamma_1-2\delta_1+3) \\
&\quad+\beta_2(\beta_2-2\gamma_2+2\delta_2)\Big] \\
(9,12) &= \frac{2}{d}\Big[\alpha_1+\alpha_2-\beta_1-\beta_2+2\gamma_2-\delta_2\Big] \\
(11,12) &= \frac{-1}{d}\Big[2\alpha_1(\alpha_1-\gamma_1-\gamma_2-3\delta_1-\delta_2+3)+\alpha_2(\alpha_2-\gamma_1+\gamma_2+\delta_1-3\delta_2) \\
&\quad+\beta_1(3\gamma_1+\gamma_2+\delta_1+\delta_2-3)+\beta_2(-\gamma_1+3\gamma_2+\delta_1-\delta_2)-2\gamma_1^2+3\gamma_1-2\gamma_2^2-2\delta_1^2+2\delta_2^2\Big]
\end{aligned}
$$

The eigenvalues of this matrix are $\pm\lambda_1,\ \lambda_2$ and 8 others, which are the remaining roots of the characteristic polynomial of $\tilde{B}$. The expression for this polynomial is too large to include here, but the distinctness of these eigenvalues is verified numerically using MATLAB® computations for the helioseismic case of a cycloidal surface.

Chapter 4

4 Numerical and Theoretical Results

4.1 Numerical Validation of Eigenvalue Distinctness

The guiding principle behind the construction of the systems of partial differential equation has been the 1939 work of Torsten Carleman [9] which requires that the matrix $B(x_1, x_2)$ for the system of the form

$$\frac{\partial \boldsymbol{w}}{\partial x_1} + \tilde{B}(x_1, x_2)\frac{\partial \boldsymbol{w}}{\partial x_2} + \tilde{C}(x_1, x_2)\boldsymbol{w} + \tilde{D}(x_1, x_2) = \mathbf{0} \tag{4.1}$$

have distinct eigenvalues. Due the complicated nature of the systems studied here, it is very difficult to prove analytically that all eigenvalues of the system are distinct since $\rho(x, \omega)$ may be any even trigonometric polynomial. Therefore we shall use numerical results specific to the application of helioseismology as justification that the matrix $B(x_1, x_2)$ has the desired properties for practical application. Following these numerical results, we shall prove a Carleman estimate which shows uniqueness and stability for the solutions to systems of the type constructed in Chapter 3.

Numerically, it is much simpler and more efficient to work with our system in the form

$$A(x_1, x_2)\frac{\partial \boldsymbol{w}}{\partial x_1} + B(x_1, x_2)\frac{\partial \boldsymbol{w}}{\partial x_2} + C(x_1, x_2)\boldsymbol{w} + D(x_1, x_2) = \mathbf{0} \tag{4.2}$$

before multiplying by A^{-1} to keep the expressions manageable. For each of the cases discussed in Chapter 3, A and B will be computed at several thousand points on a polar grid in the unit disk representing a section of the foliation described in Chapter 2. At each of these grid points $A^{-1}B$ will be computed, and then its eigenvalues determined. Finally, the eigenvalues are checked for distinctness (up to a tolerance of 10^{-8}) distance between them.

These computations were done using the unit testing framework built into MATLAB® release R2015b. Each case was tested at 100,050 polar grid points with 2001 radial grid points along a single diameter of the unit disk, this diameter was then rotated by an angle of $\pi/50$ radians as show in the figure:

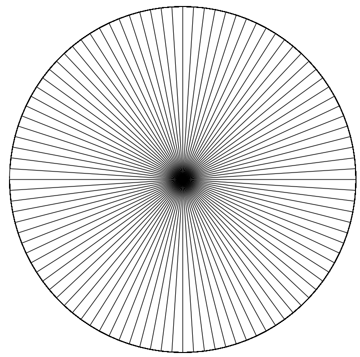


Figure 4.1: Polar grid diameters for eigenvalue tests

To make these computations, it is necessary to choose a particular weight function $\rho(x, \omega)$ and associated ray path $\varphi(|x|, r)$. As in Chapter 2, we choose (2.7)

$$\rho(x, \omega) = \sqrt{1 + (\varphi')^2 \frac{\langle x, \omega \rangle^2}{|x|^2}} \tag{4.3}$$

for which we have explicit formulas for the Fourier coefficients

$$\check{\rho}_{2n+1} = 0 \tag{4.4}$$

$$\check{\rho}_0 = \sqrt{1+q^2}\,\frac{2E(k)}{\pi} \tag{4.5}$$

$$\check{\rho}_{2n} = -\sqrt{1+q^2}\,\frac{(2n-3)!!\,k^{2n}}{2^{3n-2}\,n!}\,{}_2F_1\left(\frac{2n-1}{2}, \frac{2n+1}{2}; 2n+1; k^2\right) \tag{4.6}$$

and (2.23)

$$\varphi(|x|, r) = \begin{cases} |x| = R\,(t - \sin(t)) \\ z = R\,(1 - \cos(t)) - d \end{cases}$$

which is a cycloid and must be defined parametrically. To use our MATLAB® test for eigenvalue distinctness, the radius of a measurement ring r and the vertical shift of the ray-path must be set as parameters. A full cycloid is generated by having a rolling circle of radius R, and so the maximum height of the curve is $2R$. The total width is $2\pi R$, which means that $0 < d < 2R$ and the radius of the measurement ring r must between 0 and πR. The zero finding function `fzero` in MATLAB® is used to determine R, the radius of the generating circle, from the input parameters r and d. This computation is done for each $x \in \Omega$ to find the derivative along the surface $\varphi'(x)$ for use in (4.5) and (4.6). Fortunately, this is easy to do since cycloids obey the simple ordinary differential equation $\left(\frac{dy}{dx}\right)^2 = \frac{2R}{y} - 1$ [27]. Here is an image of the cross section of several concentric shells, showing the path along which a ray would travel between surface points.

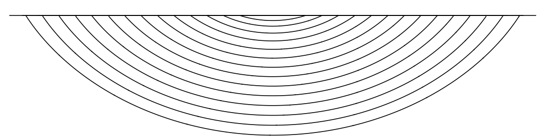


Figure 4.2: Cross section of several stacked shells

The MATLAB® used for these computations is included in the appendix and yields two very important results:

1. Every eigenvalue of the matrix $\tilde{B}$ in (4.1) is distinct.

2. The number of real eigenvalues and complex eigenvalues (with non-zero imaginary part) remain fixed for all $x \in \bar{\Omega}$.

The main estimate proven in the next section relies on these conditions, and while we do not explicitly prove that they hold for the cycloidal geodesic case, we provide in the appendices strong numerical evidence that these conditions hold here. With conditions checked numerically, we move to proving a Carleman estimate for systems of our general form (4.1).

4.2 Carleman Estimate for Systems of Partial Differential Equations

Let Ω be an open bounded domain in $\mathbb{R}^2$ with boundary $\partial\Omega \in C^1$, and P be a linear first order partial differential operator with matrix coefficients of the form

$$Pu := \partial_1 u + A\partial_2 u + Bu \tag{4.7}$$

where $A \in C^1(\bar{\Omega}; \mathbb{R}^{n\times n})$, $B \in L_\infty(\Omega; \mathbb{R}^{n\times n})$, and $u \in C^1(\bar{\Omega}; \mathbb{R}^n)$. The goal is to obtain an estimate

$$\|u\|_{L_2(\Omega;\mathbb{R}^n)} \leq C\left(\|Pu\|_{L_2(\Omega;\mathbb{R}^n)} + \|Tu\|_{W_2^1(\partial\Omega;\mathbb{R}^n)}\right) \qquad \forall u \in C^1(\bar{\Omega}; \mathbb{R}^n) \tag{4.8}$$

where $Tu := u|_{\partial\Omega}$ is the trace of the vector function u on the boundary $\partial\Omega$. We will accomplish this by proving a Carleman estimate of the type

$$\tau\|e^{\tau\phi}u\|_{L_2}^2 \leq C\|e^{\tau\phi}Pu\|_{L_2}^2 + \tau C_1\|e^{\tau\phi}Tu\|_{W_2^1}^2 \tag{4.9}$$

that holds for all $\tau \geq \tau_0$ and for all $u \in C^1(\bar{\Omega}; \mathbb{R}^n)$. Additionally, we require that the matrix A is real valued and has n distinct eigenvalues. Those eigenvalues have the structure of m complex conjugate pairs and p real values, so the total number of eigenvalues $n = 2m + p$. The eigenvectors e_j associated with each of these eigenvalues look like

$$\begin{aligned} &Ae_j = \lambda_j e_j, && \lambda_j = a_j + ib_j, && j = 1, 2, \ldots, m \\ &A\bar{e}_j = \bar{\lambda}_j \bar{e}_j, && \bar{\lambda}_j = a_j - ib_j, && j = 1, 2, \ldots, m \\ &Ae_j = \lambda_j e_j, && \lambda_j = a_j, && j = 2m+1, 2m+2, \ldots, 2m+p \end{aligned} \tag{4.10}$$

where $b_j \neq 0$ for the eigenvalues denoted as complex. It is a requirement of the theorem to be proven that the amount of real and complex eigenvalues for the matrix remain fixed.

Now let

$$Q = [e_1, e_2, \dots, e_m, \bar{e}_1, \bar{e}_2, \dots, \bar{e}_m, e_{2m+1}, \dots, e_n] \tag{4.11}$$

be an $n \times n$ matrix of the eigenvectors corresponding to A. Then we know that $AQ = Q\Lambda$ where Λ is the diagonal matrix of A's eigenvalues. Because $\det(Q) \neq 0$, its inverse exists and we have $Q^{-1}AQ = \Lambda \in C^1(\bar{\Omega}; \mathbb{R}^{n\times n})$. The roots of the characteristic polynomial of A depend smoothly on the A's coefficients when all roots are simple (distinct) for all $x \in \bar{\Omega}$. Due to this, it is sufficient to prove the Carleman estimate (4.9) for the scalar case without considering the lower order terms. Hence, we now take

$$P = \partial_1 + (a + ib)\partial_2 \tag{4.12}$$

for real-valued functions $a, b \in C^1(\bar{\Omega})$ where either

$$b(x) \equiv 0 \qquad \text{for all } x \in \bar{\Omega} \qquad \text{hyperbolic case} \tag{4.13}$$

$$b(x) \neq 0 \qquad \text{for all } x \in \bar{\Omega} \qquad \text{elliptic case} \tag{4.14}$$

Suppose $\phi(x) \in C^2(\bar{\Omega})$ is a real-valued function and choose $\tau > 0$. The notation $L_2(\Omega)$ denotes a Hilbert space of complex-valued functions with the inner product

$$\langle u, v \rangle = \int_\Omega u(x)\overline{v(x)}\, dx \tag{4.15}$$

and norm

$$\|u\|^2 = \int_\Omega |u(x)|^2\, dx. \tag{4.16}$$

Let the operator $L = e^{\tau\phi} P e^{-\tau\phi}$ and the function $v = e^{\tau\phi} u$. Then we may write

$$\|e^{\tau\phi} P u\|^2 = \|Lv\|^2 = \|L_+ v + L_- v\|^2 \tag{4.17}$$

where $L_\pm = \dfrac{L \pm L^*}{2}$ and L^* is the formal adjoint operator of L. By construction, L_+ will be symmetric and L_- will be skew-symmetric with respect to the inner product on functions $v \in C_0^1(\Omega) = C^1(\bar{\Omega}) \cap \{v = 0 \text{ on } \partial\Omega\}$:

$$\begin{aligned} L_+^* = L_+, \qquad L_-^* = -L_- \qquad \text{on } C_0^1 \\ \langle L_+ w, v\rangle = \langle w, L_+ v\rangle \\ \langle L_- w, v\rangle = -\langle w, L_- v\rangle \end{aligned}$$

The (skew) symmetry of L_- and L_+ holds on the entire space $C^1(\bar{\Omega})$ up to some additional boundary terms $B_\pm$. Simple calculations show that

$$L_+ = ib\partial_2 - \tau(\phi_1 + a\phi_2) - \frac{1}{2}(a_2 - ib_2) \tag{4.18}$$

$$L_- = \partial_1 + a\partial_2 - \tau(ib\phi_2) + \frac{1}{2}(a_2 - ib_2) \tag{4.19}$$

where $\phi_j = \partial_j \phi$, $a_j = \partial_j a$, and $b_j = \partial_j b$. Now, integration by parts allows us to work towards finding the boundary terms $B_\pm$.

$$\int_\Omega w \partial_j \bar{v}\, dx = \int_{\partial\Omega} \nu_j w \bar{v}\, ds - \int_\Omega (\partial_j w)\bar{v}\, dx \tag{4.20}$$

where $\nu = (\nu_1, \nu_2)$ is the outward unit normal to $\partial\Omega$ and find that

$$\langle w, L_- v\rangle = -\langle L_- w, v\rangle + \underbrace{\int_{\partial\Omega} (\nu_1 + a\nu_2) w\bar{v}\, ds}_{B_-(w,\bar{v})} \tag{4.21}$$

$$\langle w, L_+ v\rangle = \langle L_+ w, v\rangle + \underbrace{\int_{\partial\Omega} (-ib\nu_2) w\bar{v}\, ds}_{B_+(w,\bar{v})} \tag{4.22}$$

Here we used the fact that $\partial_1 + a\partial_2$ and $ib\partial_2$ are the principal parts of operators L_- and L_+. Returning to the right hand side of (4.17), we rewrite it as

$$\|L_+ v + L_- v\|^2 = \|L_+ v\|^2 + \|L_- v\|^2 + \langle L_+ v, L_- v\rangle + \langle L_- v, L_+ v\rangle \tag{4.23}$$

By choosing $w = L_+ v$ for $\langle L_+ v, L_- v\rangle$ and $w = L_- v$ for $\langle L_- v, L_+ v\rangle$ and applying identities (4.21) and (4.22), we achieve

$$\begin{aligned}\langle L_+ v, L_- v\rangle + \langle L_- v, L_+ v\rangle &= \langle [L_+, L_-]v, v\rangle + \int_{\partial\Omega} (\nu_1 + a\nu_2) w\bar{v}\, ds + \int_{\partial\Omega} (-ib\nu_2) w\bar{v}\, ds \\ &= \langle [L_+, L_-]v, v\rangle + (B_- + B_+) \end{aligned} \tag{4.24}$$

expressing $L_+L_- - L_-L_+$ as the commutator $[A, B] = AB - BA$.

Examining the term $B_- + B_+$, we want to carefully consider the behavior of the derivatives for these terms because they do not currently appear to be purely tangential. To do so, we express the integrands of B_- and B_+ as operators acting on a function v and find that they do simplify to a purely tangential derivative:

$$\begin{aligned}&(\nu_1 + a\nu_2)L_+ - \nu_2 ibL_- \\ &\quad = (\nu_1 + a\nu_2)\left[ib\partial_2 - \tau(\phi_1 + a\phi_2) - \frac{1}{2}(a_2 - ib_2)\right] - \nu_2 ib\left[\partial_1 + a\partial_2 - \tau(ib\phi_2) + \frac{1}{2}(a_2 + ib_2)\right] \\ &\quad = ib(\nu_1\partial_2 - \nu_2\partial_1) + M(\nu, \tau, x) \\ &\quad = ib\partial_{\nu^\perp} + M \end{aligned} \tag{4.25}$$

Here $\partial_{\nu^\perp} = (-\nu_2, \nu_1) \cdot (\partial_1, \partial_2) = \nu^\perp \cdot \nabla$ is the tangential derivative we sought and M is a just a function containing no differential operators. In this form, we may collect the boundary terms

$$B_- + B_+ = \int_{\partial\Omega} ib\partial_{\nu^\perp} v \cdot \bar{v} + M|v|^2 \, ds \tag{4.26}$$

Using the well known estimate $|2wv| \leq |v|^2 + |w|^2$, we find that

$$\left| \int_{\partial\Omega} ib\partial_{\nu^\perp} v \cdot \bar{v} + M|v|^2 \, ds \right| \leq C\tau \|v\|^2_{L_2(\partial\Omega)} \tag{4.27}$$

for the hyperbolic part when $b \equiv 0$ and

$$\left| \int_{\partial\Omega} ib\partial_{\nu^\perp} v \cdot \bar{v} + M|v|^2 \, ds \right| \leq C_1\tau \|v\|^2_{L_2(\partial\Omega)} + C_2 \|\partial_{\nu^\perp} v\|^2_{L_2(\partial\Omega)} \tag{4.28}$$

on the elliptic part when $b \neq 0$.

Moving forward, we now choose that our real-valued function ϕ depends only on the variable x_1, so $\partial_2 \phi = \phi_2 = 0$. This choice simplifies our operators L_+ and L_- to

$$L_+ = ib\partial_2 - \tau\phi_1 - \frac{1}{2}(a_2 - ib_2) \tag{4.29}$$
$$L_- = \partial_1 + a\partial_2 + \frac{1}{2}(a_2 - ib_2) \tag{4.30}$$

Furthermore, this simplifies the commutator of the operators L_+ and L_-.

$$\begin{aligned} [L_+, L_-] &= [ib\partial_2, \partial_1 + a\partial_2] + [\partial_1 + a\partial_2, \tau\phi_1] + O(1) \\ &= i\,(ba_2 - ab_2 - b_1)\,\partial_2 + \tau\phi_{11} + O(1) \end{aligned} \tag{4.31}$$

where $O(1)$ is a bounded function and $\phi_{11} = \partial_1^2 \phi$. If we are in the hyperbolic part where $b \equiv 0$, then for $\phi_{11} > 0$ and all $x \in \bar{\Omega}$, we have the term $\tau\|v\|^2$ with some positive constant which holds for all $\tau \geq \tau_0 > 0$, if τ_0 is sufficiently large.

We now perform the final set of estimates to handle our boundary terms and obtain the desired Carleman estimate (4.9). Omitting for now the terms $\|L_{\pm}v\|^2$,

$$\tau \int_{\Omega} \phi_{11}|v|^2\,dx \leq \int_{\Omega} |Lv|^2\,dx + C_1\tau \int_{\partial\Omega} |v|^2\,ds \tag{4.32}$$

and recalling that $v = e^{\tau\phi}u$ and taking $\phi_{11} \geq 1$ we have

$$\tau\|e^{\tau\phi}u\|^2 \leq \|e^{\tau\phi}Pu\|^2 + C_1\tau \int_{\partial\Omega} |e^{\tau\phi}u|^2\,ds, \tag{4.33}$$

the Carleman estimate for the hyperbolic (real eigenvalues) part of our system.

If $|b(x)| \geq b_0 > 0$ for all $x \in \bar{\Omega}$, then we can eliminate the term $c(x)\partial_2 = i(ba_2 - ab_2 - b_1)\partial_2$ from the commutator by using the term $\|L_+v\|^2$ in the following way:

$$\langle ic\partial_2 v, v\rangle = \left\langle \frac{c}{b}\underbrace{\left(ib\partial_2 - \tau\phi_1 - \frac{1}{2}(a_2 - ib_2)\right) v}_{L_+v}, v\right\rangle + \left\langle \frac{c}{b}\underbrace{\left(\tau\phi_1 + \frac{1}{2}(a_2 - ib_2)\right) v}_{Qv}, v\right\rangle \tag{4.34}$$

The first term we estimate from below

$$2\left\langle \frac{c}{2b}L_+v, v\right\rangle \geq -\epsilon\frac{\max|c(x)|}{2b_0}\|L_+v\|^2 - \frac{1}{\epsilon}\|v\|^2 \tag{4.35}$$

and choosing $\epsilon\,\dfrac{\max|c(x)|}{2b_0} = 1$ will cause the $\|L_+v\|^2$ term from (4.23) to cancel with $-\|L_+v\|^2$ from (4.35). In a similar fashion,

$$\left\langle \frac{c}{b}Qv, v\right\rangle \geq -\tau\int_{\Omega} |\phi_1|\,|v|^2\,dx - O(\|v\|^2) \tag{4.36}$$

Assuming that $\phi_{11} - |\phi_1| \geq C_0$, for some constant C_0 in the domain Ω, then for a large fixed

$\tau_0 > 0$ and all $\tau > \tau_0$ we have the estimate

$$\frac{C_0}{2}\tau\|v\|^2 \leq \|Lv\|^2 + C\left(\tau\|v\|^2_{L_2(\partial\Omega)} + C_1\int_{\partial\Omega}|\partial_{\nu^\perp}v|\cdot|v|\,ds\right) \tag{4.37}$$

We finish our estimate by returning to $u = ve^{-\tau\phi}$ and employing the ϵ-inequality $2ab \leq \epsilon a^2 + \frac{1}{\epsilon}b^2$ on $|\partial_{\nu^\perp}v|\cdot|v|$ yields our main estimate (4.9):

$$\tau\|e^{\tau\phi}u\|^2_{L_2} \leq C\|e^{\tau\phi}Pu\|^2_{L_2} + \tau C_1\|e^{\tau\phi}Tu\|^2_{W^1_2}$$

Lastly, we examine the condition on the weight function ϕ. It was necessary for it to satisfy $\phi_{11} - |\phi_1| \geq C_0$ and only depend on x_1. Choosing $\phi = e^{-\lambda x_1}$, we have $\phi_{11} = \lambda^2\phi$ and $\phi_1 = -\lambda\phi$. Hence $\phi_{11} - |\phi_1| = (\lambda^2 - \lambda)\phi = \lambda(\lambda-1)e^{-\lambda\phi} \geq C_0 > 0$ which holds for any choice of $\lambda > 1$. Selecting $\lambda = 2$, we have $C_0 = 2\exp\left(-2\max_{\bar{\Omega}}|x_1|\right)$.

Chapter 5

5 Conclusion

5.1 Main Results

We have shown uniqueness and stability for the solution to the weighted stationary two-dimensional transport equation with Cauchy data

$$\begin{aligned} Pu(x,\omega) &= \langle \omega, \nabla_x u(x,\omega)\rangle + \mu(x)u(x,\omega) = \rho(x,\omega)h(x,\omega) \\ u|_{\partial\Omega\times S} &= f \end{aligned} \tag{5.1}$$

when the weight is a finite even trigonometric polynomial. Two cases of this weight function, $N=2$ where $\rho(x,\omega) = \check{\rho}_{-2}(x)e^{2i\alpha} + \check{\rho}_0(x) + \check{\rho}_2(x)e^{-2i\alpha}$ and $N=4$ where $\rho(x,\omega) = \check{\rho}_{-4}(x)e^{4i\alpha} + \check{\rho}_{-2}(x)e^{2i\alpha} + \check{\rho}_0(x) + \check{\rho}_2(x)e^{-2i\alpha} + \check{\rho}_4(x)e^{-4i\alpha}$ where examined in both the scalar tomography $h(x,\omega) = a(x)$ and vector tomography $h(x,\omega) = \langle \omega, \boldsymbol{v}\rangle$ cases. In all cases, the Carleman estimate (4.9) holds after transforming the weighted transport equation into the form

$$A\partial_1 u + B\partial_2 u + Cu = F \tag{5.2}$$

with matrix coefficients A, B, C and known function F using both the theory of A-analytic functions and conversion of the system back into real variables. After analyzing the systems that arise in the four cases considered, we discovered additional requirements on each system beyond the basic needs that A^{-1} exists and that $A^{-1}B$ has distinct eigenvalues for all x in the domain Ω:

1. For the second approximation scalar case where $N=2$ and $h(x) = a(x)$, the term $\check{\rho}_0(x) \neq 0$, $\det(A) = 1 - 2\mathrm{Re}\,\dfrac{\check{\rho}_2(x)}{\check{\rho}_0(x)} \neq 0$, $4\mathrm{Re}\,\dfrac{\check{\rho}_2(x)}{\check{\rho}_0(x)} + 4\mathrm{Im}\,\dfrac{\check{\rho}_2(x)}{\check{\rho}_0(x)} - 1 \neq 0$ for all $x \in \Omega$.

2. For the third approximation scalar case where $N = 4$ and $h(x) = a(x)$, the term $\check{\rho}_0(x) \neq 0$ and for $\alpha_1 = \operatorname{Re} \dfrac{\check{\rho}_2(x)}{\check{\rho}_0(x)}$, $\alpha_2 = \operatorname{Im} \dfrac{\check{\rho}_2(x)}{\check{\rho}_0(x)}$, $\beta_1 = \operatorname{Re} \dfrac{\check{\rho}_4(x)}{\check{\rho}_0(x)}$, and $\beta_2 = \operatorname{Im} \dfrac{\check{\rho}_4(x)}{\check{\rho}_0(x)}$, $\det(A) = 2\alpha_1 - 2gb_1 - 1 \neq 0$, and the polynomial

$$(2\beta_1 - 2\alpha_1 + 1)x^4 + (4\alpha_2 - 8\beta_2)x^3 + (2 - 12\beta_1)x^2 + (8\beta_2 + 4\alpha_2)x + 2\beta_1 + 2\alpha_1 + 1 = 0$$

has distinct roots for all $x \in \Omega$.

3. For the second approximation vector case where $N = 2$ and $h(x) = \langle \omega, \boldsymbol{v} \rangle$, the term $\check{\rho}_2(x) \neq 0$,

$$\det(A) = [\gamma_1(x)]^2 - 4\gamma_1(x) - [\gamma_2(x)]^2 - [\delta_1(x)]^2 - 2\delta_2(x) + [\delta_2(x)]^2 + 3 \neq 0,$$

for $\gamma_1 = \operatorname{Re} \dfrac{\check{\rho}_0(x)}{\check{\rho}_2(x)}$, $\gamma_2 = \operatorname{Im} \dfrac{\check{\rho}_0(x)}{\check{\rho}_2(x)}$, $\delta_1 = \operatorname{Re} \dfrac{\check{\rho}_2(x)}{\check{\rho}_{-2}(x)}$, and $\delta_2 = \operatorname{Im} \dfrac{\check{\rho}_2(x)}{\check{\rho}_{-2}(x)}$, the polynomial

$$\begin{aligned}
&(4\gamma_1 - \gamma_1^2 + \gamma_2^2 + \delta_1^2 - 2\delta_1 - \delta_2^2 - 3)x^4 + (2\gamma_1^2 - 8\gamma_1 - 2\gamma_2^2 + 8\gamma_2 - 2\delta_1^2 + 4\delta_1 + 2\delta_2^2 + 8\delta_2 + 6)x^3 \\
&+ (2\gamma_1^2 - 2\gamma_2^2 - 2\delta_1^2 + 4\delta_1 + 2\delta_2^2 - 10)x^2 + (2\gamma_2^2 - 2\gamma_1^2 + 2\delta_1^2 + 4\delta_1 - 2\delta_2^2 + 2)x \\
&- \gamma_1^2 - 4\gamma_1 + \gamma_2^2 + \delta_1^2 - 2\delta_1 - \delta_2^2 - 3 = 0
\end{aligned}$$

has distinct roots for all $x \in \Omega$, and the attenuation $\mu(x) \neq 0$ for $\mu \in C^2(\bar{\Omega})$, and $\partial_\nu u_1$ is known.

4. For the third approximation vector case where $N = 4$ and $h(x) = \langle \omega, \boldsymbol{v} \rangle$, the term $\check{\rho}_4(x) \neq 0$, the attenuation $\mu(x) \neq 0$ for $\mu \in C^2(\bar{\Omega})$, and $\partial_\nu u_j$ for $j = 1, 3$ is known. There is a corresponding characteristic polynomial for this problem, but it is too large include here, as it is not easy to simplify, even with the aid of MATLAB®'s computer algebra system.

Under these conditions, we have obtained an estimate with boundary terms for the equation (5.2)

$$\|u\|_{L_2(\Omega;\mathbb{R}^n)} \leq C \left(\|Pu\|_{L_2(\Omega;\mathbb{R}^n)} + \|Tu\|_{W_2^1(\partial\Omega;\mathbb{R}^n)} \right) \qquad \forall u \in C^1(\bar{\Omega};\mathbb{R}^n)$$

when we have our primary condition that the eigenvalues of $A^{-1}B$ are distinct in the entire domain Ω. Using methods described in Chapter 4, these results are verified numerically in MATLAB® using the routines included in the appendix for the application of helioseismology. This estimate guarantees the uniqueness and stability of the solution to our system.

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APPENDIX

APPENDIX A

MATLAB® Programs [2]

```
classdef EigenvalueTest < matlab.unittest.TestCase

    properties (Access = protected)
        partialTest    = false;     % set false for full test
        filename       = 'eigenvalues.txt';

        d   = 0.11;              % vertical displacement of cycloid
        R   = 1;                 % radius of each cycloid
        Ne  = 3;                 % number of eigenvalues both real and complex
        n   = 1;
        N   = 1000;              % number of grid points in each radius
        tol = 1e-8;              % minimum allowable distance between eigenvalues
    end

    methods (Access = protected)
        function runEigenvalueTest(testCase, hCreateSystemFunction)

            if (nargin < 2) || isempty(hCreateSystemFunction)
                hGetSystem  = @EigenvalueTest.getSecondApproxScalarSystem;
            else
                hGetSystem  = hCreateSystemFunction;
            end

            % local function handles used to hopefully speed up parallel
            % processing

            hGetRadius  = @EigenvalueTest.getRadius;
            hVerifyTrue = @testCase.verifyTrue;

            Tol    = testCase.tol;
            phi    = rand(1);
            nRadii = testCase.n;
            nGrid  = 2*testCase.N + 1;
            nEig   = testCase.Ne;
            detA   = zeros(nRadii, nGrid);
            evA    = ones(nRadii, nGrid, nEig);
            evA    = complex(evA, zeros(size(evA)));
            vertD  = testCase.d;
```

[2] Used for numerical validation of eigenvalues

```
        cyclR  = testCase.R;
        pTest  = testCase.partialTest;

        % First Test for Expected Number of Eigenvalues
        x      = testCase.getRadius(-testCase.N, cyclR, pTest);
        [A, B] = hGetSystem(x, cyclR, vertD);
        Btilde = A\B;
        Etilde = eig(Btilde);
        NumEig = sum(~(abs(imag(Etilde)) > eps));
        fprintf('Expected Number of Real EigenValues = %g\n', NumEig);

        parfor i = 1:nRadii
            theta  = pi*i/nRadii + phi;
            rowA   = detA(i,:);
            eigen  = evA(i,:,:);

            for k = 1:nGrid
                r      = hGetRadius(k - floor(nGrid/2) - 1, cyclR, pTest);
                x      = r * exp(1i*theta);
                [A, B] = hGetSystem(x, cyclR, vertD);

                Btilde   = A\B;
                Etilde   = eig(Btilde);
                eigen(1,k,:)  = Etilde.';
                rowA(k)  = det(A);

                hVerifyTrue(sum(abs(imag(Etilde)>eps))==NumEig)
                hVerifyTrue(rowA(k) ~= 0)
                hVerifyTrue(complexunique(Etilde,Tol))
            end

            detA(i,:)  = rowA;
            evA(i,:,:) = eigen;    % picks up last set of eigenvalues
        end

        EigenvalueTest.displayResults(testCase.filename, detA, evA);
    end
end

methods(Static)
    function r = getRadius(j, R, partialTest)
        if partialTest
            if j < -99          % if tree for partial test
                r = 0.01*(-99) + (j + 99)*0.001;
            elseif j > 99
```

```
            r = 0.01*(99) + (j - 99)*0.001;
        else
            r = j * 0.01 * R;
        end
    else
        % radii for full test
        r = j * 0.001 * R;
    end
end

function [A, B] = getThirdApproxScalarSystem(x, R, d)
    rho0 = rhon(x, 0, R, d);
    alph = rhon(x, 2, R, d)./rho0;
    beta = rhon(x, 4, R, d)./rho0;

    a1   = real(alph);
    a2   = imag(alph);
    b1   = real(beta);
    b2   = imag(beta);

    A    = [1 0 0 1 0 0 0; zeros(1,4) 1 0 0;
            0 1-2*a1 0 1 0 1 0; 0 -2*a2 1 0 0 0 1;
            zeros(1,3) 1 zeros(1,3); 0 -2*b1 0 0 0 1 0;
            0 -2*b2 0 0 0 0 1];

    B    = [zeros(1,4) 1 0 0; 1 0 0 -1 0 0 0;
            0 0 -1-2*a1 zeros(1,3) 1; 0 1 -2*a2 0 0 -1 0;
            zeros(1,4) -1 zeros(1,2);
            0 0 -2*b1 zeros(1,3) -1; 0 0 -2*b2 0 0 1 0];
end

function [A, B] = getSecondApproxScalarSystem(x, R, d)
    rho0 = rhon(x, 0, R, d);
    alph = rhon(x, 2, R, d)./rho0;
    a1   = real(alph);
    a2   = imag(alph);

    A    = [1 0 0; 0 1-2*a1 0; 0 -2*a2 1];
    B    = [0 0 0; 0 0 -1-2*a1; 0 1 -2*a2];
end

function displayResults(filename, detA, evA)
    for i=1:3
        fprintf('First Eigenvalues = % +g % +gi \n', ...
                real(evA(1,1,i)), imag(evA(1,1,i)));
```

```
            end
            for i=1:3
                fprintf(' Last Eigenvalues = % +g % +gi\n', ...
                        real(evA(end,end,i)), imag(evA(end,end,i)));
            end
            fprintf('Maximum Determinant = %g\n', max(detA(:)));
            fprintf('Minimum Determinant = %g\n', min(detA(:)));

            fID = fopen(filename,'w');
            [rows, columns, values] = size(evA);

            for i = 1:rows
                for k = 1:columns
                    for j=1:values
                        fprintf(fID, '% +7g % +7gi, ', ...
                                real(evA(i,k,j)), imag(evA(i,k,j)));
                    end
                    fprintf(fID, '\n');
                end
            end
            fclose(fID);
        end
    end
end

classdef SecondScalarTest < EigenvalueTest

    methods (Test)
        function scalarTest(testCase)
            testCase.filename  =  'SecondScalarTestEV.txt';
            runEigenvalueTest(testCase);
        end
    end
end

classdef ThirdScalarTest < EigenvalueTest

    methods (Test)
        function scalarTest(testCase)
            testCase.filename  =  'ThirdScalarTestEV.txt';
            testCase.Ne        = 7;

            runEigenvalueTest(testCase, @ThirdScalarTest.getThirdApproxScalarSystem);
        end
    end
end
```

```
classdef SecondVectorTest < EigenvalueTest

    methods (Test)
        function vectorTest(testCase)
            testCase.filename  = 'SecondVectorTestEV.txt';

            testCase.Ne  = 6;
%             testCase.R   = 1;
%             testCase.d   = 0;
%             testCase.N   = 999;
%             testCase.tol = 1e-8;

            runEigenvalueTest(testCase, @SecondVectorTest.getSecondVectorSystem);
        end
    end

    methods(Static)
        function [A, B] = getSecondVectorSystem(x, R, d)
            rho2 = rhon(x, 2, R, d);
            gamm = rhon(x, 0, R, d)./rho2;
            delt = rho2./rhon(x, -2, R, d);

            l    = 2;
            c1   = real(gamm);
            c2   = imag(gamm);
            d1   = real(delt);
            d2   = imag(delt);

            A    = [1 zeros(1,5); 0 1 zeros(1,4);
                    0 0 3-c1-d1 0 -c2+d2 0;
                    0 0 0 1 0 0; 0 0 -c2-d2 0 1-c1+d1 0;
                    zeros(1,5) 1];

            B    = [l zeros(1,5); 0 -l zeros(1,4);
                    zeros(1,3) 1+c1+d1 2*(1+c1-c2+d1+d2) c2-d2;
                    0 0 -1 zeros(1,3);
                    zeros(1,3) c2+d2 2*(1-c1+c2+d1+d2) 3+c1-d1;
                    zeros(1,4) -1 0];
        end
    end
end
```

```
classdef ThirdVectorTest < EigenvalueTest

    methods (Test)
        function vectorTest(testCase)
            testCase.filename  = 'ThirdVectorTestEV.txt';

            testCase.Ne  = 12;
%             testCase.R   = 1;
%             testCase.d   = 0;
%             testCase.N   = 999;
%             testCase.tol = 1e-8;

            runEigenvalueTest(testCase, @ThirdVectorTest.getThirdVectorSystem);
        end
    end

    methods(Static)
        function [A, B] = getThirdVectorSystem(x, R, d)
            rho0  = rhon(x,  0, R, d);
            rho2  = rhon(x,  2, R, d);
            rho4  = rhon(x,  4, R, d);
            rho4n = rhon(x, -4, R, d);

            alph  = rho0./rho4;
            beta  = rho2./rho4n;
            gamm  = rho2./rho4;
            delt  = rho4./rho4n;

            l1 = 1;
            l2 = 2;

            a1 = real(alph);
            a2 = imag(alph);
            b1 = real(beta);
            b2 = real(beta);
            c1 = real(gamm);
            c2 = imag(gamm);
            d1 = real(delt);
            d2 = imag(delt);

            A  = [1 zeros(1,11);
                  0 1 zeros(1,10);
                  0 0 2 0 1 zeros(1,3) 1-a1-b1 a2-b2 a2-b2 0;
                  zeros(1,3) 1 zeros(1,8);
                  zeros(1,3) 1 1 zeros(1,3) -a2-b2 b1-a1 b1-a1 0;
```

```
                zeros(1,5) 1 zeros(1,6);
                zeros(1,6) 1 zeros(1,5);
                zeros(1,7) 1 zeros(1,4);
                0 0 1 zeros(1,5) 2-c1-d1 d2-c2 d2-c2 0;
                zeros(1,9) 1 0 0;
                zeros(1,3) 1 1 zeros(1,3) -c2-d2 -1+c1-d1 2+c1-d1 0;
                zeros(1,11) 1];

            B  = [l1 zeros(1,11);
                0 -l1 zeros(1,10);
                zeros(1,4) 2 1 0 0 a2-b2 b1-a1-1 2-2*a1-2*b1 b2-a2;
                0 0 -1 zeros(1,9);
                0 0 1 1 0 2 0 0 b1-a1 b2-a2 -2*a2-2*b2 a1-b1;
                zeros(1,4) -1 zeros(1,7);
                zeros(1,6) l2 zeros(1,5);
                zeros(1,7) -l2 zeros(1,4);
                zeros(1,3) 1 -2 zeros(1,3) c2-d2 -c1-d1 2+2*c1+2*d1 c2-d2;
                zeros(1,8) -1 zeros(1,3);
                0 0 1 0 0 -1 0 0 -1+c1-d1 -c2-d2 2*c2+2*d2 d1-c1;
                zeros(1,10) -1 0];
        end
    end
end
```

APPENDIX B

MATLAB® Support Programs [3]

```
function tf = complexunique( C,tol )

%COMPLEXUNIQUE checks if vector of complex values is unique to a tolerance

    if nargin < 2
        tol = 1e-12;
    end
    M = C(:);
    D = zeros(length(C),length(C)-1);
    for k = 2:length(C)
        M(:,k) = [M(2:end,k-1);M(1,k-1)];
        D(:,k-1) = abs(M(:,1) - M(:,k));
    end

    tf = all(D(:) > tol);

end
```

```
function x = fact2( n )

%FACT2 Computes the double factorial of n

    if n==0 || n==1
        x = 1;
    elseif mod(n,2)==0
        x = prod(2:2:n);
    elseif mod(n,2)==1
        x = prod(3:2:n);
    end

end
```

```
function I = int2n( n,k2 )

%INT2N computes the integral denoted by I_2n for the Fourier coefficients of rho

    F = hypergeom(0.5*[2*n-1 2*n+1],2*n+1,k2);
    I = (-pi*fact2(2*n-3)*k2.^n)*F./(2^(3*n-1)*factorial(n));

end
```

[3] Smaller functions necessary for main tests

```
function [dphidx,phi,xcoord,ycoord] = phiprime( x,R,d,graph )

%PHIPRIME Computes the derivative of a vertically shifted cycloid at x
%x: distance from center of cycloid
%R: distance from center to edge of circle
%d: depth adjustment for cycloid

    if nargin < 2
        R = 1;
    end
    if nargin < 3
        d = 0;
    end
    if nargin < 4
        graph = false;
    end

    %determine radius of generating circle r
    if d < 0
        error('d must be greater than 0')
    elseif d == 0;
        guess1 = 0.02;
    else
        guess1 = d/2;
    end
    if guess1 < R
            guess2 = 2*R;

    else
        guess2 = 2*d;
    end
    g = @(a) a*acos(1 - d/a) - sqrt(d*(2*a-d)) - pi*a + R;
    r = fzero(g,[guess1 guess2]);

    %switch x to cycloid orientation
    x = r*pi - abs(x);

    %find y value (phi), the depth at point x
    f = @(y) r*acos(1 - y/r) - sqrt(y*(2*r-y)) - x;
    y = fzero(f,[d/2 2*r]);

    if y - d < -1e-12
        error('y-d < 0, check input values')
    end
```

```
    %return value of phiprime
    dphidx = sqrt(2*R/y - 1);
    phi = y - d;
    xcoord = x;
    ycoord = -y;

    if graph
        t = 0:0.01:2*pi;
        figure
        hold on
        plot(r*(t-sin(t)),-r*(1-cos(t)))
        plot([0 2*pi*r],-1*[d d],'k:')
        scatter(xcoord,ycoord)
        axis equal
        grid on
    end

end

function rho = rhon( x,n,R,d )

%RHON(x,n) returns Fourier coefficient of weight function rho
% rho = rhon(x,n) takes a complex vector x with |x| < 1 and an integer n
% and determines the Fourier coefficients of the weight function
% rho = sqrt(1 + (phiprime(x) * < x, omega > / |x|)^2 )

    if nargin < 3
        R = 1;
    end
    if nargin < 4
        d = 0;
    end

    N   = length(x);
    q   = zeros(N,1);
    k   = zeros(N,1);
    rho = zeros(N,1);

    for i = 1:N

        q(i) = phiprime(x(i),R,d);
        k(i) = sqrt(q(i)^2/(1+q(i)^2));

        if mod(n,2)~=0
            disp('odd n all result in zero coefficients')
```

```
            rho(i) = 0;
        elseif n==0
            [K,E] = ellipke(k(i));
            rho(i) = sqrt(1+q(i).^2)/(2*pi).*4*E;
        else
            rho(i) = exp(n*1i*angle(x)).*sqrt(1+q(i).^2)/(2*pi).*...
                    int2n(abs(n)/2,k(i).^2);
        end

    end

end
```