

COMPOSITE OPTIMAL CONTROL FOR INTERCONNECTED SINGULARLY
PERTURBED SYSTEMS

A Dissertation by

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DEDICATION

To my parents and every member of my family

ABSTRACT

This dissertation deals with the design of a decentralized control and estimators for large-scale interconnected singularly perturbed stochastic systems for system stabilization and cost minimization. Singular perturbation theory is used to decompose the full-order systems into two, reduced-order slow and fast subsystems. It is shown that a near optimal composite control, which is obtained as a combination of a slow control and a fast control computed in separate time scales, can approximate the optimal control. The Kalman-Bucy, or simply Kalman filtering approach is utilized to derive decentralized estimators at each subsystem level when the states of the systems are not available for measurement and/or corrupted by external noise.

Because of modeling inaccuracy resulting from the simplified model of a real physical plant, certain features of systems might not actually be what they are assumed to be. Hence, a robustness analysis to guarantee stability and performance is then necessary to validate the design in the face of system uncertainties. The problem of robust control for the above system is subject to two types of uncertainties: norm-bounded nonlinear uncertainties and unknown disturbance inputs are investigated. The problem of H_∞ control in which a robust stability and robust disturbance attenuation is addressed using the Hamiltonian approach. The state feedback (SFB) gain matrices can be constructed from the positive definite (PD) solutions to a couple of linear matrix inequalities (LMIs).

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LIST OF ABBREVIATIONS

ARE	Algebraic Riccati Equation
FRE	Filter Riccati Equation
GARE	General Algebraic Riccati Equation
LMI	Linear Matrix Inequality
LQ	Linear Quadratic
LQE	Linear Quadratic Estimator
LQG	Linear Quadratic Gaussian
LQR	Linear Quadratic Regulator
LTI	Linear Time Invariant
PD	Positive Definite
PSD	Positive Semi-Definite
QLF	Quadratic Lyapunov Function
SFB	State Feedback
SGA	Successive Galerkin Approximation
SP	Singular Perturbation
SPS	Singularly Perturbed System
TTS	Two-Time-Scale

LIST OF NOTATIONS

x	Slow subsystem states variable in interconnected systems
z_i	Fast subsystem i states variable in interconnected systems
u_i	System input i
y_i	System output i
G	Feedback gain
ε	Singular perturbation small, real-value, positive parameter
$\bar{x}, \bar{z}, \bar{u}$	Quasi-steady state system variables
x^s, u^s, y^s	Reduced-order (slow) model variables
x^f, u^f, y^f	Reduced-order (fast) model variables
E	Process noise multiplier matrix
F	Measurement noise multiplier matrix
v	Process noise
w	Measurement noise
V	Intensity matrix of process noise vector v
W	Intensity matrix of measurement noise vector w
H	Observer gain matrix
P, K	Lyapunov and Riccati equations variables
$E\{.\}$	Expected value
$tr\{A\}$	Trace of matrix A
\dot{x}	First derivative of x with respect to time
$A > 0$	Matrix A is p.d.
$A \geq 0$	Matrix A is p.s.d.

LIST OF NOTATIONS (continued)

$L^2[0, \infty)$ Lebesgue space of square integrable functions

$\rho(M)$ Largest singular value of matrix M

CHAPTER 1

INTRODUCTION

1.1 Background

From cruise control to intelligent machines, control theory forms the foundation for our modern society. The purpose of control theory is to force a system, often called the plant, to obtain the desired output with the desired performance. Developing a controller using control theory techniques that are optimal, stabilizing, and robust is practical and very important in system performance. Many real-world physical systems such as computer networks, wireless telecommunication systems, and power networks are classified as large-scale systems. Control is a significant aspect of the operation of all large-scale systems. However, large-scale systems are often too enormous, and the problems are too complex to construct and analyze due to dimensionality, information structure constraints, uncertainty in information exchanges, and delays for long-distance data communication. Such complexities lead to severe difficulties that are encountered in the tasks of analyzing, designing, and implementing control strategies and algorithms [1, 2].

A system is considered large scale if it involves a number of interconnected subsystems, and it is typically characterized by a multidimensional, complex structure and multi-state and multi-input variables. From the control system perspective, a system is considered large if it exceeds the size of a single controller structure. Specifically, the control of large-scale systems consists of a multi-controller structure. Furthermore, because the control problem may increase exponentially with an increase in the complexities of the large-scale system, a control designer must develop efficient control schemes that can be implemented with minimal cost and resources. An important challenge encountered by a control designer is to ensure stability robustness and

reliable performance of a large-scale system against the failure of an individual subsystem(s). To cope with the aforementioned complexities in the analysis and synthesis tasks for large-scale systems, it is necessary to break them down into a number of smaller interconnected subsystems. This is where decentralized control is important in order to simplify the analysis and control synthesis for large-scale systems.

Decentralized control is common in large-scale systems. A decentralized system consists of several independent subsystems, each having its own input, output, and a control unit that operates locally and makes decisions for it based on its measurements. Decentralization is achieved by decomposing a large-scale system into smaller subsystems, solving these subsystems separately, and then combining their solutions as the overall solution of the original system. This can be achieved in control systems by separating the analysis and synthesis of the overall system into uncoupled subsystems in which there is no transfer of information among different local controllers. Thus, the overall system is controlled by several independent controllers, which together represent a decentralized controller. For decentralized control, each interconnected subsystem is responsible for the operation of a particular task of the overall system.

In decentralized control schemes, the control unit for each individual subsystem is designed in such a way that it makes use of a local, available, limited set of information only. The fundamental aspect of a decentralized control scheme is that it does not depend on any information from other subsystems, and therefore, the problem of corrupted information exchange does not arise at all. The reliability goal in decentralized control is to stabilize the system by designing a control unit in each control channel, such that the system can tolerate failures [3]. Decentralized control enables each local controller to be separate and independent, and thus increases the robustness of the scheme.

State feedback (SFB), or output feedback, control laws are designed to obtain a control objective [4]. Implementing the design of feedback results in the closed-loop system, and its benefits provide tremendous advantages for control systems. In cases where the states of the system are available and perfect, SFB can be used to control it. However, if the states of the system are imperfect (corrupted by disturbances or unavailable for measurement), then there are two alternative approaches to control: designing an observer (or state estimator) or implementing static output feedback.

Since dynamic systems are not only driven by controlled input but also by the inevitable presence of unknown exogenous input, such as noise in the output measurement and disturbance to the controller input, system performance is sometimes unacceptable and results in the system being unstable. Hence, a stochastic system can be defined as a system where one or more parts is associated with randomness. The stochastic approach is applied when a prior probability on the uncertain parameters is assumed. Then, the problem in optimal control is to design a control law to minimize the expected value of some performance index [5].

Among all system performance requirements, robust stability remains a dominant condition for the perfect design of system control. Since it is nontrivial to exactly model a large-scale physical system, and because of the trade-off between model complexity and model accuracy, there are model uncertainties. Such uncertainties can have a considerable impact on system performance. For instance, a nominally stable system may become unstable due to nonlinear uncertainties resulting from linearization. Hence, it is significant that a feedback control law is designed to be robust, taking into consideration system uncertainties. Robustness is the ability of the closed-loop plant to maintain stability for all uncertainties in an expected range. Furthermore, the design of a controller often begins with a plant that is subject to unknown external

disturbances. A common design goal is to reduce the effect of these disturbances to an acceptable level by making use of H_∞ control techniques.

1.2 Singular Perturbation

Mathematical modeling of a physical system is the first fundamental step in performing system analysis and control design. Modeling should strike a balance between being simple, on the one hand, and accurate with enough detailed dynamics, on the other. The realistic representation of many physical systems requires high-order ordinary differential equations. The existence of some *parasitic* parameters such as small time constants, masses, resistances, and capacitances often leads to the increased dynamic order of models and *stiffness* of these systems [2, 6]. Stiffness, which is characterized by the simultaneous occurrences of *slow* and *fast* phenomena, gives rise to time scales. A high-order system in which the suppression of a small parameter is responsible for the reduction of dimension is called a singularly perturbed system, which is a special class of more general time-scale systems [7]. Generally, for a two-time-scale (TTS) system the singular perturbation theory decouples the *slow* dynamics, which are the dominant characteristics of the system, from the *fast* dynamics, which are present for only a short period of time. However, it is important for the control designer to decide whether the system equations are of the singularly perturbed form. If small coefficients are multiplying the time derivative of some of the system states, then it is very likely that singular perturbation (SP) theory can be utilized.

Singularly perturbed systems (SPSs) have a time-scale decomposition, reduced order (*slow*) and *fast* subsystems based on a small, SP real-valued parameter. Such SPSs often involve the interacting dynamic phenomena of widely two distinct speed groups of eigenvalues. Hence, its eigenvalues ratio is adequately small. The decomposition of states into *slow* and *fast* is not an easy

modeling task but needs cautious analysis and insight [8]. The decomposition of TTS systems into separate *slow* and *fast* subsystems suggests designing separate *slow* and *fast* controllers for each subsystem, which are then combined to form a composite controller of the original full-order system. Sometimes, only the reduced (*slow*) control law is utilized when the *fast* control law is not needed [9]. Therefore, studying and analyzing system performance can be based on the reduced-order (*slow*) subsystem or the composite system, and then the results can be applied to the actual full-order system. Using SP theory provides the control designer with a simple means to obtain an approximate solution of a lower-order system than the original. The separation of time scales does not only reduce system order but also eliminates stiffness difficulties. Areas of application of SP and time-scale methods seem to increase in many fields of applied mathematics, electrical power systems, robotics, aerospace systems, and nuclear reactors, see [6] and the references therein.

The characteristics of a singularly perturbed system allow the separation of the actual full-order system into the reduced (*slow*) and *fast* subsystems by assuming that the *fast* modes are infinitely *fast* when deriving the *slow* subsystem and assuming that the *slow* modes are constant during a *fast* transient when deriving the *fast* subsystem [10].

1.3 Literature Review

Singular perturbation methodologies were largely established in the early 1960s when they first became a means to simplify computation of optimal trajectories. In the control literature, the method of SP as a model-reduction technique of finite-dimensional dynamic continuous-time systems that disregard high frequency *parasitics* was first introduced by Sannuti and Kokotovic in the late 1960s [11]. Since then, SP theory has attracted considerable attention, and rapid growth in further development of these control approaches has occurred, as shown in literature surveys [6, 9, 12]. A recent overview during the period 2002 to 2012 by Zhang et al. [13] has addressed

stability, estimation, optimal and other control problems, and applications of singularly perturbed systems. It is worth mentioning that SP theory has its roots in fluid dynamics, and wide applications are found in the area of aerospace systems, as in the survey of applications of SP on aerospace systems [14]. The necessity for order reduction related to the SP approach is most acutely felt in the design of optimal control for large-scale systems. The composite control design based on separate control law designs for slow and fast subsystems has been thoroughly reviewed by Saksena et al. [12]. A common approach used to handle SPSs is based on the so-called reduced method [8]. Chow and Kokotovic [10] designed a near-optimum state regulator for systems with slow and fast modes without knowledge of the small perturbation parameter. Haddad and Kokotovic [15] applied SP to solve the linear quadratic Gaussian (LQG) problem for stochastic systems with fast and slow dynamics. Khalil and Gajic [16] presented a new method for the decomposition and approximation of LQG estimation and control problems for SPSs. They used decoupling transformation to decompose the Kalman-Bucy filter into separate slow- and fast-mode filters. Shen et al. [17] used a SP approach to design a composite feedback controller for discrete systems subject to stochastic jump parameters. Oloomi et al. [18] derived an output feedback regulator for singularly perturbed discrete time systems. Kim et al. [19] presented algorithms to find composite control laws for singularly perturbed bilinear systems by using the successive Galerkin approximation (SGA) method. Yao [20] has proposed an iteration method to design sub-optimal static output feedback computer control for decentralized SPSs.

The problems of designing a robust controller that guarantees both closed-loop stability and given performance requirements in the presence of uncertainties for a given system have been intensively investigated in the last 35 years. The main approach in this effort has been the H_∞ robust control theory. The H_∞ control design was first introduced in 1981 by Zames [21] to

formulate feedback control to deal with disturbance and parameter uncertainty together. Later, Doyle et al. [22] published one of the first fully detailed papers on methods of simple state-space solutions for H_∞ control design. Petersen [23] designed an algebraic Riccati equation (ARE)-based state feedback controller to reduce the effect of disturbances on the output of a given linear system by attenuating the disturbances to a prespecified level. Shao and Sawan [24] discussed the robust stability of linear-invariant SPSs where the system matrix is subject to bounded parameter perturbation, and they showed that under sufficient conditions, the stability of the full-order model can be inferred from the analysis of the slow model. Shi and Dragan [25] derived a stabilizing and γ -attenuating controller for singularly perturbed linear continuous-time systems with parameter uncertainty. Shao and Sawan [26] investigated the robust stability problem of a linear time-invariant singularly perturbed system with nonlinear uncertainties, where the knowledge of norm upper bounds is the only information available for system uncertainties. Xi et al. [27] designed a robust H_∞ controller for a class of linear time-delay SPSs with parameter uncertainties. Hyun et al. [28] investigated new stability bounds for decentralized SPSs subject to parameter uncertainties, whereby the unified stability bounds were developed for the reduced (*slow*) model based on the Lyapunov matrix equation and singular value analysis.

1.4 Contribution

Given a linear time-invariant continuous interconnected singularly perturbed stochastic system, and a quadratic cost functional, the main research goal of this dissertation is to design an optimal feedback control that maintains the stability and optimizes the cost function under decentralized control. The optimal feedback control problem is decomposed into two reduced-order optimal control subproblems. The class of uncertain systems is described by a state-space model with slow and fast dynamics and unknown exogenous inputs in the states and outputs

equations. By making use of SP theory, the full-order system is decomposed into two reduced-order subsystems (slow and fast), and a separate controller is designed for each subsystem. These subcontrollers are combined to form a composite control law which will result in a near-optimal response in the closed-loop system. It is only necessary to solve the lower-order AREs, which means considerable computational savings. A theory is developed for both the SFB control design (assuming that all states are available for measurements) as well as when the states are not accessible or corrupted by noise, in which case an observer-based controller is designed to estimate those states using the Kalman-Bucy filter. Furthermore, the problem of robust control for the above system subject to norm-bounded nonlinear uncertainties and unknown disturbance inputs is also investigated. Also addressed is the problem of H_∞ control in which a robust stability and robust disturbance attenuation using the Hamiltonian approach are necessary to achieve. The SFB gain matrices can be constructed from the positive definite (PD) solutions to a couple of linear matrix inequalities (LMIs).

CHAPTER 2

PRELIMINARY BACKGROUND ON CONTROL THEORY

2.1 Linear Systems and Control Theory

Control theory involves the control of processes with inputs and outputs. The aim here is to know how to attain the desired objective for the output of a given system by appropriately choosing system inputs. This chapter presents the necessary background information from control theory.

2.1.1 Linear System Model

Most physical systems of interest to control designer can be approximated by continuous linear time-invariant (LTI) mathematical models:

$$\dot{x}(t) = Ax(t) + Bu(t) \quad (2.1)$$

$$y(t) = Cx(t) \quad (2.2)$$

where $x(t) \in R^n$ is the state vector, $u(t) \in R^m$ is the control vector, and $y(t) \in R^n$ is the output vector. The notations $A \in R^{n \times n}$, $B \in R^{n \times m}$, and $C \in R^{p \times m}$ are constant matrices of appropriate dimensions.

The two main goals of control theory are solving the following:

- i. Controller Problem: Is it possible to find the control law $u(t)$ so that the state $x(t)$ has the desired property (such as optimality, stability, etc)?
- ii. Observer Problem: If the measurement of the state $x(t)$ is not accessible or is corrupted by noise, then is it possible to estimate or reconstruct the state $x(t)$ using the available signals $u(t)$ and $y(t)$?

2.2 Optimal Control Theory

When studying the feedback control problem, one of the main difficulties is finding valid control laws. The contribution of Kalman [29] to optimal control theory is one of the most central developments that helped engineers to obtain a stabilizing control relatively easily with respect to some criterion.

2.2.1 Linear Quadratic Regulator Problem

The most well-known optimal control theory, referred to as the infinite-horizon linear quadratic regulator (LQR) problem, is a theory that develops a control that minimizes a deterministic performance index known as the “quadratic cost functional” over either a finite or an infinite-horizon timeframe. The continuous-time finite-horizon cost functional referred to as the linear quadratic (LQ) problem is given by

$$J(t) = \frac{1}{2} x^T(t_f) Q^f x(t_f) + \frac{1}{2} \int_0^{t_f} (x^T(t) Q x(t) + u^T(t) R u(t)) dt \quad (2.3)$$

where $Q^f, Q \geq 0$ and $R > 0$ are constant weighting matrices. The infinite-horizon cost functional is given by

$$J(t) = \frac{1}{2} \int_0^{\infty} (x^T(t) Q x(t) + u^T(t) R u(t)) dt \quad (2.4)$$

The design of an optimal feedback gain matrix, G^* , in the finite-horizon and infinite-horizon LQR case is linked to the solution of an associated Riccati matrix equation [30].

The finite-horizon control problem is achieved by solving the differential Riccati equation:

$$\dot{K} = KA + A^T K + Q - KBR^{-1}B^T K; \quad K(t_f) = Q^f \quad (2.5)$$

while the infinite-horizon problem involves the solution of the so-called ARE of the form

$$0 = KA + A^TK + Q - KBR^{-1}B^TK. \quad (2.6)$$

The optimal state feedback control [30, 31] is then given by

$$u^*(t) = G^*x(t) = -R^{-1}B^TKx(t) \quad (2.7)$$

where K is the unique positive-definite solution of the ARE, which minimizes the infinite-horizon cost function. The total optimal cost is given by equation (2.8)

$$J^* = \frac{1}{2}x(0)^TKx(0). \quad (2.8)$$

2.2.2 Observer-Based Controller

Consider the following linear system

$$\dot{x}(t) = Ax(t) + Bu(t), \quad y(t) = Cx(t). \quad (2.9)$$

A general SFB controller has the form

$$u(t) = Gx(t)$$

if the process measurements are available. However, when the state variables cannot be directly measured, it is possible to use

$$u(t) = G\hat{x}(t) \quad (2.10)$$

where $\hat{x}(t)$ is an estimate of $x(t)$ based on available signals $u(t)$ and $y(t)$.

The approach here is by replicating the process dynamic in \hat{x}

$$\dot{\hat{x}} = A\hat{x} + Bu. \quad (2.11)$$

By defining the state estimation error $e = x - \hat{x}$, then

$$\dot{e} = Ax - A\hat{x} = Ae \quad (2.12)$$

If the real eigenvalues of A are negative, then $\lim_{t \rightarrow \infty} e(t) = 0$ asymptotically. However, if A has some eigenvalues with nonnegative real parts, then e is unbounded, and \hat{x} grows further away from x .

To avoid such a problem, consider the observer (estimator)

$$\dot{\hat{x}} = A\hat{x} + Bu + H(y - \hat{y}) \quad (2.13)$$

where H is a given matrix, called the observer gain matrix. The error dynamic now is

$$\dot{e} = Ax - A\hat{x} - H(Cx - C\hat{x}) \quad (2.14)$$

$$\dot{e} = (A - HC)x - (A - HC)\hat{x} \quad (2.15)$$

$$\dot{e} = (A - HC)(x - \hat{x}) = (A - HC)e. \quad (2.16)$$

Hence, the observer error goes to zero asymptotically if H is chosen such that $(A - HC)$ is stable.

The closed-loop composite system under the state estimator

$$u(t) = -G\hat{x}(t) \quad (2.17)$$

is given by

$$\dot{x} = Ax - BG\hat{x} \quad (2.18)$$

$$\dot{\hat{x}} = A\hat{x} - BG\hat{x} + H(y - \hat{y}) \quad (2.19)$$

or

$$\frac{d}{dt} \begin{bmatrix} x \\ \hat{x} \end{bmatrix} = \begin{bmatrix} A & -BG \\ HC & A - HC - BG \end{bmatrix} \begin{bmatrix} x \\ \hat{x} \end{bmatrix}. \quad (2.20)$$

By changing the variables

$$\begin{bmatrix} I & 0 \\ I & -I \end{bmatrix} \begin{bmatrix} x \\ \hat{x} \end{bmatrix} = \begin{bmatrix} x \\ x - \hat{x} \end{bmatrix} = \begin{bmatrix} x \\ e \end{bmatrix} \quad (2.21)$$

the new dynamics coordinate becomes

$$\frac{d}{dt} \begin{bmatrix} x \\ e \end{bmatrix} = \begin{bmatrix} (A - BG) & BG \\ 0 & (A - HC) \end{bmatrix} \begin{bmatrix} x \\ e \end{bmatrix}. \quad (2.22)$$

Note that the dynamics of the error e do not depend on x .

Separation Principle: using the state estimator \hat{x} , a controller can be designed independently. Therefore, nominal closed-loop eigenvalues are composed of eigenvalues $(A - BG)$ and eigenvalues $(A - HC)$, separately, i.e.,

$$(eig(A - BG)) \cup (eig(A - HC)). \quad (2.23)$$

2.2.3 Linear Quadratic Estimator Problem

The linear quadratic estimator (LQE) problem is addressed here. Generally, when a system is corrupted by noise, its performance is unacceptable. The external signals cause the system to be of a stochastic type. Consider the stochastic system given by the following model:

$$\dot{x}(t) = Ax(t) + Bu(t) + v(t), \quad y(t) = Cx(t) + w(t). \quad (2.24)$$

where $v(t)$ is process disturbance, and $w(t)$ is measurement noise. The intensity density matrices V and W of $v(t), w(t)$ satisfy $V \geq 0$ and $W > 0$, respectively.

Assume that $v(t), w(t)$ are uncorrelated white zero-mean Gaussian stochastic processes, i.e.,

$$E[v(t)v(\tau)^T] = \begin{cases} V, & \text{if } t = \tau \\ 0, & \text{if } t \neq \tau \end{cases} \quad (2.25)$$

$$E[w(t)w(\tau)^T] = \begin{cases} W, & \text{if } t = \tau \\ 0, & \text{if } t \neq \tau \end{cases} \quad (2.26)$$

$$E[v(t)w(\tau)^T] = 0 \quad (2.27)$$

Because of process disturbance and measurement noise, the error dynamic of an observer equation (2.14) can be rewritten as

$$\dot{e} = Ax + Bu + v - A\hat{x} - Bu - H(Cx + w - C\hat{x}) \quad (2.28)$$

$$\dot{e} = (A - HC)e + v - Lw. \quad (2.29)$$

Due to the noise v and w , the estimation will generally not go to zero. Thus, it is desirable to keep the error small by an optimal choice of H .

The covariance P of the error is given by

$$P = E\{[e - \bar{e}][e - \bar{e}]^T\} \quad (2.30)$$

where \bar{e} is the average value of e . The expected value of the error square can be written as

$$E\{ee^T\} = \bar{e}\bar{e}^T + P \quad (2.31)$$

Therefore,

$$\text{tr } E\{ee^T\} = \text{tr } [\bar{e}\bar{e}^T] + \text{tr}[P] \quad (2.32)$$

$$\text{tr } E\{e^T e\} = \text{tr } [\bar{e}^T \bar{e}] + \text{tr}[P] \quad (2.33)$$

$$E\{e^T e\} = \bar{e}^T \bar{e} + \text{tr}[P]. \quad (2.34)$$

For the initial condition uncertainty, $E\{\bar{x}(0)\} = \bar{x}_0$ is chosen such that $\bar{e}(0) = 0$, and $\bar{e}(t) = 0$ for all t . Next $\text{tr}[P]$ is minimized, where \bar{x} is the average value of x . The optimal observer gain which minimizes $E\|e(t)\|^2$ is

$$H = PC^T W^{-1} \quad (2.35)$$

where P is the unique positive-semidefinite solution of the ARE:

$$PA^T + AP - PC^T W^{-1} CP + V = 0. \quad (2.36)$$

The observer with $H = PC^T W^{-1}$ is called the steady-state Kalman-Bucy filter, or simply Kalman filter.

In summary, estimating the state of linear dynamical systems is cast into an optimization problem. The Kalman filtering approach produces the optimal stochastic state estimator under any

performance index, given that the process and measurement noises are Gaussian [32]. Furthermore, if noises are non-Gaussian, then the Kalman filter produces the best linear estimator.

2.2.4 Linear Quadratic Gaussian Problem

A combination of an optimal output feedback controller via the Kalman filter and optimal full-state feedback gain is known as the linear quadratic Gaussian design. An LQG regulator block diagram is shown in Figure 2.1.

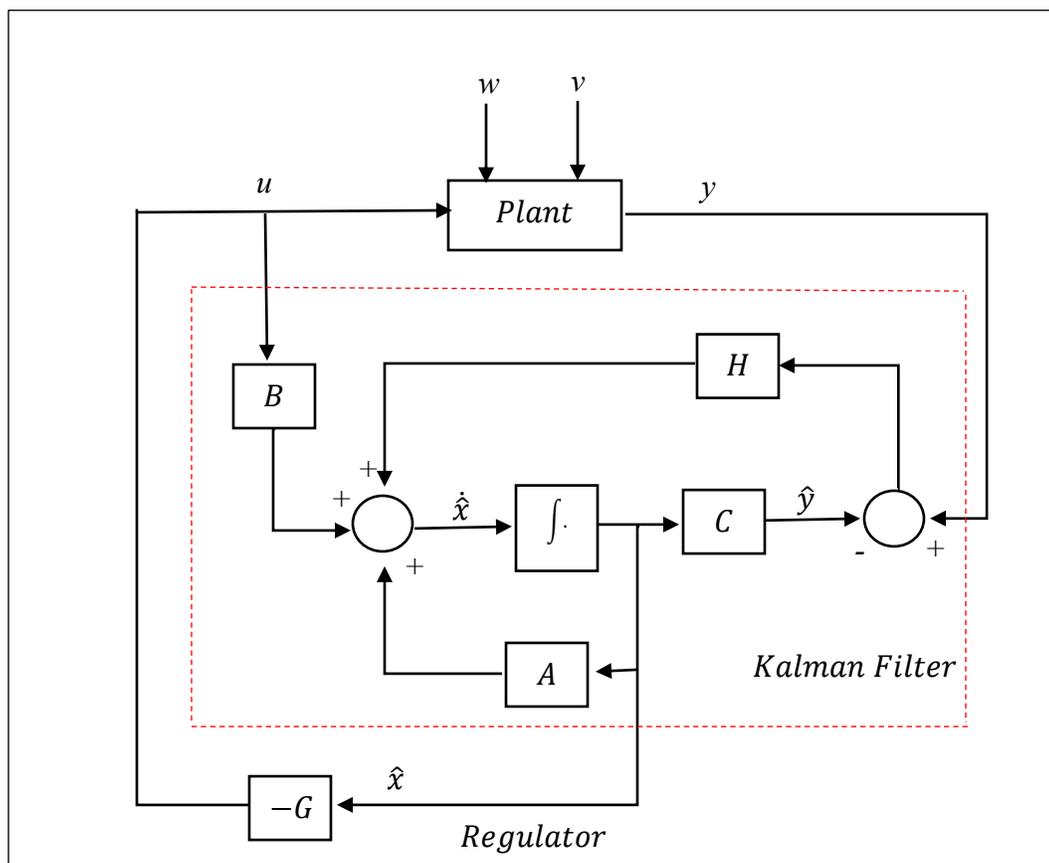


Figure 2.1. LQG regulator block diagram.

Consider the following system:

$$\dot{x} = Ax(t) + Bu(t) + Ev(t) \quad (2.37)$$

where E is process noise multiplier matrix, with the following performance index:

$$J = E \left\{ \frac{1}{2} x^T(t_f) S x(t_f) + \frac{1}{2} \int_0^{t_f} [x^T(t) Q x(t) + u^T(t) R u(t)] dt \right\}. \quad (2.38)$$

where v is the process disturbance with mean = 0 and a positive semi-definite (PSD) intensity = V . S and Q are symmetric PSD matrices, and R is a symmetric PD matrix.

Then, let

$$u(t) = Gx(t) \quad (2.39)$$

where G is a general feedback gain and can be time-varying. The closed-loop system can be written as

$$\dot{x} = A_c x(t) + Ev(t) \quad (2.40)$$

$$J = E \left\{ \frac{1}{2} x^T(t_f) S x(t_f) + \frac{1}{2} \int_0^{t_f} [x^T(t) Q_c x(t)] dt \right\} \quad (2.41)$$

where

$$\begin{aligned} A_c &= A - BG \\ Q_c &= Q + G^T R G. \end{aligned} \quad (2.42)$$

Using the following property

$$x(t) = \Phi(t)x(0) + \int_0^t \Phi(t - \tau) E v(\tau) d\tau \quad (2.43)$$

where $\Phi(t)$ is the transition function of differential equation (2.37),

the performance index J can be written as

$$J = E \left\{ \frac{1}{2} \left[\Phi(t_f)x(0) + \int_0^{t_f} \Phi(t_f - \tau)Ev(\tau)d\tau \right]^T S \left[\Phi(t_f)x(0) + \int_0^{t_f} \Phi(t_f - \tau)Ev(\tau)d\tau \right] \right. \\ \left. + \frac{1}{2} \int_0^{t_f} \left[\Phi(t)x(0) + \int_0^t \Phi(t - \tau)Ev(\tau)d\tau \right]^T Q \left[\Phi(t)x(0) + \int_0^t \Phi(t - \tau)Ev(\tau)d\tau \right] dt \right\}.$$

Assuming $x(0)$ and v are independent, this reduces to

$$J = E \left\{ \frac{1}{2} x^T(0)\Phi^T(t_f)S\Phi(t_f)x(0) + \frac{1}{2} \left[\int_0^{t_f} \Phi(t_f - \tau)Ev(\tau)d\tau \right]^T S \left[\int_0^{t_f} \Phi(t_f - \tau)Ev(\tau)d\tau \right] \right. \\ \left. + \frac{1}{2} \int_0^{t_f} x^T(0)\Phi^T(t)Q\Phi(t)x(0)dt + \frac{1}{2} \int_0^{t_f} \left[\int_0^t \Phi(t - \tau)Ev(\tau)d\tau \right]^T Q \left[\int_0^t \Phi(t - \tau)Ev(\tau)d\tau \right] dt \right\}.$$

By changing the integral limits,

$$J = \frac{1}{2} tr \left\{ \left[\Phi^T(t_f)S\Phi(t_f) + \int_0^{t_f} \Phi^T(t)Q\Phi(t)dt \right] \Sigma \right\} \\ + \frac{1}{2} tr \left\{ EVE^T \int_0^{t_f} \left[\Phi^T(t_f - \tau)S\Phi(t_f - \tau) + \int_\tau^{t_f} \Phi^T(t - \tau)Q\Phi(t - \tau)G dt \right] d\tau \right\}.$$

or

$$J = \frac{1}{2} tr[P(0)\Sigma] + \frac{1}{2} tr \left[EVE^T \int_0^{t_f} P(\tau)d\tau \right]. \quad (2.44)$$

where $\Sigma = E\{x(0)x^T(0)\}$, V is the intensity of the disturbance v , and P is the solution of the Lyapunov equation

$$-\dot{P} = PA_c + A_c^T P + Q_c \quad (2.45)$$

with the boundary condition $P(t_f) = S$. P is optimized by selecting

$$G^* = -R^{-1}B^T K \quad (2.46)$$

where K is the solution of the Riccati equation

$$-\dot{K} = KA + A^T K + Q - KBR^{-1}B^T K \quad (2.47)$$

with the boundary condition $K(t_f) = S$.

2.3 Lyapunov Stability

The Lyapunov theory was established by the Russian mathematician Aleksandr Lyapunov in the late 19th century. In systems and control theory, the notion of Lyapunov stability concerns the asymptotic performance of the state of an autonomous dynamical system. The concept of stability, asymptotic stability, and instability of such systems in terms of the existence of the so-called Lyapunov functions has been the main contribution of Lyapunov. The theory of Lyapunov functions (known as the “Second Method” or “Indirect Method”) has been a powerful tool for determining the asymptotic stability of linear and nonlinear systems without explicit knowledge of their solutions and has intensively been applied to both continuous and discrete systems. Some fundamental definitions and theorems of Lyapunov stability are introduced below:

Definition 2.1: Assume (without loss of generality) that $x(t) \equiv 0$ is an equilibrium point of a linear dynamics system $\dot{x} = Ax$. It is considered stable (in the sense of Lyapunov) if for every $\varepsilon > 0$ there exists $\delta = \delta(\varepsilon) > 0$ such that if $\|x(0)\| \leq \delta$, then $\|x(t)\| \leq \varepsilon, \forall t \geq t_0$.

Definition 2.2: An equilibrium point $x(t) \equiv 0$ of a linear dynamics system $\dot{x} = Ax$ is asymptotically stable if it is stable, and there exists $\delta > 0$ such that if $\|x(0)\| < \delta$, then

$$\lim_{t \rightarrow +\infty} x(t) = 0.$$

Definition 2.3: The function denoted by $V: R^n \rightarrow R$ for the system $\dot{x} = A(t)x$ is called Lyapunov function if the following apply:

- (1) $V[x(t)] \geq 0$.
- (2) $V[x(t)] = 0$ if and only if $x(t) = 0$.
- (3) $\dot{V}[x(t)] \equiv \frac{d}{dt}V[x(t)] \leq 0$.

Definition 2.4: A quadratic Lyapunov function (QLF) related to $\dot{x} = Ax$ is a functional $V[x(t)]: R^n \rightarrow R$ of the form

$$V[x(t)] = x^T(t)Px(t)$$

for which $\dot{V}[x(t)] \leq 0$ for all nonzero $x(t)$, where the derivative is taken along trajectories to $\dot{x} = Ax$.

Heuristically, Lyapunov's method depends on generalizing the concept of the energy in a system. The idea behind Lyapunov's Second Method is to find a function $V[x(t)]$ that can measure the energy in the system for any given solution $x(t)$. By observing the time derivative $\dot{V}[x(t)]$, Lyapunov could determine whether the energy in the system was increasing or decreasing. In instances where the energy never increases in time, it follows that the solution must remain bounded. If the energy is strictly decreasing, then the solution must reach equilibrium asymptotically.

Theorem 2.1 (Lyapunov's Second Theorem [33]): Given a dynamic system $\dot{x} = A(t)x$ with an equilibrium $x(t) \equiv 0$, if there exists an associated Lyapunov function $V(x)$, then the equilibrium is Lyapunov stable. Moreover, if $\dot{V}[x(t)] < 0$, then $x(t) \equiv 0$ is globally asymptotically stable. The word "globally" denotes that Lyapunov stability holds regardless of the initial conditions.

By selecting to consider quadratic Lyapunov functions, the first condition of definition 2.3 is trivially satisfied. Since the work here is trying to find conditions that are sufficient for global asymptotic stability of a dynamical system, the focus is on studying when P exists for which $\dot{V}[x(t)] < 0$.

CHAPTER 3

COMPOSITE OPTIMAL CONTROL OF INTERCONNECTED SINGULARLY PERTURBED SYSTEMS

3.1 Problem Formulation

3.1.1 System Description

The system under consideration is described by the block diagram shown in Figure 3.1, and the mathematical model with slow and fast dynamics is described by the linear time-invariant interconnected singularly perturbed form in equation (3.1) that follows.

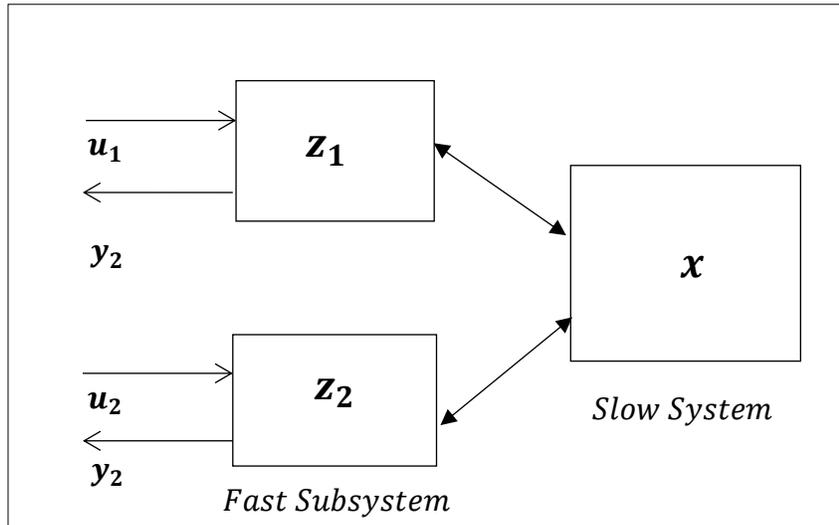


Figure 3.1 Decentralized deterministic system setup with two subsystems.

$$\begin{aligned}
 \frac{d}{dt}x(t) &= A_{00}x(t) + A_{01}z_1(t) + A_{02}z_2(t) \\
 \varepsilon \frac{d}{dt}z_1(t) &= A_{10}x(t) + A_{11}z_1(t) + B_1u_1(t) \\
 \varepsilon \frac{d}{dt}z_2(t) &= A_{20}x(t) + A_{21}z_1(t) + A_{22}z_2(t) + B_2u_2(t) \\
 y_1(t) &= C_1z_1(t) \\
 y_2(t) &= C_2z_2(t)
 \end{aligned} \tag{3.1}$$

where $x(t) \in \mathcal{R}^n$ is the state variable of the slow system; $z_1(t) \in \mathcal{R}^{p_1}$ and $z_2(t) \in \mathcal{R}^{p_2}$ are the state variables of fast subsystems $z_1(t)$ and $z_2(t)$, respectively; $u_i(t) \in \mathcal{R}^{q_i}$ are the control input vectors of subsystems $z_i(t)$; $y_i(t) \in \mathcal{R}^{r_i}$ are the control output of subsystems $z_i(t)$; A_{00} , A_{0i} , A_{i0} , A_{ii} , B_i , and C_i ; $i = 1, 2$ are constant matrices with appropriate matching dimensions; and $\varepsilon \ll 1$ is a small positive scalar denoting the singular perturbation parameter. The smallness of ε allows the classification of the variable $x(t)$ to be slow and the variables $z_i(t)$, for $i = 1, 2$ to be fast, because $z_i(t)$ may change values with the velocity of the order $O(\varepsilon^{-1})$ [34].

3.1.2 Objective

The objective in this chapter is to formulate the optimal state feedback control for interconnected singularly perturbed systems. The SP methodology is applied to decompose the TTS system into separate slow and fast subsystems. Then separate slow and fast control laws are designed for each subsystem, which are then combined to form near-optimum composite linear controller such that both stability and cost minimization are attained for the full-order system.

3.2 Full-Order LQR Solution

The following notations are introduced:

$$\tilde{x}(t) := \begin{bmatrix} x \\ z_1 \\ z_2 \end{bmatrix}, \quad u(t) := \begin{bmatrix} 0 \\ u_1(t) \\ u_2(t) \end{bmatrix}, \quad y(t) := \begin{bmatrix} 0 \\ y_1(t) \\ y_2(t) \end{bmatrix}, \quad A_\varepsilon := \begin{bmatrix} A_{00} & A_{01} & A_{02} \\ \varepsilon^{-1}A_{10} & \varepsilon^{-1}A_{11} & 0 \\ \varepsilon^{-1}A_{20} & 0 & \varepsilon^{-1}A_{22} \end{bmatrix},$$

$$B_\varepsilon := \begin{bmatrix} 0 & 0 & 0 \\ 0 & \varepsilon^{-1}B_1 & 0 \\ 0 & 0 & \varepsilon^{-1}B_2 \end{bmatrix}, \quad \text{and} \quad C := \begin{bmatrix} 0 & 0 & 0 \\ 0 & C_1 & 0 \\ 0 & 0 & C_2 \end{bmatrix}.$$

The mathematical model given in equation (3.1) can be rewritten in a compact form as

$$\begin{aligned} \frac{d}{dt} \tilde{x}(t) &= A_\varepsilon \tilde{x}(t) + B_\varepsilon u(t) \\ y(t) &= C \tilde{x}(t). \end{aligned} \tag{3.2}$$

By applying a general SFB controller

$$u(t) = G\tilde{x}(t) \quad (3.3)$$

to system (3.2), the following closed-loop system is achieved:

$$\frac{d}{dt}\tilde{x}(t) = A_{cl}(\varepsilon)\tilde{x}(t) \quad (3.4)$$

where $A_{cl}(\varepsilon) = (A_\varepsilon + B_\varepsilon G)$.

Consider the infinite-horizon quadratic performance index (cost functional) along with model equation (3.2):

$$J(\tilde{x}, u) = \frac{1}{2} \int_0^{\infty} (\tilde{x}^T(t)Q\tilde{x}(t) + u^T(t)Ru(t))dt \quad (3.5)$$

where $Q = Q^T \geq 0$, and $R = R^T > 0$ are state and input weighting matrices, respectively.

Substituting the controller equation (3.3) into equation (3.5), the quadratic performance becomes

$$J(\tilde{x}, u) = \frac{1}{2} \int_0^{\infty} \left(\tilde{x}^T(t)Q\tilde{x}(t) + (G\tilde{x}(t))^T R(G\tilde{x}(t)) \right) dt \quad (3.6)$$

$$J(\tilde{x}, u) = \frac{1}{2} \int_0^{\infty} (\tilde{x}^T(t)Q_c\tilde{x}(t))dt ; \quad Q_c = Q + G^T R G . \quad (3.7)$$

Consider the general algebraic Riccati equation:

$$A_\varepsilon^T K + K A_\varepsilon + Q - K B_\varepsilon R^{-1} B_\varepsilon^T K = 0 . \quad (3.8)$$

where $K = K(\varepsilon)$ is symmetric positive-definite solution of the ARE. The optimal state-feedback control that minimizes the quadratic index is given by

$$u^*(t) = G^*(\varepsilon)\tilde{x}(t) \quad (3.9)$$

where

$$G^* = -R^{-1} B_\varepsilon^T K \quad (3.10)$$

Moreover, the optimal cost is

$$J_{opt} = \frac{1}{2} [x^T(0) K x(0)] = \frac{1}{2} \text{tr}\{K\Sigma\} \quad (3.11)$$

where

$$\Sigma = x(0)x^T(0).$$

Hence, it is noted that the demonstration of the full-order solution (for every $\varepsilon > 0$) above is explicitly a function of the small parameter ε and may determine ill-conditioned computations of the stabilizing solution of the general algebraic Riccati equation (GARE). Therefore, it is of interest to investigate the asymptotic behavior for $\varepsilon \rightarrow 0$ of a stabilizing solution of Riccati equations associated with a near-optimum LQR problem. Such investigations result in computing Riccati equations of lower dimensions, which are independent of the small parameter ε .

3.3 Time-Scale Decomposition

The system shown in equation (3.1) is singularly perturbed, in the following subsections an order reduction and separation of time scales as $\varepsilon \rightarrow 0$ is derived. As will be shown in sections 3.3.1 and 3.3.2, the solutions obtained using SP theory, which has two remedial features—dimensional reduction and stiffness relief [2], results in separated slow and fast quadratic performance indices that do not depend upon the value of ε .

3.3.1 Slow Subsystem

By letting $\varepsilon = 0$, the fast dynamics of the system are neglected, and it is assumed that the state variables $z_1(t)$ and $z_2(t)$ have reached a quasi-steady state. From this comes the following quasi-steady state model:

$$\begin{aligned} \frac{d}{dt} \bar{x} &= A_{00} \bar{x} + A_{01} \bar{z}_1 + A_{02} \bar{z}_2 \\ 0 &= A_{10} \bar{x} + A_{11} \bar{z}_1 + B_1 \bar{u}_1 \\ 0 &= A_{20} \bar{x} + A_{22} \bar{z}_2 + B_2 \bar{u}_2 \end{aligned} \quad (3.12)$$

The second and third algebraic equations can be inverted for \bar{z}_1 and \bar{z}_2 , resulting in

$$\bar{z}_i = -A_{ii}^{-1}(A_{i0}\bar{x} + B_i\bar{u}_i); \quad i = 1,2. \quad (3.13)$$

where A_{ii} is nonsingular. Then equation (3.13) is inserted into the dynamic part of equation (3.12), yielding

$$\begin{aligned} \frac{d}{dt}\bar{x} &= (A_{00} - A_{01}A_{11}^{-1}A_{10} - A_{02}A_{22}^{-1}A_{20})\bar{x} - A_{01}A_{11}^{-1}B_1\bar{u}_1 \\ &\quad - A_{02}A_{22}^{-1}B_2\bar{u}_2. \end{aligned} \quad (3.14)$$

and

$$\bar{y}_i = C_i\bar{z}_i \quad (3.15)$$

$$\bar{y}_i = C_i(-A_{ii}^{-1}(A_{i0}\bar{x} + B_i\bar{u}_i)) = (-C_iA_{ii}^{-1}A_{i0})\bar{x} + (-C_iA_{ii}^{-1}B_i)\bar{u}_i.$$

The reduced-order (slow) dynamics is obtained as

$$\begin{aligned} \frac{d}{dt}x^s(t) &= \hat{A}x^s(t) + \sum \hat{B}_i u_i^s(t) \\ y_i^s(t) &= \hat{C}_i x^s(t) + \hat{D}_i u_i^s(t); \quad i = 1,2 \end{aligned} \quad (3.16)$$

where

$$\hat{A} = A_{00} - A_{01}A_{11}^{-1}A_{10} - A_{02}A_{22}^{-1}A_{20},$$

$$\hat{B}_i = -A_{0i}A_{ii}^{-1}B_i,$$

$$\hat{C}_i = -C_iA_{ii}^{-1}A_{i0},$$

$$\hat{D}_i = -C_iA_{ii}^{-1}B_i.$$

Hence, $\bar{x} = x^s$, $\bar{z}_i = \bar{u}_i = u_i^s$, and $\bar{y}_i = y_i^s$ are the slow components of the corresponding variables in system equation (3.1).

The quadratic performance index is given by

$$J_s = \frac{1}{2} \int_0^{\infty} ([x^s]^T Q_x x^s + \bar{z}_i^T Q_{z_i} \bar{z}_i + [u_i^s]^T R_i u_i^s) dt \quad (3.17)$$

The slow part of the fast subsystems in equation (3.13) is inserted in the cost functional given in equation (3.17) to yield

$$J_s = \frac{1}{2} \int_0^{\infty} ([x^s]^T Q_0 x^s + 2[x^s]^T M_1 u_1^s + 2[x^s]^T M_2 u_2^s + [u_1^s]^T R_{01} u_1^s + [u_2^s]^T R_{02} u_2^s) dt \quad (3.18)$$

where

$$\begin{aligned} Q_0 &= Q_x + A_{10}^T A_{11}^{-T} Q_{z1} A_{11}^{-1} A_{10} + A_{20}^T A_{22}^{-T} Q_{z2} A_{22}^{-1} A_{20} , \\ M_1 &= A_{10}^T A_{11}^{-T} Q_{z1} A_{11}^{-1} B_1 , \\ M_2 &= A_{20}^T A_{22}^{-T} Q_{z2} A_{22}^{-1} B_2 , \\ R_{01} &= R_1 + B_1^T A_{11}^{-T} Q_{z1} A_{11}^{-1} B_1 , \\ R_{02} &= R_2 + B_2^T A_{22}^{-T} Q_{z2} A_{22}^{-1} B_2 . \end{aligned}$$

To eliminate the cross-product terms ($2[x^s]^T M_i u_i^s$), let

$$u_i^s = u_{0i}^s + G_{0i} x^s \quad (3.19)$$

The preliminary feedback input [2] is given by

$$G_{0i} = -R_{0i}^{-1} M_i^T$$

Applying this feedback gain to the control input yields

$$u_i^s = u_{0i}^s + G_{0i} x^s = u_{0i}^s - R_{0i}^{-1} M_i^T x^s \quad (3.20)$$

then, quadratic equation (3.18) becomes

$$J_s = \frac{1}{2} \int_0^{\infty} ([x^s]^T Q_s x^s + [u_{0i}^s]^T R_{0i} u_{0i}^s) dt \quad (3.21)$$

where

$$\begin{aligned} Q_s &= Q_0 + 2M_1 G_{01} + 2M_2 G_{02} , \\ G_{01} &= -R_{01}^{-1} M_1^T , \quad G_{02} = -R_{02}^{-1} M_2^T . \end{aligned}$$

and the optimal feedback control of the system will take the form

$$u_i^{*s}(t) = G_{0i}^* x^s(t) \quad (3.22)$$

where

$$G_{0i}^* = -R_{0i}^{-1}(M_i^T + \hat{B}_i^T K_{si}) \quad (3.23)$$

and K_{si} is the positive-definite solution to the algebraic Riccati equation:

$$0 = K_{si} A_{cl} + A_{cl}^T K_{si} - K_{si} \hat{B}_i R_{0i}^{-1} \hat{B}_i^T K_{si} + Q_s \quad (3.24)$$

$$A_{cl} = \hat{A} - \hat{B}_i R_{0i}^{-1} M_i^T.$$

The above slow-subsystem design solution can be used to provide a reduced-order controller for the original full-order system. The reduced control that uses the original states is

$$u_r = G_r \begin{bmatrix} x \\ z_1 \\ z_2 \end{bmatrix} \quad (3.25)$$

where

$$G_r = \begin{bmatrix} -R_{0i}^{-1}(M_i^T + \hat{B}_i^T K_{si}) & 0 & 0 \end{bmatrix}.$$

3.3.2 Fast Subsystem

Fast dynamics are considered here as defined by $z^f(t) := z(t) - \bar{z}(t)$, $u^f(t) := u(t) - \bar{u}(t)$, $y^f(t) := y(t) - \bar{y}(t)$, where z^f , u^f and y^f are the fast subsystem variable, fast input vector, and fast output vector, respectively, and assuming $\frac{d}{dt} \bar{z}_i = 0$ and $x = \bar{x} = \text{constant}$ (the slow part of each variable is assumed to be constant during the transient period). Therefore,

$$\varepsilon \frac{dz_i}{dt} = A_{i0} x + A_{ii} z_i + B_i u_i \quad (3.26)$$

$$\varepsilon \frac{d\bar{z}_i}{dt} = A_{i0} \bar{x} + A_{ii} \bar{z}_i + B_i \bar{u}_i \quad (3.27)$$

Subtracting equation (3.27) from equation (3.26) yields

$$\varepsilon \frac{dz_i}{dt} - \varepsilon \frac{d\bar{z}_i}{dt} = A_{i0}(x - \bar{x}) + A_{ii}(z_i - \bar{z}_i) + B_i(u_i - \bar{u}_i) \quad (3.28)$$

$$\varepsilon \frac{dz_i^f}{dt} = A_{ii}z_i^f + B_i u_i^f \quad (3.29)$$

Similarly, the output in equation (3.29) can be subtracted from the output in equation (3.28) to yield

$$y_i^f = y_i - \bar{y}_i = C_i(z_i - \bar{z}_i) = C_i z_i^f. \quad (3.30)$$

Performance criteria for the fast subsystem is defined by

$$J_f = \frac{1}{2} \int_0^{\infty} \{ [z_i^f]^T Q_{zi} z_i^f + [u_i^f]^T R_i u_i^f \} dt \quad (3.31)$$

The optimal feedback control, which minimizes the cost functional for the fast subsystem, will take the following form:

$$u_i^{*f}(t) = G_i^* z_i^f(t) \quad (3.32)$$

where

$$G_i^* = -R_i^{-1} B_i^T K_{f_i} \quad (3.33)$$

and K_{f_i} is the positive-definite solution to the ARE:

$$0 = K_{f_i} A_{ii} + A_{ii}^T K_{f_i} - K_{f_i} B_i R_i^{-1} B_i^T K_{f_i} + Q_{zi}. \quad (3.34)$$

3.3.3 Composite Control

The designs of slow and fast controls are optimal only for their respective subsystems. A feedback control law based on the complete separation of slow and fast subsystem designs that use actual states x , z_1 , and z_2 can be formed [10]. The composite control approach has various advantages. The reduced-order slow and fast controllers are independently designed to avoid the high-dimensionality and ill-conditioned numerical problem of the full-order controller design. They may also be implemented separately in software to reduce the size of required computations

and speed up the optimization processes. In addition, the near-optimal composite control does not depend explicitly upon ε . A proof of the near optimality of the composite control law for a quadratic performance index is provided in the work of Chow and Kokotovic [10].

With solutions of the slow and fast problems in hand, the composite optimal control law for the whole system equation (2.1) is given by

$$u^c = u_i^{*s} + u_i^{*f} \quad (3.35)$$

$$= G_{0i}^* x^s(t) + G_i^* z_i^f(t) \quad (3.36)$$

This can be rewritten in terms of x and z_i by replacing x^s by x and z_i^f by $z_i - \bar{z}_i$:

$$\begin{aligned} u^c = & -[(I_r - R_i^{-1} B_i^T K_{f_i} A_{ii}^{-1} B_i) R_{0i}^{-1} (M_i^T + \hat{B}_i^T K_{s_i}) \\ & + R_i^{-1} B_i^T K_{f_i} A_{ii}^{-1} A_{i0}] x - R_i^{-1} B_i^T K_{f_i} z_i. \end{aligned} \quad (3.37)$$

CHAPTER 4

COMPOSITE OPTIMAL CONTROL OF INTERCONNECTED SINGULARLY PERTURBED STOCHASTIC SYSTEMS

4.1 Problem Formulation

4.1.1 System Description

The system under consideration here is shown in the block diagram in Figure 4.1, and the mathematical model with slow and fast dynamics is described by the linear time-invariant interconnected singularly perturbed stochastic form in equation (4.1) that follows.

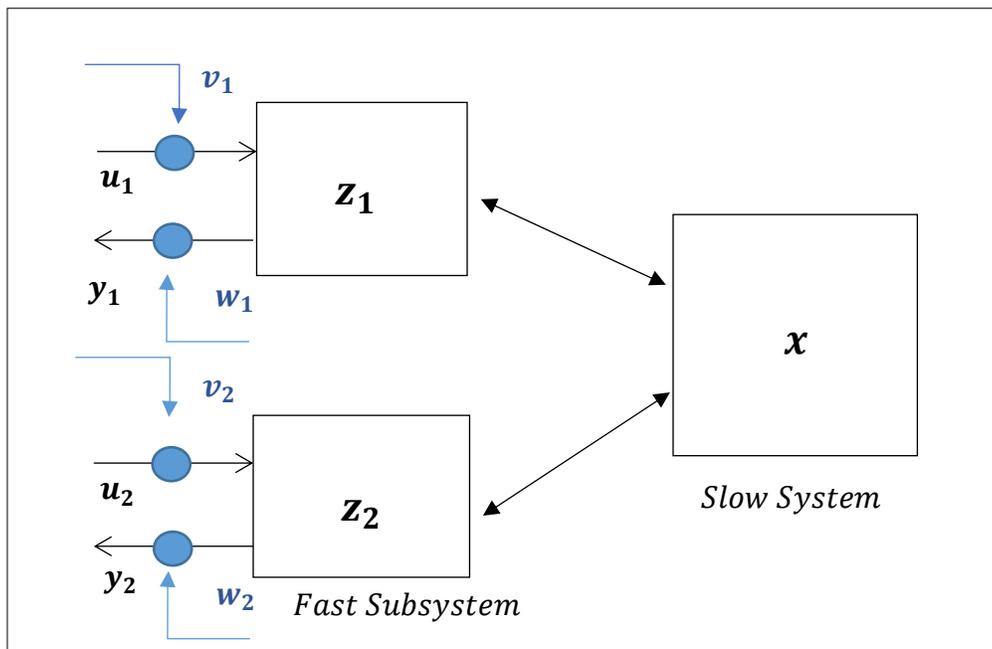


Figure 4.1 Decentralized stochastic system setup with two subsystems.

$$\begin{aligned}
\frac{d}{dt}x(t) &= A_{00}x(t) + A_{01}z_1(t) + A_{02}z_2(t) \\
\varepsilon \frac{d}{dt}z_1(t) &= A_{10}x(t) + A_{11}z_1(t) + B_1u_1(t) + E_1v_1(t) \\
\varepsilon \frac{d}{dt}z_2(t) &= A_{20}x(t) + A_{22}z_2(t) + B_2u_2(t) + E_2v_2(t) \\
y_1(t) &= C_1z_1(t) + F_1w_1(t) \\
y_2(t) &= C_2z_2(t) + F_2w_1(t)
\end{aligned} \tag{4.1}$$

where $x(t) \in \mathcal{R}^n$ is the state variable of the slow system; $z_1(t) \in \mathcal{R}^{p_1}$ and $z_2(t) \in \mathcal{R}^{p_2}$ are the state variables of fast subsystems $z_1(t)$ and $z_2(t)$, respectively; $u_i(t) \in \mathcal{R}^{q_i}$ are the control input vectors of subsystems $z_i(t)$; $y_i(t) \in \mathcal{R}^{r_i}$ are the control output of subsystems $z_i(t)$; $v_i(t)$ and $w_i(t) \in \mathcal{R}^{s_i}$ are the input disturbance and output measurement noise to subsystem i , respectively; inputs $v_i(t)$ and $w_i(t)$ are uncorrelated, stationary, white-Gaussian noises with intensity matrix $V_i \geq 0$, and $W_i > 0$, respectively; E_i and F_i are constant plant noise gain matrices to subsystem i ; A_{00} , A_{0i} , A_{i0} , A_{ii} , B_i , and C_i ; $i = 1, 2$ are real known constant matrices of appropriate matching dimensions; and $\varepsilon \ll 1$ is a small positive scalar, denoting the singular perturbation parameter.

4.1.2 Objective

The main objective of this chapter is twofold: (a) to present a procedure to design a near-optimal controller (composite controller) based on the reduced-order slow and fast subsystems of interconnected singularly perturbed systems with Gaussian noise represented in state space form, and (b) to find the optimal control input using observer-based feedback control such that both stability and cost minimization are achieved through quadratic performance criteria.

In section 4.2, the exact full-order linear quadratic Gaussian solution using SFB control is developed for the full-order model. In section 4.3, the near-optimal solution as $\varepsilon \rightarrow 0$ for this model is derived by making use of SP theory as a model simplification technique. First, the system equation (4.1) is decomposed into slow and fast modes. Then the reduced control is designed based

on the reduced-order slow subsystem. The fast control, defined as the boundary layer correction, is then developed and used in a slow and fast composite control, to obtain ε -free solution at the end. The composite feedback control law is formulated as the sum of the slow and fast controls. Since it is rarely possible to have access to all states of the system when designing full SFB in real-world control designs, Kalman filter designs are presented in section 4.4 as an optimal state estimation to estimate the states of the system. A numerical example is shown in section 4.5 to validate the efficiency of the proposed method where the exact full-order LQG solution is used as a baseline for comparison against the composite control solution, which is developed in sections 4.3 and 4.4.

4.2 Full-Order LQG Solution

First, the following notations are introduced:

$$\tilde{x}(t) := \begin{bmatrix} x \\ z_1 \\ z_2 \end{bmatrix}, \quad u(t) := \begin{bmatrix} 0 \\ u_1(t) \\ u_2(t) \end{bmatrix}, \quad y(t) := \begin{bmatrix} 0 \\ y_1(t) \\ y_2(t) \end{bmatrix}, \quad v(t) := \begin{bmatrix} 0 \\ v_1(t) \\ v_2(t) \end{bmatrix}, \quad w(t) := \begin{bmatrix} 0 \\ w_1(t) \\ w_2(t) \end{bmatrix},$$

$$A_\varepsilon := \begin{bmatrix} A_{00} & A_{01} & A_{02} \\ \varepsilon^{-1}A_{10} & \varepsilon^{-1}A_{11} & 0 \\ \varepsilon^{-1}A_{20} & 0 & \varepsilon^{-1}A_{22} \end{bmatrix}, \quad B_\varepsilon := \begin{bmatrix} 0 & 0 & 0 \\ 0 & \varepsilon^{-1}B_1 & 0 \\ 0 & 0 & \varepsilon^{-1}B_2 \end{bmatrix}, \quad E_\varepsilon := \begin{bmatrix} 0 & 0 & 0 \\ 0 & \varepsilon^{-1}E_1 & 0 \\ 0 & 0 & \varepsilon^{-1}E_1 \end{bmatrix},$$

$$C := \begin{bmatrix} 0 & 0 & 0 \\ 0 & C_1 & 0 \\ 0 & 0 & C_2 \end{bmatrix}, \quad \text{and} \quad F := \begin{bmatrix} 0 & 0 & 0 \\ 0 & F_1 & 0 \\ 0 & 0 & F_2 \end{bmatrix}.$$

With such notations, the m -dimensional ($m := n + p_1 + p_2$) system dynamics associated with equation (4.1) can be rewritten in compact form as

$$\begin{aligned} \frac{d}{dt} \tilde{x}(t) &= A_\varepsilon \tilde{x}(t) + B_\varepsilon u(t) + E_\varepsilon v(t) \\ y(t) &= C \tilde{x}(t) + F w(t) \end{aligned} \tag{4.2}$$

By coupling a general SFB controller

$$u(t) = G \tilde{x}(t) \tag{4.3}$$

to system equation (4.2), the following the closed-loop system is obtained:

$$\frac{d}{dt} \tilde{x}(t) = A_{cl}(\varepsilon) \tilde{x}(t) + E_\varepsilon v(t); \quad (4.3)$$

where $A_{cl}(\varepsilon) = (A_\varepsilon + B_\varepsilon G)$.

Associated with system equation (4.2), consider the infinite-horizon quadratic performance index:

$$J(\tilde{x}, u) = \frac{1}{2} E \left\{ \int_0^\infty (\tilde{x}^T(t) Q \tilde{x}(t) + u^T(t) R u(t)) dt \right\} \quad (4.4)$$

where $Q = Q^T \geq 0$, and $R = R^T > 0$ are constant weighting matrices.

By substituting the controller equation (4.3) into equation (4.4), the cost functional becomes

$$J(\tilde{x}, u) = \frac{1}{2} E \left\{ \int_0^\infty \left(\tilde{x}^T(t) Q \tilde{x}(t) + (G \tilde{x}(t))^T R (G \tilde{x}(t)) \right) dt \right\} \quad (4.5)$$

$$J(\tilde{x}, u) = \frac{1}{2} E \left\{ \int_0^\infty (\tilde{x}^T(t) Q_{cl} \tilde{x}(t)) dt \right\}; \quad Q_{cl} = Q + G^T R G \quad (4.6)$$

The optimal cost of the system can be rewritten

$$J_{opt} = \frac{1}{2} tr\{P\Omega\} + \frac{1}{2} tr\{PE_\varepsilon V E_\varepsilon^T\}, \quad (4.7)$$

where $\Omega = E\{\tilde{x}(0)\tilde{x}^T(0)\}$, and P is the solution of the following Lyapunov equation:

$$0 = P A_{cl}(\varepsilon) + A_{cl}^T(\varepsilon) P + Q_{cl} \quad (4.8)$$

where P is optimized by selecting

$$G^* = -R^{-1} B_\varepsilon^T K(\varepsilon) \quad (4.9)$$

where $K(\varepsilon) = K$ is the positive-definite solution of the following GARE:

$$A_\varepsilon^T K + K A_\varepsilon + Q - K B_\varepsilon R^{-1} B_\varepsilon^T K = 0. \quad (4.10)$$

The optimal control law is

$$u^*(t) = G^* \tilde{x}(t). \quad (4.12)$$

4.3 Time-Scale Decomposition

4.3.1 Slow Subsystem

The reduced model can be obtained by setting $\varepsilon = 0$:

$$\begin{aligned}
 \frac{d\bar{x}}{dt} &= A_{00}\bar{x} + A_{01}\bar{z}_1 + A_{02}\bar{z}_2 \\
 0 &= A_{10}\bar{x} + A_{11}\bar{z}_1 + B_1\bar{u}_1 + E_1v_1 \\
 0 &= A_{20}\bar{x} + A_{22}\bar{z}_2 + B_2\bar{u}_2 + E_2v_2 \\
 \bar{y}_1 &= C_1\bar{z}_1 + F_1w_1 \\
 \bar{y}_2 &= C_2\bar{z}_2 + F_2w_2
 \end{aligned} \tag{4.11}$$

where a bar indicates a continuous quasi-steady state system. The slow dynamic of the system can be presented as:

$$\begin{aligned}
 \frac{d}{dt}x^s &= \tilde{A}x^s + \sum [\tilde{B}_i u_i^s + \tilde{E}_i v_i] ; i = 1,2. \\
 y_i^s &= \tilde{C}_i x^s + \tilde{D}_i u_i^s + \bar{\bar{E}}_i v_i + F_i w_i
 \end{aligned} \tag{4.12}$$

where $\tilde{A} = A_{00} - A_{01}A_{11}^{-1}A_{10} - A_{02}A_{22}^{-1}A_{20}$,

$$\tilde{B}_i = -A_{0i}A_{ii}^{-1}B_i,$$

$$\tilde{C}_i = -C_iA_{ii}^{-1}A_{i0},$$

$$\tilde{D}_i = -C_iA_{ii}^{-1}B_i,$$

$$\tilde{E}_i = -A_{0i}A_{ii}^{-1}E_i,$$

$$\bar{\bar{E}}_i = -C_iA_{ii}^{-1}E_i.$$

Hence, $\bar{x} = x^s$, \bar{z}_i , $\bar{u}_i = u_i^s$, and $\bar{y}_i = y_i^s$ are the slow components of the corresponding variables in system equation (4.1).

The slow part of the fast subsystems is given by

$$\bar{z}_i = -A_{ii}^{-1}(A_{i0}x^s + B_i u_i^s + E_i v_i) ; i = 1,2. \tag{4.13}$$

with the quadratic performance index as

$$J_s = \frac{1}{2} E \left\{ \int_0^{\infty} ([x^s]^T Q_x x^s + \bar{z}_i^T Q_{zi} \bar{z}_i + [u_i^s]^T R_i u_i^s) dt \right\} \quad (4.14)$$

By inserting the value of $\bar{z}_i(t)$ in equation (4.11) into equation (4.12), the slow performance index is obtained as

$$J_s = \frac{1}{2} E \left\{ \int_0^{\infty} \{ [x^s]^T Q_0 x^s + 2[x^s]^T (A_{i0}^T A_{ii}^{-T} Q_{zi} A_{ii}^{-1} B_i) + [u_i^s]^T (R_i + B_i^T A_{ii}^{-T} Q_{zi} A_{ii}^{-1} B_i) u_i^s \} dt \right\} + J_v \quad (4.15)$$

where J_v is integral contains terms such as the variance of the noise v , which are ill-defined and not influenced by the control [15].

$$J_s = \frac{1}{2} E \left\{ \int_0^{\infty} \{ [x^s]^T Q_0 x^s + 2[x^s]^T M_i u_i^s + [u_i^s]^T R_{0i} u_i^s \} dt \right\} \quad (4.16)$$

where

$$Q_0 = Q_x + A_{i0}^T A_{ii}^{-T} Q_{zi} A_{ii}^{-1} A_{i0},$$

$$M_i = B_i^T A_{ii}^{-T} Q_{zi} A_{ii}^{-1} A_{i0},$$

$$R_{0i} = R_i + B_i^T A_{ii}^{-T} Q_{zi} A_{ii}^{-1} B_i.$$

To eliminate the cross-product term in equation (4.14), let

$$u_i^s = u_{0i}^s + G_{0i} x^s \quad (4.17)$$

Preliminary feedback input [2] is given by

$$G_{0i} = -R_{0i}^{-1} M_i^T \quad (4.18)$$

and the performance index will be

$$J_s = \frac{1}{2} E \left\{ \int_0^{\infty} \{ [x^s]^T Q_s x^s + [u_{0i}^s]^T R_{0i} u_{0i}^s \} dt \right\} \quad (4.19)$$

where

$$Q_s = Q_0 + 2M_i G_{0i} + G_{0i}^T R_{0i} G_{0i}$$

Since

$$G_{0i} = -R_{0i}^{-1} M_i^T$$

then

$$Q_s = Q_0 - M_i R_{0i}^{-1} M_i^T$$

The optimal cost of the system can be written as

$$J_s = \frac{1}{2} \text{tr}\{P_i \Omega\} + \frac{1}{2} \text{tr}\{P_i \tilde{E}_i V \tilde{E}_i^T\}, \quad (4.20)$$

where $\Omega = E\{x(0)x^T(0)\}$, and P_i is the solution of the following Lyapunov equation:

$$0 = P_i A_{cl} + P_i^T P + Q_{cl} \quad (4.21)$$

where $A_{cl} = \tilde{A} + \tilde{B}_i G_{0i}$, $Q_{cl} = G_{0i}^T R_{0i} G_{0i} + Q_s$, and P_i is optimized by selecting

$$G_{0i}^* = -R_{0i}^{-1} (M_i^T + \tilde{B}_i^T K_{si}), \quad (4.22)$$

and the optimal control input of the system has the form

$$u_i^{*s}(t) = G_{0i}^* x^s(t) \quad (4.23)$$

where K_{si} is the positive-definite solution to the ARE:

$$0 = K_{si} \tilde{A} + \tilde{A}^T K_{si} - K_{si} \tilde{B}_i R_{0i}^{-1} \tilde{B}_i^T K_{si} + Q_s. \quad (4.24)$$

4.3.2 Fast Subsystem

Here we formulate the solution for the fast subsystem. This solution will then be used to form the near-optimal composite control. The fast subsystem is derived by assuming that the slow part of the system is constant during the transient phase. By defining $z^f(t) := z(t) - \bar{z}(t)$, $u^f(t) := u(t) - \bar{u}(t)$, and $y^f(t) := y(t) - \bar{y}(t)$, where z^f , u^f , and y^f are the fast subsystem variable, fast input vector, and fast output vector, respectively, the fast subsystem from equations (4.1) and (4.6) can be expressed as

$$\varepsilon \frac{dz_i^f}{dt} = A_{ii}z_i^f + B_i u_i^f + E_i v_i \quad (4.25)$$

$$y_i^f = C_i z_i^f + F_i w_i \quad (4.26)$$

The performance criteria for fast subsystem is defined by

$$J_f = \frac{1}{2} E \left\{ \int_0^\infty \left\{ [z_i^f(t)]^T Q_{z_i} z_i^f(t) + [u_i^f(t)]^T R_i u_i^f(t) \right\} dt \right\} \quad (4.27)$$

The cost of the system can be written as

$$J_f = \frac{1}{2} \text{tr}\{P_i \Omega\} + \frac{1}{2} \text{tr}\{P_i E_i V E_i^T\}, \quad (4.28)$$

$\Omega = E\{z_i(0)z_i^T(0)\}$, and P_i is the solution of the following Lyapunov equation:

$$0 = P_i A_{cl} + P_i^T P + Q_{cl} \quad (4.29)$$

where $A_{cl} = A_{ii} + B_i G_i$, and $Q_{cl} = G_i^T R_i G_i + Q_{z_i}$.

However, P_i is optimized by selecting

$$G_i^* = -R_i^{-1} B_i^T K_{f_i} \quad (4.30)$$

and the optimal control input of the system has the form

$$u_i^{*f}(t) = G_i^* z_i^f(t) \quad (4.31)$$

and K_{f_i} is the positive-definite solution to the ARE:

$$0 = K_{f_i} A_{ii} + A_{ii}^T K_{f_i} - K_{f_i} B_i R_i^{-1} B_i^T K_{f_i} + Q_{z_i}. \quad (4.32)$$

4.3.3 Composite Control

As previously discussed in section 3.3.3, the separate slow and fast control laws are optimal for their respective subsystems. With solutions of the slow and fast subproblems in hand, the near-optimal composite control using the actual states x and z_i is given by

$$u^c = u_i^{*s} + u_i^{*f} \quad (4.33)$$

$$= G_{0i}^* x^s(t) + G_i^* z_i^f(t) \quad (4.34)$$

$$= -[(I_r - R_i^{-1} B_i^T K_{f_i} A_{ii}^{-1} B_i) R_{0i}^{-1} (M_i^T + \tilde{B}_i^T K_{s_i}) + R_i^{-1} B_i^T K_{f_i} A_{ii}^{-1} A_{i0}] x - R_i^{-1} B_i^T K_{f_i} z_i. \quad (4.35)$$

From Lemma 4.3 in the book of Kokotovic et al. [2], the composite control u^c can be rewritten in a more suggestive form to compare it with the exact optimal control in equation (4.12):

$$u^c = -R^{-1} \left[\left(\sum B_i^T K_{m_i}^T \right) x - \sum B_i^T K_{f_i}^T z_i \right] \quad (4.38)$$

where K_s and K_{f_i} are the solutions of the AREs, respectively:

$$0 = K_{s_i} \tilde{A} + \tilde{A}^T K_{s_i} - K_{s_i} \tilde{B}_i R_{0i}^{-1} \tilde{B}_i^T K_{s_i} + Q_{0i} \quad (4.39)$$

$$0 = K_{f_i} A_{ii} + A_{ii}^T K_{f_i} - K_{f_i} B_i R_i^{-1} B_i^T K_{f_i} + Q_{z_i} \quad (4.40)$$

and

$$K_{m_i} = -[K_{s_i} A_{0i} + A_{i0}^T K_{f_i}] x (A_{ii} - B_i R_i^{-1} B_i^T K_{f_i})^{-1} \quad (4.41)$$

Equation (4.38) can be written as

$$u^c = -G_c \chi = -G_x x - \sum G_{z_i} z_i \quad (4.42)$$

where

$$\chi = \begin{bmatrix} x \\ z_1 \\ z_2 \end{bmatrix} \quad (4.43)$$

$$G_x = -R^{-1} \sum B_i^T K_{m_i}^T \quad (4.44)$$

$$G_{z_i} = -R^{-1} B_i^T K_{f_i}^T \quad (4.45)$$

$$G_c = G_x + \sum G_{z_i} \quad (4.46)$$

The closed-loop representation of the composite system is shown in equation (4.47), and the objective here is to minimize the performance index in equation (4.48)

$$\dot{\chi} = A_{cl} \chi + E_c v_c \quad (4.47)$$

$$J_c = \frac{1}{2} E \left\{ \int_0^{\infty} \chi^T Q_{cl} \chi dt \right\} \quad (4.48)$$

where

$$A_{cl} = A_c + B G_c \quad (4.49)$$

$$A_c = \begin{bmatrix} A_{00} & A_{01} & A_{02} \\ A_{10} & A_{11} & 0 \\ A_{20} & 0 & A_{22} \end{bmatrix} \quad (4.50)$$

$$Q_{cl} = Q_c + G_c^T R G_c$$

$$Q_c = \text{BlockDiag}[Q_x \quad Q_{z_1} \quad Q_{z_2}] \quad (4.51)$$

$$E_c v_c = \varepsilon E_\varepsilon v. \quad (4.52)$$

The performance of the composite control can be shown to approximate the optimal performance to within an order of ε^2 [10]. Therefore,

$$J_c = J_{opt} + O(\varepsilon^2).$$

4.4 Kalman Filter

4.4.1 Slow Subsystem Kalman Filter Design

The slow subsystem is given by

$$\frac{d}{dt} x^s(t) = \tilde{A} x^s(t) + \sum [\tilde{B}_i u_i^s(t) + \tilde{E}_i v_i(t)] \quad ; \quad i = 1, 2.$$

$$y_i^s(t) = \tilde{C}_i x^s(t) + \tilde{D}_i u_i^s(t) + \tilde{E}_i v_i(t) + F_i w_i(t)$$

Two estimates are needed for the main system states x^s using two observers (Kalman filters) connected to each subsystem. The observed system is

$$\dot{\hat{x}}_i^s = \tilde{A} \hat{x}_i^s + \tilde{B}_i u_i^s + H(y_i^s - \hat{y}_i^s)$$

$$\hat{y}_i^s = E\{y_i^s\} = E\{\tilde{C}_i x^s + \tilde{D}_i u_i^s + w_i\} = \tilde{C}_i E_i\{x^s\} + \tilde{D}_i u_i^s \quad (4.53)$$

$$= \tilde{C}_i \hat{x}_i^s + \tilde{D}_i u_i^s.$$

where \hat{x}_i^s is the estimate of the slow variable of the main system x through the Kalman filter connected to subsystem i , and \hat{y}_i^s is the estimated output.

The estimation error is defined as $e = x - \hat{x}$, so the error dynamics of the system are as follows:

$$\dot{e}_i = \dot{x}^s - \dot{\hat{x}}_i^s = \tilde{A}x^s + \tilde{B}_i u_i^s + \tilde{E}_i v_i(t) - \left(\tilde{A}\hat{x}_i^s + \tilde{B}_i u_i^s + H(y_i^s - \hat{y}_i^s) \right) \quad (4.54)$$

$$= \tilde{A}(x^s - \hat{x}_i^s) + v_i - H(y_i^s - \hat{y}_i^s) \quad (4.55)$$

$$= \tilde{A}e_i - H(y_i^s) + H(\hat{y}_i^s) + v_i \quad (4.56)$$

$$= \tilde{A}e_i - H(\tilde{C}_i x^s + \tilde{D}_i u_i^s + w_i) + H(\tilde{C}_i \hat{x}_i^s + \tilde{D}_i u_i^s) + v_i \quad (4.57)$$

$$= \tilde{A}e_i - H\tilde{C}_i x^s + H\tilde{C}_i \hat{x}_i^s - H\tilde{D}_i u_i^s + H\tilde{D}_i u_i^s + (v_i - Hw_i) \quad (4.58)$$

$$= \tilde{A}e_i - H\tilde{C}_i e_i + (v_i - Hw_i) \quad (4.59)$$

$$= (\tilde{A} - H\tilde{C}_i)e_i + (v_i - Hw_i). \quad (4.60)$$

where \dot{e}_i is the error of the estimated state x using the Kalman filter connected to subsystem i . The second term of equation (4.60), $v_i - Hw_i$, is considered as an external input.

The symmetric error covariance is defined as

$$\Sigma = E(e_i e_i^T) \quad (4.61)$$

$$\dot{\Sigma} = E(\dot{e}_i e_i^T + e_i \dot{e}_i^T) \quad (4.62)$$

$$= E\left((\tilde{A} - H\tilde{C}_i)e_i e_i^T + (v_i - Hw_i) e_i^T + e_i e_i^T (\tilde{A}^T - \tilde{C}_i^T H^T) + e_i (v_i^T - w_i^T H^T) \right). \quad (4.63)$$

The last term is expanded using the following property:

$$e(t) = e^{(\tilde{A} - H\tilde{C}_i)t} e(0) + \int_0^t e^{(\tilde{A} - H\tilde{C}_i)(t-\tau)} (v_i(\tau) - Hw_i(\tau)) d\tau \quad (4.64)$$

Therefore,

$$\begin{aligned}
E \left(e_i (v_i^T - w_i^T H^T) \right) &= e^{(\tilde{A} - H\tilde{C}_i)t} E \left(e(0) (v_i^T - w_i^T H^T) \right) + \int_0^t e^{(\tilde{A} - H\tilde{C}_i)(t-\tau)} E \left((v_i(\tau) - Hw_i(\tau)) \right. \\
&\quad \left. (v_i^T(t) - w_i^T(t)H^T) \right) d\tau \\
&= \int_0^t e^{(\tilde{A} - H\tilde{C}_i)(t-\tau)} E \left((v_i(\tau) - Hw_i(\tau)) (v_i^T(t) - w_i^T(t)H^T) \right) d\tau \\
&= \int_0^t e^{(\tilde{A} - H\tilde{C}_i)(t-\tau)} (V_i \delta(t-\tau) + HW_i H^T \delta(t-\tau)) d\tau \\
&= \frac{1}{2} V_i + \frac{1}{2} HW_i H^T. \tag{4.65}
\end{aligned}$$

The last expression for $E \left(e_i (v_i^T - w_i^T H^T) \right)$ is symmetric, and since it appears in equation (4.63) twice, that leads to

$$\dot{\Sigma} = (\tilde{A} - H\tilde{C}_i)\Sigma + \Sigma(\tilde{A} - H\tilde{C}_i)^T + V_i - HW_i H^T \tag{4.66}$$

Setting $\dot{\Sigma}$ to zero yields

$$0 = (\tilde{A} - H\tilde{C}_i)\Sigma + \Sigma(\tilde{A} - H\tilde{C}_i)^T + V_i - HW_i H^T. \tag{4.67}$$

For this well-known Lyapunov equation, minimization is achieved by choosing the following:

$$H = S\tilde{C}_i^T W_i^{-1} \text{ (optimal state estimation gain)} \tag{4.68}$$

where S is a PSD solution of the filter Riccati equation (FRE):

$$0 = \tilde{A}S + S\tilde{A}^T + V_i - S\tilde{C}_i^T W_i^{-1} \tilde{C}_i S. \tag{4.69}$$

4.4.2 Fast Subsystem Kalman Filter Design

The fast subsystem obtained in section 4.3.2 is given by

$$\varepsilon \frac{dz_i^f}{dt} = A_{ii} z_i^f + B_i u_i^f + E_i v_i$$

$$y_i^f = C_i z_i^f + F_i w_i$$

Based on this fast subsystem shown above, the optimal estimator for z_i^f is given by the following

Kalman filter design:

$$\varepsilon \dot{z}_i^f = A_{ii} z_i^f + B_i u_i^f + H(y_i^f - \hat{y}_i^f) \quad (4.70)$$

$$\hat{y}_i^f = E(y_i^f) = E(C_i z_i^f + F_i w_i) = C_i E_i(z_i^f) = C_i \hat{z}_i^f.$$

where H is the solution of the following Lyapunov equation:

$$0 = (A_{ii} - HC_i)\Sigma + \Sigma(A_{ii} - HC_i)^T + V_i - HW_iH^T. \quad (4.71)$$

Minimization is obtained by choosing

$$H = SC_i^T W_i^{-1} \quad (4.72)$$

where S is the PSD solution of the FRE:

$$0 = A_{ii}S + SA_{ii}^T + V_i - SC_i^T W_i^{-1} C_i S. \quad (4.73)$$

4.5 Numerical Example

To demonstrate the numerical behavior of the near optimum design of singularly perturbed LQG regulators, the following interconnected singularly perturbed system is considered and solved using MATLAB:

$$\begin{aligned} \dot{x} &= \begin{bmatrix} -2 & 0 \\ 0 & -3 \end{bmatrix} x + \begin{bmatrix} 1 & -1 \\ -1 & -3 \end{bmatrix} z_1 + \begin{bmatrix} -3 & 2 \\ -4 & -2 \end{bmatrix} z_2 \\ \varepsilon \dot{z}_1 &= \begin{bmatrix} -1 & 2 \\ -4 & 3 \end{bmatrix} x + \begin{bmatrix} -5 & 4 \\ -3 & -2 \end{bmatrix} z_1 + \begin{bmatrix} 2 \\ -2 \end{bmatrix} u_1 + \begin{bmatrix} 0.1 \\ 0.1 \end{bmatrix} v_1 \\ \varepsilon \dot{z}_2 &= \begin{bmatrix} -1 & 5 \\ -3 & 2 \end{bmatrix} x + \begin{bmatrix} -2 & -1 \\ -2 & -2 \end{bmatrix} z_2 + \begin{bmatrix} -1 \\ 1 \end{bmatrix} u_2 + \begin{bmatrix} 0.1 \\ 0.1 \end{bmatrix} v_2 \\ y_1 &= [0 \quad 1] z_1 + w_1, y_2 = [1 \quad 0] z_2 + w_2 \end{aligned}$$

with $\varepsilon = 0.1$, $V_1 = V_2 = 0.2$, $R_1 = R_2 = 1$, $Q_x = Q_{z_1} = Q_{z_2} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$.

The full-order system has the following optimal feedback gain:

$$G_{full}^* = [-1.202 \quad 0.390 \quad -0.562 \quad 0.562 \quad 0.455 \quad -0.319]$$

with the solution of the Lyapunov equation P :

$$P = \begin{bmatrix} 0.337 & -0.102 & 0.015 & -0.027 & -0.027 & 0.012 \\ -0.102 & 0.327 & -0.004 & -0.003 & 0.025 & -0.024 \\ 0.015 & -0.004 & 0.011 & -0.003 & -0.001 & 0.001 \\ -0.017 & -0.003 & -0.003 & 0.025 & -0.001 & 0.001 \\ -0.027 & 0.025 & -0.001 & -0.001 & 0.032 & -0.012 \\ 0.012 & -0.024 & 0.002 & 0.001 & -0.012 & 0.021 \end{bmatrix}$$

The full-order system optimal cost is

$$J_{optimal}^* = 0.298712$$

The optimal feedback gain for the composite system is

$$F_c^* = [-1.088 \quad 0.523 \quad -0.270 \quad 0.545 \quad 0.430 \quad -0.302]$$

with the solution of the Lyapunov equation P for the closed-loop composite system given by

$$P = \begin{bmatrix} 0.260 & -0.006 & 0.018 & -0.022 & -0.005 & 0.003 \\ -0.056 & 0.192 & -0.002 & 0.015 & 0.003 & -0.012 \\ 0.018 & -0.002 & 0.019 & 0.003 & 0.001 & -0.001 \\ -0.022 & 0.015 & 0.003 & 0.029 & -0.001 & 0.001 \\ -0.005 & 0.003 & 0.001 & -0.001 & 0.017 & -0.012 \\ 0.003 & -0.012 & -0.001 & 0.001 & -0.012 & 0.028 \end{bmatrix}$$

and the closed loop composite system optimal total cost is $J_{composite}^* = 0.309377$ with system poles as follows:

$$p = \begin{bmatrix} -35.78 \\ -21.70 + 21.88i \\ -21.70 - 21.88i \\ -3.07 + 12.95i \\ -3.07 - 12.95i \\ -13.38 \end{bmatrix}$$

The response of the output and states responses of the system are shown in Figures 4.2 and 4.3, respectively, where the dotted line represents the high-order optimal response, while the solid line represents the near-optimum system. The optimal cost of the composite feedback control is very close to that of the full-order system within the order of (ϵ^2).

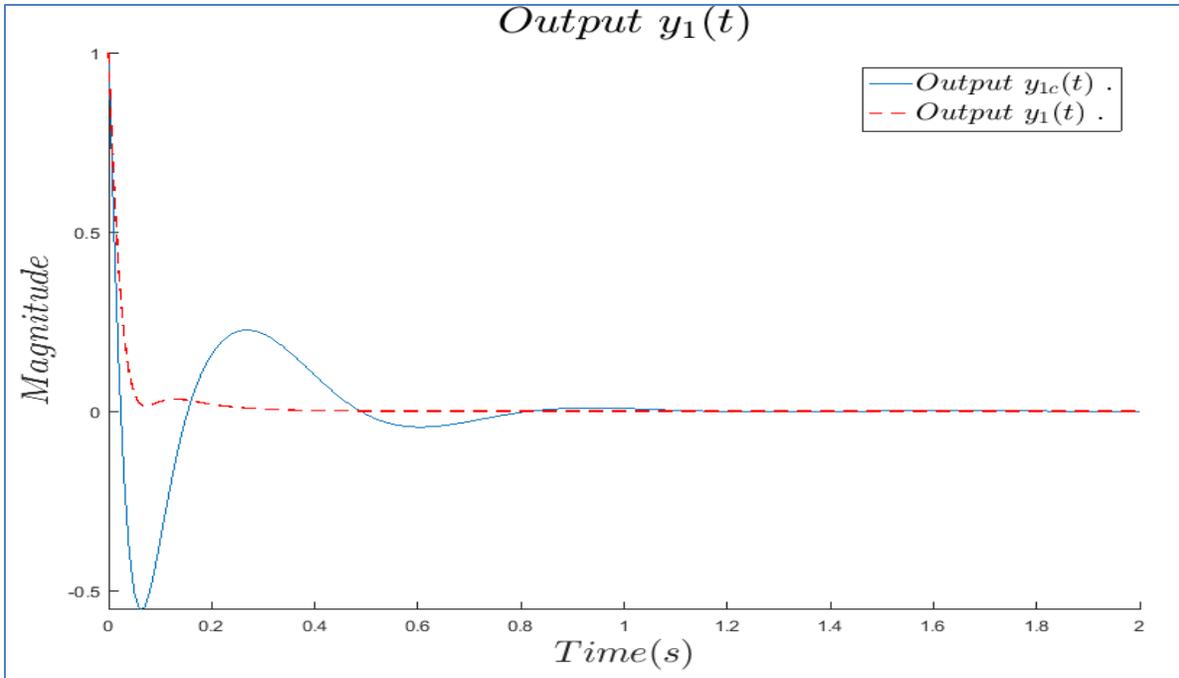


Figure 4.2. Output response for composite state feedback controller.

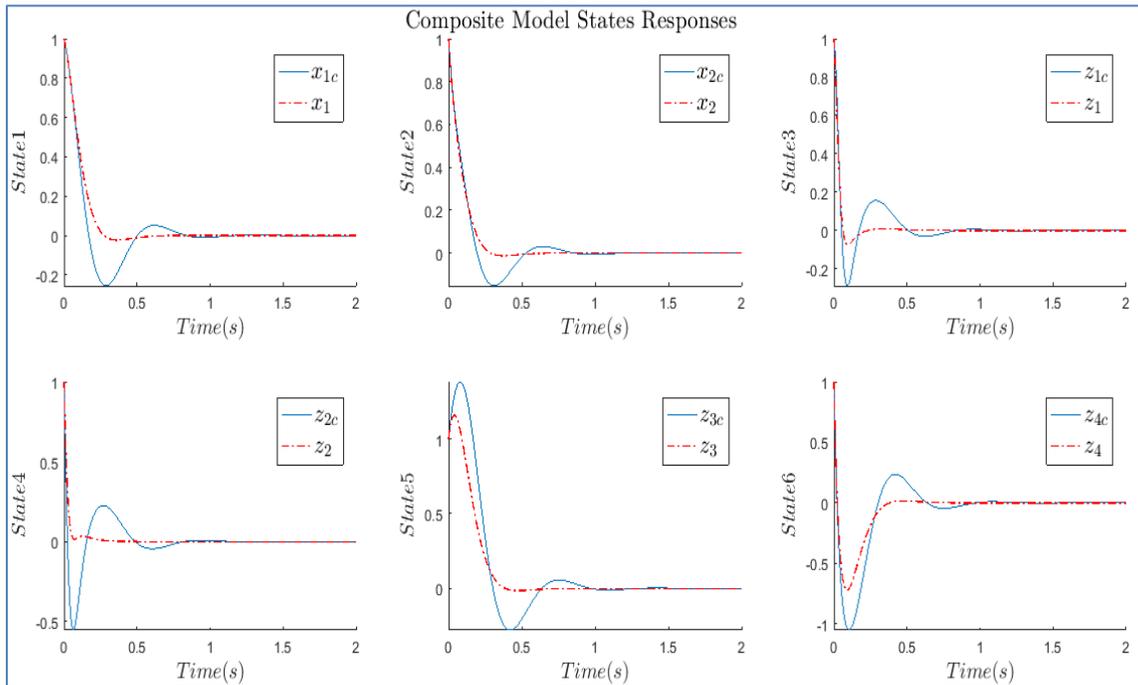


Figure 4.3. States responses for composite state feedback controller.

CHAPTER 5

ROBUST STABILITY OF INTERCONNECTED SINGULARLY PERTURBED SYSTEMS

5.1 Problem Formulation

5.1.1 System Description

Consider the slow (reduced) order interconnected singularly perturbed stochastic model discussed in Chapter 4, and subject to unknown deterministic noise and norm-bounded nonlinear uncertainties in the following form:

$$\begin{aligned} \frac{d}{dt}x^s &= \tilde{A}x^s + \sum [\tilde{B}_i u_i^s + \tilde{E}_i v_i] + \Gamma_i d_i + \Delta(x^s, u_i^s); \quad i = 1, 2. \\ y_i^s &= \tilde{C}_i x^s + \tilde{D}_i u_i^s + \bar{\bar{E}}_i v_i + F_i w_i \end{aligned} \quad (5.1)$$

where

$x^s \in \mathcal{R}^n$ is the slow state variable of the slow subsystem;

$u_i^s \in \mathcal{R}^{q_i}$ are the slow control input vectors;

$y_i^s \in \mathcal{R}^{r_i}$ are the slow control output vectors;

v_i and $w_i \in \mathcal{R}^{k_i}$ are disturbance input and output measurement noise to subsystem I ;

$d_i \in \mathcal{R}^m$ is unknown deterministic square-integrable disturbance input with finite energy,

i.e. $\int_{-\infty}^{\infty} d_i^T(t) d_i(t) < \infty$;

$\Delta(x^s, u_i^s)$ is nonlinear matrix representing norm-bounded uncertainty.

and $\tilde{A} = A_{00} - A_{01}A_{11}^{-1}A_{10} - A_{02}A_{22}^{-1}A_{20}$,

$$\tilde{B}_i = -A_{0i}A_{ii}^{-1}B_i,$$

$$\tilde{E}_i = -A_{0i}A_{ii}^{-1}E_i,$$

$$\tilde{C}_i = -C_iA_{ii}^{-1}A_{i0},$$

$$\tilde{D}_i = -C_iA_{ii}^{-1}B_i,$$

$$\bar{\bar{E}}_i = -C_iA_{ii}^{-1}E_i.$$

The admissible nonlinear uncertainty term is assumed to be norm-bounded for some constant numbers $\varepsilon_1 \geq 0$, $\varepsilon_{i+1} \geq 0$:

$$\|\Delta(x^s, u_i^s)\| \leq \varepsilon_1 \|x^s\| + \varepsilon_{i+1} \|u_i^s\| \quad (5.2)$$

$$\forall x^s \in \mathcal{R}^n, u_i^s \in \mathcal{R}^{q_i}.$$

The corresponding uncertainty set is denoted by

$$\Theta(x^s, u_i^s) = \{\Delta(x^s, u_i^s) : \|\Delta(x^s, u_i^s)\| \leq \varepsilon_1 \|x^s\| + \varepsilon_{i+1} \|u_i^s\|\}. \quad (5.3)$$

Remark 1: For general uncertain systems with nonlinear uncertainty or inaccurate parameter specifications, modeling the disturbance input as a stochastic process may not always be efficient and adequate. Another way for modelling the disturbance is to consider it as some unknown signal of bounded magnitude. The aim of the controller design then is to minimize the energy gain from the disturbance input to a certain error signal. Hence, v_i in the model equation (5.1) is neglected by setting $E_i = 0$, and d_i is introduced as unknown deterministic disturbance of bounded magnitude.

Remark 2: The matrix $\Delta(x^s, u_i^s)$ contains the uncertain parameters in the slow state and slow input matrices of the system equation (5.1). The scalars ε_1 and ε_{i+1} determine how the uncertain parameters in $\Delta(x^s, u_i^s)$ affect the nominal matrices of the system in equation (5.1).

5.1.2 Objective

The effect of uncertainties or neglected nonlinearities on the stability of the closed-loop plant is one of the important issues in the control of dynamic systems. This chapter studies the problem of robust H_∞ control of a class of uncertain interconnected singularly perturbed systems. The class of problem considered in the model equation (5.1) consists of linear reduced-order (slow) interconnected SPSs and two types of uncertainties: deterministic worst-case noise inputs, and norm-bounded nonlinear uncertainty on both the slow state and slow control inputs. It has been

shown that the type of nonlinear uncertainty set considered in this chapter has an equivalent representation by the linear uncertainty set. The main objective here is to design a state feedback control law to achieve a robust stability and guarantee a prespecified H_∞ disturbance attenuation constraint for all admissible uncertainty. In other words, the aim of the design is to make the closed-loop interconnected system asymptotically stable while attenuating the effect of the disturbance.

5.2 Main Results

Consider the state feedback control law $u_i^s = -G_i x^s$, where $G_i \in \mathcal{R}^{n \times q_i}$ are constant matrices for system equation (5.1). Then the problem can be formulated as designing a controller u_i^s such that the following apply:

- (a) The closed-loop system is globally asymptotically stable about the origin in the Lyapunov sense for all admissible nonlinear uncertainties, i.e., all $\Delta(x^s, u_i^s) \in \Theta(x^s, u_i^s)$.
- (b) Under the assumption of zero initial condition, there exists $0 \leq \gamma < \infty$ such that the performance bound is

$$\int_0^\infty \{[x^s]^T Q_0 x^s + 2[x^s]^T M_i u_i^s + [u_i^s]^T R_{0i} u_i^s\} dt \leq \gamma^2 \int_0^\infty \{d_i^T R_d d_i\} dt \quad (5.4)$$

where $Q_0 = Q_x + A_{i0}^T A_{ii}^{-T} Q_{zi} A_{ii}^{-1} A_{i0}$,

$$M_i = B_i^T A_{ii}^{-T} Q_{zi} A_{ii}^{-1} A_{i0},$$

$$R_{0i} = R_i + B_i^T A_{ii}^{-T} Q_{zi} A_{ii}^{-1} B_i.$$

and $R_d > 0$, is satisfied for all $d_i \in L^2[0, \infty)$. Hence, the controller is said to achieve robust disturbance attenuation.

The following quadratic Lyapunov function, which is commonly used to investigate robust stability [35-37], is defined as

$$V_i(x) = [x^s]^T P_i x^s. \quad (5.5)$$

where $P_i > 0$ are to be determined. The Hamiltonian function is defined as

$$H[u_i^s, d_i, \Delta(x^s, u_i^s)] = [x^s]^T Q_{0i} x^s + 2[x^s]^T M_i u_i^s + [u_i^s]^T R_{0i} u_i^s - \gamma^2 d_i^T R_d d_i + \frac{dV_i}{dt} \quad (5.6)$$

where the derivative of $V_i(x)$ is evaluated along the trajectory of the closed-loop system. It is well known in the literature that a sufficient condition to achieve robust disturbance attenuation (2) is

$$H[u_i^s, d_i, \Delta(x^s, u_i^s)] < 0, \quad \forall d_i \in L^2, \Delta(x^s, u_i^s) \in \Theta(x^s, u_i^s). \quad (5.7)$$

Furthermore, $V_i(x)$ under equation (5.7) is a strict radially unbounded Lyapunov function, and hence robust global stability of the closed-loop system is guaranteed. This chapter will involve an investigation of the conditions under which

$$\inf_{u_i^s} \sup_{\Delta(x^s, u_i^s) \in \Theta(x^s, u_i^s)} \sup_{d_i \in L^2} H[u_i^s, d_i, \Delta(x^s, u_i^s)] < 0. \quad (5.8)$$

Lemma 5.1 [38]: For $n \geq m$, suppose $v \in \mathcal{R}^n$ and $u \in \mathcal{R}^m$ have a unit norm, i.e., $\|v\| = 1$, and $\|u\| = 1$. Then, there exists a matrix $M \in \mathcal{R}^{n \times m}$ with $\rho(M) \leq 1$ such that $v = Mu$ and $M^T M = I$.

Proof: The proof for Lemma 5.1 is found in the appendix.

Lemma 5.2 [38]: There exists M_1, M_{i+1} , such that $\Theta(x^s, u_i^s) = \Theta_l(x^s, u_i^s)$, where $\Theta_l(x^s, u_i^s)$ is defined by the following linear uncertainty set

$$\begin{aligned} \Theta_l(x^s, u_i^s) = \{ & \varepsilon_1 M_1 x^s + \varepsilon_{i+1} M_{i+1} u_i^s : M_1 \in \mathcal{R}^{n \times n}, M_{i+1} \in \mathcal{R}^{n \times q_i}, \rho(M_1) \\ & \leq 1, \rho(M_{i+1}) \leq 1 \}. \end{aligned} \quad (5.9)$$

i.e., the nonlinear uncertainty set Θ has an equivalent representation by linear uncertainty sets.

Proof: The proof for Lemma 5.2 is found in the appendix.

Lemma 5.3 (Schur Complement Lemma) [39]: Given constant matrices A_1, A_2 , and A_3 , where $A_1 = A_1^T$ and $A_2 = A_2^T > 0$, then $A_1 + A_3^T A_2^{-1} A_3 < 0$ if and only if

$$\begin{bmatrix} A_1 & A_3^T \\ A_3 & -A_2 \end{bmatrix} < 0, \text{ or equivalently } \begin{bmatrix} -A_2 & A_3 \\ A_3^T & A_1 \end{bmatrix} < 0.$$

Lemma 5.4 [40]: For any matrices X and Y with appropriate dimensions. The following inequality holds for any constant $\mu > 0$:

$$X^T Y + Y^T X \leq \mu X^T X + \frac{1}{\mu} Y^T Y \quad (5.10)$$

which is obtained from the following inequality:

$$\left(\sqrt{\mu} X - \frac{1}{\sqrt{\mu}} Y \right)^T \left(\sqrt{\mu} X - \frac{1}{\sqrt{\mu}} Y \right) \geq 0.$$

Theorem 1: If there exist positive numbers $\mu_1 > 0$, $\mu_{i+1} > 0$, and PD solutions $P_1 > 0$ and $P_2 > 0$ to the following linear matrix inequalities:

$$\begin{pmatrix} Q_{0i} + P_i \tilde{A} + \tilde{A}^T P_i + \frac{\varepsilon_1}{\mu_1} I & \frac{1}{\gamma} P_i \Gamma_i & \sqrt{(\varepsilon_1 \mu_1 + \varepsilon_{i+1} \mu_{i+1})} P_i & P_i \tilde{B}_i \\ \frac{1}{\gamma} \Gamma_i^T P_i & -R_d & 0 & 0 \\ \sqrt{(\varepsilon_1 \mu_1 + \varepsilon_{i+1} \mu_{i+1})} P_i & 0 & -I & M_i \\ \tilde{B}_i^T P_i & 0 & M_i^T & \left(R_{0i} + \frac{\varepsilon_{i+1}}{\mu_{i+1}} I \right) \end{pmatrix} < 0 \quad (5.12)$$

then the control laws

$$u_i^s = G_i x^s \quad (5.13)$$

where

$$G_i = - \left(R_{0i} + \frac{\varepsilon_{i+1}}{\mu_{i+1}} I \right)^{-1} [M_i + P_i \tilde{B}_i]^T$$

achieve both robust global asymptotic stability and robust disturbance attenuation in the sense of

(a) and (b) for all $v_i \in L^2[0, \infty)$ and all $\Delta(x^s, u_i^s) \in \Theta(x^s, u_i^s)$.

Proof: This proof is similar to the one given by Wang and Zhan [38] by showing that equation (5.12) and the control feedback law equation (5.13) will guarantee the inequality of equation (5.7). Noting the definition given in equation (5.6) as

$$H[u_i^s, d_i, \Delta(x^s, u_i^s)] = [x^s]^T Q_0 x^s + 2[x^s]^T M_i u_i^s + [u_i^s]^T R_{0i} u_i^s - \gamma^2 d_i^T R_d d_i + \frac{dV_i}{dt}$$

and according to equation (5.5)

$$V_i(x) = [x^s]^T P_i x^s$$

yields

$$\begin{aligned} H[u_i^s, d_i, \Delta(x^s, u_i^s)] &= [x^s]^T Q_0 x^s + [x^s]^T (P_i \tilde{A} + \tilde{A}^T P_i) x^s - \gamma^2 d_i^T R_d d_i + [x^s]^T P_i \Gamma_i d_i \\ &\quad + d_i^T \Gamma_i^T P_i x^s + [u_i^s]^T R_{0i} u_i^s + 2[x^s]^T M_i u_i^s + [x^s]^T P_i \tilde{B}_i u_i^s + [u_i^s]^T \tilde{B}_i^T P_i x^s \\ &\quad + [x^s]^T P_i \Delta(x^s, u_i^s) + \Delta^T(x^s, u_i^s) P_i x^s \end{aligned}$$

It is easy to verify that the worst-case disturbance occurs when

$$d_i = \gamma^{-2} R_d^{-1} \Gamma_i^T P_i x^s$$

It follows that

$$\begin{aligned} H_1[u_i^s, \Delta(x^s, u_i^s)] &:= \sup_{d_i \in L^2} H[u_i^s, v_i, \Delta(x^s, u_i^s)] \\ &= F(x) + [u_i^s]^T R_{0i} u_i^s + [x^s]^T [2M_i + P_i \tilde{B}_i] u_i^s + [u_i^s]^T \tilde{B}_i^T P_i x^s \\ &\quad + [x^s]^T P_i \Delta(x^s, u_i^s) + \Delta^T(x^s, u_i^s) P_i x^s \end{aligned}$$

where

$$F(x) = [x^s]^T (Q_0 + P_i \tilde{A} + \tilde{A}^T P_i + \gamma^{-2} P_i \Gamma_i R_d^{-1} \Gamma_i^T P_i) x^s$$

According to Lemma 5.2,

$$\sup_{\Delta(x^s, u_i^s) \in \Theta(x^s, u_i^s)} H_1[u_i^s, \Delta(x^s, u_i^s)] = \sup_{\Delta(x^s, u_i^s) \in \Theta_i(x^s, u_i^s)} H_1[u_i^s, \Delta(x^s, u_i^s)]$$

Therefore, it is only necessary to consider

$$H_1[u_i^s, \Delta(x^s, u_i^s)] = F(x) + [u_i^s]^T R_{0i} u_i^s + [x^s]^T [2M_i + P_i \tilde{B}_i] u_i^s + [u_i^s]^T \tilde{B}_i^T P_i x^s \\ + [x^s]^T P_i (\varepsilon_1 M_1 x^s + \varepsilon_{i+1} M_{i+1} u_i^s) + (\varepsilon_1 M_1 x^s + \varepsilon_{i+1} M_{i+1} u_i^s)^T P_i x^s$$

Now, using Lemma 5.3, it is trivial to show that for any constants $\mu_1 > 0$ and $\mu_{i+1} > 0$,

$$[x^s]^T P_i \varepsilon_1 M_1 x^s + (\varepsilon_1 M_1 x^s)^T P_i x^s \leq \varepsilon_1 \mu_1 [x^s]^T P_i^2 x^s + \frac{\varepsilon_1}{\mu_1} [x^s]^T x^s$$

and

$$[x^s]^T P_i \varepsilon_{i+1} M_{i+1} u_i^s + (\varepsilon_{i+1} M_{i+1} u_i^s)^T P_i x^s \leq \varepsilon_{i+1} \mu_{i+1} [x^s]^T P_i^2 x^s + \frac{\varepsilon_{i+1}}{\mu_{i+1}} [u_i^s]^T u_i^s$$

Consequently,

$$\sup_{\Delta(x^s, u_i^s) \in \Theta(x^s, u_i^s)} H_1[u_i^s, \Delta(x^s, u_i^s)] \leq F(x) + (\varepsilon_1 \mu_1 + \varepsilon_{i+1} \mu_{i+1}) [x^s]^T P_i^2 x^s + \frac{\varepsilon_1}{\mu_1} [x^s]^T x^s \\ + [u_i^s]^T \left(R_{0i} + \frac{\varepsilon_{i+1}}{\mu_{i+1}} I \right) u_i^s + [x^s]^T [2M_i + P_i \tilde{B}_i] u_i^s + [u_i^s]^T \tilde{B}_i^T P_i x^s$$

The optimal feedback control law, which minimizes the right-hand side is

$$u_i^s = - \left(R_{0i} + \frac{\varepsilon_{i+1}}{\mu_{i+1}} I \right)^{-1} [M_i + P_i \tilde{B}_i]^T x^s$$

As a result,

$$\inf_{u_i^s} \sup_{\Delta(x^s, u_i^s) \in \Theta(x^s, u_i^s)} H_1[u_i^s, \Delta(x^s, u_i^s)] \leq F(x) + (\varepsilon_1 \mu_1 + \varepsilon_{i+1} \mu_{i+1}) [x^s]^T P_i^2 x^s + \frac{\varepsilon_1}{\mu_1} [x^s]^T x^s \\ - [x^s]^T M_i \left(R_{0i} + \frac{\varepsilon_{i+1}}{\mu_{i+1}} I \right)^{-1} M_i^T x^s - [x^s]^T [2M_i + P_i \tilde{B}_i] \left(R_{0i} + \frac{\varepsilon_{i+1}}{\mu_{i+1}} I \right)^{-1} \tilde{B}_i^T P_i x^s$$

which holds for every x^s ; hence, the following Riccati inequality is reached:

$$Q_0 + P_i \tilde{A} + \tilde{A}^T P_i + P_i \Gamma_i R_d^{-1} \Gamma_i^T P_i + (\varepsilon_1 \mu_1 + \varepsilon_{i+1} \mu_{i+1}) P_i^2 + \frac{\varepsilon_1}{\mu_1} I - M_i \left(R_{0i} + \frac{\varepsilon_{i+1}}{\mu_{i+1}} I \right)^{-1} M_i^T \\ - P_i \tilde{B}_i \left(R_{0i} + \frac{\varepsilon_{i+1}}{\mu_{i+1}} I \right)^{-1} \tilde{B}_i^T P_i < 0$$

This can be rewritten as

$$\begin{aligned}
& Q_0 + P_i \tilde{A} + \tilde{A}^T P_i + \frac{\varepsilon_1}{\mu_1} I + P_i (\gamma^{-2} \Gamma_i R_d^{-1} \Gamma_i^T + (\varepsilon_1 \mu_1 + \varepsilon_{i+1} \mu_{i+1}) I) \\
& - P_i^{-1} M_i \left(R_{0i} + \frac{\varepsilon_{i+1}}{\mu_{i+1}} I \right)^{-1} M_i^T P_i^{-1} - \tilde{B}_i \left(R_{0i} + \frac{\varepsilon_{i+1}}{\mu_{i+1}} I \right)^{-1} \tilde{B}_i^T P_i < 0
\end{aligned}$$

Applying the Schur complement, Riccati inequalities are transformed into the LMI form. If there exist positive-definite solutions $P_i > 0$ to the linear matrix inequalities:

$$\begin{pmatrix}
Q_0 + P_i \tilde{A} + \tilde{A}^T P_i + \frac{\varepsilon_1}{\mu_1} I & \frac{1}{\gamma} P_i \Gamma_i & \sqrt{(\varepsilon_1 \mu_1 + \varepsilon_{i+1} \mu_{i+1})} P_i & P_i \tilde{B}_i \\
\frac{1}{\gamma} \Gamma_i^T P_i & -R_d & 0 & 0 \\
\sqrt{(\varepsilon_1 \mu_1 + \varepsilon_{i+1} \mu_{i+1})} P_i & 0 & -I & M_i \\
\tilde{B}_i^T P_i & 0 & M_i^T & \left(R_{0i} + \frac{\varepsilon_{i+1}}{\mu_{i+1}} I \right)
\end{pmatrix} < 0$$

then,

$$H[u_i^s, d_i, \Delta(x^s, u_i^s)] < 0$$

for all $d_i \in L^2[0, \infty)$ and $\Delta(x^s, u_i^s) \in \Theta(x^s, u_i^s)$. \square

5.3 Numerical Example

Consider the singularly perturbed system discussed previously in section (4.5). The reduced order model is given by

$$\dot{x}^s = \begin{bmatrix} -2.71 & -4.31 \\ 3.56 & -10.76 \end{bmatrix} x^s + \begin{bmatrix} 0.59 \\ 2.24 \end{bmatrix} u_1^s + \begin{bmatrix} 0 \\ 0.1 \end{bmatrix} d_1 + \Delta(x^s, u_1^s) \quad (5.14)$$

$$y_1^s = [-1 \quad 0.53] x^s - 0.94 u_1^s$$

Let $\varepsilon_i = \mu_i = 0.01$ for $i = 1, 2$. Also, consider $\gamma = 0.1$ as the performance bound. Solving equation (5.12) using LMITOOL in MATLAB gives PD solutions $P_1 > 0$, $P_2 > 0$ as follows:

$$\begin{aligned}
P_1 &= \begin{bmatrix} 2.229 & 0.014 \\ 0.014 & 1.083 \end{bmatrix}, \\
P_2 &= \begin{bmatrix} 2.361 & -3.124 \\ -3.124 & 4.741 \end{bmatrix}.
\end{aligned}$$

The required state feedback control law is given by

$$u_i^s = G_i x^s$$

where the feedback gains are

$$G_1 = [-0.877 \quad -0.547]$$

$$G_2 = [-0.358 \quad 1.537]$$

Robust stability and disturbance attenuation of the slow (reduced-order) dynamics have been depicted for two different scenarios to see how each contributes to the closed-loop system. The first scenario considered is for the nonlinear uncertainty only ($d_1 = 0$), as shown in Figures 5.1 and 5.2. The second scenario where both disturbance (Gaussian noise) and nonlinear uncertainty are present is shown in Figures 5.3 and 5.4. Therefore, it can be concluded that system equation (5.14) can be stabilized by the control law equation (5.13).

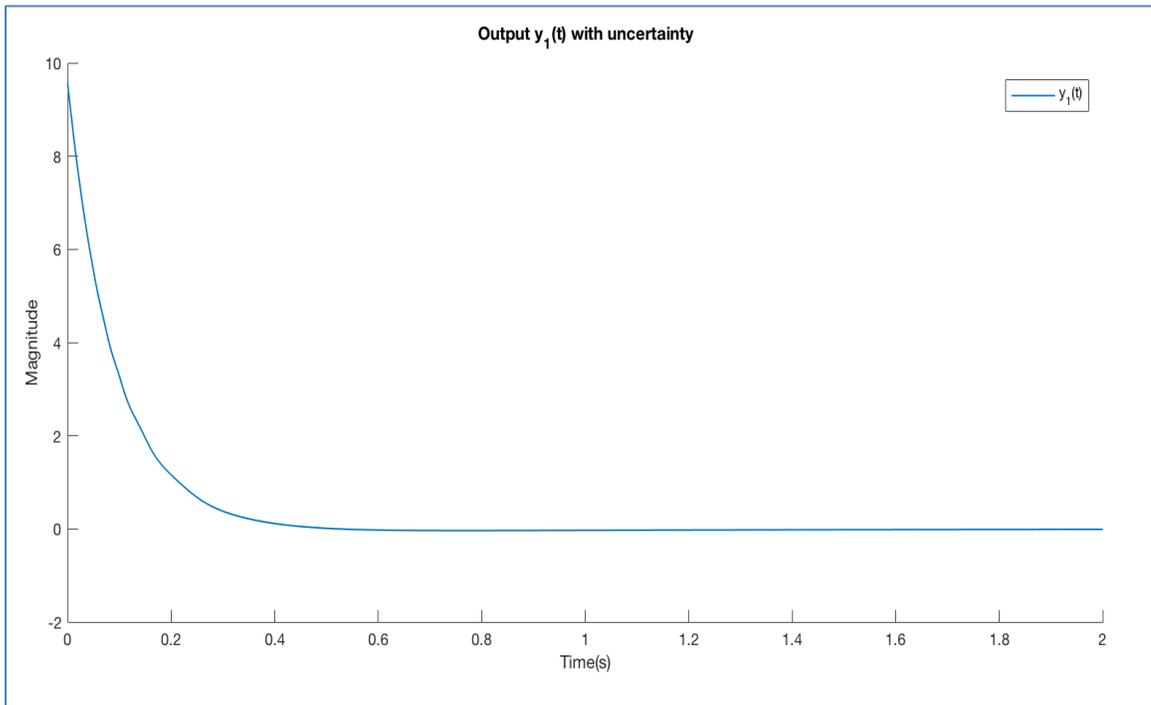


Figure 5.1. Output response for state feedback controller with uncertainty.

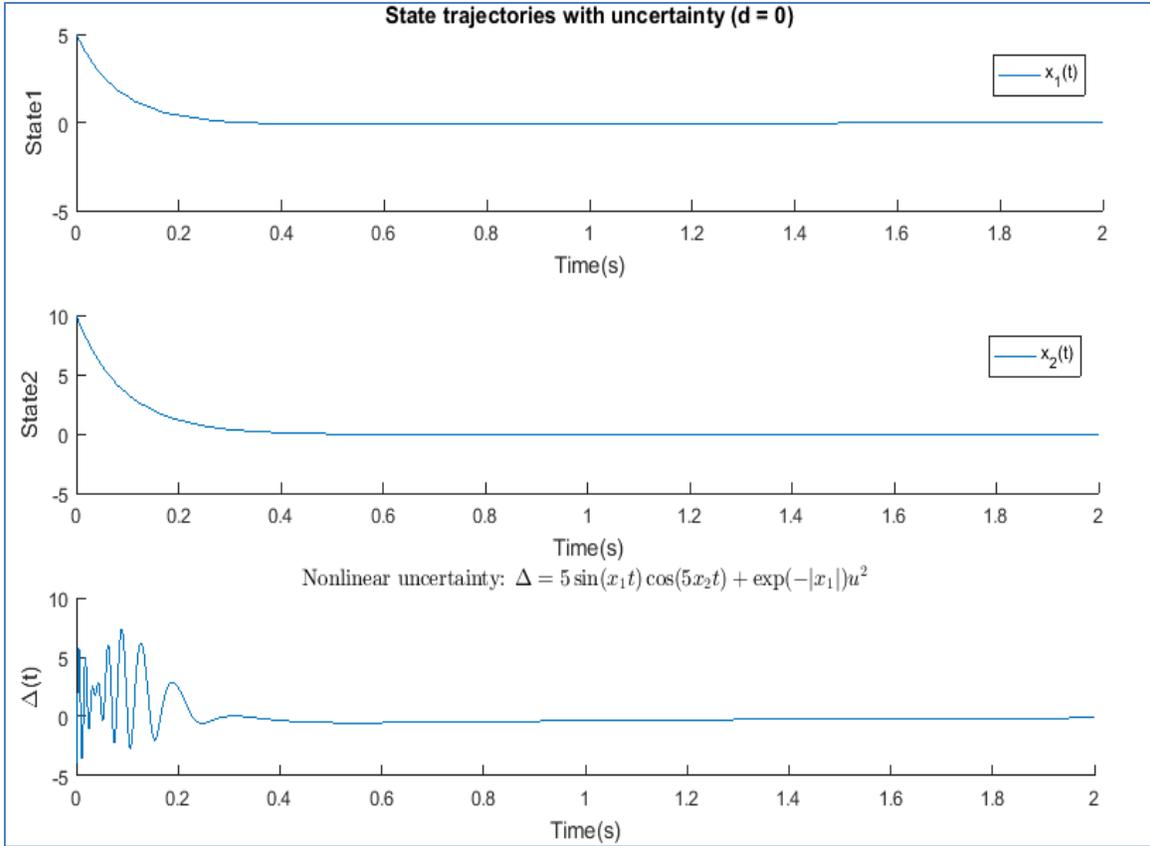


Figure 5.2. States responses for state feedback controller with uncertainty.

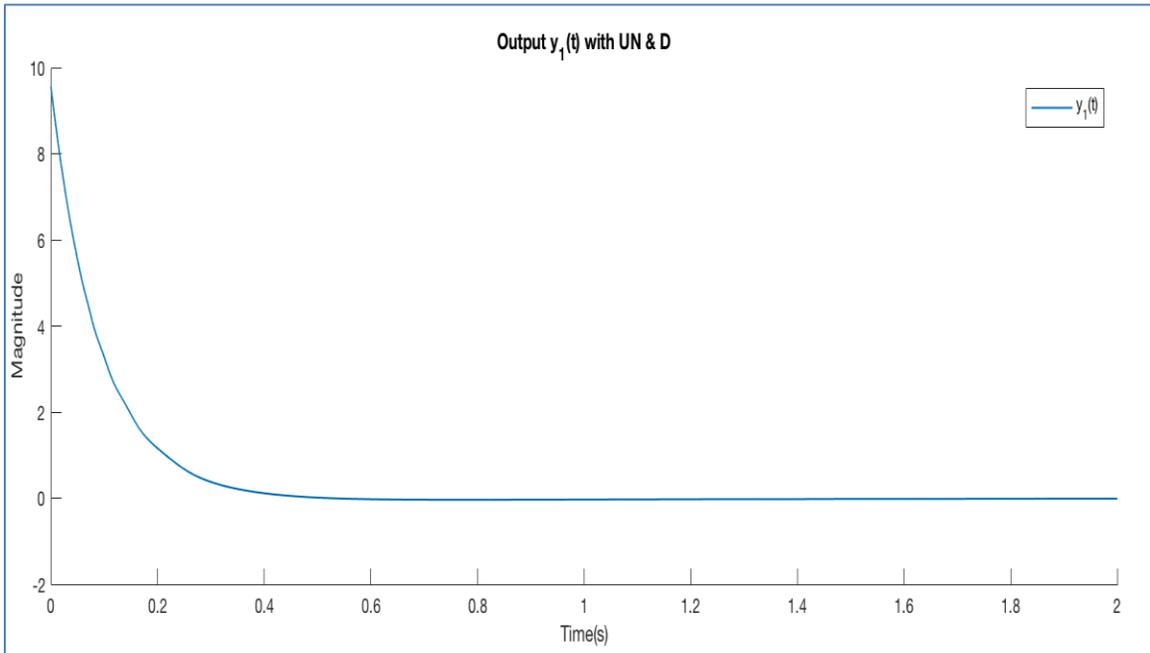


Figure 5.3. Output response for state feedback controller with uncertainty and disturbance.

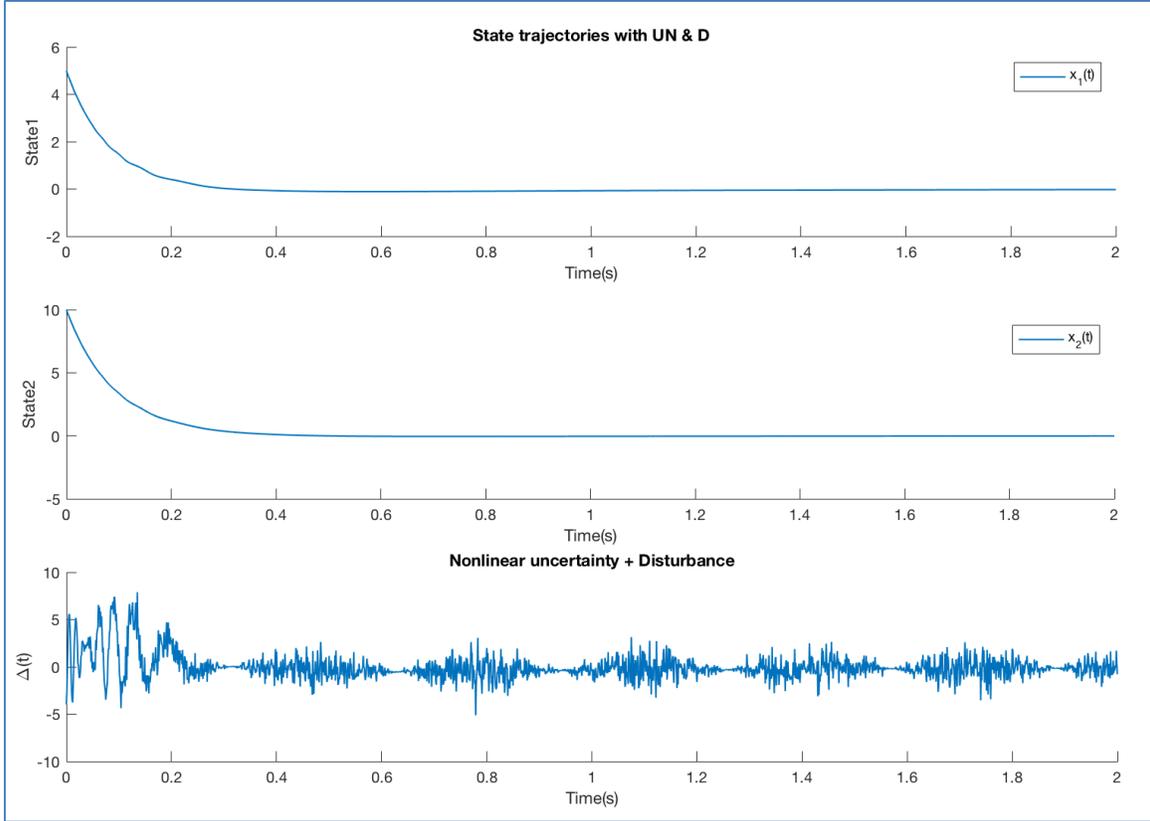


Figure 5.4. States responses for state feedback controller with uncertainty and disturbance.

To use feedback gains G_1 and G_2 to stabilize the full system, zeros following them must be added to each in order to match dimensions of the full-order model, as shown in equation (5.15).

$$G_{full,i} = [G_i \quad 0 \quad 0 \quad 0 \quad 0] \quad (5.15)$$

Feedback gains for the full-order system are

$$G_1 = [-0.877 \quad -0.547 \quad 0 \quad 0 \quad 0 \quad 0]$$

$$G_2 = [-0.358 \quad 1.537 \quad 0 \quad 0 \quad 0 \quad 0]$$

and the close loop poles are

$$p = \begin{bmatrix} -35.78 \\ -21.70 + 21.89i \\ -21.70 - 21.89i \\ -3.07 + 12.95i \\ -3.07 - 12.95i \\ -13.38 \end{bmatrix}$$

Figures 5.5 and 5.6 show the robust stability of the output response and states responses for the full-order system, respectively.

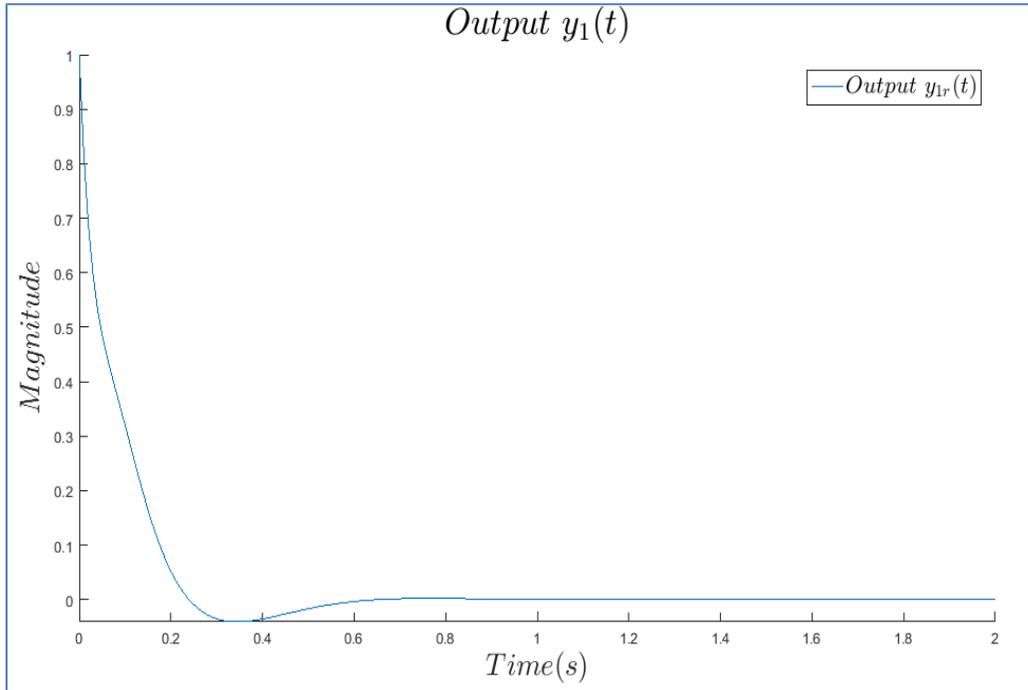


Figure 5.5. Robust output response for state feedback controller.

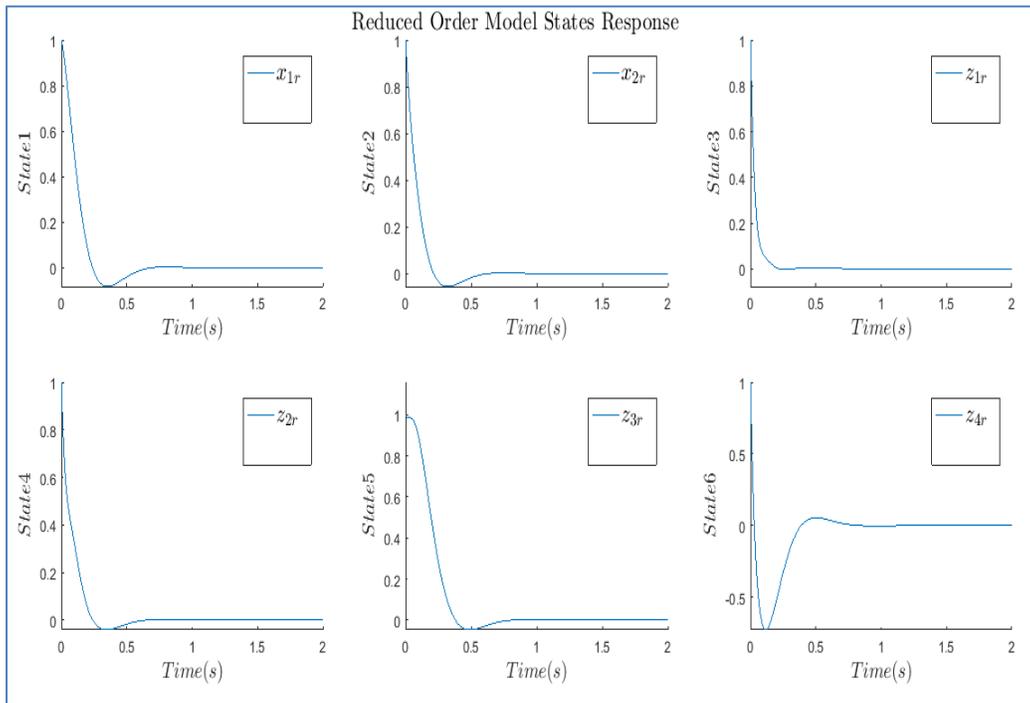


Figure 5.6. Robust states responses for state feedback controller.

CHAPTER 6

CONCLUSION AND FUTURE WORK

6.1 Conclusion

Large-scale systems do arise with great importance in many real-world physical processes. Analysis and design of large-scale systems is a difficult task due to the complex nature of interconnected systems. This dissertation addressed mainly the stability and optimization issues using decentralized control schemes for a class of interconnected large-scale systems described by linear time-invariant singularly perturbed ordinary differential equations. The systems under consideration were assumed to be either deterministic (Chapter 3) or stochastic (always a practical assumption) (Chapter 4), where the problem involved two types of uncertainties: additive white noise and unmodeled fast dynamics in the open-loop system. A common factor of these systems is that both classes of problems studied exhibit the TTS behavior and, therefore, a singular perturbation theory is utilized. Singular perturbation methodology is applied to facilitate the design of feedback control laws by separating the TTS behavior of the original systems into slow and fast dynamic subsystems. Based on these subsystems, lower-order suboptimal stabilizing controllers associated with certain quadratic performance indices (LQR/LQG) were designed. These suboptimal controllers for *slow* and *fast* subsystems were then combined to form composite feedback control laws that guarantee near-optimal performance indices for the full-order system. The main advantage is that these suboptimal controllers alleviate high dimensionality and the numerical ill-conditioning associated with computing full-order solutions to original large-scale systems.

Since fast systems are interconnected internally with the main system, a decentralized control scheme was adopted by designing an independent and separate controller in each system.

This decentralized control setup does not only reduce the order of the designed controller and implementation cost but also adds more reliability, such that if a controller has a problem, then the rest can still function and support the failure control channel. The basic assumption in the theory developed was that the states are all available for measurement. Also, an observer-based controller was designed using a Kalman filter at each subsystem level to estimate the states for the class of stochastic systems because they are often not accessible or corrupted by measurement noise. The Kalman filter is an optimal estimator that provides the best state estimation by inferring system states from indirect, inaccurate, and uncertain observations.

The stability of the closed-loop system for linear quadratic regulator and linear quadratic Gaussian designs are guaranteed. However, feedback controllers are often designed for a simplified model of the physical plant that does not take into consideration all sources of uncertainty. A robustness analysis to guarantee stability and performance is then necessary to validate the design in the face of plant uncertainties. In Chapter 5, it was assumed that the *slow* subsystem of the main stochastic system is subject to two types of uncertainties: norm-bounded nonlinear uncertainty in both the *slow* state and *slow* control inputs, and unknown disturbance inputs. It was shown that the type of norm-bounded nonlinear uncertainty set considered coincides with a set of linear uncertainty. A robust H_∞ control design methodology was investigated to achieve the robust stabilization and disturbance attenuation of the closed-loop system for all admissible uncertainty using the Hamiltonian approach. The state feedback gain matrices can be determined independently by solving two LMIs.

6.2 Future Research

This dissertation has touched upon an important stabilizing optimal problem dealing with large-scale interconnected singularly perturbed systems. An interesting subject for future research

would be the use of H_∞ control for designing a composite optimal controller and H_∞ filtering methods for state estimation where the statistics of exogenous inputs are not known. Furthermore, the stability robustness when the given system involves some other types of uncertainty, such as unpredictable abrupt structural changes, which could arise due to sudden changes in the environment conditions, component failures, or sensor failures, could be considered for future investigation.

REFERENCES

REFERENCES

- [1] L. Bakule, "Decentralized control: An overview," *Annual Reviews in Control*, vol. 32, no. 1, pp. 87-98, 2008.
- [2] P. Kokotovic, H. K. Khalil, and J. O'Reilly, *Singular perturbation methods in control: analysis and design*, London: Siam, 1999.
- [3] R. J. Veillette, J. Medanic, and W. R. Perkins, "Design of reliable control systems," *IEEE Transactions on Automatic Control*, vol. 37, no. 3, pp. 290-304, 1992.
- [4] W. L. Brogan, *Modern Control Theory*, 3rd ed., Englewood Cliffs, NJ: Prentice-Hall, 1991.
- [5] M. Aoki, *Optimization of Stochastic Systems*, New York: Academic Press, 1967.
- [6] D. Naidu, "Singular perturbations and time scales in control theory and applications: an overview," *Dynamics of Continuous Discrete and Impulsive Systems Series B*, vol. 9, pp. 233-278, 2002.
- [7] D. S. Naidu, *Singular Perturbation Methodology in Control Systems*, London: IET, 1988.
- [8] R. E. O'Malley, *Introduction to Singular Perturbations*, New York: Academic Press, 1974.
- [9] P. V. Kokotovic, R. O'Malley, and P. Sannuti, "Singular perturbations and order reduction in control theory—An overview," *Automatica*, vol. 12, no. 2, pp. 123-132, 1976.
- [10] J. Chow and P. Kokotovic, "A decomposition of near-optimum regulators for systems with slow and fast modes," in *IEEE Conference on Decision and Control including the 15th Symposium on Adaptive Processes*, 1976, pp. 219-224.
- [11] P. Sannuti and P. Kokotovic, "Near-optimum design of linear systems by a singular perturbation method," *IEEE Transactions on Automatic Control*, vol. 14, no. 1, pp. 15-22, 1969.
- [12] V. Saksena, J. O'Reilly, and P. Kokotovic, "Singular perturbations and time-scale methods in control theory: Survey 1976–1983," *Automatica*, vol. 20, no. 3, pp. 273-293, 1984.
- [13] Y. Zhang, D. S. Naidu, C. Cai, and Y. Zou, "Singular Perturbation and time scales in control theories and applications: An overview 2002–2012," *International Journal of Informaiton and Systems Sciences*, vol. 9, no. 1, pp. 1-36, 2014.
- [14] D. S. Naidu and A. J. Calise, "Singular perturbations and time scales in guidance and control of aerospace systems: A survey," *Journal of Guidance, Control, and Dynamics*, vol. 24, no. 6, pp. 1057-1078, 2001.
- [15] A. Haddad and P. Kokotovic, "Stochastic control of linear sigularly perturbed systems," *IEEE Transactions on Automatic Control*, vol. 22, no. 5, pp. 815-821, 1977.

REFERENCES (continued)

- [16] H. Khalil and Z. Gajic, "Near-optimum regulators for stochastic linear singularly perturbed systems," *IEEE Transactions on Automatic Control*, vol. 29, no. 6, pp. 531-541, 1984.
- [17] X. Shen, V. G. Gourishankar, Q. Xia, and M. Rao, "Composite control of discrete singularly perturbed systems with stochastic jump parameters," *Journal of the Franklin Institute*, vol. 331, no. 2, pp. 217-227, 1994.
- [18] H. Oloomi, A. Alyatim, and M. E. Sawan, "Output feedback design of singularly perturbed discrete-time systems," in American Control Conference, 1986.
- [19] Y. J. Kim, B. S. Kim, and M. T. Lim, "Composite control for singularly perturbed bilinear systems via successive Galerkin approximation," *IEE Proceedings-Control Theory and Applications*, vol. 150, no. 5, pp. 483-8, 2003.
- [20] K. C. Yao, "An iteration method to sub-optimal output feedback computer control of decentralized singularly-perturbed systems," in Second International Conference on Innovative Computing, Information and Control, 2007, pp. 102-102.
- [21] G. Zames, "Feedback and optimal sensitivity: Model reference transformations, multiplicative seminorms, and approximate inverses," *IEEE Transactions on Automatic Control*, vol. 26, no. 2, pp. 301-320, 1981.
- [22] J. Doyle, K. Glover, P. Khargonekar, and P. Francis, "State space solutions to standard H-2 and H-infinity," *IEEE Transactions on Automatic Control, AC*, vol. 34, pp. 831-847, 1989.
- [23] I. Petersen, "Disturbance attenuation and H^∞ optimization: A design method based on the algebraic Riccati equation," *IEEE Transactions on Automatic Control*, vol. 32, no. 5, pp. 427-429, 1987.
- [24] Z. Shao and M. Sawan, "Robust stability of singularly perturbed systems," *International Journal of Control*, vol. 58, no. 6, pp. 1469-1476, 1993.
- [25] P. Shi and V. Dragan, " H^∞ control for singularly perturbed systems with parametric uncertainties," in American Control Conference, 1997, pp. 2120-2124.
- [26] Z. Shao and M. Sawan, "Stabilisation of uncertain singularly perturbed systems," *IEE Proceedings-Control Theory and Applications*, vol. 153, no. 1, pp. 99-103, 2006.
- [27] X. Ji, J. Gao, and X. Chen, "Robust H^∞ control for uncertain linear time-delay singularly perturbed systems," in 30th Chinese Control Conference, 2011, pp. 2246-2251.
- [28] I. Hyun, M. Sawan, D. G. Lee, and D. Kim, "Robust stability for decentralized singularly perturbed unified system," in American Control Conference, 2006.

REFERENCES (continued)

- [29] R. E. Kalman, “A new approach to linear filtering and prediction problems,” *Journal of Basic Engineering*, vol. 82, no. 1, pp. 35-45, 1960.
- [30] D. E. Kirk, *Optimal Control Theory: An Introduction*, Englewood Cliffs, NJ: Prentice Hall, 1970.
- [31] B. D. Anderson and J. B. Moore, *Linear Optimal Control*, Englewood Cliffs, NJ: Prentice Hall 1971.
- [32] M. S. Grewal, “Kalman filtering,” *International Encyclopedia of Statistical Science*, pp. 705-708, Berlin, Heidelberg: Springer, 2011.
- [33] V. Lakshmikantham, S. Sivasundaram, and B. Kaymakçalan, *Dynamic Systems on Measure Chains, Mathematics and Its Applications*: Springer Science & Business Media, 1996.
- [34] P. Shi and V. Dragan, “Asymptotic H^∞ control of singularly perturbed systems with parametric uncertainties,” *IEEE Transactions on Automatic Control*, vol. 44, no. 9, pp. 1738-1742, 1999.
- [35] J. Martin, “State-space measures for stability robustness,” *IEEE Transactions on Automatic Control*, vol. 32, no. 6, pp. 509-512, 1987.
- [36] M. Rotea, M. Corless, D. Da, and I. Petersen, “Systems with structured uncertainty: Relations between quadratic and robust stability,” *IEEE Transactions on Automatic Control*, vol. 38, no. 5, pp. 799-803, 1993.
- [37] K. Zhou and P. P. Khargonekar, “Stability robustness bounds for linear state-space models with structured uncertainty,” *IEEE Transactions on Automatic Control*, vol. 32, no. 7, pp. 621-623, 1987.
- [38] L. Wang and W. Zhan, “Robust disturbance attenuation with stability for linear systems with norm-bounded nonlinear uncertainties,” *IEEE Transactions on Automatic Control*, vol. 41, no. 6, pp. 886-888, 1996.
- [39] L. Chen, R. Wang, T. Zhou, and K. Aihara, “Noise-induced cooperative behavior in a multicell system,” *Bioinformatics*, vol. 21, no. 11, pp. 2722-2729, 2005.
- [40] K. Zhou and P. P. Khargonekar, “Robust stabilization of linear systems with norm-bounded time-varying uncertainty,” *Systems & Control Letters*, vol. 10, no. 1, pp. 17-20, 1988.

APPENDIX

APPENDIX
LEMMA PROOFS

Proof Lemma 5.1

Since $v \in \mathcal{R}^n$ and $u \in \mathcal{R}^m$ have unit norms, we can construct orthonormal basis via, say, the classical Gram-Schmidt algorithm as

$$V = [v, v_2, \dots, v_n] \quad \text{for } \mathcal{R}^n$$

and

$$U = [u, u_2, \dots, u_m] \quad \text{for } \mathcal{R}^m$$

Obviously, $V_m := [v, v_2, \dots, v_m]$ satisfies $V_m^T V_m = I$. It follows that $M := V_m U^T$ satisfies $M^T M = I$, which implies $\rho(M) \leq 1$. Since $MU = V_m$, we have $v = Mu$ as required. \square

Proof Lemma 5.2:

It is obvious that

$$\Theta_i(x^s, u_i^s) \subseteq \Theta(x^s, u_i^s) \tag{A.1}$$

Moreover, for any $x^s \in \mathcal{R}^n$ and $u_i^s \in \mathcal{R}^{m_i}$

$$\Theta(x^s, u_i^s) \subseteq \Phi = \{v \in \mathcal{R}^n: \|v\| \leq \varepsilon_1 \|x^s\| + \varepsilon_{i+1} \|u_i^s\|\}. \tag{A.2}$$

Suppose $v \in \Phi$. Then there exists $a_1 \leq \varepsilon_1, a_{i+1} \leq \varepsilon_{i+1}$, such that

$$\|v\| = a_1 \|x^s\| + a_{i+1} \|u_i^s\|. \tag{A.3}$$

Decompose

$$v = \alpha_1 v + \alpha_2 v$$

where

$$a_1 \|x^s\| \tag{A.4}$$

$$\alpha_2 = \frac{a_{i+1} \|u_i^s\|}{\|v\|}$$

Without loss of generality, assume $x^s \neq 0, u_i^s \neq 0$. Then define

$$\tilde{v} = \frac{v}{\|v\|}$$

$$\tilde{x}^s = \frac{x^s}{\|x^s\|}$$

$$\tilde{u}_i^s = \frac{u_i^s}{\|u_i^s\|}$$

By Lemma 5.1, there exists $M_1 \in \mathcal{R}^{n \times n}$ and $M_{i+1} \in \mathcal{R}^{n \times m_i}$ with $\rho(M_1) \leq 1, \rho(M_{i+1}) \leq 1$ such that

$$\tilde{v} = M_1 \tilde{x}^s$$

$$\tilde{v} = M_{i+1} \tilde{u}_i^s$$

which, together with equation (A.4), leads to

$$\alpha_1 v = a_1 M_1 x^s$$

$$\alpha_2 v = a_{i+1} M_{i+1} u_i^s$$

It follows that

$$v = \alpha_1 v + \alpha_2 v$$

$$= a_1 M_1 x^s + a_{i+1} M_{i+1} u_i^s \in \Theta_l(x^s, u_i^s)$$

Since $v \in \Phi$ is arbitrary, it can be concluded that

$$\Phi \subseteq \Theta_l(x^s, u_i^s)$$

which, along with equation (A.2), implies that

$$\Theta(x^s, u_i^s) \subseteq \Theta_l(x^s, u_i^s) \tag{A.5}$$

□